OVERVIEW OF MORET-BAILLY'S THEOREM ON GLOBAL POINTS

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1. Introduction

Let K be a global field and let X_K be a geometrically integral and separated K-scheme of finite type (a "K-variety"). Let Σ be a finite non-empty set of places of K, and suppose we are given local points $x_v \in X_K(K_v)$ for each $v \in \Sigma$. It is natural to ask if we may be able to find $x \in X_K(K)$ that is arbitrarily close to x_v for the natural topology on $X_K(K_v)$ (arising from the topology on K_v) for all $v \in \Sigma$. That is, is the image of the map $X_K(K) \to \prod_{v\in\Sigma} X_K(K_v)$ dense? In general the answer is no because $X_K(K)$ may be empty. But there is a reasonable weakening of the question for which one does not see immediate counterexamples: if K'/K is a finite extension in which each place of Σ is totally split (i.e., $K' \otimes_K K_v$ is a product of copies of K_v, so K'/K is separable) then for each place v' on K' over a place $v \in \Sigma$ we get a map $X_K(K') \to X_K(K'_{v'}) = X_K(K_v)$, and so we can ask if there exists some such K'/K and a point $x' \in X_K(K')$ whose image in $X_K(K_v)$ for each place v' on K' over $v \in \Sigma$ is in a neighborhood of x_v that we specify in advance. This weaker kind of global approximation question turns out to have an affirmative answer when the x_v's lie in the smooth locus $X_K^{\rm sm}$, and this is a consequence of a stronger result that is the main theorem of Moret-Bailly in [8]. We now formulate this main theorem, and then we show how it provides an affirmative answer to the preceeding question (as well as a key result used by Taylor in [10]). In §2ff. below we shall discuss Moret-Bailly's proof.

The setup for Moret-Bailly's theorem goes as follows. We fix a global field K as above and set $B = \operatorname{Spec} R$ for R a ring of S-integers of K (with S a finite non-empty set of places of K that contains all of the archimedean places). We also fix a non-empty finite set of places Σ of K that is a proper subset of S, which is to say that Σ is non-empty and avoids the places coming from B but does not exhaust all places away from B. (In particular, S needs to have at least two elements, so $B = \mathbf{Z}$ is not permitted.) This latter condition is called *incompleteness* of Σ (with respect to B) in [8]. We also give ourselves the following geometric data: a separated surjective map $f : X \to B$ of finite type with irreducible X and geometrically irreducible generic fiber X_K, as well as finite Galois extensions L_v/K_v for each $v \in \Sigma$ and $\operatorname{Gal}(L_v/K_v)$-stable *non-empty* open subsets $\Omega_v \subseteq X_K^{\rm sm}(L_v)$ (in particular $X_K^{\rm sm}$ is non-empty, so X_K is generically K-smooth). For example, if X_K is K-smooth then we can take $L_v = K_v$ and $\Omega_v = X_K(K_v)$ for all $v \in \Sigma$ if these latter sets are non-empty. (In all interesting examples we have $L_v = K_v$, but for technical reasons related to preliminary reduction steps in the main proof we have to allow the generality indicated above.) The main theorem of [8] is:

Theorem 1.1 (Moret-Bailly)**.** *For data $(X \twoheadrightarrow B, \Sigma, \{L_v\}_{v\in\Sigma}, \{\Omega_v\}_{v\in\Sigma})$ as above, there exists an irreducible closed subset $Y \hookrightarrow X$ with Y finite over B such that $Y_K \otimes_K L_v$ is L_v-split for all $v \in \Sigma$ and each of its L_v-points lies in $\Omega_v \subseteq X_K^{\rm sm}(L_v)$.*

Example 1.2. Suppose $X_K^{\rm sm}(K_v)$ is non-empty for all $v \in \Sigma$, and let $L_v = K_v$ and $\Omega_v = X_K^{\rm sm}(K_v)$ for all such v. Taking Y as in Theorem 1.1, the normalization $\widetilde{Y}$ of the finite integral B-flat $Y_{\rm red}$ is $\operatorname{Spec} R'$ with R' the normalization of R in the function field K' of $Y_{\rm red}$. Thus, $K' \otimes_K K_v$ is a product of copies of K_v for each $v \in \Sigma$, so Σ is totally split in K'. By choosing Ω_v smaller, we can arrange that all K_v-points obtained in this way lie in a prescribed region within $X_K^{\rm sm}(K_v)$. The proof of Theorem 1.1 will give no control whatsoever on $[K' : K]$ or on Galois-theoretic properties (such as whether or not the Galois closure of K'/K is solvable

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over K). In fact, the proof of Theorem 1.1 rests on non-constructive compactness arguments, and it realizes Y_K as an irreducible component of a Cartier divisor on a certain K-curve inside of X_K.

Example 1.3. To indicate the crucial role of the incompleteness hypothesis on Σ in Theorem 1.1 (it will be used in two completely different ways in the proof), consider the following example that violates this assumption: take $B = \operatorname{Spec} \mathbf{Z}$, $X = \mathbf{G}_m$, and $\Sigma = \{\infty\}$. Also take $L_\infty = \mathbf{C}$ as an extension of $K_\infty = \mathbf{R}$, and $\Omega_\infty = \{t \in \mathbf{C}^\times \mid |t| < 1\}$. If the conclusion of Theorem 1.1 were satisfied in this case then by the previous example we would get a (totally real) number field K' and an element $u \in \mathbf{G}_m(\mathscr{O}_{K'}) = \mathscr{O}_{K'}^\times$ such that u has all archimedean absolute values < 1. This contradicts the product formula for the integral unit u.

Remark 1.4. Let us now indicate the two ways in which the incompleteness of Σ with respect to B arises in the proof of Theorem 1.1. Recall that $B = \operatorname{Spec} R$ with R the ring of S-integers of K, for S a non-empty finite set of places of K containing the archimedean places, and that Σ is a proper non-empty subset of S. The first role of incompleteness is the following consequence of the strong approximation theorem for the additive group scheme $\mathbf{G}_a$: the inclusion $R \to K_\Sigma \stackrel{\text{def}}{=} \prod_{v \in \Sigma} K_v$ has dense image. To see this denseness, we recall the usual formulation of strong approximation, namely that the inclusion of K into $\mathbf{A}_K/K_{v_0}$ has dense image for any place v_0. (This quotient is just the restricted direct product of K_v's over all places v of K distinct from v_0.) Likewise, K has dense image in $\mathbf{A}_K/(\prod_{v \in S-\Sigma} K_v)$ since $S - \Sigma$ is non-empty, so by considering open product blocks given at places $v \notin S$ by the local valuation ring $\mathscr{O}_v$ and picking open factors arbitrarily at the places of Σ the intersection with K is given by members of R satisfying the specified local conditions in the K_v's for $v \in \Sigma$. This is exactly the denseness of R in K_Σ.

The second consequence of incompleteness that we shall require is that the (typically non-Hausdorff!) topological quotient group $G = K_\Sigma^\times/R^\times$ is quasi-compact. (This is to be contrasted with $K_S^\times/R^\times$ which has a continuous surjective map onto $\mathbf{R}$ via $\log \|\cdot\|_S$, with compact kernel via the S-unit theorem.) To see this quasi-compactness from the S-unit theorem, we argue as follows. Let $H \subseteq G$ be the image of the compact group $(K_S^\times)^{\|\cdot\|_S=1}/R^\times$ under the continuous projection $K_S^\times/R^\times \to K_\Sigma^\times/R^\times$. Since G is the topological quotient of $K_S^\times/R^\times$ by the image of $K_{S-\Sigma}^\times$ and the logarithm map $\log \|\cdot\|_S$ from $K_S^\times/R^\times$ onto $\mathbf{R}$ is a topological quotient map, it follows that the topological quotient G/H is a continuous image of the topological group $\mathbf{R}/\text{image}(\log \|\cdot\|_{S-\Sigma})$ that is visibly quasi-compact (since $S - \Sigma$ is non-empty). Hence, G is a locally compact (usually not Hausdorff) topological group having a quasi-compact subgroup H for which the quotient G/H is quasi-compact. It then follows easily (via the local compactness) that G must be quasi-compact. (In fact, the local compactness can be avoided as input, but one then has to use more subtle topological arguments.)

The work of Taylor [10] uses the following corollary of Theorem 1.1:

Corollary 1.5. *Let X_K be a geometically irreducible and separated K-scheme of finite type, and let Σ be a finite non-empty set of places of K such that $X_K^{\text{sm}}(K_v) \neq \emptyset$ for all $v \in \Sigma$. Let $K^\Sigma \subseteq K^{\text{sep}}$ be the maximal subextension in which Σ is totally split.*

Then $X_K^{\text{sm}}(K^\Sigma)$ is Zariski-dense in X_K in the sense that the set of closed points $x \in X_K$ for which Σ is totally split in $K(x)$ is Zariski-dense in X_K, and for any non-empty open sets $U_v \subseteq X_K^{\text{sm}}(K_v)$ for all $v \in \Sigma$ there exists a K-finite subextension $K' \subseteq K^\Sigma$ and a point $\xi \in X_K(K')$ whose image in $X_K(K'_{v'}) = X_K(K_v)$ lies in U_v for all v' on K' over any $v \in \Sigma$.

In fact, Taylor only uses the Zariski-denseness aspect. However, this will be deduced from the topological denseness aspect.

Proof. The denseness aspect with respect to $\prod_{v \in \Sigma} X_K^{\text{sm}}(K_v) = X_K^{\text{sm}}(K_\Sigma)$ is immediate from Theorem 1.1 by varying Ω_v (with $L_v = K_v$ for all $v \in \Sigma$). To see that this implies the Zariski-denseness, assume otherwise, so there is a proper closed subset $Z_K \subseteq X_K$ such that $(Z_K \cap X_K^{\text{sm}})(K_v) = X_K^{\text{sm}}(K_v)$ for some $v \in \Sigma$ (otherwise take $U_v = (X_K - Z_K)^{\text{sm}}(K_v)$ for all such v). However, $X_K^{\text{sm}}(K_v)$ is a non-empty K_v-analytic manifold with constant dimension $\dim X_K > 0$ (positive since there is a proper closed subset) and Z_K is the zero locus of a non-zero sheaf of ideals, so $(Z_K \cap X_K^{\text{sm}})(K_v)$ has empty interior in the positive-dimensional $X_K^{\text{sm}}(K_v)$. Hence, this overlap cannot equal $X_K^{\text{sm}}(K_v)$ for any $v \in \Sigma$. ∎

2. Reduction steps

We now begin with the proof of Theorem 1.1 by sketching some important preliminary reduction steps (which are treated in detail in [8, §1-§2], though we give some explanations omitted from [8] below). Using notation as in Theorem 1.1, we can assume that the irreducible X is also reduced, so it is integral and hence flat over the Dedekind base B. In particular, if $d = \dim X_K$ then $d = \dim X_b$ for all $b \in B$ (note that all X_b's are non-empty).

There are two basic reduction steps: shrinking X_K and increasing Σ (while correspondingly shrinking B). To shrink X_K, let $U_K \subseteq X_K$ be a non-empty open subset and let $Z_K = (X_k - U_K)_{\rm red}$ be the proper closed complement. The closure Z of Z_K in X_K is B-flat (though some fibers Z_b may vanish), so when Z_b is non-empty we have $\dim Z_b = \dim Z_K < \dim X_K = \dim X_b$; this inequality also holds if $Z_b = \emptyset$. Hence, the open subset $U = X - Z$ has $U_b \neq \emptyset$ for all $b \in B$. It is a largely mechanical procedure to check that if we can solve the problem for $U \twoheadrightarrow B$ equipped with Σ, the same L_v's, and the open loci $\Omega_v \cap U(L_v)$ (which are non-empty; why?) then the solution also works for the given data on X. Since U has K-fiber equal to the initial choice of U_K, this justifies that we may replace X_K with any non-empty open subset. In particular, since X_K is generically smooth over K, by shrinking X_K we now may and do assume X_K is K-smooth and quasi-projective (e.g., affine). It then follows from general principles of "smearing out from the generic fiber" (also known as "denominator-chasing") as developed exhaustively in [4, IV, §8–§12, §17–§18] that for some dense open $B_0 \subseteq B$ the restriction of X over B_0 is smooth and quasi-projective over B_0. For this and other reasons to be indicated shortly, we are now motivated to also want to shrink B. It must be emphasized that we have to allow ourselves to throw an essentially arbitrary finite set of closed points out of B and into Σ in what follows.

Let $\Sigma_0 \subseteq \mathrm{MaxSpec}(R)$ be a finite set of closed points of B, and let $B_0 = B - \Sigma_0$ be the corresponding affine Dedekind scheme of $(S \cup \Sigma_0)$-integers of K with respect to which $\Sigma \cup \Sigma_0$ is incomplete. We want to be able to work over B_0 with the incomplete set $\Sigma \cup \Sigma_0$. To this end, we need to identify suitable choices of L_v and $\Omega_v \subseteq X_K(L_v)$ (recall X_K is now K-smooth) for $v \in \Sigma_0$. Here we will see that unavoidability of allowing $L_v \neq K_v$ and $\Omega_v \neq X_K(L_v)$. If it were the case that $X(\mathscr{O}_{K_v}) \neq \emptyset$ for all $v \in \Sigma_0$ (note that all such v *are* non-archimedean!) then taking $L_v = K_v$ and $\Omega_v = X(\mathscr{O}_{K_v})$ would suffice in the sense that any B_0-finite irreducible Y_0 that "works" in $X|_{B_0}$ for such extra restrictions over Σ_0 has irreducible closure Y in X that is B-finite (due to the way we defined Ω_v with an integrality condition for $v \in \Sigma_0 = B - B_0$!) But typically even $X(K_v)$ is empty for $v \in \Sigma_0$. There is a way out: if for each $v \in \Sigma_0$ there is a finite Galois extension L_v/K_v such that $X(\mathscr{O}_{L_v}) \neq \emptyset$ then we can take $\Omega_v = X(\mathscr{O}_{L_v})$ (visibly $\mathrm{Gal}(L_v/K_v)$-stable!) and the same closure procedure still justifies the sufficiency of working over B_0. But how are we to find such an L_v?

By expressing a separable closure $K_v^{\rm sep}$ as a directed union of finite Galois extensions of K_v, it suffices to show that $X(\mathscr{O}_{K_v^{\rm sep}}) \neq \emptyset$ for each $v \in \Sigma_0$. Now clearly $X(\mathscr{O}_{K_v^{\rm sep}}) = X(\mathscr{O}_{\overline{K}_v}) \cap X_K(K_v^{\rm sep})$ inside of $X_K(\overline{K}_v)$ for an algebraic closure $\overline{K}_v$ containing $K_v^{\rm sep}$, and $X(\mathscr{O}_{\overline{K}_v})$ is an open subset of $X(\overline{K}_v)$ (why?). Thus, to get the desired non-emptiness we just have to show that $X_K(K_v^{\rm sep})$ is a dense subset of $X(\overline{K}_v)$ and that $X(\mathscr{O}_{\overline{K}_v})$ is non-empty. To see the non-emptiness (which is not treated in detail in [8]), we use the slicing argument as in the proof of [4, IV_2, 17.16.2] as follows. Let x_0 be a closed point in the dense open Cohen-Macaulay locus of the closed fiber X_v over the closed point $v \in B$. (The Cohen-Macaulay locus of any scheme locally of finite type over a field is open by excellence and contains all generic points, so it is dense too.) We pick a regular sequence in the local ring at x_0 on X_v and lift this to a sequence in the local ring at x_0 on X; this lifted sequence is regular and has B-flat zero scheme D by the local flatness criterion (since X is B-flat). This zero scheme is quasi-finite at x_0, so by Zariski's Main Theorem and the henselian property of $\mathscr{O}_{\overline{K}_v}$ it follows that $D \otimes_R \mathscr{O}_{\overline{K}_v}$ contains a connected component that is finite flat over $\mathscr{O}_{\overline{K}_v}$ and so one of its irreducible components with reduced structure is equal to $\mathscr{O}_{\overline{K}_v}$ (since $\overline{K}_v$ is algebraically closed). In this way we get an $\mathscr{O}_{\overline{K}_v}$-point of X.

As for the denseness of $X_K(K_v^{\rm sep})$ in $X_K(\overline{K}_v)$ for $v \in \Sigma_0$, we may working Zariski-locally on the smooth X_K to get to the case where X_K is étale over some affine space $\mathbf{A}_K^n$ over K. The resulting étale map $X_K \to \mathbf{A}_K^n$ has split fibers over each $K_v^{\rm sep}$-point of the target. The analytic inverse function theorem over the algebaically closed non-archimedean complete field $\overline{K}_v^{\wedge}$ (for which there is no annoyance of non-trivial finite extensions) thereby shows that $X_K(\overline{K}_v^{\wedge}) \to \mathbf{A}^n_K(\overline{K}_v^{\wedge})$ is a local homeomorphism. Hence, the denseness problem is reduced to the case of affine n-space, and then even the affine line. This is just the statement that $K_v^{\rm sep}$ is dense in $\overline{K}_v$. Such denseness is obvious in characteristic 0, and in characteristic $p > 0$ we just have to approximate roots to inseparable polynomials by roots to separable polynomials via slight deformation of an inseparable polynomial to make it separable; see [8, Lemma 1.6.1] for details on this approximation procedure.

Now we may shrink B (and increase Σ), so we can assume that $f : X \twoheadrightarrow B$ is smooth and quasi-projective. If $\dim X_K = 0$ then $X_K = \operatorname{Spec} K$ (why?) and so f is birational, separated, and étale, so it is an isomorphism due to the normality of B (and Zariski's Main Theorem). Thus, without loss of generality we may and do focus on the more interesting case $\dim X_K \geq 1$. If $\dim X_K > 1$ then one can use Bertini-style methods to slice the problem down to the 1-dimensional case. It must be emphasized that the slicing process is not entirely formal (especially in positive characteristic) because we need to keep track of how to modify the Ω_v's during the slicing process. This in turn requires a lemma on denseness in $X_K(L_v)$ for points that are algebraic over K; see [8, Lemme 2.1] for details.

The upshot of these reduction steps is that (using further shrinking of B and increasing of Σ) we may suppose X is open in a projective flat B-scheme $\overline{f} : \overline{X} \to B$ whose fibers are geometrically integral of dimension 1; this is obtained by smearing out from the regular connected compactification $\overline{X}_K$ of X_K (which may not be smooth when $\operatorname{char}(K) > 0$). The fibral condition ensures "$\mathscr{O}_B = \overline{f}_*(\mathscr{O}_{\overline{X}})$ universally" in the usual sense via the theory of cohomology and base change for (higher) direct images of flat coherent sheaves with respect to proper maps. We likewise have that $\dim \mathrm{H}^1(X_b, \mathscr{O})$ is the same for all $b \in B$, and we denote it g since it is the arithmetic genus of the geometrically integral regular proper K-curve $\overline{X}_K$. By more shrinking of B we can assume that the complement $Z = (\overline{X} - X)_{\rm red}$ is quasi-finite over B and hence finite over B, with each of its connected components regular (and thus Dedekind). By shrinking X_K some more we can assume $Z_K \neq \emptyset$, so the common degree

$$z \stackrel{\rm def}{=} \deg_K(Z_K) = \deg(Z/B)$$

is positive.

3. Picard and Hilbert schemes

The key idea behind the proof of Theorem 1.1 will be to make Y as an irreducible component of a well-chosen relative effective Cartier divisor $D \subseteq \overline{X}$ (i.e., B-flat closed subscheme cut out by an invertible ideal sheaf) that in turn will arise from a well-chosen invertible sheaf on $\overline{X}$. In particular, a crucial aspect of the construction will be the systematic use of certain moduli schemes that parameterize invertible sheaves (Picard schemes) and flat closed subschemes (Hilbert schemes). Thus, we now quickly review some of Grothendieck's fundamental results concerning representability of Picard and Hilbert functors, focusing on aspects that are peculiar to the case of relative dimension 1 (that is what we require). It should be noted that one can get by (for the purposes of proving Theorem 1.1) with less machinery than we invoke in our discussion below, but we prefer to discuss matters in their natural general setting.

Consider a morphism of schemes $\varphi : \mathscr{X} \to T$ that is proper, flat, and finitely presented; for any T-scheme T', let $\mathscr{X}_{T'}$ denote $\mathscr{X} \times_T T'$. The functor on T-schemes defined by $T' \mapsto \operatorname{Pic}(\mathscr{X}_{T'})$ is generally not a sheaf for the *fppf* topology, nor even for the Zariski topology, since line bundles given by pullback from the base T' are Zariski-locally trivial over T' but may not be globally trivial on $\mathscr{X}_{T'}$ (so the locality axiom for sheaves is not satisfied). In particular, this functor is not representable: there can be no "universal line bundle" on $\mathscr{X}$, which is to say a T-scheme P and an invertible sheaf $\mathscr{L}$ on $\mathscr{X}_P$ such that for any T-scheme T' and line bundle $\mathscr{N}$ on $\mathscr{X}_{T'}$ there is a unique map $T' \to P$ such that pullback along $\mathscr{X}_{T'} \to \mathscr{X}_P$ carries $\mathscr{L}$ isomorphically back to $\mathscr{N}$. This problem can be rectified in two ways: somewhat brutally by sheafifying the functor (thereby giving a functor whose points have unclear global meaning), or by introducing extra rigidification to eliminate non-trivial automorphisms (so as to better localize the construction problem over the base). We shall require both points of view, so first we recall the meaning of a rigificator:

Example 3.1. Suppose that $\varphi_*(\mathscr{O}_{\mathscr{X}}) = \mathscr{O}_T$ universally (e.g., this holds if φ has geometrically integral fibers, by the usual formalism of cohomology and base change preceded by a preliminary reduction to the locally noetherian case). Assume moreover that there is a closed subscheme $\mathscr{Z} \hookrightarrow \mathscr{X}$ that is finite locally free over T with constant rank $\delta > 0$; a typical example (but not one that will generally occur in cases of relevance to us) is $\delta = 1$, which is to say an element in $\mathscr{X}(T)$ (i.e., a section to φ). Consider the refined functor $\underline{\mathrm{Pic}}_{\mathscr{X}/T,\mathscr{Z}}$ from T-schemes to abelian groups defined as follows: $\underline{\mathrm{Pic}}_{\mathscr{X}/T,\mathscr{Z}}(T')$ is the group of isomorphism classes of pairs $(\mathscr{L}, i)$ where $\mathscr{L}$ is an invertible sheaf on $\mathscr{X}_{T'}$ and $i : \mathscr{L}|_{\mathscr{Z}_{T'}} \simeq \mathscr{O}_{\mathscr{Z}_{T'}}$; the notion of isomorphism between such pairs is defined in the evident manner and the group structure is via tensor product.

The main point is that the extra data of i gets rid of non-trivial automorphisms. Indeed, first note that any automorphism of $\mathscr{L}$ is multiplication by a global unit on $\mathscr{X}_{T'}$, or equivalently a global section of $\varphi_{T'*}(\mathscr{O}^{\times}_{\mathscr{X}_{T'}}) = \mathscr{O}^{\times}_{T'}$ (recall $\varphi_*(\mathscr{O}_{\mathscr{X}}) = \mathscr{O}_T$ universally). Such a unit on T' must pull back to 1 on $\mathscr{Z}_{T'}$ under the compatibility condition with i, and so such a unit is trivial since $\mathscr{Z}_{T'} \to T'$ is a faithfully flat map. The absence of non-trivial automorphisms implies (by descent theory for quasi-coherent sheaves [3, Ch. 6]) that $\underline{\mathrm{Pic}}_{\mathscr{X}/T,\mathscr{Z}}$ is a sheaf for the *fppf* topology on the category of T-schemes.

In general, we define the *Picard functor* $\underline{\mathrm{Pic}}_{\mathscr{X}/T}$ for $\mathscr{X}$ over T to be the *fppf* sheafification of the functor $T' \mapsto \mathrm{Pic}(\mathscr{X}_{T'})$. Due to the nature of the sheafification process, it is difficult to attach much meaning to elements of $\underline{\mathrm{Pic}}_{\mathscr{X}/T}(T')$. There is always a map of groups $\mathrm{Pic}(\mathscr{X}_{T'}) \to \underline{\mathrm{Pic}}_{\mathscr{X}/T}(T')$ that is functorial in T' but it is generally neither injective nor surjective. However, *fppf*-locally on T' an element of $\underline{\mathrm{Pic}}_{\mathscr{X}/T}(T')$ does arise from a line bundle in this way. Moreover, if $\mathscr{L}$ is a line bundle on $\mathscr{X}_{T'}$ and we are given $\mathscr{Z} \hookrightarrow \mathscr{X}$ as above then by Zariski-localization on T' we can even arrange that the invertible sheaf $\mathscr{L}|_{\mathscr{Z}_{T'}}$ is trivial. This follows from:

Lemma 3.2. *If $S' \to S$ is a finite and finitely presented map of schemes and $\mathscr{F}$ is a locally free sheaf of constant rank n on S' then Zariski-locally over S it admits a trivialization.*

Proof. By chasing through direct limits, we may suppose $S = \mathrm{Spec}(A)$ is local, so $S' = \mathrm{Spec}(A')$ is semi-local. The sheaf $\mathscr{F}$ comes from a finite locally free A'-module M' with constant rank, so on the fiber over the closed point of A this is a free module over the artin ring $A'/\mathfrak{m}_A A'$. Lifting such a basis is then easily seen to give a basis of M' over A'. ■

We conclude that the morphism of *fppf* abelian sheaves $\underline{\mathrm{Pic}}_{\mathscr{X}/T,\mathscr{Z}} \to \underline{\mathrm{Pic}}_{\mathscr{X}/T}$ is locally surjective on sections and so is surjective. Though it is easier to work with the sheaf $\underline{\mathrm{Pic}}_{\mathscr{X}/T,\mathscr{Z}}$ since its points have concrete meaning, we will nonetheless find that $\underline{\mathrm{Pic}}_{\mathscr{X}/T}$ plays a crucial role in subsequent constructions. Of course, to get beyond formalism of definitions one needs some real theorems, and for us the real theorems concern representability of these functors. A fundamental result in this direction is:

Theorem 3.3 (Grothendieck)**.** *If $\mathscr{X} \to T$ is projective (in the sense of admitting a closed T-immersion into some $\mathbf{P}^N_T$) and has geometrically integral fibers then $\underline{\mathrm{Pic}}_{\mathscr{X}/T}$ is representable by a countable disjoint union of quasi-projective T-schemes.*

Proof. This is sketched in [6], and is explained in more detail in [3, §8.2]. ■

Remark 3.4. It is a rather annoying restriction to impose projectivity and geometric integrality hypotheses on the morphism. To get beyond this case one has to work in the setting of algebraic spaces and give up any hope (in general) that separatedness holds for the "representing" algebraic space. We will return to this matter of the algebraic spaces near the end of the proof of Theorem 1.1 in the case of positive characteristic. For the case of number fields, algebraic spaces are not necessary in the proof.

Let us write $\mathrm{Pic}_{\mathscr{X}/T}$ to denote the T-group scheme representing $\underline{\mathrm{Pic}}_{\mathscr{X}/T}$; we again remind the reader that the points of this group scheme have no easily-described global meaning in general. We will of course apply this in the case of our projective curve $\overline{X} \to B$ with geometrically integral fibers, but since points of $\mathrm{Pic}_{\overline{X}/B}(B)$ generally do not arise from line bundles on $\overline{X}$ we shall require a more refined group scheme to get our hands on line bundles. The more refined group scheme we will use is a representing object $\mathrm{Pic}_{\overline{X}/B,Z}$ for $\underline{\mathrm{Pic}}_{\overline{X}/B,Z}$, so let us first address the representability of this functor more generally:

Corollary 3.5. *Let $\varphi : \mathscr{X} \to T$ be a proper, finitely presented, and flat map such that $\varphi_*(\mathscr{O}_{\mathscr{X}}) = \mathscr{O}_T$ universally. Let $\mathscr{Z}$ be a closed subscheme of $\mathscr{X}$ that is finite locally free over T with constant rank $\delta > 0$. If $\underline{\mathrm{Pic}}_{\mathscr{X}/T}$ is representable by a T-group scheme $\mathrm{Pic}_{\mathscr{X}/T}$ then $\underline{\mathrm{Pic}}_{\mathscr{X}/T,\mathscr{Z}}$ is representable by a T-group scheme $\mathrm{Pic}_{\mathscr{X}/T,\mathscr{Z}}$. In such cases, the natural map of group schemes $\mathrm{Pic}_{\mathscr{X}/T,\mathscr{Z}} \to \mathrm{Pic}_{\mathscr{X}/T}$ that forgets the rigidification along $\mathscr{Z}$ is smooth and affine, and there is a short exact sequence of T-group schemes*

$$1 \to \mathrm{Res}_{\mathscr{Z}/T}(\mathbf{G}_{m,\mathscr{Z}})/\mathbf{G}_{m,T} \to \mathrm{Pic}_{\mathscr{X}/T,\mathscr{Z}} \to \mathrm{Pic}_{\mathscr{X}/T} \to 1. \tag{1}$$

Since there is generally no universal line bundle over $\mathrm{Pic}_{\mathscr{X}/T}$, this theorem cannot be proved by "putting more structure" onto a universal object. The construction instead will proceed by initially building (1) as an exact sequence of *fppf* abelian sheaves, and then using descent techniques. It should be noted that the leftmost (nontrivial) term in (1) represents the corresponding *fppf* sheaf quotient, and it is representable by a smooth affine T-group because the Weil restriction $\mathrm{Res}_{\mathscr{Z}/T}(\mathbf{G}_{m,\mathscr{Z}})$ is finitely presented and affine (by construction) and is T-smooth (by the functorial criterion for smoothness); such representability for this sheaf quotient modulo the closed subgroup $\mathbf{G}_{m,T}$ follows from the general theory of relative tori in affine group schemes; see [5, Exp. VIII, §5].

Proof. Consider the diagram of *fppf* abelian sheaves on the category of T-schemes given by

$$1 \to \mathbf{G}_{m,T} \stackrel{\alpha}{\to} \mathrm{Res}_{\mathscr{Z}/T}(\mathbf{G}_{m,\mathscr{Z}}) \stackrel{\beta}{\to} \mathrm{Pic}_{\mathscr{X}/T,\mathscr{Z}} \stackrel{\gamma}{\to} \mathrm{Pic}_{\mathscr{X}/T} \to 0,$$

where α on T'-points is pullback of units on T' to units on $\mathscr{Z}_{T'}$, γ is induced by the map that forgets rigidification (and then passes to the sheafification), and β sends a unit u on $\mathscr{Z}_{T'}$ to the pair $(\mathscr{O}_{\mathscr{X}_{T'}}, i)$ where $i : \mathscr{O}_{\mathscr{Z}_{T'}} \simeq \mathscr{O}_{\mathscr{Z}_{T'}}$ is multiplication by u. By working *fppf*-locally, this is clearly an exact sequence of sheaves. Effectivity of descent for affine morphisms implies that an *fppf*-sheaf extension of a group scheme by an affine *fppf* group scheme is necessarily representable; this is proved in [9]. Thus, we do get the representability of $\underline{\mathrm{Pic}}_{\mathscr{X}/T,\mathscr{Z}}$. The functorial interpretation of γ and the *fppf*-local nature of points of $\mathrm{Pic}_{\mathscr{X}/T}$ (as being induced by line bundles) implies that γ is an *fppf*-torsor for the smooth affine T-group $\mathrm{Res}_{\mathscr{Z}/T}(\mathbf{G}_{m,\mathscr{Z}})/\mathbf{G}_{m,T}$ and so is an *fppf* map that is moreover smooth and affine by descent theory. ■

In our situation of interest, we have a smooth affine surjection of B-groups $\mathrm{Pic}_{\overline{X}/B,Z} \to \mathrm{Pic}_{\overline{X}/B}$. The functorial criterion for smoothness and the vanishing of coherent H^2's for curves implies that $\mathrm{Pic}_{\overline{X}/B,Z}$ is smooth over B [3, 8.4/2]. If S is a B-scheme and $\mathscr{L}$ is an invertible sheaf on $\overline{X}_S$ then the function $s \mapsto \deg_{\overline{X}_s}(\mathscr{L}_s)$ is locally constant (due to local constancy of Euler characteristic in proper flat families); see [3, 9.1]. Thus, we get natural decompositions into pairwise disjoint open subschemes

$$\mathrm{Pic}_{\overline{X}/B} = \coprod_{d\in\mathbf{Z}} \mathrm{Pic}^d_{\overline{X}/B}, \;\; \mathrm{Pic}_{\overline{X}/B,Z} = \coprod_{d\in\mathbf{Z}} \mathrm{Pic}^d_{\overline{X}/B,Z}$$

where the dth piece classifies bundles with constant fibral degree d. In particular, $\mathrm{Pic}^d_{\overline{X}/B,Z}$ represents the functor of isomorphism classes of pairs $(\mathscr{L}, i)$ where $\mathscr{L}$ has constant fibral degree d.

Each Pic^d is a torsor for the group scheme Pic^0, and the geometric fibers of $\mathrm{Pic}^0_{\overline{X}/B}$ are connected because $\overline{X} \to B$ has geometrically integral fibers of dimension 1 [3, 9.2/13]. The kernel in (1) has geometrically connected fibers (since $\mathrm{Res}_{A/k}(\mathbf{G}_{m,A})$ is an extension of a torus by a unipotent group for any finite local algebra A over an algebraically closed field k), so $\mathrm{Pic}^0_{\overline{X}/B,Z}$ likewise has geometrically connected fibers over B. By Theorem 3.3 and the connectivity of B, both $\mathrm{Pic}^0_{\overline{X}/B}$ and $\mathrm{Pic}^0_{\overline{X}/B,Z}$ are therefore quasi-projective B-schemes (especially they are finite type), and so likewise for Pic^d's by Theorem 3.3.

The main idea behind the proof of Theorem 1.1 can now be formulated a bit more precisely than earlier. We will use the group structure and geometry of $\mathrm{Pic}_{\overline{X}/B}$ and $\mathrm{Pic}_{\overline{X}/B,Z}$ to make a B-point $(\mathscr{L}, i) \in \mathrm{Pic}_{\overline{X}/B,Z}(B)$ such that there is a section $s \in \mathscr{L}(\overline{X})$ satisfying $i(s|_Z) = 1$ (with $i : \mathscr{L}|_Z \simeq \mathscr{O}_Z$). In particular, s is non-zero on all fibers over B so $D = \mathrm{div}_{\mathscr{L}}(s)$ is a finite B-flat closed subscheme of $\overline{X}$ and it is disjoint from Z (since $i(s|_Z) = 1$). Hence, the B-finite D is supported in $X = \overline{X} - Z$. The section s will be carefully chosen to ensure that D is L_v-split for all $v \in \Sigma$ with all of its L_v-points in $\Omega_v \subseteq X(L_v)$. Taking Y to be an irreducible component of D will then settle the proof of Theorem 1.1.

To get our hands on the Picard schemes, we need to dominate them in a convenient way by certain Hilbert schemes that classify relative effective divisors. (But beware that though we are ultimately interested in making such relative divisors, it is really the line bundles that we largely work with in the construction: a key step in one lemma will be the flexibility to change D without changing $\mathscr{O}(D)$. Hence, it is really Picard schemes and not Hilbert schemes that will play the dominant role in the proof. They somehow manage to hide the divisor information that is too difficult to find explicitly.) In fact, Grothendieck first constructed Picard schemes out of Hilbert schemes, so the considerations that follow are rather natural from the viewpoint of Grothendieck's initial constructions. To avoid excessive terminology, we will restrict our discussion of Hilbert schemes to the case of relative curves; the degree parameters below are really to be viewed as constant polynomials, and in the case of higher relative dimension must be replaced with higher-degree Hilbert polynomials (hence the name "Hilbert scheme").

For $d \geq 1$, consider the degree-d Hilbert functor $\underline{\mathrm{Hilb}}^d_{X/B}$ on the category of B-schemes that assigns to each B-scheme S the set of closed subschemes $D \hookrightarrow X_S$ that are finite locally free of constant rank d over S. For example, if $S = \operatorname{Spec} k(b)$ for $b \in B$ then $\underline{\mathrm{Hilb}}^d_{X/B}(S)$ is the set of effective Cartier divisors on X_b with $k(b)$-degree d. We define the functor $\underline{\mathrm{Hilb}}^d_{\overline{X}/B}$ similarly. Since $\overline{X} \to B$ is projective, it follows from a theorem of Grothendieck [6] that $\underline{\mathrm{Hilb}}^d_{\overline{X}/B}$ is represented by a projective B-scheme $\mathrm{Hilb}^d_{\overline{X}/B}$. If we let

$$\overline{\mathscr{D}} \hookrightarrow \overline{X} \times_B \mathrm{Hilb}^d_{\overline{X}/B}$$

denote the universal relative effective Cartier divisor of degree d (over $\mathrm{Hilb}^d_{\overline{X}/B}$), then the closed set

$$\overline{\mathscr{D}} \cap (Z \times_B \mathrm{Hilb}^d_{\overline{X}/B})$$

has closed image in $\mathrm{Hilb}^d_{\overline{X}/B}$ whose geometric points are those effective Cartier divisors on geometric fibers of $\overline{X} \to B$ that meet Z. Hence, the open complement of this closed image in $\mathrm{Hilb}^d_{\overline{X}/B}$ represents the functor of D's not meeting Z, which is to say supported in $\overline{X} - Z = X$. That is, this open locus represents the functor $\underline{\mathrm{Hilb}}^d_{X/B}$, so we denote it $\mathrm{Hilb}^d_{X/B}$.

It should be noted that in the case of quasi-projective relative curves, the representability of the $\underline{\mathrm{Hilb}}^d$ functors can be proved by more direct methods, bypassing Grothendieck's general theory, by using dth symmetric powers of the curve. These are explained (and the link with Hilbert functors worked out) in [3, pp. 252–4] and the references therein. In [8] the language of symmetric powers is used instead of the language of Hilbert schemes (but it is still necessary to prove that such symmetric powers really represent Hilbert functors as above). Our interest in Hilbert schemes will be via the morphism

$$\varphi_d : \mathrm{Hilb}^d_{X/B} \to \mathrm{Pic}^d_{\overline{X}/B,Z}$$

that on S-points (for a B-scheme S) assigns to any S-finite locally free (of rank d) closed subscheme $D \hookrightarrow X_S$ (which is also closed in $\overline{X}_S$, due to S-finiteness) the invertible sheaf $\mathscr{O}_{\overline{X}_S}(D) = \mathscr{I}_D^{-1}$ equipped with its trivializing section 1 along the closed subscheme Z that is disjoint from D.

Example 3.6. We now work out geometric fibers of φ_d in a special case. Let $\operatorname{Spec} k \to B$ be a geometric point, and let D_0 be an effective Cartier divisor of degree d on $\overline{X}_k$ supported in X_k. For simplicity, suppose $\overline{X}_k$ is smooth. We consider $\mathscr{O}_{\overline{X}_k}(D_0)$ as a k-point of $\mathrm{Pic}^d_{\overline{X}/B,Z}$. The fiber over this under φ_d consists of all effective Cartier divisors $D \subseteq \overline{X}_k$ linearly equivalent to D_0 with $D - D_0 = \operatorname{div}(h)$ where $h|_{Z_k} = 1$. (This latter condition uniquely determines h.) In other words, this is the fiber over 1 for the linear map $\mathrm{H}^0(\overline{X}_k, \mathscr{O}(D_0)) \to \mathrm{H}^0(Z_k, \mathscr{O}_{Z_k})$ whose kernel is $\mathrm{H}^0(\overline{X}_k, \mathscr{O}(D_0 - Z_k))$. Provided $d \geq 2g + z - 1$, by Riemann–Roch this linear map is surjective and so the fiber is a torsor for the kernel group $\mathrm{H}^0(\overline{X}_k, \mathscr{O}(D_0 - Z_k))$. This description of fibers as torsors for a vector group (when $d \geq 2g + z - 1$) will be vastly generalized in Lemma 4.2 below.

4. Three lemmas and a key theorem

We record three lemmas, referring to [8] for the proofs. The first is [8, Lemme 3.3]:

Lemma 4.1. *For $d \geq 1$ and $v \in \Sigma$, let $\Omega_v^{[d]} \subseteq \mathrm{Hilb}^d_{X/B}(K_v)$ be the subset of degree-d effective divisors on X_{K_v} that are L_v-split with all points in $\Omega_v \subseteq X(L_v)$. This is an open subset, and it is non-empty if $[L_v : K_v] = \#\mathrm{Gal}(L_v/K_v)$ divides d.*

The idea behind the proof is to look at the universal relative effective divisor over the Hilbert scheme, and to show first that L_v-splitting of its fiber over one K_v-point implies such splitting over all nearby K_v-points; this is essentially a form of Krasner's Lemma, and it gives the openness claim. As for the non-emptiness when d is divisible by the order of the Galois group, this requires a K_v-rational construction, which is done using a Galois-invariant sum of degree d (possible since Ω_v is a non-empty $\mathrm{Gal}(L_v/K_v)$-stable open subset with d sufficiently divisible).

Next, using the standard formalism of cohomology and base change one can upgrade the torsor analysis in Example 3.6 to a relative situation as in the following result (which is [8, Lemme 3.6]):

Lemma 4.2. *Fix $d \geq 2g + z - 1$, let $\mathscr{L}_d$ be the universal line bundle on $\overline{X} \times_B \mathrm{Pic}^d_{\overline{X}/B,Z}$, and let π_d be the structure map to $P^d = \mathrm{Pic}^d_{\overline{X}/B,Z}$. Then $\pi_{d*}(\mathscr{L}_d(-Z_{P^d}))$ is a vector bundle on P^d with rank $(d-z)+1-g$ whose formation commutes with any base change on P^d, and φ_d is a Zariski-torsor for the additive group associated to this vector bundle.*

For the final lemma, we need some notation. If $d \geq 1$, let $W_v^{[d]} = \varphi_d(\Omega_v^{[d]}) \subseteq \mathrm{Pic}^d_{\overline{X}/B,Z}(K_v)$. This is the set of pairs $(\mathscr{L}, i)$ on X_{K_v} where $\mathscr{L}$ comes from an effective Cartier divisor $D_v \hookrightarrow X_{K_v}$ of degree d that is L_v-split with all points in $\Omega_v \subseteq X(L_v)$. If $d \geq 2g + z - 1$ then by Lemmas 4.1 and 4.2 it follows that $W_v^{[d]}$ is an open subset of $\mathrm{Pic}^d_{\overline{X}/B,Z}(K_v)$, and by Lemma 4.1 if also $[L_v : K_v] | d$ then this open subset is not empty.

The key role of group theory with Picard schemes is to allow us to change D with changing $\mathscr{O}(D)$, or more specifically to move around in the fibers of φ_d (which are Zariski torsors for vector groups, and so have lots of rational points). This group theory manifests itself in the following result [8, Lemme 3.7.2(*ii*)] that will be used in the next section:

Lemma 4.3. *If $d, d' \geq 1$ and $d \geq 2g + z$ then $W_v^{[d]} W_v^{[d']} \subseteq W_v^{[d+d']}$ inside of $\mathrm{Pic}_{\overline{X}/B,Z}(K_v)$.*

The issue in the proof is to first pick effective étale divisors D and D' on X_{K_v} inducing a given pair of line bundles, and to then move D without changing $\mathscr{O}(D)$ to arrange that D does not meet D'. Indeed, if $D + D'$ is not étale then we need some moving to find a suitable L_v-split representative divisor for $\mathscr{O}(D + D') = \mathscr{O}(D) \otimes \mathscr{O}(D')$. Considering D' as fixed, the possible choices for D are members of a torsor for a vector space over K_v, and those that are "bad" (in the sense of making $D + D'$ not be étale) occupy torsors for certain linear subspaces. By taking $d \geq 2g + z$ we can ensure (by Riemann–Roch) that these linear subspaces are proper, and so we can find lots of good choices for D.

Finally, let us state and prove the key theorem that reduces Theorem 1.1 to a construction problem with line bundles (and not divisors). This latter problem will be solved by using topological group arguments in point-groups of Picard varieties over completions of K.

Theorem 4.4. *Assume there exists $(\mathscr{L}, i) \in \mathrm{Pic}^d_{\overline{X}/B}(B)$ with $d \geq 2g + z - 1$ such that $(\mathscr{L}_{K_v}, i_{K_v}) \in W_v^{[d]}$ for all $v \in \Sigma$. There exists $s \in \mathscr{L}(\overline{X})$ such that $i(s|_Z) = 1$ and $D = \mathrm{div}_{\mathscr{L}}(s) \subseteq X$ induces a point in the non-empty open subset $\Omega_v^{[d]} \subseteq \mathrm{Hilb}^d_{X/B}(K_v)$ for all $v \in \Sigma$.*

As we have seen earlier, under the conclusion of this theorem we can take any irreducible component of D as the required Y in Theorem 1.1. In this way, the proof of Theorem 1.1 is shifted to the problem of meeting the hypotheses in Theorem 4.4.

Proof. By Lemma 4.2, $\mathscr{T} = \varphi_d^{-1}(\mathscr{L}, i)$ is a Zariski torsor over $B = \mathrm{Spec}\, R$ for the projective R-module $M = \mathrm{H}^0(\overline{X}, \mathscr{L}(-Z))$. Explicitly, points of $\mathscr{T}$ are sections of $\mathscr{L}$ (after base change) whose Z-restriction is carried to 1 by the Z-trivialization i. Hence, it suffices to prove that $\mathscr{T}(R)$ has dense image in $\prod_{v\in\Sigma}\mathscr{T}(K_v)$ (as then the image meets the open subset $\prod_{v\in\Sigma} W_v^{[d]}$ that is non-empty). But the torsor $\mathscr{T}$ is trivial because it is classified by $\mathrm{H}^1(B,\widetilde{M})=0$ (as B is affine), so $\mathscr{T}$ is B-isomorphic to the vector bundle associated to M. Hence, our problem is to prove that the natural map $M\to\prod_{v\in\Sigma}(M\otimes_R K_v)$ has dense image. But M is a projective R-module so we have $M\oplus M'=R^n$ for some R-module M' and some $n\geq 1$. In this way, we can reduce the problem to the case of the R-module R. That is, we want $R\to\prod_{v\in\Sigma}K_v$ to have dense image. This denseness follows from the incompletness of Σ with respect to $B=\operatorname{Spec} R$ and the strong approximation for the additive group, as we explained in Remark 1.4. ■

5. Arguments with non-Hausdorff topological groups

It remains to find a pair $(\mathscr{L},i)$ satisfying the hypotheses in Theorem 4.4. Let $\mathscr{L}_0$ be an ample invertible sheaf on $\overline{X}$. Passing to a sufficiently high tensor power, we can assume that $d\stackrel{\text{def}}{=}\deg_{\overline{X}_K}((\mathscr{L}_0)_K)\geq 2g+z$ and that $[L_v:K_v]|d$ for all $v\in\Sigma$. We can also likewise assume that there is a trivialization $i_0:\mathscr{L}_0|_Z\simeq\mathscr{O}_Z$ because the Dedekind affine Z has finite class group for each of its connected components. The first two of these three conditions ensure that $W_v^{[d]}$ is a non-empty open subset of $\mathrm{Pic}^d_{\overline{X}/B,Z}(K_v)$ for all $v\in\Sigma$.

Consider $\mathscr{L}=\mathscr{L}_0^{\otimes n}$ and $i=ui_0^{\otimes n}$ for a large n and a unit u on Z. We seek such n and u so that $(\mathscr{L}_{K_v},i_{K_v})\in W_v^{[nd]}$ for all $v\in\Sigma$. Since $d\geq 2g+z$, Lemma 4.3 implies that $(W_v^{[d]})^n\subseteq W_v^{[nd]}$ in $\mathrm{Pic}_{\overline{X}/B,Z}(K_v)$ for all $n\geq 1$. We let

$$p_0\in\mathrm{Pic}^d_{\overline{X}/B,Z}(K_\Sigma)=\prod_{v\in\Sigma}\mathrm{Pic}^d_{\overline{X}/B,Z}(K_v)$$

be the class of the tuple of pairs $((\mathscr{L}_0)_{K_v},i_{K_v})_{v\in\Sigma}$. We are seeking a unit u on Z and a positive n so that up_0^n lies in $\prod_{v\in\Sigma}W_v^{[nd]}$. We fix *one* point q_0 in the non-empty subset $\prod_{v\in\Sigma}W_v^{[d]}\subseteq\mathrm{Pic}^d_{\overline{X}/B,Z}(K_\Sigma)$, so $q_0^{-1}\cdot\prod_{v\in\Sigma}W_v^{[d]}$ is a neighborhood of the identity in the topological group $\mathrm{Pic}^0_{\overline{X}/B,Z}(K_\Sigma)$. It is enough to find a unit u on Z and a positive integer n so that $up_0^nq_0^{-n}$ lies in $q_0^{-1}\cdot\prod_{v\in\Sigma}W_v^{[d]}$ because then

$$up_0^n\in q_0^{n-1}\cdot\prod_{v\in\Sigma}W_v^{[d]}\subseteq\left(\prod_{v\in\Sigma}W_v^{[d]}\right)^{(n-1)+1}\subseteq\prod_{v\in\Sigma}W_v^{[nd]}.$$

It therefore suffices to prove that the sequence $\{(p_0q_0^{-1})^n\}_{n\geq 1}$ in the first-countable and locally compact (but usually non-Hausdorff) topological group

$$\mathrm{Pic}^0_{\overline{X}/B,Z}(K_\Sigma)/\mathbf{G}_m(Z)$$

has 1 as an accumulation point. Now in any first-countable quasi-compact topological group any infinite sequence has an accumulation point, so any sequence of the form $\{g^n\}_{n\geq 1}$ has the identity as an accumulation point. Thus, for the quasi-projective K-group $J_{Z_K}=\mathrm{Pic}^0_{\overline{X}_K/K,Z_K}$, we want $J_{Z_K}(K_\Sigma)/\mathbf{G}_m(Z)$ to be quasi-compact (where units on Z are embedded via β as in (1)).

Define $J=\mathrm{Pic}^0_{\overline{X}_K/K}$ (so this is to be called a Jacobian if $\overline{X}_K$ is K-smooth, as is the case for number fields, but it should not be called a Jacobian otherwise). Passing to K-fibers on (1) gives an exact sequence of smooth commutative K-groups

$$1\to\mathrm{Res}_{Z_K/K}(\mathbf{G}_{m,Z_K})/\mathbf{G}_{m,K}\to J_{Z_K}\to J\to 1$$

with $J_{Z_K}\to J$ a *smooth* morphism. Hence, for each $v\in\Sigma$ we get an induced diagram of topological groups

$$1\to(\mathrm{Res}_{Z_K/K}(\mathbf{G}_{m,Z_K})/\mathbf{G}_{m,K})(K_v)\to J_{Z_K}(K_v)\to J(K_v)$$

with the initial map a closed embedding and (by the structure theorem for smooth morphisms and the invere function theorem over K_v) the final map an open map (possibly not surjective). But open subgroups are closed, and the left term is topologically isomorphic to $\mathbf{G}_m(Z_{K_v})/K_v^\times$ by Hilbert's Theorem 90 (make sure you see why this is a homeomorphism, not just a group isomorphism). Taking the product over all $v \in \Sigma$, we get a right exact sequence of topological groups

$$\mathbf{G}_m(Z_{K_\Sigma})/\mathbf{G}_m(Z)K_\Sigma^\times \to J_{Z_K}(K_\Sigma)/\mathbf{G}_m(Z) \to G \to 1$$

where G is a closed subgroup of $J(K_\Sigma)$ and the map to G is a quotient map.

Applying Remark 1.4 to each of the Dedekind connected components of the B-finite flat Z (and the incomplete lifting of Σ to each such component), it follows that $\mathbf{G}_m(Z_{K_\Sigma})/\mathbf{G}_m(Z)$ is quasi-compact. Hence, $J_{Z_K}(K_\Sigma)$ contains a quasi-compact subgroup modulo which the quotient is closed in $J(K_\Sigma)$. Since $J_{Z_K}(K_\Sigma)$ is locally compact (but usually not Hausdorff!), it therefore suffices to prove that $J(K_\Sigma) = \prod_{v\in\Sigma} J(K_v)$ is compact. This is a special case of:

Theorem 5.1. *Let F be a local field and let C be a proper regular and geometrically integral F-curve. Let $J = Pic^0_{C/F}$. The Hausdorff topological group $J(F)$ is compact.*

Proof. If C is F-smooth (e.g., if F has characteristic 0) then J is proper due to the valuative criterion, so $J(F)$ is compact. (Briefly, this comes down to the fact that $\mathbf{P}^N(F)$ is compact, by inspection.) Hence, it remains to treat the case of positive characteristic, or more generally when F is non-archimedean. Since C is geometrically integral over F, hence generically smooth, there exists a finite separable extension F'/F such that $C(F')$ is not empty. The curve $C_{F'}$ is regular since $F \to F'$ is étale and C is regular, and since F is closed in F' it follows that $J(F)$ is closed in $J(F') = \mathrm{Pic}^0_{C_{F'}/F'}(F')$. Thus, it is harmless to replace F with such an F', so we can and do assume $C(F)$ is not empty. In particular, by using rigidification along a choice of such F-point $\xi \in C(F)$ we clearly have $\mathrm{Pic}_{C/F} = \mathrm{Pic}_{C/F,\xi}$, so now the functor represented by $\mathrm{Pic}^0_{C/F}$ has useful meaning (as classifying line bundles with fibral degree 0 and a trivialization along the section ξ).

At this point, in [8] Moret-Bailly invokes some results from [1, §8] concerning moduli-theoretic compactification of Pic^0. It seems more natural to use the integral structure theory for Pic^0, so we now give such an alternative method. (We are going to have to make essential use of algebraic spaces, so it should be noted that in characteristic 0 the entire problem was solved already due to the equivalence of regularity and smoothness for schemes locally of finite type over a field with characteristic 0.)

Let $R = \mathscr{O}_F$ and let $\mathscr{C}$ be a regular proper R-scheme with F-fiber C. The existence of such a regular proper model is a non-trivial fact in the theory of integral models of curves over Dedekind base schemes. By the valuative criterion we have $\mathscr{C}(R) = C(F)$ so if we let ξ now also denote the R-point associated to $\xi \in C(F)$ then $\underline{\mathrm{Pic}}_{\mathscr{C}/R} = \underline{\mathrm{Pic}}_{\mathscr{C}/R,\xi}$. Since the residue field k of R is perfect (even finite), by using intersection theory on the regular arithmetic surface $\mathscr{C}$ one finds (see [7, 9.1/24]) that $\mathrm{H}^0(\mathscr{C}_0, \mathscr{O}) = k$, where $\mathscr{C}_0$ denotes the closed fiber of $\pi : \mathscr{C} \to S = \mathrm{Spec}\, R$. Hence, $\pi_*(\mathscr{O}_{\mathscr{C}}) = \mathscr{O}_S$ universally (by cohomology and base change formalism). This allows us to apply Artin's work on Picard functors (as at the end of [2]) to conclude that $\underline{\mathrm{Pic}}_{\mathscr{C}/R} = \underline{\mathrm{Pic}}_{\mathscr{C}/R,\xi}$ is an algebraic space over $\mathrm{Spec}\, R$; by the functorial criteria this algebraic space is smooth (but it is not quasi-compact), and we warn that it is usually not separated. Let us write P to denote this algebraic space.

Since the points of P have functorial meaning in terms of line bundles equipped with rigidification along ξ, it is easy to see (by regularity of $\mathscr{C}$) that $P(R) \to P(F)$ is surjective. Letting $P^{[0]} \subseteq P$ denote the open subgroup kernel of the natural degree map $\deg : P \to \mathbf{Z}$ that forms fibral degree, the algebraic space group $P^{[0]}$ is R-smooth with F-fiber $\mathrm{Pic}^0_{C_F/F} = J$. The surjectivity of $P(R) \to P(F)$ implies the surjectivity of $P^{[0]}(R) \to P^{[0]}(F) = J(F)$. But usually $P^{[0]}$ is not separated, so let E be the closure of the identity section in this algebraic space group. We have $E_F = \mathrm{Spec}\, F$ since J is F-separated, and in general E is étale over R. The quotient $Q = P^{[0]}/E$ then makes sense as a smooth and separated algebraic space group over R with F-fiber $Q_F = P^{[0]}_F = J$. By separatedness, the surjective map $Q(R) \to Q(F) = J(F)$ is injective too, hence bijective. More importantly, by using intersection on the regular arithmetic surface $\mathscr{C}$ one can show that Q is *finite type* (i.e., quasi-compact); see [3, 9.5/8–11] for this latter issue.

Our problem is now intrinsic to Q, or more specifically is a special case of the following situation. Let X be a finite type and separated algebraic space over R with F-fiber X_F that is a scheme. We claim that the subset $X(R) \subseteq X(F) = X_F(F)$ is a quasi-compact subset when $X(F)$ is given its natural topology (as rational points of the locally finite type scheme X_F over the topological field F). Note that $X(R)$ is a subset of $X(F)$ precisely because of the separatedness (valuative criterion). Letting $f : X' \twoheadrightarrow X$ be a finite type and étale covering by a separated (e.g., affine) R-scheme, the map f is *separated* since X is separated. It therefore follows from Zariski's Main Theorem and the henselian property of R that $X'(R) \to X(R)$ is a surjection. Since $X'_F(F) \to X_F(F)$ is continuous, it is therefore enough to solve the problem for X' in the role of X. That is, we can assume that X is a scheme. By working Zariski-locally we can assume X is affine, and then even an affine space. Finally, we can assume X is the affine line. The problem in this case is to show that R is a compact subset of F, and this is clear. ■

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