SERRE'S CONSTRUCTION

3/15/05, Hui Xue

In this note we will describe Serre's tensor construction with the aim of applying it to abelian varieties with CM.

1. The construction

In this note R always denotes a commutative algebra (with unit), and M denotes a finite projective R-module. We let A denote an R-module scheme over S. Also, if $X \to S$ and $T \to S$ are S-schemes, X_T denotes the T-scheme $X \times_S T$.

Proposition 1.1. *The functor*

$$T \to M \otimes_R A(T) = \mathrm{Hom}_R(M^\vee, A(T))$$

on S-schemes is represented by an R-module scheme over S, denoted by $M \otimes_R A$. Here $M^\vee$ is the dual $\mathrm{Hom}_R(M, R)$ *of M.*

Proof. First if $M = R^n$ is a free R-module, then it is clear that $M \otimes_R A = A^n$. For general M, as $M^\vee$ is finite projective, we choose a finite presentation of $M^\vee$:

$$R^m \to R^n \to M^\vee \to 0.$$

The functor $\mathrm{Hom}_R(M^\vee, A)$ is the kernel of the induced morphism from $\mathrm{Hom}_R(R^n, A) = A^n$ to $\mathrm{Hom}_R(R^m, A) = A^m$, so it is representable. It is obvious that $M \otimes_R A$ carries a natural R-module structure as a functor. □

Lemma 1.2. *For $T \to S$, we have a natural isomorphism $(M \otimes_R A)_T \cong M \otimes_R A_T$ as R-module schemes over T.*

Proof. For any T-scheme X one has

$$(M \otimes_R A)_T(X) = M \otimes_R \mathrm{Hom}_S(X, A) = M \otimes_R (A_T(X)),$$

which proves the claim. □

It is obvious that for two finite projective R-modules M and N we have a natural isomorphism of R-module schemes

$$(M \oplus N) \otimes_R A \cong (M \otimes_R A) \times (N \otimes_R A). \tag{1.1}$$

Proposition 1.3. *The following properties of $A \to S$ are preserved by the tensor construction: locally finite type, quasi-compactness, locally finite presentation, separatedness, properness, smoothness, flatness, and geometric connectedness of fibers.*

Proof. Preservation of the properties of being locally finite type, quasi-compact, or locally of finite presentation are clear by construction, so we only show the preservation of the last five properties.

Suppose A is separated over S. For any valuation ring V over S, the map

$$\mathrm{Hom}_S(\mathrm{Spec}\ V, A) \to \mathrm{Hom}_S(\mathrm{Spec}\ Q_V, A)$$

is injective, where Q_V is the quotient ring of V. The map is still injective when tensoring with the flat R-module M, so $M \otimes_R A$ is proper over S. The case of properness goes similarly.

For flatness we use the following argument of Serre. As $M \otimes_R A$ is a direct factor of the S-flat scheme A^n, it suffices to check that if X and Y are S-schemes with $Y(S)$ nonempty and $X \times_S Y$ is S-flat, then X is S-flat. We may assume that $S = \mathrm{Spec}\, D$ is local and that both $X = \mathrm{Spec}\, B$ and $Y = \mathrm{Spec}\, C$ are affine. Thus, we get two ring extensions $D \to B$ and $D \to C$ with $B \otimes_D C$ flat over D. Since $D \to C$ has a section we have a splitting $C = D \oplus I$ as R-modules. Thus, B is a direct D-module summand of the flat D-module $B \otimes_D C$.

Let A be smooth over S, so $M \otimes_R A \to S$ is also locally of finite presentation. Thus, we only need to check the following functorial criterion: if $T_0 \subset T$ is a closed subscheme of an affine scheme T over S, defined by an ideal I on T with $I^2 = 0$, then

$$(M \otimes_R A)(T) \to (M \otimes_R A)(T_0) \tag{1.2}$$

is surjective. But the map $A(T) \to A(T_0)$ is surjective, so (1.2) is surjective as well (as it is obtained via the right exact functor $M\otimes$).

The geometric connectedness property for fibers is obvious, since $M \otimes_R A$ is a direct factor A^n. □

We conclude that if A/S is an abelian scheme, then $M \otimes_R A$ is also an abelian scheme. Furthermore we have the following result on the ℓ-adic Tate modules of A.

Theorem 1. *If $T_\ell(A)$ is the Tate module of A then $M \otimes_R T_\ell(A) \cong T_\ell(M \otimes_R A)$.*

Proof. It suffices to establish the fact for finite levels, namely one needs to show the natural isomorphism

$$M \otimes_R (A[\ell^n]) \cong (M \otimes_R A)[\ell^n]$$

as group schemes.

For any S-scheme T, one has an exact sequence

$$0 \to A[\ell^n](T) \to A(T) \xrightarrow{\ell^n} A(T).$$

Tensoring with the flat R-module M we get an exact sequence

$$0 \to M \otimes_R A[\ell^n](T) \to M \otimes_R A(T) \xrightarrow{\ell^n} M \otimes_R A(T),$$

which implies that $M \otimes_R (A[\ell^n]) \cong (M \otimes_R A)[\ell^n]$. □

Lemma 1.4. *Assume that $\mathrm{End}_{R,S}(A, A) = R$. The natural map*

$$M \to \mathrm{Hom}_{R,S}(S, M \otimes_R A)$$

is an isomorphism of R-modules.

Proof. When M is a free R-module the statement is clear. In general let $M \oplus N \cong R^n$, so one has

$$\begin{CD} R^n @>\sim>> \mathrm{Hom}_{R,S}(A, A^n) \\ @A{\wr}AA @A{\wr}AA \\ M \oplus N @>>> \mathrm{Hom}_{R,S}(A, M \otimes_R A) \oplus \mathrm{Hom}_{R,S}(A, N \otimes_R A) \end{CD}$$

Thus, the lower horizontal map is also an isomorphism, so its direct summand is as well. □

2. Classical theory

Now we relate the abstract tensor construction with the classical construction over the complex field $\mathbb{C}$. In this section A is a complex abelian variety with CM by the maximal order $R = \mathcal{O}_F$ in F, where F is a CM field. Let M be a projective R-module. We will be interested in the case that M is a fractional ideal of F, but for some preliminary proofs it is convenient not to restrict to the case of rank 1.

Suppose $A = V/\Lambda$ is the canonical uniformization of A, so both V and Λ are equipped with R-action (which is $\mathbb{C}$-linear on V).

Lemma 2.1. *The $R \otimes_{\mathbb{Z}} \mathbb{C}$-module $M \otimes_R V$ is a finite dimensional $\mathbb{C}$-vector space, and $M \otimes_R \Lambda$ is naturally a lattice in it.*

Proof. The claim is obvious if M is a free R-module. In general, M is a direct summand of a free one, namely, $M \oplus N \cong R^n$ for some N. Therefore the corresponding decomposition $(M \otimes_R \Lambda) \oplus (N \otimes_R \Lambda)$ proves the claim. □

By Lemma 2.1 we can form a new Lie group, $A' = (M \otimes_R V)/(M \otimes_R \Lambda)$. By identifying $T_0(G)$ with $\ker(G(\mathbb{C}[\epsilon]) \to G(\mathbb{C}))$ for any Lie group G over $\mathbb{C}$, we have by R-flatness of M that $M \otimes_R V = M \otimes_R T_0(A)$ is naturally isomorphic to $T_0(M \otimes_R A)$.

Proposition 2.2. *The exponential map*

$$M \otimes_R V = T_0(M \otimes_R A) \xrightarrow{\exp} M \otimes_R A$$

has the kernel equal to $M \otimes_R \Lambda$, so A' is isomorphic to $M \otimes_R A$.

Proof. The claim is clear if M is free. For general M one has the following commutative diagram

$$\begin{CD} T_0(A^n) = T_0(R^n \otimes_R V) @>\exp>> A^n \\ @A{\wr}AA @A{\wr}AA \\ (M \oplus N) \otimes_R V @>\exp_M \oplus \exp_N>> (M \otimes_R A) \oplus (N \otimes_R A) \end{CD}$$

As $\mathrm{Ker}(\exp) = \mathrm{Ker}(\exp_{\mathrm{M}}) \oplus \mathrm{Ker}(\exp_{\mathrm{N}})$, we have $M \otimes_R \Lambda = \mathrm{Ker}(\exp_{\mathrm{M}})$. □

Corollary 2.3. *The set $CM(\mathcal{O}_F)$ of isomorphism classes of complex abelian varieties with CM by $\mathcal{O}_F$ and a fixed CM type Φ is a principal homogeneous space under $\mathrm{Pic}(\mathcal{O}_F)$ via the Serre operation $A \mapsto \mathfrak{a} \otimes_{\mathcal{O}_F} A$.*

Proof. It is known that all such complex abelian varieties are of the form $(\mathbb{R} \otimes_{\mathbb{Q}} F)_\Phi/\mathfrak{a}$, where $\mathfrak{a}$ is a nonzero integral ideal of $\mathcal{O}_F$. By Proposition 2.2 this is $\mathfrak{a} \otimes_{\mathcal{O}_F} A$ with $A = (\mathbb{R} \otimes_{\mathbb{Q}} F)_\Phi/\mathcal{O}_F$. By Lemma 1.4 we have

$$\mathrm{Hom}_{\mathcal{O}_F, \mathbb{C}}(\mathfrak{a} \otimes A, \mathfrak{b} \otimes A) \cong \mathfrak{a}^{-1}\mathfrak{b}.$$

In particular, $\mathfrak{a}\otimes_{\mathcal{O}_F} A$ is an abelian variety with CM by $\mathcal{O}_F$ and CM type Φ. Also, $\mathfrak{a}\otimes_{\mathcal{O}_F} A$ is $\mathcal{O}_F$-linearly isomorphic to $\mathfrak{b}\otimes A$ over $\mathbb{C}$ if and only if $\mathfrak{a}^{-1}\mathfrak{b}$ has a generator as an $\mathcal{O}_F$-module, which is to say that it is trivial in $\mathrm{Pic}(\mathcal{O}_F)$. □

3. Frobenius lifting

Let $(A,\imath)$ be an abelian variety with CM by $\mathcal{O}_F$ of type Φ over a number field $K\subset\mathbb{C}$, where $\imath:\mathcal{O}_F\to\mathrm{End}_K(A)$ denotes the $\mathcal{O}_F$-action on A. Thus, K contains the reflex field E of the CM type Φ. Let $\bar{\mathbb{Q}}\subset\mathbb{C}$ be the algebraic closure of $\mathbb{Q}$. Now we take $\sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/E)$ and choose L/E a finite Galois extension inside $\bar{\mathbb{Q}}$ containing K, such that all geometric morphisms between A_L and A_L^σ are defined over L. By Chebotarev one can choose a prime $\mathfrak{P}$ of $\mathcal{O}_L$ such that $\sigma|_L=\left(\dfrac{L/K}{\mathfrak{P}}\right)$ (the Artin symbol) and A_L has good reduction at $\mathfrak{P}$. We write $k=\mathcal{O}_L/\mathfrak{P}$ and let $q=\#\mathcal{O}_K/\mathfrak{p}$ with $\mathfrak{p}=\mathcal{O}_K\cap\mathfrak{P}$, so σ reduces to the arithmetic q-Frobenius on k.

As A_L and A_L^σ are $\mathcal{O}_F$-linearly isogenous over L (since they have the same CM type and so are $\mathcal{O}_F$-linearly isogenous over $\mathbb{C}$), they both have good reduction at $\mathfrak{P}$. We write $(\mathcal{A},\imath)$ and $(\mathcal{A}^\sigma,\imath^\sigma)$ to denote the Néron models of $(A,\imath)$ and $(A^\sigma,\imath^\sigma)$ over Spec $\mathcal{O}_{L,\mathfrak{P}}$ respectively. We write $(\bar{A},\bar{\imath})$ to be denote the reduction of $(\mathcal{A},\imath)$ at $\mathfrak{P}$.

Lemma 3.1. *The reduction map*

$$\mathrm{Hom}_L((A_L,\imath),(A_L^\sigma,\imath^\sigma))=\mathrm{Hom}_{\mathcal{O}_{L,\mathfrak{P}}}((\mathcal{A},\imath),(\mathcal{A}^\sigma,\imath^\sigma))\to\mathrm{Hom}_k((\bar{A},\bar{\imath}),(\bar{A}^{(q)},\bar{\imath}^{(q)}))$$

is an isomorphism.

Proof. By Proposition 2.2, there is an $\mathcal{O}_F$-linear isomorphism $A_L^\sigma\cong\mathfrak{a}\otimes_{\mathcal{O}_F}A_L$ over L for some fractional F-ideal $\mathfrak{a}$ (strictly speaking, such an isomorphism is built over $\mathbb{C}$, but it descends to L by our choice of L). Therefore, the Néron model $\mathcal{A}^\sigma$ (over $\mathcal{O}_{L,\mathfrak{P}}$) of A_L^σ is given by the abelian scheme $\mathfrak{a}\otimes_{\mathcal{O}_F}\mathcal{A}$. Lemma 1.4 applied over $\mathcal{O}_{L,\mathfrak{P}}$ and k gives the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{O}_{L,\mathfrak{P}}}((\mathcal{A},\imath),(\mathcal{A}^\sigma,\imath^\sigma)) & \longrightarrow & \mathrm{Hom}_k((\bar{A},\bar{\imath}),(\bar{A}^{(q)},\bar{\imath}^{(q)})) \\
\uparrow\wr & & \uparrow\wr \\
\mathfrak{a} & \xrightarrow{\ \mathrm{id}\ } & \mathfrak{a}
\end{array}\tag{3.1}$$

which concludes the proof. □

Corollary 3.2. *The relative Frobenius map* $F_{\mathfrak{P}}:\ \bar{A}\to\bar{A}^{(q)}$ *over* k *can be lifted into* $\mathrm{Hom}_{\mathcal{O}_{L,\mathfrak{P}}}(\mathcal{A},\mathcal{A}^\sigma)$ $=\mathrm{Hom}_L(A_L,A_L^\sigma)$ *uniquely.*

Proof. We only need to notice that $F_{\mathfrak{P}}\in\mathrm{Hom}_k((\bar{A},\bar{\imath}),(\bar{A}^{(q)},\bar{\imath}^{(q)}))$, which is to say $F_{\mathfrak{P}}$ is $\mathcal{O}_F$-linear: this is clear due to the definition of $\bar{\imath}^{(q)}$ and the universal functoriality of relative Frobenius morphisms. □