SERRE’S CONSTRUCTION

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In this note we will describe Serre’s tensor construction with the aim of applying it to abelian varieties with CM.

1. The construction

In this note $R$ always denotes a commutative algebra (with unit), and $M$ denotes a finite projective $R$-module. We let $A$ denote an $R$-module scheme over $S$. Also, if $X \to S$ and $T \to S$ are $S$-schemes, $X_T$ denotes the $T$-scheme $X \times_S T$.

**Proposition 1.1.** The functor

$$T \to M \otimes_R A(T) = \text{Hom}_R(M^\vee, A(T))$$

on $S$-schemes is represented by an $R$-module scheme over $S$, denoted by $M \otimes_R A$. Here $M^\vee$ is the dual $\text{Hom}_R(M, R)$ of $M$.

**Proof.** First if $M = R^n$ is a free $R$-module, then it is clear that $M \otimes_R A = A^n$. For general $M$, as $M^\vee$ is finite projective, we choose a finite presentation of $M^\vee$:

$$R^n \to R^n \to M^\vee \to 0.$$ 

The functor $\text{Hom}_R(M^\vee, A)$ is the kernel of the induced morphism from $\text{Hom}_R(R^n, A) = A^n$ to $\text{Hom}_R(R^m, A) = A^m$, so it is representable. It is obvious that $M \otimes_R A$ carries a natural $R$-module structure as a functor.  

**Lemma 1.2.** For $T \to S$, we have a natural isomorphism $(M \otimes_R A)_T \cong M \otimes_R A_T$ as $R$-module schemes over $T$.

**Proof.** For any $T$-scheme $X$ one has

$$(M \otimes_R A)_T(X) = M \otimes_R \text{Hom}_S(X, A) = M \otimes_R (A_T(X)),$$

which proves the claim.  

It is obvious that for two finite projective $R$-modules $M$ and $N$ we have a natural isomorphism of $R$-module schemes

(1.1) 

$$(M \oplus N) \otimes_R A \cong (M \otimes_R A) \times (N \otimes_R A).$$

**Proposition 1.3.** The following properties of $A \to S$ are preserved by the tensor construction: locally finite type, quasi-compactness, locally finite presentation, separatedness, properness, smoothness, flatness, and geometric connectedness of fibers.
Proof. Preservation of the properties of being locally finite type, quasi-compact, or locally of finite presentation are clear by construction, so we only show the preservation of the last five properties.

Suppose $A$ is separated over $S$. For any valuation ring $V$ over $S$, the map

$$\text{Hom}_S(\text{Spec } V, A) \rightarrow \text{Hom}_S(\text{Spec } Q_V, A)$$

is injective, where $Q_V$ is the quotient ring of $V$. The map is still injective when tensoring with the flat $R$-module $M$, so $M \otimes_R A$ is proper over $S$. The case of properness goes similarly.

For flatness we use the following argument of Serre. As $M \otimes_R A$ is a direct factor of the $S$-flat scheme $A^n$, it suffices to check that if $X$ and $Y$ are $S$-schemes with $Y(S)$ nonempty and $X \times_S Y$ is $S$-flat, then $X$ is $S$-flat. We may assume that $S = \text{Spec } D$ is local and that both $X = \text{Spec } B$ and $Y = \text{Spec } C$ are affine. Thus, we get two ring extensions $D \rightarrow B$ and $D \rightarrow C$ with $B \otimes_D C$ flat over $D$. Since $D \rightarrow C$ has a section we have a splitting $C = D \oplus I$ as $R$-modules. Thus, $B$ is a direct $D$-module summand of the flat $D$-module $B \otimes_D C$.

Let $A$ be smooth over $S$, so $M \otimes_R A \rightarrow S$ is also locally of finite presentation. Thus, we only need to check the following functorial criterion: if $T_0 \subset T$ is a closed subscheme of an affine scheme $T$ over $S$, defined by an ideal $I$ on $T$ with $I^2 = 0$, then

$$(1.2) \quad (M \otimes_R A)(T) \rightarrow (M \otimes_R A)(T_0)$$

is surjective. But the map $A(T) \rightarrow A(T_0)$ is surjective, so (1.2) is surjective as well (as it is obtained via the right exact functor $M \otimes$).

The geometric connectedness property for fibers is obvious, since $M \otimes_R A$ is a direct factor $A^n$. 

We conclude that if $A/S$ is an abelian scheme, then $M \otimes_R A$ is also an abelian scheme. Furthermore we have the following result on the $\ell$-adic Tate modules of $A$.

**Theorem 1.** If $T_\ell(A)$ is the Tate module of $A$ then $M \otimes_R T_\ell(A) \cong T_\ell(M \otimes_R A)$.

**Proof.** It suffices to establish the fact for finite levels, namely one needs to show the natural isomorphism

$$M \otimes_R (A[\ell^n]) \cong (M \otimes_R A)[\ell^n]$$

as group schemes.

For any $S$-scheme $T$, one has an exact sequence

$$0 \rightarrow A[\ell^n](T) \rightarrow A(T) \rightarrow A(T).$$

Tensoring with the flat $R$-module $M$ we get an exact sequence

$$0 \rightarrow M \otimes_R A[\ell^n](T) \rightarrow M \otimes_R A(T) \rightarrow M \otimes_R A(T),$$

which implies that $M \otimes_R (A[\ell^n]) \cong (M \otimes_R A)[\ell^n]$. 

**Lemma 1.4.** Assume that $\text{End}_{R,S}(A, A) = R$. The natural map

$$M \rightarrow \text{Hom}_{R,S}(S, M \otimes_R A)$$

is an isomorphism of $R$-modules.
Proof. When $M$ is a free $R$-module the statement is clear. In general let $M \oplus N \cong R^n$, so one has

$$
\begin{array}{ccc}
R^n & \cong & \text{Hom}_{R,S}(A, A^n) \\
\downarrow & & \downarrow i \\
M \oplus N & \longrightarrow & \text{Hom}_{R,S}(A, M \otimes_R A) \oplus \text{Hom}_{R,S}(A, N \otimes_R A)
\end{array}
$$

Thus, the lower horizontal map is also an isomorphism, so its direct summand is as well. \hfill \Box

2. CLASSICAL THEORY

Now we relate the abstract tensor construction with the classical construction over the complex field $\mathbb{C}$. In this section $A$ is a complex abelian variety with CM by the maximal order $R = \mathcal{O}_F$ in $F$, where $F$ is a CM field. Let $M$ be a projective $R$-module. We will be interested in the case that $M$ is a fractional ideal of $F$, but for some preliminary proofs it is convenient not to restrict to the case of rank 1.

Suppose $A = V/\Lambda$ is the canonical uniformization of $A$, so both $V$ and $\Lambda$ are equipped with $R$-action (which is $\mathbb{C}$-linear on $V$).

**Lemma 2.1.** The $R \otimes_{\mathbb{Z}} \mathbb{C}$-module $M \otimes_R V$ is a finite dimensional $\mathbb{C}$-vector space, and $M \otimes_R \Lambda$ is naturally a lattice in it.

**Proof.** The claim is obvious if $M$ is a free $R$-module. In general, $M$ is a direct summand of a free one, namely, $M \oplus N \cong R^n$ for some $N$. Therefore the corresponding decomposition $(M \otimes_R \Lambda) \oplus (N \otimes_R \Lambda)$ proves the claim. \hfill \Box

By Lemma 2.1 we can form a new Lie group, $A' = (M \otimes_R V)/(M \otimes_R \Lambda)$. By identifying $T_0(G)$ with $\ker(G(\mathbb{Q}) \to G(\mathbb{C}))$ for any Lie group $G$ over $\mathbb{C}$, we have by $R$-flatness of $M$ that $M \otimes_R V = M \otimes_R T_0(A)$ is naturally isomorphic to $T_0(M \otimes_R A)$.

**Proposition 2.2.** The exponential map

$$
M \otimes_R V = T_0(M \otimes_R A) \xrightarrow{\exp} M \otimes_R A
$$

has the kernel equal to $M \otimes_R \Lambda$, so $A'$ is isomorphic to $M \otimes_R A$.

**Proof.** The claim is clear if $M$ is free. For general $M$ one has the following commutative diagram

$$
\begin{array}{ccc}
T_0(A^n) = T_0(R^n \otimes_R V) & \xrightarrow{\exp} & A^n \\
\downarrow & & \downarrow i \\
(M \oplus N) \otimes_R V & \xrightarrow{\exp_M \otimes \exp_N} & (M \otimes_R A) \oplus (N \otimes_R A)
\end{array}
$$

As $\text{Ker}(\exp) = \text{Ker}(\exp_M) \oplus \text{Ker}(\exp_N)$, we have $M \otimes_R \Lambda = \text{Ker}(\exp_M)$. \hfill \Box

**Corollary 2.3.** The set $CM(\mathcal{O}_F)$ of isomorphism classes of complex abelian varieties with CM by $\mathcal{O}_F$ and a fixed CM type $\Phi$ is a principal homogeneous space under $\text{Pic}(\mathcal{O}_F)$ via the Serre operation $A \mapsto a \otimes_{\mathcal{O}_F} A$.

**Proof.** It is known that all such complex abelian varieties are of the form $(\mathbb{R} \otimes_{\mathbb{Q}} F)_{\phi}/a$, where $a$ is a nonzero integral ideal of $\mathcal{O}_F$. By Proposition 2.2 this is $a \otimes_{\mathcal{O}_F} A$ with $A = (\mathbb{R} \otimes_{\mathbb{Q}} F)_{\phi}/\mathcal{O}_F$. By Lemma 1.4 we have

$$
\text{Hom}_{\mathcal{O}_F, \mathbb{C}}(a \otimes A, b \otimes A) \cong a^{-1} b.
$$
In particular, \( a \otimes_{O_F} A \) is an abelian variety with CM by \( O_F \) and CM type \( \Phi \). Also, \( a \otimes_{O_F} A \) is \( O_F \)-linearly isomorphic to \( b \otimes A \) over \( \mathbb{C} \) if and only if \( a^{-1}b \) has a generator as an \( O_F \)-module, which is to say that it is trivial in \( \text{Pic}(O_F) \).

\[ \square \]

3. FROBENIUS LIFTING

Let \((A, i)\) be an abelian variety with CM by \( O_F \) of type \( \Phi \) over a number field \( K \subset \mathbb{C} \), where \( i : O_F \to \text{End}_K(A) \) denotes the \( O_F \)-action on \( A \). Thus, \( K \) contains the reflex field \( E \) of the CM type \( \Phi \). Let \( \overline{\mathbb{Q}} \subset \mathbb{C} \) be the algebraic closure of \( \mathbb{Q} \). Now we take \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/E) \) and choose \( L/E \) a finite Galois extension inside \( \overline{\mathbb{Q}} \) containing \( K \), such that all geometric morphisms between \( A_L \) and \( A'_L \) are defined over \( L \). By Chebotarev one can choose a prime \( \mathfrak{p} \) of \( O_L \) such that \( \sigma|_L = \left( \frac{L/K}{\mathfrak{p}} \right) \) (the Artin symbol) and \( A_L \) has good reduction at \( \mathfrak{p} \). We write \( k = O_L/\mathfrak{p} \) and let \( q = \# O_K/\mathfrak{p} \) with \( \mathfrak{p} = O_K \cap \mathfrak{p} \), so \( \sigma \) reduces to the arithmetic \( q \)-Frobenius on \( k \).

As \( A_L \) and \( A'_L \) are \( O_F \)-linearly isogenous over \( L \) (since they have the same CM type and so are \( O_F \)-linearly isogenous over \( \overline{\mathbb{Q}} \), they both have good reduction at \( \mathfrak{p} \)). We write \((A, i)\) and \((A', i')\) to denote the Néron models of \((A, i)\) and \((A', i')\) over \( \text{Spec} O_L, \mathfrak{p} \) respectively. We write \((\overline{A}, \overline{i})\) to be denote the reduction of \((A, i)\) at \( \mathfrak{p} \).

**Lemma 3.1.** The reduction map

\[
\text{Hom}_L((A_L, i), (A'_L, i')) = \text{Hom}_{O_L, \mathfrak{p}}((A, i), (A', i')) \to \text{Hom}_k((\overline{A}, \overline{i}), (\overline{A}', \overline{i}'))
\]

is an isomorphism.

**Proof.** By Proposition 2.2, there is an \( O_F \)-linear isomorphism \( A'_L \cong a \otimes_{O_F} A_L \) over \( L \) for some fractional \( F \)-ideal \( a \) (strictly speaking, such an isomorphism is built over \( \mathbb{C} \), but it descends to \( L \) by our choice of \( L \)). Therefore, the Néron model \( A' \) (over \( O_L, \mathfrak{p} \)) of \( A'_L \) is given by the abelian scheme \( a \otimes_{O_F} A \). Lemma 1.4 applied over \( O_L, \mathfrak{p} \) and \( k \) gives the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_{O_L, \mathfrak{p}}((A, i), (A', i')) & \xrightarrow{\text{id}} & \text{Hom}_k((\overline{A}, \overline{i}), (\overline{A}', \overline{i}')) \\
\downarrow & & \downarrow \\
a & \xrightarrow{\text{id}} & a \\
\end{array}
\]

which concludes the proof. \[ \square \]

**Corollary 3.2.** The relative Frobenius map \( F_{\mathfrak{p}} : \overline{A} \to \overline{A}' \) over \( k \) can be lifted into \( \text{Hom}_{O_L, \mathfrak{p}}(A, A') \)

\[
= \text{Hom}_L(A_L, A'_L)
\]

uniquely.

**Proof.** We only need to notice that \( F_{\mathfrak{p}} \in \text{Hom}_k((\overline{A}, \overline{i}), (\overline{A}', \overline{i}')) \), which is to say \( F_{\mathfrak{p}} \) is \( O_F \)-linear: this is clear due to the definition of \( \overline{i}' \) and the universal functoriality of relative Frobenius morphisms. \[ \square \]