POLARIZATIONS

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If A is an abelian variety over a field, then to give a projective embedding of A is more or less to give an ample line bundle on A. Over $\mathbf{C}$, such data can be expressed in terms of a positive-definite Riemann form on the homology lattice. Hence, we consider ample line bundles on general abelian varieties to be a "positivity" structure. In a sense that will be explained, just as we view abelian varieties to be a jazzed-up sort of linear algebra datum (as the analytic uniformization makes precise over $\mathbf{C}$), we will interpret the data of ample line bundles as akin to the specification of positive-definite quadratic forms. This will be encoded in the structure to be called a *polarization* on the abelian variety. It is a fundamental fact that polarized abelian varieties have finite automorphism groups, and this makes them especially well-suited to the formulation of well-behaved moduli problems; we will say nothing more on such aspects of polarizations in these notes.

In these notes we provide quite a few references to [Mum] for some further technical details, and the earlier talks (and subsequent write-ups) on analytic and algebraic properties of abelian varieties addressed most such points. We differ slightly from the notes on the analytic theory insofar as our Appel–Humbert normalizations are presented without requiring a choice of $\sqrt{-1} \in \mathbf{C}$.

1. Correspondences and bilinear forms

Let k be a field.

Definition 1.1. A *correspondence* between an ordered pair of abelian varieties A and A' over k is a triple $(\mathscr{L}, i, i')$ where $\mathscr{L}$ is a line bundle of $A \times A'$ and

$$i : \mathscr{O}_{A'} \simeq (e \times 1)^*\mathscr{L}, \quad i' : \mathscr{O}_A \simeq (1 \times e')^*\mathscr{L}$$

are trivializations that are compatible in the sense that $e'^*(i) = e^*(i')$ as isomorphisms $\mathscr{O}_{\mathrm{Spec}\, k} \simeq (e \times e')^*\mathscr{L}$.

The notion of *isomorphism* for a pair of correspondences (between A and A') is defined in an evident manner. The isomorphisms i and i' in Definition 1.1 are unique up to multipliers $c \in \Gamma(A, \mathscr{O}_A^\times) = k^\times$ and $c' \in k^\times$ that must satisfy $cc' = 1$ in order to maintain the compatibility condition $e'^*(i) = e^*(i')$. Hence, for any two correspondences $(\mathscr{L}, i_1, i'_1)$ and $(\mathscr{L}, i_2, i'_2)$ with the same underlying line bundle, there exists an isomorphism $(\mathscr{L}, i_1, i'_1) \simeq (\mathscr{L}, i_2, i'_2)$ via multiplication by some $c \in k^\times$ on $\mathscr{L}$. This isomorphism is unique, due to the important:

Lemma 1.2. *Correspondences admit no non-trivial automorphisms.*

Proof. An automorphism of a line bundle on A is given by multiplication by an element c of $\Gamma(A, \mathscr{O}_A^\times) = k^\times$, and the compatibility with i forces $c = 1$. ∎

We define $\mathrm{Corr}(A, A')$ to be the set of isomorphism classes of correspondences $(\mathscr{L}, i, i')$ as above. This set is an abelian group via the operation of tensor product as the multiplication and formation of the dual line bundle as the inverse; the identity correspondence is $(\mathscr{O}_{A \times A'}, i_0, i'_0)$ where the trivializations i_0 and i'_0 are the evident ones. In view of what has been said above, $\mathrm{Corr}(A, A')$ maps isomorphically to the subgroup of isomorphism classes of line bundles $\mathscr{L}$ on $A \times A'$ for which $\mathscr{L}|_{e \times A'}$ and $\mathscr{L}|_{A \times e'}$ are trivial on A' and A respectively.

A key point is that the trivializations on the two trivial line bundles $(e \times 1)^*\mathscr{L}$ and $(1 \times e')^*\mathscr{L}$ do not need to be specified, as the choices that exist compatibly along $e \times e'$ are a torsor for $\mathbf{G}_m(k) = k^\times$, and $\mathbf{G}_{m/k}$ is the automorphism scheme of a line bundle on A. Put another way, we do not lose anything by passing between isomorphism classes of triples $(\mathscr{L}, i, i')$ (which have no non-trivial automorphisms) and elements of

$$\ker(\mathrm{Pic}(A \times A') \to \mathrm{Pic}(A) \times \mathrm{Pic}(A')).$$

Example 1.3. The preceding insensitivity of forgetting i and i' pervades the entirety of [Mum], and can lead to some confusion when working in a relative situation (even over a base field that may simply not be algebraically closed). The point is that in relative situations, unlike the "geometric" case, one often needs to carry out descent and so it then becomes crucial to be attentive to the presence of the rigidifying structures i and i'.

A typical example of the dichotomy between the two points of view (correspondences versus line bundles) is seen through the notions of *Poincaré bundle* and *Poincaré correspodence*. The Poincaré correspondence is correspondence

$$(\mathscr{P}_A, \iota_A, \iota'_A) \in \mathrm{Corr}(A, A^\vee)$$

that serves as a universal correspondence on products $A \times A'$ for varying abelian varieties A'. The point $0 \in A^\vee(k)$ is associated to the trivial correspondence between A and the vanishing abelian variety $\mathrm{Spec}(k)$. In view of the preceding mechanism for harmlessly forgetting the trivializing isomorphisms in the definition of a correspondence, one often works more directly with the underlying line bundle $\mathscr{P}_A$ rather than with the full correspondence. This line bundle is the Poincaré bundle, and it is usually considered as a universal object. However, to avoid the interference of ambiguous isomorphisms it is more precise to consider $\mathscr{P}_A$ as coming equipped with specific trivializations along the 0-sections of the factors, and it is only this full triple of data (the Poincaré correspondence) that has a good universal property.

When the base is not a field (and more specifically is not a local ring) then there is no doubt that the framework of correspondences is superior to that of bare line bundles (up to isomorphism) because correspondences are more "rigid" and in particular are amenable to descent. This rigidity and good behavior with respect to descent are a contrast with the case of line bundles (considered up to isomorphism), as was explained in the notes on the algebraic theory of abelian varieties (where it was seen that rigification was required to circumvent non-representability of relative Picard functors).

There is an evident isomorphism $\mathrm{Corr}(A, A') \simeq \mathrm{Corr}(A', A)$ via pullback along the flip isomorphism $A \times A' \simeq A' \times A$. Hence, in the special case $A' = A$ we get an involution on $\mathrm{Corr}(A, A)$ and so it makes sense to ask if a correspondence is invariant under this involution. That is, if $(\mathscr{L}, i_1, i_2)$ is a correspondence on $A \times A$ and if $s : A \times A \simeq A \times A$ is $s(x, y) = (y, x)$ then $(s^*\mathscr{L}, i_2, i_1)$ is a correspondence and we may ask if it is isomorphic to $(\mathscr{L}, i_1, i_2)$ (such an isomorphism being unique if it exists).

Definition 1.4. A *symmetric* correspondence in $\mathrm{Corr}(A, A)$ is a correspondence that is invariant under the natural involution.

Recall that in linear algebra, a bilinear pairing $B : V \times V \to F$ is symmetric when it is invariant under the flip involution on $V \times V$. We shall now show that in the complex-analytic case, correspondences and symmetric correspondences encode the data of certain bilinear pairings and symmetric bilinear pairings respectively (though there will also be some intervention of sesquilinear pairings over $\mathbf{C}$, due to our decision to follow Mumford's convention in [Mum] to uniformize a complex torus A as $\mathrm{H}^0(A, \Omega^1_A)^\vee/\mathrm{H}_1(A, \mathbf{Z})$ rather than as $\mathrm{H}^1(A, \mathscr{O}_A)^\vee \backslash \mathrm{H}^1_{\mathrm{dR}}(A)^\vee/\mathrm{H}_1(A, \mathbf{Z})$).

Let $k = \mathbf{C}$. We have complex-analytic uniformizations $A \simeq V/\Lambda$ and $A' \simeq V'/\Lambda'$, so there is a natural uniformization

$$A \times A' = (V \oplus V')/(\Lambda \oplus \Lambda').$$

Since a correspondence is a certain kind of line bundle on $A \times A'$, in order to describe it concretely we first recall the Appel–Humbert theorem that describes line bundles on any uniformized complex torus.

If W/L is a uniformized complex torus, then for a Hermitian form

$$H : W \times W \to \mathbf{C}$$

whose imaginary component H_{im} is $\mathbf{Z}(1)$-valued on $L \times L$ and a map of sets

$$\alpha : L \to S^1 = \{z \in \mathbf{C}^\times \mid |z| = 1\}$$

satisfying the cocycle condition $\alpha(\ell_1 + \ell_2) = e^{H_{\mathrm{im}}(\ell_1, \ell_2)/2}\alpha(\ell_1)\alpha(\ell_2) = \pm\alpha(\ell_1)\alpha(\ell_2)$, the line bundle $\mathscr{L}(H, \alpha)$ on W/L is the quotient of $\mathbf{C} \times W$ by the L-action

$$[\ell](c, w) = (\alpha(\ell)e^{H(c,\ell)/2 + H(\ell,\ell)/4}c, w + \ell)$$

covering the translation action by L on W. The set of such pairs (H, α) forms a group in an evident manner, and $(H, \alpha) \mapsto \mathscr{L}(H, \alpha)$ is a group homomorphism into $\mathrm{Pic}(W/L)$. The *Appel–Humbert theorem* says that this is an isomorphism: every line bundle on W/L is isomorphic to $\mathscr{L}(H, \alpha)$ for some Hermitian form H on W and some S^1-valued function α on L as above, with the pair (H, α) uniquely determined in this way.

We apply the preceding discussion in the case $W = V \oplus V'$ and $L = \Lambda \oplus \Lambda'$ to describe the conditions on a line bundle $\mathscr{L}$ on $A \times A'$ that encode the properties of triviality for its restrictions along the 0-sections of the two factors. We have $\mathscr{L} = \mathscr{L}(H, \alpha)$ for some

$$H : (V \oplus V') \times (V \oplus V') \to \mathbf{C}, \quad \alpha : \Lambda \oplus \Lambda' \to S^1,$$

and the triviality conditions say exactly

$$H((0, v_1'), (0, v_2')) = 0 = H((v_1, 0), (v_2, 0)), \quad \alpha(\ell, 0) = 1 = \alpha(0, \ell')$$

for $v_i \in V$, $v_j' \in V'$, $\ell \in \Lambda$, $\ell' \in \Lambda'$. Thus, the sesquilinear pairing $B : V \times V' \to \mathbf{C}$ defined by

$$B(v, v') = H((v, 0), (0, v')) \tag{1.4.1}$$

satisfies

$$H((v_1, v_1'), (v_2, v_2')) = B(v_1, v_2') + \overline{B}(v_2, v_1')$$

(thereby encoding the Hermitian property of H) and the imaginary component B_{im} is $\mathbf{Z}(1)$-valued on $\Lambda \times \Lambda'$ (thereby encoding the property that H_{im} is $\mathbf{Z}(1)$-valued on $L \times L$), with

$$\alpha(\ell, \ell') = \alpha((\ell, 0) + (0, \ell')) = \alpha(\ell, 0)\alpha(0, \ell')e^{H((\ell,0),(0,\ell'))_{\mathrm{im}}/2} = e^{B_{\mathrm{im}}(\ell,\ell')/2} = \pm 1. \tag{1.4.2}$$

Thus, the data of H and α is entirely encoded in terms of the sesquilinear form B subject only to the $\mathbf{Z}(1)$-valuedness condition for $B_{\mathrm{im}}|_{\Lambda \times \Lambda'}$.

The sesquilinear B may be expressed as a linear map $V' \to \overline{V}^{\vee}$ to the space of conjugate-linear functions ℓ on V, and the condition on B_{im} says that this linear map carries Λ into the "dual lattice" $\Lambda^{\vee}$ consisting of those conjugate-linear functionals $\ell : V \to \mathbf{C}$ for which $\ell_{\mathrm{im}}|_{\Lambda}$ is $\mathbf{Z}(1)$-valued. In other words, the data of a correspondence $(\mathscr{L}, i, i')$ on $A \times A'$ encodes exactly the data of a morphism of complex tori

$$\Phi_{\mathscr{L}} : A' = V'/\Lambda' \to \overline{V}^{\vee}/\Lambda^{\vee} = A^{\vee},$$

where we recall that the quotient $\overline{V}^{\vee}/\Lambda^{\vee}$ is an analytic model for the dual complex torus (*cf.* [Mum, §9]). (Note that $\Phi_{\mathscr{L}}$ is a map associated to a line bundle $\mathscr{L}$ on $A \times A'$, and it is not to be confused with the Mumford construction $\phi_{\mathscr{N}}$ associated to a line bundle $\mathscr{N}$ on A. The link between these two constructions will be examined later.) Since $\Phi_{\mathscr{L}}$ only depends on the isomorphism class of $\mathscr{L}$ (or, equivalently, on the isomorphism class of the correspondence $(\mathscr{L}, i, i')$) and $\Phi_{\mathscr{L} \otimes \mathscr{L}'} = \Phi_{\mathscr{L}} + \Phi_{\mathscr{L}'}$, the map $\mathscr{L} \mapsto \Phi_{\mathscr{L}}$ sets up an isomorphism of groups

$$\mathrm{Corr}(A, A') \simeq \mathrm{Hom}(A', A^{\vee}) \tag{1.4.3}$$

for complex tori A and A'.

Note that $\Phi_{\mathscr{L}}$ is an isogeny (in which case we say that $(\mathscr{L}, i, i')$ is a *non-degenerate correspondence*) if and only if the induced map $\Lambda' \to \Lambda^{\vee}$ on homology lattices is an isogeny, which is to say that the lattice pairing $B_{\mathrm{im}} : \Lambda \times \Lambda' \to \mathbf{Z}(1)$ is non-degenerate. Extending scalars from $\mathbf{Z}$ to $\mathbf{R}$, it is equivalent to say that the $\mathbf{R}$-bilinear pairing $B_{\mathrm{im}} : V \times V' \to \mathbf{R}(1)$ is a perfect duality, and since B is sesquilinear it is equivalent to say that B is a perfect pairing. It is also equivalent to say that the $\mathbf{R}$-bilinear pairing $\mathrm{Tr}_{\mathbf{C}/\mathbf{R}} \circ B$ is perfect.

Example 1.5. The line bundle $\mathscr{L}$ on $A \times A'$ is ample if and only if $H((v, v'), (v, v')) > 0$ for all nonzero $(v, v') \in V \oplus V'$, and clearly (using (1.4.1))

$$H((v, v'), (v, v')) = B(v, v') + \overline{B(v, v')} = (\mathrm{Tr}_{\mathbf{C}/\mathbf{R}} \circ B)(v, v').$$

Hence, ampleness of $\mathscr{L}$ is equivalent to positive-definiteness of the quadratic form $\mathrm{Tr}_{\mathbf{C}/\mathbf{R}} \circ B$ on the $\mathbf{R}$-vector space $V \oplus V'$. In the case that $A' = A$ and we take $V' = V$ and $\Lambda' = \Lambda$, the pullback $\Delta^* \mathscr{L}$ along the diagonal is the line bundle on $A = V/\Lambda$ with associated Appel–Humbert data

$$H_0(v_1, v_2) = B(v_1, v_2) + \overline{B}(v_2, v_1), \quad \alpha_0(\ell) = e^{B_{\mathrm{im}}(\ell,\ell)/2},$$

so this is ample if and only if $H_0(v,v) > 0$ for all nonzero $v \in V$, which is to say that that the symmetric bilinear form $\mathrm{Tr}_{\mathbf{C}/\mathbf{R}} \circ B$ on V is positive-definite.

Now recall the following variant on [Mum, p. 19]:

Lemma 1.6. *Let V be a finite-dimensional complex vector space. There is a bijective correspondence between sesquilinear pairings $\beta : V \times V' \to \mathbf{C}$ and $\mathbf{R}$-bilinear pairings $\beta_0 : V \times V' \to \mathbf{R}$ satisfying $\beta_0(iv, iv') = \beta_0(v, v')$, via $\beta_0 = \mathrm{Tr}_{\mathbf{C}/\mathbf{R}} \circ \beta$ and*

$$\beta(v, v') = \frac{\beta_0(v, v')}{2} + \frac{\beta_0(\sqrt{-1} \cdot v, v')}{2} \cdot \sqrt{-1}$$

for $(v, v') \in V \times V'$. In particular, if $V' = V$ then β is Hermitian if and only if β_0 is symmetric.

Example 1.7. Suppose $A' = A$, so it makes sense to consider the condition of symmetry for $(\mathscr{L}, i, i') \in \mathrm{Corr}(A, A')$. Let (H, α) be the Appel–Humbert data on $(V \oplus V', \Lambda \oplus \Lambda')$ associated to $\mathscr{L}$. We take $V' = V$ and $\Lambda' = \Lambda$. Symmetry of the correspondence implies

$$H((v_1, v_1'), (v_2, v_2')) = H((v_1', v_1), (v_2', v_2)) \tag{1.7.1}$$

for all $v_i \in V$ and $v_i' \in V'$, and in view of the formula for α in terms of H in (1.4.2) we see that (1.7.1) implies symmetry for α and so (1.7.1) is also sufficient for symmetry of the correspondence. The condition (1.7.1) on H may be translated in terms of B defined as in (1.4.1): it says that the sesquilinear pairing $B : V \times V \to \mathbf{C}$ satisfies $B(v_1, v_2) = \overline{B}(v_2, v_1)$, which is to say that B is Hermitian. Since B is sesquilinear, by Lemma 1.6 it is Hermitian if and only if the $\mathbf{R}$-bilinear form $\mathrm{Tr}_{\mathbf{C}/\mathbf{R}} \circ B$ is symmetric. Under this Hermitian/symmetry condition, by Example 1.5 the quadratic form on V associated to the symmetric bilinear pairing $\mathrm{Tr}_{\mathbf{C}/\mathbf{R}} \circ B$ is positive-definite on V if and only if $\mathscr{L}$ is ample on $A \times A$.

Example 1.8. Let us revisit the bijection (1.4.3) for complex tori. We wish to analyze an analogous bijection for abelian varieties over any field. Let A be an abelian variety over a field k, and let $A^\vee$ be its dual. If $(\mathscr{L}, i, i')$ is a correspondence on A then there is an associated morphism $\Phi_{\mathscr{L}} : A' \to \mathrm{Pic}_{A/k}$ characterized by $a' \mapsto (1 \times a')^*(\mathscr{L})$ on (Yoneda or geometric) points of A'. The trivialization condition from i' says $\Phi_{\mathscr{L}}(e') = 0$, so by connectivity of A' we see that $\Phi_{\mathscr{L}}$ uniquely factors through the open and closed subscheme $\mathrm{Pic}^0_{A/k}$ that is the dual abelian variety. The resulting map $A' \to A^\vee$ preserves origins and so is a map of abelian varieties. The universal property of the Picard scheme allows us to run the argument in reverse to show that any map $\Phi : A' \to A^\vee$ of abelian varieties has the form $\Phi_{\mathscr{L}}$ for a line bundle $\mathscr{L}$ on $A \times A'$ such that $(1 \times e')^* \mathscr{L}$ is trivial (this triviality is the condition $\Phi(e') = 0$). Concretely, $\mathscr{L} = (1 \times \Phi)^*(\mathscr{P}_A)$ where $\mathscr{P}_A$ is the Poincaré bundle on $A \times A'$.

It is clear that $\Phi_{\mathscr{L}} + \Phi_{\mathscr{L}'} = \Phi_{\mathscr{L} \otimes \mathscr{L}'}$ for any two line bundles $\mathscr{L}$ and $\mathscr{L}'$ that are trivial along $1_A \times e'$, but in general $\Phi_{\mathscr{L}}$ does not determine $\mathscr{L}$ up to isomorphism. Indeed, $\Phi_{\mathscr{L}} = 0$ if and only if $\mathscr{L}|_{A \times a'}$ is trivial for every geometric point a' of A, and so by the formalism of cohomology and base change this is equivalent to the condition that the coherent sheaf $\mathscr{N} = p_{2*}(\mathscr{L})$ on A is a line bundle and the natural map $p_2^* \mathscr{N} \to \mathscr{L}$ is an isomorphism. Of course, if $\mathscr{L}$ has such a form then $(e \times 1)^* \mathscr{L} \simeq \mathscr{N}$, and so we see that the possible non-triviality of $(e \times 1)^* \mathscr{L}$ is precisely the obstruction to $\Phi_{\mathscr{L}}$ determining $\mathscr{L}$ up to isomorphism. In other words, if we restrict our attention to *correspondences* then the obstruction to injectivity is trivial and we obtain an isomorphism of abelian groups

$$\mathrm{Corr}(A, A') \simeq \mathrm{Hom}_k(A', A^\vee) \tag{1.8.1}$$

that recovers (1.4.3) for $k = \mathbf{C}$. To summarize: the triviality along $1 \times e'$ corresponds to preservation of origins and triviality along $e \times 1$ ensures injectivity of (1.8.1) (and surjectivity of (1.8.1) follows from the universal property of the Picard scheme). The bijection (1.8.1) has inverse given by $\Phi \mapsto (1 \times \Phi)^*(\mathscr{P}_A)$ and so it is a restatement of the traditional universal property of the Poincaré bundle. The language of correspondences is more natural than the operationally equivalent language of maps to a dual abelian variety for the same reason that the language of bilinear pairings $W \times W' \to F$ on finite-dimensional vector spaces over a field F is more natural than the operationally equivalent language of of linear maps $W' \to W^\vee$ to a dual vector space.

Example 1.9. Let us push the preceding example one step further by considering how it interacts with double-duality for abelian varieties. The crux of the matter is that the diagram

$$\begin{array}{ccc} \mathrm{Corr}(A, A') & \xrightarrow{\simeq} & \mathrm{Hom}_k(A', A^\vee) \\ {\scriptstyle \mathrm{flip}}\downarrow & & \downarrow{\scriptstyle \mathrm{dual}} \\ \mathrm{Corr}(A', A) & \longrightarrow & \mathrm{Hom}_k(A, A'^\vee) \end{array}$$

commutes. The linear algebra analogue is easily formulated and proved, and the commutativity of the above diagram arising from A and A' is an immediate consequence of the fact that double duality for abelian varieties arises from $s_A^* \mathscr{P}_A$ serving as a Poincaré correspondence on $A^\vee \times A$ (with s_A denoting the flip isomorphism $A \times A^\vee \simeq A^\vee \times A$). It follows from the commutativity of the diagram in the case $A' = A$ that the group of symmetric correspondences on A is in natural bijection with the group of symmetric morphisms of abelian varieties $\phi : A \to A^\vee$. (Here, *symmetry* for ϕ is synonomous with being "self-dual" in the sense that $\phi^\vee \circ \iota_A = \phi$ where $\iota_A : A \simeq A^{\vee\vee}$ is the canonical double duality isomorphism whose definition rests on the unique isomorphism between the flipped correspondence $s_A^* \mathscr{P}_A$ and the Poincaré correspondence for $A^\vee$.)

To summarize, correspondences between a pair of abelian varieties are to be thought of as analogues of bilinear pairings between finite-dimensional vector spaces, non-degenerate correspondences are analogues of perfect bilinear pairings between finite-dimensional vector spaces, symmetric correspondences on $A \times A$ are analogues of symmetric bilinear forms, and symmetric correspondences whose diagonal pullback is ample are analogues of positive-definite quadratic forms.

2. Reformulation via the dual torus

Let W be a finite-dimensional vector space over a field F, and let $W^\vee$ be the dual vector space. The evaluation pairing

$$\langle \cdot, \cdot \rangle_W : W \times W^\vee \to F$$

is the universal bilinear pairing of W against another vector space in the sense that if $B : W \times W' \to F$ is any bilinear pairing then there exists a unique linear map $T_B : W' \to W^\vee$ (namely, $w' \mapsto B(\cdot, w')$) such that B is the pullback of $\langle \cdot, \cdot \rangle_W$ along $1_W \times T_B : W \times W' \to W \times W^\vee$. Double-duality and triple-duality for finite-dimensional vector spaces is the assertion that the pairing $\langle \cdot, \cdot \rangle_{W^\vee}$ is classified by a map $\iota_W : W \to W^{\vee\vee}$ that is an isomorphism and satisfies $\iota_W^\vee = \iota_{W^\vee}^{-1}$. In the special case that $W' = W$ it makes sense to ask if B is symmetric, and this is equivalent to the classifying map $T_B : W \to W^\vee$ being self-dual in the sense that $T_B^\vee \circ \iota_W = T_B$ (or, more loosely stated, $T_B = T_B^\vee$ with respect to double duality). In the case of symmetric bilinear pairings, the associated quadratic form $q_B : W \to F$ is simply the composite of B with the diagonal Δ_W, and so in the case $F = \mathbf{R}$ we see that positive-definiteness for a symmetric bilinear form B on W is encoded in terms of the pullback of $B = \langle \cdot, \cdot \rangle_W \circ (1_W \times T_B)$ along the diagonal Δ_W, or equivalently in terms of the pullback of the evaluation pairing along the composite $(1_W \times T_B) \circ \Delta_W = (1_W, T_B) : W \to W \times W^\vee$.

The preceding linear algebra formalism has straightforward analogues for abelian varieties and complex tori (and these analogues are very apt in the case of complex tori, via the complex-analytic uniformization). We consider the Poincaré bundle as the analogue of the universal evaluation pairing, and if $A = V/\Lambda$ then $\overline{V}^\vee/\Lambda^\vee$ is an analytic model for the dual with respect to which the Poincaré bundle is a correspondence whose associated Hermitian form H_A on the vector space $V \oplus \overline{V}^\vee$ arises from the sesquilinear evaluation pairing

$$B_A(v, v') = H_A((v, 0), (0, v')) = \overline{v'}(v)$$

against the conjugate dual. The universal property of the Poincaré bundle (as a classifier of correspondences) is analogous to the universal property of the evaluation pairing in linear algebra, and symmetric correspondences are thereby seen to be analogues of symmetric bilinear forms. In view of the description of positive-definiteness for symmetric $\mathbf{R}$-bilinear forms via pullback along the diagonal, we are inexorably led to make the:

Definition 2.1. A *polarization* on an abelian variety A is a symmetric correspondence $(\mathscr{L}, i, i')$ on $A \times A$ such that the line bundle $\Delta^*\mathscr{L}$ on A is ample.

Since $\mathscr{L} = (1 \times \Phi_{\mathscr{L}})^*(\mathscr{P}_A)$, by Example 1.9 we may restate the definition as follows: a polarization of A is a symmetric morphism $\phi : A \to A^\vee$ such that the line bundle

$$\Delta_A^*(1 \times \phi)^*(\mathscr{P}_A) = (1, \phi)^*\mathscr{P}_A$$

on A is ample. Such maps ϕ are necessarily isogenies. This is an analogue of the fact that a symmetric bilinear form over $\mathbf{R}$ with positive-definite associated quadratic form is automatically non-degenerate, and the proof that morphisms ϕ corresponding to polarizations are necessarily isogenies will be deduced once we work out the following example. (See Remark 2.4.)

Example 2.2. Let $\mathscr{N}$ be a line bundle on A. Let

$$\mathscr{L} = \Lambda(\mathscr{N}) \stackrel{\text{def}}{=} m^*\mathscr{N} \otimes p_1^*\mathscr{N}^{-1} \otimes p_2^*\mathscr{N}^{-1}$$

be the associated *Mumford sheaf* on $A \times A$. This is clearly a symmetric correspondence on $A \times A$ (symmetry due to the commutativity of the group law, and triviality along $e \times 1$ and $1 \times e$ due to the identity axioms for e in the group law). Hence, it is classified by a morphism $\phi_{\mathscr{N}} : A \to A^\vee$ that is necessarily *symmetric* and is given on (geometric or Yoneda) points by

$$\phi_{\mathscr{N}}(a) = (1 \times a)^*\mathscr{L} = t_a^*\mathscr{N} \otimes \mathscr{N}^{-1} \tag{2.2.1}$$

on $A \times \{a\}$. The pullback $\Delta^*\mathscr{L} = [2]^*\mathscr{N} \otimes \mathscr{N}^{\otimes(-2)}$ is isomorphic to $\mathscr{N}^{\otimes 2} \otimes \mathscr{M}$ for some $\mathscr{M}$ in $\mathrm{Pic}^0_{A/k}(k)$ (see [Mum, Cor. 3, p. 59] with $n = 2$), and so this is ample if and only if $\mathscr{N}$ is ample because tensoring against an element in $\mathrm{Pic}^0_{A/k}(k)$ does not affect the ampleness property or lack thereof. (In the complex-analytic theory this Pic^0-invariance of ampleness on A comes about because tensoring by a point in the dual torus changes only the second part of the Appel–Humbert datum (H, α), yet it is positivity of H alone that determines ampleness. In the algebraic theory, considerations with cohomological semicontinuity and the vanishing theorem in [Mum, §16] show that in a connected algebraic family of line bundles on an abelian variety, ampleness of one fiber implies ampleness of all fibers. This gives the result in the algebraic case since $\mathrm{Pic}^0_{A/k}$ is geometrically connected over k.)

In terms of our "linear algebra to abelian variety" dictionary of analogies, the passage from the line bundle $\mathscr{N}$ on A to the symmetric correspondence $\mathscr{L} = \Lambda(\mathscr{N})$ on $A \times A$ (or equivalently, to the symmetric morphism $\phi_{\mathscr{N}}$) is analogous to the passage from a bilinear form B to the symmetrized bilinear form

$$B'(x, y) = B(x + y, x + y) - B(x, x) - B(y, y) = B(x, y) + B(y, x)$$

that does not determine B (there is a skew-symmetric ambiuity), but in the case of symmetric B we see that B' is positive-definite if and only if B is positive-definite. This is a rough analogue of the equivalence between ampleness of $\mathscr{N}$ and ampleness of $\Delta^*\Lambda(\mathscr{N})$. In particular, we recover an important fact: if $\mathscr{N}$ is a line bundle on A then the symmetric morphism $\phi_{\mathscr{N}}$ is a polarization if and only if $\mathscr{N}$ is ample.

It follows from the preceding example that any abelian variety A over a field k admits a polarization. Indeed, since A is projective there exists an ample line bundle $\mathscr{N}$ on A and we therefore get a polarization $\phi_{\mathscr{N}}$. It is a fundamental fact that in the "geometric" case, every polarization arises from the $\phi_{\mathscr{N}}$-construction for a suitable ample line bundle $\mathscr{N}$:

Lemma 2.3. *If k is algebraically closed, then every symmetric morphism $\phi : A \to A^\vee$ has the form $\phi_{\mathscr{N}}$ for some line bundle $\mathscr{N}$ on A. In particular, ϕ is a polarization if and only if $\phi = \phi_{\mathscr{N}}$ for an ample line bundle $\mathscr{N}$ on A.*

Since $-\phi_{\mathscr{N}} = \phi_{\mathscr{N}^{-1}}$, it follows that if $A \neq 0$ then the negative of a polarization on A is never a polarization because the inverse of an ample line bundle on a positive-dimensional projective variety is never ample.

Proof. This is proved in [Mum, §20, Thm. 2; §23, Thm. 3]. The only step in the proof that requires k to be algebraically closed is in the analysis of the kernel of the given isogeny: the proof uses the fact (which follows from Dieudonné theory on the p-part for inseparable isogenies) that any finite commutative k-group scheme can be filtered by a chain of subgroup schemes such that the successive quotients are $\mathbf{Z}/\ell\mathbf{Z}$ for a prime ℓ or

μ_p or α_p in characteristic $p > 0$. (Such a filtration generally only exists when k is algebraically closed, and the lemma is generally false for other k.) ■

In view of Lemma 2.3, we arrive at a concrete description of polarizations over any field k: a symmetric map $\phi : A \to A^\vee$ is a polarization if and only if for some (equivalently, any) algebraically closed extension K/k, $K \otimes \phi = \phi_{\mathscr{N}}$ for an ample line bundle $\mathscr{N}$ on A_K.

Remark 2.4. Since $n\phi_{\mathscr{N}} = \phi_{\mathscr{N}^{\otimes n}}$ with $\mathscr{N}^{\otimes n} = \mathscr{O}_A(D)$ for an effective divisor D when n is sufficiently large, by [Mum, Application 1, p. 60] it follows that ampleness of $\mathscr{N}$ forces $\phi_{\mathscr{N}}$ to be an isogeny. Hence, by descent, a polarization ϕ over any field must be an isogeny.

By (2.2.1), we have now recovered the notion of polarization as introduced in an apparently *ad hoc* manner in [Mil],[Mum], and [GIT]. More specifically, by the nontrivial Lemma 2.3 we now see conceptually where the formula "$t_a^*\mathscr{N} \otimes \mathscr{N}^{-1}$" comes from in the geometric case, and we see that it really is t_a and not t_{-a} that is the translation along which we must pull back to use if we want to have morphisms ϕ such that $(1,\phi)^*\mathscr{P}_A$ is ample on A. In order to help us to better understand this point, it is instructive to work out an example that has misled a number of authors (due to the fact that $t_a^*(\mathscr{O}(D)) = \mathscr{O}(t_{-a}(D))$:

Example 2.5. Let $A = E$ be an elliptic curve and let $\mathscr{N} = \mathscr{O}(D)$ be the line bundle associated to a divisor D on E. This is ample if and only if $d = \deg D > 0$, and the associated symmetric map $\phi_{\mathscr{N}}$ is determined on (geometric or Yoneda) points by the formula

$$\phi_{\mathscr{N}}(a) = t_a^*(\mathscr{O}(D)) \otimes \mathscr{O}(-D) = \mathscr{O}(t_{-a}(D)) \otimes \mathscr{O}(-D) \simeq \mathscr{O}(t_{-a}(D) - D) \simeq \mathscr{O}([-da] - [0]) \simeq \mathscr{O}([0] - [da])$$

with the second to last isomorphism arising from the geometric description of the group law on an elliptic curve. This map is denoted ϕ_d, so $\phi_d = \phi_1 \circ [d]_E = [d]_{E^\vee} \circ \phi_1$ and in particular we see that for $d \neq 0$ the symmetric map ϕ_d is an isogeny with degree d^2 and ϕ_1 is the *unique* degree-1 polarization of E.

Note that the self-duality $E \simeq E^\vee$ used in books of Silverman, Katz–Mazur, and Serre is ϕ_{-1} and not ϕ_1, so the symmetric self-dualities ϕ used by those authors are "wrong" in the sense that $(1,\phi)^*(\mathscr{P}_E)$ is *not* ample for $\phi = \phi_{-1}$. In fact, the maps $\phi_d = [d] \circ \phi_1$ are akin to negative-definite quadratic forms for $d < 0$. By adopting the conceptual "formula-free" approach to polarizations as has been used as our foundation in these notes, with (2.2.1) derived by a calculation and not artifically introduced as a fundamental definition, we see that there is no doubt that ϕ_1 is the better self-duality to use for elliptic curves, and it is just an artifact of the passage between divisors and line bundles on a curve (which has no analogue for higher-dimensional abelian varieties) that the unique degree-1 polarization of an elliptic curve has the description $a \mapsto \mathscr{O}([-a] - [0]) \simeq \mathscr{O}([0] - [a])$ in the language of divisors. There is a similar result for Jacobians of curves of higher genus: if X is a proper, smooth, and geometrically connected curve over k and $x_0 \in X(k)$ is a rational point, then the natural map $i_{x_0} : X \to \mathrm{Pic}^0_{X/k}$ defined on points by $x \mapsto \mathscr{O}([x] - [x_0])$ has the property that $-\mathrm{Pic}^0(i_{x_0})$ rather than $\mathrm{Pic}^0(i_{x_0})$ is the (inverse of a) degree-1 polarization of the abelian variety $\mathrm{Pic}^0_{X/k}$. (Of course, if we replaced $[x] - [x_0]$ with $[x_0] - [x]$ in the definition of i_{x_0} then there would be no sign discrepancy.) We omit the verification of this claim here, as it is most conveniently done in the framework of correspondences in the generality of proper varieties (not necessarily abelian varieties) to pass between the symmetric correspondence

$$\mathscr{L} = \mathscr{O}(\Delta) \otimes p_1^*(\mathscr{O}(x_0))^{-1} \otimes p_2^*(\mathscr{O}(x_0))^{-1}$$

on $X \times X$ and a suitable symmetric correspondence on $\mathrm{Pic}^0_{X/k} \times \mathrm{Pic}^0_{X/k}$; we simply note that the key point is the calculation that the line bundle

$$\Delta_X^*\mathscr{L} \simeq (\mathscr{I}_\Delta/\mathscr{I}_\Delta^2)^{-1} \otimes \mathscr{O}(-x_0)^{\otimes 2} \simeq (\Omega^1_{X/k})^{-1} \otimes \mathscr{O}(-x_0)^{\otimes 2}$$

has degree $-(2g-2)-2 = -2g < 0$ for $g > 0$, and so it is anti-ample. Thus, it is really $\mathscr{L}^{-1}$ that corresponds to a polarization of $\mathrm{Pic}^0_{X/k}$. These annoying signs are entirely due to the translation between line bundles and divisors on curves.

It should be emphasized that although a polarization $\phi : A \to A^\vee$ over a field k acquires the form $\phi_{\mathscr{N}'}$ for an ample line bundle $\mathscr{N}'$ on $A_{k'}$ for a suitable finite extension k'/k (as we infer by descending from an algebraic closure of k via Lemma 2.3), the line bundle $\mathscr{N}'$ is not at all uniquely determined and in fact there is precisely an ambiguity of $\mathrm{Pic}^0_{A/k}(k')$ in the choice of $\mathscr{N}'$. Indeed the natural map of k-group schemes $\mathrm{Pic}_{A/k} \to \underline{\mathrm{Hom}}(A, A^\vee)$ given by $\mathscr{N} \mapsto \phi_{\mathscr{N}}$ on Yoneda-points has étale target and thus kills $\mathrm{Pic}^0_{A/k}$ for connectivity reasons, and [Mum, §8, Thm. 1] ensures that this is the entire functional kernel. The non-uniqueness of $\mathscr{N}'$ is an analogoue of the fact that the symmetrized bilinear form $B(x, y) + B(y, x)$ attached to a bilinear form B does not determine B, and this non-uniqueness prevents the possibility using descent to descend $\mathscr{N}'$ from $A_{k'}$ to A even when the map $\phi_{\mathscr{N}'}$ has been descended to a symmetric morphism ϕ over k.

There is one interesting case of a non-algebraically closed ground field for which the above non-uniqueness problem does not create obstructions to descending $\mathscr{N}'$:

Theorem 2.6. *If k is a finite field, then any symmetric isogeny $\phi : A \to A^\vee$ for an abelian variety A over k has the form $\phi = \phi_{\mathscr{N}}$ for a line bundle $\mathscr{N}$ on A.*

Proof. We begin by taking k to be a general perfect field. Let k'/k be an algebraic closure, so the map $\phi' : A_{k'} \to A^\vee_{k'}$ induced by ϕ has the form $\phi_{\mathscr{N}'}$ for some line bundle $\mathscr{N}'$ on $A_{k'}$. The non-uniqueness in $\mathscr{N}'$ is precisely given by tensoring $\mathscr{N}'$ against elements in $\mathrm{Pic}^0_{A_{k'}/k'}(k') = A^\vee(k')$, so the usual obstruction theory calculation shows that the obstruction to modifying the choice of $\mathscr{N}'$ on $A_{k'}$ so that it descends to A is given by a cohomology class in the set $\mathrm{H}^1(k'/k, A^\vee)$. In the case that k is a finite field, since $A^\vee$ is a connected smooth k-group it follows from Lang's trick that this H^1 is a singleton set. ■

3. Pairings and Rosati involution

Let $\phi : A \to A^\vee$ be a symmetric morphism, so via (1.8.1) it is associated to a symmetric correspondence $\mathscr{L} = (1 \times \phi)^*(\mathscr{P}_A)$ on $A \times A$. Double-duality of abelian varieties is compatible with Cartier duality of torsion groups up to a minus sign [Oda, Cor. 1.3(*ii*)]. That is, when we use double duality of abelian varieties then the pairings

$$\langle \cdot, \cdot \rangle_{A,n} : A[n] \times A^\vee[n] \to \mu_n, \;\; \langle \cdot, \cdot \rangle_{A^\vee,n} : A^\vee[n] \times A[n] \to \mu_n$$

are negative to each other under flip of the factors. Hence, the self-pairing

$$e_{\phi,n} = \langle \cdot, \cdot \rangle_{A,n} \circ (1 \times \phi)$$

on $A[n]$ is *skew-symmetric* when $\phi = \phi^\vee$ (as dual morphisms are adjoints with respect to the intrinsic Weil n-torsion pairings $\langle \cdot, \cdot \rangle_{A,n}$; [Oda, Thm. 1.1]). In particular, if ϕ is an isogeny then we obtain a canonical non-degenerate pairing

$$e_{\phi,\ell^\infty} : \mathrm{T}_\ell(A) \times \mathrm{T}_\ell(A) \to \mathbf{Z}_\ell(1)$$

for any prime $\ell \neq \mathrm{char}(k)$ and this pairing is skew-symmetric if and only if ϕ is symmetric. This can be seen more concretely in the complex-analytic case because the ℓ-adic pairing is (up to sign, depending on how one defines the Weil pairings) the ℓ-adic scalar extension of the $\mathbf{Z}(1)$-valued lattice pairing arising from the imaginary component of a sesquilinear form that is Hermitian if and only if ϕ is symmetric; see [Mum, §24, Thm. 1] for a proof of the comparison of Hermitian pairings and the ℓ-adic pairings.

Lemma 3.1. *If a symmetric isogeny $\phi : A \to A^\vee$ is a polarization then its degree $\deg \phi > 0$ is a perfect square.*

Proof. In the complex-analytic theory, this is proved by identifying the degree with a lattice index that is an absolute determinant of a skew-symmetric matrix (arising from the Riemann form), and the Pfaffian expresses such determinants as universal perfect squares. In the algebraic case, one can imitate this homology argument by using Tate modules (to handle the ℓ-part for $\ell \neq \mathrm{char}(k)$) and Dieudonné modules (to handle the p-part if $\mathrm{char}(k) = p > 0$) to establish that the prime-factor multiplicities of the positive integer $\deg \phi$ are all even. An alternative algebraic approach that avoids the ℓ-adic and p-adic skew-symmetric self-pairings is to extend scalars to an algebraically closed base field, where we can write $\phi = \phi_{\mathscr{N}}$ for a line bundle $\mathscr{N}$ on A, and then use the "Riemann–Roch formula" $\deg \phi_{\mathscr{N}} = \chi(\mathscr{N})^2$ [Mum, §16]. ■

Definition 3.2. A *principal polarization* of an abelian variety A over a field k is a polarization whose associated symmetric isogeny $\phi : A \to A^\vee$ is an isomorphism. A *principally polarized abelian variety* is a pair (A, ϕ) where A is an abelian variety and $\phi : A \simeq A^\vee$ is a principal polarization.

We emphasize that a principal polarization is a very special kind of self-duality: it is an isomorphism $\phi : A \simeq A^\vee$ that is not only symmetric but also gives rise to an ample pullback $(1, \phi)^*(\mathscr{P}_A)$. The negative of a principal polarization is never a principal polarization, and under the dictionary of analogies between linear algebra and abelian varieties we consider a principal polarization to be the analogue of a positive definite quadratic form over $\mathbf{Z}$ with discriminant 1 (or, in other words, a positive-definite symmetric perfect bilinear pairing on a lattice over $\mathbf{Z}$). In general, an abelian variety over a field may not admit a principal polarization over the field. An abelian variety over an algebraically closed base field always admits an isogeny to a principally polarized abelian variety [Mum, Cor. 1, p. 234]. In view of Lemma 3.1, this fact in the "geometric case" is analogous to the elementary fact that if B is a non-degenerate symmetric $\mathbf{Z}$-bilinear form on a lattice L and $\mathrm{disc}(B)$ is a square then there exists an isogenous lattice L' in $\mathbf{Q} \otimes_{\mathbf{Z}} L$ such that $B|_{L' \times L'}$ is $\mathbf{Z}$-valued with discriminant 1.

For any symmetric isogeny $\phi : A \to A^\vee$, the equality $\phi = \phi^\vee$ implies that the anti-automorphism of $\mathrm{End}_k^0(A)$ defined by

$$\lambda \mapsto \lambda^\dagger \stackrel{\text{def}}{=} \phi^{-1} \circ \lambda^\vee \circ \phi$$

is an involution (that is, $\lambda^{\dagger\dagger} = \lambda$); this is the *Rosati involution* associated to ϕ. It is clear by definition that the Rosati involution is a $\mathbf{Q}$-algebra anti-automorphism of $\mathrm{End}_k^0(A)$ whose formation is compatible with extension of the base field. Since the ordinary dual morphism is adjoint for the intrinsic Weil pairings between A and $A^\vee$, it follows that the Rosati involution computes the adjoint morphism for the non-degenerate self-pairing e_{ϕ,ℓ^∞} on $\mathrm{T}_\ell(A)$ for any prime $\ell \neq \mathrm{char}(k)$.

We are especially interested in the Rosati involution attached to a symmetric isogeny ϕ that is a polarization, which is to say for which the line bundle $(1, \phi)^*\mathscr{P}_A$ on A is ample. More precisely, a crucial ingredient in the proof of the Riemann Hypothesis for abelian varieties over finite fields is a positivity property for Rosati involutions associated to a polarization over an arbitrary field. Let us first recall a definition (and see [Mum, §19, Thm. 4] for the proof that it is $\mathbf{Z}$-valued and independent of ℓ):

Definition 3.3. Let A be an abelian variety over a field k. The linear form $\mathrm{Tr} : \mathrm{End}_k(A) \to \mathbf{Z}$ is induced by

$$\mathbf{Z}_\ell \otimes_{\mathbf{Z}} \mathrm{End}_k(A) \hookrightarrow \mathrm{End}_{\mathbf{Z}_\ell}(\mathrm{T}_\ell(A)) \stackrel{\text{trace}}{\longrightarrow} \mathbf{Z}_\ell$$

for any prime $\ell \neq \mathrm{char}(k)$.

The scalar extension $\mathrm{End}_k^0(A) \to \mathbf{Q}$ of the trace will also be denote Tr. Note that it is compatible with extension of the base field. Also, by definition this trace is insensitive to order of composition of endomorphisms, and so if $\phi : A \to A^\vee$ is any symmetric isogeny and $\lambda \mapsto \lambda^\dagger$ is the associated Rosati involution then the $\mathbf{Q}$-bilinear form

$$[\lambda_1, \lambda_2]_\phi = \mathrm{Tr}(\lambda_1 \circ \lambda_2^\dagger)$$

on $\mathrm{End}_k^0(A)$ is symmetric. In the complex analytic case, the Rosati involution associated to a symmetric isogeny is essentially an adjoint with respect to a non-degenerate Hermitian form (arising from the Appel-Humbert datum for the line bundle $(1, \phi)^*(\mathscr{P}_A)$), and so the associated symmetric $\mathbf{Q}$-bilinear trace form $[\cdot, \cdot]_\phi$ is positive-definite when ϕ is a polarization: this is because if (V, H) is a finite-dimensional Hermitian inner product space and $T : V \to V$ is a nonzero linear map then its composite TT^* against the H-adjoint is an H-normal operator that is positive-definite and hence has positive trace. By using a geometric technique, this positivity for $[\cdot, \cdot]_\phi$ in the case of polarizations on a complex torus can be proved over any ground field and thereby provides the positivity input in the proof of the Riemann Hypothesis for abelian varieties over finite fields:

Theorem 3.4. *Let A be an abelian variety over a field and let $\phi : A \to A^\vee$ be a symmetric isogeny. Let $\lambda \mapsto \lambda^\dagger$ be the associated Rosati involution on $\mathrm{End}_k^0(A)$. If ϕ is a polarization then the symmetric bilinear form is positive-definite.*

Note that this positivity condition does not characterize polarizations among all symmetric isogenies, as the endomorphism algebra may be very small (*e.g.*, it might be $\mathbf{Q}$).

Proof. We have to prove that $[\lambda, \lambda]_\phi > 0$ for any nonzero $\lambda \in \mathrm{End}_k^0(A)$, and by scaling we may restrict attention to genuine endomorphisms $\lambda \in \mathrm{End}_k(A)$. Without loss of generality, we may suppose k is algebraically closed. Hence, $\phi = \phi_{\mathscr{N}}$ for an ample line bundle $\mathscr{N}$ on A. By projectivity of A, we may write $\mathscr{N} = \mathscr{O}_A(D)$ for a divisor D on A. Letting $g = \dim A$, there is an explicit intersection-theory formula [Mum, §21]:

$$[\lambda, \lambda]_\phi = \frac{2g}{(D^g)} \cdot (D^{g-1}.\lambda^* D)$$

involving g-fold intersections among effective divisors. Such intersection numbers only depend on the linear equivalence class of the divisors, and so by general-position and Bertini-style arguments with very ample divisors we see that the intersection numbers are positive (since D is ample, and $\lambda^* D$ is an effective and non-empty divisor in A for D not containing $\lambda(A)$ because $\lambda(A)$ is a positive-dimensional closed subset of the projective variety A). ∎

Remark 3.5. A further application of integrality properties of characteristic polynomials is an important finiteness theorem that motivated Weil's interest in the concept of polarized abelian varieties: the automorphism group of a polarized abelian variety is *finite*. We refer the reader to [Mum, Thm. 5, p. 207] for this finiteness result. (This Theorem 5 proves a stronger theorem, and yields the finiteness claim as an immediate consequence.)

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