CM SEMINAR TALK NOTES

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1. Classification of Abelian Varieties with Complex Multiplication over $\mathbb{C}$

Let F_0 be a totally real number field of degree g over $\mathbb{Q}$ and let F/F_0 be a totally imaginary quadratic extension. Such an F is called a *CM field*. Let A be the ring of integers of F. Note that A is a lattice in $\mathbb{R} \otimes_{\mathbb{Q}} F$.

Definition 1.1. An *abelian variety with complex multiplication* by F (over $\mathbb{C}$) is a pair (X, i), where X is an abelian variety over $\mathbb{C}$ with dimension g and $i : F \hookrightarrow \mathrm{End}^0(X)$ is an embedding.

Remark 1.2. This is an "ad hoc" definition which is easy to use for the classification in this section. We will give a general definition of complex multiplication later.

We define two such pairs (X, i), (Y, j) to be *equivalent* if there is an isogeny $\alpha : X \to Y$ such that if $\tilde{\alpha} : \mathrm{End}^0(X) \simeq \mathrm{End}^0(Y)$ is the induced isomorphism, we have $\tilde{\alpha} \circ i = j$. It is easy to check that this is an equivalence relation. We write $X \sim Y$ if (X, i) is equivalent to (Y, j). Let $\mathcal{A}_F$ be the collection of equivalence classes. We will classify such equivalence classes.

Let D be any $\mathbb{R}$-algebra. A $\mathbb{C}$-*algebra structure* Φ on M is an $\mathbb{R}$-algebra homomorphism $\Phi : \mathbb{C} \to D$, so D can be seen as a $\mathbb{C}$-algebra via Φ. We write D_Φ to denote this $\mathbb{C}$-algebra.

Let $\mathcal{C}_F$ denote the collection of $\mathbb{C}$-algebra structures on the $\mathbb{R}$-algebra $\mathbb{R} \otimes_{\mathbb{Q}} F$.

Theorem 1.3. *There is a natural bijective map between $\mathcal{A}_F$ and $\mathcal{C}_F$.*

Proof. Let us first introduce a new collection $\mathcal{B}_F = \{(X(F, \Phi), i_\Phi)\}$. Here Φ is a $\mathbb{C}$-algebra structure on $\mathbb{R} \otimes_{\mathbb{Q}} F$, $X(F, \Phi) = (\mathbb{R} \otimes_{\mathbb{Q}} F)_\Phi / A$ is a complex torus, and $i_\Phi : A \to \mathrm{Hom}(X(F, \Phi), X(F, \Phi))$ has $i_\Phi(a)$ equal to the map induced by $1 \otimes a$ with $\mathrm{Hom}(X(F, \Phi), X(F, \Phi))$ denoting the set of endmorphisms of $X(F, \Phi)$ as a complex torus.

Obviously, there is a surjective map from the collection $\mathcal{C}_F$ to the collection $\mathcal{B}_F$. We will first construct a bijective map between $\mathcal{A}_F$ and $\mathcal{B}_F$.

Step 1: An injective map from $\mathcal{A}_F$ to $\mathcal{B}_F$.

For any pair $(X, i) \in \mathcal{A}_F$, let V be the tangent space at 0 of X and U be the kernel of exponential map from V to X, so we have a natural isomorphism $V/U \simeq X$. Then $\mathrm{End}(X)$ acts faithfully as a ring of endomorphisms of the $\mathbb{C}$-vector space V, leaving U stable. Thus, if we put $i^{-1}(\mathrm{End}(X)) = A_0 \subset F$, A_0 is an order in F and U becomes a A_0-module. Thus, $\mathbb{Q}.U \subset V$ becomes a vector space over $\mathbb{Q} \otimes_{\mathbb{Z}} A_0 = F$, and since F and $\mathbb{Q}.U$ are of dimension $2g$ over $\mathbb{Q}$, $\mathbb{Q}.U$ is a one-dimensinal F-vector space. Hence, if we choose a non-zero element $u_0 \in U$, the map $\phi : A \to \mathbb{Q}.U$ defined by $a \mapsto a.u_0$ is an injection such that U and $\phi(A)$ are commensurable lattices in V (with $\phi(A_0)$ of finite index in each). Changing X by an isogeny, we can suppose $U = \phi(A)$. The map ϕ extends to an $\mathbb{R}$-linear map which we still denote by ϕ:

$$\mathbb{R} \otimes_{\mathbb{Q}} F = \mathbb{R} \otimes_{\mathbb{Z}} A \xrightarrow{\phi} \mathbb{R} \otimes_{\mathbb{Z}} U = V$$

It follows that ϕ defines an isomorphism between *real* tori:

$$(\mathbb{R} \otimes_{\mathbb{Q}} F)/A \longrightarrow V/U = X$$

Note that if $a \in A$, then this isomorphism has been set up exactly so that the endomophism $i(a) : X \to X$ corresponds to multiplication by $1 \otimes a$ in $\mathbb{R} \otimes_{\mathbb{Q}} F$.

Next, let Φ_X denote the complex structure on the real vector space $\mathbb{R} \otimes_{\mathbb{Q}} F$ obtained by pulling back the complex structure on V via ϕ (note Φ_X is not known to be a structure of $\mathbb{C}$-algebra yet). Since multiplcation by $1 \otimes a$ in $\mathbb{R} \otimes_{\mathbb{Q}} F$ ($a \in A$) goes over via ϕ to a complex-linear map from V to V, it follows that in the complex structure Φ_X, multiplication by $1 \otimes a$ is complex-linear too. Thus, Φ_X carries elements of $\mathbb{C}$ to $\mathbb{R}$-linear endmorphism of $\mathbb{R} \otimes_{\mathbb{Q}} F$ that are also F-linear and so each elements of $\mathbb{C}$ acts through an $\mathbb{R} \otimes_{\mathbb{Q}} F$-linear endmorphism of $\mathbb{R} \otimes_{\mathbb{Q}} F$, which is a scalar multiplication. Hence, Φ_X corresponds to a map of $\mathbb{R}$-algebras $\mathbb{C} \to \mathbb{R} \otimes_{\mathbb{Q}} F$, so Φ_X is a $\mathbb{C}$-algebra structure on $\mathbb{R} \otimes_{\mathbb{Q}} F$.

The element $(X(F, \Phi_X), i_{\Phi_X}) \in \mathcal{B}_F$ clearly only depends on (X, i) up to equivalence, and so defines a map $\mathcal{A}_F \to \mathcal{B}_F$. The method of construction shows that this map is even injective, due to GAGA.

Step 2: The map constructed in step 1 is surjective.

Let us look more closely at $\mathbb{C}$-algebra structrues Φ on $\mathbb{R} \otimes_{\mathbb{Q}} F$. If σ_j $(1 \leq j \leq g)$ are the distinct embeddings of F_0 into $\mathbb{R}$, we have an isomophism of $\mathbb{R}$-algebras

$$\mathbb{R} \otimes_{\mathbb{Q}} F \longrightarrow (\mathbb{R}_{(1)} \otimes_{F_0} F) \times (\mathbb{R}_{(2)} \otimes_{F_0} F) \times \cdots \times (\mathbb{R}_{(g)} \otimes_{F_0} F) \ ,$$

$$\lambda \otimes \alpha \longmapsto (\lambda \otimes \alpha, \lambda \otimes \alpha, \cdots, \lambda \otimes \alpha,)$$

where $\mathbb{R}_{(j)} = F_0 \otimes_{F_0, \sigma_j} \mathbb{R}$. Thus, giving an $\mathbb{R}$-algebra homporphism $\Phi : \mathbb{C} \to \mathbb{R} \otimes_{\mathbb{Q}} F$ is in turn equivalent to giving $\mathbb{R}$-algbra *isomorphisms* $\Phi_j : \mathbb{C} \to \mathbb{R}_{(j)} \otimes_{F_0} F$ for $1 \leq j \leq g$. For each j, there are clearly two such possible $\mathbb{R}$-isomorphisms $\mathbb{C} \to \mathbb{R}_{(j)} \otimes_{F_0} F$. Thus, we see there are exactly 2^g possible Φ on $\mathbb{R} \otimes_{\mathbb{Q}} F$, and each Φ is uniquely determined by giving the corresponding $\mathbb{R}$-algebra isomorphisms $\Phi_j : \mathbb{C} \to \mathbb{R}_{(j)} \otimes_{F_0} F$ $(1 \leq j \leq g)$. Let $\tau_j : \mathbb{R}_{(j)} \otimes_{F_0} F \to \mathbb{C}$ be the inverse of Φ_j, so that τ_j restricted to $F \subset \mathbb{R}_{(j)} \otimes_{F_0} F$ is an embedding of F into $\mathbb{C}$ extending the embedding σ_j of F_0 into $\mathbb{R}$. Such a collection Φ or $\{\tau_j|_F\}$ of representatives for the quotient of $\mathrm{Hom}_{\mathbb{Q}-\mathrm{alg}}(F, \mathbb{C})$ modulo $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is called a *CM type* on F. (The CM type Φ_X in step 1 is called the CM type associated to (X, i).)

Fix $i = \sqrt{-1} \in \mathbb{C}$ and define $\mathrm{Im}\ z = \frac{z - \bar{z}}{2i}$ for $z \in \mathbb{C}$. Since F is CM with totally real subfield F_0, we have $F = F_0(u)$ where $u^2 = u_0 \in F_0^{\times}$ and clearly u_0 must be totally negative and u must be purely imaginary under all embeddings into $\mathbb{C}$. By weak approximation, there exists an $\epsilon_0 \in F_0^{\times}$, such that $\sigma_j(\epsilon_0) \in \mathbb{R}^{\times}$ has the same sign as $\tau_j(u)/i \in \mathbb{R}^{\times}$, so $\tau_j(\epsilon_0 u)/i \in \mathbb{R}_{>0}$ for all j. Clearly, $\alpha = N \cdot \epsilon_0 u$ lies in A for a sufficiently divisible nonzero $N \in \mathbb{Z}$, so $\alpha \in A$ satisfies $\alpha^2 \in F_0$ and $\tau_j(\alpha) = i\beta_j$ with $\beta_j \in \mathbb{R}_{>0}$ for all j.

Now let us define a Hermitian form H on the $\mathbb{C}$ -vector space $(\mathbb{R}\otimes_{\mathbb{Q}} F, \Phi)$ by putting

$$H(x,y) = 2\sum_{j=1}^{g} \beta_j \tau_j(x)\overline{\tau_j(y)}, \quad x,\ y \in \mathbb{R}\otimes_{\mathbb{Q}} F.$$

This form is clearly positive-definite, and we shall show that ImH is integral on the lattice A. Let $z \mapsto z^*$ denote the intrinsic "complex conjugation" in F. i.e., the nontrivial element in $\mathrm{Gal}(F/F_0)$. For $x, y \in A$ we have

$$\begin{aligned} \mathrm{Im}H(x,y) &= -2\mathrm{Re}\sum_{j=1}^{g} i\beta_j\tau_j(x)\overline{\tau_j(y)} \\ &= -2\mathrm{Re}\sum_{j=1}^{g}\tau_j(\alpha x y^*) \\ &= -\sum_{j=1}^{g}(\tau_j(\alpha x y^*) + \bar{\tau}_j(\alpha x y^*)) \\ &= -\mathrm{Tr}_{F/\mathbb{Q}}(\alpha x y^*) \in \mathbb{Z}. \end{aligned}$$

Thus for any complex structure Φ on $\mathbb{R}\otimes_{\mathbb{Q}} F$, and the lattice A in it, H defined above is a Riemann form and thus $X(F,\Phi)$ is an abelian variety. This completes the proof that the map from $\mathcal{A}_F$ to $\mathcal{B}_F$ is surjective.

Step 3. It remains to show that the map from $\mathcal{C}_F$ to $\mathcal{B}_F$ is injective.

Suppose there is an equivalence of $(X(F,\Phi_1), i_{\Phi_1})$ and $(X(F,\Phi_2), i_{\Phi_2})$. Then we obtain an isomorphism of $\mathbb{C}$-vector spaces $\lambda : (\mathbb{R}\otimes_{\mathbb{Q}} F)_{\Phi_1} \simeq (\mathbb{R}\otimes_{\mathbb{Q}} F)_{\Phi_2}$ such that λ is an isomorphism of F-modules. Since λ is both $\mathbb{R}$-linear and F-linear, it is easy to see λ is the multiplication by some $\alpha \in (\mathbb{R}\otimes_{\mathbb{Q}} F)^{\times}$. Since λ is $\mathbb{C}$-linear, we have $\alpha\Phi_1(c)x = \Phi_2(c)\alpha x$ for all $c \in \mathbb{C}$ and $x \in \mathbb{R}\otimes_{\mathbb{Q}} F$. Taking $x = 1$, since $\alpha \in (\mathbb{R}\otimes_{\mathbb{Q}} F)^{\times}$ and $\mathbb{R}\otimes_{\mathbb{Q}} F$ is commutative, we conclude $\Phi_1 = \Phi_2$. ☐

Remark 1.4. It is not true that $X(F,\Phi)$ is always simple. In the 3rd section, we will prove that $X(F,\Phi)$ is simple if and only if there does not exist a proper subfield L of F satisfying the following conditions:

(1) L is a quadratic extension of the totally real field $L_0 = L \cap F_0$. (So L is necessary a CM field with totally subfield L_0.)
(2) If Φ is given by the set of embeddings $\sigma'_1, \ldots, \sigma'_g$ of F into $\mathbb{C}$, then whenever $\sigma'_i|_{L_0} = \sigma'_j|_{L_0}$ we have $\sigma'_i|_L = \sigma'_j|_L$. (So in particular the set $\Phi|_L = \{\sigma'_j|_L\} \subset \mathrm{Hom}_{\,\mathbb{Q}-\mathrm{alg}}(L,\mathbb{C})$ is a CM type Ψ on L)

If such an L exists, $X(F,\Phi)$ is isogenous to $X(L,\Psi)^e$ for $e = [F:L] > 1$, where Ψ is the CM type on L in (2) above.

2. Descent to a Number Field

Theorem 2.1. *Let (X,i) be an abelian variety over $\mathbb{C}$ with complex multiplication. There exists a number field L such that (X,i) descends to L. That is, there exists an abelian variety X_0 over L with an action by F in the isogeny category of abelian varieties over L such that $X_0 \otimes_L \mathbb{C} \simeq X$ compatibly with the action of F.*

Proof. Let $\sigma_j : F_0 \hookrightarrow \mathbb{R}$, $1 \le j \le g$ be the embeddings of F_0 into $\mathbb{R}$. Since (X, i) has complex multiplication, from section 1, it has an associated CM type Φ. Clearly the $\mathbb{C} \otimes_{\mathbb{Q}} F$-module $\Gamma(X, \Omega^1_{X/\mathbb{C}}) = T_0(X)^\vee = (\mathbb{R} \otimes_{\mathbb{Q}} F)^\vee_\Phi$ is free of rank 1 over the ring $\prod_{\tau\in\Phi} \mathbb{C}_\tau$ (with $\mathbb{C}_\tau$ equal to $\mathbb{C}$ viewed as an F-algebra via τ), so

$$\Gamma(X, \Omega^1_{X/\mathbb{C}}) \simeq \prod_{j=1}^{g} V_j$$

as $\mathbb{C} \otimes_{\mathbb{Q}} F$ -module, where V_j is a rank-1 free $\mathbb{C}$-module on which F acts by the embedding $\tau \in \Phi$.

For (X', i') another abelian variety with complex multiplication by F, by section 1 we know that X' is isogeneous to X if and only if X and X' have the same CM type. Hence it is equivalent to say $\Gamma(X', \Omega^1_{X'/\mathbb{C}}) \simeq \Gamma(X, \Omega^1_{X/\mathbb{C}})$ as $\mathbb{C} \otimes_{\mathbb{Q}} F$-modules.

Claim: There exists a local noetherian subring $(R, \mathfrak{m}) \subset \mathbb{C}$ containing the Galois closure of F in $\mathbb{C}$ with R a localization of a $\mathbb{Q}$-algebra of finite type such that

(1) There exists an abelian scheme $\mathscr{X} \to \operatorname{Spec}(R)$ whose $\mathbb{C}$-fiber is isomorphic to X.
(2) $\operatorname{End}^0(\mathscr{X}) \to \operatorname{End}^0(\mathscr{X}_{\mathbb{C}}) \simeq \operatorname{End}^0(X)$ is an isomorphism, giving an action by F on $\mathscr{X}$ in the isogeny category over R.
(3) $\Gamma(\mathscr{X}, \Omega^1_{\mathscr{X}/R})$ has an $R \otimes_{\mathbb{Q}} F$-linear decompostion:

$$\Gamma(\mathscr{X}, \Omega^1_{\mathscr{X}/R}) \simeq \prod_{j=1}^{g} V'_j$$

where V'_j is a rank-1 free R-module on which F acts through $\tau_j : F \hookrightarrow R \subset \mathbb{C}$.
(4) The residue field of R is finite over $\mathbb{Q}$ and R contains a subfield L mapping isomorphically to $R/\mathfrak{m}$.

Note that the field L in (4) has to be unique and be the algebraic closure of $\mathbb{Q}$ in R (via Hensel's lemma for the henselization R^{h} or for the completion $\hat{R}$), so $L \subset \mathbb{C}$ must contain the Galois closure of F in $\mathbb{C}$.

Assume the claim is true. Let us show how claim proves the theorem.

Let $X_0 = \mathscr{X} \mod \mathfrak{m}_R$, an abelian variety of dimension g over the number field L. First note that $\operatorname{End}(\mathscr{X}) \to \operatorname{End}(X_0)$ is *injective*. Indeed, upon picking a prime l, we have an injection $\operatorname{End}(X_0) \hookrightarrow \operatorname{End}_{\mathbb{Z}_l}(T_l(X_0))$, and likewise we have an injection $\operatorname{End}(\mathscr{X}) \hookrightarrow \varprojlim \operatorname{End}(\mathscr{X}[l^n])$ since the base $\operatorname{Spec}(R)$ is connected. Because $\mathscr{X}[l^n]$ is finite étale over R, $\operatorname{End}(\mathscr{X}[l^n])$ injects into $\operatorname{End}(X_0[l^n])$, so the injectivity of $\operatorname{End}(\mathscr{X}) \hookrightarrow \varprojlim \operatorname{End}(\mathscr{X}[l^n]) \hookrightarrow \operatorname{End}_{\mathbb{Z}_l} T_l(X_0)$ and $\operatorname{End}(X_0) \hookrightarrow \operatorname{End}_{\mathbb{Z}_l}(T_l(X_0))$ give what we need. Thus, using $F \hookrightarrow \operatorname{End}(X) = \operatorname{End}(\mathscr{X})$, we conclude that X_0 has a natural structure of abelian variety with complex multiplication by F.

From (3), we see that $\Gamma(X_0, \Omega^1_{X_0/L})$ has a decompostion as an $L \otimes_{\mathbb{Q}} F$-module:

$$\Gamma(X_0, \Omega^1_{X_0/L}) \simeq \Gamma(\mathscr{X}, \Omega^1_{\mathscr{X}/R}) \otimes_R R/\mathfrak{m} = \prod_j (V'_j) \otimes_R L,$$

with F acting on the 1-dimensional L-vector space $V'_j \otimes_R L$ through $\tau_j : F \hookrightarrow L \subset \mathbb{C}$.

Make the base change by the morphism $L \hookrightarrow R \subset \mathbb{C}$, and let $X' = X_0 \otimes_L \mathbb{C}$. We see that $\Gamma(X', \Omega^1_{X'/\mathbb{C}})$ has a decompostion as a $\mathbb{C} \otimes_{\mathbb{Q}} F$-module

$$\Gamma(X', \Omega^1_{X'/\mathbb{C}}) = \prod_j (V'_j) \otimes_R L \otimes_R \mathbb{C},$$

with F acting on the 1-dimensional $\mathbb{C}$-vector space $V'_j \otimes_R L \otimes_R \mathbb{C}$ through $\tau_j : F \hookrightarrow L \subset \mathbb{C}$.

Thus, $\Gamma(X', \Omega^1_{X'/\mathbb{C}})$ has the same $\mathbb{C} \otimes_{\mathbb{Q}} F$-module decomposition as $\Gamma(X, \Omega^1_{X/\mathbb{C}})$, so X' is isogenous to X respecting the isogeny action of F on each. Let $f : X' \to X$ be an F-compatible isogeny. Let d be degree of f, so $\mathrm{Ker}(f) \subseteq X'[d]$. Note that $X'[d](\mathbb{C}) = X'[d](\overline{\mathbb{Q}})$ because char($\overline{\mathbb{Q}}$)=0, so $\mathrm{Ker}(f)$ inside $X' \simeq X_0 \otimes_L \mathbb{C}$ can be defined over a number field L'/L inside $\mathbb{C}$. Hence, the quotient $X'/\mathrm{Ker}(f)$ with its F-action desends to L', so $X'/\mathrm{Ker}(f)$ is isomophic to X respecting the F-actions.

Let us now prove the claim. Since X is a gluing of finitely many affine $\mathbb{C}$-schemes of finite type, with overlaps covered by finitely many basic open affines, there exists a finite generated $\mathbb{Q}$-subalgebra $B \subset \mathbb{C}$ such that $X \simeq \mathscr{X}_{\mathbb{C}}$ for a B-scheme $\mathscr{X}$ of finite type. The generic fiber of $\mathscr{X}$ is an abelian variety, by increasing B a bit if necessary, and by [EGA IV_3] §8-§11, we can select an affine neighborhood U of generic point of $\mathrm{Spec}(B)$ such that $\mathscr{X}|_U$ is an abelian scheme and $\mathrm{End}^0(\mathscr{X}) \hookrightarrow \mathrm{End}^0(\mathscr{X}_\eta)$ is an equality (as $\mathrm{End}^0(\mathscr{X}_\eta)$ is finitely generated over $\mathbb{Q}$). Rename U as $\mathrm{Spec}(B)$.

Let B_α be any finitely generated $\mathbb{Q}$-algebra such that $B \subseteq B_\alpha \subset \mathbb{C}$, and define $\mathscr{X}_\alpha = \mathscr{X} \otimes_B B_\alpha$. It is not hard to show that the natural maps

$$\Gamma(\mathscr{X}, \Omega^1_{\mathscr{X}/B}) \otimes_B B_\alpha \longrightarrow \Gamma(\mathscr{X}_\alpha, \Omega^1_{\mathscr{X}_\alpha/B_\alpha})$$

and $\varinjlim_\alpha \Gamma(\mathscr{X}_\alpha, \Omega^1_{\mathscr{X}_\alpha/B_\alpha}) \to \Gamma(X, \Omega^1_{X/\mathbb{C}})$ are isomorphisms. From [EGA IV_3] §8-§11 again, we can descend the $F \otimes_{\mathbb{Q}} \mathbb{C}$-module decomposition of $\Gamma(X, \Omega^1_{X/\mathbb{C}})$ to some β-level: there exists some β such that

$$\Gamma(\mathscr{X}_\beta, \Omega^1_{\mathscr{X}_\beta/B_\beta}) \simeq \prod_{\sigma_j} V_j^{(\beta)}$$

as $B_\beta \otimes_{\mathbb{Q}} F$-modules, where $V_j^{(\beta)}$ is a rank-1 free B_β-module and $V_j^{(\beta)} \otimes_{B_\beta} \mathbb{C} = V_j$ as $F \otimes_{\mathbb{Q}} \mathbb{C}$-module direct summands of $\Gamma(X, \Omega^1_{X/\mathbb{C}})$. The same holds compatibly for all $B_\alpha \supseteq B_\beta$ by base change.

Fix such a β, taking B_β sufficiently large to contain all embeddings of F into $\mathbb{C}$. Let $(R', \mathfrak{m}')$ be a local ring of B_β at a maximal ideal and let L' be the residue field of R'. By the Nullstellensatz, L' is finite over $\mathbb{Q}$.

The only problem is to lift L' back into R'. Certainly L' lifts back into the henselization R'^{h} of R', and R'^{h} is the direct limit of pointed residually-trivial local étale extensions of R', so by taking R be a sufficiently large such extension, we are done. □

3. The General Definition of Abelian Varities with Complex Multiplication

Let F be a number field with $[F : \mathbb{Q}] = 2g$.

Definition 3.1. An abelian variety with *complex multiplication by F* (over a field K) is pair (A, i), where A is an abelian variety over K with $\dim(A) = g$ and $i : F \hookrightarrow \mathrm{End}^0_K(A)$ is an embedding.

Note that we allow $\mathrm{char}(K) > 0$.

Theorem 3.2. *Let (X, i) is an abelian variety with complex multiplication by F over an arbitrary field K. Then:*

(1) *There exists a K-simple CM abelian variety X_0 over K such that X is K-isogenous to X_0^e for some $e \geq 1$. If $\mathrm{char}(K) = 0$ then $\mathrm{End}^0_K(X_0)$ is a CM field that canonically embeds F.*
(2) *If $\mathrm{char}(K) = 0$ then there exists a CM field L' with $[L' : \mathbb{Q}] = 2g$ and an embedding $i' : L' \hookrightarrow \mathrm{End}^0_K(X)$.*

Corollary 3.3. *Let (X, i) be an abelian variety with complex multiplication over $\mathbb{C}$ in the sense of the definition above Theorem 3. There exists a number field L such that X descends to L.*

Remark 3.4.

(1) In the proof of Theorem 3.2, we will see that F has to be a maximal commutative subfield of $\mathrm{End}^0_K(X)$.
(2) It is possible that a simple abelian variety over K is not simple over $\overline{K}$. However, $\mathrm{End}_{\overline{K}}(X) = \mathrm{End}_{K'}(X)$ for any sufficiently large finite separable extension K' of K. See §1, [2] for details.
(3) Suppose K is algebraically closed and $\mathrm{char}(K) = p > 0$. A simple abelian variety over K is CM if and only if X is isogenous to an abelian variety defined over a finite field. See [8] for details. Note also that if $\mathrm{char}(K) > 0$ then $\mathrm{End}^0_K(X_0)$ can fail to be commutative (e.g., supersingular elliptic curves).

Before we prove Theorem 3.2, we need to give a proof of Poincaré's complete reducibility theorem over *general ground field K*. (The proof in §12, [5] has a mistake when K is non-perfect.)

Proposition 3.5. Let X be an abelian variety over K and Y an abelian subvariety. Then there exists an abelian subvariety Z in X over K such that the natural addition map $Y \times Z \to X$ is an isogeny. In particular, the category of abelian varieties over K is artinian and semisimple.

Proof. First assume K is perfect. Let $i : Y \to X$ be the inclusion. Choose an ample invertible sheaf $\mathscr{L}$ on X and define Z to be the reduced subscheme of zero component of the kernel of $X \xrightarrow{\phi_{\mathscr{L}}} X^\vee \xrightarrow{i^\vee} Y^\vee$. Because K is *perfect*, Z is also geometrically reduced (as well as geometrically connected), so it is not hard to see that Z is an abelian variety. From the theorem on the dimension of fibres of morphisms, $\dim(Z) \geq \dim X - \dim Y$. The restriction of the morphism $X \to Y^\vee$ to Y is $\phi_{\mathscr{L}|_Y} : Y \to Y^\vee$, which has finite kernel because $\mathscr{L}|_Y$ is ample. Therefore, $Y \times_X Z$ is finite, and so $Y \times Z \to X$ is an isogeny.

Now consider the general case. Let K_p be the perfect closure of K. Let $X_p = X \otimes_K K_p$ and $Y_p = Y \otimes_K K_p$, so there exists an abelian subvariety Z' of X_p such that $Y_p \times Z'$ is K_p-isogenous to X_p. Let $f' \in \mathrm{End}_{K_p}(X_p)$ be the composite

$$X_p \to Y_p \times Z' \to Z' \hookrightarrow X_p$$

where the first step is an isogeny whose composite with $Y_p \times Z' \to X_p$ is multiplication by a nonzero integer.

According to Theorem 2.1 in [2], $\mathrm{End}_{K_p}(X_p) = \mathrm{End}_K(X)$. Thus f' can be descended to a K-endmorphism f. Let $Z = \mathrm{Im}(f)$ (scheme-theoretic closure of f). Since scheme-theoretic closure is stable under flat base change, we see $Z \otimes_K K_p \simeq \mathrm{Im}(f') = Z'$ inside of X_p, so Z is an abelian variety because Z' is in X_p. It remains to show that natural morphism $Y \times_K Z \to X$ is an isogeny. But after base change to K_p we already have that $Y_p \times_{K_p} \mathrm{Im}(f') \to X_p$ is an isogeny. Thus $Y \times_K Z \to X$ is an isogeny. $\square$

Proof of Theorem 3.2. Consider a Poincaré decomposition (in the isogeny category over K)

$$X \sim X_1^{e_1} \times \cdots \times X_r^{e_r}$$

with K-simple X_j that are pairwise non-isogenous over K. We first claim $r = 1$.

Indeed, the F-action preserves each $X_i^{e_i}$, so F can be embedded in $\mathrm{End}_K^0(X_1^{e_1})$. Note that $\mathbb{Q}_l \otimes_{\mathbb{Q}} \mathrm{End}_K^0(X_1^{e_1}) \to \mathrm{End}_{\mathbb{Q}_l}(V_l(X_1^{e_1}))$ is injective for any prime $l \neq \mathrm{char}(K)$. Thus, $F \otimes \mathbb{Q}_l$ acts faithfully on $V_l(X_1^{e_1})$, a free $\mathbb{Q}_l$-module with rank $2e_1 \mathrm{dim} X_1$. Hence, since $F \otimes_{\mathbb{Q}} \mathbb{Q}_l$ is a product of fields,

$$2\mathrm{dim}X = [F : \mathbb{Q}] = [F \otimes \mathbb{Q}_l : \mathbb{Q}_l] \leq 2e_1 \mathrm{dim} X_1 \leq 2\mathrm{dim}X.$$

Thus $[F : \mathbb{Q}] = 2e_1 \mathrm{dim}(X_1)$ and $r = 1$.

The above argument also shows that F has to be a maximal commutative subfield of $\mathrm{End}_K^0(X)$. Let $D = \mathrm{End}_K^0(X_1)$, so D is a division algebra finite-dimensional over $\mathbb{Q}$. Put $e = e_1$, $g_1 = \mathrm{dim}X_1$, $g = \mathrm{dim}X$ (so $[F : \mathbb{Q}] = 2g = 2eg_1$). For any ring A, let $\mathrm{Mat}_n(A)$ denote the $n \times n$ matrix algebra with coefficients in A, so $\mathrm{End}_K^0(X) = \mathrm{Mat}_e(D)$. We first want to show that the common $\mathbb{Q}$-dimension of the maximal commutative subfields of D is $2g_1$, so X_1 has CM over K. Since F is maximal commutative in $\mathrm{Mat}_e(D)$, it contains the center L of D (viewed as the center of $\mathrm{Mat}_e(D)$) and

$$[F : L] = \sqrt{[\mathrm{Mat}_e(D) : L]} = e\sqrt{[D : L]},$$

with $\sqrt{[D : L]}$ equal to the common L-degree of the maximal commutative subalgebras of D. Hence, since all such subalgebras must contain L it follows after multiplying both sides by $[L : \mathbb{Q}]$ that $[F : \mathbb{Q}]$ is equal to e times the common $\mathbb{Q}$-degree of the maximal commutative subalgebras of D. But $2eg_1 = [F : \mathbb{Q}]$, so cancelling e gives that $2g_1$ is indeed the common $\mathbb{Q}$-degree of the maximal commutative subfields of D.

Now assume K has characteristic 0, and let $F_0 \subseteq D$ be a maximal commutative subfield. We shall prove $F_0 = D$ and that this is a CM field. In particular, $F_0 = L$ canonically embeds F. To show $F_0 = D$, we can first shrink K (say to a finitely generated extension of $\mathbb{Q}$) so that there exists an embedding $K \hookrightarrow \mathbb{C}$. Fixing such an embedding allows us to define the topological space $X_1(\mathbb{C})$, and hence we get a $2g_1$-dimensional $\mathbb{Q}$-vector space $\mathrm{H}_1(X_1(\mathbb{C}), \mathbb{Q})$ equipped with a structure of left D-module by functoriality. But D is a division algebra, so there exists a D-basis and so $2g_1$ must be a multiple of $[D:\mathbb{Q}]$. But $[F_0:\mathbb{Q}]=2g_1$, so the equality $F_0=D$ is forced.

As for the CM property of F_0, note that the Rosati involution on $F_0=D=\mathrm{End}^0_K(X_1)$ associated to a K-polarization of X_1 is positive-definite with respect to the trace form, so $F_0=L$ is either totally real or a CM field. Using the natural embedding $L\hookrightarrow \mathrm{End}^0_{\overline{K}}(X_1\otimes_K\overline{K})$, $X_1\otimes_K\overline{K}$ is an abelian variety with complex multiplication over $\overline{K}$. Thus by applying the preceding considerations to $X_1\otimes_K\overline{K}$ and L over $\overline{K}$, $X_1\otimes_K\overline{K}\sim Y^{e'}$ for some simple abelian variety Y with complex multiplication over $\overline{K}$. Therefore, $\mathrm{End}^0_{\overline{K}}(X_1)\simeq \mathrm{Mat}_{e'}(\mathrm{End}^0_{\overline{K}}(Y))$. We know that $\mathrm{End}^0_{\overline{K}}(Y)$ canonically embeds L, so to rule out L being totally real it suffices to do the same for $\mathrm{End}^0_{\overline{K}}(Y)$. This reduces our problem to the case of a simple abelian variety with CM over an algebraically closed field, but there is an explicit classification of the possibilities for the endomorphism algebra equipped with a Rosati involution for a simple abelian variety over an algebraically closed field [7, pp. 201-2]. According to this classification, the totally real case cannot arise in our situation since $[\mathrm{End}^0(Y):\mathbb{Q}]>\dim(Y)$.

To prove the second part of Theorem 3.2, we must first recall several facts about Serre's construction from §7, [2]. Let S be a scheme and let A be an associative ring with identity. Let M be a finite projective right A-module with dual left A-module $M^\vee=\mathrm{Hom}(M,A)$ of right A-linear homomorphisms (with $(a.\phi)(m)=a\phi(m)$ for $\phi\in M^\vee$, $a\in A$, $m\in M$). Let $\mathfrak{M}$ be a left A-module scheme over S. The functor

$$T\rightsquigarrow M\otimes_A\mathfrak{M}(T)\simeq \mathrm{Hom}(M^\vee,\mathfrak{M}(T))$$

on S-schemes is represented by a commmutative group scheme over S, denoted $M\otimes_A\mathfrak{M}$. There are many properties $\mathbf{P}$ (locally finite type, proper, smooth, etc.) of S-schemes such that if $\mathfrak{M}$ satisfies propety $\mathbf{P}$ then so does $M\otimes_A\mathfrak{M}$. Especially, if X is an abelian variety over K with A embedded into $\mathrm{End}_K(X)$, then $M\otimes_A X$ is an abelian variety over K, and in the evident way M is embedded into $\mathrm{End}_K(M\otimes_A X)$ when M is moreover an associative A-algebra.

Change X up to isogeny so $\mathcal{O}_F$ acts on X. Note that $\mathcal{O}_F$ is a finite projective (generally not free) $\mathcal{O}_L$-module. Consider Serre's construction $\mathcal{O}_F\otimes_{\mathcal{O}_L}X_1$. By choosing a nonzero map $X_1\to X$ we get an $\mathcal{O}_F$-linear morphism of abelian varieties $h:\mathcal{O}_F\otimes_{\mathcal{O}_L}X_1\to X$ given functorially by

$$\mathcal{O}_F\otimes_{\mathcal{O}_L}X_1(S)\longrightarrow X(S)$$

$$f\otimes x\longrightarrow f\cdot x.$$

We claim that $\mathrm{Ker}(h)$ is finite. It suffices to prove this when K is algebraically closed. Let Y be the reduced subscheme of the identity connected component of $\mathrm{Ker}(h)$. Since $h\neq 0$, by the construction,

$$\mathrm{dim}Y<\dim(\mathcal{O}_F\otimes_{\mathcal{O}_L}X_1)=[F:L]\mathrm{dim}X_1=\mathrm{dim}X.$$

We see that Y is an abelian variety and F acts on Y. If $\mathrm{dim}Y\geq 1$ then $F\to\mathrm{End}^0(Y)$ is injective. The $\mathbb{Q}$-dimension of any commutative subfield in $\mathrm{End}^0(Y)$ is $\leq 2\mathrm{dim}Y<2g=[F:\mathbb{Q}]$, so we get a contradiction. Thus, Y has to be trivial and $\mathrm{Ker}(h)$ is finite. From the construction, as we just noted, we see that $\dim(\mathcal{O}_F\otimes_{\mathcal{O}_L}X_1)=[F:L]\dim(X_1)=\dim(X)$, so h is an isogeny.

Note that any degree-e extension of L can be embedded into $\mathrm{End}^0(X) = \mathrm{Mat}_e(L)$. Let L_0 be the maximal totally real subfield of the CM field L. It is an easy exercise via weak approximation to prove that there exists a totally real extension L_0' of L_0 such that $[L_0' : L_0] = e$. Thus, $L' = L_0' \otimes_{L_0} L$ is a CM field with

$$[L' : \mathbb{Q}] = [L_0' : L_0][L : \mathbb{Q}] = e[L : \mathbb{Q}] = [F : L][L : \mathbb{Q}] = [F : \mathbb{Q}] = 2\mathrm{dim}X$$

and $L' \hookrightarrow \mathrm{Mat}_e(L) \simeq \mathrm{End}_K^0(X)$. □

Now let us prove the remark in the end of section 1. Let $K = \mathbb{C}$ and let F be a CM field and let $X = X(F, \Phi)$ be the CM abelian variety constucted as in section 1. From Theorem 3.2 and its proof, we know that there exists a simple abelian variety X_1 over $\mathbb{C}$ with complex multiplication by a CM field $L = \mathrm{End}_{\mathbb{C}}^0(X_1)$ such that X is isogenous to X_1^e with the isomorphism $\mathrm{End}^0(X) \simeq \mathrm{End}^0(X_1^e) \simeq \mathrm{Mat}_e(L)$ identifying L with a subfield of F such that $[F : L] = e$. By the classification of section 1, there exists a $\mathbb{C}$-algebra structure Ψ on $L \otimes_{\mathbb{Q}} \mathbb{R}$ such that $X_1 \sim X(L, \Psi)$. From proof of Theorem 3.2 , there exists an F-compatible isogeny $X \sim \mathcal{O}_F \otimes_{\mathcal{O}_L} X_1$. It is easy to see that the tangent space $T_0(X) \simeq T_0(\mathcal{O}_F \otimes_{\mathcal{O}_L} X_1)$ is isomorphic to $F \otimes_L T_0(X_1)$ via the natural map. (This uses the functorial description $T_0(X) = \mathrm{Ker}(X(\mathbb{C}[\epsilon]) \to X(\mathbb{C}))$.) From the $F \otimes_{\mathbb{Q}} \mathbb{C}$-module isomorphism between $T_0(X)$ and $F \otimes_L T_0(X_1)$, we have an isomorphism of $\mathbb{C}$-algebras:

$$(F \otimes_L (L \otimes_{\mathbb{Q}} \mathbb{R}))_\Psi \simeq (F \otimes_{\mathbb{Q}} \mathbb{R})_\Phi. \tag{3.1}$$

as quotients of $F \otimes_{\mathbb{Q}} \mathbb{C}$.

Recall notations from section 1: let $\sigma_1, \ldots, \sigma_g$ be the embeddings $F_0 \hookrightarrow \mathbb{R}$, so $\Phi = (\sigma_1', \ldots, \sigma_g')$, where $\sigma_j' : F \hookrightarrow \mathbb{C}$ satisfies $\sigma_j'|_{F_0} = \sigma_j$. Note that $L_0 = L \cap F_0$ is the maximal totally real subfield of L.

Comparing the decompositions of both sides of (1), it is not hard to see that (1) holds if and only if $\Psi = (\sigma_j'|_L, 1 \leq j \leq g)$, and that this collection is a CM type on L if and only if whenever $\sigma_i'|_{L_0} = \sigma_j'|_{L_0}$ we have $\sigma_i'|_L = \sigma_j'|_L$.

4. Reduction of Abelian Varieties

Let $(R, \mathfrak{m})$ be a discrete valuation ring, k the residue field, $p = \mathrm{char}(k)$, K the fraction field of R, and A an abelian variety defined over K.

Definition 4.1. We say A has *good reduction* over K if there exists an abelian scheme $\mathscr{A}$ over R such that the generic fiber $\mathscr{A}_K = \mathscr{A} \otimes_R K$ is isomorphic to A.

Example 4.2. If E is an elliptic curve over K, then E has good reduction if and only if we can find a Weierstrass equation of E such that the discriminant $\Delta \not\equiv 0$ mod $\mathfrak{m}$. (Ch. 2, [4])

Example 4.3. By the theory of Néron models, A has good reduction if and only if its Néron model $N(A)$ is R-proper, or equivalently if and only if $N(A)$ has proper special fiber, in which case $N(A)$ is an abelian scheme with generic fiber A and is unique up to isomorphism. In particular, since the formation of $N(A)$ commutes with base change to the completion $\hat{R}$, good reduction is insensitive to base change to the completion. (See Ch. 6, [1])

Definition 4.4. We say A has *potentially good reduction* if there exists a finite extension K' over K such that $A_{K'} = A \otimes_K K'$ has good reduction over K'.

Fix $l \neq p$ a prime. Let $T_l(A) = \varprojlim A[l^n](K_s)$ be the l-adic Tate module.

Theorem 4.5 (Néron-Ogg-Shafarevich)**.** *A has good reduction if and only if an inertia group for R in $\mathrm{Gal}(K_s/K)$ acts on $T_l(A)$ trivially.*

Proof. See §1, [9]. □

Corollary 4.6.

(1) *If $R \to R'$ is an unramified local extension of discrete valuation rings with $K \to K'$ the extension of fields, then A has good reduction over R if and only if $A_{K'}$ has good reduction over R'.*
(2) *If A is isogenous to A', then A has good reduction if and only if A' has good reduction.*
(3) *If $l \neq \mathrm{char}(K)$ is a prime, then A has potentially good reduction if and only $\rho_l(I)$ is finite, where $\rho_l : \mathrm{Gal}(K_s/K) \to \mathrm{End}_{\mathbb{Z}_l}(T_l(A))$ is the l-adic representation and I is an inertia group for R.*

If A is an abelian variety defined over global field K and v a finite place of K then we say A has *good reduction at v* if A has good reduction over K_v. It is equivalent to say that A has good reduction over the "algebraic " local ring $\mathcal{O}_{K,v}$. Since A extends to an abelian scheme over the ring $\mathcal{O}_{K,S}$ for some finite set of places S of K (containing the archimedean places), the problem of studying reduction types for A over places of K is really just a problem at non-archimedean places in S (and places over these in finite separable extensions of K). Thus, the essential case remains that when the base is the fraction field of a discrete valuation ring.

Let (A, i) be an abelian variety with complex multiplication defined over the fraction field K of a discrete valuation ring R, where $i : F \hookrightarrow \mathrm{End}^0_K(A)$ is an embedding. We are going to use local class field theory to show that A has potentially good reduction over R when R is a local ring arising from a global field. (The result holds for all discrete valuation rings, but at the expense of stronger algebro-geometric input, as we shall explain at the end.)

Let $V_l(A) = \mathbb{Q}_l \otimes_{\mathbb{Z}_l} T_l(A)$ and $F_l = \mathbb{Q}_l \otimes_{\mathbb{Q}} F$, with l a prime not dividing $\mathrm{char}(K)$. Let $\mathcal{O} = F \cap \mathrm{End}_K(A)$ (an order in F) and $\mathcal{O}_l = \mathbb{Z}_l \otimes_{\mathbb{Z}} \mathcal{O}$. Note that the natural map $\mathcal{O}_l \to \mathrm{End}_{\mathbb{Z}_l}(T_l(A))$ is injective. The following theorem and corollary are valid with K an arbitrary field.

Theorem 4.7. (1) *V_l is a free rank-1 F_l-module.*
(2) *For all $\phi \in F_l$ such that $\phi T_l(A) \subset T_l(A)$, we have $\phi \in \mathcal{O}_l$.*

Proof.

(1) First note that $\mathbb{Q}_l \otimes_{\mathbb{Z}} \mathrm{End}(A) \to \mathrm{End}_{\mathbb{Q}_l}(V_l)$ is injective , so F_l acts on V_l faithfully. Since F_l and V_l are finite free $\mathbb{Q}_l$ modules and $[F_l : \mathbb{Q}_l] = [V_l : \mathbb{Q}_l] = 2g$ with F_l a product of fields, V_l is free rank-1 F_l-module.
(2) There exists an $N \geq 0$ such that $l^N \phi \in \mathcal{O}_l$. Since $\mathcal{O}$ is dense in $\mathcal{O}_l$, there exists a $\psi \in \mathcal{O}$ such that $\psi \equiv l^N \phi \mod l^N \mathcal{O}_l$. Since $\phi T_l(A) \subset T_l(A)$, clearly $\psi T_l(A) \subset l^N T_l(A)$. Hence, ψ kills $A[l^N]$, so $\psi = l^N \psi_0$ for $\psi_0 \in \mathrm{End}_K(A)$. Clearly $\psi_0 \in F$, so $\psi_0 \in \mathcal{O}$. Thus $\phi \equiv \psi_0 \mod \mathcal{O}_l$, so $\phi \in F \cap \mathcal{O}_l = \mathcal{O}$.

□

Corollary 4.8. *The commutant of* $\mathcal{O}$ *in* $\mathrm{End}(V_l)$ *(resp.* $\mathrm{End}(T_l)$, $\mathrm{End}^0_K(A) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}_K(A)$, $\mathrm{End}_K(A)$*), is* F_l *(resp.* $\mathcal{O}_l$, F, $\mathcal{O}$*).*

Proof.

(1) For each $f \in \mathrm{End}(V_l)$ such that f commutes with $\mathcal{O}$, clearly f commutes with F. By Theorem 5 (1), we have $f \in \mathrm{End}_{F\otimes\mathbb{Q}_l}(V_l) = F \otimes \mathbb{Q}_l = F_l$
(2) Combine (1) and (2) in Theorem 5.
(3) Let C be the commutant of $\mathcal{O}$ in $\mathbb{Q}\otimes\mathrm{End}_K(A)$. Note that $\mathbb{Q}_l\otimes\mathrm{End}_K(A) \to \mathrm{End}(V_l)$ is injective, so $C \otimes_{\mathbb{Q}} \mathbb{Q}_l \subset$ commutant of $\mathcal{O}$ in $\mathrm{End}(V_l)$. By step 1, this commutant is F_l. It is obvious that $F \subset C$, so we have $F \otimes_{\mathbb{Q}} \mathbb{Q}_l \subset C \otimes_{\mathbb{Q}} \mathbb{Q}_l \subset F_l$. This forces $F = C$.
(4) Obvious from step 3 and the definition $\mathcal{O} = F \cap \mathrm{End}_K(A)$.

□

Now consider the continuous representation

$$\rho_l : \mathrm{Gal}(K_s/K) \longrightarrow \mathrm{Aut}(T_l)$$

defined by the Galois module T_l. If $\sigma \in \mathrm{Gal}(K_s/K)$, it is clear that $\rho_l(\sigma)$ commutes with the elements of $\mathcal{O}$, and by Corollary 1 this implies that $\rho_l(\sigma)$ is contained in $\mathcal{O}$. Hence:

Corollary 4.9. *The representation* ρ_l *attached to* T_l *is a continuous homomorphism of* $\mathrm{Gal}(K_s/K)$ *into the group* $\mathcal{O}_l^\times$ *of invertible elements of* $\mathcal{O}_l = \mathbb{Z}_l \otimes \mathcal{O}$*. In particular,* ρ_l *has a commutative image.*

Corollary 4.10. *If* K *is a global field then there exists an finite abelian extension* K' *over* K *such that* $A_{K'}$ *has good reduction at all finite places* v' *of* K'

Proof. We know that A has good reduction everywhere except for a set S of finitely many finite places. Choose a prime l such that l is distinct from $\mathrm{char}(K)$ and the residue characteristics at all the places in S. For each $v \in S$, it suffices to show that $\rho_l(I_v)$ is finite (since for any finite Galois extension K'/K, the inertia groups at all places v' on K' over v are conjugate in $\mathrm{Gal}(K_s/K)$). Note first that $\rho_l(I_v)$ is abelian. Local class field theory then shows that the image of the inertia group I_v in the $\mathrm{Aut}(T_l)$ is a quotient of the group $\mathcal{O}_{K_v}^\times$. But $\mathcal{O}_{K_v}^\times$ is the product of a finite group and a pro-p-group P. Since $l \neq p$, the image of P in $\mathrm{Aut}(T_l)$ intersects the pro-l-group $1 + l \cdot \mathrm{End}(T_l)$ only in the neutral element, so the image of P maps injectively into finite group $\mathrm{Aut}(T_l/lT_l)$ and is thus finite. Hence the image of I_v in $\mathrm{Aut}(T_l)$ is finite. □

To prove Corollary 4.10 in the more general situation of a fraction field K of an arbitrary discrete valuation ring R (with A an abelian variety with CM over K by a number field F), we argue as follows. By the semi-stable reduction theorem of Grothendieck, there is a finite separable extension K'/K such that for the R-finite integral closure R' in K' the Néron model of $A_{K'}$ over R' has semi-abelian identity component in all closed fibers over $\mathrm{Spec}(R')$; that is, these fibers are extensions of abelian varieties by tori (and good reduction is precisely the condition that the fibral toric part vanishes). Thus, it suffices to prove (by passing to localizations at maximal ideals of R' and the relative identity component of the Néron model over such local rings) that if the Néron model of A over R has semi-abelian relative identity component $\mathcal{A}$ over R then the toric part T in the special fiber of $\mathcal{A}$ must vanish. By functoriality of the Néron model the geometric character group $\mathrm{X}_{\mathbb{Q}}(T)$ admits an action by the field F, and so this character group has a structure of F-vector space. But its $\mathbb{Q}$-dimension is at most $\dim(T) \leq \dim(A)$, so since $[F : \mathbb{Q}] > \dim(A)$ by the CM condition it follows that $\mathrm{X}_{\mathbb{Q}}(T)$ must vanish. Hence, $T = 0$ as desired.

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