1 Complex Theory of Abelian Varieties

Definition 1.1. Let k be a field. A *k-variety* is a geometrically integral k-scheme of finite type. An *abelian variety* X is a proper k-variety endowed with a structure of a k-group scheme.

For any point $x \in X(k)$ we can defined a translation map $T_x : X \to X$ by $T_x(y) = x + y$. Likewise, if K/k is an extension of fields and $x \in X(K)$, then we have a translation map $T_x : X_K \to X_K$ defined over K.

Fact 1.2. Any k-group variety G is smooth.

Proof. We may assume that $k = \overline{k}$. There must be a smooth point $g_0 \in G(k)$, and so there must be a smooth non-empty open subset $U \subseteq G$. Since G is covered by the $G(k)$-translations of U, G is smooth. □

We now turn our attention to the analytic theory of abelian varieties by taking $k = \mathbb{C}$. We can view $X(\mathbb{C})$ as a compact, connected Lie group. Therefore, we start off with some basic properties of complex Lie groups.
Let X be a Lie group over $\mathbb{C}$, i.e., a complex manifold with a group structure whose multiplication and inversion maps are holomorphic. Let $T_0(X)$ denote the tangent space of X at the identity. By adapting the arguments from the C^∞-case, or combining the C^∞-case with the Cauchy-Riemann equations we get:

Fact 1.3. For each $v \in T_0(X)$, there exists a unique holomorphic map of Lie groups $\phi_v : \mathbb{C} \to X$ such that $(d\phi_v)_0 : \mathbb{C} \to T_0(X)$ satisfies $(d\phi_v)_0(\frac{\delta}{\delta t}\mid_0) = v$. Moreover, $\phi_v(t)$ is holomorphic as a joint function of v and t.

Definition 1.4. The *exponential map* $\exp : T_0(X) \to X$ is the holomorphic map $\exp(v) = \phi_v(1)$.

We will need the following facts about the exponential map:
1. $\exp(tv) = \phi_v(t)$ by the uniqueness of ϕ_v.
2. $\exp(0) = 0$
3. $(d\exp)_0 : T_0(X) \to T_0(X)$ is the identity map.
4. Let $f : X_1 \to X_2$ be a homomorphism of complex Lie groups. Then $f(\exp_{X_1}(v)) = \exp_{X_2}((df)_0(v))$.

These are proved exactly as in the C^∞ case, or follow from the C^∞ case by relating the holomorphic exponential map with the exponential map for the underlying C^∞ Lie group.

Lemma 1.5. *For any topological group G an open subgroup is also closed. If G is connected, then any neighborhood of the identity generates G.*

Proof. Standard. $\square$

Proposition 1.6. *A compact connected complex Lie group X is commutative.*

Proof. Choose $x \in X$. Consider the conjugation automorphism $\sigma_x(y) = xyx^{-1}$, so $(d\sigma_x)_0 : T_0(X) \to T_0(X)$ is an automorphism of $T_0(X)$. The map $x \mapsto (d\sigma_x)_0$ from X to $GL(T_0(X))$ is holomorphic since the map $(x, y) \mapsto xyx^{-1}$ is analytic in both variables. Now X is compact and connected, so this map must be a constant map. Hence, $(d\sigma_x)_0 = (d\sigma_0)_0 = id_{T_0(X)}$. We use the above property 4 of the exponential map to get

$$\sigma_x(\exp v) = \exp((d\sigma_x)_0 v) = \exp v.$$

In particular, $\exp(T_0(X))$ is in the center of X. Using properties 2 and 3 and the inverse function theorem gives us that $\exp(T_0(X))$ is a neighborhood of the identity. Now use Lemma 1.5 and that X is connected to get that $\exp(T_0(X))$ generates X, and so X is commutative. $\square$

Definition 1.7. Let V be a complex vector space and Λ a lattice in V. Then we call V/Λ a *complex torus*.

Theorem 1.8. *The exponential map* $\exp : T_0(X) \to X$ *is a surjective homomorphism of complex Lie groups whose kernel is a lattice in* $T_0(X)$*. Letting* $\ker(\exp) = \Lambda$, $\exp$ *induces an isomorphism of Lie groups*

$$T_0(X)/\Lambda \xrightarrow{\sim} X.$$

Therefore X is a complex torus.

Proof. Choose $u, v \in T_0(X)$. By Proposition 1.6 we know that X is commutative, so if we define $\psi : \mathbb{C} \to X$ by $\psi(t) = (\exp(tu))(\exp(tv))$, then ψ is a homomorphism and is holomorphic. Also note that $(d\psi)_0(t) = t(u+v)$. Now choose $w \in T_0(X)$, so $\phi_w : t \mapsto \exp(tw)$ is the unique holomorphic homomorphism so that $(d\phi_w)_0(t) = tw$. This gives us that $(\exp(tu))(\exp(tv)) = \exp(t(u + v))$, and putting $t = 1$ yields that exp is a homomorphism. Since $\exp(T_0(X))$ contains a neighborhood of the identity (as in the proof above), using Lemma 1.5 again we have that $\exp(T_0(X))$ contains an open and closed

subgroup of X which must be all of X as X is connected. Let U be a neighborhood of 0 so that $\exp|_U$ is injective. Thus, $\Lambda \cap U = \{0\}$, so Λ is a discrete subgroup.
The map $\pi : T_0(X)/\Lambda \to X$ is holomorphic because exp is holomorphic. The tangent map at 0 is an isomorphism, so we can use the inverse function theorem to conclude that $X \to T_0(X)/\Lambda$ is holomorphic at the identity. Translation maps are holomorphic and bijective on X and $T_0(X)/\Lambda$, so the inverse is holomorphic everywhere. Thus $T_0(X)/\Lambda \cong X$. Since the only discrete subgroups of a vector space with compact quotients are lattices, Λ is necessarily a lattice in $T_0(X)$. □

Now since $V \to V/\Lambda$ is a covering map, and any finite dimensional $\mathbb{R}$-vector space is simply connected, we get from basic algebraic topology that $H_1(V/\Lambda, \mathbb{Z}) = \Lambda$. Therefore the above result says that $X \cong T_0(X)/H_1(X, \mathbb{Z})$. With some definition chasing it can be shown that the composite $H_1(X, \mathbb{Z}) \hookrightarrow T_0(X) = Cot_0(X)^\vee$ is $\sigma \mapsto \left(\omega_0 \mapsto \int_\sigma \omega\right)$ where ω is the translation invariant 1-form on X extending ω_0.

Serre's GAGA gives that abelian varieties are a full-subcategory of the category of $\mathbb{C}$-tori. However, in dimensions greater then 1 "most" complex tori are not abelian varieties. We will come back to this later.

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Define a multiplication by n map $[n] : X \to X$ and let $X[n]$ be the kernel of $[n]$. It is easy to see that

$$X[n] = \frac{1}{n}\Lambda/\Lambda \cong \Lambda/n\Lambda.$$

In particular, by choosing a $\mathbb{Z}$-basis for Λ, we have $X[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$.

We define the ℓ-adic Tate module by $T_\ell(X) = \varprojlim_n X[\ell^n]$, so

$$T_\ell(X) \cong \varprojlim_n \Lambda/\ell^n\Lambda \cong \varprojlim_n \left(\Lambda \otimes (\mathbb{Z}/\ell^n\mathbb{Z})\right) \cong \Lambda \otimes \mathbb{Z}_\ell$$

where we have used the fact that Λ is a finitely generated $\mathbb{Z}$-module to bring the inverse limit inside the tensor product. Thus we have that $T_\ell(X) \cong H_1(X, \mathbb{Z}) \otimes \mathbb{Z}_\ell \cong H_1(X, \mathbb{Z}_\ell)$. It is clear that $T_\ell(X)$ is a free $\mathbb{Z}_\ell$-module of rank $2g$; it will serve as our analogue of Λ when we leave the analytic picture. We can define $V_\ell(X) = T_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$; this is a $2g$-dimensional $\mathbb{Q}_\ell$ vector space.

Next we would like to discuss the group $\mathrm{Hom}(X, X')$. Thinking of X and X' as complex Lie groups as above, these are holomorphic maps from X to X' that respect the group structure. If X and X' are complex tori given by $X = V/\Lambda$ and $X' = V'/\Lambda'$ then by the functoriality of the universal cover (as a $\mathbb{C}$ manifold) we have that

$$\mathrm{Hom}(X, X') = \{f : V \to V' : f \text{ is a } \mathbb{C}\text{−linear map and } f(\Lambda) \subseteq \Lambda'.\}$$

Thus, we get an injection

$$\mathrm{Hom}(X, X') \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda') = \mathrm{Hom}_{\mathbb{Z}}(H_1(X, \mathbb{Z}), H_1(X', \mathbb{Z})) \cong M_{2g\times 2g'}(\mathbb{Z})$$

where $M_{2g\times 2g'}(\mathbb{Z})$ is the ring of $2g \times 2g'$ matrices with integer entries. This shows that $\mathrm{Hom}(X, X')$ is a finite free $\mathbb{Z}$−module of rank $\leq (2\dim X)(2\dim X')$. In particular, $\mathrm{rank}(\mathrm{End}(X)) \leq 4g^2$. We will have more to say about the endomorphisms later.

Definition 1.9. A map of complex tori $f : X \to X'$ is an *isogeny* if it is surjective and has finite kernel. The *degree* of the isogeny is the order of the kernel.

Fact 1.10. To give an isogeny $f : X \to X'$ is equivalent to giving a finite subgroup $K \subset X$. If $X = V/\Lambda$, then $K = \Lambda'/\Lambda$ for some lattice $\Lambda' \subseteq V$ containing Λ and $X' = V/\Lambda'$. In particular, $\ker(f) = \Lambda'/\Lambda$.

The map $[n]$ is an isogeny for $n \neq 0$. In fact, any isogeny between tori factors through some $[n]$ for $n \neq 0$.

Lemma 1.11. *Let X and X' be tori and let $f : X \to X'$ be an isogeny of degree d. There exists a unique isogeny $f' : X' \to X$ so that $f \circ f' = [d]_{X'}$, and $f' \circ f = [d]_X$.*

Proof. The map f is surjective because it is an isogeny, and $\ker(f) \subset \ker[d]_X$ by the definition of degree. Thus there is a unique map f' such that $f' \circ f = [d]_X$. On the other hand,

$$f \circ f' \circ f(x) = f([d]_X(x)) = [d]_{X'}(f(x))$$

so we also get $f \circ f' = [d]_{X'}$ since f is surjective. □

Let X_1, X_2, and X_3 be three complex tori. Now consider

$$\mathrm{Hom}^\circ(X, X') = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Hom}(X, X').$$

Composition of homomorphisms extends to a unique $\mathbb{Q}-$bilinear map

$$\mathrm{Hom}^\circ(X_1, X_2) \times \mathrm{Hom}^\circ(X_2, X_3) \to \mathrm{Hom}^\circ(X_1, X_3).$$

This allows us to form a category whose elements are complex tori and whose morphisms are elements of $\mathrm{Hom}^\circ(X_1, X_2)$. This is known as the category of complex tori up to isogeny. The reason for this is that by Lemma 1.11 we know that given an isogeny $f : X_1 \to X_2$, there exists a nonzero integer d and an isogeny $f' : X_2 \to X_1$ so that $f \circ f' = [d]$. Therefore, since we can invert $[d]$ in $\mathrm{Hom}^\circ(X_1, X_2)$, this shows that isogenies are isomorphisms in this new category. In the algebraic setting this notion is akin to working with $\mathbb{Q}$-vector spaces instead of confining oneself to $\mathbb{Z}$-lattices. In particular, it provides a more convenient setting for "abelian category" operations. Indeed, it follows easily from Lemma 1.11 that the category of complex tori up to isogeny is an abelian category. It is actually a semi-simple abelian category, but this is less evident. This will follow from the Poincare Irreducibility Theorem, which we will come back to later.

Our next goal is to give a classification of line bundles over a complex torus. We first recall the definition of a line bundle, also known as an invertible sheaf.

Definition 1.12. A sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules is *invertible* if it is locally free of rank 1, i.e., for every $x \in X$ there exists an open neighborhood U of x such that $\mathcal{L} \mid_U \cong \mathcal{O}_X \mid_U = \mathcal{O}_U$.

Definition 1.13. Let V be a complex vector space. A *Hermitian form* is a map $H : V \times V \to \mathbb{C}$ that is linear with respect to the first factor and satisfies $H(z, w) = \overline{H(w, z)}$.

Definition 1.14. A Hermitian form H on V is a *Riemann form* with respect to a given lattice Λ in V if the imaginary part of H, $\mathrm{Im}(H)$, takes integer values on $\Lambda \times \Lambda$.

Example 1.15. Consider the elliptic curve case with $g = 1$. Suppose our lattice is given by $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Then define

$$H(z, w) = \frac{z\overline{w}}{\mathrm{Im}(\omega_1 \overline{\omega}_2)}.$$

Then it is easy to check that H is a Riemann form.

Lemma 1.16. *Let V be a complex vector space. There is a natural bijection between Hermitian forms H on V and real alternating bilinear forms E on V satisfying $E(ix, iy) = E(x, y)$. The correspondence is given by $H \mapsto E = \operatorname{Im}(H)$ and $E \mapsto H(x, y) = E(ix, y) + iE(x, y)$.*

Proof. Exercise. □

Let $\mathcal{L}$ be a holomorphic line bundle on X. We can think of $\mathcal{L} \in H^1(X, \mathcal{O}_X^*)$. It is a theorem that all line bundles on a complex vector space are trivial, so if $\pi : V \to V/\Lambda$ is the projection map, then $\pi^*(\mathcal{L})$ is trivial. Let $\mathcal{H}^*$ be the multiplicative group of nowhere vanishing holomorphic functions on V. Then one can use the fact that $\pi^*(\mathcal{L})$ is trivial to show that $H^1(\Lambda, \mathcal{H}^*) \xrightarrow{\sim} H^1(X, \mathcal{O}_X^*)$. In particular, we can think of our line bundle $\mathcal{L}$ as a class $(\lambda \mapsto e_\lambda) \in H^1(\Lambda, \mathcal{H}^*)$.

Let $\delta : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$ be the co-boundary map coming from the exponential sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

The class $\delta(\mathcal{L})$ is called the *first chern class of* $\mathcal{L}$. One has an isomorphism $H^2(\Lambda, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$, so we can consider the first chern class in $H^2(\Lambda, \mathbb{Z})$ instead. There is also an isomorphism $H^2(\Lambda, \mathbb{Z}) \xrightarrow{\sim} \bigwedge^2 \operatorname{Hom}(\Lambda, \mathbb{Z})$ so we can further consider the first chern class as living in $\bigwedge^2 \operatorname{Hom}(\Lambda, \mathbb{Z})$. With a little work one obtains an alternating form $E : \Lambda \times \Lambda \to \mathbb{Z}$ given by

$$E(\lambda_1, \lambda_2) = f_{\lambda_2}(z + \lambda_1) + f_{\lambda_2}(z) - f_{\lambda_1}(z + \lambda_2) - f_{\lambda_1}(z), \ (z \text{ arbitrary in } V)$$

where $e_\lambda(z) = e^{2\pi i f_\lambda(z)}$. If we take an $\mathbb{R}$-scalar extension of E to V, then we also have that $E(ix, iy) = E(x, y)$. This encodes the holomorphicity of $\mathcal{L}$. Of course we can use Lemma 1.16 to associate a Riemann form to E. Therefore, given a line bundle on X we have associated a Riemann form.

Conversely, let H be a Riemann form on V. Let $\alpha : \Lambda \to \{z \in \mathbb{C}^\times : |z| = 1\}$ be a map with

$$\alpha(\lambda_1 + \lambda_2) = e^{i\pi E(\lambda_1, \lambda_2)} \alpha(\lambda_1)\alpha(\lambda_2)$$

for $\lambda_i \in \Lambda$. Such a map always exists for H as given. Let $\mathcal{L}(H, \alpha)$ be the line bundle given by the quotient of $\mathbb{C} \times V$ by the action of Λ given by

$$\lambda \cdot (z, w) = (\alpha(\lambda) e^{\pi H(w,\lambda)+\frac{1}{2}\pi H(\lambda,\lambda)} \cdot z, w + \lambda).$$

Then we have the following theorem of Appell and Humbert that classifies all line bundles on X:

Theorem 1.17. *(Appell-Humbert) Let $\mathcal{L}$ be a line bundle on the complex torus X. Then $\mathcal{L}$ is isomorphic to a $\mathcal{L}(H, \alpha)$ for a uniquely determined (H, α) as defined above.*

Proof. See Pages 21-22 of [3]. □

Our next step is to consider the algebraicity of our $g-$dimensional complex torus X. Let $\mathcal{L}(H, \alpha)$ be a line bundle on X as above. Then the sections of $\mathcal{L}(H, \alpha)$ are in 1-1 correspondence with holomorphic functions θ on V which satisfy

$$\begin{aligned} \theta(z + \lambda) &= e_\lambda(z)\theta(z) \\ &= \alpha(\lambda) e^{\pi H(z,\lambda)+\frac{1}{2}\pi H(\lambda,\lambda)} \theta(z) \end{aligned}$$

for every $z \in V$ and every $\lambda \in \Lambda$. This type of function is called a *theta function associated to* (H, α). From now on we use "section of $\mathcal{L}(H, \alpha)$" and "theta-function associated to (H, α)" interchangeably.
The main result is on algebraicity is

Theorem 1.18. *(Lefschetz) Let X be a complex torus V/Λ with $\mathcal{L} = \mathcal{L}(H, \alpha)$ the associated line bundle as above. The Riemann form H is positive-definite if and only if the space of holomorphic sections of $\mathcal{L}^{\otimes n}$ give an embedding of X as a closed complex submanifold in a projective space for each $n \geq 3$, i.e., a holomorphic map $\Theta : X \to \mathbb{CP}^d$ that is injective and induces an injective map on tangent spaces.*

Proof. (Idea of proof, for details see [3] pages 30-33) Following [3], we look at the case of $n = 3$. The other cases are handled similarly. Let θ be a section of $\mathcal{L}(H, \alpha)$ and $a, b \in V$. Then one can show quite easily by merely replacing z with $z + \lambda$ that $\theta(z - a)\theta(z - b)\theta(z + a + b)$ is a section of $\mathcal{L}(3H, \alpha^3)$. Let $z_0 \in V$ and take a non-zero section θ of $\mathcal{L}(H, \alpha)$. Then choose $a, b \in V$ such that $\theta(z_0 - a) \neq 0$, $\theta(z_0 - b) \neq 0$, and $\theta(z_0 + a + b) \neq 0$. Then put

$\phi = \theta(z-a)\theta(z-b)\theta(z+a+b)$. Then ϕ is a section of $\mathcal{L}(3H, \alpha^3)$ not vanishing at z_0. We can find such a section for any $z_0 \in V$. Let $\theta_0, \dots, \theta_d$ be a basis for the sections of $\mathcal{L}^{\otimes 3}$. Then we get a well-defined holomorphic map $\Theta : X \to \mathbb{CP}^d$ given in terms of homogeneous coordinates by

$$\Theta(\pi(z)) = (\theta_0(z), \theta_1(z), \dots, \theta_d(z)) \in \mathbb{CP}^d$$

for $z \in V$ and $\pi : V \to V/\Lambda$. The proof that this map is injective and induces an injective map on the tangent spaces is left to the reader, or consult [3]. $\square$

For an algebraic variety Y over $\mathbb{C}$, there is a canonically associated analytic space structure on the underlying set Y. We will denote this analytic space by Y^{an}. Then Serre's GAGA says that the functor from the space of proper smooth $\mathbb{C}-$schemes to the space of compact Hausdorff complex manifolds over $\mathbb{C}$ taking Y to Y^{an} is fully faithful, i.e., that for any holomorphic map between analytic spaces we get a map on the algebraic side so that the diagram

$$\begin{array}{ccc} Y_1 & \longrightarrow & Y_1^{\mathrm{an}} \\ \downarrow & & \downarrow \\ Y_2 & \longrightarrow & Y_2^{\mathrm{an}} \end{array}$$

commutes, if we are given two algebraic morphisms $f_1, f_2 : Y_1 \rightrightarrows Y_2$ with the induced maps on the analytic spaces being equal, then $f_1 = f_2$, and complex submanifolds of X^{an} have the form Y^{an} for a unique smooth closed $Y \hookrightarrow X$. This tells us that for the analytic space Y^{an} there is only one algebraic structure.

Definition 1.19. A compact Hausdorff analytic space Y is called *algebraizable* if there is an algebraic scheme X so that $Y = X^{\mathrm{an}}$.

Then as a corollary to Lefschetz's Theorem we get:

Corollary 1.20. *Let $X = V/\Lambda$ be a complex torus. There is a positive-definite Riemann form H on V if and only if X is the complex-analytic space associated to an algebraic scheme. Therefore, a complex torus is algebraizable if and only if there is a positive definite Riemann form on V.*

Proof. Let X be a complex-analytic space associated to an algebraic scheme. Then $X = A^{\mathrm{an}}$ for A an abelian variety over $\mathbb{C}$. The algebraic theory gives an embedding $A \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$. Therefore we have $\iota : X \hookrightarrow \mathbb{CP}^n$. There is an ample line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$, which pulls back via ι to an ample line bundle $\iota^*\mathcal{O}(1)$ on X. Theorem 1.17 (Appell-Humbert) then provides the positive-definite Riemann form H that we seek.
Conversely, suppose we are given a positive-definite Riemann form H on X. Using Theorem 1.18 (Lefschetz) we have an ample line bundle on X. Thus, we have an embedding $X \hookrightarrow \mathbb{CP}^n$. Chow's Theorem gives that $X = A^{\mathrm{an}}$ for a smooth A such that $A \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ a closed embedding. GAGA gives that the group laws on X are algebraizable and come from the group laws on A. Therefore, we obtain that A is an abelian variety. □

If X is a complex torus with dimension $g = 1$, then it is well known that X is algebraizable. Look at the Riemann form defined in Example 1.15. This gives the positive-definite Riemann form we need.
However, almost all complex tori of dimension greater then or equal to 2 are not algebraizable. Following [3], set $\mathrm{Pic}(X)$ to be the set of line bundles on X and $\mathrm{Pic}^{\circ}(X)$ to be the set of line bundles on X that are topologically trivial. We want to show that on almost all complex tori X, $\mathrm{Pic}(X) = \mathrm{Pic}^{\circ}(X)$. This is equivalent by Theorem 1.17 (Appell-Humbert) to the fact that there is no skew-symmetric $E : V \times V \to \mathbb{R}$ which is integral on $\Lambda \times \Lambda$ and satisfies $E(ix, iy) = E(x, y)$ for all $x, y \in V$ for $X = V/\Lambda$.
Let $T_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and $\overline{T}_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{C}-\mathrm{anti}}(V, \mathbb{C})$. Consider the map

$$\begin{aligned} \bigwedge\nolimits^2_{\mathbb{Z}} \mathrm{Hom}(\Lambda, \mathbb{Z}) &\hookrightarrow \bigwedge\nolimits^2_{\mathbb{C}} \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}) = \bigwedge\nolimits^2_{\mathbb{C}} \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \\ &= \bigwedge\nolimits^2_{\mathbb{C}} (T \oplus \overline{T}) \cong (\bigwedge\nolimits^2_{\mathbb{C}} T) \oplus (T \otimes \overline{T}) \oplus (\bigwedge\nolimits^2_{\mathbb{C}} \overline{T}). \end{aligned}$$

Taking $E_{\mathbb{Z}}$ in $\bigwedge^2_{\mathbb{C}} \mathrm{Hom}(\Lambda, \mathbb{Z})$, we really need to understand $E_{\mathbb{C}}$ on $(V \times \overline{V}) \times (V \times \overline{V})$. We can work out explicitly what this looks like.
Let $E_{\mathbb{Z}} : \Lambda \times \Lambda \to \mathbb{Z}$ be skew-symmetric, and let $E_{\mathbb{R}} : V \times V \to \mathbb{R}$ be its scalar extension by $\mathbb{Z} \to \mathbb{R}$ and similarly for $E_{\mathbb{C}}$. Let $\overline{V}$ be the conjugate space to V with respect to the complex structure. Let $c \mapsto \bar{c}$ be complex conjugation on $\mathbb{C}$, so $\overline{V}$ is V but with $c \in \mathbb{C}$ acting on $v' \in \overline{V}$ as $\bar{c}v'$. To avoid any possible confusion we write $[c]v'$ to denote the conjugate structure and omit the brackets when talking about the $\mathbb{C}$-structure.

We have a canonical $\mathbb{C}$-linear isomorphism

$$\mathbb{C} \otimes_{\mathbb{R}} V \to V \times \overline{V} \tag{1}$$

given by $c \otimes v \mapsto (cv, [c]v)$. The $\mathbb{C}$-structure on the left hand side is through the left tensor factor. To see that this map is actually an isomorphism, first note that it is compatible with direct sums in V, so we can immediately reduce to the case $V = \mathbb{C}$. Now it is just a special cause of the fact that is K/k is a finite Galois extension with $G = \mathrm{Gal}(K/k)$, then the natural map

$$\begin{aligned} K \otimes_k K &\to \prod_{g \in G} K \\ a \otimes b &\mapsto (g(a)b) \end{aligned}$$

is an isomorphism. This is a consequence of the Normal Basis Theorem. For $K = \mathbb{C}$ and $k = \mathbb{R}$, one can check this fairly easily by hand as well. The appropriate map to use and most of the relevant calculations will be discussed below anyways.

Since V and $\overline{V}$ have the same underlying $\mathbb{R}$-space, we can define an $\mathbb{R}$-linear map

$$V \times \overline{V} \to \mathbb{C} \otimes_{\mathbb{R}} V \tag{2}$$

by

$$\begin{aligned} (v, v') &\mapsto \left(\frac{1 \otimes v - i \otimes iv}{2} + \frac{1 \otimes v' - i \otimes [i]v'}{2} \right) \\ &= \left(\frac{1 \otimes v - i \otimes iv}{2} + \frac{1 \otimes v' + i \otimes iv'}{2} \right). \end{aligned}$$

This map is independent of the choice of i, so it is canonical as the map in Equation 1 was. Therefore, composing this with the map in Equation 1 gives the identity. Since the map in Equation 1 is a $\mathbb{C}$-linear isomorphism, this map must be as well. This could of course be checked by hand as well, though tedious. In particular, the map in Equation 2 must be $\mathbb{C}$-linear when using $\mathbb{C}$-structures on the two sides exactly as in Equation 1.

We are now in a position to compute $E_{\mathbb{C}}$ in the form

$$(V \times \overline{V}) \times (V \times \overline{V}) = V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}.$$

Pick a point $((v, v'), (w, w'))$ on the left hand side. This maps to the point

$$\left(\frac{1 \otimes v - i \otimes iv}{2} + \frac{1 \otimes v' + i \otimes iv'}{2}, \frac{1 \otimes w - i \otimes iw}{2} + \frac{1 \otimes w' + i \otimes iw'}{2} \right)$$

in $V_{\mathbb{C}} \times V_{\mathbb{C}}$, so by repeated application of the bilinearity this is mapped under $E_{\mathbb{C}}$ to

$$\begin{aligned} &\frac{1}{4}(E_{\mathbb{R}}(v,w) + E_{\mathbb{R}}(v,w') - E_{\mathbb{R}}(iv,iw) + E_{\mathbb{R}}(iv,iw') \\ &E_{\mathbb{R}}(v',w) + E_{\mathbb{R}}(v',w') + E_{\mathbb{R}}(iv',iw) - E_{\mathbb{R}}(iv',iw)) \\ +&\frac{1}{4}(-E_{\mathbb{R}}(v,iw) + E_{\mathbb{R}}(v,iw') - E_{\mathbb{R}}(iv,w) - E_{\mathbb{R}}(iv,w') \\ -&E_{\mathbb{R}}(v',iw) + E_{\mathbb{R}}(v',iw') + E_{\mathbb{R}}(iv',w) + E_{\mathbb{R}}(iv',w')). \end{aligned}$$

The component of $E_{\mathbb{C}}$ that is in $\bigwedge_{\mathbb{C}}^2 T$ is the restriction of $E_{\mathbb{C}}$ to an alternating $\mathbb{C}$-bilinear form on $V \times V$, which is to say

$$(v,w) \mapsto \frac{1}{4}(E_{\mathbb{R}}(v,w) - E_{\mathbb{R}}(iv,iw)) + \frac{1}{4}(-E_{\mathbb{R}}(v,iw) - E_{\mathbb{R}}(iv,w)).$$

This vanishes if and only if $E_{\mathbb{R}}(v,w) = E_{\mathbb{R}}(iv,iw)$ and $E_{\mathbb{R}}(v,iw) = -E_{\mathbb{R}}(iv,w)$. The condition that $E_{\mathbb{R}}(v,w) = E_{\mathbb{R}}(iv,iw)$ implies the condition that $E_{\mathbb{R}}(v,iw) = -E_{\mathbb{R}}(iv,w)$. Therefore, $E_{\mathbb{R}}(ix,iy) = E_{\mathbb{R}}(x,y)$ for every $x,y \in V = \bigwedge_{\mathbb{R}}$ if and only if the component of $E_{\mathbb{C}}$ in $\bigwedge_{\mathbb{C}}^2 T$ vanishes. Similarly, one can do the same for $\overline{V} \times \overline{V}$ and conclude that the resulting equations vanish when the ones in $V \times V$ does.
So in order for E to be zero it is enough for the composite map for $\bigwedge^2 \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}) \to \bigwedge^2 \mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C})$ to vanish.
So what we really need to show then is that $\bigwedge_{\mathbb{Z}}^2 \mathrm{Hom}(\Lambda,\mathbb{Z}) \to \bigwedge_{\mathbb{C}}^2 T$ is injective. $\mathrm{Hom}(\Lambda, \mathbb{Z})$ projects into a lattice in T. We would like to show that any lattice in the $\mathbb{R}$-vector space $T = \mathrm{Hom}(V, \mathbb{C})$ is the image of the $\mathbb{Z}$-dual of a suitable $\mathbb{Z}$-lattice Λ in V via the composite map

$$\Lambda^{\vee} = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V \times \overline{V}, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) = V^{\vee}.$$

Let L be an arbitrary lattice in T. Let Λ be the set of $v \in V$ such that $\mathrm{Re}(l(v))$ lies in $\mathbb{Z}$ for all $l \in L$.

Lemma 1.21. *Λ is a lattice in V and $\Lambda^{\vee}$ maps isomorphically onto L.*

Proof. First we show that Λ is a lattice. Suppose that V has dimension g over $\mathbb{C}$.
Claim: $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ as real vector spaces.
Pf: The map from $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ is given by

$$\phi \mapsto (v \mapsto \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(\phi(v)) = (v \mapsto 2\,\mathrm{Re}(\phi(v))).$$

In general, if K/k is a finite separable extension of fields and V is a finite dimensional K-vector space, then composing with the trace map gives a map

$$\mathrm{Hom}_K(V, K) \to \mathrm{Hom}_k(V, k)$$

that is an isomorphism of k-vector spaces. (The formation of the map is compatible with direct sums, so we can reduce to the case that $V = K$. The map is then injective because $\mathrm{Tr} \neq 0$ and K is a field. One can then use a dimension count to get the isomorphism.) Applying the general case we have the claim.

Therefore L is identified with a lattice in $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ via associating to each l in L its real part. Choosing a $\mathbb{Z}$-basis $\{l_1, \ldots, l_{2g}\}$ of L, we can take the the dual basis to this in V and obtain a linearly independent subset of Λ of dimension $2g$. Therefore Λ has the correct rank and is a lattice.

Now we just need to show that $\Lambda^\vee$ maps isomorphically onto L under the above maps. Let $l \in \Lambda^\vee$ and $v \in V$. We will denote the extension of l to V by $l_{\mathbb{R}}$ and the extension to $V_{\mathbb{C}}$ by $l_{\mathbb{C}}$. Now we just need to investigate what happens to l under the displayed map from $\Lambda^\vee$ to $V^\vee$ preceding the lemma. Under the injection $\Lambda^\vee \hookrightarrow \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$ we have that

$$l \mapsto l_{\mathbb{C}} : (c \otimes v \mapsto c l_{\mathbb{R}}(v)).$$

The isomorphism $\mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(V \times \overline{V}, \mathbb{C})$ is given by

$$\phi \mapsto \left((v, w) \mapsto \frac{1}{2}\phi(1 \otimes v - i \otimes iv + 1 \otimes w + i \otimes iw) \right).$$

The last map $\mathrm{Hom}_{\mathbb{C}}(V \times \overline{V}, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is just projection, i.e., $\psi \mapsto (v \mapsto \psi(v, 0))$. Putting this all together we have

$$l \mapsto \left(v \mapsto \frac{1}{2}((l_{\mathbb{R}})(v) - i(l_{\mathbb{R}})(iv)) \right).$$

Composing this map with the isomorphism given in the Claim above gives that

$$l \mapsto (v \mapsto \mathrm{Re}(l(v)))\,.$$

Note that the definition of Λ and the proof of the Claim show that upon putting L into $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ via the isomorphism in the Claim, Λ may be identified with the set of $v \in V$ under which all elements of L considered in

$\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$ are $\mathbb{Z}$-valued on v. Hence, by the simple relationship between $\mathbb{Z}$-duality for lattices and $\mathbb{R}$-duality for $\mathbb{R}$-vector spaces (via the fact that an $\mathbb{Z}$-basis of a lattice is a $\mathbb{R}$-basis of its $\mathbb{R}$-scalar extension), we can conversely say that L in $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$ is the set of $\mathbb{R}$-linear functionals on V that are $\mathbb{Z}$-valued on the lattice Λ. Since $\Lambda^\vee$ mapping to $\mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ maps isomorphically onto L if and only if the composite with the isomorphism in the Claim carries $\Lambda^\vee$ isomorphically to the image of L in $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$, our problem is to show that $\Lambda^\vee$ maps to exactly the set of elements of $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$ that are $\mathbb{Z}$-valued on Λ. We showed above that the map from $\Lambda^\vee$ to $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$ is simply $l \mapsto l_{\mathbb{R}}$, which is to say that it is the canonical injection

$$\mathrm{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$$

via extension of scalars (and the identification of $\mathbb{R}\otimes_{\mathbb{Z}}\Lambda$ with V), so our problem is to show that the image of this canonical injection is exactly the set of $\mathbb{R}$-linear functionals on V that are $\mathbb{Z}$-valued on the lattice Λ in V. This in turn is immediate upon using a $\mathbb{Z}$-basis of Λ as a $\mathbb{R}$-basis of V. □

The result now follows from

Lemma 1.22. *Let V be a g-dimensional complex vector space. Then for almost all lattices $\Lambda \subset V$, the map $\bigwedge_{\mathbb{Z}}^k \Lambda \to \bigwedge_{\mathbb{C}}^k V$ is injective for all $k \le g$.*

Proof. Consider the countably many choices of integers $q_{i_1,\dots,i_g}$ not all zero as $(i_1,\dots,i_g)$ ranges over strictly increasing sequences of integers between 1 and $2g$. For each such tuple not all zero, we get a nonzero polynomial over $\mathbb{Z}$

$$\sum_{(i_1,\dots,i_g)} q_{i_1,\dots,i_g} \det(\omega_{i_r,s})_{1\le r,s\le g}$$

where the determinant uses indeterminate coordinates $\omega_{i,j}$ for $1 \le j \le g$ of the i^{th} of the $2g$ $\mathbb{R}$-basis vectors $\lambda_1,\dots,\lambda_{2g}$ for the arbitrary choice of lattice Λ in the fixed $\mathbb{C}$ vector space V. A choice of $\omega_{i,j}$'s kills such a polynomial for the given choice of q's if and only if the non-zero vector $\sum q_{i_1,\dots,i_g}\lambda_{i_1}\wedge\cdots\wedge\lambda_{i_g}$ in $\bigwedge_{\mathbb{Z}}^g \Lambda$ has vanishing image in $\bigwedge_{\mathbb{C}}^g V$.

These are hypersurfaces in an Euclidean space on $2g^2$ parameters $\omega_{i,j}$, so it gives a countable collection of analytic sets of dimension $g(2g)-1$ in such an Euclidean space whose points are exactly the lattices that violate the injectivity condition of the Lemma. An analytic hypersurface in a complex Euclidean space has empty interior (follows from analytic continuation), so the Baire Category Theorem ensures there are lots of points not in the countable union of hypersurfaces. □

Therefore we have shown that almost all complex tori are not in fact abelian varieties.

Definition 1.23. A torus is *simple* if it does not contain any nontrivial subtori.

Theorem 1.24. *(Poincare Irreducibility Theorem) Let $X = V/\Lambda$ be an abelian variety, i.e., a complex torus with a positive definite Riemann form on V. Let Y be an abelian subvariety of X. Then there exists another abelian subvariety Y' such that $Y + Y' = X$ and $Y \cap Y'$ is finite, i.e., the map*

$$Y \times Y' \to X, \qquad (y, y') \mapsto y + y'$$

is an isogeny.

Proof. Let H be the positive definite Riemann form for X and let $E = \mathrm{Im}(H)$. Let $\Lambda_Y = T_0(Y) \cap \Lambda$ so that $Y = T_0(Y)/\Lambda_Y = V_Y/\Lambda_Y$. Then we can consider the orthogonal complement of V_Y with respect to H,

$$V_{Y'} = \{x \in V : H(x, y) = 0 \text{ for all } y \in V_Y\}.$$

Now by the definition of H, we can also write

$$V_{Y'} = \{x \in V : E(x, y) = 0 \text{ for all } y \in \Lambda_Y\}.$$

Now consider

$$\Lambda_{Y'} = \Lambda \cap V_{Y'} = \{x \in \Lambda : E(x, y) = 0 \text{ for all } y \in \Lambda_Y\}.$$

Since we know that E is nondegenerate and Λ_Y is a lattice in Y, we have

$$\mathrm{rank}\Lambda_{Y'} = \mathrm{rank}\Lambda - \mathrm{rank}\Lambda_Y = 2\dim_{\mathbb{C}} V_{Y'}.$$

This shows that $\Lambda_{Y'}$ is a lattice in $V_{Y'}$ and so $Y' = V_{Y'}/\Lambda_{Y'}$ is a complex subtorus of X, and hence an abelian subvariety by GAGA. Since $V = V_Y \oplus V_{Y'}$, we have $X = Y + Y'$ and $Y \cap Y'$ is finite. □

Corollary 1.25. *Let X be an abelian variety over $\mathbb{C}$. Then X is isogenous to a product of the form*

$$X_1^{n_1} \times \cdots \times X_r^{n_r}$$

where the X_i's are simple and pairwise nonisogenous abelian varieties. If $Y_1^{m_1} \times \cdots \times Y_s^{m_s}$ is another such decomposition, then $r = s$, X_i is isogenous to Y_j for some i, j, and $n_i = m_j$.

Proof. Exercise: Just use induction the dimension of X and the Poincare Irreducibility Theorem. □

As a corollary to all of this, we get:

Corollary 1.26. *Let X be an abelian variety with semisimple decomposition as in Corollary 1.25. Then one has*

$$\mathrm{End}^\circ(X) = \bigoplus_{i=1}^{r} M_{n_i}(\mathrm{End}^\circ(X_i))$$

where $M_{n_i}(\mathrm{End}^\circ(X_i))$ is the set of $n_i \times n_i$ matrices with entries in $\mathrm{End}^\circ(X_i)$. We also have that $\mathrm{End}^\circ(X_i)$ is a division ring.

Proof. (Sketch) The assertion that $\mathrm{End}^\circ(X_i)$ is a division ring follows immediately from the fact that a non-zero endomorphism of a simple abelian variety is necessarily an isogeny. Lemma 1.11 then shows the isogeny is invertible.

For $i \neq j$, we have that $\mathrm{Hom}(X_i, X_j) = \{0\}$, i.e., there are no non-trivial homomorphisms between simple, non-isogenous abelian varieties. Next we use the fact that if X, X', and X'' are commutative Lie groups, then maps $X \times X' \to X''$ are necessarily of the form (f, g) where $f : X \to X''$ and $g : X' \to X''$. Therefore, we can write $\mathrm{End}^\circ(X) = \bigoplus_{i=1}^{r} M_{n_i}(\mathrm{End}^\circ(X_i))$. □

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