CM SEMINAR TALK

by

Nick Ramsey

Contents

1. Topologies on points

Given an algebraic group G over a global field K, Weil has defined a topological space $G(\mathbb{A}_K)$ essentially by mimicking the construction of $\mathbb{A}_K$ itself, using only the points of G with values in various local fields. We would like to realize this in a natural way as a topological space whose underlying set is literally the set of $\mathbb{A}_K$-valued points of G in the modern sense *à la* Grothendieck.

In this spirit, let's generalize. Let R be a topological ring and let X be an affine scheme of finite type over R. We wish to endow $X(R)$ with a topology in some "nice" manner. The following Theorem is taken from ([**2**]).

Theorem 1.1. — *Let R be a topological ring. There is a unique way to topologize $X(R)$ for affine finite-type R-schemes X in a manner which is functorial in X, compatible with fiber products, takes closed immersions to topological embeddings, and for $X = \mathrm{Spec}(R[t])$ recovers the given topology on $X(R) = R$.*

It is important to note that this construction *does not* carry open immersions to open embeddings of topological spaces in general. As an example consider the open immersion

$$\mathrm{Spec}(R[t, t^{-1}]) \longrightarrow \mathrm{Spec}(R[t]),$$

which on points is simply the inclusion

$$R^\times \hookrightarrow R.$$

The problem is that it need not be the case that $R^\times$ is open in R. This fails, for example, for the ring $R = \mathbb{A}_{\mathbb{Q}}$. The point (made in ([**2**]), for example) is that this is the *only* obstruction to open immersions being carried to open embeddings. In particular, in case $R^\times$ is open in R, one can pass to affines and glue to topologize the R-points of all affine finite-type R-schemes.

Another consequence which is more relevant to us is that the naive topology one might put on the points $G(F)$ of an algebraic group G over a local field F (by embedding in $G(F) \subset \mathrm{GL}_n(F) \subset M_{n\times n}(F)$) is the right one. This "easy" topology is indeed the starting point of Weil's original method getting at the adelic points of G, which we now detail.

Let K be a global field. For any finite set S of places of K (assumed to contain the Archimedean places in the number field case), we define $\mathcal{O}_{K,S} \subseteq \mathcal{O}_K$ and $\mathbb{A}_{F,S} \subseteq \mathbb{A}_K$ to be the subrings consisting of elements that are integral outside S. Let us now fix an algebraic group G over K, a finite set S as above, and a model G_S of G over $\mathcal{O}_{K,S}$ (i.e. G_S is a flat affine $\mathcal{O}_{K,S}$-group). For any place v of K, we will denote the base change of G to K_v by G_v. For a place v of K with $v \notin S$, we will denote the base change of G_S to $\mathcal{O}_v$ by $G_{S,v}$. The following theorem is the link between points of view of Grothendieck and Weil.

Theorem 1.2. — *The natural map*

$$G(\mathbb{A}_K) \longrightarrow \varinjlim_{S' \supseteq S} \prod_{v\in S'} G_v(K_v) \times \prod_{v \notin S'} G_{S,v}(\mathcal{O}_v)$$

is is an isomorphism of topological groups.

Proof. — See [**2**]. □

Example 1.3. — Let K be a global field. Then the topology defined above on $\mathbb{G}_m(\mathbb{A}_K) = \mathbb{A}_K^\times$ agrees with the usual topology on the groups of ideles. To see this write $\mathbb{G}_m = \mathrm{Spec}(K[x,y]/(xy-1))$, so that the topology on $\mathbb{A}_K^\times$ is the one it inherits as a subset of $\mathbb{A}_K^2$ via the embedding $t \mapsto (t, t^{-1})$. This the classical idele topology as defined, for example, in [**3**].

2. Weil restriction of scalars

2.1. Generalities. — Let $S' \to S$ be a map of schemes and let X' be an S'-scheme. Consider the functor on S-schemes

$$T \longmapsto X'(T \times_S S').$$

If this functor is representable by an S-scheme X we call X the *restriction of scalars* (or Weil restriction of scalars, or simply Weil restriction) of X from S' to S, and denote it by $\mathrm{Res}_{S'/S}(X')$. When this functor is representable we will simply say that $\mathrm{Res}_{S'/S}(X)$ "exists". We note also that any two schemes representing the functor are uniquely isomorphic in a manner preserving all of the relevant data.

The following is a special case of the existence theorem given in [**1**], and will suffice for our purposes.

Theorem 2.1. — *Let A be a commutative ring and let A' be an A-algebra which is free of finite rank as an A-module. Let X' be an affine scheme of finite type over* $\operatorname{Spec}(A')$. *Then* $\operatorname{Res}_{A'/A}(X)$ *exists and is affine of finite type over A.*

Proof. — Let $\{e_1, \ldots, e_r\}$ be a basis of A' over A and let $c_{ij}^k \in A$ be the structure constants defined by

$$e_i e_j = \sum_k c_{ij}^k e_k.$$

Write

$$X' = \operatorname{Spec}(A'[T_1, \ldots, T_n]/I).$$

If C is an A-algebra, then to give a A'-algebra map

$$A'[T_1, \ldots, T_n]/I \longrightarrow C \otimes_A A'$$

is to give an n-tuple $(t_1, \ldots, t_n)$ of elements of $C \otimes_A A'$ such that $f(t_1, \ldots, t_n) = 0$ for all $f \in I$. Note that

$$C \otimes_A A' \cong \bigoplus_k C e_k \tag{1}$$

with multiplication given by the same constants $\{c_{ij}^k\}$ above. Breaking the elements t_i into their coordinates via (1) we arrive at an nr-tuple s_{ij} (s_{ij} denoting the $1 \otimes e_j$ coordinate of t_i) of elements of C.

For each $f \in I$, the condition that $f(t_1, \ldots, t_n) = 0$ translates in r separate conditions on the nr-tuple (s_{ij}) (write everything in sight in terms of the $1 \otimes e_i$'s, multiply out and collect coefficients of each of the r separate $1 \otimes e_i$'s). Let J denote the ideal in $A[S_{ij}]$ generated by these conditions as we vary $f \in I$, so we have constructed a map

$$\operatorname{Hom}_{A'}(A'[T_i]/I, C \otimes_A A') \longrightarrow \operatorname{Hom}_A(A[S_{ij}]/J, C).$$

It is clear from the construction that this map is a bijection and is compatible with change of A-algebra C, and is therefore an identification of functors. So

$$X = \operatorname{Spec}(A[S_{ij}]/J)$$

is the restriction of scalars. □

Example 2.2. — Let $A = \mathbb{Z}$ and $A' = \mathbb{Z}[\sqrt{2}]$. We can use this construction to see very concretely what the Weil restriction of an explicitly given scheme over A' looks like. For example, let

$$X' = \mathbb{A}^1_{\mathbb{Z}[\sqrt{2}]} = \operatorname{Spec}(\mathbb{Z}[\sqrt{2}][T]).$$

Then, writing $T = S_1 + S_2\sqrt{2}$ we see that the restriction is simply

$$\mathbb{A}^2_{\mathbb{Z}} = \operatorname{Spec}(\mathbb{Z}[S_1, S_2]).$$

A somewhat less trivial example is obtained by taking $X' = \operatorname{Spec}(\mathbb{Z}[\sqrt{2}])$ itself. Writing X' as

$$\operatorname{Spec}(\mathbb{Z}[T]/(T^2 - 2))$$

and using the decomposition $T = S_1 + S_2\sqrt{2}$ as above, we note that

$$T^2 - 2 = S_1^2 + 2S_2^2 - 2 + 2S_1S_2\sqrt{2}.$$

Thus the ideal of relations in $\mathbb{Z}[S_1, S_2]$ which defines the Weil restriction is $(S_1^2 + 2S_2^2 - 2, 2S_1S_2)$, and the above construction gives

$$\mathrm{Res}_{\mathbb{Z}[\sqrt{2}]/\mathbb{Z}}(\mathrm{Spec}(\mathbb{Z}[\sqrt{2}])) = \mathrm{Spec}(\mathbb{Z}[S_1, S_2]/(S_1^2 + 2S_2^2 - 2, 2S_1S_2)).$$

We record a few properties of the restriction of scalars for later use.

Theorem 2.3. — *Supposing that all of the indicated restrictions exist, we have the following.*

1. *For a map $T \to S$ and an S'-scheme X',*
$$\mathrm{Res}_{S'/S}(X') \times_S T \cong \mathrm{Res}_{S' \times_S T/T}(X' \times_{S'} (S' \times_S T))$$
2. *For S'-schemes X' and Y',*
$$\mathrm{Res}_{S'/S}(X' \times_{S'} Y') \cong \mathrm{Res}_{S'/S}(X') \times_S \mathrm{Res}_{S'/S}(Y')$$
3. *For a map $S'' \to S'$ and an S''-scheme X'',*
$$\mathrm{Res}_{S'/S}(\mathrm{Res}_{S''/S'}(X'')) \cong \mathrm{Res}_{S''/S}(X'').$$
4. *If A is a topological ring and A' is a topological A-algebra which is free of finite rank as an A-module and given the product topology, then the identification*
$$\mathrm{Res}_{A'/A}(X')(A) \cong X'(A')$$
given above in the construction of $\mathrm{Res}_{A'/A}(X')$ is a homeomorphism.

Proof. — The first three properties are easily checked by comparing functors of points. The last follows easily from the construction of $\mathrm{Res}_{A'/A}(X')$. We leave it to the reader to check the details. □

2.2. Finite separable field extensions. — Let us now specialize to the case of a finite separable extension of fields K'/K. Let X' be any K'-scheme for which the restriction of scalars $\mathrm{Res}_{K'/K}(X')$ exists (e.g. X' affine of finite type over K'). In this case the restriction of scalars has a very explicit interpretation due to Weil. Let F/K be a finite Galois extension such that F contains a Galois closure of K' over K and let $G = \mathrm{Gal}(F/K)$. We define a scheme $\tilde{X}$ over F by

$$\tilde{X} = \prod_{i:K' \hookrightarrow F} X' \otimes_{K'} F_i$$

where the product is taken over all embeddings $i : K' \hookrightarrow F$ of K-algebras, and F_i denotes F thought of as a K'-algebra via i. Note that the group G acts on the set of K-embeddings $K' \hookrightarrow F$ by post-composition. Thus, for each $\sigma \in G$, we get an isomorphism

$$\varphi_\sigma : \sigma^*\tilde{X} \xrightarrow{\sim} \tilde{X}$$

by permuting the factors in this product. The collection $\{\varphi_\sigma\}$ is easily seen to be a descent datum.

Theorem 2.4. — *This descent datum is effective and the K-scheme obtained by descent is isomorphic to $\mathrm{Res}_{K'/K}(X')$. In particular, if X' is geometrically connected (resp. smooth) over K', then so is $\mathrm{Res}_{K'/K}(X')$ over K.*

Proof. — By Galois descent it suffices to see that

$$\mathrm{Res}_{K'/K}(X') \otimes_K F \cong \prod_{i:K' \hookrightarrow F} X' \otimes_{K'} F_i$$

in a manner which takes the canonical descent datum on the left-hand side to the descent datum we have defined above on the right-hand side. By Theorem 2.3, we have

$$\begin{aligned} \mathrm{Res}_{K'/K}(X') \otimes_K F &\cong \mathrm{Res}_{K' \otimes_K F/F}(X' \otimes_{K'} (K' \otimes_K F)) \\ &\cong \mathrm{Res}_{\prod_i F_i/F}(X' \otimes_{K'} \prod_i F_i) \\ &\cong \prod_i X' \otimes_{K'} F_i. \end{aligned}$$

It is left to the reader that to check that this respects the descent data on both sides. ☐

This realization of the restriction of scalars can be used to deduce a few nice properties in the case of a finite separable field extension. Before detailing them, we give a general construction. If we begin with X/S, then we have, for any S-scheme T,

$$\mathrm{Res}_{S'/S}(X \times_S S')(T) = (X \times_S S')(T \times_S S').$$

In particular, any T-valued point of X gives rise to an T-valued point of $\mathrm{Res}_{S'/S}(X \times_S S')$ by base change (and hence in a manner that is functorial in T). Thus we have map

$$X \longrightarrow \mathrm{Res}_{S'/S}(X \times_S S').$$

Theorem 2.5. —

1. *Let K'/K be a finite Galois extension and let X' be a K'-scheme for which the restriction of scalars exists. Then*

$$\mathrm{Res}_{K'/K}(X') \otimes_K K' \cong \prod_{\sigma \in \mathrm{Gal}(K'/K)} X' \otimes_{K'} K'_\sigma.$$

2. *Let K'/K be a finite separable extension and let X be a separated K-scheme for which the restriction of scalars of $X \otimes_K K'$ exists. Then the map*

$$X \longrightarrow \mathrm{Res}_{K'/K}(X \otimes_K K')$$

defined above is a closed embedding.

Proof. —

1. This follows immediately from Theorem 2.4 since one may take $F = K'$ in the Galois case.

2. Let F be a finite Galois extension of K containing a Galois closure of K'. One checks that the morphism in question is the descent of the closed diagonal embedding
$$X \otimes_K F \longrightarrow \prod_i (X \otimes_K K') \otimes_{K'} F_i.$$

□

3. Groups with compact real points

Theorem 3.1. — *Let G be an algebraic group over $\mathbb{Q}$. If $G(\mathbb{R})$ is compact, then the embedding $G(\mathbb{Q}) \hookrightarrow G(\mathbb{A}_f)$ has discrete image.*

Proof. — We first make a general observation. Let X be a first countable Hausdorff topological space in which every convergent sequence has a subsequence which is eventually constant. We claim that X must be discrete (note that the converse is obvious). Let $x \in X$ and let $Y = X \setminus \{x\}$. If $\{y_n\}$ is a sequence in Y that converges in X, then its limit (well defined by the Hausdorff property) is also in Y since $\{y_n\}$ has a subsequence which is eventually constant. Since Y contains the limits of all of its sequences which converge in X, Y must be closed in X (this uses first countability), so that $\{x\}$ is open and X is discrete.

Note that the subspace $G(\mathbb{Q})$ of the first countable Hausdorff space $G(\mathbb{A}_f)$ is first countable and Hausdorff. Thinking of $G(\mathbb{A}_f)$ as a subspace of $G(\mathbb{A})$ and identifying $g \in G(\mathbb{Q})$ with its image *in* $G(\mathbb{A})$, we see that the image of g in $G(\mathbb{A}_f)$ is gg_∞^{-1}, where g_∞ is the infinite component of g. Let $\{g_n\}$ be a sequence in $G(\mathbb{Q})$ whose image $\{g_n g_{n,\infty}^{-1}\}$ in $G(\mathbb{A}_f)$ converges, say to g_f. Since $G(\mathbb{R})$ is compact, we may assume (by passing to a subsequence) that $g_{n,\infty} \to g_\infty \in G(\mathbb{R})$. But then $g_n \to g_f g_\infty$ in $G(\mathbb{A})$. Since $G(\mathbb{Q})$ is discrete in $G(\mathbb{A})$, the sequence $\{g_n\}$ must be eventually constant, so that the same is true of $g_n g_{n,\infty}^{-1}$. Thus $G(\mathbb{Q})$ is discrete in $G(\mathbb{A}_f)$ by the above criterion. □

Theorem 3.2. — *Let G be a connected reductive algebraic group over $\mathbb{Q}$. Suppose the inclusion $G(\mathbb{Q}) \hookrightarrow G(\mathbb{A}_f)$ has discrete image. Then the natural map*
$$G(\mathbb{Q})\backslash G(\mathbb{A}_f) \longrightarrow \varprojlim G(\mathbb{Q})\backslash G(\mathbb{A}_f)/U$$
is a topological isomorphism (the limit being taken over all compact open subgroups U of $G(\mathbb{A}_f)$), and

Proof. — Let us first note that the discreteness hypothesis implies that all of the spaces above are Hausdorff. Note also that the map in question is certainly continuous.

To check injectivity of this map, let suppose $G(\mathbb{Q})x$ and $G(\mathbb{Q})y$ have common image. That is, for each compact open U there exists $u_U \in U$ and $g_U \in G(\mathbb{Q})$ (which we will identify with its image *in* $G(\mathbb{A}_f)$) such that $x = g_U y u_U$, or $g_U = x u_U^{-1} y^{-1}$. From the topological structure of Lie groups over nonarchimedean local fields, there is a cofinal nested sequence $\{U_n\}$ of compact open subgroups in $G(\mathbb{A}_f)$ with intersection $\{1\}$. Let us denote g_{U_n} and u_{U_n} by g_n and u_n, respectively, for brevity. Then $g_n \to xy^{-1}$ by continuity, and it follows from discreteness that g_n is eventually constant, say g. Then $x = gy$, so that $G(\mathbb{Q})x = G(\mathbb{Q})y$, and the map is injective.

Since the discrete set

$$G(\mathbb{Q})\backslash G(\mathbb{A}_f)/U$$

is finite (for any compact open U), the space $G(\mathbb{A})\backslash G(\mathbb{A}_f)$ is compact since it is a finite union of cosets of U, which is compact. It follows that the image of the natural map

$$G(\mathbb{Q})\backslash G(\mathbb{A}_f) \longrightarrow \varprojlim G(\mathbb{Q})\backslash G(\mathbb{A}_f)/U$$

is closed. As this image surjects onto each of the quotients on the right, it is also dense, so this map is surjective. The result now follows since a continuous bijective map between compact Hausdorff spaces is a homeomorphism. □

4. Applications to Tori

Let $F_1 \subseteq F_2 \subseteq F_3$ be a tower of finite separable extensions of fields. Let

$$N_{F_3/F_2} : \mathrm{Res}_{F_3/F_2}(\mathbb{G}_{m/F_3}) \longrightarrow \mathbb{G}_{m/F_2}$$

denote the norm map defined on points by the usual algebra norm

$$(A \otimes_{F_2} F_3)^\times \longrightarrow A^\times$$

for any F_2-algebra A. Further restricting and using the transitivity property of Theorem 2.3, we arrive a map

$$N : \mathrm{Res}_{F_3/F_1}(\mathbb{G}_{m/F_3}) \longrightarrow \mathrm{Res}_{F_2/F_1}(\mathbb{G}_{m/F_2}).$$

Theorem 4.1. — *The map N is surjective and* $\ker(N)$ *is a torus.*

Proof. — The theorem is equivalent to the statement that the induced map on character groups

$$N^* = X(N) : X(\mathrm{Res}_{F_2/F_1}(\mathbb{G}_{m/F_2})) \longrightarrow X(\mathrm{Res}_{F_3/F_1}(\mathbb{G}_{m/F_3}))$$

is injective and has torsion-free cokernel. It suffices to check this after passing to a separable field extension of F_1. Let F be a finite Galois extension of F containing a Galois closure of F_3 (and hence one of F_2). Let H_i denote the set of F_1-embeddings of F_k into F ($k = 2, 3$). The base-change of N to F is the map

$$\prod_{i \in H_3} \mathbb{G}_{m/F} \longrightarrow \prod_{i \in H_2} \mathbb{G}_{m/F}$$

which sends the tuple $(t_i)_{i\in H_3}$ to the tuple $(s_j)_{j\in H_2}$ where s_j is product of all t_i with i restricting to j.

We have canonical identifications

$$X\left(\prod_{i\in H_k} \mathbb{G}_{m/F}\right) = \mathbb{Z}^{H_k}$$

for $k = 2, 3$. In these terms, the map N^* is

$$\begin{array}{rcl} \mathbb{Z}^{H_2} & \longrightarrow & \mathbb{Z}^{H_3} \\ f & \longmapsto & N^*(f) : (j \mapsto f(j|_{F_2})) \end{array}$$

This homomorphism is trivially injective. Moreover, the image consists of those functions $H_3 \to \mathbb{Z}$ whose value at some embedding j depends only on the restriction of j to F_2. As this property is unaffected by multiplying by some nonzero $n \in \mathbb{Z}$, so the cokernel is torsion-free. $\square$

Theorem 4.2. — *Let K be a CM field and K_0 its maximal totally real subfield. Let T be the kernel of*

$$N : \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_{m/K}) \longrightarrow \mathrm{Res}_{K'/\mathbb{Q}}(\mathbb{G}_{m/K'}).$$

Then $T(\mathbb{R})$ is compact.

Proof. — The set of real points of the kernel of N is the kernel of the map induced by N on the real points

$$\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m/K)(\mathbb{R}) \longrightarrow \mathrm{Res}_{K_0/\mathbb{Q}}(\mathbb{G}_m/K_0)(\mathbb{R}).$$

By definition of Weil restriction and compatibility with topologies (as in Theorem 2.3), this is a map

$$(\mathbb{R} \otimes_{\mathbb{Q}} K)^\times \longrightarrow (\mathbb{R} \otimes_{\mathbb{Q}} K_0)^\times$$

of topological groups.

Let σ_i $(1 \leq i \leq [K_0 : \mathbb{Q}])$ denote the distinct embeddings $K_0 \hookrightarrow \mathbb{R}$. Above each such σ_i lies a unique conjugate pair of embeddings of K into $\mathbb{C}$. Then we have the following compatible decompositions of algebras

$$\mathbb{R} \otimes_{\mathbb{Q}} K_0 \cong \prod_i \mathbb{R}_{\sigma_i}$$

and

$$\mathbb{R} \otimes_{\mathbb{Q}} K \cong (\mathbb{R} \otimes_{\mathbb{Q}} K_0) \otimes_{K_0} K \cong \prod_i \mathbb{R}_{\sigma_i} \otimes_{K_0} K.$$

The algebra norm from the latter algebra to the former is therefore the product of the the individual norms

$$(\mathbb{R}_{\sigma_i} \otimes_{K_0} K)^\times \longrightarrow \mathbb{R}_{\sigma_i}^\times.$$

Choosing one of the embeddings $K \hookrightarrow \mathbb{C}$ lying above σ_i gives an isomorphism of $\mathbb{R}_{\sigma_i}$-algebras

$$\mathbb{R}_{\sigma_i} \otimes_{K_0} K \cong \mathbb{C}.$$

In particular, the kernel of each of these individual norm maps is topologically isomorphic to the unit circle in $\mathbb{C}$. The kernel of N on real points is then isomorphic to a product of $[K_0 : \mathbb{Q}]$ circles, and hence compact. $\square$

Theorem 4.3. — *Let $T/\mathbb{Q}$ be a torus, and let $U \subset T(\mathbb{A}_f)$ be a compact open subgroup. Then the discrete space*

$$T(\mathbb{Q}) \backslash T(\mathbb{A}) / T(\mathbb{R}) U$$

is finite.

Proof. — See section 5 of [**2**] for a self-contained proof of this finiteness result for tori. $\square$

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Nick Ramsey • *E-mail :* naramsey@umich.edu