

March 18, 2004

MAZUR SEMINAR

## 1. INTRODUCTION

The goal of these notes is to show that the modular jacobian  $J_0(N)_{/\mathbf{Q}}$  has a nontrivial quotient of rank 0 for any prime  $N$  such that the genus of  $X_0(N)$  is positive.

Recall the following theorem from James' talk:

**Theorem 1.1.** *Let  $A_{/\mathbf{Q}}$  be an abelian variety with good reduction outside of  $N$  and purely toric reduction at  $N$ . Suppose moreover that  $\mathcal{A}[p]$  is admissible for some  $p \neq N$ , where  $\mathcal{A}_{/\mathbf{Z}}$  is the Néron model of  $A$ . Then  $A_{/\mathbf{Q}}$  has rank 0.*

Our aim is to construct a nonzero isogeny factor  $J_{\mathfrak{p}}$  of  $J_0(N)$  satisfying the hypotheses of this theorem. Before we construct  $J_{\mathfrak{p}}$  however, we will show that the above reduction properties of  $A$  are inherited by any isogeny factor, and will discuss the relationship between isogenies  $A \sim A' \times A''$  (with  $A, A', A''$  abelian varieties over a field  $K$ ) and idempotents in the ring of endomorphisms  $\text{End}^0(A)$  in the isogeny category of abelian varieties over  $K$ .

## 2. SEMI-ABELIAN REDUCTION

In order to apply Theorem 1.1 to the isogeny factor  $J_{\mathfrak{p}}$  of  $J_0(N)$  that we will construct, we must show that  $J_{\mathfrak{p}}$  has good reduction outside  $N$  and purely toric reduction at  $N$ , and that  $J_{\mathfrak{p}}[p]$  is admissible. We first show that all isogeny factors of  $J_{\mathfrak{p}}$  inherit these reduction properties.

Let  $K$  be the fraction field of a discrete valuation ring  $\mathcal{O}$  with normalized valuation  $v$  and residue field  $k$ . Fix a separable closure  $K_s$  of  $K$  and a prime  $v_s$  of  $K_s$  lying over  $v$ , and denote by  $I$  the inertia group of  $v_s$ . Let  $A$  be an abelian variety over  $K$ , and denote by  $\mathcal{A}$  its Néron model over  $\mathcal{O}$ , and by  $\mathcal{A}^0$  the connected component of the identity of  $\mathcal{A}$ . Recall that we say that  $A$  has *semiabelian reduction* with respect to  $\mathcal{O}$  if the reduction  $\mathcal{A}_s^0$  is an extension of an abelian variety by a torus.

The aim of this section is to prove the following proposition:

**Proposition 2.1.** *Let the notation be as above. Suppose moreover that  $A$  is isogenous to the product  $A' \times A''$  of abelian varieties  $A'$  and  $A''$  over  $K$ . Then if  $A$  has good (resp. semi-abelian) reduction, so do  $A'$  and  $A''$ .*

*Proof.* We will only prove the case of semi-abelian reduction as the other cases are easier. Choose a prime  $\ell \neq \text{char } k$ . Let  $V_\ell(A) := T_\ell(A) \otimes \mathbf{Q}$  denote the rational  $\ell$ -adic Tate module of  $A$ . First observe that

$$V_\ell(A) = V_\ell(A' \times A'') = V_\ell(A') \times V_\ell(A'').$$

Grothendieck's criterion for semi-abelian reduction (cf. [1], theorem 6, page 184) asserts that  $A$  has semiabelian reduction if and only if there exists a subspace  $V$  of  $V_\ell(A)$  stable under the inertia group  $I$  such that  $I$  acts trivially on both  $V$  and the quotient  $V_\ell(A)/V$ . (cf. [1, Theorem 6, p. 184]) Note that  $V$  can (and will) be taken to be the maximal subspace of  $V_\ell(A)$  on which the action of  $I$  is trivial.

Let  $V', V''$  be the maximal subspaces of  $V_\ell(A')$  and  $V_\ell(A'')$  respectively, on which  $I$  operates trivially. Obviously  $V' \times V'' = V$ . Moreover, as  $(V_\ell(A')/V') \times (V_\ell(A'')/V'') = V_\ell(A)/V$ , we conclude that  $I$  acts trivially on  $V_\ell(A')/V'$  and on  $V_\ell(A'')/V''$ . This finishes the proof. ■

*Remark 2.2.* Let us prove that if  $A$  has toric reduction, then so do  $A'$  and  $A''$ . Assume that  $A$  has toric reduction. First note that since  $A$  and  $A' \times A''$  are isogenous, they are both semi-abelian as their rational Tate modules are isomorphic. Denote by  $\phi_K$  an isogeny  $A \rightarrow A' \times A''$ . Then there exists an isogeny  $\psi_K : A' \times A'' \rightarrow A$  and a nonzero integer  $n$ , such that  $\phi_K \psi_K = n_{A' \times A''}$  and  $\psi_K \phi_K = n_A$  (cf. [1], p. 169). The Néron mapping property ensures that  $\phi_K$  and  $\psi_K$  extend to morphisms  $\phi$  and  $\psi$  between  $\mathcal{A}$  and the Néron model  $\mathcal{A}' \times \mathcal{A}''$  of  $A' \times A''$ . We have  $\text{Ner}(A' \times A'') = \mathcal{A}' \times \mathcal{A}''$  because of the Néron mapping property: if  $Z$  is a smooth  $\mathcal{O}$ -scheme, then

$$\begin{aligned} \text{Hom}_K(Z_K, A' \times A'') &= \text{Hom}_K(Z_K, A') \times \text{Hom}_K(Z_K, A'') \\ &= \text{Hom}_{\mathcal{O}}(Z, \mathcal{A}') \times \text{Hom}_{\mathcal{O}}(Z, \mathcal{A}'') = \text{Hom}_{\mathcal{O}}(Z, \mathcal{A}' \times \mathcal{A}''). \end{aligned}$$

We will first show that  $\bar{\phi} : \bar{\mathcal{A}} \rightarrow \bar{\mathcal{A}}' \times \bar{\mathcal{A}}''$  is an isogeny. First note that  $\bar{\psi}\bar{\phi} = \bar{\psi}\phi = [n]$ . Since  $A$  is semi-abelian,  $\bar{\mathcal{A}}^0$  is an extension of an abelian variety by a torus. As multiplication by a nonzero integer is surjective on abelian

varieties and on tori, we conclude that  $[n]$  is surjective on  $\bar{\mathcal{A}}^0$ , hence an isogeny. Since  $[n] = \bar{\psi}\bar{\phi}$ , and  $\bar{\mathcal{A}}$  and  $\bar{\mathcal{A}}' \times \bar{\mathcal{A}}''$  have the same dimension, we conclude that  $\bar{\phi}$  is surjective on the identity components and has finite kernel, hence is an isogeny.

The isogeny between  $\bar{\mathcal{A}}$  and  $\bar{\mathcal{A}}' \times \bar{\mathcal{A}}''$  induces an isogeny between  $\bar{\mathcal{A}}^0$  and  $(\bar{\mathcal{A}}' \times \bar{\mathcal{A}}'')^0 = \bar{\mathcal{A}}'^0 \times \bar{\mathcal{A}}''^0$ , so it remains to check that for connected commutative smooth algebraic groups over a field  $k$ , the property of being a torus is preserved under isogeny and under formation of direct factors. This is clear since by the structure theorem for commutative algebraic groups over  $\bar{k}$  such a group  $G$  is a torus if and only if  $G$  is affine and  $\dim(V_l(G)) = \dim(G)$ , where  $l \neq \text{char } k$  is a prime.

### 3. IDEMPOTENTS INSIDE THE ENDOMORPHISM RING OF ABELIAN VARIETIES

In this section we remark upon some generalities concerning the relationship between idempotents in the ring  $\text{End}^0(A)$  for an abelian variety  $A$  and isogenies  $A \sim A' \times A''$  with  $A', A''$  abelian varieties.

**Proposition 3.1.** *Let  $A$  be an abelian variety over a field  $K$ . There is a one-to-one correspondence between ordered pairs  $(A', A'')$  of abelian subvarieties of  $A$  such that  $A' \times_K A'' \rightarrow A$  is an isogeny and idempotents in the ring  $\text{End}^0(A)$ ; the operation  $(A', A'') \mapsto (A'', A')$  corresponds to  $e \mapsto 1 - e$ .*

*Proof.* Suppose we are given a pair  $(A', A'')$  of abelian subvarieties of  $A$  such that the natural map  $\phi : A' \times A'' \rightarrow A$  is an isogeny. Let  $\phi^{-1}$  denote the inverse of  $\phi$  inside  $\text{Hom}^0(A, A' \times A'')$ . Consider the composition

$$e : A \xrightarrow{\phi^{-1}} A' \times A'' \xrightarrow{\text{pr}} A' \xrightarrow{\iota} A' \times A'' \xrightarrow{\phi} A,$$

where  $\iota$  is inclusion and  $\text{pr}$  is projection. Note that  $e$  is an idempotent in  $\text{End}^0(A)$ , and by definition of  $\phi$ , it lifts the identity on  $A'$ .

Conversely, given an idempotent  $e \in \text{End}^0(A)$ , choose a nonzero integer  $n$  such that  $ne \in \text{End}(A)$ , and set  $A'$  to be the image of  $ne$ . Note that  $A'$  is independent of  $n$ , since multiplication by a nonzero integer is surjective on an abelian variety. The Poincare Reducibility Theorem (c.f.[4]) guarantees the existence of a unique abelian subvariety  $A''$  of  $A$ , such that  $A' \times A'' \rightarrow A$  is an isogeny (the map being addition). One checks that the two processes are inverses of each other. ■

### 4. THE OPTIMAL QUOTIENT $J_{\mathfrak{P}}$ OF $J_0(N)$

In this section we will construct a nonzero optimal quotient  $J_{\mathfrak{P}}$  of  $J_0(N)$  such that the action of  $\mathbf{T}$  on  $J_0(N)$  induces an action on  $J_{\mathfrak{P}}$ .

Fix  $N$  and let  $\mathbf{T}$  denote the Hecke ring. Let  $\mathcal{I} \subseteq \mathbf{T}$  be the Eisenstein ideal let  $\mathfrak{P} \supseteq \mathcal{I}$  be a prime of residual characteristic  $p \neq N$  (we have seen that such a  $\mathfrak{P}$  exists). Observe that  $\mathfrak{P}_p := \mathfrak{P}(\mathbf{T} \otimes \mathbf{Z}_p)$  is a prime of  $T \otimes \mathbf{Z}_p$ .

Consider the subring  $\mathbf{T} \otimes_{\mathbf{Z}} \mathbf{Q} \subseteq \text{End}^0(J_0(N))$  (we have seen that in fact equality holds, but we do not use this). By Proposition 3.1, the decomposition of  $\mathbf{T} \otimes \mathbf{Q}$  into a product of fields gives us the isogeny decomposition of  $J_0(N)$  as a product of  $\mathbf{Q}$ -simple abelian subvarieties:

$$(1) \quad J_0(N) \sim \prod_{\mathfrak{q} \in \text{MinSpec}(\mathbf{T})} J_{\mathfrak{q}}.$$

We put

$$\tilde{J}_{\mathfrak{P}} = \prod_{\substack{\mathfrak{q} \in \text{MinSpec}(\mathbf{T}) \\ \mathfrak{q} \subseteq \mathfrak{P}}} J_{\mathfrak{q}}.$$

**Definition 4.1.** Let  $A, A'$ , and  $A''$  be abelian varieties and suppose that  $A'$  and  $A''$  are quotients of  $A$ . We say that  $A'$  and  $A''$  are *isogenous as quotients* of  $A$  if there exists an isomorphism  $\phi : A' \rightarrow A''$  in the isogeny category commuting (in that category) with the quotient maps  $A \rightarrow A'$  and  $A \rightarrow A''$ .

**Claim 4.2.** *Let  $A \rightarrow A''$  be a surjective map of abelian varieties over a field  $K$ . Then there is a unique quotient abelian variety  $A_{\text{opt}}$ , isogenous to  $A''$  as a quotient of  $A$ , that is the quotient of  $A$  by an abelian subvariety.*

*Proof.* We treat existence first. By the Poincaré Reducibility Theorem over  $K$ , we see that  $A$  is isogenous to a product of simple abelian subvarieties, say  $A \leftarrow \prod_{i=1}^n A_i$ , and since  $A''$  is a quotient of  $A$  we have (renumbering if necessary) an isogeny  $A'' \sim \prod_{i=1}^k A_i$ , as quotients of  $A$ , for some  $k \leq n$ . Define  $A' = \prod_{k < i \leq n} A_i$ . Then we have a morphism

$$A' \xrightarrow{\iota} \prod_{i=1}^n A_i \xrightarrow{\varphi} A,$$

where  $\varphi$  is an isogeny. Observe that  $\varphi \circ \iota(A')$  is an abelian subvariety of  $A$ . We set

$$A_{\text{opt}} = A/\varphi \circ \iota(A').$$

It is clear that  $A_{\text{opt}}$  is the quotient of  $A$  by an abelian subvariety and it follows (again from Poincaré Reducibility) that  $A_{\text{opt}}$  is isogenous to  $A''$  as a quotient of  $A$ .

We claim that  $A_{\text{opt}}$  has the following universal property: if  $q'' : A \rightarrow \tilde{A}$  is any quotient isogenous to the quotient  $q : A \rightarrow A_{\text{opt}}$  in the sense of Definition 4.1 then  $q''$  factors uniquely through  $q$  in the isogeny category. Indeed, if  $\tilde{A}$  and  $A_{\text{opt}}$  are isogenous as quotients, there exists an isomorphism (In the isogeny category)  $\phi : A_{\text{opt}} \rightarrow \tilde{A}$ . By composing  $\phi$  with multiplication by a large enough integer, we obtain *honest* maps  $n \circ q'' : A \rightarrow \tilde{A}$  and  $\psi = n \circ \phi : A_{\text{opt}} \rightarrow \tilde{A}$  such that  $\psi \circ q = n \circ q''$ . Hence  $n \circ q''(\ker q) = 0$ . But multiplication by  $n$  is an isogeny, and hence has finite kernel, while  $q''(\ker q)$  is connected (as  $\ker q$  is an abelian subvariety of  $A$ ), so that  $q''(\ker q) = 0$ .

It follows that the map  $A \rightarrow A''$  factors uniquely through  $A_{\text{opt}}$ . From this universal property of  $A_{\text{opt}}$ , we see at once that  $A_{\text{opt}}$  is unique up to unique isomorphism.  $\blacksquare$

Applying the above claim to  $A = J_0(N)$  and the isogeny factor  $\tilde{J}_{\mathfrak{F}}$ , we obtain an optimal quotient  $J_{\mathfrak{F}}$  of  $J_0(N)$ . We claim that  $\mathbf{T}$  acts on  $J_{\mathfrak{F}}$ . This will follow from the following more general theorem:

**Theorem 4.3.** *Let  $\pi : A \rightarrow A'$  be a surjective map of abelian varieties over a field  $K$  of characteristic 0 having an abelian variety kernel  $B$  and let  $T \in \text{End}(A)$ . Assume there exists  $T' \in \text{End}(A')^0$  such that the following diagram commutes in the isogeny category:*

$$(2) \quad \begin{array}{ccc} A & \xrightarrow{\quad} & A' \\ & \searrow \pi & \downarrow T' \\ T \downarrow & & A' \\ A & \xrightarrow{\quad \pi} & A' \end{array}$$

Then  $T' \in \text{End}(A')$ .

*Proof.* We have  $T_0 := nT' \in \text{End}(A')$  for some nonzero integer  $n$ , and by the universal property of the quotient map  $A' \rightarrow A'$  having kernel  $A'[n]$ , we have  $T' \in \text{End}(A')$  if and only if  $T_0$  kills  $A'[n]$ . Since  $T_0\pi = nT'\pi = n\pi T = \pi Tn$  in the isogeny category, the genuine maps  $T_0\pi$  and  $\pi Tn$  agree so that  $T_0\pi$  kills  $A'[n]$  as  $\pi Tn$  obviously does. To conclude that  $T_0$  kills  $A'[n]$ , it therefore suffices to show that  $\pi$  is faithfully flat on  $n$ -torsion, for which it is enough to show surjectivity on  $\bar{K}$ -points (as finite  $K$ -groups are étale so  $A[n] \xrightarrow{\pi} A'[n]$  is faithfully flat if and only if it is surjective on  $\bar{K}$ -points). But we have the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & B(\bar{K}) & \longrightarrow & A(\bar{K}) & \longrightarrow & A'(\bar{K}) \longrightarrow 0 \\ & & n \downarrow & & n \downarrow & & n \downarrow \\ 0 & \longrightarrow & B(\bar{K}) & \longrightarrow & A(\bar{K}) & \longrightarrow & A'(\bar{K}) \longrightarrow 0 \end{array}$$

and since  $A, A', B$  are abelian varieties and  $K$  is of characteristic 0, the vertical maps are all surjective, so by the Snake lemma, the map on  $n$ -torsion  $A[n] \xrightarrow{\pi} A'[n]$  is surjective as desired. This completes the proof.  $\blacksquare$

Applying Theorem 4.3 to the optimal quotient  $J_{\mathfrak{F}}$  of  $J_0(N)$  shows that we have an action of  $\mathbf{T}$  on  $J_{\mathfrak{F}}$ .

## 5. ADMISSIBILITY

In Brian's talk, it was explained why  $J_{\mathfrak{P}}[p]$  is an admissible group scheme over  $\mathbf{Z}$ . In this section we will recall the proof of this fact in more detail.

Since  $\mathbf{T}$  is a finite  $\mathbf{Z}$ -module, the ring  $\mathbf{T} \otimes_{\mathbf{Z}} \mathbf{Z}_p$  is a finite  $\mathbf{Z}_p$ -module and hence is semi-local. Moreover, we have seen in section 4 that  $\mathbf{T}$  acts on  $J_{\mathfrak{P}}$  in a manner that respects the quotient map  $J_0(N) \rightarrow J_{\mathfrak{P}}$  so we obtain an action of  $\mathbf{T} \otimes_{\mathbf{Z}} \mathbf{Z}_p$  on the Tate modules  $T_p(J_0(N))$  and  $T_p(J_{\mathfrak{P}})$  as  $G_{\mathbf{Q}} := \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ -modules. By 8.7 and 8.15 of [3], there is a canonical isomorphism

$$(3) \quad \mathbf{T} \otimes_{\mathbf{Z}} \mathbf{Z}_p \simeq \prod_{\mathfrak{m} \in \text{MaxSpec}(\mathbf{T} \otimes_{\mathbf{Z}} \mathbf{Z}_p)} (\mathbf{T} \otimes_{\mathbf{Z}} \mathbf{Z}_p)_{\mathfrak{m}}.$$

**Claim 5.1.** *The action of  $\mathbf{T} \otimes_{\mathbf{Z}} \mathbf{Z}_p$  on  $T_p(J_{\mathfrak{P}})$  factors through  $\mathbf{T}_{\mathfrak{P}} := (\mathbf{T} \otimes_{\mathbf{Z}} \mathbf{Z}_p)_{\mathfrak{P}}$ , so the induced action of  $\mathbf{T} \otimes_{\mathbf{Z}} \mathbf{Z}_p$  on  $J_{\mathfrak{P}}[p]$  is through the quotient  $\mathbf{T}_{\mathfrak{P}}$ .*

*Proof.* First note that the functoriality of the idempotent decomposition of  $\mathbf{T} \otimes \mathbf{Q}$  implies that

$$V_p(J_{\mathfrak{P}}) = \prod_{\substack{\mathfrak{q} \in \text{Spec}(\mathbf{T} \otimes \mathbf{Q}) \\ \mathfrak{q} \subset \mathfrak{P}}} V_p(J_{\mathfrak{q}}) = \prod_{\substack{\mathfrak{q} \in \text{Spec}(\mathbf{T} \otimes \mathbf{Q}) \\ \mathfrak{q} \subset \mathfrak{P}}} V_p(J)_{\mathfrak{q}},$$

where the objects  $V_p(J)_{\mathfrak{q}}$  denote the localizations of the  $\mathbf{T} \otimes \mathbf{Q}$ -module  $V_p(J)$  at primes  $\mathfrak{q}$ . (We will use the same letter  $\mathfrak{q}$  to denote both a prime ideal of  $\mathbf{T} \otimes \mathbf{Q}$  and its inverse image in  $\mathbf{T}$ ). From this we see that the action of  $\mathbf{T}$  on  $V_p(J_{\mathfrak{P}})$  factors through  $\prod_{\mathfrak{q} \subset \mathfrak{P}} (\mathbf{T} \otimes \mathbf{Q})_{\mathfrak{q}}$  and since every element of  $\mathbf{T} - \mathfrak{P}$  is mapped to a unit of  $\prod_{\mathfrak{q} \subset \mathfrak{P}} (\mathbf{T} \otimes \mathbf{Q})_{\mathfrak{q}}$  (since  $\mathfrak{q} \subset \mathfrak{P}$ ), the map  $\mathbf{T} \rightarrow \prod_{\mathfrak{q} \subset \mathfrak{P}} (\mathbf{T} \otimes \mathbf{Q})_{\mathfrak{q}}$  factors through the localization  $\mathbf{T}_{\mathfrak{P}}$ . The claim now follows after tensoring with  $\mathbf{Z}_p$  and noting that  $T_p(J_{\mathfrak{P}}) \subseteq V_p(J_{\mathfrak{P}})$ . ■

We now prove:

**Lemma 5.2.** *Let  $\mathcal{I} \subseteq \mathbf{T}$  be the Eisenstein ideal and  $\mathfrak{P} \supset \mathcal{I}$  be any prime of  $\mathbf{T}$  having residue characteristic  $p \neq N$ . Then  $\mathfrak{P}^r$  kills  $J_{\mathfrak{P}}[p]$  for some  $r > 0$ .*

*Proof.* The action of  $\mathbf{T} \otimes_{\mathbf{Z}} \mathbf{Z}_p$  on  $J_{\mathfrak{P}}[p]$  factors through  $\mathbf{T}_{\mathfrak{P}}/p\mathbf{T}_{\mathfrak{P}}$ . As  $\mathbf{T} \otimes_{\mathbf{Z}} \mathbf{Z}_p$  is a finite  $\mathbf{Z}_p$ -module, we see that  $\mathbf{T}_{\mathfrak{P}}/p\mathbf{T}_{\mathfrak{P}}$  is a finite  $\mathbf{F}_p$ -module, and consequently is a finite local ring. It follows that its maximal ideal is nilpotent. Thus, for some  $r > 0$  we have  $\mathfrak{P}^r \subseteq p\mathbf{T}_{\mathfrak{P}}$ , so  $\mathfrak{P}^r J_{\mathfrak{P}}[p] = 0$  as claimed. ■

It follows from Lemma 5.2 that we have a filtration of  $\mathbf{T}_{\mathfrak{P}}[G_{\mathbf{Q}}]$  modules

$$(4) \quad J_{\mathfrak{P}}[p] \supseteq \mathfrak{P} J_{\mathfrak{P}}[p] \supseteq \mathfrak{P}^2 J_{\mathfrak{P}}[p] \supseteq \dots \supseteq \mathfrak{P}^r J_{\mathfrak{P}}[p] = 0.$$

Observe that each quotient  $\mathfrak{P}^i J_{\mathfrak{P}}[p] / \mathfrak{P}^{i+1} J_{\mathfrak{P}}[p]$  is killed by  $\mathfrak{P}$ . We will now show that this implies that the quotients are all admissible. Recall from [2] (Corollary 1.6) that admissibility of a group scheme over  $\mathbf{Z}$  can be checked on the corresponding Galois module of  $\overline{\mathbf{Q}}$  points. It then follows that  $J_{\mathfrak{P}}[p]$  is admissible, by general results on Jordan–Hölder series.

**Lemma 5.3.** *Let  $G$  be a finite discrete  $G_{\mathbf{Q}}$ -module of  $p$ -power order on which  $\mathbf{T}$  acts and let  $\mathfrak{P} \supseteq \mathcal{I}$  be a prime of  $\mathbf{T}$  having residual characteristic  $p \neq N$  and containing the Eisenstein ideal. Assume that for any prime  $\ell \nmid Np$  that the inertia group  $I_{\ell}$  at  $\ell$  acts trivially on  $G$ , and hence that we have an action of  $\text{Frob}_{\ell} \in \text{Gal}(\overline{\mathbf{F}}_{\ell}/\mathbf{F}_{\ell})$  on  $G$ . Suppose that the Eichler–Shimura relation*

$$\text{Frob}_{\ell}^2 - T_{\ell} \text{Frob}_{\ell} + \ell = 0$$

*holds on  $G$  for all such  $\ell$ . If  $\mathfrak{P}$  kills  $G$  then  $G$  has a filtration by admissible closed subgroups with successive quotients  $\mathbf{Z}/p\mathbf{Z}$  or  $\mu_p$ .*

*Proof.* Let  $\Gamma$  be the discrete finite quotient of  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$  through which the Galois action on  $G$  factors. By hypothesis, for  $\ell \nmid Np$  we have that  $\text{Frob}_{\ell}$  acts on  $G$  and

$$(5) \quad \text{Frob}_{\ell}^2 - T_{\ell} \text{Frob}_{\ell} + \ell = 0$$

on  $G$ . But as  $\mathfrak{P} \supseteq \mathcal{I}$  kills  $G$  and  $T_{\ell} \equiv \ell + 1 \pmod{\mathcal{I}}$ , we see that  $\text{Frob}_{\ell}$  acts on  $G$  with eigenvalues contained in  $\{1, \ell\}$ . Now any  $\sigma \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$  also acts on  $G^{\vee} = \text{Hom}(G, \mu_p)$  via  $f^{\sigma}(g) = \sigma f(g^{\sigma^{-1}})$ . Observe that if  $g$  is an eigenvector of  $\sigma = \text{Frob}_{\ell}$  with eigenvalue  $\ell$  then  $f^{\sigma}(g) = \sigma f(\ell^{-1}g) = \ell f(\ell^{-1}g) = f(g)$ , so that  $f(g)$  is an eigenvector

with eigenvalue 1. Similarly, if  $g$  has eigenvalue 1 then  $f(g)$  has eigenvalue  $\ell$ . It follows that the eigenvalues  $\{1, \ell\}$  of  $\text{Frob}_\ell$  on  $G \times G^\vee$  occur with the same multiplicity. Thus, the characteristic polynomial of  $\text{Frob}_\ell$  on  $G \times G^\vee$  over  $\mathbf{F}_p$  is  $(X - 1)^d(X - \ell)^d$  for some  $d$  (which is the dimension of  $G$  as over  $\mathbf{F}_p$ ).

Consider now the  $\mathbf{F}_p[G_{\mathbf{Q}}]$ -module  $(\mathbf{Z}/p\mathbf{Z})^d \times \mu_p^d$ . By extending  $\Gamma$  if necessary (and still preserving the finiteness of  $\Gamma$ ) we can regard  $(\mathbf{Z}/p\mathbf{Z})^d \times \mu_p^d$  as a finite discrete  $\Gamma$ -module. Since  $\text{Frob}_\ell$  acts on  $\mu_p$  with eigenvalue  $\ell$ , we see that the characteristic polynomial of  $\text{Frob}_\ell$  on  $(\mathbf{Z}/p\mathbf{Z})^d \times \mu_p^d$  is also  $(X - \ell)^d(X - 1)^d$ . Now by the Tchebotarev density theorem, every  $\gamma \in \Gamma$  is the image of  $\text{Frob}_\ell$  for some  $\ell \neq N, p$ . Thus, every  $\gamma \in \Gamma$  has the same characteristic polynomial on the two  $\mathbf{F}_p[\Gamma]$ -modules  $(\mathbf{Z}/p\mathbf{Z})^d \times \mu_p^d$  and  $G$ . Applying the Brauer–Nesbitt theorem, we conclude that these  $\Gamma$ -modules have the same semisimplifications. Since  $(\mathbf{Z}/p\mathbf{Z})^d \times \mu_p^d$  has a filtration as a Galois module with successive quotients isomorphic to  $\mathbf{Z}/p\mathbf{Z}$  or  $\mu_p$ , so does  $G$ . ■

**Corollary 5.4.** *Let  $N, p, \mathfrak{F}$  be as before. Then  $J_{\mathfrak{F}}[p]$  is admissible.*

*Proof.* By [2] (Corollary 1.6), we need only check that the Galois module  $J_{\mathfrak{F}}[p]$  has a  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ -stable filtration by  $\mu_p$ 's and  $\mathbf{Z}/p\mathbf{Z}$ 's. Using Proposition 2.1 and Remark 2.2, we see that  $J_{\mathfrak{F}}$  has toric reduction at  $N$  and good reduction outside  $N$ , so the inertia group at any  $\ell \nmid Np$  acts trivially on the Galois module  $J_{\mathfrak{F}}[p](\overline{\mathbf{Q}})$ . Moreover, the Eichler–Shimura relations hold on  $J_{\mathfrak{F}}[p]$ , as they do in  $J_0(N)$  and hence on the isogeny factor  $J_{\mathfrak{F}}$ . We may therefore apply Lemma 5.3 to the situation discussed after the proof of Lemma 5.2 to conclude that  $J_{\mathfrak{F}}[p]$  is admissible. ■

**Corollary 5.5.** *The nonzero isogeny factor  $J_{\mathfrak{F}}$  of  $J_0(N)$  has rank 0.*

*Proof.* By Proposition 2.1 and Remark 2.2, we know that  $J_{\mathfrak{F}}$  has good reduction away from  $N$  and purely toric reduction at  $N$ . As  $J_{\mathfrak{F}}[p]$  is admissible by Corollary 5.4, we see that  $J_{\mathfrak{F}}$  satisfies the hypotheses of Theorem 1.1 and hence has rank 0. ■

#### REFERENCES

- [1] S. Bosch, W. Lutkeböhmer, M. Raynaud, *Néron models*, Springer-Verlag Berlin Heidelberg 1990.
- [2] Mazur, B. Modular curves and the Eisenstein ideal. Publ. math. I.H.E.S., **47**, (2), 1977. pp. 33–186.
- [3] Matsumura, H. *Commutative Ring Theory*, Cambridge University Press, 1986.
- [4] Milne, J. Abelian Varieties, in *Arithmetic Geometry*.
- [5] Mumford, D. *Abelian Varieties*, Oxford University Press, 1970.