Families of tori

Although SGA3 develops a rather general relative theory of tori, for Deligne's purposes in §5 it is not at all necessary to delve into such technology. What he truly needs to construct are some families of tori over specific $\mathbf{Q}$-schemes such that, on the level of geometric points with values in algebraically closed fields K, we get certain explicit sets of tori (and auxiliary data) in a manner which is functorial in K. These can be constructed without requiring the general relative theory of tori, as we shall explain.

Following standard notation, we write $\mathfrak{z}(v)$ to denote $\ker \mathrm{ad}(v)$, the centralizer of an element v in a Lie algebra $\mathfrak{g}$. If $\mathfrak{g} = \mathrm{Lie}(G)$ for an algebraic group G, then in characteristic 0 or for semisimple v we have $\mathfrak{z}(v) = \mathrm{Lie}(Z_G(v))$ where $Z_G(v) = \{g \in G \mid \mathrm{Ad}(g)(v) = v\}$ is a k-smooth group (see 9.1 in Borel's book). We use this fact several times.

We make one further definition, simply for the purpose of explaining how some of the constructions below admit stronger properties than we shall prove. A *torus* over a scheme S is a smooth affine commutative group scheme $T \to S$ whose geometric fibers are tori in the sense of algebraic groups. If $G \to S$ is a smooth group scheme over a base scheme S and $T \hookrightarrow G$ is a closed subgroup scheme which is a torus over S, we say T is a *maximal torus* in $G_{/S}$ if it is maximal on geometric fibers (in the sense of algebraic groups).

1. A universal torus

Let G be a reductive algebraic group over a field k (of any characteristic). Pick a maximal torus T_0 in G defined over k. Not all such tori are $G(k)$-conjugate (unless k is algebraically closed), but this won't matter. We let $N_0 = N_G(T_0)$ be the normalizer, and $W_0 = N_0/T_0$ be the finite étale Weyl group. Since W_0 splits over a finite Galois extension k', various quotients below by the W_0-action may be readily proved to exist by first working over k' where W_0 is a finite constant group, and then using Galois descent to get down to k (the relevant quotients we need to work with will be made from finite étale groups acting on disjoint unions of quasi-projective schemes over a field, so no fancy technology is needed to establish the existence of such quotients and their basic properties).

Consider the universal conjugation map $G \times_k G \to G \times_k G$ defined by $(h, g) \mapsto (ghg^{-1}, g)$. This is clearly an isomorphism. Thus, if we compose with the closed immersion $T_0 \times G \hookrightarrow G \times G$ obtained by base change on $T_0 \hookrightarrow G$, we get a closed immersion. This composite closed immersion

$$T_0 \times G \hookrightarrow G \times G \tag{1.1}$$

is described by $(t, g) \mapsto (gtg^{-1}, g)$. The map (1.1) is equivariant for right multiplication by T_0 on the second factor G on both source and target, so we may pass to the quotient to get a map

$$T_0 \times (G/T_0) \simeq (T_0 \times G)/T_0 \to G \times (G/T_0). \tag{1.2}$$

This map is a closed immersion too, since it recovers the closed immersion (1.1) upon applying the faithfully flat base change $G \to G/T_0$ along the second factor.

The map (1.2) is equivariant for a free right action of the finite étale Weyl group $W_0 = N_G(T_0)/T_0$ via $(t, \overline{g}) \mapsto (n^{-1}tn, gn)$ on the source and $(g_1, \overline{g}_2) \mapsto (g_1, \overline{g}_2 n)$ on the target. Thus, passage to the quotient by this action yields a map

$$\mathscr{T} \to G \times \mathrm{Tor}_G \tag{1.3}$$

where $\mathrm{Tor}_G \overset{\mathrm{def}}{=} G/N_0$. This map fits into the bottom row of the following commutative diagram:

$$\begin{array}{ccc} T_0 \times (G/T_0) & \longrightarrow & G \times (G/T_0) \\ \Big\downarrow & & \Big\downarrow \\ \mathscr{T} & \longrightarrow & G \times \mathrm{Tor}_G \end{array} \tag{1.4}$$

The top map is (1.2), and the columns are finite étale W_0-torsor maps. The rows respect this torsor structure, so the diagram has to be cartesian. In particular, since the top arrow is a closed immersion we deduce the same for the bottom row by descent. Thus, (1.3) is a *closed immersion* of Tor_G-group schemes.

Since we are aiming to avoid using the relative theory of tori, we should record the property we really need: the fibers of (1.3) over geometric points of Tor_G have a simple description. For any algebraically closed field K/k, what is the fiber $\mathscr{T}_x$ over an element $x \in \mathrm{Tor}_G(K)$, with $\mathscr{T}_x$ viewed *as a closed subscheme of* G_K? Since formation of the quotient $\mathrm{Tor}_G = G/N_0$ commutes with arbitrary extension of the ground field and non-empty algebraic schemes over an algebraically closed field have a rational point (by the Nullstellensatz), we have $(G/N_0)(K) = G(K)/N_0(K)$. Thus, x corresponds to a coset $gN_{0/K}$ with $g \in G(K)$. The fiber $\mathscr{T}_x$ is the torus $gT_{0/K}g^{-1}$ inside of G_K. Although all of these tori are abstractly K-isomorphic as we vary g, they are *not* the same torus inside of G_K. In fact, as g varies over $G(K)/N_0(K)$ we get precisely all maximal tori of G_K without repetition: this is because of the general theorem that in a reductive algebraic group over an algebraically closed field, all maximal tori are conjugate (here we are implicitly using the fact that a maximal torus in an algebraic group over an algebraically closed field, such as T_0 inside of G, remains maximal upon base change to any larger algebraically closed field, a property which I leave as a delightful exercise; use the "smearing out" method in the proof of Lemma 4.1).

The conclusion we draw is that the diagram

$$\begin{array}{ccc} \mathscr{T} & \longrightarrow & G \times \mathrm{Tor}_G \\ & \searrow & \downarrow \\ & & \mathrm{Tor}_G \end{array}$$

is a "universal torus" in the sense that for any algebraically closed field K/k, the fibers $\mathscr{T}_x \hookrightarrow G_K$ over all $x \in \mathrm{Tor}_G(K)$ run over all maximal tori in G_K without repetition. We will have no need for any stronger property; however, the functoriality in K (which is obvious) is clear from our construction and is quite crucial for Deligne's arguments.

Remark 1.1. In fact, Grothendieck proved that the above construction represents the functor that assigns to any k-scheme S the set of maximal tori in the S-group $G \times_k S$; that is, it is a universal maximal torus in G. Grothendieck really proved a much more general result without requiring that the situation started life over a field; see Cor. 1.10 in Exp. XII of SGA3.

2. Regular vectors and another family

Let $V = \mathrm{Lie}(G)$ be the Lie algebra of G, viewed as an affine space over k. Let n be its dimension. Consider the adjoint representation $\mathrm{ad} : V \to \mathfrak{gl}(V) = \mathrm{End}(V)$ defined by $v \mapsto [v, \cdot]$. This is a linear map of affine spaces, with matrix entries determined by some universal polynomial expressions in the structure constants of the Lie algebra (relative to a choice of basis of V). Thus, if we form the characteristic polynomial of $\mathrm{ad}(v)$ we get a degree n polynomial $p_v = T^n + a_{n-1}(v)T^{n-1} + \cdots + a_0(v)$ whose coefficients $a_j(v)$ are (homogeneous) polynomial functions on V (depending on the structure constants of the Lie algebra, and some determinantal mess).

It might happen that $a_0 = 0$, but in any case there is some least $r \le n$ such that a_r is nonzero (let $a_n = 1$); note that this vanishing may be verified by just checking on k'-points for one infinite extension of k (in case k was possibly finite), since a polynomial in several variables over an infinite field is zero if and only if it vanishes at all rational points of the corresponding affine space. In terms of the Jordan decomposition $v = v_{\mathrm{ss}} + v_{\mathrm{n}}$, $\mathrm{ad}(v_{\mathrm{ss}})$ is the semisimple part of $\mathrm{ad}(v)$, so these have the same characteristic polynomial. Thus, $\mathfrak{z}(v_{\mathrm{ss}})$ has dimension equal to the multiplicity of 0 as a root of the characteristic polynomial of $\mathrm{ad}(v)$. Thus, r is the minimal dimension of $\mathfrak{z}(v)$ for variable semisimple v in $\mathrm{Lie}(G)$ (again, when k is finite this must be interpreted in the sense of v varying over $\mathrm{Lie}(G)$ as an algebraic variety and not as a k-vector space).

The locus $\mathrm{Lie}(G)^{\mathrm{reg}}$ of *regular vectors* is defined to be the nonempty Zariski open locus in $V = \mathrm{Lie}(G)$ where $a_r \neq 0$. This definition makes sense for any algebraic group over k, but for later purposes it is important to describe both r and the locus of regular vectors for reductive groups (such as our G). Since this issue is not explicitly addressed in basic books such as by Borel and Springer, we give proofs for convenience of the reader.

Theorem 2.1. *Let d be the dimension of maximal tori in the reductive group G. Then r above is equal to d. Moreover, a vector $v \in \mathrm{Lie}(G)$ is regular and semisimple if and only if $Z_G(v)^0_{\mathrm{red}}$ is a maximal torus. In fact, if k is infinite then such v can be found in $\mathrm{Lie}(T)$ for any maximal torus T in G defined over k.*

Using Proposition 9.1(2) in Borel's book, for semisimple v it is automatic that $Z_G(v)_{\mathrm{red}}$ is k-smooth (in Borel's terminology, he says $Z_G(v)$ is "defined over k", but this means the k-smoothness of the underlying reduced closed subscheme). In characteristic zero, the same arguments (or an appeal to Cartier's theorem on smoothness of locally finite type group schemes in characteristic 0) gives the smoothness without any restrictions on v.

Proof. Due to the fact that semisimple v have $Z_G(v)_{\mathrm{red}}$ automatically smooth, hence of formation compatible with change of the ground field, for the proof of the lemma we may take k to be algebraically closed. By 11.8 in Borel, $v \in \mathfrak{g} = \mathrm{Lie}(G)$ is semisimple if and only if lies in $\mathfrak{t} = \mathrm{Lie}(T)$ for some torus T in G, and we can then certainly take T to be maximal. Since T is commutative of dimension d, we see that $\mathfrak{z}(v)$ contains the d-dimensional $\mathrm{Lie}(T)$. For general (perhaps non-semisimple) $v \in \mathfrak{g}$, we have seen above that the characteristic polynomial of $\mathrm{ad}(v)$ has 0 as a root with the same multiplicity as for $\mathrm{ad}(v_{\mathrm{ss}})$. Thus, $r \geq d$.

It remains to find some semisimple v for which $\dim \mathfrak{z}(v) = d$. Pick a maximal torus T and consider the root space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(T,G)} \mathfrak{g}_\alpha, \tag{2.1}$$

with 1-dimensional root spaces $\mathfrak{g}_\alpha$ for the pairwise distinct non-trivial roots α describing the adjoint representation of T on $\mathfrak{g}$. A technical point for positive characteristic (trivially verified in characteristic zero) is that the differential $\mathrm{d}\alpha : \mathfrak{t} \to \mathrm{End}(\mathfrak{g}_\alpha)$ of the adjoint action of T on $\mathfrak{g}_\alpha$ is nonzero for each root α; this follows from 13.19–13.21 in Borel's book, for example. Thus, we can pick $v \in \mathfrak{t}$ for which $\mathrm{d}\alpha(v) \neq 0$ for all α. It follows that $\mathrm{ad}(v)$ acts by a non-zero scalar multiplier on each root line $\mathfrak{g}_\alpha$, so $\mathfrak{z}(v) = \mathfrak{t}$. This has dimension d, so $r = d$.

Now we check that $v \in \mathrm{Lie}(G)$ is regular and semisimple if and only if $Z_G(v)^0_{\mathrm{red}}$ is a maximal torus. If v is regular and semisimple, then not only is $Z_G(v)^0_{\mathrm{red}}$ is a connected k-smooth closed subgroup of G (as is true for any v), but semisimplicity allows us to find a maximal torus T with $v \in \mathrm{Lie}(T)$. Since T is commutative, we therefore have an inclusion $T \hookrightarrow Z_G(v)$ as closed subschemes of G, so $T \hookrightarrow Z_G(v)^0_{\mathrm{red}}$. To get equality for this latter inclusion, we just need to know that the k-smooth $Z_G(v)_{\mathrm{red}}$ has dimension d. This dimension can be computed via the Lie algebra, thanks to smoothness, and the regularity of v ensures that the Lie-theoretic centralizer $\mathfrak{z}(v)$ of v in $\mathrm{Lie}(G)$ has dimension d. Thus, it suffices to know that $\mathfrak{z}(v)$ is the Lie algebra of the group centralizer $Z_G(v)_{\mathrm{red}}$; this Lie algebra computation follows from the semisimplicity of v (using 9.1(2) in Borel's book).

Finally, we prove that if $Z_G(v)^0_{\mathrm{red}} = T$ is a maximal torus, then v is regular and semisimple. We again appeal to the root space decomposition (2.1) for $\mathfrak{g}$ under the adjoint action of T, from which we see that $\mathrm{Lie}(T)$ is the full invariant subspace for the T-action on $\mathfrak{g}$. But v is T-invariant since $T = Z_G(v)^0_{\mathrm{red}}$, so therefore $v \in \mathrm{Lie}(T)$. In particular, v is semisimple in $\mathrm{Lie}(G)$ (this is the easy direction of 11.8 in Borel). But with v now known to be semisimple, the Lie-theoretic centralizer $\mathfrak{z}(v)$ of v is the Lie algebra of the smoothed group centralizer $Z_G(v)^0_{\mathrm{red}}$. This latter group is T, of dimension d, so its Lie algebra $\mathfrak{z}(v)$ is also d-dimensional. This says that the semisimple v is regular.

To prove that affine linear subspace $\mathrm{Lie}(T)$ inside of the affine space $\mathrm{Lie}(G)$ meets the Zariski open $\mathrm{Lie}(G)^{\mathrm{reg}}$ for any maximal k-torus T in G when k is infinite, it suffices to consider the case of algebraically closed k (since a non-empty Zariski open in an affine space over an infinite field must contain a rational point). But all maximal tori are $G(k)$-conjugate for algebraically closed k and the regularity property of $v \in \mathrm{Lie}(G)$ is invariant under the adjoint action of $G(k)$, so the known existence of a regular vector lying in the Lie algebra of some maximal torus implies the existence for any torus.

■

Recalling Cartier's theorem that locally finite type group schemes in characteristic zero are automatically smooth, when k has characteristic zero we get a smooth closed subgroup $Z_G(v)$ in G for any vector $v \in \mathrm{Lie}(G)$. The preceding theorem is important for our purposes due to its role in the proof of:

Corollary 2.2. *If k has characteristic zero, then any regular $v \in \mathrm{Lie}(G)$ is semisimple and the centralizer $Z_G(v)$ is connected. Thus, an arbitrary $v \in \mathrm{Lie}(G)$ is regular if and only if $Z_G(v)$ is a maximal torus.*

Proof. The second part of the corollary is immediate from the first part, thanks to the preceding theorem. Let $v = v_{\mathrm{ss}} + v_{\mathrm{n}}$ be the Jordan decomposition of an arbitrary vector v. Since $\mathrm{ad}(v_{\mathrm{ss}})$ is the semisimple part of $\mathrm{ad}(v)$, v_{ss} is regular if and only if v is regular. Thus, when v is regular then $Z_G(v_{\mathrm{ss}})^0$ is a maximal torus T (by the preceding theorem). Uniqueness of Jordan decomposition ensures that $Z_G(v)$ is contained in $Z_G(v_{\mathrm{ss}})$, so $Z_G(v)^0$ is contained in T. Thus, $\mathrm{Lie}(Z_G(v)^0)$ is contained in $\mathrm{Lie}(T)$. But because we are in characteristic zero, the discussion in 9.1 in Borel ensures that the Lie algebra of $Z_G(v)$ (or equivalently, of $Z_G(v)^0$) is exactly the Lie-theoretic centralizer $\mathfrak{z}(v)$, which contains v. Thus, $v \in \mathrm{Lie}(T)$. It follows that v must be semisimple.

The remaining task is to prove that for regular (and hence semisimple) v, the centralizer $Z_G(v)$ is connected. We may assume k is algebraically closed. Let $T = Z_G(v)^0$, a maximal torus. Clearly $Z_G(v)$ normalizes T, so $Z_G(v)$ is intermediate between $T = Z_G(T)$ and the normalizer $N_G(T)$, so it corresponds to a subgroup of the Weyl group $W = W(T,G) = N_G(T)/T$, with this subgroup trivial precisely when $Z_G(v)$ is connected (i.e., is equal to T). Thus, the connectivity of $Z_G(v)$ says exactly that the action of W on $\mathfrak{t} = \mathrm{Lie}(T)$ has trivial stabilizer for the vector v. Thus, we must show that each non-trivial $w \in W$ moves v. Using the Lie algebra decomposition

$$\mathrm{Lie}(Z_G) \times \mathrm{Lie}(G') \simeq \mathrm{Lie}(G),$$

it suffices to treat the group G'. The group G' is semisimple, and in characteristic zero the Lie algebra of a semisimple algebraic group is semisimple. Thus, we are faced with a problem concerning semisimple Lie algebras $\mathfrak{g}$ over an algebraically closed field of characteristic zero: for any Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{g}$ (i.e., a maximal commutative subalgebra whose elements are semisimple in $\mathfrak{g}$) with associated roots $\Phi \subseteq \mathfrak{t}^\vee$ and Weyl group W acting on $\mathfrak{t}$, we shall prove that a vector $v \in \mathfrak{t}$ has $\alpha(v) \neq 0$ for all $\alpha \in \Phi$ if and only if $w(v) \neq v$ for all non-trivial $w \in W$. I am grateful to James Parson for suggesting the following elegant argument.

It suffices to prove that the intersection of the non-vanishing loci of the operators $w - 1$ for non-trivial $w \in W$ coincides with the intersection of the non-vanishing loci of the roots $\alpha \in \Phi$. This is a purely linear algebra assertion, so its truth is unaffected by extension of (or descent through) the ground field without affecting the semisimplicity property. We may therefore descend to an algebraic closure of a finitely generated subfield of k, and then can embed this into $\mathbf{C}$ to reduce to the case $k = \mathbf{C}$. Every semisimple Lie algebra over $\mathbf{C}$ arises from a unique "split form" $\mathfrak{g}_0$ over $\mathbf{R}$, and we can pick a split Cartan subalgebra $\mathfrak{t}_0$ inside of $\mathfrak{g}_0$. Let Φ_0 be the resulting set of roots and W_0 the Weyl group.

The root system arising from the choice of $\mathfrak{t}_0$ gives rise to a natural $\mathbf{R}$-vector space $X_{\mathbf{R}}$ equipped with a positive definite inner product on which W_0 acts faithfully through actions generated by reflections through various "root hyperplanes" H_α indexed by the roots $\alpha \in \Phi_0$. This $\mathbf{R}$-vector space is W_0-equivariantly identified with $\mathfrak{t}_0$ and the kernel hyperplane of the root functional α on $\mathfrak{t}_0$ is exactly the hyperplane H_α. Thus, the problem comes down to one about root systems: the elements of $X_{\mathbf{R}}$ fixed by a non-trivial element of W_0 are exactly the points lying in the union of the root hyperplanes. Since reflections through the hyperplanes are elements of $W_0 \subseteq \mathrm{Aut}(X_{\mathbf{R}})$, the part which requires an argument is to prove that a point of $X_{\mathbf{R}}$ not in any root hyperplane cannot be fixed by a non-trivial element of W_0. But from the general theory of root systems, the Weyl group action is simply transitive on the Weyl chambers (the connected components of the complement of the union of the root hyperplanes). Thus, a point of this complement cannot be fixed by a non-trivial element of W_0. ■

Definition 2.3. Let $H \to S$ be a smooth group scheme with identity section e, so $e^*\Omega^1_{H/S}$ is a locally free quasi-coherent sheaf with finite rank. We define the *relative Lie algebra* $\mathbf{Lie}(H)$ to be the vector bundle corresponding to the dual of $e^*(\Omega^1_{H/S})$.

The formation of the relative Lie algebra commutes with base change on S. One can show via a nice functorial argument (given in the Néron models book, for example) that on any group scheme over any base scheme at all, any relative cotangent vector along the identity section may be uniquely extended to a left-invariant relative 1-form. For smooth schemes the relative Ω^1 is a locally free sheaf of finite rank, so for smooth group schemes we may dualize to conclude that any section of the relative Lie algebra may be uniquely extended to a left-invariant vector field (which in turn may be identified with certain derivations on the structure sheaf of the group). Thus, by using the usual bracket construction on vector fields, we can naturally endow $\mathbf{Lie}(H)$ with a structure of sheaf (or bundle) of Lie algebras compatibly with base change on S and functorially in H. When $S = \mathrm{Spec}(k)$, this is just the usual construction for algebraic groups.

As an example, we can take $H = G \times \mathrm{Tor}_G$ as a smooth group over Tor_G, and we can form $\mathbf{Lie}(H)$. This is rather boring, since it is just $\mathrm{Lie}(G) \times \mathrm{Tor}_G$ with its Lie bracket coming from $\mathrm{Lie}(G)$. That is, this is just a "constant family" of Lie algebras parameterized by Tor_G. The example $H = \mathscr{T} \to \mathrm{Tor}_G$ is more interesting, since it is a quotient of a constant family $T_0 \times_k (G/T_0)$ by the action of the finite étale Weyl group $W_0 = W(T_0, G)$ (so $\mathbf{Lie}(\mathscr{T}) \to \mathrm{Tor}_G$ is a "twisted" family of copies of $\mathrm{Lie}(T_0)$).

Functoriality with respect to the closed immersion $\mathscr{T} \hookrightarrow G \times \mathrm{Tor}_G$ defines a closed immersion of Lie algebras over Tor_G:

$$\mathbf{Lie}(\mathscr{T}) \hookrightarrow \mathbf{Lie}(G \times \mathrm{Tor}_G) \tag{2.2}$$

For an algebraically closed field K/k and $g \in G(K)$, the fiber of this embedding over $\overline{g} \in G(K)/N_0(K)$ is the Lie algebra of $gT_{0/K}g^{-1}$ inside of the Lie algebra of G_K for $g \in G(K)$ representing $\overline{g}$, or in other words it "is" the image of $\mathrm{Lie}(T_0)$ embedded into $\mathrm{Lie}(G)$ via the adjoint representation action $\mathrm{Ad}(g)$; the actual embedding map on $\mathrm{Lie}(T_0)$ depends on the choice of representative g modulo $T_0(K) = Z_G(T_0)(K)$, and there is no canonical isomorphism $\mathrm{Lie}(\mathscr{T}_{\overline{g}}) \simeq \mathrm{Lie}(T_0)_K$.

Inside of $\mathbf{Lie}(G \times \mathrm{Tor}_G) \simeq \mathrm{Lie}(G) \times \mathrm{Tor}_G$ there is the fiberwise dense open subset $\mathrm{Lie}(G)^{\mathrm{reg}} \times \mathrm{Tor}_G$. Define U to be the (fiberwise non-empty!) open locus in $\mathbf{Lie}(\mathscr{T})$ over Tor_G which, via (2.2), lies in $\mathrm{Lie}(G)^{\mathrm{reg}} \times \mathrm{Tor}_G$. For an algebraically closed field K, an element $u \in U(K)$ consists of a pair (T, v) where T is a maximal torus in G_K (this comes from the image of u in $\mathrm{Tor}_G(K)$) and $v \in \mathrm{Lie}(T)$ is a Lie vector along T which is regular as an element in $\mathrm{Lie}(G_K)$. Thus, one should think of U as another universal family: it classifies pairs (T, v) over algebraically closed extensions K of k, *functorially in* K. A key point is that we have made these arise from a geometric object U which we constructed over k. In fact, if one grants the stronger universal property for $\mathscr{T} \hookrightarrow G \times_k \mathrm{Tor}_G$ as recorded at the end of §1, it is clear that U represents the functor that, to any k-scheme S, assigns the set of pairs (T, v) where T is a maximal torus in $G \times_k S$ and v is a section of $\mathbf{Lie}(T) \to S$ which is fiberwise regular when viewed as a section of $\mathbf{Lie}(G \times_k S) \simeq \mathrm{Lie}(G) \otimes_k \mathscr{O}_S$.

3. Hom functors of tori

We want to make a scheme that classifies morphisms between two tori. It will actually very much clarify the situation to prove a genuine representability theorem in this case (not just to make a construction and make a description of geometric points functorially in an algebraically closed field); this will be simplify the task of keeping track of residue fields later on. To keep things simple, we will only consider the case of tori which begin life over a field. This is adequate for our purposes, though one can prove much stronger results by essentially the same methods, provided one knows more about the relative theory of tori from SGA3. Let T and T' be two k-tori. Consider the functor $\underline{\mathrm{Hom}}(T, T')$ on k-schemes whose value on S is the group of S-group scheme maps $T_S \to T'_S$. I claim this is represented by a (very large) étale k-scheme. To keep the picture clear, let us first consider the case in which T and T' are both split. For this we could even work over $\mathbf{Z}$, but we avoid this for simplicity. Suppose $T = \mathbf{G}_m^g$ and $T' = \mathbf{G}_m^{g'}$ over k. Let

$$H = \coprod_{m \in \mathrm{Mat}_{g' \times g}(\mathbf{Z})} \mathrm{Spec}\, k,$$

and consider the map $f : T_H \to T'_H$ whose component map $f_m : T \to T'$ over the component $\mathrm{Spec}\, k$ indexed by the matrix m is precisely the map in

$$\mathrm{Hom}_k(\mathbf{G}_m^g, \mathbf{G}_m^{g'}) \simeq \mathrm{Mat}_{g' \times g}(\mathrm{Hom}_k(\mathbf{G}_m, \mathbf{G}_m)) = \mathrm{Mat}_{g' \times g}(\mathbf{Z})$$

corresponding to the matrix m.

Since we will be interested in tori that might not be split, for the present circumstances it seems most efficient to actually prove the representability theorem lurking in the background. This stronger result will enable us to descend to the non-split case without unpleasant tedium.

Theorem 3.1. *The map $f : T_H \to T'_H$ is universal. That is, for any k-scheme S and any map $\phi : T_S \to T'_S$ of S-group schemes, there exists a unique map $S \to H$ under which the pullback of f is ϕ.*

Proof. For each $s \in S$, the fiber map ϕ_s between split tori is given by some matrix $m(s) \in \mathrm{Mat}_{g' \times g}(\mathbf{Z})$. Hence, for each $g' \times g$ integer matrix m we can form the locus U_m in S consisting of those $s \in S$ for which $m(s) = m$, and we have a set-theoretic covering of S by the U_m's. I claim that U_m is open in S and that over each U_m the map ϕ is actually given by the matrix m. Once this is shown, then it follows that the map $S \to H$ sending U_m to the point $\operatorname{Spec} k$ indexed by m does pull f back to ϕ and it is the only such map $S \to H$ with this property.

Pick $s \in S$, and let $R = \mathscr{O}_{S,s}$ be the local ring. Let $m = m(s)$. Since m defines a map $T_S \to T'_S$, to show that this map agrees with ϕ in a neighborhood of s (thereby proving openness of the various U_m loci) it suffices (by viewing $\mathscr{O}_{S,s}$ as a direct limit of coordinate rings of open affines around s) to show that the two maps agree over R. This would also show that $\phi|_{U_m}$ is given by m for each choice of $g' \times g$ integral matrix m. Thus, we are reduced to a problem for split tori over a local ring R: if two maps agree on the closed fiber, they must be equal. By a direct limit argument, we can assume R is a local noetherian ring (express R as a direct limit of localized noetherian subrings with local transition maps, and descend ϕ to some such subring using that it is described using only finitely many elements of R). To check equality of two elements in the coordinate ring of a torus, it suffices to check equality modulo each power of the maximal ideal of R (look at the coefficient of a fixed monomial and use Krull's intersection theorem). Thus, we may assume R is an artin local ring.

With R an artin local ring, pick a positive integer n which is a unit in R, and consider the n-torsion on the two split tori of interest. This is finite étale over R. Since R is an artin local ring, passing to the quotient by its maximal ideal sets up an equivalence of categories between finite étale R-schemes and finite étale schemes over the residue field. Thus, the equality of two R-torus map on the closed fiber implies that the maps over R coincide on the finite étale n-torsion. The problem is therefore reduced to proving that two maps $\mathbf{G}^g_{m/R} \rightrightarrows Y$ over R which coincide on the closed subschemes $Z_n = \mu^g_{n/R}$ for all $n > 0$ invertible in R must in fact be the same map. Over the residue field this is clear, as such torsion is dense in $\mathbf{G}^g_m$ over a field. For the situation over R, since the Z_n's are R-flat we can use Grothendieck's theory of relative schematic density (see 11.10.10 in EGA IV) to infer the result in general. ∎

Now we improve the situation in Theorem 3.1 to handle non-split tori. Let T and T' be arbitrary k-tori, and let F be the functor $F(S) = \mathrm{Hom}_S(T_S, T'_S)$ on k-schemes. Let k'/k be a finite Galois extension over which T and T' become split. By Theorem 3.1, the restriction F' of F to the category of k'-schemes is represented by an enormous constant étale k'-scheme with components indexed by matrices. Let (f', H') denote the universal structure over k'. For each $\sigma \in \mathrm{Gal}(k'/k)$, the pullback $(\sigma^*(f'), \sigma^*(H'))$ is also a universal family representing F' on the category of k'-schemes (because σ is a k-map). Thus, Yoneda provides us with a unique isomorphism $\xi_\sigma : (f', H') \to (\sigma^*(f'), \sigma^*(H'))$ between these two universal families. The uniqueness ensures the compatibility condition $\tau^*(\xi_\sigma) \circ \xi_\tau = \xi_{\tau\sigma}$. In other words, we have Galois descent data on the situation.

But each Galois orbit in H' is just a finite disjoint union of copies of $\operatorname{Spec} k'$ (hence is affine), so the descent problem has a (necessarily unique) solution H which we could even write down by hand if so inclined. Explicitly, each matrix map m over k' can be viewed as a map of split tori $T_{k'} \to T'_{k'}$ and as such has a smallest field of definition $k(m) \subseteq k'$ containing k (here we work with *fixed* splittings on these two k'-tori). In the Galois orbits of such a map (using the k-structures on T and T'), we select one representative and use a copy of its $\operatorname{Spec} k(m)$ as a component of H. In this way, we also see that we can make a unique H-map $f : T_H \to T'_H$ which induces the universal f' over H'. Because (H, f) was made as a solution to the descent problem, we can prove the following result which improves Theorem 3.1:

Theorem 3.2. *The map $f : T_H \to T'_H$ is universal. That is, for any k-scheme S and any map $\phi : T_S \to T'_S$ of S-groups, there is a unique map $S \to H$ under which the pullback of f is ϕ.*

Proof. Let $\underline{H}$ denote the functor represented by H on k-schemes, so the structure f defines an element α in $F(H)$, where F is the Hom-functor for T and T' on the category of k-schemes. We want to show that α is universal. That is, we wish to show that the map of functors $\rho : \underline{H} \to F$ is an isomorphism. Recalling how (H, f) was constructed by descending the universal structure on H' that represented F', the restriction $\rho' : \underline{H'} \to F'$ to the category of k'-schemes is our earlier isomorphism. For any k-scheme S, Galois descent identifies $H(S)$ with the set of elements in $\underline{H}'(S') = \underline{H}(S')$ which are invariant under the action of $\mathrm{Gal}(k'/k)$ (where $S' = k' \otimes_k S$). Likewise, since the functor F classifies morphisms of schemes, Galois descent for scheme maps ensures that $F(S)$ is the set of Galois-invariant elements in $F'(S') = F(S')$.

Since the map $\underline{H}(S') \simeq F(S')$ is *functorial* in the S' as a k-scheme and is an *isomorphism* since S' is a k'-scheme, we conclude by functoriality with respect to the k-action of $\mathrm{Gal}(k'/k)$ on S' that the isomorphism $\underline{H}(S') \simeq F(S')$ must be Galois-equivariant, so the map $\underline{H}(S) \to F(S)$ is identified with the induced bijection on Galois-invariant subsets. Thus, we conclude that $\underline{H}(S) \to F(S)$ is an isomorphism for any k-scheme S. This is what we needed to prove. ■

Corollary 3.3. *The functor $\underline{\mathrm{Hom}}(\mathbf{G}_m, \mathscr{T})$ on Tor_G-schemes is represented by a countable disjoint union of finite étale Tor_G-schemes.*

Proof. We first work over the finite étale W_0-torsor $\mathrm{Tor}'_G = G \times (G/T_0)$ over $\mathrm{Tor}_G = G \times (G/N_0)$. Arguing as in our verification that (1.4) is cartesian, we get an isomorphism $\mathscr{T}' = \mathscr{T} \times_{\mathrm{Tor}_G} \mathrm{Tor}'_G \simeq T_0 \times (G/T_0)$ with the base change on the k-torus T_0 via the structure map for Tor_G as a k-scheme. For any map of schemes $S' \to S$, the restriction of a representable functor $\mathrm{Hom}_S(\cdot, M)$ to the subcategory of S'-schemes is naturally identified with the representable functor $\mathrm{Hom}_{S'}(\cdot, M \times_S S')$ (i.e., restricting a representable functor to a "slice subcategory" translates into base change on a representing object). Applying this to the functor $\underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)$ on k-schemes and the base change $\mathrm{Tor}'_G \to \mathrm{Spec}\, k$, we conclude that for H as in the preceding theorem (representing $\underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)$ on the category of k-schemes), $H \times_k \mathrm{Tor}'_G$ represents $\underline{\mathrm{Hom}}(\mathbf{G}_m, \mathscr{T}')$ on the category of Tor'_G-schemes. Since H was made as a countable disjoint union of finite étale k-schemes, we conclude that$H \times_k \mathrm{Tor}'_G$ is a countable disjoint union of finite étale Tor'_G-schemes.

There is a natural action of the finite étale Weyl group $W_0 = N_0/T_0$ on $\underline{\mathrm{Hom}}(\mathbf{G}_m, \mathscr{T}')$, hence on a representing object $H \times_k \mathrm{Tor}'_G$, over its free action on the smooth quasi-projective k-scheme $\mathrm{Tor}'_G = G/T_0$. Thanks to the freeness of the action on the base, the quotient of $\underline{\mathrm{Hom}}(\mathbf{G}_m, \mathscr{T}')$ by this action is readily checked to be a representing object for $\underline{\mathrm{Hom}}(\mathbf{G}_m, \mathscr{T})$ on the category of Tor_G-schemes (the point is that the quotient maps are finite étale, due to the freeness aspect of the W_0-action). The end result has countably many connected components, each finite étale over Tor_G, since this property descends from what we have established over the finite étale cover Tor'_G. ■

4. The main construction

Now fix an algebraically closed extension K of k and a map $h : \mathbf{G}_{m/K} \to G_K$ (Deligne uses $k = \mathbf{Q}$ and $K = \mathbf{C}$, with h equal to what Deligne would call $(h \circ r)_{\mathbf{C}}$). We are interested in studying pairs (T, s) where T is a maximal torus in G_F for an algebraically closed extension F of k and $s : \mathbf{G}_{m/F} \to T$ is a map whose composite $\mathbf{G}_{m/F} \to G_F$ is conjugate to h over some common extension of K and F (over k). To make sure this problem is meaningful, we pick k-embeddings of an algebraic closure $\overline{k}$ into K and F and prove:

Lemma 4.1. *Let K and F be algebraically closed extensions of an algebraically closed field k_0. Let T be a torus over k_0 and $h_K : T_K \to G_K$ and $h_F : T_F \to G_F$ two maps of algebraic groups, with G a reductive algebraic group over k_0. Let F' be a common algebraically closed extension of K and F over k_0. The property of the F'-base changes of h_K and h_F being $G(F')$-conjugate is independent of the choice of F'.*

Proof. The problem only depends on the $G(K)$-conjugacy class of h_K and the $G(F)$-conjugacy class of h_F. Let T_0 be a maximal torus in G over k_0. Since all maximal tori in G_F are $G(F)$-conjugate and all maximal tori in G_K are $G(K)$-conjugate, and moreover the maps h_F and h_K each factor through maximal tori over F

and K respectively, by applying suitable conjugations we may assume that these maps factor through $T_{0/F}$ and $T_{0/K}$ respectively. But two k_0-tori (such as T and T_0) don't acquire any new maps over an extension of k_0 since k_0 is algebraically closed. Thus, with these modifications the maps h_F and h_K are defined over k_0. We are therefore reduced to the problem of proving that if $h_1, h_2 : T \rightrightarrows G$ are maps over k_0 which become conjugate over some algebraically closed extension F' of k_0, then h_1 and h_2 are k-conjugate.

We use a standard direct limit trick via the Nullstellensatz. Write $F' = \varinjlim A_i$ as a direct limit of finite type k_0-subalgebras. Pick $g \in G(F')$ which conjugates $h_{1/F'}$ into $h_{2/F'}$. We can pick large i so that g comes from $g' \in G(A_i)$ and such that the two A_i-maps h_{2/A_i} and $g' h_{1/A_i} {g'}^{-1}$ coincide (at the expense of enlarging i a little bit we can certainly reach this situation). Now for the trick: reduce modulo a maximal ideal of A_i. Such a quotient of A_i is the algebraically closed field k_0 (Nullstellensatz!), so the resulting reduction of the "smeared out" g' provides an element in $G(k')$ which does the job. ■

Motivated by this lemma, we make a definition:

Definition 4.2. For two algebraically closed fields K and F over k, with $\overline{k}_K$ and $\overline{k}_F$ the algebraic closures of k in K and F respectively, maps $\mathbf{G}_{m/K} \to G_K$ and $\mathbf{G}_{m/F} \to G_F$ are *geometrically conjugate relative to* k if there exists a k-isomorphism $\sigma : \overline{k}_K \simeq \overline{k}_F$ such that the maps become conjugate over some (hence any, by the preceding lemma) common algebraically closed extension of K and F over the isomorphism σ.

We must warn the reader to not forget that this definition merely requires conjugacy for some choice of σ; not all choices will necessarily work (unless k happens to be separably closed, so algebraic closures of k have no non-trivial k-automorphisms). For example, we might have $K = F = \overline{k}$ a common algebraic closure of k, and then two maps $h_1, h_2 : \mathbf{G}_{m/\overline{k}} \rightrightarrows G_{\overline{k}}$ are geometrically conjugate over k if and only if h_1 is $G(\overline{k})$-conjugate to $\sigma^*(h_2)$ for some $\sigma \in \mathrm{Aut}(\overline{k}/k)$ (i.e., the Galois orbits of the conjugacy classes of h_1 and h_2 coincide). To demand that this work for all σ would amount to requiring that h_1 be $G(\overline{k})$-conjugate to $\sigma^*(h_1)$ for all $\sigma \in \mathrm{Gal}(\overline{k}/k)$, a condition which is far too strong (it is analogous to demanding that a reflex field equal $\mathbf{Q}$).

Recalling our fixed map $h : \mathbf{G}_m \to G_K$, we would like to study triples (T, s, v) consisting of a maximal torus T in G_F for algebraically closed F/k, a map $s : \mathbf{G}_{m/F} \to T_F$ whose composite into G_F is geometrically conjugate to h over k, and a Lie vector $v \in \mathrm{Lie}(T)$ which is regular in $\mathrm{Lie}(G_F)$. What we really want to do is make a structure over k which geometrically realizes all such triples without repetition when passing to F-points for any algebraically closed F over k.

Consider the product scheme $P = \underline{\mathrm{Hom}}_{\mathrm{Tor_G}}(\mathbf{G}_m, \mathscr{T}) \times_{\mathrm{Tor_G}} U$, where $U \subseteq \mathbf{Lie}(\mathscr{T}) \simeq \mathbf{Lie}(T_0 \times_k \mathrm{Tor}_G)$ is the fiberwise nonempty (over Tor_G) open locus from the end of §2; the F-points of U for an algebraically closed field F/k classify pairs (T, v), where T is a maximal torus in G_F and $v \in \mathrm{Lie}(T)$ is regular in $\mathrm{Lie}(G_F)$. We naturally have P as open inside of $\underline{\mathrm{Hom}}_{\mathrm{Tor_G}}(\mathbf{G}_m, \mathscr{T}) \times_{\mathrm{Tor_G}} \mathbf{Lie}(\mathscr{T})$. By Corollary 3.3 we conclude that P is a countable disjoint union of smooth quasi-projective k-schemes of pure dimension equal to that of G.

For F algebraically closed over k, an F-point of P is a triple (T, s, v) where (T, v) is as above and $s : \mathbf{G}_{m/F} \to T$ is a map of F-tori. That is, the F-points of P are in bijection with such triples (T, s, v) over F and this bijection is *functorial* in F. This is of course quite weak from the perspective of representable functors, but it is good enough for our purposes. What we would like to study are those F-points of P corresponding to triples (T, s, v) for which the composite $s' : \mathbf{G}_m \xrightarrow{s} T \to G_F$ is geometrically conjugate to the original $h : \mathbf{G}_m \to G_K$ over k. We wish to naturally cut out an open and closed subscheme P_h in P whose geometric points are such triples.

Over the étale Tor_G-scheme $\mathscr{H} = \underline{\mathrm{Hom}}(\mathbf{G}_m, \mathscr{T})$, we have a universal map $\mathbf{G}_m \times_k \mathscr{H} \to \mathscr{T} \times_{\mathrm{Tor_G}} \mathscr{H}$, and we can compose this with the base change to $\mathscr{H}$ of canonical closed immersion $\mathscr{T} \hookrightarrow G \times_k \mathrm{Tor}_G$ over Tor_G. This provides us with a map

$$\beta : \mathbf{G}_m \times_k \mathscr{H} \to G \times_k \mathscr{H}.$$

For any algebraically closed field F over k and $x \in \mathscr{H}(F)$ over $T \in \mathrm{Tor}_G(F)$, the fiber $\beta_x : \mathbf{G}_m \to G_F$ is the map s'_x arising from the F-map $s_x : \mathbf{G}_m \to T$ classified by x. We are interested in the locus P_h of points x in $\mathscr{H}$ for which the map β_x over $k(x)$ is geometrically conjugate to h over k (this is a property which may be checked by working over any algebraically closed extension of $k(x)$).

We claim that P_h is *open and closed* in $\mathscr{H}$. In fact, we claim something much more precise. There is a natural map $\mathscr{H} \to \underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)/W_0$ obtained by passage to the W_0-quotient on the natural smooth projection

$$\underline{\mathrm{Hom}}_{\mathrm{Tor}'_G}(\mathbf{G}_m, \mathscr{T}') \simeq \underline{\mathrm{Hom}}(\mathbf{G}_m, T_0) \times_k (G/T_0) \to \underline{\mathrm{Hom}}(\mathbf{G}_m, T_0),$$

and for algebraically closed F over k we have a natural bijection

$$(\underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)/W_0))(F) = \mathrm{Hom}_F(\mathbf{G}_{m/F}, T_{0/F})/W_0(F) \simeq \mathrm{Hom}_F(\mathbf{G}_{m/F}, G_F)/G(F) \tag{4.1}$$

onto the set of $G(F)$-conjugacy classes of maps from $\mathbf{G}_{m/F}$ to G_F (the final bijection in (4.1) uses the $G(F)$-conjugacy of maximal tori in G_F and the fact that centralizers of arbitrary tori in reductive groups are again reductive; cf. Corollary 2 in §13.17 in Borel).

Remark 4.3. Due to (4.1), our étale k-scheme $\underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)/W_0$ is what Deligne calls $\underline{\mathrm{Hom}}(\mathbf{G}_m, G)/G$, the "scheme of G-conjugacy classes of homomorphisms $\mathbf{G}_m \to G$". To put Deligne's point of view in perspective, we note (but will not use) that by Prop. 3.12 and Cor. 4.2 in Exp. XI of SGA3, for any smooth affine k-group H the functor $S \mapsto \mathrm{Hom}_{S-\mathrm{gp}}(\mathbf{G}_m, H_S)$ on k-schemes is represented by a smooth affine k-scheme, usually denoted $\underline{\mathrm{Hom}}(\mathbf{G}_m, H)$ (actually, Grothendieck proves something far more general concernings morphisms from tori to other group schemes over any base, without reference to an initial ground field).

The significance of (4.1) is that it then says that in our situation, the functorial map of smooth affine moduli schemes $\underline{\mathrm{Hom}}(\mathbf{G}_m, T_0) \to \underline{\mathrm{Hom}}(\mathbf{G}_m, G)$ induces a morphism of smooth quasi-projective quotients

$$\underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)/W_0 \to \underline{\mathrm{Hom}}(\mathbf{G}_m, G)/G$$

which is bijective on geometric points. In fact, as one might expect, this is an isomorphism. To see this, it is enough to prove both sides are étale over the base. For the source this follows from our construction of $\underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)$ as an étale k-scheme, though it can also be proved by pure thought using functorial criteria and infinitesimal properties of tori. For the target, we just have to check unramifiedness (since the base is a field, so flatness is automatic). This may be checked using points with values in an artin local ring with algebraically closed residue field, and then Thm. 3.2 in Exp IX of SGA3 provides the necessarily conjugacy result for maps from tori.

The reason we mention this fact is that we make frequent use of $\underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)/W_0$ below, where it would really be more natural to work with $\underline{\mathrm{Hom}}(\mathbf{G}_m, G)/G$ as Deligne does; however, our slightly weaker foundation (sans SGA3) and our concrete description of various constructions enables us to get by using the quotient $\underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)/W_0$ without requiring that this quotient really has a stronger functorial interpretation (as a quotient of a representing object for $\underline{\mathrm{Hom}}(\mathbf{G}_m, G)$ by the functorial G-action through conjugation).

Using (4.1), the map h is identified with a geometric point of the étale k-scheme $\underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)/W_0$; we write $z(h) = \mathrm{Spec}\, k(z(h))$ to denote the corresponding (open and closed) image point on $\underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)/W_0$. The locus P_h is nothing more or less than the fiber over $z(h)$ for the map

$$P \to \underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)/W_0 \tag{4.2}$$

(so P_h has a natural structure of $k(z(h))$-scheme). We leave it as a pleasant exercise to check that this description of P_h as a fiber is correct; the key is to use *how we defined geometric conjugacy.* Recalling how $\mathscr{H}$ breaks up as a disjoint union of finite étale Tor_G-schemes, we see that the fiber P_h over $z(h)$ is finite type (not just locally finite type) over $z(h)$, and of course it is also smooth over k (since it is open and closed in P which is k-smooth).

Theorem 4.4. *Assume k has characteristic zero. The composite map*

$$\pi : P_h \to \mathbf{Lie}(\mathscr{T}) \hookrightarrow \mathbf{Lie}(G \times_k \mathrm{Tor}_G) = \mathrm{Lie}(G) \times_k \mathrm{Tor}_G \to \mathrm{Lie}(G)$$

is étale and surjective onto $\mathrm{Lie}(G)^{\mathrm{reg}}$.

Proof. Let $\overline{k}$ be the algebraic closure of k inside of the algebraically closed extension K over which h was given. Replacing h with a $G(K)$-conjugate so that it factors through $T_{0/K}$, and hence is defined over $\overline{k}$, we may assume h is given over $\overline{k}$. Since a point of the étale k-scheme $\underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)$ is naturally identified with an $\mathrm{Aut}(\overline{k}/k)$-orbit of $\overline{k}$-points (where $\overline{k}$ is a fixed algebraic closure of k), a point of $\underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)/W_0$ is identified with a $W_0(\overline{k})$-orbit of such $\mathrm{Aut}(\overline{k}/k)$-orbits.

By the regularity condition in the definition of π, it has image inside of $\mathrm{Lie}(G)^{\mathrm{reg}}$. To check étaleness and surjectivity, it suffices to work over $\overline{k}$ and to examine the map on $\overline{k}$-points. More specifically, we consider the map

$$\mathrm{Lie}(T_0) \times_k (G/T_0) \to \mathrm{Lie}(G) \tag{4.3}$$

defined functorially by $(v,g) \mapsto \mathrm{Ad}(g)(v)$. It suffices to prove that if $\mathrm{Ad}(g)(v)$ is a regular vector then this map is étale at (v,g), and every regular vector in $\mathrm{Lie}(G_{\overline{k}})$ arises in the form $\mathrm{Ad}(g)(v)$. Since any regular vector in $\mathrm{Lie}(G)$ is semisimple (as we are in characteristic zero, so Corollary 2.2 applies), any such vector lies in the Lie algebra of a maximal torus. The $G(\overline{k})$-conjugacy of maximal tori thereby provides the desired surjectivity of $P_h(\overline{k})$ onto $\mathrm{Lie}(G_{\overline{k}})^{\mathrm{reg}}$ (we can always enhance a pair (gTg^{-1}, v) with a suitable choice of s so the triple comes from $P_h(\overline{k})$).

To prove that π is étale at (v,g) if $\mathrm{Ad}(g)(v)$ is regular, the k-smoothness of the source and target makes the problem equivalent to the tangent space map $\mathrm{d}\pi$ being an isomorphism at (v,g). Since (4.3) is equivariant under the left action of G on G/T_0 and the adjoint action of G on $\mathrm{Lie}(G)$, we can use transitivity of the G-action on G/T_0 to reduce to studying points $(v,1)$ with $v \in \mathrm{Lie}(T_{0/\overline{k}})$ regular in $\mathrm{Lie}(G_{\overline{k}})$. The differential

$$\mathrm{d}\pi(v,1) : \mathrm{Lie}(T_{0/\overline{k}}) \times \mathrm{Lie}(G_{\overline{k}}/\mathrm{Lie}(T_{0/\overline{k}}) \to \mathrm{Lie}(G_{\overline{k}})$$

is readily computed to be

$$(x,y) \mapsto [v,y] + x.$$

The image of the set of points $(x,0)$ is the Lie algebra of T_0, and the image of the set of points $(0,y)$ with $y \in \mathfrak{g}_\alpha$ is $\mathfrak{g}_\alpha$ since $[v,y] = \mathrm{d}\alpha(v) \cdot y$ with $\mathrm{d}\alpha(v) \neq 0$ since v is regular. Hence, the differential of π at $(v,1)$ is clearly an isomorphism (using the root decomposition of the Lie algebra of $G_{\overline{k}}$ with respect to the maximal torus T).

■

Corollary 4.5. *Assume k has characteristic zero. Let $z = z(h)$ be the point on $\underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)/W_0$ over which P_h is the fiber of P via (4.2). Then P_h is geometrically irreducible over $k(z)$ and the étale surjection $\pi : P_h \to \mathrm{Lie}(G)^{\mathrm{reg}}$ is finite.*

This corollary, coupled with the preceding theorem, completes the verification of Deligne's Lemma 5.1.2 and various properties asserted in the discussion preceding that lemma. Of course, to make the translation back to his picture one has to show that when our h is taken to be Deligne's $(h \circ r)_{\mathbf{C}}$ (using $k = \mathbf{Q}$ and $K = \mathbf{C}$), then what we are calling $k(z)$ is precisely Deligne's reflex field $E(G,h)$ (as an abstract finite extension of $k = \mathbf{Q}$, well-defined as a subfield of $\mathbf{C}$ only up to conjugation). But the reflex field was defined as a "field of definition" of the $G(\mathbf{C})$-conjugacy class of Deligne's $h_{\mathbf{C}}$, and this only depends on $(h \circ r)_{\mathbf{C}}$ since by Deligne's axioms $h \circ w$ lands in the center of G. The bijection (4.1), functorial in F, provides everything one needs to make the identification of our $k(z)$ with $E(G,h)$ in Deligne's situation.

Proof. We first check finiteness, and then we prove geometric irreducibility. From the preceding theorem we know that P_h is finite type (even quasi-projective, from the construction of P) and is étale surjective over $\mathrm{Lie}(G)^{\mathrm{reg}}$). In particular, π is quasi-finite, separated, and étale. By making base change to sufficiently small étale neighborhoods over $\overline{k}$-points and using the structure theorem for quasi-finite separated étale maps over a strictly henselian local ring, to establish finiteness of π it suffices to show that the étale fibers of π over $\overline{k}$-points of the target all have the same (necessarily finite) number of $\overline{k}$-points.

Pick a regular vector v in $\mathrm{Lie}(G_{\overline{k}})$. The $\overline{k}$-points of $\pi^{-1}(v)$ correspond bijectively to triples (T,s,v) where the composite s' of $s : \mathbf{G}_m \to T$ and $T \hookrightarrow G_{\overline{k}}$ is $G(\overline{k})$-conjugate to a point in the $\mathrm{Aut}(\overline{k}/k)$-orbit of h. Since T is a maximal torus which centralizes the regular vector v, the characteristic zero hypothesis allows us to apply Corollary 2.2 to conclude that v is semisimple. Thus, $T_v \stackrel{\mathrm{def}}{=} Z_G(v)^0_{\mathrm{red}}$ is a maximal torus by Theorem 2.1 (and is equal to $Z_G(v)$ by Corollary 2.2, though we do not need this fact here). It follows that $T = T_v$, so $\pi^{-1}(v)$ is is bijection with the set of maps $s : \mathbf{G}_m \to T_v$ whose composite $s' : \mathbf{G}_m \to G_{\overline{k}}$ is $G(\overline{k})$-conjugate to some $\mathrm{Aut}(\overline{k}/k)$-twist of h. For any maximal torus T of $G_{\overline{k}}$, define Σ_T to be the set of maps $s : \mathbf{G}_m \to T$ for which s' is $G(\overline{k})$-conjugate to an $\mathrm{Aut}(\overline{k}/k)$-twist of h. Thus, $\pi^{-1}(v)$ is in bijection with the set Σ_{T_v}. But all T's are $G(\overline{k})$-conjugate, and picking such a conjugation between two such tori T and T' obviously sets up a bijection between Σ_T and $\Sigma_{T'}$. This concludes the proof of finiteness.

Next, we check the geometric irreducibility. Rather than work on the geometric fiber over a point z in $\underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)/W_0$, it suffices to pick a point z' over this on the étale k-scheme $\underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)$ and to study the situation over a geometric point dominating $k(z')$. Pick a k-embedding of $k(z')$ into K. By replacing h with a suitable $G(K)$-conjugate, we can assume it factors through $T_{0/K}$, and the induced map $s_h : \mathbf{G}_{m/K} \to T_{0/K}$ has to descend to the universal morphism over $k(z')$. The map h therefore descends to $k(z')$. Fix a k-embedding of $k(z')$ into $\overline{k}$ (thereby giving a k-embedding of the subfield $k(z)$ into $\overline{k}$). We want to prove that $P_h \times_{k(z)} \overline{k}$ is irreducible. Let us consider its $\overline{k}$-points. Such points correspond to triples (T, s, v) where T is a maximal torus in $G_{\overline{k}}$, $s : \mathbf{G}_{m/\overline{k}} \to T$ is a map whose composite $s' : \mathbf{G}_{m/\overline{k}} \to G_{\overline{k}}$ is $G(\overline{k})$-conjugate to an $\mathrm{Aut}(\overline{k}/k(z))$-twist of h, and $v \in \mathrm{Lie}(T)$ is regular in $\mathrm{Lie}(G_{\overline{k}})$. It is *crucial* that we only have $\mathrm{Aut}(\overline{k}/k(z))$-twists here and *not* merely $\mathrm{Aut}(\overline{k}/k)$-twists!

Since we have modified h to be defined over $k(z')$ and the $\mathrm{Aut}(\overline{k}/k(z))$-twists of h are geometrically conjugate to h, we conclude that the triples of interest are (T, s, v) where T is the unique maximal torus centralizing the regular semisimple v and s' is $G(\overline{k})$-conjugate to h. To prove the geometric irreducibility of P_h over $k(z)$, we will use the smooth k-scheme $\mathscr{H} = \underline{\mathrm{Hom}}_{\mathrm{Tor}_G}(\mathbf{G}_m, \mathscr{T})$ of pairs (T, s) consisting of a maximal torus T and a map $s : \mathbf{G}_m \to T$. There is a natural functorial left action of G on $\mathscr{H}$ covering its transitive left action on $\mathrm{Tor}_G = G/N_0$, the orbits of which are the fibers of the smooth map

$$p : \mathscr{H} = \underline{\mathrm{Hom}}_{\mathrm{Tor}_G}(\mathbf{G}_m, \mathscr{T}) \to \underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)/W_0. \tag{4.4}$$

The geometric irreducibility of G over k therefore implies the geometric irreducibility of the fibers of p. Hence, the G-orbits in $\mathscr{H}$ are pairwise disjoint open and closed subschemes which are geometrically irreducible over their image point in the étale k-scheme $\underline{\mathrm{Hom}}(\mathbf{G}_m, T_0)/W_0$.

There is a natural smooth projection from P_h to $\mathscr{H}$ whose image is the open and closed subscheme $p^{-1}(z)$ which we have just shown to be geometrically irreducible over $k(z)$. Thus, $P_h \to p^{-1}(z)$ is a smooth surjection with target which is smooth and geometrically irreducible over $k(z)$. Since smooth surjections are quotient maps, we conclude that P_h is geometrically irreducible over $k(z)$ if and only if its geometric fibers over $p^{-1}(z)$ are connected. The fiber of P_h over a geometric point (T, s) of $p^{-1}(z)$ is naturally isomorphic to the locus in $\mathrm{Lie}(T)$ consisting of vectors which are regular in $\mathrm{Lie}(G)$. This locus is a non-empty Zariski open in the affine space $\mathrm{Lie}(T)$, so it is connected.

■

5. An openness result and an application

Partly due to the connectivity conclusion in Corollary 2.2, Deligne needs to study smooth affine group schemes $T \to Y$ with Y of finite type over $\mathbf{R}$ and T having torus geometric fibers. He needs the following basic fact:

Theorem 5.1. *The locus of $y \in Y(\mathbf{R})$ for which $T_y(\mathbf{R})$ is compact is open in the usual analytic topology on $Y(\mathbf{R})$.*

Proof. Without loss of generality, we may assume T has constant fiber dimension d and Y (hence T) is separated. Fix a positive integer n. The multiplication map $n : T \to T$ over Y is fiberwise finite étale of degree n^d, so since T is Y-flat (even Y-smooth) we conclude that $n : T \to T$ is finite étale of degree n^d. Thus, $T[n] \to Y$ is finite étale of degree n^d. I claim that an étale map of algebraic $\mathbf{R}$-schemes induces a local homeomorphism on topological spaces of $\mathbf{R}$-points. Granting this, the induced map $T[n](\mathbf{R}) \to Y(\mathbf{R})$ is a local isomorphism of topological spaces. Since fibers have size at most n^d and the source and target are Haudorff (being $\mathbf{R}$-points of separated $\mathbf{R}$-schemes), it follows that for any $y \in Y(\mathbf{R})$ with fiber of size n^d, all nearby points in $Y(\mathbf{R})$ have the same property.

It remains to prove the general fact that for an étale map $Y' \to Y$ between algebraic $\mathbf{R}$-schemes, the map $Y'(\mathbf{R}) \to Y(\mathbf{R})$ of topological spaces (with the usual analytic topology) is a local homeomorphism. If Y is $\mathbf{R}$-smooth then this can be proved using the structure theorems for smooth maps and étale maps, together with the inverse function theorem from multivariable calculus. This is the only case Deligne requires. For the general case, one uses the point of view of real analytic spaces and the fact that the completion of the analytic local ring is the completion of the algebraic local ring (or, as an alternative, by identifying **R**-points with Gal(**C**/**R**)-invariant **C**-points, it suffices to treat the situation over **C**, where one may instead appeal to the theory of complex analytic spaces).

■

The preceding theorem provide a class of open loci to which the following theorem is applied by Deligne (the geometric irreducibility condition is verified in Deligne's situation by applying Corollary 4.5). Here, for the first time, we require special ground fields of characteristic zero:

Theorem 5.2. *Let E be a number field. Consider a commutative diagram*

$$\begin{array}{ccc} W & \xrightarrow{f} & V \\ {\scriptstyle p}\downarrow & & \downarrow \\ \operatorname{Spec} E & \longrightarrow & \operatorname{Spec} \mathbf{Q} \end{array}$$

with V a dense open in an affine space over $\mathbf{Q}$, W reduced and geometrically irreducible over E, and f quasi-finite and dominant. Let $U \subseteq V(\mathbf{R})$ be a non-empty open set in the analytic topology. Let F/E be a finite extension.

There exists $v \in V(\mathbf{Q}) \cap U$ such that $F \otimes_E f^{-1}(v)$ is the spectrum of a field.

The same method works with **Q** replaced by any number field and **R** replaced by its completion at any place; for global function fields with positive characteristic, the theorem requires additional conditions of étaleness which are automatic in characteristic zero.

Proof. Since f is a quasi-finite and dominant map between finite type integral schemes over a field of characteristic zero, it is generically étale on W. Thus, by shrinking on both V and W we may assume f is finite étale; this shrinking cannot be disjoint from $U(\mathbf{R})$, since a proper Zariski closed set in an affine space over **Q** cannot contain a non-empty analytic open set of **R**-points (hence, the shrinking does not harm any of the hypotheses). Likewise, replacing W with $W \times_E F$ does not affect any of the assumptions or the condition of f being finite étale. In particular, this base change does *not* affect the irreducibility because of the hypothesis that W is geometrically irreducible over E. Thus, we may rename F as E so as to reduce to finding $v \in V(\mathbf{Q}) \cap U$ such that the étale $k(v)$-scheme $f^{-1}(v)$ is a single point (hence is the spectrum of a field). We will no longer make use of geometric irreducibility; just irreducibility will be used.

By the primitive element theorem, we may write $k(W) \simeq k(V)[T]/(g)$ for $g \in k(V)[T]$ a monic separable polynomial. Shrinking V around its generic point, we may assume $V = \operatorname{Spec} A$ with $A = \mathbf{Q}[x_1, \dots, x_n][1/h]$ and $W = \operatorname{Spec} B$ with $B = A[T]/(g)$ with $g \in \mathbf{Q}[x_1, \dots, x_n, T]$ involving T and irreducible. Thus, it suffices to show that inside of $\mathbf{R}^n$, there is a dense set of point elements $(q_1, \dots, q_n) \in \mathbf{Q}^n$ for which $g(q_1, \dots, q_n, T) \in \mathbf{Q}[T]$ is irreducible (for denseness allows us to find such a rational point which lies in the non-empty open locus $U \subseteq \mathbf{R}^n$). The classical Hilbert irreducibility theorem says that for irreducible $g \in \mathbf{Q}[X, T]$ involving T, there are infinitely many $x \in \mathbf{Q}$ for which $g(x, T) \in \mathbf{Q}[T]$ is irreducible; a mild strengthening is that the set of such x's is dense relative to any finite set of absolute values on **Q** (such as denseness in **R**). What we need is the multivariable version. In the algebraic theory, one says that a field H is *Hilbertian* if for any irreducible $g \in H[X, T]$ there are infinitely many $x \in H$ such that $g(x, T) \in H[T]$ is irreducible, and it is a basic theorem that when H is Hilbertian then similar statements (with Zariski denseness conditions) hold for $g \in H[x_1, \dots, x_n, T]$. However, we require a stronger multivariable generalization over number fields (or even just **Q**) in which one encodes a density condition relative to a given finite set of places. This is exactly what is proved in Lang's Diophantine Geometry book. ■

6. A descent result

We end by proving a general form of Deligne's Proposition 5.3:

Theorem 6.1. *Let K/k be an algebraically closed extension of a perfect field k, and let X be a k-scheme. Let Y be a reduced closed subscheme of X_K.*

Assume there is given an indexed family of algebraic extensions k_i of k inside of K with intersection k and k_i-schemes M_i and k_i-maps $v_i : M_i \to X_{k_i}$ such that $v_i : M_i \times_{k_i} K \to X_K$ factors through Y and has dense image in Y. Then Y is defined over k in the sense that there exists a closed reduced subscheme $Y_0 \hookrightarrow X$ such that $Y_0 \times_k K = Y$ inside of X_K.

Proof. The problem is local on X, so we may assume $X = \operatorname{Spec} A$ is affine, with I the radical ideal of Y in $A_K = K \otimes_k A$. We want to prove that $I = K \otimes_k I_0$ for a (necessarily unique) ideal I_0 in A. We may likewise assume each M_i is a disjoint union of affines $\operatorname{Spec} B_{i,\alpha}$, so we are given k_i-algebra maps $k_i \otimes_k A \to B_{i,\alpha}$ such that the map $K \otimes_k A \to \prod_\alpha (K \otimes_{k_i} B_{i,\alpha})$ has kernel equal to I. Since this map is $\operatorname{Aut}(K/k_i)$-invariant, we conclude that $I \subseteq K \otimes_k A$ is $\operatorname{Aut}(K/k_i)$-invariant for all i. If $I = K \otimes_k I_0$ for a k-vector space I_0 inside of A, then I_0 is an ideal since I is an ideal (and K is faithfully flat over k).

Thus, we are reduced to a problem in linear algebra: if V is a vector space over k and W is a K-subspace of $V_K = K \otimes_k V$ such that W is $\operatorname{Aut}(K/k_i)$-invariant for all i, then $W = K \otimes_k V'$ for a (necessarily unique) k-subspace V' in V. Since each k_i is perfect, so k_i is the subfield of $\operatorname{Aut}(K/k_i)$-invariants in K, a standard argument via writing nonzero elements of V_K as sums of tensors in a minimal manner implies that $W^{\operatorname{Aut}(K/k_i)} \neq 0$ when $W \neq 0$ and more specifically $W = K \otimes_{k_i} W^{\operatorname{Aut}(K/k_i)}$ inside of $V_K = K \otimes_{k_i} (k_i \otimes_k V)$ for each i. The problem is therefore to show that if W is a K-subspace in V_K and $W = K \otimes_{k_i} W_i$ for k_i-subspaces inside of $k_i \otimes_k V$ for all i, then $W = K \otimes_k W'$ for a k-subspace W' inside of V.

Let $\overline{k}$ be the algebraic closure of k inside of K, so by faithful flatness we see that W is defined over $\overline{k}$ compatibly with the W_i's over k_i, so we may assume $K = \overline{k}$. The $\overline{k}$-subspace W inside of $V_{\overline{k}}$ is $\operatorname{Gal}(\overline{k}/k_i)$-invariant for each i (with continuous Galois action for the discrete topology, since elements of $V_{\overline{k}}$ are finite sums of elementary tensors. Thus, W is invariant under the closure of the subgroup of $\operatorname{Gal}(\overline{k}/k)$ generated by the subgroups $\operatorname{Gal}(\overline{k}/k)$. This subgroup is obviously $\operatorname{Gal}(\overline{k}/k)$ since $\cap k_i = k$. Using the normal basis theorem (or really, finite Galois descent for well-chosen subspaces of finite dimension), a discrete $\overline{k}$-vector space with a continuous action of $\operatorname{Gal}(\overline{k}/k)$ is naturally the extension of scalars on its k-vector subspace of $\operatorname{Gal}(\overline{k}/k)$-invariants. This says that W is defined over k as a $\overline{k}$-subspace of $V_{\overline{k}}$. ■