AKSHAY ON SHIMURA VARIETIES

NOTES/ERRORS BY JEREMY BOOHER AND SAM LICHTENSTEIN

1. Lecture 1 (12/7/12)

1.1. Shimura varieties are like a brightly colored toy.

The motivation for their study comes from considering moduli spaces of Hodge structures.

Let X be a smooth, complex, projective variety of dimension n. The lattice $\Lambda = \mathrm{H}^n(X, \mathbf{Z})/\text{torsion}$ (the singular cohomology in middle degree) carries extra structures beyond that of a free abelian group, in virtue of the fact that X is a complex variety.

(1) It has a **pure Hodge structure of weight** n: in other words, $\Lambda_{\mathbf{C}} = \Lambda \otimes_{\mathbf{Z}} \mathbf{C}$ has an action of the algebraic $\mathbf{R}$-group $\mathbf{C}^\times$; equivalently, $\Lambda_{\mathbf{C}}$ decomposes as a direct sum

$$\Lambda_{\mathbf{C}} = \bigoplus_{p+q=n} \Lambda_{pq}$$

satisfying $\Lambda_{pq} = \overline{\Lambda}_{qp}$, where $z \in \mathbf{C}^\times$ acts on Λ_{pq} as scaling by $z^p \overline{z}^q$.[1]

One can reformulate this data perhaps more vividly as a *rotation structure*: in addition to the numerical invariant n, one only needs to keep track of the $U(1)$-action on the real vector space $\Lambda_{\mathbf{R}}$, coming from the complex structure on X. Namely, we can view a cohomology class (with real coefficients) as a differential form, i.e. something which eats elements of $\bigwedge^\bullet T_X$, and $U(1) \subset \mathbf{C}^\times$ acts by scaling on the exterior algebra $\bigwedge^\bullet T_X$, since this is a complex vector space; to get the action of $U(1)$ on the cohomology class, simply precompose with this scaling action.

The irreducible real representations of $U(1)$ are indexed by non-negative natural numbers m. Each is 2-dimensional, given by the rule

$$\theta \mapsto \begin{pmatrix} \cos m\theta & \sin m\theta \\ -\sin m\theta & \cos m\theta \end{pmatrix}.$$

Thus $\Lambda_{\mathbf{R}}$ decomposes as a direct sum of isotypic components V_m for the $U(1)$-action. The fact that X is a smooth algebraic variety imposes one constraint on this decomposition: if any $V_m \neq 0$, then m must be congruent to $n = \dim_{\mathbf{C}} X$ modulo 2. The pair $(n, \Lambda_{\mathbf{R}} = \bigoplus_{m \equiv n \pmod 2} V_m)$ is data equivalent to a pure weight n Hodge structure on Λ as defined above.

(2) There is a **polarization** on Λ. This is the data of a $U(1)$-equivariant, bilinear form

$$q : \Lambda \times \Lambda \to \mathbf{Z},$$

satisfying certain symmetry and definiteness constraints. Namely, if n is even, then q must be symmetric. If n is odd, then q must be skew-symmetric. On the real-ification $\Lambda_{\mathbf{R}} = \bigoplus_{m \equiv n \pmod 2} V_m$, the form $q_{\mathbf{R}}$ must be positive (resp. negative) definite on $V_0, V_4, V_8, \ldots$ (resp. $V_2, V_6, \ldots$) if n is even. If n is odd, the skew-symmetric form $q_{\mathbf{R}}$ gives rise to a symmetric form $q(v, Xv)$ for any $X \in \operatorname{Lie} U(1)$; this must be alternate between being positive and negative definite on $V_1, V_3, V_5, \ldots$ for any X.

[1] Actually, Deligne uses the sign convention $z^{-p}\overline{z}^q$.

If one fixes the discrete invariants n (the weight) and $\{h_{pq} = \dim \Lambda_{pq}\}_{p+q=n}$ (the Hodge numbers), then one can consider a "moduli space" $\mathcal{M}$ of all polarized Hodge structures with these invariants. We would like a family $\mathcal{X} \to \mathcal{M}$ such that the Hodge structure corresponding to $s \in M$ is that of the middle cohomology of the variety $\mathcal{X}_s$. This is basically never going to happen; as we study variation of Hodge structure we will see what additional conditions must be imposed to make something like this true.

We next turn to several examples of moduli spaces of Hodge structures which have natural structures of complex algebraic/analytic varieties and/or locally symmetric spaces.

1.2. *Smooth quartic surfaces.* For a degree 4 homogeneous polynomial f in four variables, let X_f be the corresponding surface in $\mathbf{P}^3$. It is easy to calculate that there is a 34-dimensional projective space of quartics f, on which the 15-dimensional group PGL_4 acts by linear changes of variable. So there is a 19-dimensional moduli space of smooth quartic surfaces X_f. (This is an open set in the moduli space of all K3 surfaces.)

Let X_f be any such. Slicing X_f with hyperplanes, an elementary argument[2] using Euler characteristics allows one to compute

$$\mathrm{rk}\, \mathrm{H}^2(X_f, \mathbf{Z}) = 22.$$

The 21-dimensional orthogonal complement (with respect to the Poincaré duality pairing) of the fundamental class of a hyperplane section of X_f is called the **primitive** cohomology of X_f. This lattice Λ carries a polarized weight-2 Hodge structure with Hodge numbers

$$h^{20} = h^{02} = 1, \qquad h^{11} = 19.$$

(We might say Λ is "of K3 type".) This gives us a map

$$\{\text{smooth quartics } f\} \to \mathcal{S} := \{\text{polarized wt 2 Hodge strs of type (1,1,19)}\}.$$

(This turns out to be an open immersion, whose complement is a **Special Divisor**, i.e. the image of some lower-dimensional Shimura variety... but this is a story for another time.)

We can and should ask: *What does $\mathcal{S}$ look like?*

Let Λ we a rank 21 lattice with a symmetric bilinear form q of real signature $(2, 19)$ that polarizes a generic weight 2 Hodge structure of type (1,1,19) on Λ. Let Γ be the arithmetic group $\mathrm{SO}(\Lambda, q) \subset \mathrm{SO}(q_{\mathbf{R}})$. Then it is easy to see that the Hodge structure conditions mean that both Λ_{02} and Λ_{11} can be recovered from the *line* Λ_{20}. (By, respectively, taking the complex conjugate and then an orthogonal complement.)

So $\mathcal{S} = \{\text{lines } \mathbf{C}v \subset \Lambda_{\mathbf{C}} \text{ such that } q(v, v) = 0, q(v, \overline{v}) > 0\}/\Gamma$. This is evidently something like a complex manifold, since the "numerator" is an open subset of a quadric hypersurface in $\mathbf{P}^{20}$.

We could also describe $\mathcal{S}$ in terms of the "rotations structure" viewpoint, as

$$\mathcal{S} = \{\text{2-dimensional subspaces of } \Lambda_{\mathbf{R}} \text{ such that the restriction of } q \text{ is positive definite }\}/\Gamma.$$

By Witt's theorem on quadratic forms, $\mathrm{SO}(2, 19)$ acts transitively on such 2-dimensional subspaces. The upshot of these considerations is that

$$\mathcal{S} = \mathrm{O}(19) \times \mathrm{SO}(2) \backslash \mathrm{SO}(19, 2)/\Gamma,$$

which gives a nice group theoretic description of $\mathcal{S}$ (albeit one which obscures the complex structure seen above).

[2]which we omit. FIXME: add this

1.3. *Six points in* $\mathbf{P}^1$. This is an example of a *unitary* Shimura variety (as opposed to the orthogonal one above).

For a sextuple $\alpha = (\alpha_1, \dots, \alpha_6) \in (\mathbf{P}^1)^6$, let

$$X_\alpha : y^3 = \prod_{i=1}^{6} (x - \alpha_i)$$

be the corresponding genus 4 plane curve. This has a μ_3-action by scaling y, which induces an action of the Eisenstein integers $\mathcal{O} = \mathbf{Z}[e^{2\pi i/3}]$ on $\mathrm{H}^1(X_\alpha, \mathbf{Z})$. It turns out that this lattice is $\mathcal{O}$-free of rank 4. This gives rise to a map

$$\phi : \{6 \text{ points in } \mathbf{P}^1\}/\operatorname{PGL}_2 \to \mathcal{S} := \mathrm{SU}(3)\backslash\mathrm{SU}(3,1)/\Gamma$$

for a lattice $\Gamma =$ " $\mathrm{SU}_{3,1}(\mathcal{O})$".

A nice thing about the target, here, is that is has extremely nice geometry: it is the quotient of the unit ball in $\mathbf{C}^3$ by a discrete group.

We now describe ϕ. To α we obtain an $\mathcal{O}$-lattice $\mathrm{H}^1(X_\alpha, \mathbf{Z})$ of rank 4 in the complex vector space $\mathrm{H}^1(X_\alpha, \mathbf{C}) = V^+ \oplus V^-$, where $V^\pm$ are the eigenspaces for the μ_3-action. The complexification

$$\Lambda \otimes_{\mathcal{O}} \mathbf{C} = V^+$$

inherits from the intersection form a *Hermitian* form $H(x) = \langle x, \overline{x} \rangle$ which turns out to have signature (3,1). The $\mathbf{C}^\times$-action on V^+ cuts out exactly the subspaces V_{10} of dimension 3 and V_{01} of dimension 1 on which H is definite.[3] To α, we associate the negative line V_{01}. The target of ϕ can be described as $\{\text{negative lines in } \Lambda \otimes_{\mathcal{O}} \mathbf{C}\}/\Gamma$ where $\Gamma = \operatorname{Aut}(\Lambda, H)$. (Where I guess we are using that H is already "defined over $\mathcal{O}$", so $\operatorname{Aut}(\Lambda, H)$ is naturally the $\mathcal{O}$-points of an $\mathcal{O}$-module for the complex algebraic group $\mathrm{SU}(V^+, H)$.

Why is the signature of H (3,1)? This is just a computation. Basically, you need to check it by hand using an explicit basis

$$\frac{\mathrm{d}\,x}{y}, \frac{\mathrm{d}\,x}{y^2}, \frac{x\,\mathrm{d}\,x}{y^2}, \frac{x^2\,\mathrm{d}\,x}{y^2}$$

for $H^0(X_\alpha, \Omega^1)$. The first basic vector is the negative part; the latter three span the positive part.[4]

2. Lecture 2 (1/11/13)

2.1. *Hermitian Symmetric Domains.* Hermitian symmetric domains will generalize the notion of the upper half plane or unit ball, which we took quotients of last time.

2.2. **Definition.** A Hermitian symmetric domain $U \subset \mathbf{C}^N$ is an open connected bounded set with the property that for every $x \in U$, there is a holomorphic isomorphism $r_x : U \to U$ such that $r_x^2 = \mathrm{id}$ and x is an isolated fixed point.

Note that r_x acts by -1 on the tangent space $T_x U$.

Here are a few examples.

1. The unit disc in $\mathbf{C}$ is a Hermitian symmetric domain.
2. The space of $p \times q$ complex matrices with norm at most 1. Here the norm refers to the operator norm on functions $\mathbf{C}^p \to \mathbf{C}^q$.

[3] *Caveat*: V^+ is not itself a Hodge structure, but just a piece of one.

[4] FIXME: Expand on this; i.e. do the calculations.

(3) The space of $p \times p$ complex matrices with norm at most 1 that are also symmetric (or skew-symmetric).

It is a fact that on a Hermitian symmetric domain U there is a canonical Kähler metric preserved by the holomorphic automorphisms of U.

The group $G = \mathrm{Hol}(U)^\circ$, i.e. the connected component of the group of holomorphic automorphisms of U, is a semisimple real Lie group which acts transitively on U. The stabilizer of a point K is a maximal compact subgroup.

Now consider the group $U(p,q)$, which consists of linear transformations stabilizing the form $|z_1|^2 + \ldots + |z_p|^2 - |z_{p+1}|^2 - \ldots - |z_{p+q}|^2$. $U(p,q)$ modulo its center acts on the Hermitian symmetric domain of $p \times q$ matrices of norm at most 1 via fractional linear transformations. Writing an element of $U(p,q)$ as a block matrix with blocks of size p and q,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = \frac{AZ+B}{CZ+D}$$

(Of course, we would need to check that $CZ+D$ is invertible.) This is a illustration of the following theorem. A good reference for this is Helgason's book on symmetric spaces[1].

2.3. **Theorem.** Any Hermitian symmetric domain U is isomorphic to a product $\prod G_i/K_i$ where G_i is a connected, simple, adjoint-type[5] Lie group and K_i is a maximal compact subgroup.

If $Z(K_i)$ is infinite, then in fact $Z(K_i)^\circ = S^1$. At the point fixed by K_i, the S^1 gives a way to rotate so we get an almost complex structure on G_i/K_i and in fact a complex structure. Furthermore, G_i/K_i embeds in a flag variety for $G_i^{\mathbf{C}}$.

We can fully classify all such pairs (G,K). The list is as follows, where in all examples we first quotient by the center.

(1) $G = U(p,q)$, $K = U(p) \times U(q)$.
(2) $G = Sp(2n, \mathbf{R})$, $K \simeq U(n)$
(3) $G = SO^*(2n)$, $K = U(n)$
(4) $G = SO(p,2)^\circ$, $K = SO(p) \times SO(2)$
(5) Two exceptional examples involving E_6 and E_7.

2.4. **Remark.** The reflections r_x in our initial definition of Hermitian symmetric domains correspond to Cartan involutions for the pairs (G,K).

2.5. *Arithmetic Varieties.* A **lattice** Λ in a group G is a discrete subgroup so that the quotient G/Λ has finite volume (ie there is a $G-$invariant measure on G/Λ). An **arithmetic lattice** is approximately a lattice Λ where there exists a $\mathbf{Q}$-group $\underline{G}$ such that $\underline{G}(\mathbf{R})$ surjects onto G with compact kernel and the image of $\underline{G}(\mathbf{Z})$ is Λ.[6] An **arithmetic variety** (over $\mathbf{C}$) is a quotient of a Hermitian symmetric space X by an arithmetic lattice $\Lambda \subset \mathrm{Aut}(X)$. One important result is the Bailey-Borel theorem.

2.6. **Theorem.** An arithmetic variety X/Λ has a unique structure of a quasiprojective variety over $\mathbf{C}$.

[5]This means G_i is isomorphic to the image of its own adjoint representation; the center of G_i is trivial.
[6]This was intentionally left imprecise in the lecture.

When X is compact, it turns out that $\mathcal{K}_X$ is ample (think about the curvature with respect to the canonical metric). Although Hermitian symmetric domains need not be compact, this is good motivation. Another big result is due to Kazhdan.

2.7. **Theorem.** If Y is an arithmetic variety and $\sigma \in \mathrm{Aut}(\mathbf{C})$, then Y^σ is also an arithmetic variety.

The proof uses a more algebraic classification of arithmetic varieties. This has the following consequence, which can also be seen in other ways in most cases (by writing down a moduli problem):

2.8. **Corollary.** The arithmetic variety Y can be defined over a number field. □

2.9**.** *A Digression on p-adic Uniformization* Let F be a real quadratic field, ∞_1 and ∞_2 the archimedian places, and v an inert prime. Let D_{∞_1,∞_2} be the quaternion algebra over F which is ramified at ∞_1 and ∞_2, and $D_{\infty_1,v}$ and $D_{\infty_2,v}$ be defined likewise. Recall that for a quaternion F-algebra D to be ramified at w means that $\mathrm{inv}_w(D) = \frac{1}{2}$, i.e. $D \otimes F_w$ is the nontrivial quaternion F_w-algebra, rather than $\mathrm{Mat}_2(F_w)$. Let Γ_{w_1,w_2} be the norm 1 units in D_{w_1,w_2}.[7]

Then using the embedding ∞_2, $\Gamma_{\infty_1,v}$ embeds into $\mathrm{SL}_2(\mathbf{R})$. Likewise for $\Gamma_{\infty_2,v}$. Also, $\Gamma_{\infty_1,\infty_2}$ embeds into $\mathrm{SL}_2(F_v)$. Then the upper half plane modulo $\Gamma_{\infty_1,v}$ and $\Gamma_{\infty_2,v}$ are two compelx Shimura curves X_1, X_2.

By work of Doi, Naganuma and Cerednik (cf. also Varshavsky's papers on this subject), there exists an F-variety Z such that $(Z \otimes_{\infty_i} \mathbf{C})^{\mathrm{an}} = X_i$, while the p-adic analytic space $(Z \otimes_v \overline{\mathbf{Q}}_p)^{\mathrm{an}}$ is the quotient of the p-adic upper half-plane by $\Gamma_{\infty_1,\infty_2}$. In other words, one has a p-adic uniformization entirely analogous to the "classical" complex-analytic uniformization of Z.

(For Shimura curves this is Cerednik's old result. Varshavsky generalized it to higher dimensional unitary Shimura varieties. The "point"[8] is that in Varshavsky's second paper on this subject, he uses Kazhdan's result, or at least ideas from its proof, to prove the existence of the p-adic uniformization.)

2.10**.** *Connection with Period Domains.* Last time we talked about period mappings and moduli problems. For example, we constructed a map from the moduli space of smooth quartic surfaces in $\mathbf{P}^3$ to $\Gamma\backslash SO(19,2)/S0(19) \times SO(2)$ (see section 1.2). We recognize $SO(19,2)/S0(19) \times SO(2)$ as a Hermitian symmetric domain. The map constructed was an open immersion with the complement of its image a Shimura subvariety.

The general philosophy about period domains and Shimura varieties is as follows. Whenever we are given a family $X \to B$ of algebraic varieties, we get a map from B to the period domain by looking at the Hodge structures. This is very far from being surjective, except in special cases. Dan will talk about this later. But when it is close to being surjective, we are essentially forced to be in a Shimura variety situation.

Here is a more precise statement. Let V be a real vector space, and ω a nondegenerate symmetric or skew-symmetry bilinear form on V. We consider all Hodge structures that respect ω, ie all $\varphi : S^1 \to \mathrm{Aut}(\omega)$ which satisfy the conditions to be a Hodge structure. (Possibly we also impose an additional linear algebraic conditions, such as that φ preserves

[7] More precisely, $\Gamma_{\infty_i,v}$ is the norm 1 units in the *maximal order* of $D_{\infty_i,v}$, while $\Gamma_{\infty_1,\infty_2}$ is the norm 1 units in $D_{\infty_1,\infty_2}[\frac{1}{p}]$.

[8] I think...

some collection of tensors on V.) Call this collection of Hodge structures $\mathcal{F}$. Then for every $x \in \mathcal{F}$, we get a polarized ($\mathbf{R}$-)Hodge structure on V. It is a fact that there is a canonical complex structure on $\mathcal{F}$. If this family satisfies the Griffiths transversality theorem (wait for Dan's talk) then $\mathcal{F}$ is biholomorphic to a Hermitian symmetric domain.

The moral here is that Hermitian symmetric domains arise naturally even if you only care *a priori* about studying Hodge structures and families thereof.

2.11**.** *Some Words on the Proof of the Proof of Kazhdan's Theorem.* Let X be a Hermitian symmetric domain, Γ an arithmetic lattice acting on X, and $Y = X/\Gamma$ the corresponding arithmetic variety. For $\sigma \in \mathrm{Aut}(\mathbf{C})$, we wish to prove that Y^σ is also of the form X'/Γ' for a Hermitian symmetric domain X' and an arithmetic lattice Γ'.

We begin with a warmup. Let X be the disc in $\mathbf{C}$. Let Γ be an arithmetic lattice in $SL_2(\mathbf{R})$, and σ an automorphism of $\mathbf{C}$. We want to show that $(X/\Gamma)^\sigma$ is of the form X/Γ' for Γ' an arithmetic lattice. By the uniformization of curves, we do know that $(X/\Gamma)^\sigma$ is of the form X/Γ' where Γ' is a not necessarily arithmetic lattice.

The question is why Γ' must be arithmetic. This uses the following theorem of Margulis to relate arithmeticity to the commensurator. The commensurator is defined to be $\mathrm{Comm}(\Gamma) = \{g \in G : g\Gamma g^{-1}, \Gamma \text{ are commensurable}\}$.

2.12. **Theorem.** A lattice $\Gamma \subset G$ (for a semisimple Lie group G) is arithmetic if and only if the index of Γ in the commensurator $\mathrm{Comm}(\Gamma)$ is infinite.

For $g \in \mathrm{Comm}(\Gamma)$, we get two finite maps from $X/(\Gamma \cap g^{-1}\Gamma g)$ to X/Γ that give correspondences on X/Γ. This property – having *lots* of algebraic correspondences – behaves well with respect to the action of field automorphisms, so Y's being arithmetic must be preserved by σ.

Now for a slightly harder example. Let X be the ball $\{|z_1|^2 + |z_2|^2 < 1\}$, and $G = PU(2,1)$. Let Γ be an arithmetic subgroup, and σ an automorphism. A theorem of S. T. Yau says that $(X/\Gamma)^\sigma$ is still uniformized by X,[9] so the above argument shows that $(X/\Gamma)^\sigma$ an arithmetic variety.

Now let X be a general Hermitian symmetric domain, Γ an arithmetic subgroup, $Y = X/\Gamma$ the arithmetic variety, and σ a complex automorphism. Our goal is to show that Y^σ is an arithmetic variety by showing its universal cover Z is a Hermitian symmetric domain, and then show there are many correspondences so Y is the quotient by an arithmetic subgroup. The general strategy is as follows:

(1) Construct a canonical metric on the universal cover Z.
(2) Construct a group Δ of isometries of Z with dense orbit.
(3) Show that the isometry group of Z is a Lie group that acts transitively on Z.

We now sketch some of the ideas needed to prove this.

To construct Δ, remember that elements of $\mathrm{Comm}(\Gamma)$ give Hecke correspondences on Y and also on Y^σ. These lift to maps $Z \to Z$. The lifts are certainly holomorphic isomorphisms: once we understand the metric, it will be automatic that they must preserve it. So let us construct the metric now.

[9] The point is that the existence of a uniformization is a metric property that Yau showed is characterized in topological terms. The relevant topological invariants (certain Chern numbers) can be computed *algebraically* using étale cohomology, and are thus invariant under automorphisms of $\mathbf{C}$. Also, this argument probably requires that X/Γ is compact, so let's assume we are in this easier situation.

The general construction is inspired by the following construction of the Arakelov metric. Let X be a Riemann surface of genus $g \geq 1$. Pick an orthogonal basis $\omega_1, \ldots \omega_g$ for $H^0(X, \Omega^1)$. (The inner product is $(\alpha, \beta) = \int_X \alpha \wedge \overline{\beta}$.) The tensor $\sum \omega_i \otimes \overline{\omega_i}$ defines a metric.

If X were a disc and hence not compact, we would use L^2 holomorphic 1-forms. The 1-forms $\omega_n = \frac{\sqrt{2n+2}}{\sqrt{-4\pi i}} z^n dz$ form a basis (think about Taylor series), and a calculation with the inner product shows they are orthonormal. Then we use

$$(-2\pi i) \sum \omega_n \otimes \overline{\omega_n} = \frac{2\pi i}{-4\pi i} \sum_{n \geq 0} |z|^{2n}(2n+2) dz \otimes d\overline{z} = \frac{dz \otimes d\overline{z}}{(1-|z|^2)^2}$$

as the metric. The constants are not particularly important

2.13. **Remark.** On a compact Riemann surface, this is not the hyperbolic metric. But if one passes to a large cover, does this construction and pushes down, this will converge to the hyperbolic metric.

The above techniques rely on it being a Riemann surface: in higher dimensions, analogous constructions do not give an element of the 1-forms. Instead, we use the Bergmann metric.

Suppose that X is compact for now of dimension n, with many n-forms (to be made precise soon). We can put a norm on $H^0(X, \Omega^n)$. Then we can obtain a Hermitian metric on Ω^n by taking an n form at a point and asking for the smallest norm of a lift of the form to $H^0(X, \Omega_n)$. Note that this process is preserved by holomorphic maps. This metric on the line bundle Ω^n gives a $1,1$ form which we hope will be positive definite.

What happens if X is an open bounded domain in $\mathbf{C}^n$? Let $\{f_i dz_1 \wedge \ldots \wedge dz_n\}_i$ be an orthonormal basis for the L^2 holomorphic n-forms on X. Again, we use the quotient metric, and see that the norm of $dz_1 \wedge \ldots \wedge dz_n$ is

$$(|f_1(x)|^2 + |f_2(x)|^2 + \ldots)^{-\frac{1}{2}}$$

The corresponding 1-form is $\partial\overline{\partial} \log(f_1(x)^2 + \ldots)$ and is called the Bergmann metric.

For the unit disc, we can use the orthonormal basis from before and get a metric of $\partial\overline{\partial} \log(\frac{1}{(1-|z|^2)^2})$.

Finally, there are arguments involving separating tangent vectors to show it is positive definite, and showing there are enough differential n-forms on a Hermitian symmetric domain to apply this sort of argument.

References

1. S. Helgason, *Differential geometry, lie groups, and symmetric spaces*, Graduate Studies in Mathematics, American Mathematical Society, 1978.