THE ANALYTIC AND ADELIC GALOIS ACTION ON CM POINTS

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For any Shimura datum (G, X) for a connected reductive group G over $\mathbf{Q}$ and any arithmetic subgroup Γ inside $G(\mathbf{Q})$, if Γ is torsion-free then there is a natural complex manifold structure on X for which the left action of $G(\mathbf{Q})$ is by holomorphic automorphisms. Hence, there is a natural complex manifold structure on $\Gamma\backslash X$ since the Γ-action on X is properly discontinuous and has trivial isotropy groups (by the torsion-freeness hypothesis). If we drop the torsion-free hypothesis on Γ then there is a finite-index normal subgroup $\Gamma' \subset \Gamma$, so the recipe $(\Gamma/\Gamma')\backslash(\Gamma'\backslash X)$ provides a structure of normal complex-analytic space on $\Gamma\backslash X$ that is easily seen to be independent of the choice of Γ' and in fact to be a categorical quotient of X for its Γ-action in the category of complex-analytic spaces. Consequently, for any compact open subgroup K of $G(\mathbf{A}_f)$ we have normal complex-analytic spaces

$$G(\mathbf{Q})\backslash(X \times G(\mathbf{A}_f))/K$$

that (as Sam explained) are finite disjoint unions of quotients $\Gamma_j\backslash X$ for various arithmetic subgroups Γ_j of $G(\mathbf{Q})$.

In Mike's talk we saw that such analytic spaces $\Gamma\backslash X$ admit (up to unique isomorphism) a unique algebraization as normal quasi-projective schemes over $\mathbf{C}$, even though they are generally non-compact (so GAGA does not apply); the result of Baily–Borel included a crucial functorial aspect as well: for any quasi-projective scheme Y over $\mathbf{C}$, every analytic map $\Gamma\backslash X \to Y^{\mathrm{an}}$ arises from a uniquely determined $\mathbf{C}$-morphism from the algebraization of $\Gamma\backslash X$ to Y. We defined $\mathrm{Sh}_K(G, X)$ to be the algebraization of $G(\mathbf{Q})\backslash(X \times G(\mathbf{A}_f))/K$ obtained in this way, so as a normal quasi-projective $\mathbf{C}$-scheme is it functorial with respect to maps between the analytifications. In particular, as we vary K the collection $\{\mathrm{Sh}_K(G, X)\}$ form an inverse system of normal quasi-projective $\mathbf{C}$-schemes; we denote it as $\mathrm{Sh}(G, X)$.

Now consider the special case $G = \mathrm{GSp}_{2g}$ and the "usual" $X^{\pm}$ as discussed in Brian's talk (we will review this below). These notes define a "Galois action" on the inverse system $\mathrm{Sh}(\mathrm{GSp}_{2g}, X^{\pm})$ over $\mathbf{C}$ and show how to descend this tower over $\mathbf{C}$ to one over $\mathbf{Q}$ (in a sense to be made precise, since the individual $\mathrm{Sh}_K(\mathrm{GSp}_{2g}, X^{\pm})$'s won't necessarily be equipped with a $\mathbf{Q}$-structure in this way, only a descent to a number field inside $\mathbf{C}$ that may depend on K). The main input is the main theorem of complex multiplication, which we review, and the fact that $A_{g,d,n}$ is representable by a smooth quasi-projective scheme over $\mathbf{Q}$ (as discussed by Rebecca), together with the identification of its analytification (as discussed by Arnav).

The strategy is to interpret the points of $\mathrm{Sh}_K(\mathrm{GSp}_{2g}, X^{\pm})$ as parameterizing abelian varieties over $\mathbf{C}$ equipped with extra structure, define a "Galois action" using this moduli interpretation, and use a special choice of $K \subset \mathrm{GSp}_{2g}(\mathbf{A}_f)$ to relate it to the normal quasi-projective moduli schemes $A_{g,d,n}$ over $\mathbf{Q}$. An important step is to rephrase the "Galois action" in terms of the double coset description of $\mathrm{Sh}_K(\mathrm{GSp}_{2g}, X^{\pm})$ to obtain an adelic description that depends only on the geometry and group theory of GSp_{2g} and not the moduli interpretation. This will be important for generalizing this to Shimura data for other reductive groups.

Two main references are *Travaux de Shimura* [2], especially section 4, and Milne's *Introduction to Shimura Varieties* [3].

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1. Background on CM Abelian Varieties

The material in this section is discussed at length in last year's seminar, and can also be found in Appendix 2 of [1] (and many other places).

Definition 1.1. Let A be an abelian variety defined over a field K of characteristic zero. Let $\mathrm{End}(A)$ be the endomorphism ring of A. It is a finitely generated torsion-free $\mathbf{Z}$-module. Define $\mathrm{Hom}^0(A, B) = \mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{Hom}(A, B)$ for abelian varieties A and B over K (it being understood that we only consider homomorphisms over K). For $B = A$ we write $\mathrm{End}^0(A)$. These are "homomorphisms in the isogeny category".

Recall that if $f \in \mathrm{Hom}(A, B)$ is an isogeny then it has a "quasi-inverse" $g \in \mathrm{Hom}(B, A)$ in the sense that $f \circ g$ and $g \circ f$ equal multiplication by some nonzero $n \in \mathbf{Z}$. It follows that $\mathrm{Hom}(A, B)$ meets $\mathrm{Isom}^0(A, B)$ (the set of isomorphisms in the isogeny category) in precisely the set of isogenies from A to B in the usual sense.

If A is nonzero and has no nonzero proper abelian subvarieties over K, it is called *K-simple*. A version of Schur's lemma states that $\mathrm{Hom}(B, B') = 0$ if B, B' are simple non-isogenous abelian varieties over K, and $\mathrm{End}^0(B)$ is a division algebra for K-simple B (equivalently, every non-zero homomorphism $B \to B$ is an isogeny).

Theorem 1.2 (Corollary of Poincaré Reducibility)**.** *Any abelian variety A over K is K-isogenous to a product of K-simple abelian varieties.*

As a consequence, we can find an isogeny from A to $\prod B_i^{e_i}$ with K-simple B_i that are pairwise non-isogenous, and the B_i are unique up to K-isogeny, with the image of $B_i^{e_i}$ in A also uniquely determined; we call this the *B_i-isotypic part* of A). We say A is *isotypic* if it has only one K-simple factor, which is to say it is isogenous to a power of a K-simple abelian variety. Clearly

$$\mathrm{End}^0(A) \simeq \mathrm{End}^0(\prod B_i^{e_i}) = \prod \mathrm{Mat}_{e_i}(\Delta_i)$$

where $\Delta_i = \mathrm{End}^0(B_i)$ are division algebras (whose centers are various number fields).

This motivates focusing on K-simple abelian varieties A. In that case, $\mathrm{End}^0(A)$ is a division algebra.

Definition 1.3. A CM field is a a quadratic extension of a totally real number field with no real places. [i.e., adjoin the square root of a totally negative element]

Note that CM fields have an intrinsic "conjugation", induced by any embedding into $\mathbf{C}$.

Theorem 1.4. *a) Let A be a K-simple abelian variety of dimension g. Any commutative subfield $L \subseteq \mathrm{End}^0(A)$ satisfies $[L : \mathbf{Q}] \leq 2g$, and when equality holds then L is a CM field.*

b) Let A be an abelian variety of dimension $g > 0$. Any commutative semisimple $\mathbf{Q}$-subalgebra $P \subseteq \mathrm{End}^0(A)$ satisfies $[P\colon \mathbf{Q}] \leq 2g$. If this equality holds for some P, then there exists a commutative semisimple $\mathbf{Q}$-subalgebra $P' \subseteq \mathrm{End}^0(A)$ such that $P' = \prod L_i$ and $A \sim \prod A_i$ where A_i are isotypic and L_i are CM fields of dimension $2\dim(A_i)$ inside $\mathrm{End}^0(A_i)$.

Definition 1.5. An abelian variety A over K of dimension g is called *of CM type* if $\mathrm{End}^0(A)$ contains a commutative semisimple $\mathbf{Q}$-subalgebra P of dimension $2g$. A *CM structure* of A is a choice of such embedding $P \hookrightarrow \mathrm{End}^0(A)$. The ring $P \bigcap \mathrm{End}(A)$ (regarded as a subring of P) is called a *CM order*.

If A is K-simple, a CM structure is given by an embedding of a CM field $L \hookrightarrow \mathrm{End}^0(A)$, and the CM order is the subring $L \bigcap \mathrm{End}(A)$. It is an order in the field L, so it is contained in $\mathcal{O}_L$; but it need not be the whole $\mathcal{O}_L$.

Note that $L \otimes_{\mathbf{Q}} \mathbf{R} \simeq \prod_{\sigma : L \hookrightarrow \mathbf{C}} \mathbf{C}_\sigma$, where the g embeddings of L into each copy of $\mathbf{C}$ are described by a choice of a member of each of the g pairs of conjugate embeddings of L into $\mathbf{C}$. If we want to pick one out of each pair, we will denote the obtained $\mathbf{C}$-algebra as $(L \otimes_{\mathbf{Q}} \mathbf{R})_\Phi$; Φ describing these g choices.

Now assume A is a K-simple abelian variety with a CM structure $L \hookrightarrow \mathrm{End}^0(A)$. Then the CM order $\mathcal{O}$ acts K-linearly on the tangent space $T_0(A)$, so $L = \mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Q}$ does as well. This makes $V = T_0(A)$ into a $K \otimes_{\mathbf{Q}} L$ module. This, of course commutes with base change; i.e., extending K. Let $\overline{V}$ be the $K \otimes_{\mathbf{Q}} L$-module obtained by conjugating the action of L, i.e. $V \otimes_{L,c} L$ where $c : L \simeq L$ is the conjugation.

Theorem 1.6. *As a $\overline{\mathbf{Q}} \otimes_{\mathbf{Q}} L$-module, $V_{\overline{\mathbf{Q}}} \oplus \overline{V}_{\overline{\mathbf{Q}}}$ is free of rank 1, which is to say that it is isomorphic to $\overline{\mathbf{Q}} \otimes_{\mathbf{Q}} L$.*

By faithful flatness of $\overline{\mathbf{Q}} \to \mathbf{C}$ (or more elementary considerations), it suffices to prove this result after scalar extension from $\overline{\mathbf{Q}}$ to $\mathbf{C}$. By Hodge theory, $\overline{T}_0(A) \simeq \mathrm{H}(A, \mathcal{O}_A)$ functorially in A, and consequently $T_0(A) \oplus \overline{T}_0(A) \simeq \mathbf{C} \otimes_{\mathbf{Q}} \mathrm{H}_1(A(\mathbf{C}), \mathbf{Q})$ functorially in A. Since $\mathrm{H}_1(A(\mathbf{C}), \mathbf{Q})$ is naturally an L-vector space whose underlying $\mathbf{Q}$-vector space has dimension $2g = [L : \mathbf{Q}]$, it is 1-dimensional over L. This completes the proof. As an immediate consequence, we obtain:

Theorem 1.7. *As an $\overline{\mathbf{Q}} \otimes_{\mathbf{Q}} L$-module, V is isomorphic to $\oplus_{\sigma \in \Phi} \overline{\mathbf{Q}}^\sigma$ for Φ a set of embeddings $L \hookrightarrow \overline{\mathbf{Q}}$ that contains exactly one of each pair of conjugate embeddings.*

The set Φ is called the *CM type* of A. Two CM abelian varieties that are L-isogenous (the L-action commutes with the isogeny) must obviously have the same type: base change to $\overline{\mathbf{Q}}$ and look at the L-map of tangent spaces.

Example 1.8. Over $\mathbf{C}$, up to isogeny all simple abelian varieties with CM type (L, Φ) are of the form $(L \otimes_{\mathbf{Q}} \mathbf{R})_\Phi / \Lambda$ where $\Lambda \subset L$ is an $\mathcal{O}$-module for an order $\mathcal{O}$ in $\mathcal{O}_L$. In particular, it is isogenous to the variety $(L \otimes_{\mathbf{Q}} \mathbf{R})_\Phi / \mathcal{O}_L$.

Any abelian variety over $\mathbf{C}$ with complex multiplication uniquely descends (along with all of its endomorphisms) to $\overline{\mathbf{Q}}$, so we consider such data now over $\overline{\mathbf{Q}}$. Such data over $\mathbf{Q}$ equipped with a chosen CM structure descends with its CM type to a number field. While we don't know how to descend it to a specific minimal possible number field (which might not be possible), we know how to descend its "tangent space" to a minimal possible number field, as we now make precise.

Suppose A over $\overline{\mathbf{Q}}$ has CM type (L, Φ), with $\Phi \subset \mathrm{Hom}(L, \overline{\mathbf{Q}})$. If A and its action by L descend to a number field K then A is L-isomorphic to A^σ for any $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/K)$. It is easy to see that A^σ has type $\Phi^\sigma := \sigma \circ \Phi$. Thus we need σ to fix Φ inside $\mathrm{Hom}(L, \overline{\mathbf{Q}})$.

The finite-index open subgroup of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ fixing Φ is of the form $\mathrm{Gal}(\overline{\mathbf{Q}}/E)$ for a number field E. It turns out E is CM (look at the conjugation). By Galois descent, is also the unique minimal field among all subfields F of $\overline{\mathbf{Q}}$ such that $T_0(A)$ descends to an $L \otimes_{\mathbf{Q}} F$-module.

Definition 1.9. The *reflex field* associated with the CM type Φ is the fixed field of all $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with $\Phi^\sigma = \Phi$.

Note that whereas the field K over which the abelian variety is defined is an abstract field, as is the CM field L, the reflex field is by definition is a specific subfield of $\overline{\mathbf{Q}}$.

Definition 1.10. The *reflex norm* $N_\Phi \colon R_{E/\mathbf{Q}}(\mathbf{G}_m) \to R_{L/\mathbf{Q}}(\mathbf{G}_m)$ is defined on R-points by condition that for a $\mathbf{Q}$-algebra R, $N_\Phi \colon (R \otimes_{\mathbf{Q}} E)^\times \to (R \otimes_{\mathbf{Q}} L)^\times$ is the determinant of the action of $R \otimes_{\mathbf{Q}} E$ on $V_0 \otimes_{\mathbf{Q}} R$ (which is a free $L \otimes_{\mathbf{Q}} B$-module of dimension g).

The composition $N_{L/L^+} \circ N_\Phi$ equals $N_{E/\mathbf{Q}}$.

If A has a CM-structure of type Φ, its dual $A^\vee$ naturally inherits an action of L and obtains a CM structure of type $\overline{\Phi}$. In particular, if we compose the action of L on $A^\vee$ with conjugation, $A^\vee$ gets CM type Φ. We call this the *dual* CM structure, so it has the same CM type and hence has the possibility to be isogenous to A (as it automatically is over $\overline{\mathbf{Q}}$ since the CM type determines the L-linear isogeny class over an algebraically closed ground field).

Definition 1.11. An *L-linear polarization* $\phi\colon A \to A^\vee$ is a polarization that commutes with the L-action. If $f\colon A \to B$ is an L-isogeny and $\psi\colon B \to B^\vee$ is an L-linear polarization then $f^*\phi$ is the L-isogeny of A defined by the following diagram:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "f"'] & A^\vee \\ B \arrow[r, "\psi"'] & B^\vee \arrow[u, "f^\vee"'] \end{tikzcd}$$

Although $f^*\phi$ might not be a genuine morphism, it is a $\mathbf{Q}_{>0}^\times$-multiple of an L-linear polarization; we call such maps L-linear $\mathbf{Q}$-*linear* $\mathbf{Q}$-*polarizations*.

Definition 1.12 (Serre tensor)**.** Let R be a ring and A a commutative group scheme over a base S with an action of R (as an S-group scheme). If M is a finitely generated projective R-module, the scheme $A \otimes_R M$ is defined to be the scheme representing the functor

$$T \to M \otimes_R A(T)$$

A short argument is required to show the functor is representable.

This has a concrete description for abelian varieties over $\mathbf{C}$:

Example 1.13. Let A be a CM abelian variety over $\mathbf{C}$ with CM type (L, Φ), with A^{an} identified with V/Λ where $V = (L \otimes_{\mathbf{Q}} \mathbf{R})_\Phi$ and Λ an $\mathcal{O}$-module for an order $\mathcal{O}$ in $\mathcal{O}_L$. Then $A \otimes_{\mathcal{O}} M$ is identified with $(V \otimes_{\mathcal{O}} M)/(\Lambda \otimes_{\mathcal{O}} M)$. In particular, when $\mathcal{O} = \mathcal{O}_L$ and $M = \mathfrak{a}$ is a fractional ideal of L, we identify $A \otimes_{\mathcal{O}_L} \mathfrak{a}$ with $V/(\Lambda \cdot \mathfrak{a})$.

Definition 1.14. Let $\mathfrak{a}$ be an ideal of L such that $\mathfrak{a}\bar{\mathfrak{a}} = (\alpha)$ with $\alpha \in \mathbf{Q}_{>0}$. (This is the case if $\mathfrak{a}$ is in the image of the reflex norm, for example.) Let ϕ be an L-linear $\mathbf{Q}$-polarization of A. The $\mathbf{Q}$-polarization $\phi_{\mathfrak{a}}$ of $A \otimes_{\mathcal{O}_L} \mathfrak{a}$ is defined to be $\frac{1}{\alpha}(\mathfrak{j}_{\mathfrak{a}}^{-1})^*(\phi)$ where $\mathfrak{j}_{\mathfrak{a}}$ is the rational isogeny $A \to A \otimes_{\mathcal{O}_L} \mathfrak{a}$ corresponding to the "$\mathbf{Q}$-map" $\mathcal{O}_L \to \mathfrak{a}$.

Remark 1.15. We can describe the $\mathbf{Q}$-polarization $(\mathfrak{j}_{\mathfrak{a}}^{-1})^*(\phi)$ very neatly over $\mathbf{C}$. If we identify A^{an} with $(L \otimes_{\mathbf{Q}} \mathbf{R})_\Phi/\Lambda$ where $\Lambda \subset L$ is a lattice with an action of $\mathcal{O}_L$ (i.e. a fractional ideal), then ϕ amounts to a bilinear pairing $\Lambda \times \Lambda \to \mathbf{Q}(1)$ that is restricted from a bilinear pairing $L \times L \to \mathbf{Q}(1)$ satisfying certain positivity and symmetry properties.

Since $(A \otimes_{\mathcal{O}_L} \mathfrak{a})^{\mathrm{an}} = (L \otimes_{\mathbf{Q}} \mathbf{R})/(\mathfrak{a}\Lambda)$ and $\Lambda, \mathfrak{a}\Lambda$ live in the same ambient space L, we see that $\mathfrak{a}\Lambda$ generates the same $\mathbf{Q}$-vector space as Λ does. The L-linear $\mathbf{Q}$-polarization $(\mathfrak{j}_{\mathfrak{a}})^{-1}\phi$ is the bilinear map $\mathfrak{a}\Lambda \times \mathfrak{a}\Lambda \to \mathbf{Q}(1)$ obtained by restricting the "same" $L \times L \to \mathbf{Q}(1)$ to $\mathfrak{a}\Lambda \times \mathfrak{a}\Lambda$. In other words, $(\mathfrak{j}_{\mathfrak{a}})^{-1}\phi$ is "the same" polarization but with a different domain.

Since we change the domain, the degree of the $\mathbf{Q}$-polarization will change. The positive rational number $[\mathfrak{a}\mathfrak{a}^*]$ (i.e., unique positive generator of the fractional L-ideal $\mathrm{a}\mathfrak{a}^*$) is exactly what we will need to control this change of degree.

The rational Tate module has an action of the finite rational adeles, and combined with the L-action it becomes an $L \otimes_{\mathbf{Q}} \mathbf{A}_{\mathbf{Q},f} = \mathbf{A}_{L,f}$-module. For dimension reasons, we see:

Proposition 1.16. *The rational Tate module $V_f(A) = T_f(A) \otimes_{\mathbf{Z}} \mathbf{Q}$ is free over $\mathbf{A}_{L,f}$ of rank* 1*.*

Suppose $(A, i\colon L \hookrightarrow \mathrm{End}^0(A), L)$ is a CM abelian variety over $\overline{\mathbf{Q}}$, with reflex field E. Then for $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/E)$, (A^σ, i^σ, L) is a CM abelian variety of the same type. The main theorem of CM describes the structure of this abelian variety in terms of (A, i, L) as we now review. Last year, we proved a version when A is principal; i.e., $\mathcal{O} = \mathcal{O}_L$.

Via the Artin reciprocity map, let σ correspond to $s \in \mathbf{A}_E^\times$ (i.e., the reciprocity map carries s to $\sigma|_{E^{\mathrm{ab}}}$). Let $[N_\Phi s]$ be the L-fractional ideal associated to the L-idele $N_\Phi s$.

Theorem 1.17. *With the notations above, assuming A has principal CM structure, there exists a unique isomorphism*

$$\theta_{\sigma,s}\colon [N_\Phi s]^{-1} \otimes_{\mathcal{O}_L} A \to A^\sigma$$

such that under $\theta_{\sigma,s}$ the Tate module map $\sigma\colon V_f(A) \to V_f(A^\sigma)$ corresponds to

$$V_f(A) \overset{\cdot N_{\Phi(s)^{-1}}}{\to} V_f(A \otimes_{\mathcal{O}_L} [N_\Phi(s)]^{-1}) \simeq V_f(A) \otimes_{\mathcal{O}_L} [N_\Phi(s)]^{-1}$$

In addition, it behaves well with respect to polarizations: if $\phi\colon A \to A^\vee$ is an L-linear $\mathbf{Q}$-polarization, then $\theta_{\sigma,s}$ intertwines ϕ^σ and $\phi_{[N_\Phi(s)]^{-1}}$.

This is the form of the main theorem of complex multiplication we proved last year. There are several other forms of the theorem which work for non-principal varieties (see [1, A.2.8]). Later we will be working with complex abelian varieties, so we present an analytic version of the theorem which has the advantage of concreteness and avoids the principality hypothesis (which is not functorial in the isogeny category). This requires the notion of "adelic operations" on the 1-dimensional L-vector space $\Lambda_{\mathbf{Q}}$, which we now explain.

Consider a nonzero finite-dimensional L-vector space W and a lattice $\Lambda \subset W$ of full rank that is stable under the action of $\mathcal{O}' \subset \mathcal{O}_L$. Such a Λ is called an *order lattice*, and the *endomorphism order* of Λ is the largest $\mathcal{O}'$ for which Λ is stable. The quotient W/Λ is a torsion $\mathbf{Z}$-module, so W/Λ is a torsion $\mathcal{O}'$ module, and will decompose as a product indexed by primes in $\mathcal{O}'$. An element $\mathbf{A}_{L,f}^\times$ will then act component-wise on these pieces.

More formally, for v' be a place (maximal ideal) of $\mathcal{O}'$, let $\mathcal{O}'_{v'}$ denote the completion of the order. Beware that there may be several ideals v of $\mathcal{O}_L$ that contract to the same ideal v' of $\mathcal{O}'$. For a place v' of $\mathcal{O}'$, define

$$L_{v'} := \prod_{v|v'} L_v, \qquad W_{v'} := \prod_{v|v'} L_v \otimes W, \quad \text{and} \quad \Lambda_{v'} := \mathcal{O}'_{v'} \otimes_{\mathcal{O}'} \Lambda.$$

The following properties are discussed just before Lemma A.2.8.1 of [1]:

- $W_{v'}$ is a free $L_{v'}$ module of rank 1.
- $W_{v'}/\Lambda_{v'}$ is the submodule of W/Λ consisting of $\mathfrak{m}'_{v'}$-power torsion.
- $W/\Lambda = \oplus_{v'} W_{v'}/\Lambda_{v'}$.
- If $\mathcal{O}''$ is an order inside $\mathcal{O}'$, and v'' a place of $\mathcal{O}''$, then

  $$W_{v''} = \prod_{v'|v''} W_{v'} \quad \text{and} \quad \Lambda_{v''} = \prod_{v'|v''} \Lambda_{v'}.$$

  In particular, this shows the decomposition is well-behaved with respect to changing the order, so it makes sense to work with the maximal endomorphism order in proofs if we wish (but everything works the same way with the *same meaning* even if we do not use the maximal endomorphism order!).

To define an adelic operation on the set of order lattices, the idea is to look at the quotient W/Λ and have each finite component s_v of the adele act on the v-torsion. Carrying this out carefully yields:

Lemma 1.18. *Let $\mathcal{O}'$ be the endomorphism order for $\Lambda \subset W$. For $s, s' \in \mathbf{A}_{L,f}^\times$,*

- *there is a unique $\mathcal{O}'$-stable order lattice $s\Lambda$ in W such that $(s\Lambda)_{v'} = s_{v'}\Lambda_{v'}$ in W for all places v' of $\mathcal{O}'$, and its endomorphism order is also $\mathcal{O}'$;*
- *there is a unique $\mathcal{O}'$-linear isomorphism $W/\Lambda \simeq W/s\Lambda$ such that on v'-factors it is given by multiplication by s'_v.*
- *the action satisfies $s'(s\Lambda) = (s's)\Lambda$.*

Proof. See [1, Lemma A.2.8.1]. □

Remark 1.19. If $\mathcal{O}' = \mathcal{O}_L$, then the adelic action is given by multiplying by the fractional ideal associated to s.

This adelic action on lattices allows us to state a complex-analytic version of the main theorem of CM for an abelian variety A defined over $\overline{\mathbf{Q}}$ with CM type (L, Φ). As before, let $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/E)$ correspond to $s \in \mathbf{A}_{E,f}^\times$, let ϕ be an L-linear $\mathbf{Q}$-polarization of A, and set $\Lambda = \mathrm{H}_1(A(\mathbf{C}), \mathbf{Z})$. Let the Riemann form on $H_1(A(\mathbf{C}), \mathbf{Q})$ associated to a polarization ϕ be denoted by Ψ_ϕ.

Theorem 1.20. *With the previous notation, there is a unique $\mathbf{C} \otimes L$ linear isomorphism between $V = \mathrm{Lie}(A)$ and $V_\sigma = \mathrm{Lie}(A^\sigma)$ such that the exponential uniformization $V = V_\sigma \to A^\sigma(\mathbf{C})$ identifies $\mathrm{H}_1(A^\sigma(\mathbf{C}), \mathbf{Z})$ with $N_\Phi(1/s)\Lambda$ inside V and the following diagram commutes*

$$\begin{tikzcd}
\Lambda_{\mathbf{Q}}/\Lambda \arrow[r, "\simeq"] \arrow[d, "N_\Phi(1/s)"] & A(\mathbf{C})_{\mathrm{tor}} & A(\overline{\mathbf{Q}}) \arrow[l, "\simeq"'] \arrow[d, "\sigma"] \\
\Lambda_{\mathbf{Q}}/N_\Phi(1/s)\Lambda \arrow[r, "\simeq"] & A^\sigma(\mathbf{C})_{\mathrm{tor}} & A^\sigma(\overline{\mathbf{Q}}) \arrow[l, "\simeq"']
\end{tikzcd}$$

Furthermore, under the identification of $\mathrm{H}_1(A^\sigma(\mathbf{C}), \mathbf{Q})$ with $\Lambda_{\mathbf{Q}}$, the Riemann form Ψ_{ϕ^σ} equals $q_s\Psi_\phi$, where q_s is the unique positive generator of the fractional $\mathbf{Q}$-ideal $[N_{E/\mathbf{Q}}(s)]$.

Remark 1.21. Note that because σ acts via multiplication by $N_\Phi(1/s)$ on the $\Lambda_{\mathbf{Q}}/\Lambda$ description of the torsion, it acts the same way on $V_f(A) = \Lambda_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{A}_f$.

2. The Siegel Modular Variety

The case of Shimura data for the symplectic similitude group is described in Example 3.4 of the notes from Brian's talk. Let V be a $2g$-dimensional $\mathbf{Q}$-vector space with fixed symplectic form Ψ, and let $G = \mathrm{GSp}(V, \Psi) \simeq \mathrm{GSp}_{2g}$. A complex structure on $V_{\mathbf{R}}$ is the same as a Hodge structure of type $(-1, 0)$ and $(-1, 0)$ on $V_{\mathbf{R}}$. Let $h : \mathbf{C}^\times \to \mathrm{GL}(V)(\mathbf{R})$ give the complex structure (it is the natural inclusion on $\mathbf{R}^\times$). The condition that this Hodge structure is polarized by $\pm\Psi$ means that the form $\Psi_{\mathbf{R}}(v, h(i)w)$ is definite on $V_{\mathbf{R}}$ and satisfies

$$\Psi_{\mathbf{R}}(v, h(i)w) = \Psi_{\mathbf{R}}(w, h(i)v).$$

This symmetry is equivalent to h having image contained in $G(\mathbf{R})$.

There is only one $G(\mathbf{R})$-conjugacy classes, denoted $X^\pm$, of algebraic homomorphisms $h : \mathbf{C}^\times \to G(\mathbf{R})$ satisfying these conditions. Brian checked that $(G, X^\pm)$ is a Shimura datum. The *Siegel modular variety* (for (V, Ψ)) is the name given to the associated Shimura variety. Sam gave a description of the points in terms of double cosets: for a compact open subgroup $K \subset G(\mathbf{A}_f)$,

$$\mathrm{Sh}_K(G, X^\pm) = G(\mathbf{Q})\backslash(X^\pm \times G(\mathbf{A}_f))/K$$

The Shimura variety $\mathrm{Sh}(G, X^\pm)$ is then defined as the limit over K: most of our work will be done with $\mathrm{Sh}_K(G, X^\pm)$.

For $x \in X^\pm$ and $a \in G(\mathbf{A}_f)$, we use the notation $[x, a]_K$ to denote the double coset represented by (x, a). We will give several interpretations of the points of $\mathrm{Sh}_K(G, X^\pm)$ in terms of various alternate types of data.

Definition 2.1. Let $\mathcal{H}_K$ denote the set of triples $((W, h), s, \eta K)$ where

- (W, h) is a rational Hodge structure with type $(-1, 0)$, $(0, -1)$.
- $\pm s$ is a polarization for (W, h).
- ηK is a K-orbit of $\mathbf{A}_f$-linear isomorphisms $W(\mathbf{A}_f) \simeq V(\mathbf{A}_f)$ that intertwine s with an $\mathbf{A}_f^\times$-multiple of Ψ.

Two such triples are said to be *isomorphic* provided that (W,h) and (W',h') are isomorphic as rational Hodge structures in a manner that makes s and s' agree up to an element of $\mathbf{Q}^\times$ and makes ηK and $\eta' K$ agree.

As discussed above, the first two conditions can be interpreted in terms of a complex structure on $W(\mathbf{R})$ and symmetry and definiteness conditions on $s_{\mathbf{R}}(v,h(i)w)$.

Given such a triple, the above analysis showed that the image of h lies in $\mathrm{GSp}(W,\pm s)$. It is also clear that $\dim W = \dim V$, and so there is an isomorphism $a : W \to V$ that identifies s with a rational multiple of Ψ. Define h^a to be the map $\mathbf{C}^\times \to \mathrm{GL}(V)$ given by $z \mapsto ah(z)a^{-1}$. Let $a \circ \eta$ denote the composition of the map $W(\mathbf{A}_f) \to V(\mathbf{A}_f)$ induced by a with η.

Proposition 2.2. *There is a natural bijection between $\mathcal{H}_K/\simeq$ and $\mathrm{Sh}_K(G,X^\pm)$ defined by*

$$((W,h),s,\eta K) \to [h^a, a\circ\eta]_K.$$

Proof. Note that $[h,g]_K$ is the image of the triple $((V,h),\Psi,gK)$ constructed using our choice (V,Ψ). The rest is elementary verifications. We first check this is well defined. Using a different isomorphism $a' : W \to V$ produces an isomorphic triple: $a'a^{-1}$ gives an element of $G(\mathbf{Q})$ (since the symplectic form is preserved up to rational multiple) which is lost in the quotient. Changing the representative of ηK changes the second component by an element of K, which is also lost in the quotient.

Isomorphic triples are sent to the same element of $\mathrm{Sh}_K(G,X^\pm)$: given an isomorphism $b : W \to W'$ that induces an isomorphism of triples, use ab^{-1} as the isomorphism $W' \to V$. Then $((W',h'),s',\eta' K)$ maps to $[h'^{ab^{-1}}, ab^{-1}\circ\eta']_K$. But $ab^{-1}h'ba^{-1} = (b^{-1}h'b)^a = h^a$ and $b^{-1}\circ\eta' = \eta$.

The map is injective, since equality in the quotient means that there are $q \in G(\mathbf{Q})$ and $k \in K$ such that

$$h^a = q \circ h'^{a'} \quad \text{and} \quad a\circ\eta = q\circ a'\circ\eta'\circ k.$$

This shows that $a \circ (a')^{-1}$ gives an isomorphism between the triples. $\square$

The next step is to further reinterpret the points in terms of abelian varieties. Recall that an abelian variety over $\mathbf{C}$ can be described as a complex torus $A = V/\Lambda$ admitting a Riemann form. To define a Riemann form, note that as $\Lambda\otimes\mathbf{R} = V$, $\Lambda\otimes\mathbf{R}$ has a complex structure and in this way multiplication by i induces an automorphism J. A Riemann form is an alternating form $s : \Lambda\times\Lambda \to \mathbf{Z}$ such that $s_{\mathbf{R}}(Ju,Jv) = s_{\mathbf{R}}(u,v)$ and $s_{\mathbf{R}}(Ju,u) > 0$ for $u \neq 0$. The version of this fact in the isogeny category is the following.

Theorem 2.3. *There is an equivalence of categories between the isogeny category of $\mathbf{Q}$-polarized complex abelian varieties and the category of polarized rational Hodge structures of type $(-1,0)$ and $(0,-1)$, defined by sending A to $\mathrm{H}_1(A(\mathbf{C}),\mathbf{Q})$.*

Recall that $T_l(A) = \Lambda\otimes\mathbf{Z}_l$ and hence $V_f(A) = \Lambda\otimes_{\mathbf{Z}}\mathbf{A}_f$.

Definition 2.4. Let $\mathcal{A}_K$ denote the set of triples $(A,s,\eta K)$ where

- A is a complex abelian variety.
- $\pm s$ is a $\mathbf{Q}$-polarization on A.
- ηK is a K-conjugacy class of isomorphisms $V(\mathbf{A}_f) \to V_f(A)$ that sends Ψ to a $\mathbf{A}_f^\times$-multiple of s.

Two triples are *isomorphic* if A and A' are isogenous, and the isogeny identifies s and s' up to $\mathbf{Q}^\times$-multiples and $\eta' K$ with ηK.

Again, note that $2\dim A = \dim \mathrm{H}_1(A(\mathbf{C}),\mathbf{Q}) = \dim V$.

Proposition 2.5. *There is a natural bijection between $\mathcal{A}_K/\simeq$ and $\mathcal{H}_K/\simeq$ given by sending $(A,s,\eta K)$ to $(\mathrm{H}_1(A(\mathbf{C}),\mathbf{Q}),s,\eta K)$.*

Proof. Given the above facts about abelian varieties, this equivalence is again elementary. To check the map is well defined, let $(A', s', \eta' K)$ be an isomorphic triple, with isogeny $\alpha : A \to A'$. Then $\mathrm{H}_1(\alpha)$ is an isomorphism on rational homology which certainly respects the Hodge structure, so $\mathrm{H}_1(A(\mathbf{C}), \mathbf{Q})$ and $\mathrm{H}_1(A'(\mathbf{C}), \mathbf{Q})$ are isomorphic polarized Hodge structure. It is immediate that the other components match under $\mathrm{H}_1(\alpha)$. Injectivity and surjectivity follow from the equivalence of categories. □

This is also connected to the moduli problem $A_{g,d,n}$ Rebecca discussed. Recall its $\mathbf{C}$-points were complex abelian varieties of dimension g with a polarization of degree d^2 and full level structure of degree n. This is related to $\mathcal{A}_K$ when K is the principal congruence subgroup

$$K_n := \ker(\mathrm{GSp}_{2g}(\widehat{\mathbf{Z}}) \to \mathrm{GSp}_{2g}(\mathbf{Z}/n\mathbf{Z}))$$

Proposition 2.6. *There is an isomorphism between $A_{g,1,n}(\mathbf{C})$ and $\mathcal{A}_{K_n}/\simeq$.*

Proof. Given an Abelian variety with degree 1 polarization given as a Riemann form ψ on $H_1(A, \mathbf{Q})$ and a full level n structure, we construct the K_n oribt of isomorphisms $\eta : V(\mathbf{A}_f) \to V_f(A)$ by working at each prime l separately. For $l \not| n$, we need to specify a $\mathrm{GSp}_{2g}(\mathbf{Z}_l)$ orbit of isomorphisms $V \otimes \mathbf{Q}_l$ with $V_l(A)$. Since $\mathrm{GSp}_{2g}(\mathbf{Z}_l)$ are the automorphisms of a lattice with symplectic form, such an isomorphism is equivalent to specifying such a lattice to correspond to $V_{\mathbf{Z}} \otimes \mathbf{Q}_l$. Use $H_1(A(\mathbf{C}), \mathbf{Z}) \otimes \mathbf{Z}_l \subset H_1(A(\mathbf{C}), \mathbf{Z}) \otimes \mathbf{Q}_l = V_l(A)$. For $l|n$, an orbit is equivalent to a lattice with symplectic form plus a basis for the l^r torsion, where r is the largest integre so l^r divides n. Use $H_1(A(\mathbf{C}), \mathbf{Z}) \otimes \mathbf{Z}_l$ together with the level n structure of A to specify this.

Conversely, given a triple $(A, s, \eta K_n)$, again view η as a collection of isomorphisms for each l. Note that these triples are only considered up to isogeny of A. However, we are looking for a degree 1 polarization, so this tells us which isogenous variety to use. Now η_l gives an identification of $T_l(A) = H_1(A(\mathbf{C}), \mathbf{Z}) \otimes \mathbf{Q}_l$ with $V \otimes \mathbf{Q}_l$, and our fixed $V_{\mathbf{Z}} \subset V$ corresponds to a lattice $\Lambda_l \subset H_1(A(\mathbf{C}), \mathbf{Z}) \otimes \mathbf{Q}_l$. The K_n orbit does not change this. For $l|n$, we furthermore have a basis for the l^r torsion where l^r exactly divides n. This data determines a lattice $\Lambda \subset V$: using V/Λ as the complex Abelian variety gives an isogenous variety A' with $H_1(A'(\mathbf{C}), \mathbf{Z}) = \Lambda$ and a level n structure. Furthermore, s (only a $\mathbf{Q}$-bar polarization) corresponds to a $\mathbf{Q}$-multiple of Ψ, which had degree 1. So use Ψ as the required degree one polarization. This specifies an element of $A_{g,1,n}(\mathbf{C})$.

It is now elementary to check these proceedures are well-defined inverses. □

Remark 2.7. Since points of $\mathrm{Sh}_K(G, X^{\pm})$ correspond to abelian varieties over $\mathbf{C}$ with extra structure related to polarizations and endomorphisms, and the functor from the category of abelian varieties over $\overline{\mathbf{Q}}$ to the category of abelian varieties over $\mathbf{C}$ is fully faithful (as for any extension of algebraically closed fields in place of $\overline{\mathbf{Q}} \to \mathbf{C}$), it makes sense to speak of a point of $\mathrm{Sh}_K(G, X^{\pm})$ being $\overline{\mathbf{Q}}$-*point*: this is the condition the abelian variety part of the data descends to $\overline{\mathbf{Q}}$ (in which case the descent is unique up to unique isomorphism and all of the additional structure descends uniquely as well in a compatible way).

We shall now define an action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the set of such $\overline{\mathbf{Q}}$-points $(A, s, \eta K)$. Given $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, it is easy to define the abelian variety $A^\sigma = A \otimes_{\overline{\mathbf{Q}}, \sigma} \overline{\mathbf{Q}}$ and a map $\eta^\sigma : V(\mathbf{A}_f) \to V_f(A^\sigma)$ by

$$[\sigma] \circ \eta : V(\mathbf{A}_f) \to V_f(A) \simeq V_f(A^\sigma).$$

In a similar way via scalar extension, s^σ is a $\mathbf{Q}$-polarization on A^σ. We define

$$\sigma(A, s, \eta K) := (A^\sigma, s^\sigma, \eta^\sigma K).$$

Therefore, if a triple $(A, s, \eta K)$ corresponds to $[h, g] \in \mathrm{Sh}_K(G, X^{\pm})$, the natural problem is to describe the point of $\mathrm{Sh}_K(G, X^{\pm})$ corresponding to $\sigma(A, s, \eta K)$. We will get a partial answer for special points corresponding to CM abelian varieties, using the adelic-operation formulation of the main theorem for complex multiplication.

3. Description of the Action on CM Points

A point $[h,g]_K \in \mathrm{Sh}_K(G,X^{\pm})$ is called a *special point* or a *CM point* if it corresponds to a triple $(A,s,\eta K) \in \mathcal{A}_K$ such that A is an abelian variety with CM type (L,Φ) and s gives an L-linear $\mathbf{Q}$-polarization. Note that these are $\overline{\mathbf{Q}}$-points in the sense defined above! If we need to note the dependence on h, we will use subscripts, so the CM algebra would be denoted $L = L_h$.

For a CM point, Akshay showed that the Zariski closure of $h(\mathbf{C}^\times) \subset G(\mathbf{R})$ will be a torus defined over $\mathbf{Q}$. (For more general Shimura varieties, this will be taken to be the definition of a special point). Call this torus T, and let F be a number field. Given any cocharacter $\mu : \mathbf{G}_{m,F} \to T_F$, define a map r_μ by combining Weil restriction with the norm map:

$$s_\mu = N_{F/\mathbf{Q}} \circ \mathrm{R}_{F/\mathbf{Q}}(\mu) : \mathrm{R}_{F/\mathbf{Q}}(\mathbf{G}_{m,F}) \to \mathrm{R}_{F/\mathbf{Q}}\, T_F \to T.$$

Now let j be the map $\mathbf{G}_{m,\mathbf{C}} \to (\mathrm{R}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_m))_{\mathbf{C}}$ given by sending $z \to (z,1)$. An example of such a μ is given by (any descent) of the composition

$$\mu_h = h \circ j : \mathbf{G}_{m,\mathbf{C}} \to (\mathrm{R}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_m))_{\mathbf{C}} \to T_{\mathbf{C}}$$

It is in fact easy to find an explicit torus containing $h(\mathbf{C}^\times)$. In the case A has CM type (L,Φ) where L is a field, let $T = \mathrm{R}_{L/\mathbf{Q}}\,\mathbf{G}_m$. We may take $V = H_1(A,\mathbf{Q})$ as usual. Define a map $T_{\overline{\mathbf{Q}}} \to G_{\overline{\mathbf{Q}}}$ on $\overline{\mathbf{Q}}$ algebras R by having $T_{\overline{\mathbf{Q}}}(R) = (R \otimes_{\mathbf{Q}} L)^\times$ act on $R \otimes V_{\overline{\mathbf{Q}}}$ via the CM action on $V \otimes \overline{\mathbf{Q}}$. It obviously descends to an inclusion $T \to G$. Furthermore, $T_{\mathbf{R}}(\mathbf{R}) = (L \otimes \mathbf{R})^\times \simeq (\mathbf{C}^\Phi)^\times$. By definition of CM type, the action of $\mathbf{C}^\times = \mathrm{R}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_m)(\mathbf{R})$ on $V(\mathbf{R})$ is precisely given by the action of $\mathbf{C}^\times$ on $(\mathbf{C}^\Phi)$, so h factors through $T_{\mathbf{R}}$ and hence the Zariski closure of $h(\mathbf{C}^\times)$ is contained in T. For a general CM Abelian variety, it is isogenous to a product of abelian varieties with complex multiplication by CM fields. So apply the field case to each factor.

It is also easy to describe s_{μ_h}. The torus $T_{\mathbf{C}}$ is $\prod_{\sigma\in\Phi} \mathbf{G}_{m,\sigma} \times \prod_{\sigma\in\overline{\Phi}} \mathbf{G}_{m,\sigma}$, and μ_h maps $\mathbf{G}_{m,\mathbf{C}}$ into the first product via the identity and the second product trivially. This can be defined over the reflex field, as E is the fixed field of Φ. For a $\mathbf{Q}$-algebra R, the Weil restriction has $r \otimes x \in \mathrm{R}_{E/\mathbf{Q}}\,\mathbf{G}_m(R) = (R \otimes E)^\times$ act on $V \otimes R \otimes E) = \prod_{\sigma\in\Phi} E_\sigma \otimes R$ via multiplication, corresponding to the element $r \otimes x \in \mathrm{R}_{E/\mathbf{Q}}\, T_E(R) = (R \otimes E)^\times$. Taking $R = \mathbf{A}_f$, denote the map by r_h:

$$r_h : \mathbf{A}_{E,f}^\times = \mathrm{R}_{E/\mathbf{Q}}\,\mathbf{G}_m(\mathbf{A}_f) \to T(\mathbf{A}_f) = \mathbf{A}_{L,f}^\times$$

Since norms can be defined by taking the determinant of the associated multiplication map, for $a \in \mathbf{A}_{E,f}^\times$ the composition with the inclusion $T(\mathbf{A}_f)$ into $G(\mathbf{A}_f)$ sends a the transformation given by the reflex norm $N_\Phi(a)$ acting on $V(\mathbf{A}_f)$ via the CM datum.

Remark 3.1. The above construction can be phrased so as to ignore the details of the CM points coming from the modulie interpretation. All that is important is the existence of the torus containing $h(\mathbf{C}^\times)$ together with a cocharacter of this torus, which is purely geometric. Brian will show how to define r_h in general, which will allow us to define an analogue of the Galois action without the moduli interpretation.

Now consider $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/E)$ and let $a \in \mathbf{A}_{E,f}^\times$ be the finite part of a representative for the class in the idèle class group that corresponds to $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/E)^{\mathrm{ab}} = \mathrm{Gal}(E^{\mathrm{ab}}/E)$. The map r_h allows us to rewrite the Galois action in terms of the double coset description of $\mathrm{Sh}_K(G,X^{\pm})$:

Proposition 3.2. *Let $[h,g]_K \in \mathrm{Sh}_K(G,X^{\pm})$ be a CM point, with σ and a as above. Then*

$$\sigma[h,g]_K = [h, r_h(a)g]_K.$$

Proof. Using the identifications in Section 2, $[h,g]_K$ corresponds to the triple (A,s,gK) that is described analytically as follows: $A^{\mathrm{an}} = V_{\mathbf{R}}/\Lambda$ with

$$V_{\mathbf{R}} = \mathrm{Lie}(A^{\mathrm{an}}), \Lambda = \mathrm{H}_1(A^{\mathrm{an}},\mathbf{Z}), V_f(A) = \Lambda \otimes_{\mathbf{Z}} \mathbf{A}_f,$$

and ηK corresponds to gK giving the identification between $V(\mathbf{A}_f)$ and $V_f(A)$. By definition, $\sigma[h,g]_K$ corresponds to $\sigma(A,s,\eta K) = (A^\sigma, s^\sigma, \eta^\sigma K) \in \mathcal{A}_K$. Let

$$V_\sigma = \mathrm{Lie}(A^\sigma(\mathbf{C})), \Lambda_\sigma = \mathrm{H}_1(A^\sigma(\mathbf{C}), \mathbf{Z}), \dots.$$

The main theorem of complex multiplication says that we can identify the uniformizations $V_{\mathbf{R}}$ and $\mathrm{Lie}(A^\sigma)$ so that the action of L given by the CM data is preserved. This identifies Λ_σ with $N_\Phi(1/a)\Lambda$, and the action of σ on torsion points (note $\Lambda_{\mathbf{Q}}/\Lambda = V/\Lambda$) corresponds to

$$N_\Phi(1/a) : V/\Lambda \to V/N_\Phi(1/a)\Lambda.$$

In particular, as $V_f(A)$ is an inverse limit of torsion points, and since $V = \Lambda_{\mathbf{Q}} = (N_\Phi(1/a)\Lambda)_{\mathbf{Q}}$, the map $V_f(A) \to V_f(A^\sigma)$ induces the map

$$N_\Phi(1/a) : \mathbf{A}_f \otimes_{\mathbf{Q}} V \to \mathbf{A}_f \otimes_{\mathbf{Q}} V$$

given by multiplication by $N_\Phi(1/a)$. Finally, the main theorem gives that s^σ is identified with a rational multiple of the L-linear $\mathbf{Q}$-polarization s. Therefore, the pair $(A^\sigma, s^\sigma, \eta K)$ is isomorphic to the pair $(A, s, N_\Phi(a)\eta K)$. Now $N_\Phi(a)$ is an element of $\mathbf{A}_{f,L}^\times$, and it acts on $V(\mathbf{A}_f)$ via the $L \otimes \mathbf{A}_f$-action given by the complex multiplication. Therefore $N_\Phi(a)\eta$ is, as an element of $G(\mathbf{A}_f)$ just $r_h(a)g$, so the double coset $[h, r_h(a)g]_K$ corresponds to $(A, s, N_\Phi(a)\eta K)$. □

4. The Canonical Model

We will now show that the tower $\mathrm{Sh}(\mathrm{GSp}_{2g}, X^\pm)$ can be descends to $\mathbf{Q}$ in a precise sense, by relating it to the moduli schemes $A_{g,d,n}$ over $\mathbf{Q}$. Keep in mind that a given $\mathbf{C}$-scheme or $\overline{\mathbf{Q}}$-scheme generally admits many non-isomorphic descents to a number field, even when considering moduli schemes:

Example 4.1. Consider the moduli problem for elliptic curves $E \to S$ over $\mathbf{Q}$-schemes equipped with an inclusion of finite étale S-groups $(\mathbf{Z}/N\mathbf{Z})_S \to E[N]$. There is a related problem using μ_N instead of $\mathbf{Z}/N\mathbf{Z}$. Over any field containing $\mathbf{Q}(\zeta_N)$, these are the same moduli problem and the problem is represented by $Y_1(N)$. However, over $\mathbf{Q}$ the moduli problems are different, and the associated moduli schemes are non-isomorphic *as descents* of the $\mathbf{Q}(\zeta_N)$-scheme $Y_1(N)$.

We can avoid this problem for $\mathrm{Sh}_K(\mathrm{GSp}_{2g}, X^\pm)$ by using the CM points, since they are dense:

Lemma 4.2. *Let $[h,g]_K$ be a CM point. The set $\{[h,g']_K : g' \in G(\mathbf{A}_f)\}$ is analytically dense in $\mathrm{Sh}_K(G, X^\pm)$. In particular, CM points are dense for the analytic topology, hence for the Zariski topology over $\mathbf{C}$, even those CM-points with a specified reflex field $E \subset \overline{\mathbf{Q}} \subset \mathbf{C}$.*

Proof. Note that $G(\mathbf{Q})$ is dense in $G(\mathbf{R})$: this is the real approximation theorem used in the last lecture, and can also be seen directly in the GSp_{2g} case (as for any split connected reductive $\mathbf{Q}$-group, via computations with SL_2's). Therefore, the *right* $G(\mathbf{Q})$-translates of a given $h \in X^\pm$ are dense for the (analytic) topology on $X^\pm$. Therefore for $h \in X^\pm$ that gives a complex structure leading to an abelian variety with complex multiplication, the points $[h,g]_K$ with varying $g \in G(\mathbf{A}_f)$ are dense in $\mathrm{Sh}_K(G, X^\pm)$ for the analytic topology. □

The next few results underlie how we will use CM points to remove the ambiguity in Galois descent.

Proposition 4.3. *Let K/k be an extension of algebraically closed fields. Let X and Y be k-schemes with X reduced and Y separated. Let $\Sigma \subset X(k)$ be a subset with Σ Zariski-dense in X_K. Any K-morphism $f \colon X_K \to Y_K$ that takes Σ to $Y(k)$ descends uniquely to a k-morphism $f_0 \colon X \to Y$.*

Proof. The uniqueness of the descent is immediate from the fact that $k \to K$ is faithfully flat. To prove existence, we will work with the closed subscheme $(1, f) : X_K \hookrightarrow X_K \times Y_K$ that is the graph of f; we denote it as Γ_0. (This is a closed immersion due to the separatedness.) Since k is algebraically closed, X_K is reduced.

Define $\Sigma_f = \{(x, f(x)) \in \Sigma \times Y(k) \,|\, x \in \Sigma\} \subset X(k) \times Y(k)$, and let Γ_0 be the Zariski closure of Σ_f in $X \times Y$. This is a reduced closed subscheme of $X \times Y$. We will show that $(\Gamma_0)_K = \Gamma$. Note that since the projection $\mathrm{pr}_1 : \Gamma \to X_K$ is an isomorphism, it would then follow by faithfully flat descent that $\mathrm{pr}_1 : \Gamma_0 \to X$ is an isomorphism, so its inverse composed with $\mathrm{pr}_2 : \Gamma_0 \to Y$ provides an $f_0 : X \to Y$ whose graph must coincide with Γ_0 and hence f_0 descends f.

Since taking Zariski closure commutes with field extension, $(\Gamma_0)_K$ is the Zariski closure in $X_K \times Y_K$ of the set of points $(x, f(x))$ viewed inside $X(K) \times Y(K)$. Thus, it visibly lies inside the reduced Γ as a reduced closed subscheme. To prove equality, it therefore suffices to show that $(\Gamma_0)_K$ contains a Zariski-dense set of K-points of Γ. But $\mathrm{pr}_1 : \Gamma \to X_K$ is an isomorphism that carries Γ_0 onto Σ that is dense in X_K by hypothesis. □

Corollary 4.4. *Let X be a reduced separated $\mathbf{C}$-scheme. If Σ is a Zariski-dense subset of $X(\mathbf{C})$ then there is at most one $\overline{\mathbf{Q}}$-descent X_0 of X such that $\Sigma \subset X_0(\overline{\mathbf{Q}})$ inside $X(\mathbf{C})$. Moreover, if Y is any separated $\overline{\mathbf{Q}}$-scheme and $f : X \to Y_{\mathbf{C}}$ is a $\mathbf{C}$-morphism carrying Σ into $Y(\overline{\mathbf{Q}})$ then f descends to a $\overline{\mathbf{Q}}$-morphism $X \to Y$.*

Proof. It suffices to prove the second assertion (since we can apply it to two descends to get a unique $\overline{\mathbf{Q}}$-isomorphism between them as descents). But this second assertion is a special case of the preceding proposition. □

Corollary 4.5. *There exists a unique $\overline{\mathbf{Q}}$-descent of $\mathrm{Sh}_K(\mathrm{GSp}_{2g}, X^{\pm})$ for which the CM-points are $\overline{\mathbf{Q}}$-points.*

Proof. The uniqueness follows from the Zariski-density of CM-points in $\mathrm{Sh}_K(\mathrm{GSp}_{2g}, X^{\pm})$. For the existence first consider the special case when $K = K_n$ with $n \geq 3$.

In this case, Arnav explained in his talk that the analytic space $\mathrm{Sh}_{K_n}(\mathrm{GSp}_{2g}, X^{\pm})^{\mathrm{an}}$ is the analytification of the $\mathbf{C}$-fiber of the moduli scheme $A_{g,1,n}$, so $(A_{g,1,n})_{\overline{\mathbf{Q}}}$ provides a $\overline{\mathbf{Q}}$-descent. Moreover, since CM abelian varieties over $\mathbf{C}$ are always defined over $\overline{\mathbf{Q}}$, in this $\overline{\mathbf{Q}}$-descent all CM-points are $\overline{\mathbf{Q}}$-points. This solves the problem for $K = K_n$ with $n \geq 3$.

In general, $\mathrm{GSp}_{2g}(\widehat{\mathbf{Z}})$ is a maximal compact subgroup of $\mathrm{GSp}_{2g}(\mathbf{A}_f)$ (as for any split connected reductive $\mathbf{Q}$-group in place of GSp_{2g}, by general facts from Bruhat–Tits theory[1]). Right multiplication by $g \in \mathrm{GSp}_{2g}(\mathbf{A}_f)$ gives an isomorphism between $\mathrm{Sh}_K(\mathrm{GSp}_{2g}, X^{\pm})$ and $\mathrm{Sh}_{g^{-1}Kg}(\mathrm{GSp}_{2g}, X^{\pm})$, so by conjugating we may assume $K \subset \mathrm{GSp}_{2g}(\widehat{\mathbf{Z}})$ and hence K contains K_n for sufficiently divisible $n \geq 3$. By Baily–Borel, the natural analytic action of the finite group K/K_n on $\mathrm{Sh}_{K_n}(\mathrm{GSp}_{2g}, X^{\pm})^{\mathrm{an}}$ arises from an algebraic action of K/K_n on $\mathrm{Sh}_{K_n}(\mathrm{GSp}_{2g}, X^{\pm})$, and this action visibly carries CM-points to CM-points. Since CM-points are a Zariski-dense set of $\overline{\mathbf{Q}}$-points, by Proposition 4.3 this action respects the quasi-projective $\overline{\mathbf{Q}}$-structure. The quotient by this finite group action exists over $\overline{\mathbf{Q}}$ by quasi-projectivity, and the formation of finite group quotients for quasi-projective schemes commutes with ground field extension and with analytification. Hence, we get the existence of the desired $\overline{\mathbf{Q}}$-structure for any K. □

Our remaining problem is to descend the $\overline{\mathbf{Q}}$-structures to specific number fields (dictated by K). For this, we require a variant of Proposition 4.3 adapted to the setting of descent through Galois extensions:

[1] In particular, for k a non-archimedean local field and G a smooth affine $\mathcal{O}_k$-group with split connected reductive fibers, the subgroup $G(\mathcal{O}_k)$ inside $G(k)$ is a maximal compact subgroup.

Proposition 4.6. *Let K/k be a Galois extension of fields, and X a reduced separated finite type K-scheme of finite type. Let $\Sigma \subset X(K)$ be a Zariski-dense subset of K-rational points in X. Suppose there is given an action of* $\mathrm{Gal}(K/k)$ *on Σ. There is up to unique isomorphism at most one k-descent $(X_0, (X_0)_K \simeq X)$ of X relative to which the* $\mathrm{Gal}(K/k)$*-action on $X(K)$ induces the given action on Σ. More generally, if (X', Σ') is another such pair and also admits a descent X'_0 to k then any map $f : X \to X'$ carrying Σ into Σ' and respecting the natural* $\mathrm{Gal}(K/k)$*-actions on Σ and Σ' descends to a k-morphism $f_0 : X_0 \to X'_0$.*

Proof. By descent theory, f_0 is unique if it exists and X_0 is separated and finite type over k (since X is over K). The map f descends to *some* finite Galois extension of k since X_0 is finite type over k, so $f^\gamma = f$ for all γ in an open normal subgroup of $\mathrm{Gal}(K/k)$. Thus, by Galois descent for morphisms, the existence of f_0 is equivalent to the condition $f^\gamma = f$ for all $\gamma \in \mathrm{Gal}(K/k)$. But a graph argument shows that such an equality for K-morphisms $X \rightrightarrows X'$ is equivalent to equality on a Zariski-dense set of K-points of the reduced X (as separatedness of X' ensures closedness of the graphs of such maps). We compare f^γ and f on Σ: for any $x \in \Sigma$ we have

$$f^\gamma(x) = \gamma(f(\gamma^{-1}(x))) = \gamma(\gamma^{-1}(f(x))) = f(x),$$

where the second equality uses the Galois-equivariance of $f|_\Sigma : \Sigma \to \Sigma'$ (since this restricted Galois action has been specified at the outset, and is part of the conditions on the k-descents X_0 of X and X'_0 of X'). □

A *model* M_K for $\mathrm{Sh}_K(\mathrm{GSp}, X^\pm)$ over a number field $F \subset \overline{\mathbf{Q}}$ is an F-descent of the $\overline{\mathbf{Q}}$-structure that we have built above. But we don't want a random model; rather, we want one for which the resulting $\mathrm{Gal}(\overline{\mathbf{Q}}/E)$-action on the set of $\overline{\mathbf{Q}}$-points is given on the Zariski-dense set of CM-points by the recipe in Proposition 3.2. To be precise:

Definition 4.7. A model M_K over a number field $F \subset \overline{\mathbf{Q}}$ is a *canonical model relative to* F if for any CM point $[h, g]_K$ with CM by L and reflex field E, and any $\sigma \in \mathrm{Gal}(FE^{\mathrm{ab}}/FE)$ corresponding to $a \in \mathbf{A}_E^\times$, the $\overline{\mathbf{Q}}$-point $([h, g]_K)$ of $\mathrm{Sh}_K(\mathrm{GSp}, X^\pm)$ is a $F \cdot E^{\mathrm{ab}}$-point of the F-scheme M_K, and in $M_K(FE^{\mathrm{ab}})$ we have $\sigma[h, g]_K = [h, r_h(a)g]_K$.

Remark 4.8. This is phrased in terms of r_h because the description of the Galois action in terms of abelian varieties is not available for other Shimura varieties, while this condition (with an appropriate definition of r_h) will work in the general case.

Since the CM-points with a given reflex field are Zariski-dense (see Lemma 4.2), Proposition 4.6 ensures that there is at most one such canonical model (up to unique isomorphism) relative to a given number field F. For each $\mathrm{Sh}_K(\mathrm{GSp}_{2g}, X^\pm)$ we aim to prove the existence and uniqueness of a canonical model relative to a specific number field (determined by K), but we want to do more: we want to shown that in a precise sense the entire tower $\mathrm{Sh}(\mathrm{GSp}_{2g}, X^\pm)$ is *defined over* $\mathbf{Q}$.

We first prove a lemma related to varying K.

Lemma 4.9. *Let K_m be a finite index subgroup of K, and let $Q = K/K_m$ be the quotient. Let M be a canonical model for* $\mathrm{Sh}_{K_m}(\mathrm{GSp}_{2g}, X^\pm)$ *defined over* $\mathbf{Q}$*. Then Q acts on M, and the quotient M/Q is a canonical model for* $\mathrm{Sh}_K(\mathrm{GSp}_{2g}, X^\pm)$.

Proof. The action of the finite group Q on $\mathrm{Sh}_{K_m}(\mathrm{GSp}_{2g}, X^\pm)$ respects the $\overline{\mathbf{Q}}$-structure since it carries CM-points to CM-points, and it also respects the $\mathbf{Q}$-structure coming from M. Indeed, by Proposition 4.6 it suffices to check that this action respects the Galois action on CM-points. This compatibility is a direct calculation: given a CM point $[h, g]_{K_m} \in \mathrm{Sh}_{K_m}(\mathrm{GSp}_{2g}, X^\pm)$ with CM by L, $\sigma \in \mathrm{Gal}(E^{\mathrm{ab}}/E)$ corresponding to $a \in \mathbf{A}_{E,f}^\times$, and $[g'] \in Q$, σ sends $[h, g]_{K_m}$ to $[h, r_h(a)g]_{K_m}$. This element is sent to $[h, r_h(a)gg']_{K_m}$ by the action of Q. On the other hand, the group action sends $[h, g]_{K_m}$ to $[h, gg']_{K_m}$ which σ sends to $[h, r_h(a)gg']_{K_m}$. This shows the group action is Galois equivariant, so Q acts (obviously freely) on M. Therefore we can pass to the quotient by this action on the quasi-projective $\mathbf{Q}$-structure to get a $\mathbf{Q}$-descent of $\mathrm{Sh}_K(\mathrm{GSp}_{2g}, X^{\pm})$: call this descent M'. The fact that M' is a canonical model will follow from that fact that M is one. Now $(M')_{\overline{\mathbf{Q}}}$ is the quotient of $\mathrm{Sh}_{K_n}(\mathrm{GSp}_{2g}, X^{\pm})$ by Q. In particular, the $\overline{\mathbf{Q}}$ points M' are the Q orbits of the $\overline{\mathbf{Q}}$ points of $\mathrm{Sh}_{K_m}(\mathrm{GSp}_{2g}, X^{\pm})$. The description of the Galois action on the CM points of $\mathrm{Sh}_{K_m}(\mathrm{GSp}, X^{\pm})$ therefore shows that M' is a canonical model for $\mathrm{Sh}_K(\mathrm{GSp}_{2g}, X^{\pm})$. □

The key example from which everything else follows is $K = K_n$ for $n \geq 3$, in which case we will make a model over $F = \mathbf{Q}$.

Theorem 4.10. *For any compact open $K \subset \mathrm{GSp}_{2g}(\mathbf{A}_f)$, $\mathrm{Sh}_{K_n}(\mathrm{GSp}_{2g}, X^{\pm})$ admits a canonical model over $\mathbf{Q}$.*

Proof. As in Corollary 4.5, after conjugation we may pick a finite index subgroup $K_n \subset K$ with $n \geq 3$. Observe the isomorphism of Shimura varieties preserves the property of being canonical as it is given by right multiplication. The previous lemma reduces the theorem to producing a canonical model for K_n.

Our problem is to find a $\mathbf{Q}$-descent of the given $\overline{\mathbf{Q}}$-structure relative to which any CM-point with reflex field $E \subset \overline{\mathbf{Q}}$ is an E^{ab}-point on which $\mathrm{Gal}(E^{\mathrm{ab}}/E)$ acts in accordance with the desired adelic recipe. The fine moduli scheme $A_{g,1,n}$ over $\mathbf{Q}$ provides what we need. Recall the $\overline{\mathbf{Q}}$ stucture on $\mathrm{Sh}_{K_n}(\mathrm{GSp}_{2g}, X^{\pm})$ comes from $(A_{g,1,n})_{\overline{\mathbf{Q}}}$, which represented the functor of abelian varieties of dimension g with degree 1 polarization and full level n structure. The action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on $A_{g,1,n}(\overline{\mathbf{Q}})$ matches the obvious action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on this moduli space description. But Proposition 3.2 shows that this Galois action is given, on CM Abelian varieties, by the formula $\sigma[h, g]_{K_n} = [h, r_h(a)g]_{K_n}$. Therefore $A_{g,1,n}$ is a canonical model, defined over $\mathbf{Q}$, for $\mathrm{Sh}_{K_n}(\mathrm{GSp}_{2g}, X^{\pm})$. □

Finally, we define what it means for $\mathrm{Sh}(\mathrm{GSp}_{2g}, X^{\pm})$ to have a canonical model.

Definition 4.11. A canonical model over a number field N for the Shimura variety $\mathrm{Sh}(\mathrm{GSp}_{2g}, X^{\pm})$ is a collection $\{M_K\}$ of canonical models for $\mathrm{Sh}_K(\mathrm{GSp}_{2g}, X^{\pm})$ where K runs over compact open subgroups of $\mathrm{GSp}_{2g}(\mathbf{A}_f)$ along with maps $M_K \to M_{K'}$ defined over N for $K' \subset K$ that descend the natural quotients $\mathrm{Sh}_K(\mathrm{GSp}_{2g}, X^{\pm}) \to \mathrm{Sh}_{K'}(\mathrm{GSp}_{2g}, X^{\pm})$.

Remark 4.12. As Sam discussed, $\mathrm{Sh}(G, X)$ does not always have a nice adelic description, so we define canonical models in terms of the inverse system to allow generalization.

The work we did above gave canonical models over $\mathbf{Q}$. The proof of the lemma also shows the quotient maps descend to maps of the canonical models defined over $\mathbf{Q}$. Therefore $\mathrm{Sh}(\mathrm{GSp}_{2g}, X^{\pm})$ has a canonical model over $\mathbf{Q}$.

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