HERMITIAN SYMMETRIC DOMAINS

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1. Introduction

Warning: these are rough notes based on the two lectures in the Shimura varieties seminar. After the unitary Hermitian symmetric spaces example, the lectures closely followed Brian Conrad's notes on "Complex structures and Shimura data" available on the seminar website so the reader will hopefully forgive the lack of details in places.

In an earlier lecture, we defined a *Hermitian symmetric domain* $\mathcal{U} \subset \mathbb{C}^N$ to be an open connnected bounded subset such that for every $x \in \mathcal{U}$, there exists a holomorphic involution $r_x : \mathcal{U} \to \mathcal{U}$ such that x is an isolated fixed point of r_x. It is a theorem then that all Hermitian symmetric domains are isomorphic (as real manifolds) to G/K, where G is a semi-simple Lie group and K is a maximal compact subgroup of G. In particular, the holomorphic automorphism group $\mathrm{Hol}(\mathcal{U})$ acts transitively and the stabilizer of any point is a maximal compact subgroup.

Instead of focusing on this theorem, in these notes, we take the opposite approach and ask the question, given a semi-simple algebraic group $\mathbf{G}$ over $\mathbb{R}$ and K a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$, when does $\mathbf{G}(\mathbb{R})/K$ have complex analytic structure for which the translation action is holomorphic? This will be used to motivate the axioms for a Shimura datum. We will also discuss how to build the Hermitian symmetric domain $\mathbf{G}(\mathbb{R})/K$ in a way that is internal to the group.

2. Unitary Shimura varieties

We begin with the example of unitary groups. By hand (in detail), we will make the translation from Hermitian symmetric domain to "moduli" of Hodge structures to Shimura data.

Let $V = \mathbb{C}^p$ and $W = \mathbb{C}^q$ equipped with the standard positive definite Hermitian inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$ respectively. Define a Hermitian form on $V \times W$ by

$$\langle (v_1, w_1), (v_2, w_2) \rangle = \langle v_1, v_2 \rangle_V - \langle w_1, w_2 \rangle_W$$

for any $v_1, v_2 \in V$ and $w_1, w_2 \in W$.

Definition 2.1. The *unitary group* $\mathrm{U}(p,q)$ *of type* (p,q) is an algebraic group over $\mathbb{R}$ such that $\mathrm{U}(p,q)(\mathbb{R})$ is the subgroup of $\mathrm{GL}_{p+q}(\mathbb{C})$ which respects $\langle \cdot, \cdot \rangle$. We define the unitary similitude group $\mathrm{GU}(p,q)(\mathbb{R})$ to be the subgroup of $\mathrm{GL}_{p+q}(\mathbb{C})$ which respects $\langle \cdot, \cdot \rangle$ up to a scaling factor.

For now, we won't be using the algebraic structure of $\mathrm{U}(p,q)$ just its structure as a Lie group so we will just write $\mathrm{U}(p,q)(\mathbb{R})$ as $\mathrm{U}(p,q)$.

It will be convenient to think of $M \in \mathrm{U}(p,q) \subset \mathrm{GL}_{p+q}(\mathbb{C})$ in block diagonal form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

so that

$$M.(v,w) = (Av + Bw, Cv + Dw)$$

where A is $p \times p$ matrix, B is $p \times q$ matrix, C is $q \times p$ matrix and D is $q \times q$ matrix.

Now, consider the complex vector space $\mathrm{Hom}(W,V)$. For any $Z \in \mathrm{Hom}(W,V)$, the operator norm of Z is

$$||Z|| := \sup_{w \in W, w \neq 0} \frac{||Zw||_V}{||w||_W}.$$

Then, define $X := \{Z \in \mathrm{Hom}(W,V) \mid ||Z|| < 1\}$. It is a bounded open subset of $\mathrm{Hom}(W,V)$. In fact, it is the Hermitian symmetric domain for the group $\mathrm{U}(p,q)$ as we will see shortly.

Proposition 2.2. *The group $\mathrm{U}(p,q)$ acts transitively on X through holomorphic automorphisms by the formula*

$$Z \rightsquigarrow (AZ + B)(CZ + D)^{-1}$$

for any $M \in \mathrm{U}(p,q)$.

Proof. Once we show that the action is well-defined the holomorphicity is clear and the transitivity is left as an exercise for the reader.

We first claim that $(CZ + D)$ is invertible for any $Z \in X$. Assume it wasn't. Then, there would exist a nonzero $w \in W$ with $CZw + Dw = 0$. Consider the formula

$$M.(Zw, w) = (AZw + Bw, CZw + Dw). \tag{1}$$

Let $Q((v,w)) = \langle (v,w), (v,w) \rangle$ denote the diagonal form on $V \times W$ defined by $\langle \cdot, \cdot \rangle$. If $CZw + Dw = 0$, then

$$Q(M.(Zw,w)) = ||AZw + Bw||_V^2 \geq 0.$$

On the other hand, $M \in \mathrm{U}(p,q)$ preserves the Hermitian form so

$$Q(M.(Zw,w)) = Q((Zw,w)) = ||Zw||_V^2 - ||w||_W^2 < 0$$

since the operator norm of Z is less than 1. This is a contradiction.

If we define $M(Z) := (AZ+B)(CZ+D)^{-1}$ for $M \in \mathrm{U}(p,q)$ and $Z \in X$, then we claim that $M(Z) \in X$. We leave it to the reader to check that this follows from the fact for any $w \in W$

$$(M(Z).w, w) = M.(Zy, y) \tag{2}$$

where $y = (CZ+D)^{-1}(w)$. □

Equation (2) suggest another possible description of X. Define

$$X' := \{W' \subset V \times W \mid \dim_{\mathbb{C}} W' = q, \langle \cdot, \cdot \rangle|_{W'} \text{ is negative definite}\}.$$

This is an open subset of the Grassmanian of q-planes in the $(p+q)$-dimensional $V \times W$. Note that $\mathrm{U}(p,q)$ acts on X' via its action of $V \times W$.

Proposition 2.3. *The map $\Psi : X \to X'$ which sends Z to the graph Γ_Z of Z is an isomorphism of complex manifolds which is furthermore equivariant for the action of* $\mathrm{U}(p,q)$.

Proof. The map is well-defined because for any $Z \in X$ and $w \in W$ with $w \neq 0$,

$$\langle ((Zw, w), (Zw, w)) \rangle = ||Zw||_V^2 - ||w||_W^2 < 0$$

so that the form restricted to Γ_Z is negative definite.

Given any $W' \in X'$, i.e., $W' \subset V \times W$ the projection $\pi : W' \to W$ is an isomorphism (anything in the kernel would lie in V violating negative definiteness). If we let $\mathrm{pr}_V : V \times W \to V$ be the projection to V, then $\mathrm{pr}_V \circ \pi^{-1} \in \mathrm{Hom}(V, W)$. The map $W' \mapsto \mathrm{pr}_V \circ \pi^{-1}$ is inverse to Ψ.

The fact that Ψ is equivariant the $\mathrm{U}(p,q)$ actions can be deduced from equation (2). □

It is general phenomenon that Hermitian symmetric domains can be embedded in complex flag varieties. Later, we will use such an embedding to prove that an almost complex structure on G/K is, in fact, integrable.

Next, we discuss a third variant on the unitary Hermitian symmetric space. This one will be "intrinsic" to the group $\mathrm{U}(p,q)$, in the sense that the final description will make no reference to V or W.

Let $\mathbb{C}^*$ be the multiplicative group of $\mathbb{C}$ (really as an algebraic group over $\mathbb{R}$ but we won't need that here). Define a $\mathbb{C}^*$-action h on $V \times W$ by $h(a).(v, w) = (av, \bar{a}w)$ for any $a \in \mathbb{C}^*$. Note that the $\mathbb{R}$-action is just the ordinary scaling action. Furthermore,

$$\langle h(a).(v_1, w_1), h(a).(v_2, w_2) \rangle = a\bar{a} \langle (v_1, w_1), (v_2, w_2) \rangle$$

so that $h(a) \in \mathrm{GU}(p,q)$. Thus, h defines a homomorphism $h : \mathbb{C}^* \to \mathrm{GU}(p,q)$.

Let X_h be the $\mathrm{GU}(p,q)$-conjugacy class of h. Being the quotient of $\mathrm{GU}(p,q)$ by the stabilizer Z_h of h, X_h is a real manifold.

Proposition 2.4. *There is a natural bijection $X_h \to X'$ (even diffeomorphism) given by*

$$h' \rightsquigarrow W_{h'} := \{x \in V \times W \mid h'(i).x = -ix\}.$$

Proof. For surjectivity, note that $h \mapsto W$ and that map is $\mathrm{U}(p,q)$-equivariant which suffices since $\mathrm{U}(p,q)$ acts transitively on X'.

For any $h' \in X_h$, $h'(\mathbb{R})$ acts by scaling so any two homomorphisms $h_1, h_2 \in X_h$ are the same if and only if $h_1(i) = h_2(i)$. If $W_{h_1} = W_{h_2}$, then $h_1(i)$ and $h_2(i)$ agree on W_{h_1} as well as on $W_{h_1}^{\perp}$. Since $\langle \cdot, \cdot \rangle$ is non-degenerate on $W_{h_1} \in X'$, the span of W_{h_1} and $W_{h_1}^{\perp}$ is all of $V \times W$. □

Since the only difference between $\mathrm{GU}(p,q)$ and $\mathrm{U}(p,q)$ is the center, we can just as well consider X_h as the $\mathrm{U}(p,q)$ orbit of h under conjugation.

Proposition 2.5. *The stablizer of h in $\mathrm{U}(p,q)$ is $\mathrm{U}(p) \times \mathrm{U}(q)$ which is a maximal compact of $\mathrm{U}(p,q)$.*

Proof. Any element $g \in \mathrm{GL}(V \times W)$ which commutes with h preserves both W (the $-i$-eigenspace of $h(i)$) and V (the i-eigenspace of $h(i)$) so is an element of $\mathrm{GL}(V) \times \mathrm{GL}(W)$. If g also preserves the Hermitian form, then $g \in \mathrm{U}(p) \times \mathrm{U}(q)$, and $\mathrm{U}(p) \times \mathrm{U}(q)$ is a maximal compact of $\mathrm{U}(p,q)$. □

Our "intrinsic" description X_h of the Hermitian symmectric domain X is deficient in one key respect. We only described the real structure on X_h (as a quotient of $\mathrm{U}(p,q)$). So far, the only way to describe the complex structure is via the relationship to either X or X'. In the remainder of the lecture, we will "identify" the complex structure on X_h.

Let $K := \mathrm{U}(p) \times \mathrm{U}(q)$ and let $\mathfrak{g} := \mathrm{Lie}(\mathrm{U}(p,q))$. We can identify the real tangent space of X_h at h with $\mathfrak{g}/\mathrm{Lie}(K)$. Consider the action of $\mathrm{ad}(h(i))$ on $\mathfrak{g}$. Since $h(-1)$ is in the center of $\mathrm{U}(p,q)$, $\mathrm{ad}(h(i))$ is an involution so we can decompose

$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-.$$

Since K is the centralizer of h, we deduce that $\mathrm{Lie}(K) = \mathfrak{g}^+$ so that $\mathfrak{g}^-$ maps isomorphically onto $T_h(X_h)$. Thus, $J_h := \mathrm{ad}(h(e^{\frac{\pi i}{4}}))$ defines a complex structure on $T_h(X_h)$.

This complex structure can be transported around to all of X_h using translation by $\mathrm{U}(p,q)$ since J_h is preserved by stabilizer K of h. Explicitly, multiplication by i on $T_{h'}(X_h)$ is given by $\mathrm{ad}(h'(e^{\frac{\pi i}{4}}))$. This defines an almost complex structure on X_h. We will now show that this is almost complex structure is exactly the complex structure defined by identifying X_h with the Hermitian symmetric domain X.

Proposition 2.6. *With respect to the isomorphism $T_h(X_h) \cong T_0(\mathrm{Hom}(W,V))$ of real vector spaces induced by the diffeomorphism $X_h \cong X$, the endomorphism J_h corresponds to multiplication by i.*

Proof. For any $a \in \mathbb{C}^*$, the conjugation action of $h(a)$ on $\mathrm{U}(p,q)$ descends to an automorphism of $\mathrm{U}(p,q)/K$ since $h(a)$ centralizes K. The diffeomorphism $X_h \cong X$ is $\mathrm{U}(p,q)$-equivariant so we just have to compute the derivative at 0 of $\mathrm{ad}(h(a))$ acting on X.

I leave it as an exercise to the reader that for any $Z \in X$, there exists an $A \in \mathrm{Mat}_{p\times p}(\mathbb{C})$ and $C \in \mathrm{Mat}_{q\times p}(\mathbb{C})$ such that

$$M = \begin{pmatrix} A & Z \\ C & I \end{pmatrix} \in \mathrm{U}(p,q)$$

where I is the identity matrix. Since $M([0]) = Z$, we have that $\mathrm{ad}(h(a)).Z = (\mathrm{ad}(h(a))(M))(Z)$ and one computes that

$$\mathrm{ad}(h(a))(M) = \begin{pmatrix} A & a\overline{a}^{-1}Z \\ a^{-1}\overline{a}C & I \end{pmatrix}.$$

Thus, $\mathrm{ad}(h(a))(Z) = a\overline{a}^{-1}Z$, and when $a = e^{\frac{\pi i}{4}}$, then $a\overline{a}^{-1} = i$. □

One remark on the proof of Proposition 2.6 is that the formula $a\overline{a}^{-1}$ which appears is related in a more general context to a Hodge decomposition (with type $(-1,1)$ playing an important role).

3. Axioms for Shimura data

In this lecture, we start with a reductive algebraic group $\mathbf{G}$ over $\mathbb{R}$ and an "algebraic" cocharacter $h : \mathbb{C}^* \to \mathbf{G}(\mathbb{R})$. As in our unitary example, we will consider the $\mathbf{G}(\mathbb{R})$-conjugacy class of X which is a priori a real manifold. We will discuss the condition under which we can give X a complex structure for which $\mathbf{G}(\mathbb{R})$ acts by holomorphic automorphisms. There is considerable overlap between these notes and Brian's handout on Shimura data available on the seminar webpage. We will occasionally refer to this handout for details.

Definition 3.1. A Lie group homomorphism $h : \mathbb{C}^* \to \mathbf{G}(\mathbb{R})$ is *algebraic* if the restriction to $\mathbb{R}^*$ is induced by an algebraic map $\mathbb{G}_m \to \mathbf{G}$. Equivalently, h is induced by an algebraic homomorphism from the Weil restriction $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ to $\mathbf{G}$.

For any representation $\rho : \mathbf{G} \to \mathrm{GL}(V)$ and any algebraic homomorphism h valued in $\mathbf{G}$, the composition $\rho \circ h$ restricted to $\mathbb{R}^*$ defines a grading

$$V = \oplus_{n\in\mathbb{Z}} V_{n,h},$$

such that $v \in V_{n,h}$ if and only if $\rho(h(t)).v = t^n v$ for all $t \in \mathbb{R}^*$. The $V_{n,h}$ are called *weight spaces*.

Fix an algebraic homorphism $h : \mathbb{C}^* \to \mathbf{G}(\mathbb{R})$ and let X denote the $\mathbf{G}(\mathbb{R})$ conjugacy class of h. If Z_h is the centralizer of h (a closed subgroup of $\mathbf{G}(\mathbb{R})$) then the orbit map identifies X with

$\mathbf{G}(\mathbb{R})/Z_h$. We give X the structure of a real-analytic manifold via this identification. Under certain conditions, we would like to put a "natural" complex structure on X.

Remark 3.2. The motivation for the "conditions" (Shimura data axioms) comes from thinking of X as parametrixing families of Hodge structures. We will be vague about this to avoid an unnecessary digression into Tannakian formalism. Given a representation (V, ρ) of $\mathbf{G}$ and any $h' \in X$, we note that $\rho \circ h'$ defines a real Hodge structure on V, that is,

$$V \otimes_{\mathbb{R}} \mathbb{C} = \oplus_{p,q} V^{p,q},$$

where $V^{p,q} := \{v \in V_{\mathbb{C}} \mid \rho(h'(a)).v = a^{-p}\overline{a}^{-q}v\}$.

Proposition 3.3. *All of the following are equivalent:*

1. *For all representations (V, ρ) of $\mathbf{G}$, the weight decomposition of V induced by $h' \in X$ is independent of the h';*
2. *For any $h' \in X$, $h'(\mathbb{R}^*)$ is in the center of $\mathbf{G}(\mathbb{R})$;*
3. *For any $h' \in X$, the real Hodge structure on* $\mathrm{Lie}(\mathbf{G})$ *is pure of weight 0.*

Proof. See Lemma 5.1 in Brian's handout. □

Assume X satisfies the properties in Proposition 3.3, then for any (V, ρ), the weight spaces $V_{n,h'}$ are independent of h' and so we have trivial complex vector bundles

$$X \times (V_n)_{\mathbb{C}} \to X$$

for each V and n. Furthermore, for any $h' \in X$, the Hodge structure on $V_{\mathbb{C}}$ induced by $\rho \circ h'$ defines a filtration $F^p_{n,h'} \subset (V_n)_{\mathbb{C}}$ indexed by p. As h' varies, this defines a subbundle $\mathcal{F}^p_n \subset X \times (V_n)_{\mathbb{C}}$.

Whatever complex structure we put on X, we want $\mathcal{F}^p_n$ to be a holomorphic subbundle. Furthermore (and this motivates Axiom I), we want the $\mathcal{F}^p_n$ to satisfy Griffiths transversality with respect to the trivial connection on $X \times (V_n)_{\mathbb{C}} \to X$. Griffiths transversality holds for an algebraic family of Hodge structures, and so Axiom I is motivated by the fact that we expect to X to be an algebraic variety with a "universal family of Hodge structures" (see Proposition 5.3 in Brian's handout for a more precise statement).

The full list of axioms on $(\mathbf{G}, X)$ to be a Shimura datum is given in Definition 3.3 of Brian's seminar notes on real algebraic groups from this seminar or Definition 5.13 of Brian's handout. We focus on Axiom I. For a full treatment of Cartan involutions and Axiom II see Brian's seminar notes on real algebraic groups. Recall that the type of a Hodge structure $V_{\mathbb{C}} = \oplus_{p,q} V^{p,q}$ is the set of pairs (p, q) for which $V^{p,q}$ is non-empty.

(Axiom I) The Hodge structure on $\mathrm{Lie}(\mathbf{G})_{\mathbb{C}}$ given by $\mathrm{ad}\circ h'$ for any $h' \in X$ (equivalently for one $h \in X$) is of type $\{(-1,1),(0,0),(1,-1)\}$.

Assume $(\mathbf{G}, X)$ satisfies Axiom I. An immediate consequence of Axiom I is that $\mathrm{ad}(h(\mathbb{R}^*))$ is trivial and so X satisfies the properties of Proposition 3.3. For simplicity, we assume in our arguments that X is connected. It is both possible and relevant that X may be disconnected, but then everything can be done one component at a time. We save space and notation by making this assumption.

For any representation V of $\mathbf{G}$, by the discussion after Proposition 3.3 we have a real-analytic map

$$X \xrightarrow{\phi} \prod_n \mathrm{Flag}(V_{n,\mathbb{C}})$$

where $\mathrm{Flag}(V_{n,\mathbb{C}})$ is an appropriate flag variety which depends on the ranks of the bundles $\mathcal{F}_n^p$. Note that if X has a complex structure, then the $\{\mathcal{F}_n^p\}$ are holomorphic subbundles exactly when ϕ is a holomorphic map.

Let $\mathfrak{g} := \mathrm{Lie}(\mathbf{G})$. Denote the Hodge structure on $\mathfrak{g}$ induced by $\mathrm{ad}\circ h$ by $\{\mathfrak{g}_{\mathbb{C}}^{p,q}\}$. This will play an important role in what follows.

Lemma 3.4. *There is a natural isomorphism $T_h(X) \cong \mathfrak{g}/\mathfrak{g}^{0,0}$ where $\mathfrak{g}^{0,0}$ is the real descent of $\mathfrak{g}_{\mathbb{C}}^{0,0}$.*

Proof. Using the isomorphism, $\mathbf{G}(\mathbb{R})/Z_h \cong X$, we can identify $T_h(X)$ with $\mathfrak{g}/\mathrm{Lie}(Z_h)$. By functorialities with Lie algebras, $\mathrm{Lie}\, Z_h = \mathfrak{g}^{\mathrm{ad}h=1}$. The real descent $\mathfrak{g}^{0,0}$ is the real subspace of $\mathfrak{g}$ such that $(\mathfrak{g}^{0,0})_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{0,0}$ (which exists because $\mathfrak{g}_{\mathbb{C}}^{0,0} = \overline{\mathfrak{g}_{\mathbb{C}}^{0,0}}$. It is an easy exercise to check that $\mathfrak{g}^{0,0}$ is exactly the fixed points of $\mathrm{ad}\circ h$. □

Lemma 3.5. *If (V,ρ) is faithful, then both ϕ and $d(\phi)$ are injective.*

Proof. Choose $h, h' \in X$. We claim that if $\phi(h) = \phi(h')$ then $\rho(h) = \rho(h')$ and hence $h = h'$. For any $a \in \mathbb{C}$, the $\mathbb{C}$-linear extensions of $\rho(h(a))$ and $\rho(h(a'))$ on $V_{\mathbb{C}}$ are determined by the decomposition $V = \oplus_{p,q} V_{p,q}$ which is determined by $\phi(h)$ and $\phi(h')$.

Define a filtration on $\mathrm{End}_{\mathbb{C}}(V_{n,\mathbb{C}})$ by

$$F^i(\mathrm{End}_{\mathbb{C}}(V_{n,\mathbb{C}})) := \{A \in \mathrm{End}_{\mathbb{C}}(V_{n,\mathbb{C}}) \mid A(F_{n,h}^a(V_{n,\mathbb{C}}) \subset F_{n,h}^{a+i}(V_{n,\mathbb{C}})\}.$$

It is standard fact that $T_{\phi(h)}(\prod_n \mathrm{Flag}(V_{n,\mathbb{C}})) \cong \prod_n \mathrm{End}(V_{n,\mathbb{C}})/F^0(\mathrm{End}(V_{n,\mathbb{C}}))$. One can check furthermore that

$$d(\phi) : T_h(X) \to T_{\phi(h)}(\prod_n \mathrm{Flag}(V_{n,\mathbb{C}}))$$

is induced by the natural map $\mathfrak{g} \hookrightarrow \mathrm{End}(V) \subset \mathrm{End}(V_{\mathbb{C}})$. Injectivity of $d(\phi)$ is thus equivalent to the following equality:

$$\mathfrak{g} \cap F_h^0(\mathfrak{g}_{\mathbb{C}}) = \mathfrak{g}^{0,0} \tag{3}$$

using that the map $\mathfrak{g}_{\mathbb{C}} \hookrightarrow \mathrm{End}(V_{\mathbb{C}})$ respects the Hodge decompositions (and hence is a strict map of filtrations).

Now, $F_h^0(\mathfrak{g}_{\mathbb{C}}) = \oplus_{p \geq 0} \mathfrak{g}_{\mathbb{C}}^{p,-p}$ (using only the assumption that $\mathfrak{g}$ is pure of weight 0). Since $\mathfrak{g}_{\mathbb{C}}^{p,-p}$ is complex conjugate to $\mathfrak{g}_{\mathbb{C}}^{-p,p}$, only $\mathfrak{g}_{\mathbb{C}}^{0,0}$ intersects the underlying real vector space $\mathfrak{g}$. This proves (3). $\square$

Remark 3.6. Given that a complex structure on X exists for which the $\mathcal{F}_n^p$ are holomorphic subbundles (for a single faithful V), then the complex structure is uniquely determined by this property by injectivity of $d(\phi)$. In Theorem 3.7, we will construct a complex structure from a related embedding.

Griffiths transversality for the subbundles $\mathcal{F}_{n,h}^p \subset X \times (V_{n,\mathbb{C}})$ is the condition that the trivial connnection

$$\nabla : (V_{n,\mathbb{C}}) \otimes_{\mathbb{C}} \mathcal{O}_X \xrightarrow{1 \otimes d} (V_{n,\mathbb{C}}) \otimes_{\mathbb{C}} \Omega_X^1$$

sends $\mathcal{F}_{n,h}^p$ into $\mathcal{F}_{n,h}^{p-1} \otimes_{\mathcal{O}_X} \Omega_X^1$. The connection ∇ is equivalent to a map from the tangent bundle $\mathrm{Tan}(X)$ to $\mathrm{End}(V_{n,\mathbb{C}}) \otimes_{\mathbb{C}} \mathcal{O}_X$. From this perspective, Griffiths transversality is the condition that for every $h \in X$, the image of $T_h(X)$ is contained in $F_h^{-1}(\mathrm{End}(V_{n,\mathbb{C}}))$ where F_h^{-1} is the filtration defined in the proof of Lemma 3.5. Since $\mathfrak{g}_{\mathbb{C}} \to \prod_n \mathrm{End}(V_{n,\mathbb{C}})$ respects the Hodge decompositions, Axiom I guarantees that $\mathfrak{g}_{\mathbb{C}} \subset \prod_n F_{n,h}^{-1}\mathrm{End}(V_{n,\mathbb{C}})$ and so the $\mathcal{F}_{n,h}^p$ satisfy Griffiths transversality. For more details, see Proposition 5.3 in Brian's handout.

Finally, we are ready to define a complex structure on X. For any $h \in X$, let $\mathrm{Ad}(h(i))$ be the conjugation action of $h(i)$ on X. This fixes the point $h \in X$ and so we can consider the action on the tangent space $T_h(X)$. $\mathrm{Ad}(h(i))$ acts by -1 on $T_h(X)$ because

$$T_h(X) = \mathfrak{g}/\mathfrak{g}^{0,0} \subset \mathfrak{g}_{\mathbb{C}}^{1,-1} \oplus \mathfrak{g}_{\mathbb{C}}^{-1,1}$$

under projection from $\mathfrak{g}_{\mathbb{C}}$. (Note that i acts by -1 on components of type $(1,-1)$ and $(-1,1)$ in a Hodge decomposition).

For any $h \in X$, define J_h to be the action of $\mathrm{Ad}\, h(e^{\frac{\pi i}{4}})$ on $T_h(X)$. Since $J_h^2 = -1$, this defines a complex structure on $T_h(X)$. One can check that varying over h defines an almost complex structure on X which is preserved under translation by $\mathbf{G}(\mathbb{R})$.

Theorem 3.7. *Let* $\mathbf{G}$ *be a reductive group over* $\mathbb{R}$ *and let* X *be a* $\mathbf{G}(\mathbb{R})$*-conjugacy class of algebraic homomorphisms* $\mathbb{C}^* \to \mathbf{G}(\mathbb{R})$*. If* $(\mathbf{G}, X)$ *satisfies Axiom I, then the almost complex structure defined* *by $\{J_h\}$ is integrable. Moreover, for any representation (V, ρ) of $\mathbf{G}$ and any integers n and p, $\mathcal{F}^p_{n,h}$ is a holomorphic vector bundle on X with respect to this complex structure.*

Proof. Our strategy is to construct an analytic isomorphism from X onto an open subset of a complex flag variety $\mathbf{G}(\mathbb{C})/P$ such that the almost complex structure $\{J_h\}$ is induced by the complex structure on $\mathbf{G}(\mathbb{C})/P$.

Pick a faithful representation (V, ρ) of $\mathbf{G}$ and let F_h denote the filtration on $V_{\mathbb{C}}$ induced by $\rho \circ h$. Let P be the closed subgroup of $\mathbf{G}(\mathbb{C})$ of elements which preserve F_h so P is a parabolic subgroup of $\mathbf{G}(\mathbb{C})$. The orbit map $\mathbf{G}(\mathbb{C})/P \to \mathrm{Flag}(V_{\mathbb{C}})$ is a closed immersion of complex manifolds.

We first note that the map $\phi : X \to \mathrm{Flag}(V_{\mathbb{C}})$ defined before Lemma 3.4 factors through the flag variety $\mathbf{G}(\mathbb{C})/P$ (because $\mathbf{G}(\mathbb{R}) \subset \mathbf{G}(\mathbb{C})$ acts transitively on X). Thus, we get a real-analytic morphism

$$X \xrightarrow{\phi_P} \mathbf{G}(\mathbb{C})/P.$$

Furthermore, by Lemma 3.5 both ϕ_P and $d(\phi_P)$ are injective.

We claim that ϕ_P is an open immersion. To see this, it suffices to check that $d(\phi_P)$ is surjective at h (by transitivity of the group action). By Axiom 1, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{-1,1} \oplus \mathfrak{g}_{\mathbb{C}}^{0,0} \oplus \mathfrak{g}_{\mathbb{C}}^{1,-1}$. Identifying $T_h(X)$ with $\mathfrak{g}/\mathfrak{g}^{0,0}$ (see Lemma 3.4) and $T_{\phi_P(h)}(\mathbf{G}(\mathbb{C})/P)$ with $\mathfrak{g}_{\mathbb{C}}/\,\mathrm{Lie}(P)$, we see that $d(\phi_P)_h$ is induced by the inclusion

$$\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}.$$

A calculation as in Lemma 3.5 shows that $\mathrm{Lie}(P) = F^0(\mathfrak{g}_{\mathbb{C}}) = \mathfrak{g}_{\mathbb{C}}^{(0,0)} \oplus \mathfrak{g}_{\mathbb{C}}^{1,-1}$ so we can identify $T_{\phi_P(h)}(\mathbf{G}(\mathbb{C})/P)$ with $\mathfrak{g}_{\mathbb{C}}^{-1,1}$. Then,

$$\dim_{\mathbb{R}} T_{\phi_P(h)}(\mathbf{G}(\mathbb{C})/P) = 2\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^{-1,1} = \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} - \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^{0,0} = \dim_{\mathbb{R}} \mathfrak{g} - \dim_{\mathbb{R}} \mathfrak{g}^{0,0} = \dim_{\mathbb{R}} T_h(X).$$

As open subset of the complex manifold $\mathbf{G}(\mathbb{C})/P$, X inherits a complex structure. To see that this is the complex structure defined by $\{J_h\}$, note that J_h is given by the adjoint action of $h(e^{\frac{\pi i}{4}})$ which acts on $\mathfrak{g}_{\mathbb{C}}^{-1,1}$ by multiplication by $e^{\frac{\pi i}{2}} = i$.

Finally, one can check that for any representations (V, ρ) of $\mathbf{G}$, the map $\phi : X \to \mathrm{Flag}(V_{\mathbb{C}})$ factors through ϕ_P and so the filtrations $F^p_{n,h}$ vary holomorphically. □

We have seen then that Axiom 1 is sufficient to give X a complex structure. Axiom 2 as discussed in Brian's seminar notes implies compactness of the stabilizer of h (when $\mathbf{G}$ is semi-simple) which becomes important in order to take quotients by the action of arithmetic subgroups of $\mathbf{G}(\mathbb{R})$.