Baily-Borel compactification

Michael Lipnowski

0 Introduction

We set up some notation to be enforced throughout.
Let $\mathbf{G}$ be a semisimple group over a number field F. Let $G = \mathbf{G}(F_{\mathbb{R}})$. Let $K \subset G$ be a maximal compact subgroup and $X = G/K$ its associated symmetric space. We consider locally symmetric spaces of the form $\Gamma\backslash G/K$ for (finite covolume) arithmetic subgroups $\Gamma \subset \mathbf{G}(F)$. There will be no essential loss in assuming that $\Gamma = \mathbf{G}(F) \cap U$ for some compact open $U \subset \mathbf{G}(\mathbb{A}_F^{\mathrm{fin}})$. The goal of this note is to outline a proof of the following theorem:

Theorem 0.0.0.1 (Baily-Borel)**.** *Let X be Hermitian, so that $Y = \Gamma\backslash X$ canonically admits the structure of a complex space. Then there is a unique quasi-projective variety $\mathbf{Y}/\mathbb{C}$ satisfying $\mathbf{Y}^{\mathrm{an}} = Y$.*

0.0.1 What if Y is compact?

Note that X has a canonical hermitian metric, given by the Bergman kernel. Because the Bergman kernel is invariant under complex automorphisms of X, it descends to an invariant Kahler form ω on Y. On the other hand, the Chern form of the canonical bundle of Y is also an invariant 2-form. Because the space of invariant 2-forms on Y is 1 dimensional, it follows that an appropriate multiple $c \cdot \omega$ - in fact, $c = \frac{1}{2\pi i}$ - of the Kahler form equals the Chern form. Therefore, $c \cdot \omega$ is an integral Kahler class on Y.

By the Kodaira embedding theorem, Y admits a holomorphic embedding into some projective space. By GAGA, it follows that Y admits a unique structure of a complex projective variety.

Remark 0.0.1.1. A projective embedding of Y can alternatively be constructed more directly using the methods outlined in §4.

0.0.2 Strategy for general Y

We would like to mimic this argument for general Y, but there is one obvious snag: Y is not, in general, compact! Baily and Borel rectify this problem in three broad steps:

(1) Construct a topological compactification $\overline{Y}$ of Y, called the *Baily-Borel compactification.* This requires the full force of reduction theory.

(2) Endow $\overline{Y}$ with the structure of a normal complex analytic space.

(3) Use automorphic forms to embed the complex analytic space $\overline{Y}$ into some projective space, using a sufficiently high power of the canonical bunle of $\overline{Y}$. The boundary $\partial\overline{Y} = \overline{Y} - Y$ is a closed analytic subspace, which is therefore algebraic. Therefore, its complement $\mathbf{Y} = \overline{Y} - \partial\overline{Y}$ admits the structure of a quasiprojective variety with $\overline{Y}^{\mathrm{an}} = Y$.

Remark 0.0.2.1. Interestingly, it is not a priori clear that sections of $\Omega_{\overline{Y}}$ - the unique extension of the canonical sheaf of $\Gamma\backslash X$ to $\overline{Y}$ - separate tangent vectors on the boundary, though we can separate points using Eisenstein series. We will produce a map $\overline{Y} \to Z$ to a normal projective variety Z, a subvariety of the ambient projective space, which is a bijection on points. We then use the a priori information that $\overline{Y}$ is *normal* to conclude that $\overline{Y} \to Z$ is an isomorphism.

0.0.3 Uniqueness of algebraic structure

Unlike for compact Y, it is not clear that the algebraic structure on Y is unique. However, some remarkable extension theorems from hyperbolic complex analysis will save the day.

Theorem 0.0.3.1 (Borel, Kwack, Kobayashi)**.** *Let $f : (\mathbb{D}^\times)^a \times \mathbb{D}^b \to Y$ be any holomorphic mapping. Then f extends to a holomorphic mapping $f : \mathbb{D}^a \times \mathbb{D}^b \to \overline{Y}^{\mathrm{BB}}$.*

Now assuming this result, we can prove uniqueness of the algebraic structure.

Proof. (of *uniqeness*) Let $\mathbf{Y}'$ denote a second quasiprojective variety with $(\mathbf{Y}')^{\mathrm{an}} = Y$. By Hironaka's resolution of singularities, there is a projective variety $\overline{\mathbf{Y}'}$ with such that $\mathbf{Y}' \subset \overline{\mathbf{Y}'}$ is open with complement a normal crossings divisor. The identity map $\mathbf{Y}'^{\mathrm{an}} \xrightarrow{f} Y$ is holomorphic. Furthermore, for every $x \in \overline{Y'}^{\mathrm{an}}$, there is a Euclidean neighorhood N for which $N \cap (\mathbf{Y}')^{\mathrm{an}}$ is biholomorphic to $(\mathbb{D}^\times)^a \times \mathbb{D}^b$, for some a, b. By Borel's extension theorem, f can be extended to a holomorphic map on N. By uniqueness of analytic continuation, these extensions glue to a unique holomorphic map $\bar{f} : \overline{\mathbf{Y}'}^{\mathrm{an}} \to \overline{Y}^{\mathrm{an}}$.

By GAGA, there is a unique algebraic map $\bar{g} : \overline{\mathbf{Y}'} \to \overline{\mathbf{Y}}$ for which $\bar{g}^{\mathrm{an}} = \bar{f}$. Restricting g to $\mathbf{Y}'$ gives an algebraic map $\mathbf{Y}' \xrightarrow{g} \mathbf{Y}$ whose analytification is an isomorphism. Therefore, g itself is an isomorphism. ∎

1 An orienting example: Hilbert modular surfaces

Our discussion of Hilbert modular surfaces closely follows that of [AMRT, §1].

Let $F/\mathbb{Q}$ be real quadratic with real embeddings ϕ_1, ϕ_2. We sometimes use the pair $\Phi = (\phi_1, \phi_2)$, which in context will always have a clear meaning. Let

$$U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\} \subset B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\} \subset \Gamma = SL_2(O_F). \tag{1}$$

The Hilbert modular group $SL_2(O_F)$ acts properly discontinuously on $\mathbb{H} \times \mathbb{H}$ by

$$\gamma \cdot (z_1, z_2) = (\phi_1(\gamma) \cdot z_1, \phi_2(\gamma) z_2).$$

with quotient $Y = SL_2(O_F)\backslash\mathbb{H} \times \mathbb{H}$. The cusps are in bijection with $SL_2(O_F)$ conjugacy classes of rational parabolic subgroups of $R_{F/\mathbb{Q}}(SL_2)$, parameterized in this case by Γ orbits on $\Gamma/B = \mathbb{P}^1(F)$ of which there are $h_F =$ class number of F.

1.1 Topology of Y

Consider $W_{\geq d} = \{(z_1, z_2) \in \mathbb{H} \times \mathbb{H} : y_1 y_2 \geq d\}$ and define $W_{>d}$ and $W_{=d}$ similarly. $W_\bullet$ is invariant under B. By reduction theory, $W_{>d}/B$ it is an open neighborhood of the cusp $(i\infty, i\infty)$ of Y. There is a fiber bundle

$$W_{=d}/B \to \{(y_1, y_2) \in \mathbb{R}^2_{>0} : y_1 y_2 = d\}/\Phi(O_F^\times) \cong \mathbb{S}^1$$

with fibers $\mathbb{R}^2/\Phi(O_F) \cong \mathbb{S}^1 \times \mathbb{S}^1$.
The map

$$W_{\geq d}/B \to [d, \infty)$$
$$(z_1, z_2) \mapsto y_1 y_2$$

then realizes $W_{\geq d}/B$ (by Ehressmann) as a topological product

$$W_{\geq d}/B = W_{=d} \times [d, \infty).$$

The topological Baily-Borel compactification $\overline{Y}^{BB}$ is formed by taking the one-point compactification of $W_{\geq d}/B$, and its analogues for other cusps. The one point compactification of $W_{\geq d}/B$ is of course the cone over $W_{=d}$.

Remark 1.1.0.2. The compactification $\overline{Y}^{BB}$ is necessarily singular. Indeed, the links of neighborhoods of the cusps are $\mathbb{S}^1 \times \mathbb{S}^1$ bundles over $\mathbb{S}^1$. For smooth manifolds, the links would be homeomorphic to $\mathbb{S}^3$.

1.2 A toroidal compactification of Y

We use torus embeddings to compactify $\mathbb{H} \times \mathbb{H}/U$, which we realize as an analytically open subset of a torus, containing $W_{>d}/U$ as a smaller open subset. Then quotienting by the action of B/U yields a compactification of $W_{>d}/B$ near the cusp. Peforming this partial compactification to each cusp and blowing down the boundaries for each cusp to points yields the Baily-Borel compactification.

We can identify $\mathbb{H} \times \mathbb{H}/U$ as the open subset of $T = \mathbb{C} \times \mathbb{C}/\Phi(O_F)$ where both imaginary parts are > 0. Note that the cocharacter group $N = N(T)$ can be identified with $\Phi(O_F)$. We add on some analytic boundary component $\mathcal{E}$ onto this torus so that $B/U \cong \langle \gamma_0 \rangle$ continues to act on $T \cup \mathcal{E}$. Note that γ_0 preserves the cone of totally positive elements in $N_\mathbb{R}$. We can find (many) rational polyhedral decompositions

$$O_F^+ = \cup_{i \in \mathbb{Z}} \sigma_i$$

for which

- σ_i and σ_j intersect either along a face or at the origin. Their intersection is a face exactly when $|i - j| = 1$.
- $\gamma_0(\sigma_i) = \sigma_{i+d}$ for all i.

The rational polyhedral decomposition $\{\sigma_i\}$ gives rise to a toric variety X_σ for which T is an equivariantly open subset (induced by the inclusion $0 \to \cup\sigma_i$). We'll now use some aspects of the dictionary between rational polyhedral decompositions and toric varieties. (see [KKMS-D])

Toric dictionary.

rational cones and f.g. semigroups	**toric varieties X_σ**
$\check{\sigma} \cap M$ saturated	X_σ normal
faces of σ	orbits of T acting on X
$\sigma_1 \subset \sigma_2$	$\overline{\mathcal{O}^{\sigma_1}} \supset \mathcal{O}^{\sigma_2}$
$\dim \sigma$	$\dim T - \dim \mathcal{O}^\sigma$
$\sigma \cap N$ contains a $\mathbb{Z}$-basis for N	X_σ non-singular

$X_{\{\sigma_i\}}$ is a toric variety over $\mathbb{C}$ which is locally of finite type and is acted on by γ_0. We can express

$$X_{\{\sigma_i\}} = T \cup \mathcal{E}, \mathcal{E} = \cup_{i\in\mathbb{Z}} E_i,$$

where each E_i is the rational curve corresponding to the orbit of each half line, and they intersect transversally at the orbits corresponding to the 2-dimensional faces. We compactify $W_{>d}/U$ to $W_{>d}/U \cup \mathcal{E}$, which is a B/U-equivariant open subset of $X_{\{\sigma_i\}}$. The quotient by B/U is a compactification of $W_{>d}/B$, given set theoretically as $W_{>d}/B \cup \mathcal{E}'$, where $\mathcal{E}'$ is a d-gon of rational curves.

It turns out that the self-intersections of the E_i can be read off lattice-theoretically. For each rational cone σ_i, we let e_i, e_{i+1} denote the smallest integral points on the faces of σ_i, in counterclockwise order. Then,

$$E_i \cdot E_i = -[\Phi(O_F) : \mathrm{span}_{\mathbb{Z}}\{e_{i-1}, e_{i+1}\}].$$

By the analytic counterpart to a theorem of Artin on blowing-down divisors on algebraic surfaces, the boundary can be blown down to a single point giving the structure of a normal complex analytic space because the intersection matrix is negative definite. Applying this procedure to each cusp, we finally arrive at $\overline{Y}^{BB}$, the Baily-Borel compactification.

Remarks.

- By building the $\sigma_i = \mathrm{span}_{\mathbb{R}_{\geq 0}}\{e_{i-1}, e_i\}$ from the corners $\{e_i\}$ of the convex hull of O_F^+, we obtain a minimal non-singular resolution. The boundary vertices are related to best rational approximations to the numbers $\sqrt{d}$. This resolution, expressed in terms of the continued fraction expansion of $\sqrt{d}$, was first discovered by Hirzebruch.

- It is not obvious, from this procedure, that $\overline{Y}^{BB}$ is a projective variety. We have only constructed it as a compact complex analytic space.

- Using the heuristic that "the more negative definite the intersection matrix of the minimal resolution, the more singular the blow-down", we might suspect that the singularities of $\overline{Y}^{BB}$ are quite bad. Indeed, they are.

- The boundary of $\overline{Y}^{BB}$ has very high codimension. For any Hilbert modular variety for totally real F, for example, the boundary would be a union of points, of codimension $[F : \mathbb{Q}]$.

2 Compactifications as topological spaces

2.1 A "uniform method" for compactification [BJ]

Let $Y = \Gamma\backslash X$ be a locally symmetric space, as above.

- Attach to X certain boundary components $e(\mathbf{P})$ indexed by a Γ-invariant collection $\mathcal{P}$ of rational parabolic subgroups $P \subset \mathbf{G}$. The $e(\mathbf{P})$ will be related to the Langlands decomposition of $\mathbf{P}$. In the case of Baily-Borel compactification, we take $\mathcal{P} = \mathcal{P}_{\max}$, the collection of maximal rational parabolic subgroups.
- Glue the boundary components of $X \sqcup e(\mathbf{P})$ together in an appropriate way, to form $\overline{X}_{\mathbb{Q}} := X \sqcup_{\sim} e(\mathbf{P})$. Prove that the action of $\mathbf{G}(\mathbb{Q})$ on X extends continuously to $\overline{X}_{\mathbb{Q}}$.
- Prove that the quotient $\Gamma\backslash\overline{X}_{\mathbb{Q}}$ is compact and Hausdorff.

2.1.1 Reductive Borel-Serre compactification ([BJ], III.10.2)

Apply the uniform method to the corners
$e(\mathbf{P}) = X(\mathbf{P}) = \mathbf{P}(\mathbb{R})/N(\mathbf{P})A(\mathbf{P})K(\mathbf{P}), K(\mathbf{P}) = K \cap M(\mathbf{P})$ where $\mathcal{P}$ denotes the collection of all rational parabolic subgroups. We define $\overline{X}_{\mathbb{Q}} := X \sqcup_{\sim} e(\mathbf{P})$, where the equivalence relation $\sim$ and the topology are characterized as follows:

(1) For a rational parabolic subgroup $\mathbf{P}$, an unbounded sequence $\{(n_j, a_j, z_j)\} \subset X$, expressed in horospherical coordinates, converges to $z \in X(\mathbf{P})$ exactly when

 (a) $\alpha(a_j) \to +\infty$ for all $\alpha \in \Phi(A(\mathbf{P}), \mathbf{P})$

 (b) $z_j \to z$ in $X(\mathbf{P})$.

(2) Suppose $\mathbf{P} \subset \mathbf{Q}$. The boundary face $e(\mathbf{P})$ is attached to $e(\mathbf{Q})$ in an analogous way. Namely, let $\mathbf{P}'$ be the parabolic subgroup of $M(\mathbf{Q})$ determined by $\mathbf{P}$. Then there is a relative horospherical decomposition

$$X(\mathbf{Q}) = N(\mathbf{P}') \times A(\mathbf{P}') \times X(\mathbf{P}).$$

Say that $(n_j, a_j, z_j) \in e(\mathbf{Q})$ converges to $z \in e(\mathbf{P})$ if

 (a) $\alpha(a_j) \to +\infty$ for all $\alpha \in \Phi(A(\mathbf{P}'), \mathbf{P}')$ and

 (b) $z_j \to z$ in $X(\mathbf{P})$.

The group $\mathbf{G}(\mathbb{Q})$ acts continuously on $\overline{X}_{\mathbb{Q}}$ by acting in the usual way on X and by letting $g = kman, k \in K, m \in M(\mathbf{P}), n \in N(\mathbf{P})$, act on the boundary components by

$$\begin{aligned} e(\mathbf{P}) &\to e(g\mathbf{P}g^{-1}) \\ hK(\mathbf{P}) &\mapsto k(mh)k^{-1}K(g\mathbf{P}g^{-1}). \end{aligned}$$

Theorem 2.1.1.1. *For a congruence subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$, the quotient $\overline{\Gamma\backslash X}^{RBS} := \Gamma\backslash\overline{X}_{\mathbb{Q}}$ is a compact Hausdorff space with $\Gamma\backslash X$ a dense, open subset. It is called the* reductive Borel-Serre compactification *of $\Gamma\backslash X$.*

2.1.2 Baily-Borel compactification

We can also obtain the Baily-Borel compactification by the uniform method, where $\mathcal{P} = \mathcal{P}_{\max}$, the collection of maximal parabolic rational subgroups. The corners are closely related to $e_{RBS}(\mathbf{P}) = X(\mathbf{P})$ defined in the reductive Borel-Serre compactification. However, we must modify them to admit complex space structure.

Example 2.1.2.1. Consider $\mathbf{G} = Sp_{2g}$. The maximal parabolic subgroup $\mathbf{P}$ corresponds to removing a single vertex from the C_g Dynkin diagram. The resulting disconnected Dynkin diagram is a disjoint union of the A_{g-n-1} and C_n Dynkin diagrams. Let

$$e_{BB}(\mathbf{P}) = Sp_{2n}(\mathbb{R})/U_n \subset e_{RBS}(\mathbf{P}) = Sp_{2n}(\mathbb{R})/U_n \times SL_{g-n}(\mathbb{R})/SO_{g-n}(\mathbb{R}).$$

By analyzing the classification of simple Hermitian symmetric domains case-by-case, Baily-Borel show that for maximal parabolics $\mathbf{P}$, the reductive Borel-Serre corner $e_{RBS}(\mathbf{P})$ always contains a unique factor which is a Hermitian symmetric domain; they define $e_{BB}(\mathbf{P})$ much as in the above example. These boundary components must be joined along their "common intersection within X". This then defines the partial compactification $\overline{X}_{\mathbb{Q}}^{BB}$.

Theorem 2.1.2.2. $\mathbf{G}(\mathbb{Q})$ *acts continuously on* $\overline{X}_{\mathbb{Q}}^{BB}$*. The quotient* $\overline{\Gamma\backslash X}^{BB} := \Gamma\backslash\overline{X}_{\mathbb{Q}}^{BB}$ *by a congruence subgroup* $\Gamma \subset \mathbf{G}(\mathbb{Q})$ *is a compact Hausdorff space with* $\Gamma\backslash X$ *a dense open subset. It is called the* Baily-Borel compactification *of* $\Gamma\backslash X$.

The construction of both the Baily-Borel and reductive Borel-Serre compactifications by the same uniform method makes them relatively straightforward to compare.

Lemma 2.1.2.3. *The reductive Borel-Serre compactification of* $\Gamma\backslash X$ *dominates its Baily-Borel compactification, i.e. the identify map of* $\Gamma\backslash X$ *extends uniquely to a continuous map*

$$\overline{\Gamma\backslash X}^{RBS} \to \overline{\Gamma\backslash X}^{BB}.$$

Proof. (Idea) The map $e_{RBS}(\mathbf{P}) \to e_{BB}(\mathbf{P})$, for maximal parabolic $\mathbf{P}$, which projects onto the Hermitian factor, is a continuous extension of the identity map. $\square$

3 Compactification as an analytic space

3.1 Statement of an abstract analyticity criterion

We describe a criterion of Baily-Borel for endowing certain topological spaces of the form $V = \sqcup V_i$ (finite union), which are unions of complex analytic spaces, with a complex analytic structure.

Definition 3.1.0.4. The *sheaf of $\mathcal{A}$-functions on V* is defined, on an open subset $U \subset V$, to be the ring of continuous functions whose restriction to each $U \cap V_i$ is analytic.

Theorem 3.1.0.5. *Let* $V = \sqcup V_i$*, where each* V_i *is a complex analytic space. Assume*

(1) For each d, the union $V_{(d)}$ of the V_i of dimension $\leq d$ is closed. Assume $\dim V_0 = \dim V$*, that* V_0 *is dense in* V*, and that* $\dim V_i < \dim V$ *for each* i.

(2) Every point of V has a basis of open neighborhoods whose intersection with V_0 is connected.

(3) Each of the restriction maps, from $\mathcal{A}$-functions on V to analytic functions on V_i, are surjective maps of sheaves.

(4) Every $v \in V$ has an open neighborhood U on which the $\mathcal{A}$-functions of V separate points.

Then $(V, \mathcal{A})$ is an irreducible, normal, complex analytic space. For each $d \leq \dim V, V_{(d)}$ is an analytic subspace of dimension equal to $d = \max\{\dim V_i : \dim V_i \leq d\}$.

Remark 3.1.0.6. In non-pathological situations such as ours, the hardest condition to verify in practice is (4). The conditions from $(1), (2)$ are meant to guarantee that $(V, \mathcal{A})$ is irreducible.

Proof. (Very rough idea) To give a flavor of the proof, we consider the following special case. Assume that

- $V = V_0 \sqcup V_1, V_1 = \{x_0\}$ is a single point.
- Furthermore, assume that there is a neighborhood U of x_0 for which finitely many $\mathcal{A}$-functions $f_1, ..., f_s$ separate all points of U and for which the associated map $f = (f_1, ..., f_s) : U \to N \subset \mathbb{C}^s$ is proper; this can always be arranged under the hypotheses of $(1), (2), (3), (4)$.

Remark. While these two assumptions are restrictive, they are readily satisfied in the case of Hilbert modular varieties. So even this very special case is not devoid of content.

- **Step 1**: $\mathcal{A}_{x_0}$ is integrally closed domain.

  A_{x_0} is a domain because x_0 has a basis of neighborhoods which are dense and irreducible. Suppose that $y = f/g$ satisfies the equation

  $$P(y, x) = y^n + a_{n-1}(x)y^{n-1} + ... + a_0(x) = 0. \tag{2}$$

  By restricting to $V_0 \cap U$, we know there is an analtyic function h on $V_0 \cap U$ satisfying $hg = f$. We claim that h is continuous at x_0. Indeed, h satisfies $P(h(x), x) = 0$. Because the a_i are continuous, $h(x)$ is bounded on $U \cap V_0$ and so $h(x)$ does have accumulation points as $x \to x_0$. Any accumulation point is a root of the polynomial $P(y, x_0)$. Let $I_1, ..., I_m$ be a union of disjoint discs centered at the roots of $P(y, x_0)$. By continuity of the a_i, shrinking U if necessary, we may assume that $U \cap V_0 \subset h^{-1}(I_1) \cup ... \cup h^{-1}(I_n)$. For any $U' \subset U$ for which $U' \cap V_0$ is irreducible, we must therefore have that $U' \cap V_0 = h^{-1}(I_k)$ for a single k. It follows that the only possible accumulation point of $h(x)$, as $x \to x_0$, is the root of $P(y, x_0)$ centering I_k. Thus, h is continuous at x_0. Thus, $h = f/g$ is an $\mathcal{A}$-function.

  Remark 3.1.0.7. In the more general situation, where the V_i are higher dimensional, it follows that h is an $\mathcal{A}$ function because, on any normal analytic space, the ring of analytic functions is integrally closed in the ring of continuous functions. This follows by the inverse function theorem and the Riemann extension theorem.

- **Step 2**: identification of $\mathcal{A}|_U$ with a normal complex analytic space.

  $f : U - \{x_0\} \to N - \{x_0\}$ is a proper mapping. In fact, because f is injective, it is a homeomorphism onto its image. By the direct image theorem, $f(U - x_0) \subset N - \{f(x_0)\}$ is a closed analytic subspace of dimension $\dim V > 0$. By the Remmert-Stein theorem, its closure $Y = f(U) \subset N$ is a closed analytic subspace. It is clear that pullback $O_{Y,f(u)} \to \mathcal{A}_u$ is injective, so that Y is reduced and irreducible.

  Let $\pi : \tilde{Y} \to Y$ be the normalization of Y; the stalk of its structure sheaf at y is the integral closure of the stalk $O_{Y,y}$. We will show that for each $y \in Y$, the pullback map

  $$f^* : O_{\tilde{Y},f(y)} \to \mathcal{A}_y$$

  is an isomorphism.

  - Injectivity is clear.
  - Let $a \in \mathcal{A}_y$. We would like to show that $a \circ f^{-1}$ is an analytic in a neighborhood of $f(y)$. But this follows by the miraculous extension theorems of complex analysis.

    Indeed, $a \circ f^{-1}$ is continuous in N and analytic in $\pi^{-1}\{f(V_0) \cap N\}$, except possibly at the singularities of V_0. Therefore, because $\tilde{Y}$ is normal, it extends to an analytic mapping on all of $\pi^{-1}\{f(V_0) \cap N\}$. Furthermore, $\pi^{-1}\{[f(U) - f(U \cap V_0)] \cap N\}$ is a proper analytic subset of $\tilde{Y}$. Therefore, by normality of $\tilde{Y}$, this extends to an analytic function on all of $\tilde{Y}$. Its pullback by f is clearly equal to a, thus proving surjectivity.

□

Corollary 3.1.0.8. *The sheaf of $\mathcal{A}$-functions on V is given by the sheaf of continuous functions whose restriction to V_0 is analytic.*

Proof. This follows by the Riemann extension theorem, because V is a normal complex analytic space. □

4 Abundance of automorphic forms

The most difficult requirement to satisfy in the abstract analyticity criterion 3.1.0.5 is (4) : having sufficiently many $\mathcal{A}$-functions to separate points. In this section, we outline an "explicit" construction of an abundance of automorphic forms, namely Poincaré series and Eisenstein series, which extend continuously to the boundary of the Baily-Borel compactification.

4.1 Poincaré Series

Poincaré first provided the first systematic construction of automorphic forms, called *Poincaré series*, as follows:

Theorem 4.1.0.9 (Harish-Chandra)**.** *Let ϕ be a function in $L^1(G)$ which is $\mathcal{Z} = \mathrm{center}(\mathcal{U}(\mathfrak{g}_{\mathbb{C}}))$ finite. Let*

$$p_\phi(x) = \sum_{\gamma\in\Gamma} \phi(\gamma x). \tag{3}$$

Then

(i) If ϕ is K-finite on the right, the series converges absolutely, locally uniformly to a Γ-automorphic form which belongs to $L^1(\Gamma\backslash G)$.

(ii) If ϕ is K-finite on the left, then p_ϕ converges absolutely, uniformly and compact sets and is bounded.

The existence of functions ϕ as above, for a given G, is not automatic; it is equivalent to the existence of discrete series representations for G. However, we wil lconstruct explicit examples of ϕ using the realization of X as a bounded symmetric domain.

Let $X^\flat$ denote the bounded realization of X. Let p denote a polynomial function on $X^\flat$. Let $\mu_{X^\flat}(x, g)$ denote the determinant of the Jacobian, at x, of translation by g acting on $X^\flat$; to make sense of determinant, we make the canonical identification of tangent spaces available from $X^\flat$ being a domain in some $\mathbb{C}^n$.

Theorem 4.1.0.10 (Poincaré)**.** *Let f be a polynomial function on $X^\flat, m$ be an integer ≥ 4. Then the series*

$$P_{f,m}(w) := \sum_{\gamma\in\Gamma} \mu_{X^\flat}(\gamma, w)^m f(\gamma \cdot w) \tag{4}$$

defines a holomorphic automorphic form of "weight m". The function $g \mapsto f(g\cdot 0)\mu_{X^\flat}(g, 0)^{-m}$ is bounded if f is a polynomial.

Remark 4.1.0.11. The relevant growth assumptions for applying theorem 4.1.0.9 are proved by observing that

$$|\mu_{X^\flat}(x, 0)|^{-2}\omega = dx,$$

where dx is an invariant volume element on $X^\flat$ and ω is the Euclidean volume element of the bounded realization. Because the automorphy factor $|\mu_{X^\flat}(0, x)|$ is bounded on $X^\flat$, it follows that $|\mu_{X^\flat}(0, g)|^m$ is integrable for any $m \geq 2$.

This construction, for sufficiently large m, gives rise to automorphic forms with any specified local behavior at finitely many points.

Theorem 4.1.0.12. *For any Γ-inequivalent points $a_1, ..., a_n$, we can find f such that $P_{f,m}(a_i) = b_i, ..., P_{f,m}(a_n) = b_n$ for m sufficiently large.*

Proof. (from [C], p.170-171) The image of the map $f \mapsto P_{f,m}(a) = (P_{f,m}(a_1), ..., P_{f,m}(a_n))$ is a linear subspace of $\mathbb{C}^n$. We will show that for $m > m(\epsilon)$ and $b \in \mathbb{C}^n$ fixed, we can find a polynomial f for which $P_{f,m}(a)$ is within ϵ of b. By letting b range through a fixed basis for $\mathbb{C}^n$ and letting ϵ be small enough, this will imply that for m sufficiently large, the image of $f \mapsto P_{f,m}(a)$ contains a basis for $\mathbb{C}^n$ and so is surjective.

Recall that $\mu_{X^\flat}$ is square integrable. Therefore, for any $0 < u < 1$, the set Γ_u for which $|\mu_{X^\flat}(\gamma, a_i)| \geq u$ for some i is finite. Choose a polynomial f satisfying $f(a_i) = b_i$ for all i and for which $f_i(\gamma \cdot a_i) = 0$ for all $\gamma \in \Gamma_u$. We readily see that

$$|b - P_{f,m}(a)| = O_f(u^m).$$

The claim follows. $\square$

4.2 Eisenstein series

The hermitian structure of X gives rise to a decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}^+ + \mathfrak{p}^-$, where $\mathfrak{p}^\pm$ denotes the $\pm i$-eigenspaces of the complex structure. Let

$$\mu : K \to \mathbb{C}^\times = \det\{Ad(k)|\mathfrak{p}^+\}^{-1}.$$

A holomorphic section of $\omega_Y^{\otimes m}$ can be thought of as:

- $f : G \to \mathbb{C}$ which is left Γ-invariant: $f(\gamma g) = f(g)$.
- $Y \cdot f = 0$ for every $Y \in \mathfrak{p}^-$. This corresponds to holomorphy.
- $f(gk) = \mu(k)^m \cdot f(g)$.
- f has moderate growth.

To go back and forth between this definition and the classical definition, just multiply by the appropriate automorphy factor:

$$F(z) \text{ satisfying } F(\gamma \cdot z) = \mu_{X^\flat}(z, \gamma)^m F(z) \leftrightarrow T_{\mathbf{G}} F := f(g) = \mu_{X^\flat}(0, g)^{-m} \cdot F(g \cdot 0).$$

Let $c_{\mathbf{P}}$ denote the constant term operator and r "restriction to the Hermitian factor". Let F be any holomorphic section as above.

Proposition 4.2.0.13. *The function*

$$\Phi(F) := (T_{\mathbf{P}}^{-1} c_{\mathbf{P}} T_{\mathbf{G}} F)_{\{\mathbf{P}\}} : \overline{X}_{\mathbb{Q}}^{BB} \to \mathbb{C}$$

is continuous, Γ-equivariant, and holomorphic on each boundary component.

Proof. A matter of unravelling definitions. $\square$

Lemma 4.2.0.14. *The canonical bundle of $\Gamma\backslash X$ extends to locally-free $\mathcal{A}$-module $\omega_{\overline{\Gamma\backslash X}^{BB}}$ of rank 1 on $\overline{\Gamma\backslash X}^{BB}$. For F of weight m, the functions $\Phi(F)$ patch together to a global section of $\omega^{\otimes m}_{\overline{\Gamma\backslash X}^{BB}}$.*

Remark 4.2.0.15. Think of the line bundle $\omega_{\Gamma\backslash X}$ as given by a GL_1-valued cocycle. In a neighborhood N of a boundary point, the transition maps over $N \cap \Gamma\backslash X$ are given by holomorphic functions $N \cap \Gamma\backslash X \to \mathbb{C}^\times$. Morally, because the boundary is high codimension, Hartog's theorem should allow this to extend to $N \cap \Gamma\backslash X \to \mathbb{C}$. There is further content to the statement that the map lands inside of $\mathbb{C}^\times \subset \mathbb{C}$.

Proposition 4.2.0.16. *The operator $F \mapsto \Phi(F)_{\mathbf{P}}$ maps the space of Poincaré series for $\mathbf{G}$ onto the space of Poincaré series for $M'(\mathbf{P})$, where $M'(\mathbf{P})$ is the Hermitian factor of $M(\mathbf{P})$.*

Proof. The surjectivity statement follows by directly computing the constant term, with respect to maximal parabolics $\mathbf{P}$ of Eisenstein series for $\mathbf{P}$: $c_{\mathbf{P}} E_{\mathbf{P}} f$. $\square$

4.3 Analytic space and algebraic variety structure

Corollary 4.3.0.17. *The sheaf of $\mathcal{A}$-functions on $\overline{\Gamma\backslash X}^{BB}$ separate points.*

Proof. Poincaré series separate points on each boundary component. The operator $F \mapsto \Phi(F)_{\mathbf{P}}$ maps surjectively onto the Poincaré series for $M'(\mathbf{P})$. $\square$

Corollary 4.3.0.18. *The ringed space $(\overline{\Gamma\backslash X}^{BB}, \mathcal{A})$ admits the structure of a compact, normal, irreducible complex analytic space.*

Proof. All of the conditions of theorem 3.1.0.5 can be satisfied. In particular, the results of the sections on Eisenstein series and Poincaré seires prove the existence of an abundance of $\mathcal{A}$-functions, enough to separate points and thus satisfy (4). The space is irreducible because $\Gamma\backslash X$ is open, dense, and smooth. $\square$

Finally, we are ready to outline a proof of the main theorem.

Theorem 4.3.0.19 (BB, 10.10)**.** $\overline{\Gamma\backslash X}^{BB}$ *uniquely admits the structure of a projective algebraic variety.*

Proof. Let $\overline{Y} = \overline{\Gamma\backslash X}^{BB}$.

The $\mathcal{A}$-sections of $\omega_{\overline{Y}}^{\otimes m}$ separate points, for some m. Because $\overline{Y}^{BB}$ is a compact complex analytic space, it satisfies the descending chain condition for closed analtyic subsets. Therefore, there are a finite number $E_0, ..., E_N$ of $\mathcal{A}$-sections of $\omega_{\overline{Y}}^{\otimes m}$ which separate points. This defines an injective analytic map,

$$\overline{Y} \xrightarrow{f=[E_0:...:E_N]} \mathbb{P}^N.$$

Without loss of generality, we assume that the automorphic forms of degree a multiple of m are generated in degree m and the $E_0, ..., E_N$ is a complete basis of the homogeneous elements of degree exactly m. Let I denote the ideal of relations among the E_i, so that f factors through $V(I)$.

The map of analytic spaces $\overline{Y} \xrightarrow{f} V(I)$ is injective by construction. On the other hand, because f is proper, the image of f either equals all of $V(I)$ or is a proper analytic subset. In the latter case, there would be a polynomial P which vanishes on the image of f, implying that P lies in I, a contradiction. Therefore, f is a bijective holomorphic map. The map f factors uniquely through the normalization $\widetilde{V(I)}$:

$$\overline{Y} \xrightarrow{\tilde{f}} \widetilde{V(I)}.$$

Because normalizations are bijections on underlying topological spaces, $\tilde{f}$ is a bijection between normal complex analytic spaces. Therefore, $\tilde{f}$ is an isomorphism between $\overline{\Gamma\backslash X}^{BB}$ and a normal projective variety. $\square$

Remark 4.3.0.20. Note that we did, at the start, have any way of separating tangent vectors on the boundary components of $\overline{\Gamma\backslash X}^{BB}$. Rather, this argument relied crucially on the a priori knowledge that $\overline{\Gamma\backslash X}^{BB}$ is a normal complex analytic space.

5 Hyperbolic Complex Analysis

The extension property of theorem 0.0.3.1 is a big generalization of the big Picard theorem. One source of such extension theorems is nicely explained by the theory Kobayashi-pseudodistance, an elaboration of the fact that holomorphic maps of the unit disc to itself do not increase hyperbolic distance.

5.1 Reformuation of the Schwarz Lemma

The usual Schwarz Lemma. For any holomorphic $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0, f'(0) = 1$, there is an inequality $|f(z)| \leq |z|$.

Proof. Maximum principle. □

The Schwarz Lemma has a very striking reformulation in terms of the usual hyperbolic distance ρ on the unit disc.

Proposition 5.1.0.21. *Let $f : \mathbb{D} \to \mathbb{D}$ be any holomorphic map. Then $\rho(f(z), f(w)) \leq \rho(z, w)$.*

Proof. Pre and post compose f with linear fractional transformations as necessary so the transformed function satisfies $f(0) = 0, f'(0) = 1$ to verify that

$$\left|\frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}}\right| \leq \left|\frac{z - w}{1 - z\bar{w}}\right|.$$

But one calculates that the hyperbolic distance is simply

$$\rho(z, w) = \tanh^{-1}\left|\frac{z - w}{1 - z\bar{w}}\right|.$$

□

5.2 Intrinsic pseudo-distances on complex spaces

The above distance decreasing property of holomorphic self-maps of the unit disc gives content to the ensuing definitions.

Definition 5.2.0.22. Let M be a complex analytic space (think manifold). Let $a, b \in M$. The *Kobayashi (pseudo)-distance* $k(a, b)$ is defined as

$$k(a, b) = \inf\{\rho(a_1, b_1) + ... + \rho(a_n, b_n) : a_0 = a, b_n = b, f_1, ..., f_n : \mathbb{D} \to M \text{ holomorphic }, f_i(b_i) = f_{i+1}(a_i)\}.$$

To parse: the $\{f_i\}$ quantified over is nothing more than a chain of holomorphic discs in M. The K distance also arises by integrating a certain pseudo-norm.

Definition 5.2.0.23. The *Kobayashi pseudo-norm* is defined, for a holomorphic tangent vector $\xi \in T_m(M)$, by

$$k_{M,m}(\xi) := \inf\{\lambda : \text{ there exists } f : \mathbb{D} \to M, f(0) = m, \lambda f'(0) = \xi\}.$$

We can also define a "dual" intrinsic distance function.

Definition 5.2.0.24. Let M be a complex analytic space, $a, b \in M$. The *Caratheodory (pseudo)-distance* $c(a, b)$ is defined to be

$$c(a, b) = \sup\{\rho(f(a), f(b)) : f : M \to \mathbb{D} \text{ holomorphic }\}.$$

The Kobayashi distance and Caratheodory distance can be compared.

Lemma 5.2.0.25. *There is a comparison $c_M(p, q) \leq k_M(p, q)$ for every M.*

Proof. Choose $f_1, ..., f_n : \mathbb{D} \to M, a_1, b_1, ..., a_n, b_n \in \mathbb{D}$ as in the definition of Kobayashi distance and let $f : M \to \mathbb{D}$ be any holomorphic map.

$$\begin{aligned} \sum \rho(a_i, b_i) &\geq \sum \rho(f \circ f_i(a_i), f \circ f_i(b_i)) \\ &\geq \sum \rho(a_i, b_i) \\ &\geq \rho(f(p), f(q)). \end{aligned}$$

The claim follows. □

Key properties of Caratheodory and Kobayashi pseudo-distance

- Almost by definition, these pseudo-distances satisfy a distance decreasing property; for $f : M \to N$ a holomorphic map, $k_N(f(a), f(b)) \leq k_M(a, b)$. This implies, in particular, that biholomorphisms are isometric for these intrinsic distances.
- The above lemma showed that $c_M(a, b) \leq k_M(a, b)$.
- Let $\pi : \tilde{M} \to M$ be a covering. Then $k_M(a, b) = \inf\{d_{\tilde{M}}(\tilde{a}, \tilde{b}) : \tilde{a}, \tilde{b} \text{ lift } a, b\}$. As we'll see shortly, the analogous property is not true for c_M.

Some examples

- $k_{\mathbb{D}} = \rho = c_{\mathbb{D}}$. So C and K distances recover the usual distance on $\mathbb{D}$.
- $k_{\mathbb{C}} \equiv 0$. Indeed, there is a holomorphic map $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = 0, f(r) = 1$. Letting $r \to 0$, we see that

  $$k_{\mathbb{C}}(0, 1) \leq \rho(0, r) \to 0.$$

  This implies that $c_{\mathbb{C}} \equiv 0$. This examples shows that c, k are generally not distance functions but only pseudo-distances. This proves Liouville's theorem that every holomorphic map $\mathbb{C} \to \mathbb{D}$ is constant.
- For a Riemann surface X, k_X is the hyperbolic distance, i.e. the distance that arises from uniformization. But by the maximum modulus principle on the other hand, $c_X \equiv 0$.

Evidently, the Kobayashi metric k_M being a definite distance function is a special property.

Definition 5.2.0.26. A complex analytic space M is called *hyperbolic* if k_M is a definite distance function.

Examples and non-examples of hyperbolic spaces.

- If c_M is definite, then k_M is definite too. Therefore, any bounded domain $U \subset \mathbb{C}^n$ is hyperbolic.

- Much more is in fact true for bounded domains. The inequality $c_M \leq k_M$ shows that two nearby points in the K distance are close in c_M distance, which defines the usual Euclidean topology. On the other hand, for any two points p, q in a complex ball $B \subset M$ satisfy $k_M(p,q) \leq k_B(p,q)$ by the distance-decreasing property of K distance. Therefore, the Euclidean topology and the k_M topology on M are equivalent.

  Even more is true for bounded symmetric domains X. Let g_X denote the invariant Hermitian metric. Fix $x_0 \in X$. Because the K distance topology and d_X distance topology are equivalent, the unit balls $\{\xi : k_{X,x_0}(\xi) < 1\}, \{\xi : g_{X,x_0}(\xi) < 1\} \subset T_{x_0}(X)$ are comparable, i.e. there are some constants r, R such that

  $$\{\xi : k_{X,x_0}(\xi) < r\} \subset \{\xi : g_{X,x_0}(\xi) < 1\} \subset \{\xi : k_{X,x_0}(\xi) < R\}$$

  Because both the k_X and g_X pseudonorms are G-invariant, the same equality holds replacing x_0 by any $x \in X$. Therefore,

  $$r \cdot k_X(p,q) \leq d_X(p,q) \leq R \cdot k_X(p,q).$$

- M is hyperbolic $\iff$ $\tilde{M}$ is hyperbolic for any covering $\tilde{M}$ of M.

- Suppose M admits a non-constant map $\mathbb{C} \xrightarrow{f} M$. Because f is K distance decreasing, k_M is not definite.

In fact, this last non-example is the main obstruction to hyperbolicity.

Theorem 5.2.0.27 (Brody)**.** *A compact complex manifold M is hyperbolic iff there are no non-constant holomorphic maps $\mathbb{C} \to M$.*

5.3 An application to monodromy

In situations where k_X is non-degenerate, its distance decreasing property can have serious geometric consequences.

Example 5.3.0.28. Consider the Legendre family $\pi : \mathcal{E}_\lambda = \{y^2 = x(x-1)(x-\lambda)\} \to \mathbb{P}^1_\lambda - \{0,1,\infty\}$. The locally sheaf of free abelian groups $R^1\pi_*\mathbb{Z}$ over $\mathbb{P}^1 - \{0,1,\infty\}$ defines a polarized variation of Hodge structure. Let $\gamma_0, \gamma_1, \gamma_\infty$ denote small counterclockwise loops around $0, 1, \infty$ respectively. One computes that the monodromy action of these generators of $\pi_1(\mathbb{P}^1 - \{0,1,\infty\})$ is given by

$$\gamma_\infty = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \gamma_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \gamma_0 = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}.$$

We observe that $\gamma_\infty, \gamma_1, \gamma_0$ have characteristic polynomials $(t-1)^2, (t-1)^2, (t+1)^2$ respectively. Therfore, these local monodromies are alll quasi-unipotent. As we'll see next, this is no accident.

Theorem 5.3.0.29 (Borel)**.** *Let V denote an integral polarized variation of Hodge structures over $\mathbb{D}^\times$. The monodromy action of $\gamma \in \pi_1(\mathbb{D}^\times)$ acting on the fibers of V is quasi-unipotent.*

Proof. Let $X = G/H$ denote the classifying space for variations of polarized Hodge structure, with discrete data determined by V. The variation V is classified by a holomorphic period map $f : \mathbb{D}^\times \to \Gamma\backslash X$. This lifts to holomorphic $\tilde{f} : \mathbb{H} \to X$ satisfying $\tilde{f}(z+1) = \gamma\tilde{f}(z)$. Let $\tilde{f}(i \cdot n) = g_n H$. Then

$$\begin{aligned} k_X(g_n^{-1}\gamma g_n H, H) &= k_X(\gamma g_n H, g_n H) \\ &= k_X(\tilde{f}(i \cdot n), \tilde{f}(i \cdot n + 1)) \\ &\leq k_{\mathbb{H}}(i \cdot n, i \cdot n + 1) \\ &= \frac{1}{n} \to 0. \end{aligned}$$

Assume that the k_X topology equals the Euclidean topology, this implies that the conjugacy class of γ has an accumulation point in the compact group K. Therefore, all eigenvalues of γ acting on the fiber of V have absolute value 1. On the other hand, because V is an *integral* variation of Hodge structures, all eigenvalues of the γ-action are algebraic integers. Therefore, all eigenvalues are actually roots of unity. □

Remark 5.3.0.30. There are certainly cases where the k_X-topology equals the Euclidean topology on X, e.g. when X is biholomorphic to a bounded domain. I'm not sure whether such bounded realization can always be achieved for $X = G/H$ as in the proof of the theorem. That being said, this theorem is true in maximum generality. The Griffiths transversality condition determines a subbundle of the holomorphic tangent bundle of X, called the *horizontal tangent bundle* , to which all classifying maps for variations of Hodge structure are tangent. The holomorphic sectional curvatures of the horizontal tangent bundle are bounded above by a strictly negative constant. This fact implies that $f^* d_X \leq d_{\mathbb{H}}$, which can be used to prove the above theorem in general, i.e. not just for period domains which admit bounded realizations.

5.4 Warmup for later extension theorems: the big Picard theorem

We use the big Picard theorem as an illustration that the theory of Kobayashi distance is genuinely powerful. It also gives a useful preview of the ultimate extension theorem.

Theorem 5.4.0.31. *Any holomorphic map $f : \mathbb{D}^\times \to \mathbb{P}^1 - \{0, 1, \infty\}$ extends to a holomorphic map $\tilde{f} : \mathbb{D} \to \mathbb{P}^1$.*

Proof. Because $\mathbb{P}^1$ is compact, any infinite sequence of points in $\mathbb{P}^1$ has an accumulation point.

- Suppose that for any sequence of points $z_i \to 0$ the sequence $f(z_i)$ has no accumulation point in $\mathbb{P}^1 - \{0, 1, \infty\}$. This implies that there are some small disjoint discs D_1, D_2, D_3 centered at $0, 1, \infty$ and some $r > 0$ for which $f(\mathbb{D}_r^\times) \subset D_1 \cup D_2 \cup D_3$. Because D_1, D_2, D_3 are dics, f then extends uniquely to a holomorphic map $\tilde{f} : \mathbb{D} \to \mathbb{P}^1$ by the Riemann extension theorem.

- Otherwise, choose a sequence $z_i \to 0$ for which $f(z_i) \to m \in \mathbb{P}^1 - \{0, 1, \infty\}$.

  Both the punctured disc $\mathbb{D}^\times$ and $\mathbb{P}^1 - \{0, 1, \infty\}$ are covered by the unit disc $\mathbb{D}$. Since $k_{\mathbb{D}}$ defines the usual topology on $\mathbb{D}$, the (quotient) metrics $k_{\mathbb{D}^\times}$ and $k_{\mathbb{P}^1 - \{0,1,\infty\}}$ define the usual topologies on $\mathbb{D}^\times$ and $\mathbb{P}^1 - \{0, 1, \infty\}$ respectively.

Consider the loop γ_i centered at 0 of radius $|z_i| = r_i$. Its length in the K metric is approximately $1/\log(1/r_i) \to 0$. Because the K metric is distance decreasing with respect to holomorphic maps, and beause $f(z_i) \to m$, the loop $f \circ \gamma_i$ must eventually - for sufficiently large i - be contained in a fixed contractible neighborhood of m. Therefore, f acts trivially on fundamental groups and so lifts to a map $\tilde{f} : \mathbb{D}^\times \to \mathbb{D}$. By the Riemann extension theorem, $\tilde{f}$ extends to a unique map $\mathbb{D} \xrightarrow{\tilde{f}} \mathbb{D}$. Then $\pi \circ \tilde{f}$ provides the desired extension of f.

□

This argument is a perfect prototype for the more general extension principle to be presented in the next section: use hyperbolicity and the distance decreasing property of maps to find a continuous extension of f, then apply the Riemann extension theorem.

5.5 Hyperbolically embeddings and a general extension theorem

We will be able to prove a generalization of the big Picard theorem for a special class of embeddings $M \subset Y$, e.g. $M = \mathbb{P}^1 - \{0, 1, \infty\}, Y = \mathbb{P}^1$.

Definition 5.5.0.32. Let Y be a complex space and M a relatively compact complex hyperbolic subspace. We say that M is *hyperbolically embedded in* Y if, for any boundary points p, q of M and any sequences $p_n \to p, q_n \to q$,

$$k_M(p_n, q_n) \to 0 \text{ implies } p = q.$$

Loosely, this can be thought of as saying that the definite distance function k_M on M extends to a definite distance function on $\partial M \subset Y$.

Example 5.5.0.33. We will later prove that

$$M = \Gamma \backslash X \subset Y = \overline{\Gamma \backslash X}^{BB}$$

is hyperbolically embedded.

The miraculous fact is that any hyperbolic embedding gives rise to an extension theorem.

Theorem 5.5.0.34 (Kobayashi, Kwack, Kiernan)**.** *Let M be hyperbolically embedded in Y. Let $f : \mathbb{D}^\times \to M$ be a holomorphic map. Let $\{z'_k\}$ be a sequence in $\mathbb{D}^\times$ with $z_k, z'_k \to 0$. Suppose that $f(z'_k) \to q \in Y$. Then*

(i) Let $\{z_k\}$ be any other sequence in $\mathbb{D}^\times$ for which $z_k \to 0$. Then $f(z_k) \to q$.

(ii) f extends to a holomorphic map $f : \mathbb{D} \to Y$ for which $f(0) = q$.

Proof. The below proof directly from [Kie] theorem 1, which remarks that the clever winding number argument from below is due to Grauert.

(i) Suppose that $f(z_k) \to p \neq q$ and assume that $|z_k| \leq |z'_k|$. Then there is some neighborhood $p \in U \subset Y$ which is an analytic subset of a ball

$$U \subset W_2 = \{(w_1, ..., w_n) : |w_1|^2 + ... + |w_n|^2 < 2\}, q \notin U.$$

Let ρ_k be the curve $\rho_k(t) = z_k e^{2\pi i t}$. Note that ρ_k has $k_{\mathbb{D}^\times}$ diameter approaching zero and so, because f is Kobayashi distance decreasing, $f(\rho_k)$ has k_M diameter approaching 0. By definition of hyperbolic embedding, $f(\rho_k)$ must converge to p, i.e. we may assume that $f(\rho_k)$ is contained in the smaller ball W_1. Let R_k denote the largest annulus centered at 0 containing ρ_k for which $f(R_k) \subset W_1$.

Since $f(z'_k) \to q \notin U$, there must be points on the boundary of $\overline{R_k}$ which escape W_1, i.e. there are a_k on the inner circle and b_k on the outer circle for which $|f(a_k)| = |f(b_k)| = 1$. Because the boundary of $\overline{W_1}$ is compact, we may assume by passing to a subsequence that $a_k \to q' \in Y$ and $b_k \to q'' \in Y$, neither of which equal $p = (0, ..., 0)$.

Let $\sigma_k(t) = a_k e^{2\pi i t}, \tau_k(t) = e^{2\pi i t}$. Using the hyperbolic embedding assumption exactly as above, we see that $f(\sigma_k) \to q', f(\tau_k) \to q''$. Summarizing, the curve $f(\rho_k)$ is contained in a small neighborhood U_0 of 0, and $f(\sigma_k), f(\tau_k)$ in a small neighborhoods U', U'' of q', q'', both of which are certainly disjoint from U_0. Let $f = (f_1, ..., f_n)$. Rotating our coordinates as necessary, we can assume that $f_1(\sigma_k)$ and $f_1(\tau_k)$ have winding number 0 around $f_1(z_k)$. Therefore,

$$I_{\text{inner}} = \int_{\sigma_k} \frac{f_1(z)}{f_1(z) - f_1(z_k)} dz = \int_{f_1(\sigma_k)} \frac{dw}{w - f_1(z_k)} = 0,$$

$$I_{\text{outer}} = \int_{\tau_k} \frac{f_1(z)}{f_1(z) - f_1(z_k)} dz = \int_{f_1(\tau_k)} \frac{dw}{w - f_1(z_k)} = 0.$$

On the other hand, by Cauchy's integral formula,

$$I_{\text{outer}} - I_{\text{inner}} = 2\pi i(N - P),$$

where N and P respectively denote the number of zeros and the number of poles of $f_1(z) - f_1(z_k)$ inside the annulus R_k. But $N > 0$ and $P = 0$, a contradiction. Therefore, $f(z_k) \to q$ after all.

(ii) The above argument proved that f extends uniquely to a continuous map $\mathbb{D} \xrightarrow{f} Y$. By the Riemann extension theorem, this extension is holomorphic.

□

Remark 5.5.0.35. When you meditate carefully on the above proof, we could replace the single map f by a sequence of maps f_k. The distance decreasing property of holomorphic maps, one key feature of the above proof, continues to hold uniformly for a sequence f_k. This gives slightly more flexibility and allows, by an inductive argument to show that any holomorphic map $(\mathbb{D}^\times)^a \times \mathbb{D}^b \to M$ extends to a holomorphic map $\mathbb{D}^a \times \mathbb{D}^b \to Y$.

Corollary 5.5.0.36 (of slightly more general statement from remark 5.5.0.35)**.** *Let X be a smooth complex manifold with normal crossings divisor $A \subset X$. Any holomorphic map $f : X - A \to M$ extends uniquely to a holomorphic map $f : X \to Y$.*

Corollary 5.5.0.37. *Let S be any non-singular quasi-projective algebraic variety over $\mathbb{C}$. Any holomorphic map $f : S^{\text{an}} \to \Gamma\backslash X$ is the analytification of a unique algebraic map $f^{\text{alg}} : S \to \Gamma\backslash X$.*

Proof. Use resolution of singularities to realize S as the complement of a normal crossings divisor in a smooth projective variety. By corollary 5.5.0.35, the theorem follows. $\square$

Let us grant, for now, that the inclusion $M = \Gamma\backslash X \subset Y = \overline{\Gamma\backslash X}^{BB}$ is a hyperbolic embedding; this will be proven in §5.6. Then

Corollary 5.5.0.38. *The algebraic structure on* $\Gamma\backslash X$ *is unique.*

Proof. See the introduction. $\square$

Corollary 5.5.0.39. *The automorphism group of any* $\Gamma\backslash X$ *is finite.*

Proof. Because $\Gamma\backslash X$ is a quasi-projective variety, its automorphism scheme H, is locally of finite type over $\mathbb{C}$ and smooth because $\mathbb{C}$ has characteristic 0. Because $\Gamma\backslash X$ is hyperbolic, it admits no non-constant maps from $\mathbb{C}$. Therefore, H^0 acts trivially and so H is discrete.

On the other hand, we can readily identify the $\mathbb{C}$-points of H with $N_G(\Gamma)/\Gamma$. Because $\Gamma\backslash X$ has finite volume, the space $N(\Gamma)\backslash X$ which is covered by $\Gamma\backslash X$ has finite volume too. Therefore, the degree of the cover must be finite, i.e. $|H| = |N_G(\Gamma)/\Gamma|$ must be finite. $\square$

5.6 $\Gamma\backslash X \subset \overline{\Gamma\backslash X}^{BB}$ is a hyperbolic embedding

As discussed in §5.2, the Kobayashi distance k_X on $\Gamma\backslash X$ is comparable to the invariant Riemannian distance d_X, i.e. there are constants r, R such that

$$r \cdot k_{\Gamma\backslash X}(p,q) \leq d_{\Gamma\backslash X}(p,q) \leq R \cdot k_{\Gamma\backslash X}(p,q). \tag{5}$$

Definition 5.6.0.40. We call a compactification $\overline{M}$ of a metric space M *hyperbolic* if for any sequences $p_n \to p \in \overline{M}, q_n \to q \in \overline{M}$,

$$d(p_n, q_n) \to 0 \text{ implies } p = q.$$

Clearly, the equation (5) implies that the inclusion $\Gamma\backslash X \subset \overline{\Gamma\backslash X}^{BB}$ is hyperbolic with respect to $k_{\Gamma\backslash X}$ iff it is "hyperbolic" with respect to $d_{\Gamma\backslash X}$. Furthermore, we have the following two lemmas, the first straightforward and the second not:

Lemma 5.6.0.41. *If* $\overline{M}_1$ *and* $\overline{M}_2$ *are compactifications of* M *and* $\overline{M}_1$ *dominates* $\overline{M}_2$*, then*

$$M \subset \overline{M}_1 \textit{ hyperbolic} \implies M \subset \overline{M}_2 \textit{ hyperbolic.}$$

According to this lemma and lemma 2.1.2.3, in order to prove that $\Gamma\backslash X \subset \overline{\Gamma\backslash X}^{BB}$ is a hyperbolic, it suffices to prove that $\Gamma\backslash X \subset \overline{\Gamma\backslash X}^{RBS}$ is hyperbolic.

Theorem 5.6.0.42 (BJ, III.22.11)**.** *The reductive Borel-Serre compactification* $\Gamma\backslash X \subset \overline{\Gamma\backslash X}^{RBS}$*, with the invariant Riemannian distance on* $\Gamma\backslash X$*, is hyperbolic.*

Proof. The proof of [BJ], III.22.11 is recorded here for convenience, with only minor modifications.

Let p, q be boundary points of $\overline{\Gamma\backslash X}^{RBS}$ satisfying $p_j \to p, q_j \to q$. Suppose that $p \neq q$.

Suppose that $z \in X(\mathbf{P})$ projects to p. Convergence to p implies that there is a lift $\tilde{p}_j = (n_j, a_j, z_j)$, in $\mathbf{P}$ horospherical coordinates, for which

$$z_j \to z, n_j \text{ is bounded }, \alpha(a_j) \to +\infty \text{ for all } \alpha \in \Phi(A_{\mathbf{P}}, \mathbf{P}).$$

On the other hand, because no subsequence of $\{q_j\}$ converges to p, there is some j_0 such that for any $j \geq j_0$ and any lifts $\tilde{q}_j = (n'_j, a'_j, z'_j)$ of q_j,

$$(a'_j, z'_j) \notin B_{A(\mathbf{P})}(a_j, \epsilon_0) \times B_{X(\mathbf{P})}(z, \epsilon_0).$$

Now in $\mathbf{P}$-horospherical coordinates $X = N(\mathbf{P}) \times A(\mathbf{P}) \times X(\mathbf{P})$, the invariant Riemannian metric on X, associated to the Killing form, can be expressed as

$$dx^2 = dz^2 + da^2 + \sum_{\alpha \in \Phi(A(\mathbf{P}), \mathbf{P})} \alpha^{-2}(a) h_\alpha(z),$$

where $h_\alpha(z)$ is a metric on the root lie algebra $\mathfrak{g}_\alpha$ which depends smoothly on z. Therefore, for $\tilde{p} = (n_p, a_p, z_p), \tilde{q} = (n_q, a_q, z_q)$,

$$d_X(\tilde{p}, \tilde{q}) \geq \max\{d_{X(\mathbf{P})}(z_p, z_q), d_{A(\mathbf{P})}(a_p, a_q)\}.$$

But for sufficiently large j_0, we may assume $z_j \in B_{X(\mathbf{P})}(z, \epsilon_0/2)$. Then,

$$d_X(\tilde{p}, \tilde{q}) \geq \epsilon_0/2.$$

This proves the desired result. □

6 References

[AMRT] Ash, Mumford, Rapoport, Tai. *Smooth compactifications of locally symmetric varieties.*

[BB] Baily, Borel. *Compactification of arithmetic quotients of bounded symmetric domains.*

[BJ] Borel, Ji. *Compactifications of symmetric and locally symmetric spaces.*

[C] Cartan. *Fonctions automorphes et séries de Poincaré.*

[GR] Grauert, Remmert. emphCoherent analytic sheaves.

[KKMS-D] Kempf, Knudsen, Mumford, Saint-Donat. *Toroidal embeddings I.*

[Kie] Kiernan. *Extensions of holomorphic mappings.*

[K] Kobayashi. *Hyperbolic manifolds and holomorphic mappings.*

[M] McNamara. *The Baily-Borel compactification.*