UNIVERSAL PROPERTY OF NON-ARCHIMEDEAN ANALYTIFICATION

BRIAN CONRAD

1. Introduction

1.1. **Motivation.** Over $\mathbf{C}$ and over non-archimedean fields, analytification of algebraic spaces is defined as the solution to a quotient problem. Such analytification is interesting, since in the proper case it beautifully explains the essentially algebraic nature of proper analytic spaces with "many" algebraically independent meromorphic functions. (See [A] for the complex-analytic case, and [C3] for the non-archimedean case.) Working with quotients amounts to representing a covariant functor. Our aim is to characterize analytification of algebraic spaces via representing a contravariant functor, generalizing what is done for schemes.

In the remainder of §1.1, we review the situation in the case of schemes, and then address the difficulties which arise for algebraic spaces (especially over non-archimedean fields). In particular, we will explain why a certain naive approach to the non-archimedean case (using functors on affinoid algebras) is ultimately not satisfactory. Our main theorem is stated in §1.2.

For a scheme X locally of finite type over $\mathbf{C}$, the analytification X^{an} can be defined in two ways. In the concrete method, we choose an open affine cover $\{U_i\}$ and use a closed immersion of each U_i into an affine space to define U_i^{an} as a zero locus of polynomials in a complex Euclidean space. These are glued together, and the result is independent of $\{U_i\}$. A more elegant approach, pushing open affines into the background and functoriality into the foreground, is to use a map $i_X : X^{\mathrm{an}} \to X$ that exhibits X^{an} as the solution to a universal mapping problem: it is final among all morphisms $Z \to X$ where Z is a complex-analytic space and morphisms are taken in the category of locally ringed spaces of $\mathbf{C}$-algebras. In other words, (X^{an}, i_X) represents the contravariant functor $\mathrm{Hom}(\cdot, X)$ on the category of complex-analytic spaces.

Example 1.1.1. If X is affine n-space $\mathbf{A}^n_{\mathbf{C}}$ for some $n \geq 1$, the functorial criterion yields $X^{\mathrm{an}} = \mathbf{C}^n$ with its evident i_X because (i) $\mathbf{C}^n$ equipped with its standard coordinate functions is universal among complex-analytic spaces equipped with an ordered n-tuple of global functions, (ii) $\mathbf{A}^n_{\mathbf{C}}$ satisfies the analogous property in the category of all locally ringed spaces over $\mathbf{C}$ (by [EGA, II; $\mathrm{Err_I}$, 1.8.1]).

For fields k complete with respect to a non-archimedean absolute value, if $|k^\times| \neq 1$ then it goes similarly since the categories of algebraic k-schemes and rigid-analytic spaces over k are full subcategories of the category of locally ringed G-spaces of k-algebras (where a *G-space* is a Grothendieck topology whose "opens" are subset inclusions, coverings are distinguised set-theoretic coverings, and fiber products are set-theoretic intersections); see [C1, §5.1]. The analogue for k-analytic Berkovich spaces is in [Ber2, 2.6].

One merit of introducing the map i_X is that it underlies various "analytification" operations:

Example 1.1.2. Let $f : X \to Y$ be a map between locally finite type schemes over $\mathbf{C}$. By the universal property of i_Y, there is a unique analytic map filling in the top row of a commutative diagram:

$$\begin{array}{ccc} X^{\mathrm{an}} & \xrightarrow{f^{\mathrm{an}}} & Y^{\mathrm{an}} \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Date: August 23, 2010.

1991 *Mathematics Subject Classification.* Primary 14G22; Secondary 14D15.

This work was partially supported by NSF grant DMS-0917686. I am very grateful to J. Lurie for helpful discussions which influenced and clarified this work in numerous ways.

The same happens in the non-archimedean cases. This illustrates the role of the functorial characterization: it not only defines $f^{\rm an}$ in an elegant manner, but also underlies the definition $\mathscr{F}^{\rm an} := i_X^*(\mathscr{F})$ for any coherent sheaf $\mathscr{F}$ on X (as arises in the formulation of GAGA over $\mathbf{C}$; see [Se, Prop. 2, Def. 2] for the implicit use of i_X in Serre's original work). A similar procedure is used in the formulation of the Artin comparison isomorphism between topological and algebraic étale cohomology for finite type schemes over $\mathbf{C}$ [SGA4, XVI, 4.1], as well as its non-archimedean analogue with Berkovich spaces ([Ber1, §3.4], [Ber2, 7.5.1]).

In a previous joint work with M. Temkin [CT], a theory of analytification was developed for (quasi-separated) algebraic spaces $\mathscr{X}$ locally of finite type over a field k complete with respect to a non-archimedean absolute value. We considered analytification in two senses, using rigid-analytic spaces (when k has a non-trivial absolute value) and using k-analytic spaces in the sense of Berkovich. Briefly, the idea is to choose an arbitrary étale chart in schemes $\mathscr{R} \rightrightarrows \mathscr{U}$ for $\mathscr{X}$, and to consider a quotient $\mathscr{X}^{\rm an}$ for the analytified equivalence relation $\mathscr{R}^{\rm an} \rightrightarrows \mathscr{U}^{\rm an}$, where these latter analytifications are either taken to be rigid-analytic spaces (with $|k^\times| \neq 1$) or k-analytic Berkovich spaces; when defining the quotient (if it exists!), we use the Tate-étale topology from [CT, §2.1] in the rigid-analytic case and the usual étale topology in the Berkovich case. The same analytified quotient procedure is used over $\mathbf{C}$.

The existence or not of $\mathscr{X}^{\rm an}$, as well as its functoriality, turn out to be independent of the choice of étale scheme chart (in a sense made precise in [CT, §2], especially [CT, §2.2]). The main theorem in [CT] is that $\mathscr{X}^{\rm an}$ exists in both the rigid-analytic and Berkovich senses when $\mathscr{X}$ is separated (along with a compatibility between these two kinds of analytification, as well as for fiber products and extension of the ground field).

Remark 1.1.3. To emphasize the importance of the global hypothesis of separatedness for the existence of $\mathscr{X}^{\rm an}$, even though it may seem that the existence should be a more local problem, we note that in [CT, §3.1] there are examples of non-separated smooth algebraic spaces $\mathscr{S}$ of dimension 2 over $\mathbf{Q}$ whose diagonal $\Delta_{\mathscr{S}/\mathbf{Q}}$ is an affine immersion and whose scalar extension to any non-archimedean field of characteristic 0 *does not* admit an analytification in either sense yet for which $\mathscr{S}_{\mathbf{C}}^{\rm an}$ *does* exist.

Remark 1.1.4. In the complex-analytic and both non-archimedean cases, a necessary condition for $\mathscr{X}^{\rm an}$ to exist (or as we shall say, for $\mathscr{X}$ to be *analytifiable*) is that the quasi-compact $\Delta_{\mathscr{X}/k}$ is an immersion [CT, 2.2.5, 4.1.4]. It is elementary to prove sufficiency of this condition over $\mathbf{C}$ [Kn, I, 5.18], but Remark 1.1.3 shows that such sufficiency fails in the non-archimedean case. (The dichotomy stems from the nature of fiber products in these various cases.) In the non-archimedean case it a very difficult problem to exhibit sufficient conditions for the existence of $\mathscr{X}^{\rm an}$ which are weaker than separatedness

Question. In the complex-analytic and especially both non-archimedean cases, can we characterize the analytification $\mathscr{X}^{\rm an}$ in a manner which avoids the crutch of an étale scheme chart (much as the characterization of $X^{\rm an}$ in the scheme case via i_X avoids the crutch of affine open subschemes)? The aim is not to give a new construction of (or practical existence criterion for) $\mathscr{X}^{\rm an}$, but rather to describe the contravariant functor of points $\mathrm{Hom}(\cdot, \mathscr{X}^{\rm an})$ in a manner which also characterizes when $\mathscr{X}^{\rm an}$ exists, akin to the scheme case. (The quotient approach describes the covariant functor $\mathrm{Hom}(\mathscr{X}^{\rm an}, \cdot)$.)

The answer to this Question is provided by our main result in Theorem 1.2.1 below. It involves topoi, so we now consider a more concrete attempt in the non-archimedean case. The essential problem when $\mathscr{X}$ is an algebraic space is that it is (by definition) a functor on schemes, so it is not obvious how to define a concept of *morphism* $Z \to \mathscr{X}$ with Z an analytic space over the ground field. In the rigid-analytic case, if $Z = \mathrm{Sp}(A)$ is affinoid then it is tempting to make the *ad hoc* definition that a morphism $Z \to \mathscr{X}$ is an element in $\mathscr{X}(A) = \mathrm{Hom}_k(\mathrm{Spec}\, A, \mathscr{X})$. But is this local on Z? More specifically, if $\{\mathrm{Sp}(A_i)\}$ is a finite affinoid cover of Z and $A' := \prod A_i$ then is the diagram of sets

$$\mathscr{X}(A) \to \mathscr{X}(A') \rightrightarrows \mathscr{X}(A' \widehat{\otimes}_A A') \tag{1.1.1}$$

exact? This is not clear, because although fpqc descent holds for (quasi-separated) algebraic spaces [LMB, A.4], so

$$\mathscr{X}(A) \to \mathscr{X}(A') \to \mathscr{X}(A' \otimes_A A')$$

is exact, it is not evident if $\mathscr{X}(A' \otimes_A A') \to \mathscr{X}(A' \widehat{\otimes}_A A')$ is injective since $A' \otimes_A A' \to A' \widehat{\otimes}_A A'$ is generally not faithfully flat (as $A' \otimes_A A'$ is generally not even noetherian).

In the special case that $A = k'$ is a finite extension of k and $\mathscr{X}$ is analytifiable, functoriality of analytification defines a map

$$\mathscr{X}(k') = \mathrm{Hom}_k(\mathrm{Spec}\, k', \mathscr{X}) \to \mathrm{Hom}((\mathrm{Spec}\, k')^{\mathrm{an}}, \mathscr{X}^{\mathrm{an}}) = \mathrm{Hom}(\mathrm{Sp}(k'), \mathscr{X}^{\mathrm{an}}) = \mathscr{X}^{\mathrm{an}}(k')$$

which is bijective (by [CT, Ex. 2.3.2]). A similar argument works whenever $\dim A = 0$. For general k-affinoid A, there is a "relative analytification" functor for schemes locally of finite type over A (assigning to any such $\mathscr{Y}$ a rigid-analytic space $\mathscr{Y}^{\mathrm{an}}$ over $\mathrm{Sp}(A)$, with $(\mathrm{Spec}\, A)^{\mathrm{an}} = \mathrm{Sp}(A)$). This can be extended to the case of separated algebraic spaces locally of finite type over A, and for $\mathscr{X}$ locally of finite type over k we naturally have $(\mathscr{X}_A)^{\mathrm{an}} \simeq \mathscr{X}^{\mathrm{an}} \times \mathrm{Sp}(A)$. Thus, in the separated case we get a natural map

$$\mathscr{X}(A) = \mathscr{X}_A(A) \to (\mathscr{X}_A)^{\mathrm{an}}(\mathrm{Sp} A) = (\mathscr{X}^{\mathrm{an}} \times \mathrm{Sp}(A))(\mathrm{Sp}(A)) = \mathscr{X}^{\mathrm{an}}(\mathrm{Sp}(A)) \tag{1.1.2}$$

which is functorial in $\mathscr{X}$ and A.

The bijectivity of (1.1.2) is not evident in general, essentially because $\mathrm{Sp}(A)$ is rather different from $\mathrm{Spec}\, A$ when $\dim A > 0$. One source of inspiration for expecting such a bijection to exist is that when the algebraic space $\mathscr{X}$ arises as a moduli space for some class of "polarized" structures, bijectivity often has natural meaning in terms of rigid-analytic GAGA over an affinoid base. In the appendix, which is logically independent from the rest of the paper, we prove that (1.1.2) and its Berkovich space analogue are bijective for separated $\mathscr{X}$. This provides a concrete description of the functor of points of $\mathscr{X}^{\mathrm{an}}$ on affinoid objects when $\mathscr{X}$ is separated. It rests on the recently proved Nagata compactification theorem for separated algebraic spaces [CLO] (applied over the base scheme $\mathrm{Spec}\, A$). One consequence is the exactness of (1.1.1) for separated $\mathscr{X}$ and any faithfully flat map of k-affinoid algebras $A \to A'$, since representable functors on the category of rigid-analytic spaces are sheaves for the Tate-fpqc topology [C2, Cor. 4.2.5].

We do not regard the concrete viewpoint via (1.1.2) as an adequate one to answer the above Question (though it is interesting!). First of all, it does not have a useful analogue over $\mathbf{C}$. More importantly, we do not want to have to assume the existence of $\mathscr{X}^{\mathrm{an}}$ (which is implies by the separatedness hypothesis), but rather we wish to characterize even its existence in terms of the representability of a contravariant functor defined in terms of $\mathscr{X}$, at least in the Berkovich case (which is more natural than the rigid-analytic case for global construction problems). For this to be interesting, we must relax the separatedness hypothesis in order to incorporate some examples (such as in Remark 1.1.3) for which the analytification in the sense of quotients does not exist.

1.2. **Main result.** In the non-archimedean case, if $\mathscr{X}^{\mathrm{an}}$ exists as a rigid-analytic space and we give it the Tate-étale topology [CT, §2.1] then there is a natural map of locally ringed topoi

$$i_{\mathscr{X}} = (i_{\mathscr{X}*}, i_{\mathscr{X}}^*) : \widetilde{(\mathscr{X}^{\mathrm{an}})}_{\text{ét}} \to \widetilde{\mathscr{X}_{\text{ét}}}$$

over k even though there is no "morphism" $\mathscr{X}^{\mathrm{an}} \to \mathscr{X}$ in a naive sense. The description of the pushforward is quite simple: $(i_{\mathscr{X}})_*(\mathscr{F})(\mathscr{U}) = \mathscr{F}(\mathscr{U}^{\mathrm{an}})$ for any scheme $\mathscr{U}$ étale over $\mathscr{X}$. (Here, the content is that if $\mathscr{U}' \to \mathscr{U}$ is an étale cover of schemes then $\mathscr{U}'^{\mathrm{an}} \to \mathscr{U}^{\mathrm{an}}$ admits sections locally for the Tate-étale topology; see [C2, Thm. 4.2.2].) The functor $i_{\mathscr{X}}^*$ is characterized as the left adjoint of $i_{\mathscr{X}*}$, and it can be constructed by a procedure that is similar to the topological case.

The formulation and proof of non-archimedean and complex-analytic GAGA for proper morphisms between algebraic spaces is expressed in terms of maps of ringed topoi such as $i_{\mathscr{X}}$ in [CT, §3.3] since there are no actual "maps of spaces" as in the scheme case. (The same applies to the non-archimedean étale cohomology comparison morphism for algebraic spaces.) By using a suitable generalization of the notion of locally ringed topos (strictly henselian topoi, introduced by J. Lurie in [DAGV] and reviewed in Definition 2.2.2), we prove that the map $i_{\mathscr{X}}$ satisfies a universal property very similar in spirit to the one used in the scheme case. This is part of our main result, describing the contravariant functor of points of $\mathscr{X}^{\mathrm{an}}$ (under a mild diagonal hypothesis):

Theorem 1.2.1. *Let $\mathscr{X}$ be an algebraic space locally of finite over a field k that is either* **C** *or is complete with respect to a non-archimedean absolute value. In the non-archimedean case, use k-analytic spaces in the sense of Berkovich.*

(1) *Consider the contravariant functor $\mathrm{Hom}_k(\cdot, \mathscr{X})$ on the category of k-analytic spaces which assigns to every Z the set of isomorphism classes of morphisms of locally ringed topoi $f : \widetilde{Z_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{ét}}}$ over k. A representing object for this functor must be an analytification of $\mathscr{X}$.*

Conversely, if $\mathscr{X}$ is analytifiable and $\Delta_{\mathscr{X}/k}$ is affine then $\mathscr{X}^{\text{an}}$ represents $\mathrm{Hom}_k(\cdot, \mathscr{X})$: for every f as above, there is a unique map of k-analytic spaces $f^{\text{an}} : Z \to \mathscr{X}^{\text{an}}$ over k such that $i_{\mathscr{X}} \circ \widetilde{f^{\text{an}}}$ is naturally isomorphic to f.

(2) *If $\mathscr{X}$ is separated and k is non-archimedean with $|k^\times| \neq 1$ then the analytification $\mathscr{X}^{\text{an}}$ in the sense of rigid-analytic spaces satisfies the universal mapping property analogous to* (1) *on the rigid-analytic category.*

It was noted earlier that if $\Delta_{\mathscr{X}/k}$ is a closed immersion then $\mathscr{X}^{\text{an}}$ exists. We do not know if the affine hypothesis in the converse part of (1) can be replaced with the condition of being an immersion (which would be a bit more natural, in view of Remark 1.1.4, though Remark 1.1.3 provides interesting examples with affine immersive diagonal which are not analytifiable in either sense over non-archimedean fields). The complex-analytic case in Theorem 1.2.1 is a simpler framework in which we can explain the main ideas of the proof, without the complications of the non-archimedean case.

Theorem 1.2.1(1) shows that (under mild assumptions on $\Delta_{\mathscr{X}/k}$ weaker than separatedness) the definition of analytification through quotients of étale equivalence relations is equivalent to a definition through representing a contravariant functor defined in terms of $\mathscr{X}$. When it comes to the task of *constructing* $\mathscr{X}^{\text{an}}$, the quotient approach seems to be unavoidable; however, the characterization as in Theorem 1.2.1 in terms of representing a contravariant functor is more elegant (and is closer to the spirit of the characterization in the scheme case).

Remark 1.2.2. In an unpublished work [Lur] (to be incorporated into [DAGVIII]), J. Lurie has proved a result which has a similar flavor to Theorem 1.2.1: for any strictly henselian topos $(T, \mathscr{O})$ and (quasi-separated) algebraic space (or more general stack) $\mathscr{X}$, there is an equivalence between morphisms of locally ringed topoi $(T, \mathscr{O}) \to (\widetilde{\mathscr{X}_{\text{ét}}}, \mathscr{O}_{\mathscr{X}_{\text{ét}}})$ and tensor functors $\mathrm{Qcoh}(\mathscr{X}) \to \mathrm{Mod}_T(\mathscr{O})$ in the opposite direction. This is stated precisely in [Lur, Thm. 5.11], and the methods used there carry over to the non-archimedean analytic case [Lur, Rem. 10.4]. However, neither the main results in the present work nor in [Lur] have logical consequences for the other; they simply complement each other.

In the same spirit as in [Lur], one could consider carrying over the concept of Deligne–Mumford stack to the non-archimedean analytic setting (which amounts to working more directly with diagrams such as $\mathscr{R}^{\text{an}} \rightrightarrows \mathscr{U}^{\text{an}}$ rather than with an analytic quotient space), and then asking for a version of Theorem 1.2.1 with Deligne–Mumford stacks instead of algebraic spaces. (Beware that the Deligne–Mumford version, specialized back to the case of algebraic spaces, is not concerned with the issue of existence the quotient $\mathscr{U}^{\text{an}}/\mathscr{R}^{\text{an}}$ as an analytic space, much as using algebraic spaces entails giving up on representing functors by "spaces" and instead doing geometry with suitable functors.) We leave this task to the interested reader; our arguments should carry over to handle that generality.

2. Strictly henselian topoi and morphisms

We begin by reviewing some concepts related to Grothendieck's definition of locally ringed topoi [SGA4, IV, Exer. 13.9]. The formulation will differ a bit from Grothendieck's in that it makes the role of the Zariski topology more explicit. That will enable us (following Lurie) to replace appearances of the Zariski topology with the étale topology (for affine schemes) to get a "strictly henselian" variant that is the main goal of this section.

Fix a (small) site X, and let T be the corresponding topos (category of sheaves of sets on X) and $\mathscr{O}$ a commutative ring object in the category T (i.e., the functor of points of $\mathscr{O}$ on T is ring-valued, or equivalently $\mathscr{O}$ is a sheaf of rings on X). We assume that *all representable functors on X are sheaves*. That is, for all objects U in X, the functor $\underline{U} := \mathrm{Hom}_X(\cdot, U)$ on the site X satisfies the sheaf axioms. In other words, the topology on the site is subcanonical.

Subcanonicity holds in many familiar cases, such as if X is the site associated to a topological space, or a scheme or algebraic space equipped with the étale topology (and it is not assumed in Grothendieck's definition of locally ringed topoi, but that is irrelevant for the cases of interest).

2.1. **Spec in a ringed topos.**

It is important for what follows that the the étale topology sites arising in non-archimedean geometry are subcanonical:

Example 2.1.1. If X is the Tate-étale site of a rigid-analytic space (see [C2, Def. 4.2.1]) then the sheaf property for representable functors follows from [C2, Cor. 4.2.5]. If X is the étale site of a k-analytic space $\mathscr{X}$ in the sense of Berkovich over a non-archimedean field k (see [Ber2, 4.1]), then the sheaf property follows from combining [Ber2, 4.1.3, 4.1.5] and [CT, 4.1.2]. In fact, these references give more for Berkovich spaces: the functor $\mathrm{Hom}_k(\cdot, \mathscr{X}') = \mathrm{Hom}_{\mathscr{X}}(\cdot, \mathscr{X} \times \mathscr{X}')$ on $X = \mathscr{X}_{\text{ét}}$ is a sheaf for *any* k-analytic space $\mathscr{X}'$ (not necessarily an étale $\mathscr{X}$-space).

We also note for later purposes that coherent sheaves on rigid-analytic spaces satisfy effective descent for the Tate-étale topology, and coherent sheaves for the G-topology on k-analytic Berkovich spaces satisfy effective descent for the étale topology on such spaces; in the rigid-analytic case this is [C2, Thm. 4.2.8], and in the Berkovich case it is [CT, 4.2.4]. Applying this to coherent ideal sheaves inside of the structure sheaf (for the G-topology in the Berkovich case), relative to these étale topologies there is effective descent for closed immersions, and hence likewise for Zariski-open immersions.

The preceding examples (including algebraic spaces and complex-analytic spaces) are the only ones which interest us, so the reader may safely restrict attention to sites X which admit fiber products. Thus, by [MM, III.4, Prop. bis], the sheaf axioms on the topos T can be expressed in familiar terms without digressing into the language of sieves.

Remark 2.1.2. We will write $\widetilde{X}$ to denote $(T, \mathscr{O})$ (or to denote the underlying topos T if the context makes it clear). In the special case that X arises from a scheme $\mathscr{X}$ with the Zariski (resp. étale) topology, we will also write $\widetilde{X}$ (resp. $\widetilde{\mathscr{X}_{\text{ét}}}$) to denote $(\widetilde{\mathscr{X}}, \mathscr{O})$ (resp. $(\widetilde{\mathscr{X}_{\text{ét}}}, \mathscr{O})$); the analogous convention will be used for algebraic spaces, complex-analytic spaces, and non-archimedean spaces as in Example 2.1.1.

For any commutative ring R and object U in X, define a *morphism* $f : U \to \mathrm{Spec}\, R$ (or more precisely, $(U, \mathscr{O}) \to \mathrm{Spec}\, R$) to be a ring homomorphism $R \to \mathscr{O}(U)$. Observe the role of $\mathscr{O}$ in this definition. Composition of f with morphisms $U' \to U$ and morphisms $\mathrm{Spec}\, R \to \mathrm{Spec}\, R'$ is defined in an evident "associative" manner (using $\mathscr{O}(U) \to \mathscr{O}(U')$).

Example 2.1.3. If X is the site associated to a topological space and $\mathscr{O}$ has local stalks (so each $(U, \mathscr{O}|_U)$ is a locally ringed space) then this notion of morphism naturally coincides (functorially in U and R) with the usual notion of a morphism $(U, \mathscr{O}|_U) \to \mathrm{Spec}\, R$ of locally ringed spaces (by assigning to any morphism of the latter sort the induced ring map between rings of global functions). This is [EGA, II, $\mathrm{Err}_{\mathrm{I}}$, 1.8.1].

Definition 2.1.4. Consider an object U in X and a morphism $U \to \mathrm{Spec}\, R$ for a commutative ring R. For any R-algebra R', the functor $\underline{U}_{R'/R}$ on X assigns to any V the set of commutative diagrams

$$\begin{array}{ccc} V & \xrightarrow{f} & U \\ {\scriptstyle s}\downarrow & & \downarrow \\ \mathrm{Spec}\, R' & \longrightarrow & \mathrm{Spec}\, R \end{array}$$

The functor $\underline{U}_{R'/R}$ should be viewed as a replacement for $U \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, R'$. It is contravariant in the R-algebra R' and covariant in U in an evident manner. In particular, $\underline{U}_{R'/R} \to \underline{U}_{R/R} = \underline{U}$ is the map which forgets s. We will only be interested in cases when R' is R-étale.

Example 2.1.5. If $\operatorname{Spec} R' \to \operatorname{Spec} R$ is an open immersion then there is at most one possibility for s. This is most easily seen by using the equivalent expression

$$\begin{array}{ccc} \operatorname{Spec} \mathscr{O}(V) & \xrightarrow{f^*} & \operatorname{Spec} \mathscr{O}(U) \\ {\scriptstyle s^*}\downarrow & & \downarrow \\ \operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R \end{array}$$

with affine schemes. In particular, the forgetful map $\underline{U}_{R'/R} \to \underline{U}$ is a subfunctor inclusion when $\operatorname{Spec} R' \to \operatorname{Spec} R$ is an open immersion.

To work more effeciently with $\underline{U}_{R'/R}$ and subsequent generalizations, it will be convenient (following Lurie) to introduce another way to work with morphisms $U \to \operatorname{Spec} R$. This rests on:

Proposition 2.1.6. *For any $\mathscr{F}$ in T define $\mathscr{O}(\mathscr{F})$ to be the ring $\operatorname{Hom}_T(\mathscr{F}, \mathscr{O})$. Let R be a commutative ring.*

1. *The contravariant functor $H : \mathscr{F} \rightsquigarrow \operatorname{Hom}(R, \mathscr{O}(\mathscr{F}))$ is represented by an object $\mathscr{O}(R)$ in T. Explicitly, $\mathscr{O}(R)(\underline{U}) = \mathscr{O}(R)(U) = \operatorname{Hom}(R, \mathscr{O}(U))$ for $U \in X$.*
2. *For any ring R and map of topoi $f : T' \to T$ there is a canonical morphism $f^*(\mathscr{O}(R)) \to (f^*\mathscr{O})(R)$ in T' that is functorial in $\mathscr{O}$ and R, as well as in f. When R is finitely generated over $\mathbf{Z}$, this canonical morphism is an isomorphism.*
3. *If $\operatorname{Spec} R' \to \operatorname{Spec} R$ is an open immersion then $\mathscr{O}(R') \to \mathscr{O}(R)$ is a subobject.*

Loosely speaking, $\mathscr{O}(R)$ serves as a substitute for $\operatorname{Spec} R$ when working in $(T, \mathscr{O})$.

Proof. Since $\mathscr{O}$ is a sheaf on X, it is trivial to verify that the functor $U \rightsquigarrow \operatorname{Hom}(R, \mathscr{O}(U))$ satisfies the sheaf axioms and hence is an object $\mathscr{O}(R)$ in T. Likewise, if

$$\mathscr{R} \rightrightarrows \mathscr{F}' \twoheadrightarrow \mathscr{F}$$

is a cokernel presentation in T (i.e., $\mathscr{R} \to \mathscr{F}' \times \mathscr{F}'$ has image $\mathscr{F}' \times_{\mathscr{F}} \mathscr{F}'$) then

$$\mathscr{O}(\mathscr{F}) \to \mathscr{O}(\mathscr{F}') \rightrightarrows \mathscr{O}(\mathscr{R})$$

is an exact sequence of rings. Applying $\operatorname{Hom}(R, \cdot)$ then gives an exact sequence of sets

$$H(\mathscr{F}) \to H(\mathscr{F}') \rightrightarrows H(\mathscr{R}).$$

Likewise, from the definitions

$$H(\coprod \mathscr{F}_i) = \operatorname{Hom}(R, \mathscr{O}(\coprod \mathscr{F}_i)) = \operatorname{Hom}(R, \prod \mathscr{O}(\mathscr{F}_i)) = \prod H(\mathscr{F}_i).$$

The functor $\operatorname{Hom}_T(\cdot, \mathscr{O}(R))$ satisfies the same properties as just verified for H, so to identify these functors on T it suffices to do so as functors on the full subcategory of representable sheaves. But $\mathscr{O}(\underline{U}) = \mathscr{O}(U)$, so by definition of $\mathscr{O}(R)$ we have proved (1).

For (2), first observe that the bijection

$$\operatorname{Hom}_T(\mathscr{F}, \mathscr{O}(R)) = \operatorname{Hom}(R, \mathscr{O}(\mathscr{F})) = \operatorname{Hom}(R, \operatorname{Hom}_T(\mathscr{F}, \mathscr{O}))$$

with $\mathscr{F} = \mathscr{O}(R)$ provides a canonical ring map $\phi_R : R \to \operatorname{Hom}_T(\mathscr{O}(R), \mathscr{O})$. (Explicitly, $\phi_R(r)$ is the family of evaluation maps $\operatorname{Hom}(R, \mathscr{O}(U)) \to \mathscr{O}(U)$ at r, for varying U in X.) For any $\mathscr{F}$ in T, applying f^* defines a map of sets

$$\mathscr{O}(\mathscr{F}) = \operatorname{Hom}_T(\mathscr{F}, \mathscr{O}) \to \operatorname{Hom}_{T'}(f^*(\mathscr{F}), f^*\mathscr{O}) = (f^*\mathscr{O})(f^*\mathscr{F})$$

that is a map of rings, and setting $\mathscr{F} = \mathscr{O}(R)$ then defines a ring map

$$R \overset{\phi_R}{\to} \mathscr{O}(\mathscr{O}(R)) \to (f^*\mathscr{O})(f^*(\mathscr{O}(R))).$$

This is an element in

$$\operatorname{Hom}(R, (f^*\mathscr{O})(f^*(\mathscr{O}(R)))) = \operatorname{Hom}_{T'}(f^*(\mathscr{O}(R)), (f^*\mathscr{O})(R)),$$

so we have constructed a canonical map $f^*(\mathscr{O}(R)) \to (f^*\mathscr{O})(R)$ in T', as desired. By construction, it has the asserted functorial properties. (Explicitly, the map

$$f^*(\mathscr{O}(R))(U) \to (f^*\mathscr{O})(R)(U) = \mathrm{Hom}(R, (f^*\mathscr{O})(U))$$

carries $h \in f^*(\mathscr{O}(R))(U)$ to the map $r \mapsto f^*(\phi_R(r))(h)$.)

Now suppose R is finite type over $\mathbf{Z}$ and choose an isomorphism $R \simeq \mathbf{Z}[t_1, \dots, t_n]/(h_1, \dots, h_m)$. Then for any ring S, the set $\mathrm{Hom}(R, S)$ is identified with the zero locus in S^n for the h_j's. Taking $S = \mathscr{O}(U)$ for U in X, $\mathscr{O}(R)(U)$ is the zero locus in $\mathscr{O}(U)^n$ for the h_j's. That is, as a sheaf of sets, $\mathscr{O}(R)$ is the intersection of the kernels in $\mathscr{O}^n$ of the polynomial maps $h_j : \mathscr{O}^n \to \mathscr{O}$. This is an expression for $\mathscr{O}(R)$ as an iterated fiber product in T, so by left-exactness of f^* it follows that $f^*(\mathscr{O}(R))$ is the intersection of the kernels in $(f^*\mathscr{O})^n$ of the polynomial maps $h_j : (f^*\mathscr{O})^n \to f^*\mathscr{O}$. But this latter description also yields $(f^*\mathscr{O})(R)$, so we have identified $f^*(\mathscr{O}(R))$ and $(f^*\mathscr{O})(R)$ as objects in T'. We claim that this identification is the canonical morphism $f^*(\mathscr{O}(R)) \to (f^*\mathscr{O})(R)$. Since the canonical morphism is functorial in R, by the construction of the identification just given and the left-exactness of f^* it suffices to treat the case $R = \mathbf{Z}[t_1, \dots, t_n]$ (with its evident presentation by itself). This case is an easy calculation.

Finally, (3) is trivial because $\mathscr{O}(R')(U) \to \mathscr{O}(R)(U)$ is identified with the map of sets

$$\mathrm{Hom}(\mathrm{Spec}\,\mathscr{O}(U), \mathrm{Spec}\,R') \to \mathrm{Hom}(\mathrm{Spec}\,\mathscr{O}(U), \mathrm{Spec}\,R).$$

■

For any U in X, $f \in \mathscr{O}(R)(U) = \mathrm{Hom}(\underline{U}, \mathscr{O}(R))$, and R-algebra R', there is a natural isomorphism

$$\underline{U}_{R'/R} \simeq \underline{U} \times_{\mathscr{O}(R)} \mathscr{O}(R')$$

functorial in U, $\mathscr{O}$, and R'. This is seen by evaluating both sides on any V in X and applying Example 2.1.5, and recovers the fact that $\underline{U}_{R'/R}$ is a subfunctor of $\underline{U}$ when $\mathrm{Spec}\,R' \to \mathrm{Spec}\,R$ is an open immersion. It also shows that in general the functor $\underline{U}_{R'/R}$ is a sheaf; i.e., it belongs to the topos T (as can be seen directly from the sheaf properties of $\underline{U}$ and $\mathscr{O}$ as well.)

Example 2.1.7. Consider any of the following categories $\mathscr{C}$: schemes, algebraic spaces, complex-analytic spaces, or rigid-analytic or Berkovich spaces over a field k complete with respect to a non-archimedean absolute value (assumed to be non-trivial in the rigid-analytic case). Let $R \to R'$ be a finite type map of rings, and U an object in $\mathscr{C}$ equipped with a ring map $R \to \mathscr{O}(U)$. We define the functor $\underline{U}_{R'/R}$ on $\mathscr{C}$ in the evident manner, analogous to Definition 2.1.4.

We claim that $\underline{U}_{R'/R}$ is represented by an object $U_{R'/R}$ in $\mathscr{C}$, and that the resulting canonical map $U_{R'/R} \to U$ is an open immersion (resp. étale) when $\mathrm{Spec}\,R' \to \mathrm{Spec}\,R$ is an open immersion (resp. étale). Likewise, we claim that if $\{\mathrm{Spec}\,R_\alpha \to \mathrm{Spec}\,R\}$ is a Zariski-open covering (resp. étale cover) then so is $\{U_{R_\alpha/R} \to U\}$.

In the case of schemes or algebraic spaces, everything is obvious since the "morphism" $U \to \mathrm{Spec}\,R$ corresponds to an actual morphism of schemes or algebraic spaces and so the fiber product $U \times_{\mathrm{Spec}\,R} \mathrm{Spec}\,R'$ makes sense and represents $\underline{U}_{R'/R}$.

We will now handle the case of complex-analytic spaces, and the non-archimedean cases will go in exactly the same way since in all cases analytic affine spaces $\mathbf{A}^n$ and zero-spaces of coherent sheaves have the expected universal mapping properties. Choose an R-algebra isomorphism $R[t_1, \dots, t_n]/J \simeq R'$ for some ideal J. The "morphism" $U \to \mathrm{Spec}\,R$ corresponds by definition to a map of rings $\varphi : R \to \mathscr{O}(U)$, so we get a ring homomorphism

$$R[t_1, \dots, t_n] \to \mathscr{O}(U)[t_1, \dots, t_n] \to \mathscr{O}(U \times \mathbf{C}^n)$$

also denoted φ. Inside of $U \times \mathbf{C}^n$ it makes sense to form the zero locus of $\varphi(J)$, and its functor of points clearly coincides with $\underline{U}_{R'/R}$. (Note that there is no problem if J is not finitely generated, since rising chains of coherent ideals on complex-analytic spaces locally terminate; the analogue for rigid-analytic or Berkovich spaces is elementary.) It follows that this construction is independent of the presentation and is functorial in R' (independently of a choice of presentation); it is denoted $U_{R'/R}$. For any morphism $V \to U$, we clearly have $U_{R'/R} \times_U V = V_{R'/R}$; in the non-archimedean setting we likewise have compatibility with extension of the ground field.

It is clear that if $R' \to R''$ is another finite type algebra then $(U_{R'/R})_{R''/R'} = U_{R''/R}$, so the formation of $U_{R'/R}$ behaves in the expected manner with respect to replacing R' with a basic open affine algebra $R'[1/r']$. Thus, to prove that $U_{R'/R} \to U$ is an open immersion when $\operatorname{Spec} R' \to \operatorname{Spec} R$ is an open immersion, and that $\{U_{R_i/R} \to U\}$ is an open covering when $\{\operatorname{Spec} R_i \to \operatorname{Spec} R\}$ is an open cover, it suffices to consider R' and R_i which are basic open affine R-algebras. These cases are clear (e.g., if $\{r_i\}$ in R generates 1, the images of the r_i under a ring map $R \to \mathscr{O}(U)$ generate 1).

Finally, consider étale R-algebras R'. To prove that $U_{R'/R} \to U$ is étale, we can first work Zariski-locally to reduce to the case when R' is a standard étale R-algebra: $R' = (R[t]/(f))_{f'}$ for a monic $f \in R[t]$. Then by construction $U_{R'/R}$ is the non-vanishing locus of $\varphi(f') = \varphi(f)' \in \mathscr{O}(U)[t]$ on the zero-space of the monic $\varphi(f) \in \mathscr{O}(U)[t]$ in $U \times \mathbf{C}$. Thus, the inverse function theorem for complex-analytic spaces implies that $U_{R'/R} \to U$ is étale. If $\{\operatorname{Spec} R_i \to \operatorname{Spec} R\}$ is an étale cover, to prove that $\{U_{R_i/R} \to U\}$ is an étale cover we can use functoriality in U (and compatibility with change of the ground field in the non-archimedean setting) to reduce to the case when U is a single rational point. The problem is then to show that some $U_{R_i/R}$ is non-empty. By functoriality in the map $R \to R_i$, we can replace R with the field $k := \mathscr{O}(U)$ and R_i with the étale $\mathscr{O}(U)$-algebra $R_i \otimes_R k$ (since $\{\operatorname{Spec} R_i \to \operatorname{Spec} R\}$ is an étale cover, so it remains as such after scalar extension by $R \to k$). Thus, $U_{R_i/R}$ is the 0-dimensional space associated to the finite étale k-algebra R_i. One of the R_i is non-zero, so the corresponding $U_{R_i/R}$ is non-empty.

An unsatisfying feature of the definition of the object $\underline{U}_{R'/R}$ in T is that it involves the site X. It is more elegant to give a characterization of $\underline{U}_{R'/R}$ expressed in terms of T, its ring object $\mathscr{O}$, and its object $\underline{U}$ without reference to X. We will generalize the site-dependent definition of $\underline{U}_{R'/R}$ to the case when the representable $\underline{U}$ is replaced with any object $\mathscr{F}$ in T, and then reformulate things to avoid mentioning X.

For any $\mathscr{F}$ in T, define the "$\mathscr{F}$-valued points" of $\mathscr{O}$ to be the ring $\mathscr{O}(\mathscr{F}) := \operatorname{Hom}_T(\mathscr{F}, \mathscr{O})$. (If $\mathscr{F} = \underline{U}$ then this is identified with $\mathscr{O}(U)$ naturally in U.) Define a *morphism* $\mathscr{F} \to \operatorname{Spec} R$ (or more precisely, a morphism $(\mathscr{F}, \mathscr{O}) \to \operatorname{Spec} R$) to be a ring homomorphism $R \to \mathscr{O}(\mathscr{F})$; observe the role of $\mathscr{O}$ in this definition. This notion of morphism has an evident "associative" notion of composition with morphisms $\mathscr{F}' \to \mathscr{F}$ in T and morphisms $\operatorname{Spec} R' \to \operatorname{Spec} R$ of affine schemes. Note that we have used $(T, \mathscr{O})$ and have suppressed the mention of X.

For any $\mathscr{F}$ in T, morphism $\mathscr{F} \to \operatorname{Spec} R$, and R-algebra R', the functor $\mathscr{F}_{R'/R}$ on X is defined exactly like $\underline{U}_{R'/R}$, replacing the arrow $V \to U$ (i.e., an element of $\underline{U}(V)$) with an arrow $\underline{V} \to \mathscr{F}$ (i.e., an element of $\mathscr{F}(V)$). For representable $\mathscr{F} = \underline{U}$ this recovers $\underline{U}_{R'/R}$ as defined above (due to the equality $\mathscr{O}(\underline{V}) = \mathscr{O}(V)$). The functor $\mathscr{F}_{R'/R}$ on X depends covariantly on $\mathscr{F}$ and contravariantly on R' in the evident manner. As in the case of representable $\mathscr{F}$ considered above, we have

$$\mathscr{F}_{R'/R} = \mathscr{F} \times_{\mathscr{O}(R)} \mathscr{O}(R') \tag{2.1.1}$$

(so this is always a sheaf, and (2.1.1) provides a definition of $\mathscr{F}_{R'/R}$ that is intrinsic to $\mathscr{F}$ as an object in $(T, \mathscr{O})$). By Proposition 2.1.6(3), $\mathscr{F}_{R'/R}$ is a subfunctor of $\mathscr{F}$ on T via the forgetful map when $\operatorname{Spec} R' \to \operatorname{Spec} R$ is an open immersion.

There is a natural map $\mathscr{F}_{R'/R} \to \operatorname{Spec} R'$, or equivalently a ring homomorphism $R' \to \mathscr{O}(\mathscr{F}_{R'/R})$, functorial in the R-algebra R', via

$$\mathscr{F}_{R'/R} = \mathscr{F} \times_{\mathscr{O}(R)} \mathscr{O}(R') \to \mathscr{O}(R') \to \operatorname{Spec} R'. \tag{2.1.2}$$

Concretely, for any object V in X, any element $(f, s) \in \mathscr{F}_{R'/R}(V)$ gives rise to a map $s : V \to \operatorname{Spec} R'$ (i.e., a ring map $s^* : R' \to \mathscr{O}(V)$) *naturally in* V. Thus, from any $r' \in R'$ we obtain a system of set maps $\mathscr{F}_{R'/R}(V) \to \mathscr{O}(V)$ via $(f, s) \mapsto s^*(r')$ which are functorial in V, and the resulting map of sets $R' \to \mathscr{O}(\mathscr{F}_{R'/R})$ corresponds to the composition in (2.1.2).

Now we give the topos-theoretic characterization of $\mathscr{F}_{R'/R}$ as a kind of fiber product $\mathscr{F} \times_{\operatorname{Spec} R} \operatorname{Spec} R'$:

Proposition 2.1.8. *For any $\mathscr{G}$ in T, "composition" with $\mathscr{F}_{R'/R} \to \operatorname{Spec} R'$ and with the forgetful map $\mathscr{F}_{R'/R} \to \mathscr{F}$ identifies $\operatorname{Hom}_T(\mathscr{G}, \mathscr{F}_{R'/R})$ with the set of commutative diagrams*

$$\begin{array}{ccc} \mathscr{G} & \xrightarrow{f} & \mathscr{F} \\ {\scriptstyle s}\downarrow & & \downarrow \\ \operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R \end{array}$$

This proposition gives a characterization of $\mathscr{F}_{R'/R}$ in terms of $(T, \mathscr{O}, \mathscr{F}, R'/R)$ without mentioning X.

Proof. By expressing the problem in terms of the rings $\mathscr{O}(\mathscr{F})$ and $\mathscr{O}(\mathscr{G})$, the problem reduces to a routine diagram chase using Proposition 2.1.6(1) and the identification of $\mathscr{F}_{R'/R}$ with $\mathscr{F} \times_{\mathscr{O}(R)} \mathscr{O}(R')$. ■

Remark 2.1.9. The formation of $\mathscr{F}_{R'/R}$ is left-exact in $\mathscr{F}$, carries epimorphisms in $\mathscr{F}$ to epimorphisms, and naturally commutes with the formation of "disjoint union" (i.e., coproducts) in $\mathscr{F}$ in the sense that the natural map $\coprod(\mathscr{F}_i)_{R'/R} \to (\coprod \mathscr{F}_i)_{R'/R}$ is an isomorphism. These assertions are easily deduced from the description $\mathscr{F}_{R'/R} = \mathscr{F} \times_{\mathscr{O}(R)} \mathscr{O}(R')$, from which we also see that $\mathscr{F}_{R'/R}$ respects the formation of fiber products in $\mathscr{F}$.

2.2. **Locality for topoi.** Consider a collection of open immersions $\{\operatorname{Spec} R_\alpha \to \operatorname{Spec} R\}$ which is a covering for the Zariski topology. For any $\mathscr{F}$ in T, the objects $\mathscr{F}_\alpha := \mathscr{F}_{R_\alpha/R}$ all map to $\mathscr{F}$ as subobjects. Since we may think of the $\mathscr{F}_\alpha$ as fiber products $\mathscr{F} \times_{\operatorname{Spec} R} \operatorname{Spec} R_\alpha$, which is to say as "pullback" along the constituents of an open cover of $\operatorname{Spec} R$, it is natural to ask if the $\mathscr{F}_\alpha$ "cover" $\mathscr{F}$ in the sense that the natural map $\coprod \mathscr{F}_\alpha \to \mathscr{F}$ is an epimorphism in the topos T. In general this amounts to a restriction on $\mathscr{O}$, as the following example using representable $\mathscr{F}$ shows.

Example 2.2.1. Assume that T has enough points (e.g., the topos associated to a topological space), and let P be a conservative set of points. We claim that $\mathscr{O}_x$ is either 0 or local for all $x \in P$ if and only if $\{\mathscr{F}_\alpha \to \mathscr{F}\}$ is a covering for $\mathscr{F}$ in T, maps $\mathscr{F} \to \operatorname{Spec} R$ (i.e., all ring homomorphisms $R \to \mathscr{O}(\mathscr{F})$), and affine Zariski-open coverings $\{\operatorname{Spec} R_\alpha \to \operatorname{Spec} R\}$.

It clearly suffices to work with finite Zariski-open covering by basic open affines, which is to say that we can assume $R_\alpha = R[1/r_\alpha]$ for a finite set of r_α which generate 1 in R. We have $\sum r'_\alpha r_\alpha = 1$ for some $r'_\alpha \in R$. Define $A \subseteq R$ to be the $\mathbf{Z}$-subalgebra generated by the r_α and r'_α, and $A_\alpha = A[1/r_\alpha]$. By Proposition 2.1.8, clearly $\mathscr{F}_{A_\alpha/A} = \mathscr{F}_{R_\alpha/R}$, so it is *equivalent* to restrict to the case that R is finitely generated over $\mathbf{Z}$.

Since a map $\mathscr{F} \to \operatorname{Spec} R$ is a ring homomorphism $R \to \mathscr{O}(\mathscr{F})$, or equivalently a map $\mathscr{F} \to \mathscr{O}(R)$, by taking the universal case $\mathscr{F} = \mathscr{O}(R)$ (with the canonical map $\mathscr{O}(R) \to \operatorname{Spec} R$) it suffices to check that the maps $\{\mathscr{O}(R_\alpha) \to \mathscr{O}(R)\}$ are a covering in T precisely when each $\mathscr{O}_x$ is 0 or local. The covering property may be checked on stalks at $x \in P$, and so amounts to the condition that the maps of sets $\mathscr{O}(R_\alpha)_x \to \mathscr{O}(R)_x$ are collectively surjective. But R and each $R_\alpha = R[1/r_\alpha]$ is now finitely generated over $\mathbf{Z}$, so by Proposition 2.1.6(2) applied to x^* we are reduced to the case when T is the topos of sets.

Now $\mathscr{O}$ corresponds to a ring A, and the claim is that A is either 0 or local precisely if and only if for *any* Zariski-open covering $\{\operatorname{Spec} R_\alpha \to \operatorname{Spec} R\}$, every map $\operatorname{Spec} A \to \operatorname{Spec} R$ factors through some $\operatorname{Spec} R_\alpha$. The "only if" direction is obvious, and for the converse we take $R = A$ and argue by contradiction: if there are distinct closed points $\xi, \xi' \in \operatorname{Spec} A$ then we can take the $\operatorname{Spec} R_\alpha$ to be open affines each of which omits at least one of ξ or ξ'.

The preceding example motivates Grothendieck's definition of a locally ringed topos and Lurie's generalization of it to the étale topology. We first give a definition involving the site X, and then we express it in terms of T.

Definition 2.2.2. The ringed topos $(T, \mathscr{O})$ is *locally ringed* if $\{\mathscr{F}_\alpha \to \mathscr{F}\}$ is a covering in T for any $\mathscr{F}$ in T, morphism $(\mathscr{F}, \mathscr{O}) \to \operatorname{Spec} R$, and affine Zariski-open covering $\{\operatorname{Spec} R_\alpha \to \operatorname{Spec} R\}$. (In such cases, we say $\mathscr{O}$ is *Zariski-local*.) If the same holds using affine étale covers, then $(T, \mathscr{O})$ is *strictly henselian* and we say that $\mathscr{O}$ is *étale-local*.

For the reader who is concerned with the apparent quantification over "too many" things in Definition 2.2.2, note that it suffices (by the method of proof of Yoneda's Lemma) to check the definition only when the maps $R \to \mathscr{O}(\mathscr{F})$ are identity maps, and moreover if X is a (small) site giving rise to T then it suffices to restrict to representable $\mathscr{F}$ (as we saw in Example 2.2.1 by a formal argument which applies equally well with étale covers as it does with Zariski covers).

In the Zariski case of Definition 2.2.2 it suffices to use only Zariski covers by basic affine opens, in which case we recover a trivially equivalent form of Grothendieck's definition of a locally ringed topos [SGA4, IV, Exer. 13.9]. For later purposes, it is useful to give an alternative characterization for when $\mathscr{O}$ is Zariski-local (resp. étale-local). This rests on the following result.

Proposition 2.2.3. *Let $(T, \mathscr{O})$ be a ringed topos.*

1. *The ring object $\mathscr{O}$ is Zariski-local (resp. étale-local) if and only if for any collection of maps $\{\operatorname{Spec} R_\alpha \to \operatorname{Spec} R\}$ that is a Zariski covering (resp. étale covering), the induced collection of maps $\{\mathscr{O}(R_\alpha) \to \mathscr{O}(R)\}$ is a covering in T. Moreover, it suffices to restrict to R which are of finite type over $\mathbf{Z}$.*
2. *If T has enough points and P is a conservative set of points then $\mathscr{O}$ is étale-local if and only if for all $x \in P$ the stalk $\mathscr{O}_x$ is 0 or a strictly henselian local ring.*

Proof. Let X be a (small) site giving rise to T, and fix an object U in X and a morphism $U \to \operatorname{Spec} R$. This is exactly an element $f \in \mathscr{O}(R)(U)$, and for any V in X and ring map $R \to R'$ we see that $\underline{U}_{R'/R}(V)$ is identified with the fiber of $\mathscr{O}(R')(V) \to \mathscr{O}(R)(V)$ over the image of f under $\mathscr{O}(R)(U) \to \mathscr{O}(R)(V)$. Thus, the condition on a collection of ring maps $\{R \to R_\alpha\}$ that the maps $\underline{U}_{R_\alpha/R} \to \underline{U}$ are a covering in T is that for any map $\phi : V \to U$ in X there is a covering $\{V_i \to V\}$ such that the element $\mathscr{O}(\phi)(f) \in \mathscr{O}(R)(V)$ has restriction to each $\mathscr{O}(R)(V_i)$ that lifts to some $\mathscr{O}(R_{\alpha_i})(V_i)$. By considering the special case when ϕ is the identity map of V, it is necessary and sufficient that the collection of maps $\{\mathscr{O}(R_\alpha) \to \mathscr{O}(R)\}$ is a covering in T. Thus, (1) is proved, apart from the sufficiency of using R of finite type over $\mathbf{Z}$.

Since we may certainly restrict attention to finite collections of R_α, as étale maps have open image (and $\operatorname{Spec} R$ is quasi-compact), by finite presentation of the R_α over R we can always descend the covering $\{\operatorname{Spec} R_\alpha \to \operatorname{Spec} R\}$ to a covering $\{\operatorname{Spec} A_\alpha \to \operatorname{Spec} A\}$ (of the same type) over a finite type $\mathbf{Z}$-subalgebra A in R. Then, akin to the argument in Example 2.2.1, we have

$$\mathscr{O}(R_\alpha) = \mathscr{O}(R) \times_{\mathscr{O}(R)} \mathscr{O}(R_\alpha) = \mathscr{O}(R)_{R_\alpha/R} = \mathscr{O}(R)_{A_\alpha/A} = \mathscr{O}(R) \times_{\mathscr{O}(A)} \mathscr{O}(A_\alpha).$$

It therefore suffices to prove that $\{\mathscr{O}(A_\alpha) \to \mathscr{O}(A)\}$ is a covering, so the final claim in (1) is proved.

The proof of (2) is similar to the Zariski case in Example 2.2.1, up to two small changes. First, to reduce to the case when T is the topos of sets, by the local structure theorem for étale morphisms [EGA, IV_4, 18.4.6(ii)] we replace argumentsusing just basic affine opens with the analogous arguments using a larger class of finitely presented algebras: basic affine opens in standard étale algebras (i.e., $(B[t]/(h))[1/h']$ for monic $h \in B[t]$) over basic affine opens. Then it remains to check that if A is a ring then it is either 0 or strictly henselian local if and only if for any collection of maps $\{\operatorname{Spec} R_\alpha \to \operatorname{Spec} R\}$ which is an étale covering, any map $\operatorname{Spec} A \to \operatorname{Spec} R$ factors through some $\operatorname{Spec} R_\alpha$. Restricting this latter condition to the special case of Zariski coverings and using the known Zariski-local case allows us to restrict attention to the case when A is either local or 0. Then our hypothesis on A (when it is local) is exactly one of the characterizations of strictly henselian local rings among all local rings in [EGA, IV_4, 18.8.1(c)]. ■

Corollary 2.2.4. *Let $f : T' \to T$ be a map of topoi, and $\mathscr{O}$ a ring object in T. If $\mathscr{O}$ is Zariski-local (resp. étale-local) in T, then so is $f^*\mathscr{O}$ in T'.*

Proof. This is immediate from Proposition 2.2.3(1) and Proposition 2.1.6(2). ■

Since Zariski covers are étale covers, obviously any strictly henselian topos is locally ringed. By Example 2.1.7, the following ringed topoi are strictly henselian: schemes and algebraic spaces with the étale topology, complex-analytic spaces, and Example 2.1.1.

2.3. **Locality for morphisms.** We need to introduce a property of morphisms between strictly henselian topoi that refines the notion of morphism of locally ringed topoi (to be reviewed in Definition 2.3.4). This will enable us to faithfully put algebraic spaces and non-archimedean analytic spaces into a common (bi)category so that it makes sense to speak of morphisms between them.

Remark 2.3.1. For a pair of topoi X' and X, the collection of morphisms $\mathrm{Hom}(X', X)$ is not a set, but rather is a category. Namely, if $f, g : X' \rightrightarrows X$ are two morphisms (i.e., pairs of adjoint functors (f_*, f^*) and (g_*, g^*) satisfying the usual axioms), then $\mathrm{Hom}(f, g)$ is the set of natural transformations $F : f_* \to g_*$, or equivalently the set of natural transformations $F' : g^* \to f^*$ (which is a set due to the exactness of g^* and f^* as well as the fact that we work with topoi arising from small sites). For example, if we work with the topoi associated to topological spaces, and take f and g to respectively arise from inclusions of the 1-point space onto points x and y of X such that $x \neq y$ but y is in the closure of x then there is a natural map of stalk functors $g^* \to f^*$ and hence a natural transformation $f_* \to g_*$.

In this way, the right framework for considering morphisms between topoi is that of 2-categories, or really bicategories, in which the 2-morphisms between a pair of maps among objects may not be invertible and the composition law

$$\mathrm{Hom}(X'', X') \times \mathrm{Hom}(X', X) \to \mathrm{Hom}(X'', X)$$

viewed as a bifunctor satisfies "associativity" and "identity element" axioms expressed in terms of auxiliary isomorphism data. We will not state the axioms on such isomorphism data here, but simply note that in the cases of interest to us below they are the expected isomorphisms expressing associativity of pushforwards and pullbacks. We will not need any facts from, or even the existence of, the theory of bicategories (see [Bén, §1, §4] for further details on the basic definitions), and the word "bicategory" is used below because it is the convenient and appropriate thing to do. The reader who dislikes this will lose nothing by always thinking in terms of the specific topoi that arise (étale topoi on algebraic spaces, Berkovich spaces, etc.).

The bicategory of ringed topoi is defined in the expected manner, with a *morphism* from $(f, f^\#)$ to $(g, g^\#)$ being a natural transformation $f_* \to g_*$ such that $f_*\mathscr{O}' \to g_*\mathscr{O}'$ is an $\mathscr{O}$-algebra map (via the $\mathscr{O}$-structures $f^\#$ and $g^\#$). Rather than speaking of a pair of morphisms of topoi (or ringed topoi) $X' \rightrightarrows X$ being *equal*, we only speak of them being *equivalent* (or *isomorphic*) in the sense of natural equivalence of functors.

Beware that even though we are working with small sites, it can happen (e.g., when using classifying topoi) that with this notion of equivalence, the collection of equivalence classes of morphisms between two topoi X' and X (i.e., the discrete category arising from the category $\mathrm{Hom}(X', X)$) is too large to be a set. An analogue is the fact that any particular abelian group is a direct limit of its finitely generated subgroups, and finitely generated abelian groups up to isomorphism constitute a set, but the category of abelian groups is not small. As a consequence of the intervention of bicategories, when contemplating a "universal mapping property" for morphisms between topoi we do not work with the usual Yoneda Lemma in terms of Hom-sets or representable Set-valued functors. Nonetheless, Yoneda-style reasoning is still useful, and we will state in explicit terms the specific universal mapping properties that we require.

Definition 2.3.2. A covariant functor $F : C \to C'$ between bicategories is *faithful* if the associated natural transformation $h_{F,X,Y} : \mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_{C'}(F(X), F(Y))$ is fully faithful for any pair of objects X and Y. The functor F is *fully faithful* when $h_{F,X,Y}$ is an equivalence of categories for any X, Y.

The case of most interest to us will be when C is an ordinary category (i.e., $\mathrm{Hom}_C(X, Y)$ is a set), in which case Definition 2.3.2 takes on a more concrete form as follows. The functor F is faithful precisely when $\mathrm{Hom}_{C'}(F(f), F(g))$ is empty if $f \neq g$ and is $\{\mathrm{id}\}$ when $f = g$. Likewise, such an F is full faithful precisely when it is faithful and every natural transformation $F(X) \to F(Y)$ is (necessarily uniquely) isomorphic to $F(f)$ for a (necessarily unique) morphism $f : X \to Y$ in C.

Example 2.3.3. A very important example is the case of sober topological spaces (those in which every irreducible closed set has a unique generic point, such as schemes and locally Hausdorff topological spaces). By [MM, IX, §3, Cor. 4] and [MM, IX, §5, Prop. 2], if $\mathscr{X}'$ and $\mathscr{X}$ are sober and T' and T are their associated topoi, then any map of topoi $T' \to T$ is uniquely isomorphic to the map $\widetilde{f}$ arising from a unique continuous map $f : \mathscr{X}' \to \mathscr{X}$; the uniqueness of the isomorphism is due to the fact that the site associated to a topological space has no non-identity endomorphisms among its objects. It follows that for any ring objects $\mathscr{O}$ and $\mathscr{O}'$ in T and T' respectively, any map of ringed topoi $(T', \mathscr{O}') \to (T, \mathscr{O})$ is uniquely isomorphic to the map arising from a unique map of ringed spaces $(f, f^{\#}) : (\mathscr{X}', \mathscr{O}') \to (\mathscr{X}, \mathscr{O})$.

Beware that the functor from the category of sober topological spaces to the bicategory of topoi is not faithful in the sense of Definition 2.3.2. For example, when $\mathscr{X}'$ is a 1-point space this corresponds to the fact that for the stalk functors i_x and i_y associated to points $x, y \in \mathscr{X}$ respectively, there is a morphism $i_x \to i_y$ whenever x is in the closure of y. A similar problem arises for ringed spaces and the bicategory of ringed topoi. To get faithfulness results for interesting locally ringed spaces we will need to work with locally ringed topoi, as will be considered in Proposition 3.1.1.

Let X and X' be (small) sites on which all representable functors are sheaves, and choose ring objects $\mathscr{O}$ and $\mathscr{O}'$ in the associated topoi T and T'. Consider a morphism $f : (T', \mathscr{O}') \to (T, \mathscr{O})$. In particular, there is a given map $\mathscr{O} \to f_*\mathscr{O}'$ of ring objects in T and hence a map $f^*\mathscr{O} \to \mathscr{O}'$ of ring objects in T'. Assume that $\mathscr{O}$ and $\mathscr{O}'$ are locally ringed.

Consider $\mathscr{F}$ in T equipped with a morphism $\mathscr{F} \to \operatorname{Spec} A$ for a ring A (i.e., a ring map $A \to \mathscr{O}(\mathscr{F})$). For any A-algebra B, we have defined the object $\mathscr{F}_{B/A}$ in T. The functor f^* and the map of ring objects $f^*\mathscr{O} \to \mathscr{O}'$ yield a map of rings

$$\mathscr{O}(\mathscr{F}) := \operatorname{Hom}_T(\mathscr{F}, \mathscr{O}) \xrightarrow{f^*} \operatorname{Hom}_{T'}(f^*\mathscr{F}, f^*\mathscr{O}) \to \operatorname{Hom}_{T'}(f^*\mathscr{F}, \mathscr{O}') =: \mathscr{O}'(f^*\mathscr{F}), \tag{2.3.1}$$

so composition with the chosen $A \to \mathscr{O}(\mathscr{F})$ yields a morphism $f^*\mathscr{F} \to A$. We likewise define $f^*(\mathscr{F}_{B/A}) \to \operatorname{Spec} B$, and unraveling the definitions shows that the diagram

$$\begin{array}{ccc} f^*(\mathscr{F}_{B/A}) & \longrightarrow & f^*\mathscr{F} \\ \downarrow & & \downarrow \\ \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A \end{array}$$

commutes (i.e., both compositions around the diagram define the same ring map $A \to \mathscr{O}'(f^*(\mathscr{F}_{B/A}))$). Hence, Proposition 2.1.8 yields a map

$$\theta_{\mathscr{F},B/A} : f^*(\mathscr{F}_{B/A}) \to (f^*\mathscr{F})_{B/A}$$

in T' respecting the natural morphisms from both sides to $\operatorname{Spec} B$. In more suggestive terms, this is a map

$$f^*(\mathscr{F} \times_{\operatorname{Spec} A} \operatorname{Spec} B) \to f^*\mathscr{F} \times_{\operatorname{Spec} A} \operatorname{Spec} B$$

in T' over $\operatorname{Spec} B$, so it is a kind of "base change morphism" for f^*. This leads to:

Definition 2.3.4. The morphism f is *local for the Zariski topology* (or is a *morphism of locally ringed topoi*) if $\theta_{\mathscr{F},B/A}$ an isomorphism for every $\mathscr{F}$ in T, morphism $\mathscr{F} \to \operatorname{Spec} A$, and Zariski-open immersion $\operatorname{Spec} B \to \operatorname{Spec} A$.

If $\mathscr{O}$ and $\mathscr{O}'$ are strictly henselian in the sense of Definition 2.2.2 then f is *local for the étale topology* (or is a *strictly henselian morphism*) if the analogous isomorphism condition holds whenever B is A-étale.

To explain why there are no set-theoretic quantification issues in Definition 2.3.4, first note that since B is finitely presented over A, by the same trick that was used in Example 2.2.1 and the proof of Proposition 2.2.3 we may restrict attention to A which are finite type over $\mathbf{Z}$. Likewise, by exactness of f^* and the formula $\mathscr{F}_{B/A} = \mathscr{F} \times_{\mathscr{O}(A)} \mathscr{O}(B)$, it suffices to treat the case of representable $\mathscr{F}$ when we have chosen a small site X giving rise to T. This is sometimes convenient in practice. If we restrict attention to basic affine opens $B = A[1/a]$ in Definition 2.3.4 then we recover Grothendieck's definition of a morphism of locally ringed topoi [SGA4, IV, Exer. 13.9]. It is clear conversely that this special case of the definition implies the general case, so the condition of being local for the Zariski topology in the sense of Definition 2.3.4 is equivalent to Grothendieck's notion of morphism of locally ringed topoi.

Moreover, if we unravel the definitions then $\theta_{\mathscr{F},B/A}$ factors as the composition

$$f^*(\mathscr{F} \times_{\mathscr{O}(A)} \mathscr{O}(B)) \simeq f^*(\mathscr{F}) \times_{f^*(\mathscr{O}(A))} f^*(\mathscr{O}(B)) \to f^*(\mathscr{F}) \times_{(f^*\mathscr{O})(A)} (f^*\mathscr{O})(B) \to f^*(\mathscr{F}) \times_{\mathscr{O}'(A)} \mathscr{O}'(B)$$

where the second map uses the natural morphism $f^*(\mathscr{O}(R)) \to (f^*\mathscr{O})(R)$ from Proposition 2.1.6(2) (which is an isomorphism for R of finite type over $\mathbf{Z}$) and the third map uses the natural morphism $f^*\mathscr{O} \to \mathscr{O}'$. Hence, when we restrict to A of finite type over $\mathbf{Z}$, it is equivalent for the maps

$$f^*\mathscr{F} \times_{(f^*\mathscr{O})(A)} (f^*\mathscr{O})(B) \to f^*\mathscr{F} \times_{\mathscr{O}'(A)} \mathscr{O}'(B)$$

to be isomorphisms. But this is a fiber product of $f^*\mathscr{F}$ over $(f^*\mathscr{O})(A)$ against the map

$$(f^*\mathscr{O})(B) \to (f^*\mathscr{O})(A) \times_{\mathscr{O}'(A)} \mathscr{O}'(B) \tag{2.3.2}$$

(which in turn is exactly the special case $\mathscr{F} = \mathscr{O}(A)$ when A is finite type over $\mathbf{Z}$). This final condition only involves the ring map $f^*\mathscr{O} \to \mathscr{O}'$ in T', so by recalling Corollary 2.2.4 we arrive at:

Proposition 2.3.5. *Suppose that $(T, \mathscr{O})$ and $(T', \mathscr{O}')$ are ringed topoi, with $\mathscr{O}$ and $\mathscr{O}'$ both Zariski-local (resp. étale-local), so $f^*\mathscr{O}'$ has the same property.*

1. *A map of ringed topoi $f : (T', \mathscr{O}') \to (T, \mathscr{O})$ is local for the Zariski topology (resp. strictly henselian) if and only if the map $f^*\mathscr{O} \to \mathscr{O}'$ in T' (i.e., the morphism $(T', \mathscr{O}') \to (T', f^*\mathscr{O})$) has the same locality property.*
2. *It is equivalent in (1) that for any map of affines $\operatorname{Spec} B \to \operatorname{Spec} A$ that is a Zariski-open immersion (resp. étale), the induced map (2.3.2) in T' is an isomorphism. Moreover, it suffices to consider A of finite type over $\mathbf{Z}$.*

Example 2.3.6. Let $f : (\mathscr{X}', \mathscr{O}') \to (\mathscr{X}, \mathscr{O})$ be a morphism of locally ringed spaces. Example 2.1.3 (which is trivial in the case of schemes) implies that the associated map $\widetilde{\mathscr{X}'} \to \widetilde{\mathscr{X}}$ of ringed topoi is local for the Zariski topology (i.e., it is a morphism of locally ringed topoi). Thus, $(\mathscr{X}, \mathscr{O}) \rightsquigarrow \widetilde{\mathscr{X}}$ defines a functor from the category of locally ringed spaces (with local morphisms) to the bicategory of locally ringed topoi (with Zariski-local morphisms).

Proposition 2.3.7 (Lurie)**.** *For strictly henselian ringed topoi $(T, \mathscr{O})$ and $(T', \mathscr{O}')$, morphisms $(T', \mathscr{O}') \to (T, \mathscr{O})$ as locally ringed topoi are automatically strictly henselian.*

Proof. I am grateful to Lurie for providing the following concrete version of an argument given in a more general framework in [DAGV]. By Proposition 2.3.5(1) we may assume $T' = T$ and that the underlying map of topoi is the identity. Thus, by Proposition 2.3.5(2), we are given a map $\mathscr{O} \to \mathscr{O}'$ between strictly henselian ring objects in T such that

$$\mathscr{O}(B) \to \mathscr{O}(A) \times_{\mathscr{O}'(A)} \mathscr{O}'(B) \tag{2.3.3}$$

is an isomorphism in T for any Zariski-open immersion $\operatorname{Spec} B \to \operatorname{Spec} A$, and we seek to prove the same when B is merely étale over A.

For an étale A-algebra B, the image of $\operatorname{Spec} B$ in $\operatorname{Spec} A$ is open and quasi-compact, so it is covered by basic affine opens $\operatorname{Spec} A[1/a_i]$ for finitely many $a_i \in A$. Letting $b_i \in B$ denote the image of a_i, the b_i generate 1, so $\{\operatorname{Spec} B[1/b_i]\}$ is a Zariski covering of $\operatorname{Spec} B$. Thus, since $\mathscr{O}'$ is Zariski-local, the collection of maps $\mathscr{O}'(B[1/b_i]) \to \mathscr{O}'(B)$ is a covering in T. Thus, to prove the isomorphism property of (2.3.3) it suffices to do so after pullback to each $\mathscr{O}'(B[1/b_i])$: this yields the maps

$$\mathscr{O}(B) \times_{\mathscr{O}'(B)} \mathscr{O}'(B[1/b_i]) \to \mathscr{O}(A) \times_{\mathscr{O}'(A)} \mathscr{O}'(B[1/b_i]) = (\mathscr{O}(A) \times_{\mathscr{O}'(A)} \mathscr{O}'(A[1/a_i])) \times \mathscr{O}'(A[1/a_i]) \mathscr{O}'(B[1/b_i]).$$

Applying the isomorphism property in (2.3.3) for the basic open affine algebras $A \to A[1/a_i]$ and $B \to B[1/b_i]$ then identifies the above map with (2.3.3) for the étale algebra $A[1/a_i] \to B[1/b_i]$ that is faithfully flat. Thus, we may now assume that $\operatorname{Spec} B \to \operatorname{Spec} A$ is an étale *covering*.

Since $\mathscr{O}$ is strictly henselian, $\mathscr{O}(B) \to \mathscr{O}(A)$ is a covering in T and so to prove that the map (2.3.3) over $\mathscr{O}(A)$ in T is an isomorphism, it suffices to do so after pullback along $\mathscr{O}(B) \to \mathscr{O}(A)$. This is the natural map

$$\mathscr{O}(B) \times_{\mathscr{O}(A)} \mathscr{O}(B) \to \mathscr{O}(B) \times_{\mathscr{O}'(A)} \mathscr{O}'(B) = \mathscr{O}(B) \times_{\mathscr{O}'(B)} (\mathscr{O}'(B) \times_{\mathscr{O}'(A)} \mathscr{O}'(B)).$$

The natural maps

$$\mathscr{O}(B \otimes_A B) \to \mathscr{O}(B) \times_{\mathscr{O}(A)} \mathscr{O}(B), \quad \mathscr{O}'(B \otimes_A B) \to \mathscr{O}'(B) \times_{\mathscr{O}'(A)} \mathscr{O}'(B)$$

are isomorphisms, due to the general definition $\mathscr{O}(R)(U) = \mathrm{Hom}(R, \mathscr{O}(U))$, so we can replace $A \to B$ with the first factor inclusion $B \to B \otimes_A B$ to reduce to the case when there is a section $s : B \to A$. Hence, since B is étale over A, s identifies $\mathrm{Spec}\, A$ with an open subscheme of $\mathrm{Spec}\, B$ *over* $\mathrm{Spec}\, A$. The Zariski-local hypothesis on $\mathscr{O} \to \mathscr{O}'$ then implies that the natural map

$$\mathscr{O}(A) \to \mathscr{O}(B) \times_{\mathscr{O}'(B)} \mathscr{O}'(A)$$

over $\mathscr{O}'(A)$ is an isomorphism. The resulting composite map

$$\mathscr{O}(B) \to \mathscr{O}(A) \times_{\mathscr{O}'(A)} \mathscr{O}'(B) \simeq (\mathscr{O}(B) \times_{\mathscr{O}'(B)} \mathscr{O}'(A)) \times_{\mathscr{O}'(A)} \mathscr{O}'(B) = \mathscr{O}(B)$$

is readily checked to be the identity map, so the first step is an isomorphism. ■

In the examples we need, it will be easy to directly verify the strictly henselian property for the morphisms arising from geometric maps. Thus, for our purposes the main point of Proposition 2.3.7 is to simplify the statements of results, rather than to simplify proofs.

3. Construction of morphisms

We now take up two primary tasks. In §3.1 we faithfully (and sometimes fully faithfully) embed various categories of geometric objects into (bi)categories of strictly henselian topoi. Then in §3.2 we prove universal mapping properties for complex-analytifications in terms of such topoi, thereby answering the complex-analytic case of the Question raised in §1.1. We conclude in §3.3 by adapting those arguments to the non-archimedean case, completing the proof of Theorem 1.2.1.

3.1. **Faithfulness results.** Building on the consideration of ringed topoi in Example 2.3.3, we have:

Proposition 3.1.1. *Let $(\mathscr{X}', \mathscr{O}')$ and $(\mathscr{X}, \mathscr{O})$ be locally ringed topoi, and $f : (\mathscr{X}', \mathscr{O}') \to (\mathscr{X}, \mathscr{O})$ a morphism of ringed topoi.*

(1) *The map f is a map of locally ringed spaces if and only if the corresponding map $\widetilde{f}$ of ringed topoi is local for the Zariski topology.*
(2) *The functor from sober locally ringed spaces to locally ringed topoi is* fully faithful *when restricted to the full subcategory of schemes, as well as the full subcategory of locally Hausdorff objects.*

Part (2) fixes the lack of faithfulness in Example 2.3.3.

Proof. It is clear that if f is locally ringed, then $\widetilde{f}$ is local for the Zariski topology. For the converse we have to show that if $\widetilde{f}$ is local for the Zariski topology then f respects the local structure of the stalks. Assume not, so for some $x' \in \mathscr{X}'$ the map $\mathscr{O}_{f(x')} \to \mathscr{O}'_{x'}$ is not local. By shrinking around x' and x, we can arrange that there exists $a \in \mathscr{O}(\mathscr{X})$ which is a non-unit at $f(x')$ such that its image under $\mathscr{O} \to f_*(\mathscr{O}')$ is a unit in $(f_*\mathscr{O}')(\mathscr{X}) = \mathscr{O}'(\mathscr{X}')$.

Consider the morphism $(\mathscr{X}, \mathscr{O}) \to \mathrm{Spec}\, \mathbf{Z}[t]$ corresponding to $t \mapsto a$, and the Zariski-open subscheme $\mathrm{Spec}\, \mathbf{Z}[t, 1/t] \hookrightarrow \mathrm{Spec}\, \mathbf{Z}[t]$. Since $\widetilde{f}$ is local for the Zariski topology, $f^*(\mathscr{O}_{\mathbf{Z}[t,1/t]/\mathbf{Z}[t]}) = \mathscr{O}'_{\mathbf{Z}[t,1/t]/\mathbf{Z}[t]}$. But $\mathscr{O}_{\mathbf{Z}[t,1/t]/\mathbf{Z}[t]}$ is represented by the maximal open subspace $\mathscr{X}_a$ of $\mathscr{X}$ on which a is a unit, and likewise $\mathscr{O}'_{\mathbf{Z}[t,1/t]/\mathbf{Z}[t]}$ is represented by $\mathscr{X}'_{f^*(a)}$. Thus, $f^*(\mathscr{O}_{\mathbf{Z}[t,1/t]/\mathbf{Z}[t]})$ is represented by $f^{-1}(\mathscr{X}_a)$, so we conclude that $f^{-1}(\mathscr{X}_a) = \mathscr{X}'_{f^*(a)} = \mathscr{X}'$. This says that $f(\mathscr{X}') \subseteq \mathscr{X}_a$, which is absurd since $f(x') \notin \mathscr{X}_a$.

Finally, we prove that when the functor from sober locally ringed spaces to locally ringed topoi is restricted to either (i) schemes or (ii) locally Hausdorff objects then it is fully faithful. In view of Example 2.3.3, the problem is to prove that if $f, g : (\mathscr{X}', \mathscr{O}') \rightrightarrows (\mathscr{X}, \mathscr{O})$ are maps between sober locally ringed spaces then in both cases (i) and (ii) the set $\mathrm{Hom}(\widetilde{f}, \widetilde{g})$ is empty if $f \neq g$ and is $\{\mathrm{id}\}$ if $f = g$.

Assume there is a natural transformation $F : \widetilde{f} \to \widetilde{g}$, so there is a natural transformation $f_* \to g_*$, or equivalently $g^* \to f^*$. The latter implies that $g^{-1}(U) \subseteq f^{-1}(U)$ for all open $U \subseteq \mathscr{X}$, and the existence of

F implies that the diagram

$$\begin{array}{ccc} \mathscr{O}(U) & \xrightarrow{f^\#} & \mathscr{O}'(f^{-1}(U)) \\ & \searrow^{g^\#} & \downarrow \text{res} \\ & & \mathscr{O}'(g^{-1}(U)) \end{array}$$

commutes. Thus, to prove $f = g$ it suffices to prove equality on underlying topological spaces. For $x' \in \mathscr{X}'$ we may compose with $(\{x'\}, k(x')) \to (\mathscr{X}', \mathscr{O}')$ to reduce to the case that $\mathscr{X}'$ has a single point x'. We have $f(x') \in U$ whenever $g(x') \in U$, so $g(x')$ is in the closure of $f(x')$ and passing to the limit on U containing $g(x')$ yields a commutative diagram

(3.1.1)
$$\begin{array}{ccc} \mathscr{O}_{g(x')} & \longrightarrow & \mathscr{O}_{f(x')} \\ {\scriptstyle g^\#}\downarrow & & \downarrow{\scriptstyle f^\#} \\ k(x') & = & k(x') \end{array}$$

in which the vertical maps are local. In the locally Hausdorff case, $g(x') = f(x')$ since all points are closed. In the scheme case the map $\mathscr{O}_{g(x')} \to \mathscr{O}_{f(x')}$ is the natural localization map, and by chasing kernel ideals in (3.1.1) it follows that this localization map carries $\mathfrak{m}_{g(x')}$ into $\mathfrak{m}_{f(x')}$, so again $f(x') = g(x')$. Hence, in cases (i) and (ii) we have $\mathrm{Hom}(\widetilde{f}, \widetilde{g})$ is empty except when $f = g$. Since objects in the site associated to a topological space have no nontrivial endomorphisms, it is clear that $\mathrm{Hom}(\widetilde{f}, \widetilde{f}) = \{\mathrm{id}\}$. ■

Example 3.1.2. By Example 2.1.7 (and the sufficiency of using representable $\mathscr{F}$ in Definition 2.3.4), the ringed topos associated to any complex-analytic space is strictly henselian. Likewise, any morphism $\mathscr{X}' \to \mathscr{X}$ between complex-analytic spaces induces a strictly henselian morphism over $\mathbf{C}$. The functor $\mathscr{X} \rightsquigarrow \widetilde{\mathscr{X}}$ from the category of complex-analytic spaces to the bicategory of strictly henselian topoi over $\mathbf{C}$ is fully faithful, due to Proposition 3.1.1.

To extend Proposition 3.1.1 to the case of algebraic spaces equipped with their étale topoi, let $f : (\mathscr{X}', \mathscr{O}') \to (\mathscr{X}, \mathscr{O})$ be a morphism of algebraic spaces and consider the associated étale sites (whose objects may be either schemes or algebraic spaces; it will work the same either way). This includes the case of a morphism of schemes equipped with their étale topologies. In particular, $\mathscr{O}'$ and $\mathscr{O}$ are strictly henselian. By Example 2.1.7 and the sufficiency of using representable $\mathscr{F}$ in Definition 2.3.4, $(\mathscr{X}, \mathscr{O}) \rightsquigarrow \widetilde{\mathscr{X}_{\text{ét}}}$ is a functor from the category of algebraic spaces to the bicategory of strictly henselian topoi (equipped with strictly henselian morphisms). In the spirit of Remark 2.3.1, we recall what this means: for any map of algebraic spaces $f : (\mathscr{X}', \mathscr{O}') \to (\mathscr{X}, \mathscr{O})$ we have an associated morphism of strictly henselian topoi

$$\widetilde{f} = (f_*, f^*) : \widetilde{\mathscr{X}'_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{ét}}},$$

and if $f' : (\mathscr{X}'', \mathscr{O}'') \to (\mathscr{X}', \mathscr{O}')$ is another such map of algebraic spaces then there is a specified isomorphism between $\widetilde{f' \circ f}$ and $\widetilde{f'} \circ \widetilde{f}$ satisfying certain "associativity" conditions.

Theorem 3.1.3. *The functor $(\mathscr{X}, \mathscr{O}) \rightsquigarrow \widetilde{\mathscr{X}_{\text{ét}}}$ from the category of algebraic spaces to the bicategory of strictly henselian topoi is fully faithful in the sense that (i) every morphism $\widetilde{\mathscr{X}'_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{ét}}}$ as locally ringed topoi is isomorphic to $\widetilde{f}$ for a unique map of algebraic spaces $f : (\mathscr{X}', \mathscr{O}') \to (\mathscr{X}, \mathscr{O})$, (ii) $\mathrm{Hom}(\widetilde{f}, \widetilde{g})$ is empty if $f \neq g$ and consists of only the identity when $f = g$.*

By Proposition 2.3.7, the morphisms of locally ringed topoi considered in this theorem are automatically strictly henselian morphisms (in the sense of Definition 2.3.4).

Proof. For a map of algebraic spaces $f : (\mathscr{X}', \mathscr{O}') \to (\mathscr{X}, \mathscr{O})$ and an étale map $U \to \mathscr{X}$ from an algebraic space, $f^*(\underline{U})$ is represented by $U' := \mathscr{X}' \times_{\mathscr{X}} U$. Also, the map of algebraic spaces $U' \to U$ induced by f has associated map $\widetilde{U'_{\text{ét}}} \to \widetilde{U_{\text{ét}}}$ of ringed topoi that is induced by viewing them as respective subcategories of $\widetilde{\mathscr{X}'_{\text{ét}}}$ and $\widetilde{\mathscr{X}_{\text{ét}}}$. Hence, by taking U to be a scheme étale over $\mathscr{X}$, if we can settle faithfulness when $\mathscr{X}$ is a scheme then it will hold in general. Thus, to prove faithfulness we can assume $\mathscr{X}$ is a scheme, and then we can assume $\mathscr{X}'$ is a scheme. We may likewise reduce to the case when $\mathscr{X}$ is affine. In this case $\widetilde{f}$ uniquely determines f (by evaluating $\mathscr{O} \to f_*(\mathscr{O}')$ on $\mathscr{X}$).

To settle faithfulness, we just have to prove that $\mathrm{Hom}(\widetilde{f}, \widetilde{g})$ is empty when $f \neq g$ and is the identity when $f = g$. If there is such a map then the same holds for the restricted functors on the underlying locally ringed Zariski topoi, which forces $f = g$ by Proposition 3.1.1. To prove that $\mathrm{Hom}(\widetilde{f}, \widetilde{f})$ has no non-identity elements we may work étale-locally to again reduce back to the Zariski topos case which was settled by Proposition 3.1.1.

Now we prove full faithfulness, so choose a strictly henselian morphism $F : \widetilde{\mathscr{X}'_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{ét}}}$ between the associated strictly henselian topoi. The aim is to prove that $F \simeq \widetilde{f}$ for some f, in which case we will call F *geometric*. Consider an étale cover $\{U_i \to \mathscr{X}\}$ by affine schemes, so $\{\underline{U}_i \to \underline{\mathscr{X}}\}$ is a cover of the final object in the topos. Hence, $\{F^*(\underline{U}_i)\}$ is a cover of $\underline{\mathscr{X}}'$. Let $\{\underline{U}'_{ij} \to F^*(\underline{U}_i)\}_{j \in J_i}$ be a cover of $F^*(\underline{U}_i)$ with U'_{ij} a scheme étale over $\mathscr{X}'$. Hence, $\{U'_{ij}\}_{i,j}$ is an étale cover of $\mathscr{X}'$ by schemes, so F is dominated by a collection of strictly henselian morphisms $F_{ij} : (U'_{ij})^{\sim}_{\text{ét}} \to (U_i)^{\sim}_{\text{ét}}$ associated to the schemes U'_{ij} and U_i.

Assume we can settle full faithfulness in the case of schemes, with $\mathscr{X}$ affine (using the étale topoi). Thus, for the schemes $U' = \coprod_{ij} U'_{ij}$ and $U = \coprod_i U_i$ that are respectively étale covers of $\mathscr{X}'$ and $\mathscr{X}$ we thereby get a map of schemes $h : U' \to U$ such that the diagram

$$\begin{array}{ccc} \widetilde{U'_{\text{ét}}} & \xrightarrow{\widetilde{h}} & \widetilde{U_{\text{ét}}} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ \widetilde{\mathscr{X}'_{\text{ét}}} & \xrightarrow[F]{} & \widetilde{\mathscr{X}_{\text{ét}}} \end{array}$$

commutes up to *canonical* equivalence (expressed in terms of descent theory with pullback sheaves). The composite map $\widetilde{U'_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{ét}}}$ across the top and right sides is geometric (as each step is geometric). By using factorization through the left and bottom sides it follows from the established faithfulness that the resulting map $U' \to \mathscr{X}$ yields the same composition with both projections $p'_1, p'_2 : U' \times_{\mathscr{X}'} U' \rightrightarrows U'$. Hence, by étale descent for morphisms of algebraic spaces, we obtain a morphism $f : \mathscr{X}' \to \mathscr{X}$ of algebraic spaces such that there is an isomorphism $\xi : \widetilde{f} \circ \pi \simeq F \circ \pi$. But any object V' in $\mathscr{X}'_{\text{ét}}$ is *functorially* determined by $\pi^*(\underline{V}')$ *equipped with* the isomorphism ${p'_1}^*(\pi^*(\underline{V}')) \simeq {p'_2}^*(\pi^*(\underline{V}'))$ satisfying the cocycle condition. It follows that ξ descends to an isomorphism $F \simeq \widetilde{f}$, so F is geometric.

It remains to prove full faithfulness in the special case that $\mathscr{X}'$ and $\mathscr{X}$ are schemes equipped with the étale topology and $\mathscr{X}$ is affine. By [SGA4, VIII, Prop. 6.1], the subobjects of the final object of the étale topos of a scheme are precisely the functors represented by open subschemes (for the Zariski topology). Moreover, by [SGA4, VIII, 7.9], the points of the étale topos of a scheme are naturally identified (up to isomorphism) with stalk functors at the geometric points in the usual sense (i.e., separable closures of residue fields at physical points, taken up to isomorphism). Under the natural bijection between points of the étale topos (up to isomorphism) and physical points in the scheme case (applied to $\mathscr{X}$), it follows that for a strictly henselian morphism $f : \widetilde{\mathscr{X}'_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{ét}}}$ the induced map on geometric points arises from a uniquely determined continuous map $|f| : |\mathscr{X}'| \to |\mathscr{X}|$ between the topological spaces. In particular, since subobjects of the final object of $\widetilde{\mathscr{X}'_{\text{ét}}}$ are represented by open subspaces of $\mathscr{X}'$, $U' := |f|^{-1}(U)$ represents $f^*(\underline{U})$ for any open subset $U \subseteq |\mathscr{X}|$.

For any $\mathscr{F}'$ in $\mathscr{X}'_{\text{ét}}$

$$(f_*(\mathscr{F}'))(U) = \mathrm{Hom}(\underline{U}, f_*(\mathscr{F}')) = \mathrm{Hom}(f^*(\underline{U}), \mathscr{F}') = \mathrm{Hom}(\underline{U}', \mathscr{F}') = \mathscr{F}'(U')$$

naturally in U. That is, the restriction of $f_*(\mathscr{F}')$ to $\mathscr{X}_{\text{Zar}}$ coincides with $|f|_*(\mathscr{F}'|_{\mathscr{X}'_{\text{Zar}}})$. In particular, the restriction of $\mathscr{O} \to f_*(\mathscr{O}')$ to $\mathscr{X}_{\text{Zar}}$ is a map of sheaves of rings

$$\mathscr{O}_{\mathscr{X}_{\text{Zar}}} \to |f|_*(\mathscr{O}'_{\mathscr{X}'_{\text{Zar}}}).$$

Since f is a map of locally ringed topoi, the argument in the proof of Proposition 3.1.1 shows that the composite maps

$$\mathscr{O}_{\mathscr{X}_{\mathrm{Zar}, f(x')}} \to |f|_*(\mathscr{O}'_{\mathscr{X}'_{\mathrm{Zar}}})_{f(x')} \to \mathscr{O}'_{\mathscr{X}'_{\mathrm{Zar}, x'}}$$

induced by the map of sheaves of rings on $\mathscr{X}_{\mathrm{Zar}}$ are local for all $x' \in |\mathscr{X}'|$. Thus, we have obtained a map between the schemes $\mathscr{X}'$ and $\mathscr{X}$ as locally ringed spaces; call this map f_{Zar}.

This map of schemes (with their Zariski topologies) promotes in the evident manner to a strictly henselian map $\widetilde{\mathscr{X}'_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{ét}}}$ which coincides on underlying (ringed) Zariski topoi with the given f. Also, left-exactness of f^* implies that the unique $f^*(\underline{\mathscr{X}}) \to \underline{\mathscr{X}'}$ to the final object in $\widetilde{\mathscr{X}'_{\text{ét}}}$ is an isomorphism. But for any affine object

$$U = \operatorname{Spec} A \to \operatorname{Spec} R := \mathscr{X}$$

in $\mathscr{X}_{\text{ét}}$, obviously $\underline{\mathscr{X}}_{A/R} = \underline{U}$. Thus, since f is a strictly henselian morphism, we have

$$f^*(\underline{U}) = f^*(\underline{\mathscr{X}}_{A/R}) = f^*(\underline{\mathscr{X}})_{A/R} \xleftarrow[\simeq]{\theta_{\mathscr{X}, A/R}} \underline{\mathscr{X}'}_{A/R}$$

over A, naturally in $U = \operatorname{Spec} A$. By the scheme case of Example 2.1.7, $\underline{\mathscr{X}'}_{A/R}$ is represented by the étale $\mathscr{X}'$-scheme $\mathscr{X}' \otimes_{f_{\mathrm{Zar}}, \operatorname{Spec} R} \operatorname{Spec} A$.

In other words, the functor f^* coincides with pullback along the scheme morphism f_{Zar} on affine objects in $\mathscr{X}_{\text{ét}}$. The affine objects are sufficient to compute the étale topos, so the given abstract morphism f and the morphism $\widetilde{f_{\mathrm{Zar}}}$ define equivalent pullback functors. The functor f_* is determined up to equivalence by f^* via adjointness, so we conclude that f on underlying topoi (ignoring ring objects) arises from a scheme morphism which agrees with f on underlying *ringed* Zariski topoi. It remains to prove that the additional data of the maps $f^{\#}, (f_{\mathrm{Zar}})^{\#}_{\text{ét}} : \mathscr{O} \rightrightarrows f_*\mathscr{O}'$ coincide on $\mathscr{X}_{\text{ét}}$ (and not merely on $\mathscr{X}_{\mathrm{Zar}}$).

For any affine étale $U = \operatorname{Spec} A \to \mathscr{X}$ and its pullback $U' = \mathscr{X}' \times_{\mathscr{X}} \operatorname{Spec} A$ under the scheme morphism $f_{\mathrm{Zar}} : \mathscr{X}' \to \mathscr{X}$, evaluation on U applied to $f^{\#} : \mathscr{O} \to f_*\mathscr{O}'$ (arising from f) defines a map of rings

$$A = \mathscr{O}(U) \to (f_*\mathscr{O}')(U) = \operatorname{Hom}(\underline{U}, f_*\mathscr{O}') \to \operatorname{Hom}(f^*(\underline{U}), \mathscr{O}') = \mathscr{O}'(U')$$

and we have to prove that this coincides with the canonical map. Let $(f^{\#})' : f^*\mathscr{O} \to \mathscr{O}'$ denote the map adjoint to $f^{\#}$.

The identification of $f^*(\underline{U})$ with $\underline{U'}$ is defined via the isomorphism $\theta_{\underline{\mathscr{X}}, A/R}$ over A, so the desired compatibility of A-structures reduces to the commutativity of the diagram

$$\begin{CD}
(f_*\mathscr{O}')(U) @>>> \mathscr{O}'(f^*\underline{U}) \\
@A{f^{\#}}AA @AAA \\
\mathscr{O}(U) @= A
\end{CD}$$

in which the bottom side is a definition, the top side is adjunction, and the right side uses (2.3.1) with $\mathscr{F} = \underline{U}$. An equivalent description in terms of $\underline{U}$ rather than U is the commutativity of the diagram

$$\begin{CD}
\mathscr{O}(\underline{U}) @>{f^*}>> (f^*\mathscr{O})(f^*\underline{U}) \\
@V{f^{\#}}VV @VV{(f^{\#})'}V \\
(f_*\mathscr{O}')(\underline{U}) @= \mathscr{O}'(f^*\underline{U})
\end{CD}$$

in which the bottom side is adjunction and the right side is the $f^*\underline{U}$-evaluation of the map $(f^{\#})'$ adjoint to $f^{\#}$.

In more general categorical terms, if $F_* : C' \to C$ is a covariant functor between two categories and $F^* : C \to C'$ is a left adjoint, then we claim that for any objects X, Y of C and object Y' of C' equipped with a morphism $h : Y \to F_* Y'$ having adjoint $h' : F^* Y \to Y'$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}(X, Y) & \xrightarrow{F^*} & \mathrm{Hom}(F^* X, F^* Y) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ \mathrm{Hom}(X, F_* Y') & = & \mathrm{Hom}(F^* X, Y') \end{array}$$

commutes (with bottom row equal to the adjunction bijection). We simply diagram chase using any $f \in \mathrm{Hom}(X, Y)$. This is carried to $h \circ f$ in the lower left term, and the adjunction along the bottom is computed by applying F^* and composing with $F^* F_* \to \mathrm{id}$ on Y'. Since the top side carries f to $F^*(f)$, and F^* is covariant, the problem thereby reduces to checking that the map h' adjoint to h is the composite map

$$F^* Y \stackrel{F^* h}{\to} F^* F_* Y' \to Y'.$$

But this equality of maps is well-known general nonsense in the basic theory of adjointness. ■

We now consider variants on Theorem 3.1.3 for non-archimedean analytic spaces. In contrast with Example 3.1.2, we will only assert a faithfulness result:

Lemma 3.1.4. *Let k be a field complete with respect to a non-archimedean absolute value. For any k-analytic Berkovich space $\mathscr{X}$, the ringed topos $(\widetilde{\mathscr{X}_{\text{ét}}}, \mathscr{O})$ is strictly henselian. If $f : \mathscr{X}' \to \mathscr{X}$ is a k-analytic morphism then the associated map $\widetilde{f}$ between ringed topoi over k is strictly henselian. The resulting functor from the category of k-analytic Berkovich spaces to the bicategory of strictly henselian topoi over k is faithful in the sense of Definition* 2.3.2.

If k has non-trivial absolute value then the same holds for rigid-analytic spaces over k equipped with their Tate-étale topoi.

We do not know if full faithfulness holds.

Proof. The same argument as in the complex-analytic case in Example 3.1.2 (using Example 2.1.7) shows that these topoi are strictly henselian and that all k-analytic maps between them induce strictly henselian morphisms of ringed topoi over k. Thus, assigning the ringed topos (over k) for the étale site in the Berkovich case and for the Tate-étale site in the rigid-analytic case defines a functor from these various categories of k-analytic objects to the bicategory of strictly henselian ringed topoi (over k).

It remains to prove that this functor to the bicategory of strictly henselian ringed topoi (over k) is faithful. Let $f, g : \mathscr{X}' \rightrightarrows \mathscr{X}$ be k-analytic maps between either Berkovich spaces or rigid-analytic spaces (over k). We have to prove that if $\mathrm{Hom}(\widetilde{f}, \widetilde{g})$ is non-empty then $f = g$ and the only endomorphism of $\widetilde{f}$ is the identity. A "continuity" argument by contradiction (using f^* and g^* applied to functors represented by suitable "open" subsets of $\mathscr{X}$) shows that if there is a map $\widetilde{f} \to \widetilde{g}$ then f and g must coincide on underlying sets of points. Thus, $f_* = g_*$, so the maps on underlying sets coincide.

To check equality of f and g as analytic morphisms, we can use the set-theoretic equality to reduce to the case of affinoid $\mathscr{X}$ as follows. In the rigid-analytic case, for any admissible affinoid open $\mathscr{V}$ in $\mathscr{X}$ we observe that there is a common preimage $\mathscr{V}' = f^{-1}(\mathscr{V}) = g^{-1}(\mathscr{V})$, so f and g restrict to maps $\widetilde{\mathscr{V}'_{\text{ét}}} \rightrightarrows \widetilde{\mathscr{V}_{\text{ét}}}$ between full subcategories. Thus, composing with the inclusions $\mathscr{V}' \hookrightarrow \mathscr{X}'$ allows us to replace $\mathscr{X}$ with each such $\mathscr{V}$ and $\mathscr{X}'$ with the corresponding $\mathscr{V}'$. In the Berkovich case, a similar argument allows us to first reduce to the case when $\mathscr{X}$ is Hausdorff, so k-analytic affinoid domains in $\mathscr{X}$ are closed. Then we work with k-analytic affinoid domains $\mathscr{V}$ in $\mathscr{X}$ as follows. Although such a $\mathscr{V}$ is typically not étale over $\mathscr{X}$, by [Ber2, 4.3.4] the functor $\widetilde{\mathscr{V}_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{ét}}}$ is an equivalence onto the full subcategory of sheaves of sets on $\mathscr{X}_{\text{ét}}$ with empty stalk outside of $|\mathscr{V}|$. (The statement of [Ber2, 4.3.4(ii)] is given in terms of abelian sheaves and vanishing stalks away from a closed set, but the argument works verbatim using sheaves of sets and empty stalks outside of a closed set.) Thus, for the common preimage $\mathscr{V}' = f^{-1}(\mathscr{V}) = g^{-1}(\mathscr{V})$, the given morphism $\widetilde{f} \to \widetilde{g}$ restricts to an morphism between the functors $\widetilde{\mathscr{V}'_{\text{ét}}} \rightrightarrows \widetilde{\mathscr{V}_{\text{ét}}}$ on full subcategories induced by the respective k-analytic maps $f, g : \mathscr{V}' \rightrightarrows \mathscr{V}$. Hence, we can replace $(\mathscr{X}, \mathscr{X}')$ with $(\mathscr{V}, \mathscr{V}')$ for varying $\mathscr{V}$ to reduce to the case when $\mathscr{X}$ is affinoid.

Now that $\mathscr{X}$ is affinoid, we exploit that the given map $\widetilde{f} \to \widetilde{g}$ is one of morphisms of *ringed* topoi (over k), not just as morphisms of topoi. It follows that $\mathscr{O} \rightrightarrows f_*\mathscr{O}' = g_*\mathscr{O}'$ coincides on global sections. In view of the universal property of affinoid spaces in both the rigid-analytic and Berkovich cases, we deduce that $f = g$. The preceding reduction steps likewise imply that the only endomorphism of $\widetilde{f}$ is the identity. ■

Remark 3.1.5. If Y is any k-analytic Berkovich space, the set of (isomorphism classes of) points of $\widetilde{Y_{\text{ét}}}$ is naturally identified with the underlying set $|Y|$ of Y by assigning to each $y \in Y$ the corresponding stalk functor at a geometric point. The proof of this is very similar to the case of schemes [SGA4, VIII, 7.9]. The properties which make the scheme proof carry over are: (i) k-analytic spaces are locally Hausdorff (hence sober), (ii) étale k-analytic maps are open, (iii) an étale k-analytic map is *finite étale* locally on the source and target, with the category of germs of finite étale covers of a pointed k-analytic space (X, x) naturally isomorphic (via the formation of x-fibers) to the category of finite étale covers of $\operatorname{Spec} \mathscr{H}(x)$, where $\mathscr{H}(x)$ denotes the completed residue field at x [Ber2, 3.4.1]. This characterization of points does not seem to significantly simplify any later proofs, nor help in addressing the full faithfulness aspect of Lemma 3.1.4.

Now we summarize the conclusion of our preparations. First, we have shown that the natural map from the category of algebraic spaces to the bicategory of strictly henselian topoi is fully faithful (in the sense of Definition 2.3.2), and so likewise for the subcategories of objects over a base ring (such as $\mathbf{C}$, or any field). The same holds for the functor from the category of complex-analytic spaces to the bicategory of strictly henselian topoi over $\mathbf{C}$. Finally, for a field k complete with respect to a non-archimedean absolute value, the category of k-analytic Berkovich spaces sits faithfully as a subcategory of the bicategory of strictly henselian topoi over k, as does the category of rigid-analytic spaces over k when $|k^\times| \neq 1$.

3.2. **Universal mapping property: complex-analytic case.** Our main interest is a topos-theoretic interpretation of analytification in the non-archimedean case, but it will simplify the presentation to first work out the complex-analytic analogue. The definitions, statements of results, and especially methods of proof will carry over to both the Berkovich and rigid-analytic cases. However, we will need to make some modifications to account for extra difficulties in the non-archimedean setting (e.g., the abundance of "non-classical" points in the Berkovich setting, and the lack of a general existence theorem for quotients by separated étale equivalence relations in the rigid-analytic setting). The reader should not ignore the complex-analytic case, since many of the arguments given in that case are specifically written to work for non-archimedean spaces (and so will not be repeated there).

Inspired by the universal property which characterizes the analytification of schemes locally of finite type over $\mathbf{C}$, we are led to the following variant for algebraic spaces. Let $\mathscr{X}$ be a quasi-separated algebraic space locally of finite type over $\mathbf{C}$. Does there exist a complex-analytic space $\mathscr{X}^{\text{an}}$ and a morphism $i : (\mathscr{X}^{\text{an}})^{\sim} \to \widetilde{\mathscr{X}_{\text{ét}}}$ of locally ringed topoi over $\mathbf{C}$ which is "final" among all such maps? That is, for any complex-analytic space Z and morphism $\varphi : \widetilde{Z} \to \widetilde{\mathscr{X}}$ of locally ringed topoi over $\mathbf{C}$ we demand that there is a unique map $f : Z \to \mathscr{X}^{\text{an}}$ of complex-analytic spaces such that $i \circ \widetilde{f}$ is naturally isomorphic to φ. If such a pair $(\mathscr{X}^{\text{an}}, i)$ exists then $\mathscr{X}^{\text{an}}$ is unique up to unique isomorphism and is functorial in $\mathscr{X}$. Note that even if $\mathscr{X}$ is a scheme, it is not obvious that its usual analytification satisfies this property!

As we reviewed in §1.1, in [CT, §2] a more geometric notion of analytifiability is defined in terms of étale scheme charts: for an étale cover $\mathscr{U} \to \mathscr{X}$ by a scheme and the scheme $\mathscr{R} := \mathscr{U} \times_{\mathscr{X}} \mathscr{U}$, does the analytic étale equivalence relation $\mathscr{R}^{\text{an}} \rightrightarrows \mathscr{U}^{\text{an}}$ admits a quotient in the category of complex-analytic spaces? The discussion in [CT, §2] makes precise the sense in which $\mathscr{X}^{\text{an}} := \mathscr{U}^{\text{an}}/\mathscr{R}^{\text{an}}$ is then independent of the choice of étale scheme cover and is functorial in $\mathscr{X}$. (Strictly speaking, [CT, §2] consider only the non-archimedean setting. However, it applies verbatim in the complex-analytic case, where it coincides with the definition used in [Kn, Ch. I, 5.17*ff*].)

Definition 3.2.1. Let $\mathscr{X}$ be a quasi-separated algebraic space locally of finite type over $\mathbf{C}$. It is *analytifiable in the sense of topoi* if there exists a complex-analytic space $\mathscr{X}^{\text{an}}$ and map of locally ringed topoi $i : \widetilde{\mathscr{X}^{\text{an}}} \to$

$\widetilde{\mathscr{X}_{\text{ét}}}$ over $\mathbf{C}$ such that $(\mathscr{X}^{\text{an}}, i)$ satisfies the universal property as formulated above. If the analytic quotient $\mathscr{U}^{\text{an}}/\mathscr{R}^{\text{an}}$ exists for an étale scheme chart $\mathscr{R} \rightrightarrows \mathscr{U}$ for $\mathscr{X}$, then $\mathscr{X}$ is *analytifiable in the sense of charts.*

Analytifiability in the sense of charts is a tangible geometric property, whereas analytifiability in the sense of topoi is a more "intrinsic" property. It is therefore natural to try to relate these notions, and this is the aim of the remainder of this section.

First, due to lack of a reference, we digress to record a general lemma on points of the étale topos of an algebraic space. Let $\mathscr{X}$ be a quasi-separated algebraic space. There is an associated topological space $|\mathscr{X}|$ defined in two equivalent ways. The more concrete definition rests on the fact that every field-valued point of $\mathscr{X}$ factors uniquely through a monic field-valued point [Kn, II, 6.2] (this is false without quasi-separatedness). The latter are called *atoms* by Knutson, and $|\mathscr{X}|$ is the set of atoms of $\mathscr{X}$ (taken up to isomorphism). It is equipped with the quotient topology from any étale scheme cover of $\mathscr{X}$. Another definition is given in [LMB, 5.5] which applies more broadly to quasi-separated Artin stacks (with separated diagonal), and it uses an equivalence relation on "geometric points"; in the case of algebraic spaces it is naturally isomorphic to Knutson's construction. By [LMB, 5.7.2], $|\mathscr{X}|$ is sober. It defines the Zariski site $\mathscr{X}_{\text{Zar}}$ of $\mathscr{X}$.

Lemma 3.2.2. *Let $\mathscr{X}$ be a quasi-separated algebraic space. Assigning to each $x \in |\mathscr{X}|$ the stalk functor associated to $\overline{x} = \operatorname{Spec} k(x)_s \to \mathscr{X}$ defines an bijection from $|\mathscr{X}|$ to the set of isomorphism classes of points of $\widetilde{\mathscr{X}_{\text{ét}}}$. The inverse is given by composition with the map $\widetilde{\mathscr{X}_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{Zar}}}$.*

Proof. Choose distinct $x, x' \in |\mathscr{X}|$. Since $|\mathscr{X}|$ is sober and $x \neq x'$, one of them is not in the closure of the other. Thus, there is open subspace $\mathscr{V}$ in $\mathscr{X}$ containing one but not the other. Hence, precisely one of $\underline{\mathscr{V}}_{\overline{x}}$ or $\underline{\mathscr{V}}_{\overline{x}'}$ is empty, so the stalk functors at $\overline{x}$ and $\overline{x}'$ are not isomorphic. This proves injectivity.

Now consider any point φ of the topos $\widetilde{\mathscr{X}_{\text{ét}}}$. Its composition with $\widetilde{\mathscr{X}_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{Zar}}}$ is the stalk functor at a unique point $x \in |\mathscr{X}|$ since $|\mathscr{X}|$ is sober. It remains to prove that the exact functor $\varphi^* : \widetilde{\mathscr{X}_{\text{ét}}} \to \text{Set}$ is isomorphic to $\overline{x}^*$ (equivalently, isomorphic to the functor of global sections of pullback to the strict henselization of $\mathscr{X}$ at $\overline{x}$). This is shown for schemes in the proof of [SGA4, VIII, 7.9], and the proof there works verbatim for quasi-separated algebraic spaces because (i) for any affine schemes U' and U'' étale over $\mathscr{X}$, their maps to $\mathscr{X}$ are separated (as U' and U'' are separated) and the fiber product $U' \times_{\mathscr{X}} U''$ is a scheme, (ii) exactly as for schemes, every field-valued point of $\mathscr{X}$ factors through a unique monic field-valued point, and the latter points constitute $|\mathscr{X}|$. ■

Next we turn to the relationship between analytifiability in the sense of charts and in the sense of topoi for quasi-separated algebraic spaces $\mathscr{X}$ locally of finite type over $\mathbf{C}$. The following result provides an equivalence between the two sense of analytification.

Theorem 3.2.3. *Let $\mathscr{X}$ be a quasi-separated algebraic space locally of finite type over $\mathbf{C}$.*

(1) *Assume there exists an analytification $(\mathscr{X}^{\text{an}}, i)$ in the sense of topoi. Choose any étale scheme cover $\pi : \mathscr{U} \to \mathscr{X}$ and consider the associated étale equivalence relation in schemes $\mathscr{R} := \mathscr{U} \times_{\mathscr{X}} \mathscr{U} \rightrightarrows \mathscr{U}$. Let $\mathscr{U}^{\text{an}}$ and $\mathscr{R}^{\text{an}}$ denote the usual analytifications of the schemes $\mathscr{U}$ and $\mathscr{R}$. Define the map $\pi^{\text{an}} : \mathscr{U}^{\text{an}} \to \mathscr{X}^{\text{an}}$ via the universal property of $(\mathscr{X}^{\text{an}}, i)$ applied to*

$$\widetilde{\mathscr{U}^{\text{an}}} = \widetilde{(\mathscr{U}^{\text{an}})_{\text{ét}}} \xrightarrow{\widetilde{\pi_{\text{ét}}}} \widetilde{\mathscr{U}_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{ét}}},$$

and define $\delta : \mathscr{R}^{\text{an}} \to \mathscr{U}^{\text{an}} \times_{\mathscr{X}^{\text{an}}} \mathscr{U}^{\text{an}}$ similarly.

The map π^{an} is an étale cover and δ is an isomorphism. In particular, the quotient $\mathscr{U}^{\text{an}}/\mathscr{R}^{\text{an}}$ exists and is identified with $\mathscr{X}^{\text{an}}$, so $\mathscr{X}$ is analytifiable in the sense of charts.

(2) *Assume there exists an analytification $\mathscr{X}^{\text{an}}$ of $\mathscr{X}$ in the sense of charts. If $\Delta_{\mathscr{X}/\mathbf{C}}$ is affine (e.g., a closed immersion) then there exists a morphism of locally ringed topoi $i : \widetilde{\mathscr{X}^{\text{an}}} \to \widetilde{\mathscr{X}_{\text{ét}}}$ over $\mathbf{C}$ such that $(\mathscr{X}^{\text{an}}, i)$ is an analytification in the sense of topoi.*

Proof. Throughout this proof, we use Proposition 2.3.7 without comment. For any finite local $\mathbf{C}$-algebra A, $\text{Sp}(A) := \text{Spec}(A)^{\text{an}}$ is a 1-point space with $\mathbf{C}$-algebra of global functions A, so it is really $\text{Spec}(A)$ by another name. The algebraic and analytic étale sites of $\text{Spec}(A)$ are likewise identified.

The underlying set $|\mathscr{X}^{\mathrm{an}}|$ of $\mathscr{X}^{\mathrm{an}}$ is identified with the set of analytic maps $\mathrm{Sp}(\mathbf{C}) \to \mathscr{X}^{\mathrm{an}}$, and by the universal property of $(\mathscr{X}^{\mathrm{an}}, i)$ the set of such maps is in natural bijection with the set of isomorphism classes of strictly henselian $\mathbf{C}$-maps $\widetilde{\mathrm{Sp}(\mathbf{C})}_{\text{ét}} \to \widetilde{\mathscr{X}_{\text{ét}}}$. By Theorem 3.1.3, this latter set is naturally identified with the set of $\mathbf{C}$-maps $\mathrm{Spec}(\mathbf{C}) \to \mathscr{X}$, so i naturally identifies $|\mathscr{X}^{\mathrm{an}}|$ with $\mathscr{X}(\mathbf{C})$ as sets. Hence, π^{an} on underlying sets is identified with π on $\mathbf{C}$-points, so π^{an} is surjective.

Choose $u \in \mathscr{U}(\mathbf{C}) = |\mathscr{U}^{\mathrm{an}}|$ and let $x = \pi(u) \in \mathscr{X}(\mathbf{C}) = |\mathscr{X}^{\mathrm{an}}|$. For any finite local $\mathbf{C}$-algebra A, we wish to prove that the set of $\mathbf{C}$-maps $\mathrm{Sp}(A) \to \mathscr{U}^{\mathrm{an}}$ lifting u is carried bijectively under π^{an} to the set of $\mathbf{C}$-maps $\mathrm{Sp}(A) \to \mathscr{X}^{\mathrm{an}}$ lifting x. By the universal property of $(\mathscr{X}^{\mathrm{an}}, i)$, the latter set of maps is identified with the set of (isomorphism classes of) strictly henselian $\mathbf{C}$-maps $\widetilde{\mathrm{Sp}(A)}_{\text{ét}} \to \widetilde{\mathscr{X}_{\text{ét}}}$ lifting $\widetilde{x}$, so then applying Theorem 3.1.3 identifies this with the set of $\mathbf{C}$-maps $\mathrm{Spec}(A) \to \mathscr{X}$ lifting x. Likewise, the set of $\mathbf{C}$-maps $\mathrm{Sp}(A) \to \mathscr{U}^{\mathrm{an}}$ lifting u is identified with the set of $\mathbf{C}$-maps $\mathrm{Spec}(A) \to \mathscr{U}$ lifting u. Since $\mathscr{U} \to \mathscr{X}$ is an étale map carrying u to x, the desired bijection of A-points follows. This proves that π^{an} is an étale cover.

A very similar argument shows that the natural map $\mathscr{R}^{\mathrm{an}} \to \mathscr{U}^{\mathrm{an}} \times_{\mathscr{X}^{\mathrm{an}}} \mathscr{U}^{\mathrm{an}}$ is a complex-analytic étale bijection, hence an isomorphism. This settles part (1).

Now we consider part (2), so assume $\mathscr{X}$ is analytifiable in the sense of charts: $\mathscr{U}^{\mathrm{an}} \to \mathscr{X}^{\mathrm{an}}$ is an étale cover which is a quotient by the equivalence relation $\mathscr{R}^{\mathrm{an}}$. In particular, the category $(\mathscr{X}^{\mathrm{an}})_{\text{ét}}$ is equivalent to the category of objects in $(\mathscr{U}^{\mathrm{an}})_{\text{ét}}$ equipped with descent data relative to $\mathscr{R}^{\mathrm{an}} \rightrightarrows \mathscr{U}^{\mathrm{an}}$. Thus, using étale descent for sheaves of sets, the pullback component of the strictly henselian map of ringed topoi over $\mathbf{C}$ corresponding to the composite map of sites

$$\mathscr{U}^{\mathrm{an}} \to \mathscr{U}_{\text{ét}} \to \mathscr{X}_{\text{ét}}$$

canonically through a functor between topoi (ignoring ring objects) $i^* : \widetilde{\mathscr{X}^{\mathrm{an}}} = ((\mathscr{X}^{\mathrm{an}})_{\text{ét}})^\sim \to (\mathscr{X}_{\text{ét}})^\sim$.

The construction of i^* via descent implies that it is left exact and provides a map of ring objects $i' : i^*\mathscr{O}_{\mathscr{X}_{\text{ét}}} \to \mathscr{O}_{\mathscr{X}^{\mathrm{an}}}$. To construct i_*, for $\mathscr{F}$ in $\widetilde{\mathscr{X}^{\mathrm{an}}}$ let $\mathscr{F}'$ and $\mathscr{F}''$ denote the respective pullbacks of $\mathscr{F}$ to $\widetilde{\mathscr{U}^{\mathrm{an}}}$ and $\widetilde{\mathscr{R}^{\mathrm{an}}}$. Let

$$p : \widetilde{\mathscr{U}^{\mathrm{an}}} \to \widetilde{\mathscr{U}_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{ét}}}, \quad q : \widetilde{\mathscr{R}^{\mathrm{an}}} \to \widetilde{\mathscr{R}_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{ét}}}$$

denote the canonical maps, and likewise for $\pi_1, \pi_2 : \widetilde{\mathscr{R}^{\mathrm{an}}} \rightrightarrows \widetilde{\mathscr{U}^{\mathrm{an}}}$. Using the canonical isomorphisms

$$\pi_1^*\mathscr{F}' \simeq \mathscr{F}'' \simeq \pi_2^*\mathscr{F}'$$

for $\mathscr{F}$ in $\widetilde{\mathscr{X}^{\mathrm{an}}}$, we get a pair of maps $p_*\mathscr{F}' \rightrightarrows q_*\mathscr{F}''$ defined by

$$p_*\mathscr{F}' \to p_*\pi_{j*}\pi_j^*\mathscr{F}' = q_*\pi_j^*\mathscr{F}' \simeq q_*\mathscr{F}''$$

for $j = 1, 2$. Defining $i_* : \widetilde{\mathscr{X}^{\mathrm{an}}} \to \widetilde{\mathscr{X}_{\text{ét}}}$ by

$$i_*\mathscr{F} := \ker(p_*\mathscr{F}' \rightrightarrows q_*\mathscr{F}''),$$

it is easy to check that i_* is right adjoint to i^*.

The triple $i = (i_*, i^*, i')$ is a map of ringed topoi over $\mathbf{C}$, and it follows from the descent construction of (i^*, i') that i is strictly henselian. We claim that $(\mathscr{X}^{\mathrm{an}}, i)$ is an analytification in the sense of topoi. Consider a complex-analytic space Z and a strictly henselian morphism $\varphi : \widetilde{Z} \to \widetilde{\mathscr{X}_{\text{ét}}}$ over $\mathbf{C}$. We need to prove that there exists a unique map of complex-analytic spaces $f : Z \to \mathscr{X}^{\mathrm{an}}$ such that φ is equivalent to $i \circ \widetilde{f}$. First we establish the uniqueness of f.

Given such an f, to prove its uniqueness we first check that f is uniquely determined on underlying sets. For any $z \in Z$, $f(z) \in \mathscr{X}^{\mathrm{an}}$ gives rise to a point of the topos $\widetilde{\mathscr{X}^{\mathrm{an}}}$ (i.e., a stalk functor), and composing this point with i^* yields the point $\varphi(z)$ of $\widetilde{\mathscr{X}_{\text{ét}}}$ since φ is equivalent to $i \circ \widetilde{f}$. Since $\varphi(z)$ does not involve f, it suffices to check that if two physical points $x, x' \in \mathscr{X}^{\mathrm{an}}$ satisfy $(i^*)_x \simeq (i^*)_{x'}$ as functors on $\widetilde{\mathscr{X}_{\text{ét}}}$ then $x = x'$. By the descent construction of i^*, if we choose $u, u' \in \widetilde{\mathscr{U}^{\mathrm{an}}}$ over x, x' respectively, then $(i^*)_x = (p^*)_u$ and $(i^*)_{x'} = (p^*)_{u'}$. The points u_0, u_0' of $\widetilde{\mathscr{U}_{\text{ét}}}$ obtained from u and u' satisfy $(\pi_{\text{ét}}^*)_{u_0} \simeq (\pi_{\text{ét}}^*)_{u_0'}$ as points of $\widetilde{\mathscr{X}_{\text{ét}}}$. Thus, the $\mathbf{C}$-valued (hence monic) points $\pi(u_0)$ and $\pi(u_0')$ of $\mathscr{X}$ coincide, by Lemma 3.2.2, so $(u_0, u_0') \in (\mathscr{U} \times \mathscr{U})(\mathbf{C})$ lies in $\mathscr{U} \times_{\mathscr{X}} \mathscr{U} = \mathscr{R}$.

Under the map $\mathscr{U}^{\mathrm{an}} \times \mathscr{U}^{\mathrm{an}} \to \mathscr{U} \times \mathscr{U}$, (u, u') is carried to the **C**-point (u_0, u_0') of $\mathscr{R}$. But the preimage of $\mathscr{R}$ in $\mathscr{U}^{\mathrm{an}} \times \mathscr{U}^{\mathrm{an}}$ is $\mathscr{R}^{\mathrm{an}}$ due to the universal property of analytification of locally finite type **C**-schemes, so $(u, u') \in \mathscr{R}^{\mathrm{an}}$. Hence, the points $x, x' \in \mathscr{X}^{\mathrm{an}}$ coincide, as desired. This completes the proof that any analytic morphism $f : Z \to \mathscr{X}^{\mathrm{an}}$ satisfying $i \circ \widetilde{f} \simeq \varphi$ is determined on underlying sets. For the uniqueness of f as an analytic map it therefore remains to show that $f_z^{\#} : \mathscr{O}_{\mathscr{X}^{\mathrm{an}}, f(z)} \to \mathscr{O}_{Z,z}$ is uniquely determined for each $z \in Z$. Since $\varphi \simeq i \circ \widetilde{f}$ as strictly henselian maps of ringed topoi, we have a commutative diagram of *local* maps

$$\begin{array}{ccc} \mathscr{O}_{\mathscr{X}_{\text{ét}}, \varphi(z)} & \xrightarrow{\varphi_z^{\#}} & \mathscr{O}_{Z,z} \\ {\scriptstyle i^{\#}}\downarrow & \nearrow {\scriptstyle f_z^{\#}} & \\ \mathscr{O}_{\mathscr{X}^{\mathrm{an}}, f(z)} & & \end{array}$$

between local noetherian rings. The map along the left induces an isomorphism between completions, so $f_z^{\#}$ is uniquely determined and hence so is f (if it exists).

Beware that to prove that f exists, the uniqueness does not formally permit us to work locally on Z for existence, since there is an isomorphism ambiguity in the universal property. In our eventual construction of the desired analytic map $f : Z \to \mathscr{X}^{\mathrm{an}}$ and equivalence $\xi : \varphi \simeq i \circ \widetilde{f}$, the non-canonical choice of étale scheme cover $\pi : \mathscr{U} \to \mathscr{X}$ will play a key role in the construction of ξ. To establish the existence of (f, ξ), we will use the hypothesis that $\mathscr{X}$ has affine diagonal. The key is:

Lemma 3.2.4. *For any étale map $\mathscr{V} = \operatorname{Spec} A \to \mathscr{X}$ from an affine scheme, $\varphi^*(\underline{\mathscr{V}})$ is represented by an étale separated analytic map $V \to Z$.*

Proof. By left exactness of φ^* we have $\varphi^*(\underline{\mathscr{X}}) = \underline{Z}$ (preservation of final objects), and we can choose an analytic étale map $\pi : Z' \to Z$ such that there exists an epimorphism $s : \underline{Z'} \to \varphi^*(\underline{\mathscr{V}})$. In $\widetilde{Z}$ we have

$$\underline{Z'} \times \varphi^*(\underline{\mathscr{V}}) = \underline{Z'} \times_{s, \varphi^*(\underline{\mathscr{V}}), p_1} (\varphi^*(\underline{\mathscr{V}}) \times_{\underline{Z}} \varphi^*(\underline{\mathscr{V}})) = \underline{Z'} \times_{s, \varphi^*(\underline{\mathscr{V}}), p_1} \varphi^*(\underline{\mathscr{V}} \times_{\underline{\mathscr{X}}} \underline{\mathscr{V}}) \tag{3.2.1}$$

Since $\mathscr{X}$ has affine diagonal and $\mathscr{V} = \operatorname{Spec} A$ is affine, it follows that $\mathscr{V} \times_{\mathscr{X}} \mathscr{V}$ is also affine, say $\operatorname{Spec} B$, and B is an étale A-algebra via either of the projections $\mathscr{V} \times_{\mathscr{X}} \mathscr{V} \rightrightarrows \mathscr{V}$. For specificity, we make B into an A-algebra using the first projection (due to the appearance of this projection in (3.2.1)).

The definitions of A and B imply $\underline{\mathscr{V}}_{B/A} = \underline{\mathscr{V}} \times_{\underline{\mathscr{X}}} \underline{\mathscr{V}}$, so the strictly henselian property of φ implies

$$\varphi^*(\underline{\mathscr{V}} \times_{\underline{\mathscr{X}}} \underline{\mathscr{V}}) = (\varphi^*(\underline{\mathscr{V}}))_{B/A}.$$

By Proposition 2.1.8 applied to $\mathscr{F} := \varphi^*(\underline{\mathscr{V}})$, we obtain from (3.2.1) the isomorphism

$$\underline{Z'} \times \varphi^*(\underline{\mathscr{V}}) \simeq \underline{Z'}_{B/A}$$

where the right side rests on the "map" $Z' \to \operatorname{Spec} A$ over **C** defined by the composition

$$\underline{Z'} \xrightarrow{s} \varphi^*(\underline{\mathscr{V}}) \to \operatorname{Spec} A$$

whose second step corresponds to the **C**-algebra map

$$A = \operatorname{Hom}(\underline{\mathscr{V}}, \mathscr{O}_{\mathscr{X}}) \to \operatorname{Hom}(\underline{\mathscr{V}}, \varphi_* \mathscr{O}_Z) = \operatorname{Hom}(\varphi^*(\underline{\mathscr{V}}), \mathscr{O}_Z). \tag{3.2.2}$$

By Example 2.1.7, $\underline{Z'}_{B/A}$ is represented by an analytic space étale over Z' and hence étale over Z.

We have proved that the object $\underline{Z'} \times \varphi^*(\underline{\mathscr{V}})$ in $\widetilde{Z}$ is representable, and the projection $\underline{Z'} \times \varphi^*(\underline{\mathscr{V}}) \to \varphi^*(\underline{\mathscr{V}})$ is an epimorphism since s is. For $Z'' := Z' \times_Z Z'$ either of the maps $Z'' \rightrightarrows Z'$ make $\underline{Z''}$ a covering of $\varphi^*(\underline{\mathscr{V}})$, so by replacing Z' with Z'' above we see that $\underline{Z''} \times \varphi^*(\underline{\mathscr{V}})$ in $\widetilde{Z}$ is also represented by an analytic space étale over Z. But $\underline{Z''} \times \varphi^*(\underline{\mathscr{V}})$ is the fiber square of $\underline{Z'} \times \varphi^*(\underline{\mathscr{V}}) \to \varphi^*(\underline{\mathscr{V}})$, so we conclude that $\varphi^*(\underline{\mathscr{V}})$ is the quotient of an étale equivalence relation in analytic spaces étale over Z. This quotient is represented by an étale analytic space over Z provided that the diagonal of the equivalence relation is a topological embedding (due to the criterion in [Gr, Prop. 5.6]), and then it is moreover separated over Z if the diagonal of the equivalence relation is a closed immersion. It therefore suffices to prove that the map $\varphi^*(\underline{\mathscr{V}}) \to \varphi^*(\underline{\mathscr{V}}) \times \varphi^*(\underline{\mathscr{V}})$ in $\widetilde{Z}$ is relatively representable in closed immersions.

Since $\underline{Z}' \to \varphi^*(\underline{\mathscr{V}})$ is an epimorphism, so $\varphi^*(\underline{\mathscr{V}})$ has empty stalks away from the open image of the étale map $Z' \to Z$, and analytic étale descent is effective for closed immersions, it is equivalent to prove that the $\underline{Z}'$-map

$$\underline{Z}' \times \varphi^*(\underline{\mathscr{V}}) \to \underline{Z}' \times (\varphi^*(\underline{\mathscr{V}}) \times \varphi^*(\underline{\mathscr{V}}))$$

between representable sheaves on $Z_{\text{ét}}$ is a closed immersion. This map is a section to either of the evident projections over Z', so it suffices to prove that the representing object $Z'_{B/A}$ for $\underline{Z}' \times \varphi^*(\underline{\mathscr{V}}) \in \widetilde{Z}$ is separated over Z'. By construction of $Z'_{B/A}$ and the universal property of analytification of affine $\mathbf{C}$-schemes of finite type, $Z'_{B/A} = Z' \times_{\mathrm{Spec}(A)^{\mathrm{an}}} \mathrm{Spec}(B)^{\mathrm{an}}$ (where $Z' \to \mathrm{Spec}(A)^{\mathrm{an}}$ arises from the map $Z' \to \mathrm{Spec}(A)$ over $\mathbf{C}$ associated to the $\mathbf{C}$-algebra map $A \to Z'$ constructed above). This is visibly separated over Z'. ■

Let $\varphi^*(\mathscr{V})$ denote the analytic space representing $\varphi^*(\underline{\mathscr{V}})$ as in Lemma 3.2.4. If $\{\mathscr{V}_\alpha = \mathrm{Spec}\, A_\alpha \to \mathscr{X}\}$ is an étale cover by affines, right exactness of φ^* implies that $\{\varphi^*(\mathscr{V}_\alpha)\}$ is an étale cover of Z since $\varphi^*(\underline{\mathscr{X}}) = \underline{Z}$. The essential point in the construction of the desired $f : Z \to \mathscr{X}^{\mathrm{an}}$ and especially the equivalence $\xi : \varphi \simeq i \circ \widetilde{f}$ is to show that if $\mathscr{V} = \mathrm{Spec}\, A \to \mathscr{X}$ is étale and *admits a factorization* through $\pi : \mathscr{U} \to \mathscr{X}$ then there is an analytic map $f_{\mathscr{V}} : \varphi^*(\mathscr{V}) \to \mathscr{X}^{\mathrm{an}}$ and an equivalence $\xi_{\mathscr{V}}$ from $i \circ \widetilde{f}_{\mathscr{V}}$ to the composite map

$$\widetilde{\varphi^*(\mathscr{V})} \to \widetilde{Z} \overset{\varphi}{\to} \widetilde{\mathscr{X}_{\text{ét}}}$$

(in the bicategory of strictly henselian topoi over $\mathbf{C}$) such that $(f_{\mathscr{V}}, \xi_{\mathscr{V}})$ is *functorial* in $\mathscr{V}$ over $\mathscr{X}$. The functorial aspect will then be applied to globalize and solve the original problem for Z. (The pair $(f_{\mathscr{V}}, \xi_{\mathscr{V}})$ is precisely a solution to our original factorization problem with Z replaced by $\varphi^*(\mathscr{V})$, so $f_{\mathscr{V}}$ will be uniquely determined. The key issue is to arrange $\xi_{\mathscr{V}}$ to be functorial in $\mathscr{V}$.)

Fix an étale object $h : \mathscr{V} = \mathrm{Spec}\, A \to \mathscr{X}$ for which there is an $\mathscr{X}$-map $j : \mathscr{V} \to \mathscr{U}$. Thus, we obtain an analytic map $\pi^{\mathrm{an}} \circ j^{\mathrm{an}} : \mathscr{V}^{\mathrm{an}} \to \mathscr{X}^{\mathrm{an}}$. This map is *independent* of the choice of j. Indeed, if $j' : \mathscr{V} \to \mathscr{U}$ is another $\mathscr{X}$-map then the pair (j, j') defines an $\mathscr{X}$-map $\mathscr{V} \to \mathscr{U} \times_{\mathscr{X}} \mathscr{U} =: \mathscr{R}$ whose projections recover j and j', so we get an analytic map $\mathscr{V}^{\mathrm{an}} \to \mathscr{R}^{\mathrm{an}}$ whose composites with $\pi_1^{\mathrm{an}}, \pi_2^{\mathrm{an}} : \mathscr{R}^{\mathrm{an}} \rightrightarrows \mathscr{U}^{\mathrm{an}}$ recover j^{an} and j'^{an}. Since the maps $\pi^{\mathrm{an}} \circ \pi_1^{\mathrm{an}}$ and $\pi^{\mathrm{an}} \circ \pi_2^{\mathrm{an}}$ from $\mathscr{R}^{\mathrm{an}}$ to $\mathscr{U}^{\mathrm{an}}/\mathscr{R}^{\mathrm{an}} =: \mathscr{X}^{\mathrm{an}}$ coincide, the independence of $\pi^{\mathrm{an}} \circ j^{\mathrm{an}}$ with respect to the choice of j is proved. Thus, it is well-posed to define $h^{\mathrm{an}} := \pi^{\mathrm{an}} \circ j^{\mathrm{an}}$, and the independence of j implies that h^{an} is *functorial* in $\mathscr{V}$ over $\mathscr{X}$.

By (3.2.2) we get a $\mathbf{C}$-algebra map $A \to \mathscr{O}(\varphi^* \mathscr{V})$, which in turn defines an analytic map $\varphi^*(\mathscr{V}) \to (\mathrm{Spec}\, A)^{\mathrm{an}} = \mathscr{V}^{\mathrm{an}}$ due to the universal property of analytification for schemes locally of finite type over $\mathbf{C}$. We now check that the resulting diagram of strictly henselian morphisms over $\mathbf{C}$

(3.2.3)
$$\begin{array}{ccccc}
\widetilde{\varphi^*(\mathscr{V})} & \longrightarrow & \widetilde{\mathscr{V}^{\mathrm{an}}} & \xrightarrow{\widetilde{h^{\mathrm{an}}}} & \widetilde{\mathscr{X}^{\mathrm{an}}} \\
\downarrow & & & & \downarrow{\scriptstyle i} \\
\widetilde{Z} & & \xrightarrow{\quad\varphi\quad} & & \widetilde{\mathscr{X}_{\text{ét}}}
\end{array}$$

commutes up to a specific equivalence $\xi_{\mathscr{V}}$ which is canonical in $h : \mathscr{V} \to \mathscr{X}$.

Upon choosing an $\mathscr{X}$-map $j : \mathscr{V} \to \mathscr{U}$, the definition of i^* in terms of the étale chart $\mathscr{R} \rightrightarrows \mathscr{U}$ for $\mathscr{X}$ implies (by consideration of pullback functors) that the diagram of strictly henselian morphisms over $\mathbf{C}$

$$\begin{array}{ccccccc}
\widetilde{\varphi^*(\mathscr{V})} & \longrightarrow & \widetilde{\mathscr{V}^{\mathrm{an}}} & \longrightarrow & \widetilde{\mathscr{U}^{\mathrm{an}}} & \longrightarrow & \widetilde{\mathscr{X}^{\mathrm{an}}} \\
& \searrow & \downarrow & & \downarrow & & \downarrow{\scriptstyle i} \\
& & \widetilde{\mathscr{V}_{\text{ét}}} & \longrightarrow & \widetilde{\mathscr{U}_{\text{ét}}} & \longrightarrow & \widetilde{\mathscr{X}_{\text{ét}}}
\end{array}$$

commutes up to a specific equivalence which is canonical in the choice of $j : \mathscr{V} \to \mathscr{U}$ (where the left triangle in the diagram uses the universal property of $\mathscr{V}^{\mathrm{an}}$ and the definition of the map of locally ringed spaces $\varphi^*(\mathscr{V}) \to \mathrm{Spec}\, A = \mathscr{V}$ over $\mathbf{C}$ underlying the construction of the diagonal arrow). Exactly as in our proof that h^{an} is independent of j, if we drop the appearance of the two $\mathscr{U}$-terms in the diagram then we obtain canonicity in $\mathscr{V} \to \mathscr{X}$ (i.e., independence of j) for the equivalence expressing commutativity of the resulting diagram of strictly henselian morphisms of topoi because any two $\mathscr{X}$-maps $\mathscr{V} \rightrightarrows \mathscr{U}$ are obtained from the projections of a map $\mathscr{V} \to \mathscr{U} \times_{\mathscr{X}} \mathscr{U} = \mathscr{R}$ with $\mathscr{R}$ a **C**-scheme locally of finite type and $\mathscr{X}^{\mathrm{an}} := \mathscr{U}^{\mathrm{an}}/\mathscr{R}^{\mathrm{an}}$.

To complete our analysis of (3.2.3) for a fixed étale $\mathscr{V} = \operatorname{Spec} A \to \mathscr{X}$ which factors through $\mathscr{U} \to \mathscr{X}$, we prove the commutativity (up to a specific equivalence functorial in $\mathscr{V} \to \mathscr{X}$) of the diagram of strictly henselian morphisms over **C**

(3.2.4)
$$\begin{array}{ccc} \widetilde{\varphi^*(\mathscr{V})} & \longrightarrow & \widetilde{\mathscr{V}_{\text{ét}}} \\ \downarrow & & \downarrow \\ \widetilde{Z} & \xrightarrow[\varphi]{} & \widetilde{\mathscr{X}_{\text{ét}}} \end{array}$$

We will first consider the situation on underlying topoi (ignoring ring objects) by canonically identifying the composite pullback functors in both directions.

The category $\widetilde{\varphi^*(\mathscr{V})}$ is identified with the slice category over $\varphi^*(\underline{\mathscr{V}})$ inside of $\widetilde{Z}$, so pulling back along the bottom and left in (3.2.4) carries $\mathscr{F} \in \widetilde{\mathscr{X}_{\text{ét}}}$ to $\varphi^*(\underline{\mathscr{V}}) \times_{\underline{Z}} \varphi^*(\mathscr{F}) = \varphi^*(\underline{\mathscr{V}} \times_{\underline{\mathscr{X}}} \mathscr{F})$. But $\underline{\mathscr{V}} \times_{\underline{\mathscr{X}}} \mathscr{F}$ is the pullback of $\mathscr{F}$ along the right side, so commutativity of the diagram of topoi (3.2.4) up to a specific equivalence (canonical in $\mathscr{V} \to \mathscr{X}$) is reduced to canonically identifying pullback along the top with the functor $\mathscr{G} \rightsquigarrow \varphi^*\mathscr{G}$ into the slice category over $\varphi^*(\underline{\mathscr{V}})$. This problem is easily reduced to the case of $\mathscr{G}$ represented by an affine (necessarily étale over $\mathscr{V} = \operatorname{Spec} A$), in which case it follows from the hypothesis that φ is *strictly henselian*.

Finally, letting $g : \widetilde{\varphi^*(\mathscr{V})} \to \widetilde{\mathscr{X}_{\text{ét}}}$ denote the composite map of underlying topoi either way around (3.2.4), we check that the two resulting maps $\mathscr{O}_{\mathscr{X}_{\text{ét}}} \rightrightarrows g_*\mathscr{O}_{\varphi^*(\mathscr{V})}$ of ring objects over **C** coincide. It suffices to compare the maps when evaluating on affines $\mathscr{V}' = \operatorname{Spec} A'$ étale over $\mathscr{X}$, and since $\varphi^*(\mathscr{V}) \times_Z \varphi^*(\mathscr{V}') = \varphi^*(\mathscr{V} \times_{\mathscr{X}} \mathscr{V}')$ with $\mathscr{V} \times_{\mathscr{X}} \mathscr{V}'$ also affine (as $\Delta_{\mathscr{X}/\mathbf{C}}$ is affine), the problem reduces to one of functoriality: we claim that for an arbitrary étale map $\mathscr{W} = \operatorname{Spec} B \to \mathscr{X}$ from an affine, the associated map $B \to \mathscr{O}_Z(\varphi^*(\mathscr{W}))$ as in (3.2.2) is functorial in $\mathscr{W} \to \mathscr{X}$. (This functoriality is applied to the first projection $\mathscr{V} \times_{\mathscr{X}} \mathscr{V}' \to \mathscr{V}'$ over $\mathscr{X}$.) Such functoriality is immediate from naturality of $\varphi^\# : \mathscr{O}_{\mathscr{X}} \to \varphi_*\mathscr{O}_Z$ on $\mathscr{X}_{\text{ét}}$ and functoriality of the adjunction between φ^* and φ_*.

We have solved the existence problem for each $\varphi^*(\mathscr{V})$ in place of Z, where $h : \mathscr{V} \to \mathscr{X}$ is any étale map from an affine such that h factors through $\mathscr{U}$ over $\mathscr{X}$, and by uniqueness the resulting analytic maps $f_{\mathscr{V}} : \varphi^*(\mathscr{V}) \to \mathscr{X}^{\mathrm{an}}$ all arise from a common analytic map $f : Z \to \mathscr{X}^{\mathrm{an}}$. The equivalence $\xi_{\mathscr{V}} : \varphi|_{\widetilde{\varphi^*(\mathscr{V})}} \simeq i \circ \widetilde{f}_{\mathscr{V}}$ was constructed to be functorial in $\mathscr{V}$ over $\mathscr{X}$ (not just $\mathscr{V}$ over $\mathscr{U}$), so the canonical isomorphisms $\varphi^*(\mathscr{V}) \times_Z \varphi^*(\mathscr{V}') \simeq \varphi^*(\mathscr{V} \times_{\mathscr{X}} \mathscr{V}')$ with *affine* $\mathscr{V} \times_{\mathscr{X}} \mathscr{V}'$ allow us to use étale descent with pullback functors to globalize the $\xi_{\mathscr{V}}$ to an equivalence $\xi : \varphi \simeq i \circ \widetilde{f}$ as we let $\mathscr{V}$ vary through an affine étale cover of $\mathscr{U}$ (e.g., a Zariski affine open covering). ■

3.3. **Universal mapping property: non-archimedean case.** Let k be a field complete with respect to a non-archimedean absolute value, and let $\mathscr{X}$ be a quasi-separated algebraic space locally of finite type over k. We use [CT, §2] to define the concepts of *analytifiability in the sense of charts* for both Berkovich k-analytic spaces and rigid-analytic spaces over k (assuming $|k^\times| \neq 1$ in the latter case), similarly to Definition 3.2.1. By [CT, 2.3.5] and its easier k-analytic variant, these properties are preserved under any extension of the ground field (respecting the absolute value).

We also define the notion of analytifiability in the sense of topoi in both the Berkovich and rigid-analytic cases similarly to the complex-analytic case in Definition 3.2.1: in the Berkovich (resp. rigid-analytic) case we use the étale topos (resp. Tate-étale topos) $\widetilde{Z_{\text{ét}}}$ and require that $i : \widetilde{(\mathscr{X}^{\mathrm{an}})_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{ét}}}$ satisfies the expected universal property relative to maps $\varphi : \widetilde{Z_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{ét}}}$ of locally ringed topoi over k. (Such φ are strictly henselian morphisms, by Proposition 2.3.7.) In the case of k-schemes $\mathscr{X}$ locally of finite type, the analytification $\mathscr{X}^{\mathrm{an}}$ in the sense of locally ringed spaces over k as in [Ber2, 2.6] is characterized by a universal property among *good* k-analytic spaces (considered as locally ringed spaces over k). So there is some work to be done even in the scheme case, and we carry this out now to streamline later arguments when $\mathscr{X}$ is just an algebraic space.

Proposition 3.3.1. *Let $\mathscr{X}$ be a scheme locally of finite type over k, and $\mathscr{X}^{\text{an}}$ its good analytification in the sense of good k-analytic Berkovich spaces. Let $i : \widetilde{(\mathscr{X}^{\text{an}})_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{ét}}}$ be the map of locally ringed topoi induced by the morphisms of sites $(\mathscr{X}^{\text{an}})_{\text{ét}} \to \mathscr{X}_{\text{ét}}$ defined by analytification of schemes. Then $(\mathscr{X}^{\text{an}}, i)$ is an analytification in the sense of topoi.*

The rigid-analytic case (when $|k^\times| \neq 1$) is similar but easier, so we leave that case to the reader.

Proof. The idea of the proof of the desired universal property of $(\mathscr{X}^{\text{an}}, i)$ is to systematically adapt the proof of Theorem 3.1.3, except we have to make some modifications to address technical issues such as the intervention of non-good spaces Z and the fact that inclusions of affinoid domains are generally not étale in the k-analytic case.

Let Z be a k-analytic Berkovich space equipped with a morphism $\varphi : \widetilde{Z_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{ét}}}$ of locally ringed topoi over k. We claim there exists a unique k-analytic map $f : Z \to \mathscr{X}^{\text{an}}$ such that $i \circ \widetilde{f} \simeq \varphi$. By Proposition 2.3.7, i and φ are strictly henselian. Since étale k-analytic maps are open, subobjects of the final object $\underline{Z}$ in $\widetilde{Z_{\text{ét}}}$ are represented by open subspaces of Z (the proof for schemes in [SGA4, VIII, Prop. 6.1] carries over verbatim to the k-analytic case). To prove uniqueness of f, we can argue exactly as in the proof of faithfulness in Theorem 3.1.3 to reduce to the case when $\mathscr{X} = \operatorname{Spec} A$ is affine (with its étale topology), and then we can compose with inclusions from k-analytic affinoid domains in Z to reduce to the case when Z is k-affinoid and hence good. Then the map $f : Z \to \mathscr{X}^{\text{an}}$ is uniquely determined by the induced k-algebra map $A \to \mathscr{O}(Z)$ due to the universal property of the map of locally ringed spaces $\mathscr{X}^{\text{an}} \to \mathscr{X}$ relative to good k-analytic spaces, yet $A \to \mathscr{O}(Z)$ is precisely $\varphi^\sharp$ on global sections with $\varphi \simeq i \circ \widetilde{f}$.

To prove the existence of f, we can argue exactly as in the proof of Theorem 3.1.3 to reduce to the case when $\mathscr{X}$ is affine and Z is Hausdorff. Hence, for any closed k-analytic domain Y in Z, $\widetilde{Y_{\text{ét}}} \to \widetilde{Z_{\text{ét}}}$ is an equivalence onto the full subcategory of objects with empty stalks over $Z - Y$ (as we noted near the end of the proof of Lemma 3.1.4). This applies with Y any k-affinoid domain in Z. The following lemma for gluing étale sheaves along k-analytic domains allows us to reduce to the case when Z is k-affinoid (and hence good) by the same argument used to reduce to the case that $\mathscr{X}'$ is a scheme in the proof of Theorem 3.1.3.

Lemma 3.3.2. *Let Z be a Hausdorff k-analytic space, and $\{Z_i\}$ a collection of closed k-analytic domains in Z such that any point $z \in Z$ has a neighborhood covered by finitely many Z_i. For any $\mathscr{F}$ in $\widetilde{Z_{\text{ét}}}$, let $\mathscr{F}_i = \mathscr{F}|_{Z_i} \in \widetilde{Z_{i,\text{ét}}}$ and let $\theta_{ji} : \mathscr{F}_i|_{Z_{ij}} \simeq \mathscr{F}_j|_{Z_{ij}}$ be the evident isomorphism of étale sheaves over $Z_{ij} = Z_i \cap Z_j$. Then $\mathscr{F} \rightsquigarrow (\{\mathscr{F}_i\}, \{\theta_{ji}\})$ is an equivalence from $\widetilde{Z_{\text{ét}}}$ to the category of étale sheaves on the Z_i equipped with "descent data".*

Proof. By considering maps of stalks, faithfulness is clear and the proof of full faithfulness together with essential surjectivity is easily reduced to the case when $\{Z_i\}$ is finite. Then for any étale $U \to Z$, with U_i the preimage of Z_i, for any $(\{\mathscr{F}_i\}, \{\theta_{ij}\})$ define

$$\mathscr{F}(U) := \{(s_i) \in \prod \mathscr{F}_i(U_i) \,|\, \theta_{ji}(s_i) = s_j\}.$$

This is clearly an étale sheaf on Z, and there is are evident maps $\theta_i : \mathscr{F}|_{Z_i} \to \mathscr{F}_i$ over $Z_{i,\text{ét}}$ (adjoint to the maps $\mathscr{F} \to (\eta_i)_*(\mathscr{F}_i)$, where $\eta_i : Z_i \to Z$ is the inclusion). It is easy to check that $\theta_{ji} \circ \theta_i = \theta_j$ over $(Z_i \cap Z_j)_{\text{ét}}$, so provided that each θ_i is an isomorphism it follows from consideration of stalks that we have inverted the given functor and so have established the required equivalence of categories.

To prove that each θ_i is an isomorphism, it suffices to check on stalks at each $z \in Z$. More generally, if $\eta : Y \to Z$ is the inclusion of a closed k-analytic domain and $\mathscr{G}$ is an étale sheaf on Y then we claim that $\eta_*\mathscr{G}$ has empty stalk outside of the closed set $\eta(Y)$ and that the natural map $(\eta_*\mathscr{G})_{\eta(y)} \to \mathscr{G}_y$ is bijective for all $y \in Y$. The former is obvious and the latter is shown in the proof of [Ber2, 4.3.4(i)]. ■

Now we have arranged that Z is good (even k-affinoid). Since subobjects of $\underline{Z}$ in $\widetilde{Z_{\text{ét}}}$ are represented by open subspaces, φ gives rise to a map of locally ringed spaces $\varphi_{\text{Zar}} : Z \to \mathscr{X}$ over k by the same procedure used in the proof of Theorem 3.1.3 to construct the map denoted there as f_{Zar}. By the universal property of $\mathscr{X}^{\mathrm{an}}$ among good k-analytic spaces, there is a unique k-analytic map $\varphi_{\mathrm{Zar}}^{\mathrm{an}} : Z \to \mathscr{X}^{\mathrm{an}}$ which, viewed as a map of locally ringed spaces over k, recovers φ_{Zar} after composition with the map of locally ringed spaces $\mathscr{X}^{\mathrm{an}} \to \mathscr{X}$ over k. We have to check that $\varphi \simeq i \circ \widetilde{\varphi_{\mathrm{Zar}}^{\mathrm{an}}}$ as maps between strictly henselian ringed topoi over k. Such an isomorphism merely as maps of topoi is deduced from the strictly henselian property of φ, exactly as in the proof of Theorem 3.1.3 (using the Berkovich case of Example 2.1.7 in place of the scheme case). The further comparison at the level of ring objects over k goes exactly as near the end of the proof of Theorem 3.1.3, using that a k-map of locally ringed spaces from a good k-analytic space to an affine k-scheme of finite type is uniquely determined by the induced map on global sections of structure sheaves. ■

Exactly as in the complex-analytic case, over k analytifications in the sense of topoi are naturally functorial in the algebraic space. (Functoriality for analytifiability in the sense of charts is [CT, 2.2.3].) By [CT, §4.2], if $\mathscr{X}$ is separated then it is analytifiable in the sense of charts for both the Berkovich and rigid-analytic cases over k, and in the Berkovich case $\mathscr{X}^{\mathrm{an}}$ is good (and strictly k-analytic). Before stating the non-archimedean version of Theorem 3.2.3, we discuss some useful compatibility properties of topos-theoretic analytification with respect to locally closed immersions. (The reason for interest in this point is that the algebraic space $\mathscr{X}$ admits a locally finite stratification by locally closed subspaces which are *schemes*. This will be crucial for overcoming difficulties created by the abundance of "non-classical" points on Berkovich k-analytic spaces.)

Suppose that in the Berkovich sense there exists an analytification $(\mathscr{X}^{\mathrm{an}}, i)$ in the sense of topoi. We claim that if $\mathscr{Z} \to \mathscr{X}$ is a locally closed immersion then an analytification $(\mathscr{Z}^{\mathrm{an}}, i')$ in the sense of topoi also exists, and in a natural manner $\mathscr{Z}^{\mathrm{an}}$ is a locally closed subspace of $\mathscr{X}^{\mathrm{an}}$ for the analytic Zariski topology (in the sense of being a closed subspace of a Zariski-open subspace). To see this, it suffices to separately treat the case of closed immersions and open immersions into $\mathscr{X}$.

Let $\mathscr{Z} \hookrightarrow \mathscr{X}$ be the closed immersion corresponding to a coherent ideal $\mathscr{I}$ on $\mathscr{X}$, and consider the ideal sheaf $\mathscr{I}^{\mathrm{an}}$ in $\mathscr{O}_{\mathscr{X}^{\mathrm{an}}}$ that is generated by the image of $i^*(\mathscr{I}) \to i^*(\mathscr{O}_{\mathscr{X}}) \to \mathscr{O}_{\mathscr{X}^{\mathrm{an}}}$. Since $\mathscr{I}$ is locally of finite presentation and i^* is exact, $i^*(\mathscr{I})$ is locally of finite presentation over $i^*(\mathscr{O}_{\mathscr{X}})$. Hence, $\mathscr{I}^{\mathrm{an}}$ is coherent, so its zero space in $\mathscr{X}^{\mathrm{an}}$ makes sense. By the universal property of $\mathscr{X}^{\mathrm{an}}$, for any k-analytic space Z a k-analytic map $f : Z \to \mathscr{X}^{\mathrm{an}}$ factors through the zero space of $\mathscr{I}^{\mathrm{an}}$ if and only if the composite map $i \circ \widetilde{f} : \widetilde{Z_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{ét}}}$ factors through the fully faithful functor $\widetilde{\mathscr{Z}_{\text{ét}}} \hookrightarrow \widetilde{\mathscr{X}_{\text{ét}}}$; the resulting map $\widetilde{Z_{\text{ét}}} \to \widetilde{\mathscr{Z}_{\text{ét}}}$ is easily seen to be strictly henselian and unique up to equivalence (since étale sheaves of sets on a closed k-analytic subspace are precisely those on the ambient k-analytic space with empty pullback over the complementary open subspace; see [Ber2, 4.3.4(ii)] for the analogue with abelian sheaves, whose proof adapts to sheaves of sets.

By consideration of structure sheaves, we conclude that the zero space of $\mathscr{I}^{\mathrm{an}}$ in $\mathscr{X}^{\mathrm{an}}$ serves as an analytification $\mathscr{Z}^{\mathrm{an}}$ of $\mathscr{Z}$ in the sense of topoi. A simpler version of the same argument shows that $\mathscr{X}^{\mathrm{an}} - \mathscr{Z}^{\mathrm{an}}$ is an analytification of $\mathscr{X} - \mathscr{Z}$ in the sense of topoi. Combining the two, we see that the formation of $\mathscr{X}^{\mathrm{an}}$ "commutes" with passage to locally closed subspaces of $\mathscr{X}$.

Here is our main result.

Theorem 3.3.3. *Let $\mathscr{X}$ be a quasi-separated algebraic space locally of finite type over k.*

(1) *If there exists an analytification $(\mathscr{X}^{\mathrm{an}}, i)$ in the sense of topoi for k-analytic Berkovich spaces then $\mathscr{X}^{\mathrm{an}}$ is an analytification of $\mathscr{X}$ in the sense of charts, precisely in the same manner as in the complex-analytic case in Theorem* 3.2.3(1)*.*
(2) *Assume there exists an analytification $\mathscr{X}^{\mathrm{an}}$ of $\mathscr{X}$ in the sense of charts for k-analytic Berkovich spaces. If $\Delta_{\mathscr{X}/k}$ is affine* (*e.g., a closed immersion*) *then there exists a strictly henselian morphism $i : \widetilde{\mathscr{X}^{\mathrm{an}}} \to \widetilde{\mathscr{X}_{\text{ét}}}$ over k such that $(\mathscr{X}^{\mathrm{an}}, i)$ is an analytification in the sense of topoi.*
(3) *If $\mathscr{X}$ is separated and $|k^{\times}| \neq 1$ then its analytification in the sense of charts for rigid-analytic spaces is an analytification in the sense of topoi for rigid-analytic spaces, exactly as in part* (1)*.*

The interested reader can easily verify from our construction of i below that it recovers the maps of ringed topoi (over k) used in [CT, §3.3] to discuss relative GAGA and comparison morphisms for coherent sheaves in the rigid-analytic case.

Proof. Our argument will be a modification of the one used in the complex-analytic case, adapted to the special features of non-archimedean analytic spaces. Curiously, although (1) is much easier than (2) in the complex-analytic case due to the identification of $|\mathscr{X}^{\rm an}|$ with $\mathscr{X}(\mathbf{C})$, in the k-analytic case (1) will require more work because (i) $\mathscr{X}^{\rm an}$ typically has far more points than $\mathscr{X}$ even when $\mathscr{X}$ is a scheme, and (ii) the proof of the complex-analytic analogue of (2) will work essentially unchanged for k-analytic spaces.

Proof of (1). Consider an étale scheme cover $\pi : \mathscr{U} \to \mathscr{X}$ and define the scheme $\mathscr{R} = \mathscr{U} \times_{\mathscr{X}} \mathscr{U}$. The problem is precisely to prove that the canonical map $\pi^{\rm an} : \mathscr{U}^{\rm an} \to \mathscr{X}^{\rm an}$ is an étale cover and that the natural map $\mathscr{R}^{\rm an} \to \mathscr{U}^{\rm an} \times_{\mathscr{X}^{\rm an}} \mathscr{U}^{\rm an}$ is an isomorphism.

Lemma 3.3.4. *The map* $\pi^{\rm an}$ *is surjective and quasi-finite.*

We refer the reader to [Ber2, 3.1] for the notion of quasi-finiteness for k-analytic morphisms $f : Y' \to Y$, and note that by [Ber2, 3.1.10] this property near a point $y' \in Y'$ is equivalent to y' being isolated in $f^{-1}(f(y))$ and lying in the relative interior $\mathrm{Int}(Y'/Y)$ (see [Ber2, 1.5.4]).

Proof. Since $\mathscr{X}$ is a quasi-separated locally noetherian algebraic space, it has a locally finite stratification in reduced locally closed sub*schemes*. By Lemma 3.2.2 the underlying topological space $|\mathscr{X}|$ naturally coincides with the set of points of $\widetilde{\mathscr{X}_{\text{ét}}}$, so for any $x \in \mathscr{X}^{\rm an}$ the point $\overline{x}^* \circ i^*$ of $\widetilde{\mathscr{X}_{\text{ét}}}$ corresponds to a point $i(x) \in |\mathscr{X}|$. Thus, there is a locally closed subscheme $\mathscr{Y}$ in $\mathscr{X}$ such that $i(x) \in |\mathscr{Y}|$ inside of $|\mathscr{X}|$. The preceding construction of $\mathscr{Y}^{\rm an}$ inside of $\mathscr{X}^{\rm an}$ shows that $i(x) \in \mathscr{Y}$ implies $x \in \mathscr{Y}^{\rm an}$. But $\mathscr{Y}$ is a scheme, so $\mathscr{Y}^{\rm an}$ coincides with the usual analytification of $\mathscr{Y}$ (by Proposition 3.3.1). Since $\mathscr{U}_{\mathscr{Y}} := \mathscr{U} \times_{\mathscr{X}} \mathscr{Y} \to \mathscr{Y}$ is an étale scheme cover, analytification yields an étale cover of k-analytic spaces [Ber2, 2.6.2, 2.6.8, 3.3.11]. Thus, x lifts to a point $u \in (\mathscr{U} \times_{\mathscr{X}} \mathscr{Y})^{\rm an}$. Under the closed immersion of this latter analytification into $\mathscr{U}^{\rm an}$, the image of u is carried by $\pi^{\rm an}$ onto x. This proves the surjectivity of $\pi^{\rm an}$.

Consideration of zero spaces of coherent sheaves implies $(\pi^{\rm an})^{-1}(\mathscr{Y}^{\rm an}) = (\mathscr{U}_{\mathscr{Y}})^{\rm an}$, so $\pi^{\rm an}$ has quasi-finite fibers over $\mathscr{Y}^{\rm an}$ since the map $\mathscr{U}_{\mathscr{Y}} \to \mathscr{Y}$ of locally finite type k-schemes is étale and hence locally quasi-finite (so its analytification is quasi-finite [Ber2, 3.1.7]). Thus, to prove that $\pi^{\rm an}$ is quasi-finite at the points of ${\pi^{\rm an}}^{-1}(x)$ it suffices to show that all such points lie in the relative interior of $\mathscr{U}^{\rm an}$ over $\mathscr{X}^{\rm an}$. Since $\mathscr{U}_{\mathscr{Y}}$ is Zariski-open in a closed subspace of $\mathscr{U}$, the same holds after analytification. Hence, by [T, Cor. 4.6], $\mathrm{Int}(\mathscr{U}^{\rm an}/\mathscr{X}^{\rm an})$ meets $(\mathscr{U}_{\mathscr{Y}})^{\rm an}$ in $\mathrm{Int}((\mathscr{U}_{\mathscr{Y}})^{\rm an}/\mathscr{Y}^{\rm an})$. But this latter relative interior equals $(\mathscr{U}_{\mathscr{Y}})^{\rm an}$ since $\mathscr{U}_{\mathscr{Y}} \to \mathscr{Y}$ is a map of locally finite type k-schemes. Hence, $\pi^{\rm an}$ is quasi-finite at all points. ■

Since $\pi^{\rm an}$ is quasi-finite in the k-analytic sense, by [Ber2, 3.2.8, 3.3.10] the locus of points of $\mathscr{U}^{\rm an}$ near which it is étale is Zariski-open. A non-empty Zariski-closed set in the good and strictly k-analytic $\mathscr{U}^{\rm an}$ must contain points u whose completed residue field is k-finite, so to prove that $\pi^{\rm an}$ is étale it suffices to work at points with k-finite completed residue field. Fix such a point $x \in \mathscr{X}^{\rm an}$. The universal property of $(\mathscr{X}^{\rm an}, i)$ applied to $Z := \mathscr{M}(R)$ for local k-finite R implies that the quasi-finite k-analytic space ${\pi^{\rm an}}^{-1}(x)$ is identified with $\pi^{-1}(x)^{\rm an}$. Hence, the étale property of π reduces the task of proving that $\pi^{\rm an}$ is étale to verifying that at all points $u \in {\pi^{\rm an}}^{-1}(x)$, the quasi-finite $\pi^{\rm an}$ is flat at u in the sense of [Ber2, 3.2.5]. But using k-finite artinian points allows us to apply the universal property of $(\mathscr{X}^{\rm an}, i)$ to verify that for any k-affinoid domain $V := \mathscr{M}(A) \subset \mathscr{X}^{\rm an}$ through x and V-finite k-affinoid domain $V' := \mathscr{M}(B) \subset {\pi^{\rm an}}^{-1}(V)$ through u, the finite map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is formally étale at the maximal ideal corresponding to u. It follows by further shrinking of V' around u that $\pi^{\rm an}$ is étale at every u over x, so (by varying the k-finite x) $\pi^{\rm an}$ is étale. This finishes the proof that $\pi^{\rm an}$ is an étale cover.

To complete the proof of (1), we have to show that the natural map $\delta : \mathscr{R}^{\rm an} \to \mathscr{U}^{\rm an} \times_{\mathscr{X}^{\rm an}} \mathscr{U}^{\rm an}$ is an isomorphism. This map is a monomorphism since $\mathscr{R} \to \mathscr{U} \times \mathscr{U}$ is a monomorphism (and monicity for a k-morphism between locally finite type k-schemes is preserved under analytification, due to its equivalence with the relative diagonal being an isomorphism). But δ is a $\mathscr{U}^{\rm an}$-map between k-analytic spaces étale over $\mathscr{U}^{\rm an}$, so δ is étale. A k-analytic étale monomorphism is an isomorphism onto its open image (as we see by proving triviality of the completed residue field extensions and using [Ber2, 3.4.1]), so δ is an open immersion. To prove it is surjective, we pick $x \in \mathscr{X}^{\rm an}$ and check the surjectivity of δ on x-fibers as follows. By the same stratification trick as in the proof of Lemma 3.3.4, we find a locally closed subscheme $\mathscr{Y} \subset \mathscr{X}$ through $i(x)$ and can then replace $\mathscr{R} \rightrightarrows \mathscr{U}$ with its pullback over $\mathscr{Y}$ to reduce to the known case when $\mathscr{X}$ is a scheme.

Proof of (2). Choose an étale scheme cover $\pi : \mathscr{U} \to \mathscr{X}$ and assume that $\mathscr{X}^{\rm an} := \mathscr{U}^{\rm an}/\mathscr{R}^{\rm an}$ exists (where $\mathscr{R} = \mathscr{U} \times_{\mathscr{X}} \mathscr{U}$). Exactly as in the complex-analytic case, we construct a strictly henselian k-morphism $i : \widetilde{(\mathscr{X}^{\rm an})_{\rm ét}} \to \widetilde{\mathscr{X}_{\rm ét}}$. By using the existence of quotients by *separated* étale equivalence relations for k-analytic spaces [CT, 4.2.2] in place of the appeal to [Gr, Prop. 5.6] in the complex-analytic case (in the proof of Lemma 3.2.4), the only aspect of the proof of (2) in the complex-analytic case which does not carry over verbatim to the k-analytic case is the verification of *uniqueness* of a k-analytic map $f : Z \to \mathscr{X}^{\rm an}$ such that $i \circ \widetilde{f} : \widetilde{Z_{\rm ét}} \to \widetilde{\mathscr{X}_{\rm ét}}$ is equivalent to a specified k-morphism. But to prove uniqueness we can compose with the inclusion of k-analytic affinoid domains in Z to reduce to the case when Z is affinoid, hence good. Since $\mathscr{X}^{\rm an}$ is also good, the k-analytic map f is then uniquely determined by the underlying map of locally ringed spaces over k [Ber2, 1.5.2ff], so it suffices to show that f is determined on underlying sets and on local stalks of the structure sheaves.

The formation of the analytification $\mathscr{X}^{\rm an}$ in the sense of charts is compatible with passage to locally closed immersions $\mathscr{Y} \hookrightarrow \mathscr{X}$ (due to the observations about closed and Zariski-open immersions at the end of Example 2.1.1). Hence, we can use Lemma 3.2.2 and the stratification trick to reduce the uniqueness of f on underlying sets to the case when $\mathscr{X}$ is a scheme (with Z a good k-analytic space). But then we can compute $\mathscr{X}^{\rm an}$ using the obvious chart $\mathscr{U} = \mathscr{X}$, so $\mathscr{X}^{\rm an}$ has the expected universal property among good k-analytic spaces and thus f is unique in such cases. This proves the uniqueness of f on underlying sets in general.

To prove the uniqueness of $f_z^{\#} : \mathscr{O}_{\mathscr{X}^{\rm an}, f(z)} \to \mathscr{O}_{Z,z}$, since $\mathscr{O}_{Z,z}$ is noetherian it suffices to check uniqueness modulo powers of some proper ideal. Once again choosing a locally closed subscheme $\mathscr{Y}$ in $\mathscr{X}$ through $\varphi(z)$, we can then use the infinitesimal neighborhoods of $\mathscr{Y}$ in $\mathscr{X}$ to reduce to the case when $\mathscr{X}_{\rm red}$ is a scheme. But then $\mathscr{X}$ is a scheme [Kn, III, Thm. 3.3], and we just saw that the k-analytic map f is unique when $\mathscr{X}$ is a scheme.

Proof of (3). Assume $|k^{\times}| \neq 1$. Since $\mathscr{X}$ is assumed to be separated, there exists a Berkovich analytification $\mathscr{X}^{\rm an,Ber}$ in the sense of charts and it is a Hausdorff (even separated and good) strictly k-analytic space. By construction in [CT], the analytification of $\mathscr{X}$ in the sense of charts for rigid-analytic spaces over k is the rigid-analytic space $\mathscr{X}_0^{\rm an,Ber}$ associated to $\mathscr{X}^{\rm an,Ber}$ under the fully faithful functor $X \rightsquigarrow X_0$ from Hausdorff strictly k-analytic spaces to quasi-separated rigid-analytic spaces as in [Ber2, 1.6.1]. The argument near the end of [CT, §4.3] shows that the map $\mathscr{U}_0^{\rm an,Ber} \to \mathscr{X}_0^{\rm an,Ber}$ admits local sections for the Tate-étale topology, so $U \rightsquigarrow U_0$ defines a map of ringed sites $(\mathscr{X}_0^{\rm an,Ber})_{\rm ét} \to (\mathscr{X}^{\rm an,Ber})_{\rm ét}$. The resulting map i_0 of ringed topoi is easily seen to be strictly henselian (this amounts to Example 2.1.7 being compatible with the functor $U \rightsquigarrow U_0$), so

$$i_{\rm rig} \stackrel{\rm def}{=} i \circ i_0 : (\mathscr{X}_0^{\rm an,Ber})_{\rm ét}^{\sim} \to \widetilde{\mathscr{X}_{\rm ét}}$$

is a strictly henselian morphism over k.

Consider a map $\varphi_0 : \widetilde{(Z_0)_{\rm ét}} \to \widetilde{\mathscr{X}_{\rm ét}}$ of locally ringed topoi over k (so φ_0 is strictly henselian, by Proposition 2.3.7), with Z_0 a rigid-analytic space over k. We must prove there is a unique rigid-analytic map $f_0 : Z_0 \to \mathscr{X}_0^{\rm an,Ber}$ such that $i_{\rm rig} \circ \widetilde{f_0} \simeq \varphi_0$. By [CT, Ex. 2.3.2], $|\mathscr{X}_0^{\rm an,Ber}|$ is identified with the set of closed points of $|\mathscr{X}|$ and completed stalks of the structure sheaves of $\mathscr{X}_0^{\rm an,Ber}$ and $\mathscr{X}$ at such matching k-finite points are naturally identified. Thus, the uniqueness of f_0 is proved very similarly to the complex-analytic case except that we have to work throughout with points valued in finite extensions of k (as a substitute for $\mathbf{C}$-valued points). The proof of existence of f_0 is identical to the complex-analytic case once we verify the analogue of Lemma 3.2.4 in the rigid-analytic case. The only subtlety in this is that there is no rigid-analytic analogue of the general existence theorem for quotients by separated étale equivalence relations for k-analytic Berkovich spaces [CT, 4.2.2] (which served as a substitute for the much easier complex-analytic quotient result used in the proof of Lemma 3.2.4), so we need to look more closely at the specific rigid-analytic quotient problem for the Tate-étale topology which arises in the rigid-analytic version of the proof of Lemma 3.2.4.

For the purpose of constructing the required quotient we may work locally on Z, so we can assume $Z = {\rm Sp}(R)$ for a strictly k-affinoid algebra R. Then the separated étale quotient problem is easily identified with the "underlying rigid-analytic counterpart" of a separated étale quotient problem in k-analytic Berkovich spaces over $\mathscr{M}(R)$ (using Berkovich analytification of some affine k-schemes of finite type). Hence, we can apply the separated étale quotient existence theorem in the Berkovich setting and then pass to underlying rigid-analytic spaces to get the required quotient for the Tate-étale topology with rigid-analytic spaces. (Here we use the argument at the end of [CT, §4.3] to get the required sections locally for the Tate-étale topology so as to ensure we really have constructed a quotient for that topology.) ■

Appendix A. Affinoid points of algebraic spaces

Let k be a field complete with respect to a non-archimedean absolute value. This appendix addresses the definition and bijectivity of a certain natural map $\mathscr{X}(A) \to \mathscr{X}^{\mathrm{an}}(\mathrm{Sp}(A))$ for strictly k-affinoid A and separated $\mathscr{X}$ locally of finite type over k when $|k^\times| \neq 1$, a problem which arose in §1.1. It will be convenient to use the algebraic space $\mathscr{X}_A$ over $\operatorname{Spec} A$, so we first address the notion of relative analytification for algebraic spaces. This is reviewed for schemes in §A.1, extended to the case of algebraic spaces in §A.2. It is then applied to the problem of relating $\mathscr{X}(A)$ and $\mathscr{X}^{\mathrm{an}}(\mathrm{Sp}(A))$ in §A.3 (as well as a Berkovich space analogue).

A.1. **Review of relative analytification for schemes.** Let X be an A-scheme locally of finite type. In the unpublished [Kö] there is developed a theory of "relative analytification" $X \rightsquigarrow X^{\mathrm{an}}$ valued in the category of rigid-analytic spaces over $\mathrm{Sp}(A)$ when $|k^\times| \neq 1$ and A is strictly k-affinoid. In [Ber2, §2.6] (generalizing [Ber1, §3.4–§3.5] for $A = k$) the same procedure is explained for good k-analytic Berkovich spaces (allowing $|k^\times| = 1$, and not requiring A to be strictly k-affinoid). The idea is identical to the case $A = k$, namely that algebraic affine n-space over $\operatorname{Spec} A$ analytifies to analytic affine n-space over $\mathrm{Sp}(A)$ (or over $\mathscr{M}(A)$ in the Berkovich case), and appropriate use of closed immersions and gluing takes care of the rest. As a trivial but useful example, $(\operatorname{Spec} A)^{\mathrm{an}} = \mathrm{Sp}(A)$ in the rigid-analytic case (and it is $\mathscr{M}(A)$ in the Berkovich case). There is a characterization in terms of a universal mapping property over A, akin to the "classical" case $A = k$.

Example A.1.1. Assume $|k^\times| \neq 1$, and let $A \to B$ is a map of strictly k-affinoid algebras. For any scheme X locally of finite type over A, by applying the universal property of analytification to the composite map of locally ringed G-spaces $(X_B)^{\mathrm{an}} \to X_B \to X$ over A, we obtain a canonical analytic map

$$(X_B)^{\mathrm{an}} \to X^{\mathrm{an}} \times_{\mathrm{Sp}(A)} \mathrm{Sp}(B)$$

which is easily prove to be an isomorphism.

The special case $B = A/\mathfrak{m}^n$ for maximal ideals $\mathfrak{m}$ in A and $n \geq 1$ enables us to get a handle on the infinitestimal structure of X^{an}. In particular, if $x \in X$ is a closed point (so x is closed in a fiber of $X \to \operatorname{Spec} A$ over a closed point, since A is Jacobson) then $\mathscr{O}_{X,x} \to \mathscr{O}_{X^{\mathrm{an}},x}$ induces an isomorphism on completsions.

All of the familiar properties of this functor in the case $A = k$ (such as behavior under fiber products and open immersions, and preservation of properness, étaleness, and smoothness) carry over with the same proofs for general A; see [Ber2, 2.6.9] for the Berkovich case. Also, the GAGA theorems hold in this setting. This was proved in [Kö] in the rigid-analytic case (also see [C2, Ex. 3.2.6] for an alternative proof), and in [Ber1, §3.4–§3.5] in the case of (good) Berkovich spaces. The main aspects of GAGA in the relative setting over an affinoid are exactly as in the situation over $\mathbf{C}$, namely isomorphisms for higher direct images of coherent sheaves, and functorial equivalences between categories of coherent sheaves (from which GAGA correspondences for closed immersions and morphisms are deduced exactly as Serre did over $\mathbf{C}$ in [Se]).

Example A.1.2. Using the relative analytification functor, we get a natural map of sets $X(A) \to X^{\mathrm{an}}(\mathrm{Sp}(A))$ (and $X(A) \to X^{\mathrm{an}}(\mathscr{M}(A))$ in the Berkovich case); this is functorial in X and A. As an important special case, if X is locally of finite type over k, we get a natural map of sets

$$X(A) = (X_A)(A) \to (X_A)^{\mathrm{an}}(\mathrm{Sp}(A)) = (X^{\mathrm{an}} \times \mathrm{Sp}(A))(\mathrm{Sp}(A)) = X^{\mathrm{an}}(\mathrm{Sp}(A))$$

that is functorial in X and A.

A.2. **The case of algebraic spaces.** The entire development of analytification via quotients of étale schemes charts in [CT] carries over verbatim to the relative setting over an affinoid. To be precise, if $\mathscr{X}$ is an algebraic space locally of finite type over a strictly k-affinoid algebra A (with $|k^\times| \neq 1$) and $\mathscr{R} \rightrightarrows \mathscr{U}$ is an étale scheme chart for $\mathscr{X}$, then $\mathscr{R}$ and $\mathscr{U}$ are locally finite type A-schemes. Thus, we can form the étale equivalence relation $\mathscr{R}^{\rm an} \rightrightarrows \mathscr{U}^{\rm an}$ in relative analytifications and hence the quotient sheaf $\mathscr{U}^{\rm an}/\mathscr{R}^{\rm an}$ for the Tate-étale topology on the category of rigid-analytic spaces over $\mathrm{Sp}(A)$.

We call $\mathscr{X}$ *analytifiable* if this sheaf is represented by a rigid-analytic space over $\mathrm{Sp}(A)$, in which case the representing object is canonically independent of the choice of $\mathscr{R} \rightrightarrows \mathscr{U}$ and is naturally functorial in $\mathscr{X}$; the arguments for this as given in [CT, §2.2] when $A = k$ are entirely formal and so work as written in the general case. In the special case $\dim A = 0$ (equivalently, A is k-finite), we may naturally identify the relative analytifications of X over A and over k (ultimately due to the special case $X = \mathrm{Spec}\, A$).

The argument for compatibility with change of the ground field in [CT, 2.3.5] carries over to prove that if $A \to B$ is a map of k-affinoid algebras and $\mathscr{X}$ is analytifiable over A then $\mathscr{X}_B$ is analytifiable over B and naturally $(\mathscr{X}_B)^{\rm an} \simeq \mathscr{X}^{\rm an} \times_{\mathrm{Sp}(A)} \mathrm{Sp}(B)$. Likewise, the entire discussion of properties of the relative analytification functor for analytifiable algebraic spaces as in [CT, §2.3] carries over without change. The main theorem in [CT] remains true in the relative setting:

Theorem A.2.1. *If $\mathscr{X}$ is a separated algebraic space locally of finite type over A then $\mathscr{X}^{\rm an}$ exists. The same is true for analytification in the sense of k-analytic Berkovich spaces, allowing $|k^\times| = 1$ and not requiring A to be strictly k-affinoid.*

Proof. By the same argument as at the end of [CT, §4.3], the rigid-analytic case is reduced to the k-analytic Berkovich case. The main existence theorem for quotients in [CT] is for the k-analytic quotient of *any* separated étale equivalence relation (including preservation of goodness and strict k-analyticity under the formation of the quotient). In the relative setting, the analytified equivalence relation is of this type, so the required quotient exists (and inherits an $\mathscr{M}(A)$-structure by the universal property of such quotients in the k-analytic category). ■

As a special case, proper algebraic spaces over $\mathrm{Spec}\, A$ are always analytifiable, and the relative GAGA theorems carry over with the same proofs for such algebraic spaces, exactly akin to the arguments used in the scheme case; see [CT, §3.3].

Example A.2.2. The procedure in Example A.1.2 also carries over: if $\mathscr{Y}$ is a separated algebraic space locally of finite type over A then we get a natural map $\mathscr{Y}(A) \to \mathscr{Y}^{\rm an}(\mathrm{Sp}(A))$. As a special case, for a separated algebraic space $\mathscr{X}$ locally of finite type over k we get a natural map $\mathscr{X}(A) \to \mathscr{X}^{\rm an}(\mathrm{Sp}(A))$ by using relative analytification for $\mathscr{Y} = \mathscr{X}_A$.

A.3. **Comparison of A-valued points.** Now we come to the main purpose of this appendix:

Proposition A.3.1. *Assume $|k^\times| \neq 1$. For any strictly k-affinoid algebra A and separated algebraic space $\mathscr{Y}$ locally of finite type over A, the natural map $\mathscr{Y}(A) \to \mathscr{Y}^{\rm an}(\mathrm{Sp}(A))$ is bijective. In particular, $A' \rightsquigarrow \mathscr{Y}(A')$ is a sheaf for the Tate-fpqc topology on affinoid A-algebras, and if $\mathscr{X}$ is a separated algebraic space locally of finite type over k then the natural map $\mathscr{X}(A) \to \mathscr{X}^{\rm an}(\mathrm{Sp}(A))$ as in Example A.2.2 is bijective.*

The same holds in the Berkovich case using $\mathscr{M}(A)$, allowing $|k^\times| = 1$ and not requiring A to be strictly k-affinoid.

The Tate-fpqc topology is defined in [CT, §2.1].

Proof. We give the argument in the rigid-analytic case; the Berkovich case goes the same way. By using suitable quasi-compact Zariski open subspaces $\mathscr{V}$ of $\mathscr{Y}$ (e.g., containing the image of any two A-maps $\mathrm{Spec}\, A \rightrightarrows \mathscr{Y}$, or for which $\mathscr{V}^{\rm an}$ contains the image of an A-map $\mathrm{Sp}(A) \to \mathscr{Y}^{\rm an}$), we may assume that $\mathscr{Y}$ is quasi-compact, and hence of finite type. The Nagata compactification theorem for separated maps of finite type between algebraic spaces was recently proved (see [CLO], as well as forthcoming work of D. Rydh and Temkin–Temkin), so there is an open immersion $\mathscr{Y} \hookrightarrow \overline{\mathscr{Y}}$ into a proper algebraic space over $\mathrm{Spec}\, A$. We will show that the result for $\overline{\mathscr{Y}}$ implies the result for $\mathscr{Y}$, and then handle the proper case using relative GAGA.

Let $\mathscr{Z}$ be the closed complement $\overline{\mathscr{Y}} - \mathscr{Y}$ with the reduced structure, so likewise $\mathscr{Z}^{\rm an}$ is the reduced analytic set in $\overline{\mathscr{Y}}^{\rm an}$ complementary to the Zariski-open subspace $\mathscr{Y}^{\rm an}$. Thus, $\mathscr{Y}(A)$ is identified with the set of $f \in \overline{\mathscr{Y}}(A)$ such that $f^{-1}(\mathscr{Z})$ is empty, and similarly $\mathscr{Y}^{\rm an}({\rm Sp}(A))$ is identified with the set of $F \in \overline{\mathscr{Y}}^{\rm an}({\rm Sp}(A))$ such that $F^{-1}(\mathscr{Z}^{\rm an})$ is empty. In the special case $F = f^{\rm an}$, we have $F^{-1}(\mathscr{Z}^{\rm an}) = f^{-1}(\mathscr{Z})^{\rm an}$. But it is obvious that analytification preserves the property of being empty or not, so the desired bijectivity result for $\mathscr{Y}$ is reduced to the same for $\overline{\mathscr{Y}}$.

We may now assume that $\mathscr{Y}$ is proper over ${\rm Spec}\, A$. By relative GAGA applied to closed subspaces, $\mathscr{Z} \mapsto \mathscr{Z}^{\rm an}$ is a bijection between closed immersions into $\mathscr{Y}$ and closed immersions into $\mathscr{Y}^{\rm an}$. Thus, it remains to check that if $\mathscr{Z}$ is an analytifiable (quasi-separated) algebraic space of finite type over ${\rm Spec}\, A$ then the structure map $f : \mathscr{Z} \to {\rm Spec}\, A$ is an isomorphism if and only if its analytification $f^{\rm an} : \mathscr{Z}^{\rm an} \to {\rm Sp}(A)$ is an isomorphism. This equivalence when $A = k$ is [CT, 2.3.1] for the property of being an isomorphism, and the same proof works in the relative setting. (For the case of interest, we can give a more direct proof of the interesting implication: by the Jacobson property of ${\rm Spec}\, A$ and semi-continuity of fiber dimension, the proper f is quasi-finite and hence finite. The case of finite f is trivial.) ■

References

[A] M. Artin, *Algebraization of formal moduli: II. Existence of modifications*, Annals of Math., **91** no. 1 (1970), pp. 88–135.

[Bén] J. Bénabou, "Introduction to bicategories" in *Reports of the midwest category seminar*, Lecture Notes in Math **47**, 1967.

[Ber1] V. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, 1990.

[Ber2] V. Berkovich, *Étale cohomology for non-Archimedean analytic spaces*, Publ. Math. IHES, **78** (1993), pp. 7–161.

[C1] B. Conrad, *Irreducible components of rigid spaces*, Annales de L'institut Fourier, **49** (1999), pp. 473–541.

[C2] B. Conrad, *Relative ampleness in rigid-analytic geometry*, Annales Inst. Fourier, **56** (2006), pp. 1049–1126.

[C3] B. Conrad, *Moishezon spaces in rigid geometry*, submitted, 2010.

[CLO] B. Conrad, M. Lieblich, M. Olsson, *Nagata compactification for algebraic spaces*, to appear, 2010.

[CT] B. Conrad, M. Temkin, *Non-archimedean analytification of algebraic spaces*, Journal of Algebraic Geometry **18** (2009), pp. 731–788.

[EGA] J. Dieudonné, A. Grothendieck, *Éléments de géométrie algébrique*, Publ. Math. IHES, **4, 8, 11, 17, 20, 24, 28, 32**, (1960-7).

[Gr] A. Grothendieck, *Techniques de construction en géométrie analytique* IV*: formalisme général foncteurs représentables*, Exposé 11, Séminaire H. Cartan 1960/61.

[SGA4] A. Grothendieck, *Théorie des topos et cohomologie étale des schémas*, Lecture Notes in Math. **269, 270, 305**, Springer-Verlag, New York, 1972-73.

[Kn] D. Knutson, *Algebraic spaces*, Lecture Notes in Math. **203**, Springer-Verlag, New York, 1971.

[Kö] U. Köpf, *Über eigentliche familien algebraischer varietäten über affinoiden räumen*, Schriftenreihe Math. Inst. Münster, 2. Serie. Heft **7**, (1974).

[LMB] G. Laumon, L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete **39**, Springer-Verlag, Berline (2000).

[Lur] J. Lurie, *Tannaka duality for geometric stacks*, unpublished preprint, 2006.

[DAGV] J. Lurie, *Derived algebraic geometry* V*: structured spaces*, in preparation, 2010.

[DAGVIII] J. Lurie, *Derived algebraic geometry* VIII*: quasi-coherent sheaves and Tannaka duality*, in preparation, 2010.

[MM] S. MacLane, I. Moerdijk, *Sheaves in geometry and logic*, Universitext, Springer-Verlag, New York, 1992.

[Se] J-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier **6** (1956), pp. 1–42.

[T] M. Temkin, *On local properties of non-Archimedean analytic spaces*, Math. Ann., **318** (2000), pp. 585–607.

Department of Mathematics, Stanford University, Stanford, CA 94305, USA

E-mail address: conrad@math.stanford.edu