Units on product varieties

1. Introduction

The purpose of this note is to prove a very interesting result of Rosenlicht, in a generalized form as is stated without proof in 4.1, VIII, SGA7:

Theorem 1.1. *Let X and Y be geometrically connected and geometrically reduced schemes locally of finite type over a field k, and assume $X(k), Y(k) \neq \emptyset$. Every unit on $X \times Y$ is a product of pullbacks of a unit on X and a unit on Y. Equivalently, the sequence of groups*

$$1 \to k^\times \xrightarrow{\delta} \mathbf{G}_m(X) \times \mathbf{G}_m(Y) \to \mathbf{G}_m(X \times Y) \to 1$$

is short exact, where $\delta(c) = (c, 1/c)$.

Actually, even this version from SGA7 can be generalized: the hypotheses concerning existence of k-rational points can be dropped. This will be treated in §4; the real work will be to first prove Theorem 1.1. Before we prove the theorem, we illustrate it in two ways: Rosenlicht's original application to group varieties, and the necessity of a hypothesis concerning nilpotents in the statement of the theorem. Rosenlicht's application was:

Corollary 1.2. *Let G be a smooth connected group scheme over a field k, and let $f : G \to T$ be a k-scheme map to a torus. If $f(e) = 1$ then f is a group morphism.*

Proof. Since $G(k)$ is non-empty, G is geometrically connected over k. We may therefore extend scalars to an algebraic closure so that k is algebraically closed. We then have $T \simeq \mathbf{G}_m^r$ for some $r > 0$, so by composing f with members of a basis of the character group of T we can assume $T = \mathbf{G}_m$. The meaning of the corollary in this case is that if u is a unit on G and $u(e) = 1$ then $u(gg') = u(g)u(g')$. Since $(g, g') \mapsto u(gg')$ is a unit on $G \times G$, by the theorem we get $u(gg') = u_1(g)u_2(g')$ for some units u_1 and u_2 on G. Hence, by setting $g' = e$ and then $g = e$ we get $u = c_1u_1$ and $u = c_2u_2$ where $c_1 = u_2(e) \in k^\times$ and $c_2 = u_1(e) \in k^\times$. This yields $u(gg') = cu(g)u(g')$ where $c = 1/c_1c_2$. Evaluating at $g = g' = 1$ gives $c = 1$. ∎

To show the necessity of a hypothesis concerning nilpotents, let $A = k[x, y]/(x^2, xy)$ with k any field. Let $X = Y = \mathrm{Spec}(A)$; this is the gluing of an extra tangent vector onto the origin of the affine y-line (so X and Y are even generically smooth: $A[1/y] = k[y, 1/y]$). Since $A_{\mathrm{red}} = k[y]$, clearly every unit in A has the unique form $c(1 + c'x)$ with $c \in k^\times$ and $c' \in k$. As groups, this gives $A^\times = k^\times \times k$. By writing

$$A \otimes_k A = k[x, y, x', y']/(x^2, xy, {x'}^2, x'y'),$$

the subgroup of $(A \otimes_k A)^\times$ generated by units of the two tensor factors is the group of elements

$$b(1 + cx)(1 + c'x') = b(1 + cx + c'x' + cc'xx').$$

But the entire unit group of $A \otimes_k A$ clearly consists of elements of the form

$$b(1 + cx + c'x' + c''xx')$$

with unique $b \in k^\times$ and $c, c', c'' \in k$. Taking cases with $c'' \neq cc'$ gives "extra" units. For example, the unit $1 + xx'$ is such an "extra" unit.

Rosenlicht proved the Corollary 1.2 without Theorem 1.1, instead using some group-theoretic considerations resting on ideas in §3 below (with Weil-style language that somehow encoded Lemma 3.1). Hence, though he never stated Theorem 1.1, it is not unreasonable to credit the result to him.

2. Reduction steps

To prepare for the proof of Theorem 1.1, we make some preliminary observations. If we pick $x_0 \in X(k)$ and $y_0 \in Y(k)$, then Theorem 1.1 is equivalent to the identity

$$u(x, y) = u(x, y_0)u(x_0, y)u(x_0, y_0)^{-1} \tag{1}$$

on $X \times Y$. Indeed, such an identity certainly implies Theorem 1.1, and if $u(x,y) = u_X(x)u_Y(y)$ for units u_X and u_Y on X and Y respectively then $u(x_0,\cdot) = u_X(x_0)u_Y$ and $u(\cdot,y_0) = u_Y(y_0)u_X$, so $u(x,y) = u(x,y_0)u(x_0,y)(u_X(x_0)u_Y(y_0))^{-1}$. Since $u_X(x_0)u_Y(y_0) = u(x_0,y_0)$, we get the desired identity. Of course, whether or not (1) holds is unaffected by extension of the base field. Hence, to prove Theorem 1.1 it suffices to treat the case when k is algebraically closed.

It is also enough to prove (1) over opens $X' \times Y'$ for quasi-compact connected opens $X' \subseteq X$ and $Y' \subseteq Y$ around x_0 and y_0 respectively, so it is harmless to assume that X and Y are of finite type. Note also that if $\{U, V\}$ are connected opens covering X then $U \cap V \neq \emptyset$ since X is connected, and so upon choosing $x_0 \in U \cap V$ and working with (1) it is enough to separately treat the problem on $U \times Y$ and $V \times Y$. Hence, since we have the flexibility to choose whatever x_0 and y_0 we please (in our present setup over an algebraically closed base field), it follows from connectivity considerations and induction on the size of an open affine covering that it is enough to handle the case when X and Y are affine. (The crux is that if $\{U_1, \dots, U_n\}$ is an open cover of X by connected affines with $n > 1$ then we can relabel so that $V = U_2 \cup \dots \cup U_n$ is connected. This amounts to the elementary fact that any finite connected graph with at least two vertices must have a vertex such that removing it and all edges touching it leaves a connected graph.) More generally, it suffices to handle the case when X and Y are quasi-projective.

Next, we pass to the irreducible case as follows. If X is reducible, say with irreducible components $X_1, \dots, X_n$, then (again using that a finite connected graph with at least two vertices has a non-disconnecting vertex) we can relabel if necessary so that $Z = X_2 \cup \dots \cup X_n$ (with its reduced structure) is connected; by connectivity of X we must have $X_1 \cap Z \neq \emptyset$. Hence, if Theorem 1.1 is known for the pairs $\{X_1, Y\}$ and $\{Z, Y\}$ then by choosing $x_0 \in X_1 \cap Z$ we see that the desired identity (1) on $X \times Y$ does hold upon restriction to $X_1 \times Y$ and $Z \times Y$. By reducedness, we therefore get the identity on $X \times Y$ as well. In this way, we can induct on the number of irreducible components of X without changing Y so as to reduce to the case when X is irreducible. Repeating the argument again with the roles of X and Y switched, we can reduce to the case when both X and Y are irreducible.

The verification of (1) can be checked after pullback along surjective maps from varieties covering X and Y, so via normalization we can assume that X and Y are also *normal*. Hence, we may finally assume that we are in the case that there are open immersions $X \subseteq \overline{X}$ and $Y \subseteq \overline{Y}$ into projective *normal* varieties. Likewise, $X \times Y$ is open in the projective product variety $\overline{X} \times \overline{Y}$ that is also normal (as k is algebraically closed). The idea is to study units by considering orders along codimension-1 irreducible components (if any exist!) of $\overline{X} - X$ in $\overline{X}$, $\overline{Y} - Y$ in $\overline{Y}$, and $\overline{X} \times \overline{Y} - X \times Y$ in $\overline{X} \times \overline{Y}$.

3. Study of divisors at infinity

Using notation as above, let $\{X_i\}_{i \in I}$ and $\{Y_j\}_{j \in J}$ be the codimension-1 irreducible components complementary to X in $\overline{X}$ and to Y in $\overline{Y}$ respectively; either or both of these collections may be empty. The codimension-1 irreducible components complementary to $X \times Y$ in $\overline{X} \times \overline{Y}$ are the $X_i \times \overline{Y}$'s and $\overline{X} \times Y_j$'s. By normality and projectivity of $\overline{X} \times \overline{Y}$, we therefore get an exact sequence

$$1 \to k^\times \to \mathbf{G}_m(X \times Y) \to \mathbf{Z}^I \oplus \mathbf{Z}^J \tag{2}$$

where the final map is built from orders along each $X_i \times \overline{Y}$ and each $\overline{X} \times Y_j$. (If $I = J = \emptyset$ then $\mathbf{G}_m(X \times Y) = k^\times$, so there is nothing to do.)

The key to everything is compatibility of pole-order and generic specialization:

Lemma 3.1. *Choose nonzero $f \in \mathscr{O}(X \times Y)$. There is a Zariski-dense open $U \subseteq Y$ such that for all $y \in U(k)$ we have $f|_{X \times \{y\}} \neq 0$ and*

$$\mathrm{ord}_{X_i \times \overline{Y}}(f) = \mathrm{ord}_{X_i}(f|_{X \times \{y\}})$$

for all i. Likewise, there is a Zariski-dense open $V \subseteq X$ such that for all $x \in V(k)$ we have $f|_{\{x\} \times Y} \neq 0$ and

$$\mathrm{ord}_{\overline{X} \times Y_j}(f) = \mathrm{ord}_{Y_j}(f|_{\{x\} \times Y})$$

for all j.

Both U and V may depend on f.

Proof. By irreducibility and symmetry, it suffices to find a U that works for X_1. Since $\overline{X}$ is normal, its smooth locus has complement in $\overline{X}$ with codimension at least 2. Thus, the codimension-1 subvariety X_1 has smooth locus meeting that of $\overline{X}$, whence we can find a smooth open $W \subseteq \overline{X}$ around the generic point η_1 of X_1 such that $W \cap X_1$ is smooth and is cut out by a global function t on W. Hence, $p_1^*(t)$ on $W \times \overline{Y}$ cuts out $X_1 \times \overline{Y}$ in $\overline{X} \times \overline{Y}$ near the generic point of $X_1 \times \overline{Y}$.

Let $n = \mathrm{ord}_{X_1 \times \overline{Y}}(f)$, so $p_1^*(t)^{-n} f$ is a unit on an open $P \subseteq \overline{X} \times \overline{Y}$ around the generic point of $X_1 \times \overline{Y}$. Let $\overline{U} \subseteq \overline{Y}$ be the open image of P in $\overline{Y}$, and let $U = \overline{U} \cap Y$. For $y \in U(k)$ we then have that the regular function $t^{-n} f|_{X \times \{y\}}$ on X is a unit on the nonempty open $P \cap (X \times \{y\})$ in $X \times \{y\} = X$ that contains η_1. This forces $f|_{X \times \{y\}}$ to be nonzero on $X \times \{y\} = X$ with order n along $X_1 \subseteq \overline{X} - X$. ■

Consider a unit u on $X \times Y$, and let $V \subseteq X$ and $U \subseteq Y$ be associated to u as in Lemma 3.1, so for $x_0 \in V(k)$ and $y_0 \in U(k)$ the units $u_X = u|_{X \times \{y_0\}}$ and $u_Y = u|_{\{x_0\} \times Y}$ on X and Y satisfy

$$\mathrm{ord}_{X_i \times \overline{Y}}(u) = \mathrm{ord}_{X_i}(u_X), \;\; \mathrm{ord}_{\overline{X} \times Y_j}(u) = \mathrm{ord}_{Y_j}(u_Y)$$

for all $i \in I$ and $j \in J$. Consider $p_X^*(u_X) \in k(\overline{X} \times \overline{Y})^\times$. This is a unit on $X \times \overline{Y}$, so it is a unit near the generic point of each $\overline{X} \times Y_j$, and by Lemma 3.1 it has order along each $X_i \times \overline{Y}$ equal to $\mathrm{ord}_{X_i}(u_X)$ (since $p_X^*(u_X)$ restricted to any $X \times \{y\} = X$ is u_X). Likewise, $p_Y^*(u_Y)$ is a unit on $\overline{X} \times Y$, so it is a unit near the generic point of each $X_i \times \overline{Y}$, and it has order along each $\overline{X} \times Y_j$ equal to $\mathrm{ord}_{Y_j}(u_Y)$. Thus, the unit $u \cdot p_X^*(u_X)^{-1} \cdot p_Y^*(u_Y)^{-1}$ on $X \times Y$ has order 0 along every codimension-1 irreducible component complementary to $X \times Y$ in $\overline{X} \times \overline{Y}$. Using the exactness of (2), we conclude that $u \cdot p_X^*(u_X)^{-1} \cdot p_Y^*(u_Y)^{-1} \in k^\times$. Hence, u is a product of a nonzero scalar and pullback of units from X and Y. Absorbing the scalar into either of these latter two units concludes the proof of Theorem 1.1.

4. Further generalization

In this final section, we somewhat weaken the assumptions in Theorem 1.1:

Theorem 4.1. *Let X and Y be non-empty geometrically connected and geometrically reduced schemes locally of finite type over a field k. The sequence of groups*

$$1 \to k^\times \xrightarrow{\delta} \mathbf{G}_m(X) \times \mathbf{G}_m(Y) \to \mathbf{G}_m(X \times Y) \to 1$$

is short exact, where $\delta(c) = (c, 1/c)$. In particular, every unit on $X \times Y$ is a product of pullbacks of a unit on X and a unit on Y.

Some connectivity assumption is necessary, as we see by taking X and Y to be disjoint unions of copies of $\mathrm{Spec}\, k$. It is not enough to assume connectivity alone, as it clear by taking $X = Y = \mathrm{Spec}\, k'$ for a non-trivial finite Galois extension k'/k. (Indeed, then $(k' \otimes_k k')^\times = \prod_{g \in G} k'^\times$ with $G = \mathrm{Gal}(k'/k)$, and a unit $(c_g)_g$ with $c_1 = 1$ then comes from $k'^\times \times k'^\times$ if and only if $c_g = g(c')/c'$ for some $c' \in k'^\times$. That is, $g \mapsto c_g$ has to be a 1-cocycle. In particular, each $c_g \in k'^\times$ has norm 1. By treating finite fields separately, there are always elements with norm distinct from 1.)

Let us now explain the left-exactness of the sequence. Say u_X and u_Y are units on X and Y respectively such that $p_X^*(u_X) = p_Y^*(u_Y)$ on $X \times Y$, where p_X and p_Y are the standard projections from $X \times Y$ to its factors. We seek to prove $u_X \in k^\times$ (so then also $u_Y \in k^\times$). We first observe that k has no non-trivial extension fields within the ring $\mathscr{O}(X)$. Indeed, otherwise the structure map $X \to \mathrm{Spec}\, k$ factors through $\mathrm{Spec}\, k' \to \mathrm{Spec}\, k$ where k'/k is a non-trivial extension, so geometric connectivity forces k'/k to be purely inseparable. But then $X \otimes_k k'$ is faithfully flat over the non-reduced one-point scheme $\mathrm{Spec}(k' \otimes_k k')$, so $X \otimes_k k'$ is everywhere non-reduced; this contradicts the hypothesis that X is geometrically reduced over k. By generic smoothness we can find a finite separable extension k'/k such that $Y(k') \neq \emptyset$, so the pullback of u_X to $X \otimes_k k' = X'$ is equal to $u_Y(y_0) \in k'^\times$ for $y_0 \in Y(k')$. Hence, u_X satisfies a monic irreducible polynomial f over k inside $\mathscr{O}(X)$ (as it does so in $\mathscr{O}(X')$), so u_X lies in a subfield $k[t]/(f) \subseteq \mathscr{O}(X)$. Such a subfield must equal k, so $u_X \in k^\times$.

Having established left-exactness in general, we turn to the more interesting matter of exactness on the right, which is to say that any unit u on $X \times Y$ has the form $p_X^*(u_X)p_Y^*(u_Y)$ for units u_X and u_Y on X and Y respectively. Let k'/k be a finite Galois extension such that $X(k'), Y(k') \neq \emptyset$, so Theorem 1.1 applies to $X' = X \otimes_k k'$ and $Y' = Y \otimes_k k'$ over k'. Thus, $u_{k'}$ on $X' \times_{\mathrm{Spec}\, k'} Y'$ has the form $p_{X'}^*(u_1')p_{Y'}^*(u_2')$ for some units u_1' and u_2' on X' and Y' respectively. By Galois equivariance, for every $g \in G = \mathrm{Gal}(k'/k)$ we then have

$$p_{X'}^*(g(u_1'))p_{Y'}^*(g(u_2')) = g(u_{k'}) = u_{k'} = p_{X'}^*(u_1')p_{Y'}^*(u_2').$$

Hence, $p_{X'}^*(g(u_1')/u_1') = p_{Y'}^*(u_2'/g(u_2'))$. This forces $g(u_1')/u_1', u_2'/g(u_2') \in {k'}^\times$ for all $g \in G$. Hence, we get functions $G \rightrightarrows {k'}^\times$ defined by $g \mapsto g(u_1')/u_1', g(u_2')/u_2'$ that are 1-cocycles. By Hilbert's multiplicative Theorem 90, we get $c_1', c_2' \in {k'}^\times$ such that $g(u_j')/u_j' = g(c_j')/c_j'$, so u_1'/c_1' is G-invariant on X' and u_2'/c_2' is G-invariant on Y'. But G-invariant units on X' and Y' are precisely pullbacks of units on X and Y respectively, so we obtain units u_1 and u_2 on X and Y such that the units $p_X^*(u_1)p_Y^*(u_2)$ and u on $X \times Y$ have ratio whose pullback to $X' \times_{\mathrm{Spec}\, k'} Y'$ is in ${k'}^\times$. This forces the ratio on the geometrically connected and geometrically reduced $X \times Y$ to lie in $k^\times$.