CMI SUMMER SCHOOL NOTES ON p-ADIC HODGE THEORY (PRELIMINARY VERSION)

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Contents

Part I. First steps in p-adic Hodge theory 4
1. Motivation 4
1.1. Tate modules 4
1.2. Galois lattices and Galois deformations 6
1.3. Aims of p-adic Hodge theory 7
1.4. Exercises 9
2. Hodge–Tate representations 10
2.1. Basic properties of $\mathbf{C}_K$ 11
2.2. Theorems of Tate–Sen and Faltings 12
2.3. Hodge–Tate decomposition 15
2.4. Formalism of Hodge–Tate representations 18
2.5. Exercises 25
3. Étale φ-modules 26
3.1. p-torsion representations 27
3.2. Torsion and lattice representations 32
3.3. $\mathbf{Q}_p$-representations of G_E 41
3.4. Exercises 43
4. Better ring-theoretic constructions 45
4.1. From gradings to filtrations 45
4.2. Witt vectors and universal Witt constructions 48
4.3. Properties of R 52
4.4. The field of p-adic periods B_{dR} 56
4.5. Exercises 64
5. Formalism of admissible representations 66
5.1. Definitions and examples 66
5.2. Properties of admissible representations 67
5.3. Exercises 72

Part II. Period rings and functors 73

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6. de Rham representations 73
6.1. Basic definitions 73
6.2. Filtered vector spaces 74
6.3. Filtration on D_{dR} 76
6.4. Exercises 82
7. Why filtered isocrystals? 83
7.1. Finite flat group schemes 84
7.2. p-divisible groups and Dieudonné modules 90
7.3. Motivation from crystalline and de Rham cohomologies 95
7.4. Exercises 99
8. Filtered (ϕ, N)-modules 101
8.1. Newton and Hodge polygons 102
8.2. Weakly admissible modules 110
8.3. Twisting and low-dimensional examples 116
8.4. Exercises 126
9. Crystalline and semistable period rings 128
9.1. Construction and properties of B_{cris} 128
9.2. Construction of B_{st} 135
9.3. Finer properties of crystalline and semistable representations 146
9.4. Exercises 153

Part III. Integral p-adic Hodge theory 155
10. Categories of linear algebra data 155
10.1. Modules with φ and connection 155
10.2. The equivalence of categories 159
10.3. Slopes and weak admissibility 164
10.4. Integral theory 168
10.5. Exercises 172
11. $\mathfrak{S}$-modules and applications 173
11.1. Étale φ-modules revisited 174
11.2. $\mathfrak{S}$-modules and G_{K_∞}-representations 177
11.3. Applications to semistable and crystalline representations 181
11.4. Exercises 185
12. Applications to p-divisible groups and finite group schemes 186
12.1. Divided powers and Grothendieck-Messing theory 186
12.2. S-modules 188
12.3. From S to $\mathfrak{S}$ 191
12.4. Finite flat group schemes and strongly divisible lattices 192
12.5. Exercises 196

Part IV. (φ, Γ)-modules and applications 198
13. Foundations 198
13.1. Ramification estimates 198
13.2. Perfect norm fields 206

13.3. Imperfect fields of norms: construction 209
13.4. Imperfect norm fields: Galois equivalence 216
13.5. Some rings in characteristic zero 221
13.6. (φ, Γ)-modules 224
13.7. Exercises 227
14. The Tate-Sen formalism and initial applications 233
14.1. The Tate-Sen conditions 234
14.2. Consequences of the Tate–Sen axioms 243
14.3. Descent of cohomology from G_K to Γ 250
14.4. Exercises 255
15. p-adic representations and formal linear differential equations 257
15.1. Classical Sen theory 257
15.2. Sen theory over B^+_{dR}: the descent step 265
15.3. Sen theory over B^+_{dR}: decompletion 271
15.4. Fontaine's functor $\mathrm{D}_{\mathrm{dif}}$ 278
15.5. Exercises 286
16. Overconvergence of p-adic representations (to be added in!) 288
References 288

Part I. First steps in p-adic Hodge theory

1. Motivation

1.1. **Tate modules.** Let E be an elliptic curve over a number field F, and fix an algebraic closure $\overline{F}/F$ and a prime number p. A fundamental arithmetic invariant of E is the $\mathbf{Z}$-rank of its finitely generated Mordell-Weil group $E(F)$ of rational points over F. This is conjecturally encoded in (and most fruitfully studied via) the p-adic representation of $G_F := \mathrm{Gal}(\overline{F}/F)$ associated to E. Let us review where this representation comes from, as well as some of its interesting properties.

For each $n \geqslant 1$ we can choose an isomorphism of abelian groups

$$\iota_{E,n} : E(\overline{F})[p^n] \simeq (\mathbf{Z}/p^n\mathbf{Z})^2$$

in which G_F acts on the left side through the finite Galois group quotient $\mathrm{Gal}(F(E[p^n])/F)$ associated to the field generated by coordinates of p^n-torsion points of E. By means of $\iota_{E,n}$ we get a representation of this finite Galois group (and hence of G_F) in $\mathrm{GL}_2(\mathbf{Z}/p^n\mathbf{Z})$. As n grows, the open kernel of this representation shrinks in G_F. It is best to package this collection of representations into a single object: we can choose the $\iota_{E,n}$'s to be compatible with respect to reduction modulo p-powers on the target and the multiplication map $E[p^{n+1}] \to E[p^n]$ by p on the source to get an isomorphism of $\mathbf{Z}_p$-modules

$$\mathrm{T}_p(E) := \varprojlim E(\overline{F})[p^n] \simeq \mathbf{Z}_p^2$$

on which G_F acts through a *continuous* representation

$$\rho : G_F \to \mathrm{GL}_2(\mathbf{Z}_p);$$

passing to the quotient modulo p^n recovers the representations on torsion points as considered above.

For any prime $\wp$ of F we choose an embedding of algebraic closures $\overline{F} \hookrightarrow \overline{F_\wp}$ (i.e., we lift the $\wp$-adic place of F to one of $\overline{F}$) to get a decomposition subgroup $G_{F_\wp} \subseteq G_F$, so we may restrict ρ to this subgroup to get a continuous representation $\rho_\wp : G_{F_\wp} \to \mathrm{GL}_2(\mathbf{Z}_p)$ that encodes local information about E at $\wp$. More specifically, if $I_\wp \subseteq G_{F_\wp}$ denotes the inertia subgroup and we identify the quotient $G_{F_\wp}/I_\wp$ with the Galois group $G_{k(\wp)}$ of the finite residue field $k(\wp)$ at $\wp$ then we say that $\rho_\wp$ (or ρ) is *unramified* at $\wp$ if it is trivial on $I_\wp$, in which case it factors through a continuous representation $G_{k(\wp)} \to \mathrm{GL}_2(\mathbf{Z}_p)$. In such cases it is natural to ask about the image of the (arithmetic) Frobenius element $\mathrm{Frob}_\wp \in G_{k(\wp)}$ that acts on $\overline{k(\wp)}$ by $x \mapsto x^{q_\wp}$, where $q_\wp := \#k(\wp)$.

Theorem 1.1.1. *If $\wp \nmid p$ then E has good reduction at $\wp$ (with associated reduction over $k(\wp)$ denoted as $\overline{E}$) if and only if $\rho_\wp$ is unramified at $\wp$. In such cases, $\rho_\wp(\mathrm{Frob}_\wp)$ acts on $\mathrm{T}_p(E)$ with characteristic polynomial $X^2 - a_{E,\wp}X + q_\wp$, where $a_{E,\wp} = q_\wp + 1 - \#\overline{E}(k(\wp)) \in \mathbf{Z} \subseteq \mathbf{Z}_p$.*

Remark 1.1.2. Observe that $a_{E,\wp}$ is a rational integer that is independent of the choice of p (away from $\wp$). By Hasse's theorem, $|a_{E,\wp}| \leqslant 2\sqrt{q_\wp}$. If we had only worked with the representation $\rho \bmod p^n$ on p^n-torsion points rather than with the representation ρ that encodes all p-power torsion levels at once then we would only obtain $a_{E,\wp} \bmod p^n$ rather than $a_{E,\wp} \in \mathbf{Z}$. By the Hasse bound, this sufficies to recover $a_{E,\wp}$ when $q_\wp$ is "small" relative to p^n (i.e., $4\sqrt{q_\wp} < p^n$).

It was conjectured by Birch and Swinnerton-Dyer that $\mathrm{rank}_{\mathbf{Z}}(E(F))$ is encoded in the behavior at $s = 1$ of the Euler product

$$L_{\mathrm{good}}(s, E/F) = \prod_{\mathrm{good}\,\wp} (1 - a_{E,\wp} q_\wp^{-s} + q_\wp^{1-2s})^{-1};$$

this product is only known to make sense for $\mathrm{Re}(s) > 3/2$ in general, but it has been meromorphically continued to the entire complex plane in many special cases (by work of Taylor-Wiles and its generalizations). For each p, the G_F-representation on $\mathrm{T}_p(E)$ encodes all Euler factors at primes $\wp$ of good reduction away from p by Theorem 1.1.1. For this reason, the theory of p-adic representations of Galois groups turns out to be a very convenient framework for studying the arithmetic of L-functions.

Question 1.1.3. Since the notion of good reduction makes sense at $\wp$ without any reference to p, it is natural to ask if there is an analogue of Theorem 1.1.1 when $\wp | p$.

This question was first answered by Grothendieck using p-divisible groups, and his answer can be put in a more useful form by means of some deep results in p-adic Hodge theory: the property of being unramified at $\wp$ (for $\wp \nmid p$) winds up being replaced with the property of being a *crystalline* representation at $\wp$ (when $\wp | p$). This latter notion will be defined much later, but for now we wish to indicate why unramifiedness cannot be the right criterion when $\wp | p$. The point is that the determinant character $\det \rho_\wp : G_{F_\wp} \to \mathbf{Z}_p^\times$ is infinitely ramified when $\wp | p$. In fact, this character is equal to the p-adic cyclotomic character of $F_\wp$, a character that will be ubiquitous in all that follows. We therefore now recall its definition in general (and by Example 1.1.5 below this character is infinitely ramified on $G_{F_\wp}$).

Let F be a field with a fixed separable closure F_s/F and let p be a prime distinct from $\mathrm{char}(F)$. Let $\mu_{p^n} = \mu_{p^n}(F_s)$ denote the group of p^nth roots of unity in $F_s^\times$, and let μ_{p^∞} denote the rising union of these subgroups. The action of G_F on μ_{p^∞} is given by $g(\zeta) = \zeta^{\chi(g)}$ for a unique $\chi(g) \in \mathbf{Z}_p^\times$: for $\zeta \in \mu_{p^n}$ the exponent $\chi(g)$ only matters modulo p^n, and $\chi(g) \bmod p^n \in (\mathbf{Z}/p^n\mathbf{Z})^\times$ describes the action of g on the finite cyclic group μ_{p^n} of order p^n. Thus, $\chi \bmod p^n$ has open kernel (corresponding to the finite extension $F(\mu_{p^n})/F$) and χ is continuous. We call χ the *p-adic cyclotomic character* of F.

Remark 1.1.4. Strictly speaking we should denote the character χ as $\chi_{F,p}$, but it is permissible to just write χ because p is always understood from context and if F'/F is an extension (equipped with a compatible embedding $F_s \to F'_s$ of separable closures) then $\chi_{F,p}|_{G_{F'}} = \chi_{F',p}$.

Example 1.1.5. Let F be the fraction field of a complete discrete valuation ring R with characteristic 0 and residue characteristic p. Hence, $\mathbf{Z}_p \subseteq R$, so we may view $\overline{\mathbf{Q}}_p \subseteq \overline{F}$. In this case $F(\mu_{p^\infty})/F$ is infinitely ramified, or in other words $\chi : G_F \to \mathbf{Z}_p^\times$ has infinite image on the inertia subgroup $I_F \subseteq G_F$. Indeed, since $e := \mathrm{ord}_F(p)$ is finite $F(\mu_{p^n})$ has ramification degree e_n over F satisfying $e_n \cdot e \geqslant \mathrm{ord}_{\mathbf{Q}_p(\mu_{p^n})}(p) = p^{n-1}(p-1)$, so $e_n \to \infty$.

1.2. **Galois lattices and Galois deformations.** Moving away from elliptic curves, we now consider a wider class of examples of p-adic representations arising from algebraic geometry, and we shall formulate a variant on Question 1.1.3 in this setting.

Let X be an algebraic scheme over a field F; the case of smooth projective X is already very interesting. For a prime $p \neq \mathrm{char}(F)$, the étale cohomology groups $\mathrm{H}^i_{\text{ét}}(X_{F_s}, \mathbf{Z}_p)$ are finitely generated $\mathbf{Z}_p$-modules that admit a natural action by $G_F = \mathrm{Gal}(F_s/F)$ (via pullback-functoriality of cohomology and the natural G_F-action on $X_{F_s} = X \otimes_F F_s$), and these modules need not be torsion-free. Hence, the G_F-action on them is not described via matrices in general, but satisfies a continuity condition in the sense of the following definition.

Definition 1.2.1. Let Γ be a profinite group. A *continuous representation of* Γ on a finitely generated $\mathbf{Z}_p$-module Λ is a $\mathbf{Z}_p[\Gamma]$-module structure on Λ such that the action map $\Gamma \times \Lambda \to \Lambda$ is continuous (or, equivalently, such that the Γ-action on the finite set $\Lambda/p^n\Lambda$ has open kernel for all $n \geqslant 1$). These form a category denoted $\mathrm{Rep}_{\mathbf{Z}_p}(\Gamma)$, and $\mathrm{Rep}_{\mathbf{F}_p}(\Gamma)$ is defined similarly.

Example 1.2.2. If a $\mathbf{Z}_p[\Gamma]$-module Λ is finite free as a $\mathbf{Z}_p$-module then $\Lambda \in \mathrm{Rep}_{\mathbf{Z}_p}(\Gamma)$ if and only if the matrix representation $\Gamma \to \mathrm{GL}_n(\mathbf{Z}_p)$ defined by a choice of $\mathbf{Z}_p$-basis of Λ is a continuous map.

Example 1.2.3. Let F be a number field and consider the action by G_F on $\mathrm{H}^i_{\text{ét}}(X_{F_s}, \mathbf{Z}_p)$ for a smooth proper scheme X over F. This turns out to always be a finitely generated $\mathbf{Z}_p$-module whose G_F-action is continuous, but it is generally *not* a free $\mathbf{Z}_p$-module. It is unramified at all "good reduction" primes $\wp \nmid p$ of F (i.e., $I_\wp \subseteq G_F$ acts trivially) due to general base change theorems for étale cohomology. However, if X has good reduction (appropriately defined) at a prime $\wp | p$ then this p-adic representation is rarely unramified at $\wp$. We would like a nice property satisfied by this p-adic representation at primes $\wp | p$ of good reduction for X, replacing unramifiedness. Such a replacement will be provided by p-adic Hodge theory.

Example 1.2.4. A *finite* Γ*-module* is a finite abelian group M equipped with a continuous (left) Γ-action relative to the discrete topology. (That is, each $m \in M$ has an open stabilizer, so M is just a Γ/U-module for some open normal subgroup $U \subseteq \Gamma$.) In case M is a p-group, this is just an object in $\mathrm{Rep}_{\mathbf{Z}_p}(\Gamma)$ with finite $\mathbf{Z}_p$-length. A basic example of interest is $E[p^n](F_s)$ for an elliptic curve E over a field F with $p \neq \mathrm{char}(F)$ and $\Gamma = \mathrm{Gal}(F_s/F)$.

A *finite* Γ*-set* is a finite set Σ equipped with a continuous (left) Γ-action relative to the discrete topology. This is just a finite set with an action by Γ/U for an open normal subgroup $U \subseteq \Gamma$. A basic example of interest is $X(F_s)$ for a finite F-scheme X, with $\Gamma = \mathrm{Gal}(F_s/F)$. The main reason for interest in finite Γ-sets is given in Lemma 7.1.10.

Galois representations as in Example 1.2.3 are the source of many interesting representations, such as those associated to modular forms, and Wiles developed techniques to prove that various continuous representations $\rho : G_F \to \mathrm{GL}_n(\mathbf{Z}_p)$ not initially related to modular forms in fact arise from them in a specific manner. His technique rests on deforming ρ; the simplest instance of a deformation is a continuous representation

$$\widetilde{\rho} : G_F \to \mathrm{GL}_n(\mathbf{Z}_p[\![x]\!])$$

that recovers ρ at $x = 0$ and is unramified at all but finitely many primes of F. A crucial part of Wiles' method is to understand deformations of $\rho|_{G_{F_\wp}}$ when $\wp | p$, and some of the most important recent improvements on Wiles' method (e.g., in work of Kisin [30], [31]) focus on precisely such $\wp$. For these purposes it is essential to work with Galois representations having coefficients in $\mathbf{Z}_p$ or $\mathbf{F}_p$ as a prelude to considerations with $\mathbf{Q}_p$-coefficients. Much of p-adic Hodge theory focuses on the case of $\mathbf{Q}_p$-coefficients, and so we are led to make the following definition.

Definition 1.2.5. A *p-adic representation* of a profinite group Γ is a representation $\rho : \Gamma \to \mathrm{Aut}_{\mathbf{Q}_p}(V)$ of Γ on a finite-dimensional $\mathbf{Q}_p$-vector space V such that ρ is continuous (viewing $\mathrm{Aut}_{\mathbf{Q}_p}(V)$ as $\mathrm{GL}_n(\mathbf{Q}_p)$ upon choosing a basis of V, the choice of which does not matter). The category of such representations is denoted $\mathrm{Rep}_{\mathbf{Q}_p}(\Gamma)$.

One source of objects in $\mathrm{Rep}_{\mathbf{Q}_p}(\Gamma)$ is scalar extension to $\mathbf{Q}_p$ of objects in $\mathrm{Rep}_{\mathbf{Z}_p}(\Gamma)$ (see Exercise 1.4.3). This is essentially the universal example, due to the next lemma.

Lemma 1.2.6. *For $V \in \mathrm{Rep}_{\mathbf{Q}_p}(\Gamma)$, there exists a Γ-stable $\mathbf{Z}_p$-lattice $\Lambda \subseteq V$ (i.e., Λ is a finite free $\mathbf{Z}_p$-submodule of V and $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda \simeq V$).*

Proof. Let $\rho : \Gamma \to \mathrm{Aut}_{\mathbf{Q}_p}(V)$ be the continuous action map. Choose a $\mathbf{Z}_p$-lattice $\Lambda_0 \subseteq V$. Since $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda_0$, we naturally have $\mathrm{Aut}_{\mathbf{Z}_p}(\Lambda_0) \subseteq \mathrm{Aut}_{\mathbf{Q}_p}(V)$ and this is an open subgroup. Hence, the preimage $\Gamma_0 = \rho^{-1}(\mathrm{Aut}_{\mathbf{Z}_p}(\Lambda))$ of this subgroup in Γ is open in Γ. Such an open subgroup has finite index since Γ is compact, so Γ/Γ_0 has a finite set of coset representatives $\{\gamma_i\}$. Thus, the finite sum $\Lambda = \sum_i \rho(\gamma_i)\Lambda_0$ is a $\mathbf{Z}_p$-lattice in V, and it is Γ-stable since Λ_0 is Γ_0-stable and $\Gamma = \coprod \gamma_i \Gamma_0$. ∎

1.3. **Aims of p-adic Hodge theory.** In the study of p-adic representations of $G_F = \mathrm{Gal}(\overline{F}/F)$ for F of finite degree over $\mathbf{Q}_p$, it is very convenient in many proofs if we can pass to the case of an algebraically closed residue field. In practice this amounts to replacing F with the completion $\widehat{F^{\mathrm{un}}}$ of its maximal unramified extension inside of $\overline{F}$ (and replacing G_F with its inertia subgroup I_F; see Exercise 1.4.4(1) below). Hence, it is convenient to permit the residue field k to be either finite or algebraically closed, and so allowing perfect residue fields provides a good degree of generality.

Definition 1.3.1. A *p-adic field* is a field K of characteristic 0 that is complete with respect to a fixed discrete valuation that has a perfect residue field k of characteristic $p > 0$.

Most good properties of p-adic representations of G_K for a p-adic field K will turn out to be detected on the inertia group I_K, so replacing K with $\widehat{K^{\mathrm{un}}}$ is a ubiquitious device in the theory (since $I_K := G_{K^{\mathrm{un}}} = G_{\widehat{K^{\mathrm{un}}}}$ via Exercise 1.4.4(2); note that K^{un} is not complete if $k \neq \overline{k}$). The goal of p-adic Hodge theory is to identify and study various "good" classes of p-adic representations of G_K for p-adic fields K, especially motivated by properties of p-adic representations arising from algebraic geometry over p-adic fields.

The form that this study often takes in practice is the construction of a dictionary that relates good categories of p-adic representations of G_K to various categories of semilinear algebraic objects "over K". By working in terms of semilinear algebra it is often easier to deform, compute, construct families, etc., than is possible by working solely with Galois representations. There are two toy examples of this philosophy that are instructive before we take up the development of the general theory (largely due to Fontaine and his coworkers), and we now explain both of these toy examples (which are in fact substantial theories in their own right).

Example 1.3.2. The theory of *Hodge–Tate representations* was inspired by Tate's study of $\mathrm{T}_p(A)$ for abelian varieties A with good reduction over p-adic fields, and especially by Tate's question as to how the p-adic representation $\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) := \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p)$ arising from a smooth proper K-scheme X is related to the Hodge cohomology $\oplus_{p+q=n} \mathrm{H}^p(X, \Omega^q_{X/K})$. This question concerns finding a p-adic analogue of the classical Hodge decomposition

$$\mathbf{C} \otimes_{\mathbf{Q}} \mathrm{H}^n_{\text{top}}(Z(\mathbf{C}), \mathbf{Q}) \simeq \bigoplus_{p+q=n} \mathrm{H}^p(Z, \Omega^q_Z)$$

for smooth proper $\mathbf{C}$-schemes Z.

In §2 we will define the notion of a Hodge–Tate representation of G_K, and the linear algebra category over K that turns out to be related to Hodge–Tate representations of G_K is the category $\mathrm{Gr}_{K,f}$ of finite-dimensional graded K-vector spaces (i.e., finite-dimensional K-vector spaces V equipped with a direct sum decomposition $V = \oplus_q V_q$, and maps $T : V' \to V$ that are K-linear and satisfy $T(V'_q) \subseteq V_q$ for all q).

Example 1.3.3. A more subtle class of representations arises from the Fontaine–Wintenberger theory of norm fields, and gives rise to the notion of an *étale φ-module* that will arise repeatedly (in various guises) throughout p-adic Hodge theory. The basic setup goes as follows. Fix a p-adic field K and let K_∞/K be an infinitely ramified algebraic extension such that the Galois closure K'_∞/K has Galois group $\mathrm{Gal}(K'_\infty/K)$ that is a p-adic Lie group. The simplest such example is $K_\infty = K'_\infty = K(\mu_{p^\infty})$, in which case K_∞/K is infinitely ramified by Example 1.1.5 and the infinite subgroup $\mathrm{Gal}(K_\infty/K) \subseteq \mathbf{Z}_p^\times$ that is the image of the continuous p-adic cyclotomic character $\chi : G_K \to \mathbf{Z}_p^\times$ is closed and hence open. (Indeed, the p-adic logarithm identifies $1 + p\mathbf{Z}_p$ with $p\mathbf{Z}_p$ for odd p and identifies $1 + 4\mathbf{Z}_2$ with $4\mathbf{Z}_2$ for $p = 2$, and every nontrivial closed subgroup of $\mathbf{Z}_p$ is open.) Another interesting example that arose in work of Breuil and Kisin is the non-Galois extension $K_\infty = K(\pi^{1/p^\infty})$ generated by compatible p-power roots of a fixed uniformizer π of K, in which case $\mathrm{Gal}(K'_\infty/K)$ is an open subgroup of $\mathbf{Z}_p^\times \ltimes \mathbf{Z}_p$.

For any K_∞/K as above, a theorem of Sen ensures that the closed ramification subgroups of $\mathrm{Gal}(K'_\infty/K)$ in the upper numbering are of finite index, so in particular K_∞ with its natural absolute value has residue field k' that is a finite extension of k. The Fontaine–Wintenberger theory of norm fields ([24], [51]) provides a remarkable functorial equivalence between the category of separable algebraic extensions of K_∞ and the category of separable algebraic extensions of an associated local field E of equicharacteristic p (the "field of norms" associated to K_∞/K). The residue field of E is naturally identified with k', so non-canonically we have $E \simeq k'(\!(u)\!)$.

The theory of norm fields will be discussed in §13 in a self-contained manner for the special case when $K_\infty = K(\mu_{p^\infty})$. Logically the norm field formalism precedes p-adic Hodge theory, but it is sufficiently intricate in its constructions that without some knowledge of how p-adic Hodge theory works it is difficult to digest. Fortunately, the main kind of result which we need from the theory of norm fields can be easily stated and used without knowing its proof: upon choosing a separable closure of K_∞, the theory of norm fields yields a separable closure for E and an associated canonical topological isomorphism of the associated absolute Galois groups

$$G_{K_\infty} \simeq G_E. \tag{1.3.1}$$

This is really amazing: the Galois group of an infinitely ramified field of characteristic 0 is naturally isomorphic to the Galois group of a discretely-valued field of equicharacteristic p. In §13.4 this will be proved when $K_\infty = K(\mu_{p^\infty})$; see Theorem 13.4.3. For the general case, see [51].

Because E has equicharacteristic p, we will see in §3 that the category $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ is equivalent to a category of semilinear algebra objects (over a certain coefficient ring depending on E) called étale φ-modules. This equivalence will provide a concrete illustratration of many elementary features of the general formalism of p-adic Hodge theory.

If K_∞/K is Galois with Galois group Γ then G_K-representations can be viewed as G_{K_∞}-representations equipped with an additional "Γ-descent structure" that encodes the descent to a G_K-representation. In this way, (1.3.1) identifies $\mathrm{Rep}_{\mathbf{Z}_p}(G_K)$ with the category of (φ, Γ)-*modules* that consists of étale φ-modules endowed with a suitable Γ-action encoding the descent of an object in $\mathrm{Rep}_{\mathbf{Z}_p}(G_E) = \mathrm{Rep}_{\mathbf{Z}_p}(G_{K_\infty})$ to an object in $\mathrm{Rep}_{\mathbf{Z}_p}(G_K)$. The category of (φ, Γ)-modules gives a remarkable and very useful alternative description of the entire category $\mathrm{Rep}_{\mathbf{Z}_p}(G_K)$ in terms of objects of semilinear algebra. It will be discussed in §13.

1.4. **Exercises.**

Exercise 1.4.1. Let Γ be a profinite group and Λ an object in $\mathrm{Rep}_{\mathbf{Z}_p}(\Gamma)$. Define

$$G = \mathrm{Aut}_{\mathbf{Z}_p}(\Lambda), \quad G_n = \mathrm{Aut}_{\mathbf{Z}_p}(\Lambda/p^n\Lambda),$$

so there are natural "reduction" maps $G \to G_n$ and $G_n \to G_m$ whenever $n \geqslant m$.

(1) Show that if $\Lambda = \mathbf{Z}_p \oplus \mathbf{Z}/p\mathbf{Z}$ then the maps $G_n \to G_1$ and $G \to G_1$ are not surjective.
(2) Prove that the natural map of groups $G \to \varprojlim G_n$ is an isomorphism. Use this to give G a structure of profinite group. Show that a base of opens around the identity consists of the $\ker(G \to G_n)$'s, and that the kernels $\ker(G_n \to G_1)$ are p-groups (hint: show p-power torsion to avoid messy counting). Deduce that G contains an open normal subgroup that is pro-p..
(3) Prove the equivalence of the two definitions given in Definition 1.2.1, and that these are equivalent to the condition that the map $\Gamma \to G = \mathrm{Aut}_{\mathbf{Z}_p}(\Lambda)$ is continuous.
(4) Prove that a continuous map from a pro-ℓ group to a pro-p group is trivial when $\ell \neq p$, and deduce by (3) that if Γ is pro-ℓ then an open subgroup of Γ must act trivially on Λ; in particular, Γ has *finite* image in $\mathrm{Aut}_{\mathbf{Z}_p}(\Lambda)$ in such cases.

Exercise 1.4.2. Let Λ be a finitely generated $\mathbf{Z}_p$-module equipped with a continuous representation by $G_F = \mathrm{Gal}(F_s/F)$ for the fraction field F of a complete discrete valuation ring A. Let $P \subseteq I$ be the wild inertia group and inertia group respective. Let $\rho : G_F \to \mathrm{Aut}_{\mathbf{Z}_p}(\Lambda)$ be the associated homomorphism.

(1) Prove that $\ker \rho$ is closed in G_F, and let F_∞ be the corresponding fixed field; we call it the *splitting field* of ρ. In case ρ is the Tate module representation of an elliptic curve

E over F with $\mathrm{char}(F) \neq p$, prove that the splitting field of ρ is the field $F(E[p^\infty)$ generated by the coordinates of the p-power torsion points.

(2) Let ℓ be the residue characteristic of A. If $\ell \neq p$, prove that the wild inertia group P acts on Λ with an open kernel (so $\rho(P)$ is finite in such cases). Using Tate curves, show by example that this is not necessarily true for the action of I.

(3) Let F_n be the (F-finite) splitting field of $\ker(\rho \bmod p^n\Lambda)$. Prove that $F_\infty = \cup F_n$ and show that $\rho(I)$ is finite if and only if the ramification degree $e(F_n/F)$ is bounded as $n \to \infty$, in which case $\#\rho(I) = \max_n e(F_n/F)$. Deduce that $\rho(I)$ is infinite if and only if the valuation on F_∞ is non-discrete; we then say ρ is *infinitely ramified*. Formulate a related criterion for $\rho(P)$ and wild ramification.

(4) Suppose F is a number field. Using Exercise 1.4.1, prove that $\rho|_{P_v}$ is trivial for all but finitely many places v of F, where P_v is the wild inertia subgroup of I_v. (Ramakrishna constructed examples of ρ that are ramified at infinitely many v.)

Exercise 1.4.3. For $\Lambda \in \mathrm{Rep}_{\mathbf{Z}_p}(\Gamma)$, prove that the scalar extension $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda$ lies in $\mathrm{Rep}_{\mathbf{Q}_p}(\Gamma)$.

Exercise 1.4.4. Let K be a p-adic field with residue field k.

(1) Explain why the valuation ring of K is naturally a local extension of $\mathbf{Z}_p$, and prove that $[K : \mathbf{Q}_p]$ is finite if and only if k is finite.

(2) Prove that every algebraic extension of K admits a unique valuation extending the one on K, and that the maximal unramified extension K^{un}/K inside of $\overline{K}$ (i.e., the compositum of all finite unramified subextensions over K) is not complete when k is not algebraically closed.

(3) Prove that the completion $\widehat{K^{\mathrm{un}}}$ is naturally a p-adic field with residue field $\overline{k}$ that is an algebraic closure of k, and use Krasner's Lemma to prove that $I_K := G_{K^{\mathrm{un}}}$ is naturally isomorphic to $G_{\widehat{K^{\mathrm{un}}}}$ as profinite groups. More specifically, prove that $L \rightsquigarrow L \otimes_{K^{\mathrm{un}}} \widehat{K^{\mathrm{un}}}$ is an equivalence of categories from finite extensions of K^{un} to finite extensions of $\widehat{K^{\mathrm{un}}}$, with $L \otimes_{K^{\mathrm{un}}} \widehat{K^{\mathrm{un}}} \simeq \widehat{L}$.

2. Hodge–Tate representations

From now on, K will always denote a p-adic field (for a fixed prime p) in the sense of Definition 1.3.1, and we fix a choice of algebraic closure $\overline{K}/K$. The Galois group $\mathrm{Gal}(\overline{K}/K)$ is denoted G_K, and we write $\mathbf{C}_K$ to denote the completion $\widehat{\overline{K}}$ of $\overline{K}$ endowed with its unique absolute value extending the given absolute value $|\cdot|$ on K. Generally π will denote a uniformizer of K.

Sometimes we will normalize the absolute value by the requirement that $\mathrm{ord}_K := \log_p |\cdot|$ on $K^\times$ (base p logarithm) satisfies $\mathrm{ord}_K(p) = 1$, and we also write $|\cdot|$ and ord_K to denote the unique continuous extensions to $\mathbf{C}_K$ and $\mathbf{C}_K^\times$ respectively; we define $\mathrm{ord}_K(0) = \infty$. When working with many valuations at once (as will happen in our later study of (φ, Γ)-modules in §13) we may write v instead of ord_K.

Historically, the first class of "good" p-adic representations of G_K were those of Hodge–Tate type; this class was discovered by Serre and Tate in their study of p-adic representations arising from abelian varieties with good reduction over p-adic fields, and in this section we will examine this class of representations with the benefit of hindsight provided by subsequent developments.

The most basic ingredient in the story is the p-adic cyclotomic character from §1.1, which appears through its twisting action on everything in sight. Hence, before we begin it seems best to make some remarks on this character. The p-adic Tate module $\varprojlim \mu_{p^n}(\overline{K})$ of the group GL_1 over K is a free $\mathbf{Z}_p$-module of rank 1 and we shall denote it as $\mathbf{Z}_p(1)$. This does not have a canonical basis, and a choice of basis amounts to a choice of compatible system $(\zeta_{p^n})_{n\geqslant 1}$ of primitive p-power roots of unity (satisfying $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all $n \geqslant 1$). The natural action of G_K on $\mathbf{Z}_p(1)$ is given by the $\mathbf{Z}_p^\times$-valued p-adic cyclotomic character $\chi = \chi_{K,p}$ from §1.1, and sometimes it will be convenient to fix a choice of basis of $\mathbf{Z}_p(1)$ and to thereby view $\mathbf{Z}_p(1)$ as $\mathbf{Z}_p$ endowed with a G_K-action by χ.

For any $r \geqslant 0$ define $\mathbf{Z}_p(r) = \mathbf{Z}_p(1)^{\otimes r}$ and $\mathbf{Z}_p(-r) = \mathbf{Z}_p(r)^\vee$ (linear dual: $M^\vee = \mathrm{Hom}_{\mathbf{Z}_p}(M, \mathbf{Z}_p)$ for any finite free $\mathbf{Z}_p$-module M) with the naturally associated G_K-actions (from functoriality of tensor powers and duality), so upon fixing a basis of $\mathbf{Z}_p(1)$ we identify $\mathbf{Z}_p(r)$ with the $\mathbf{Z}_p$-module $\mathbf{Z}_p$ endowed with the G_K-action χ^r for all $r \in \mathbf{Z}$. If M is an arbitrary $\mathbf{Z}_p[G_K]$-module, we let $M(r) = \mathbf{Z}_p(r) \otimes_{\mathbf{Z}_p} M$ with its natural G_K-action, so upon fixing a basis of $\mathbf{Z}_p(1)$ this is simply M with the modified G_K-action $g.m = \chi(g)^r g(m)$ for $g \in G_K$ and $m \in M$. Elementary isomorphisms such as $(M(r))(r') \simeq M(r+r')$ (with evident transitivity behavior) for $r, r' \in \mathbf{Z}$ and $(M(r))^\vee \simeq M^\vee(-r)$ for $r \in \mathbf{Z}$ and M finite free over $\mathbf{Z}_p$ or over a p-adic field will be used without comment.

2.1. **Basic properties of $\mathbf{C}_K$.** The theory of Hodge–Tate representations will involve studying the G_K-action on $\mathbf{C}_K \otimes_{\mathbf{Q}_p} V$ for a p-adic representation V of G_K (where $g(c \otimes v) = g(c) \otimes g(v)$). Thus, we now discuss two fundamental facts about $\mathbf{C}_K$, the first of which we will use all the time, and the second of which will play an important role later in the theory of norm fields in §13.3.

Proposition 2.1.1. *The field $\mathbf{C}_K$ is algebraically closed.*

Proof. By scaling the variable suitably, it suffices to construct roots for monic non-constant polynomials over $\mathscr{O}_{\mathbf{C}_K}$. Write such a polynomial as

$$P = X^N + a_1 X^{N-1} + \cdots + a_N \in \mathscr{O}_{\mathbf{C}_K}[X]$$

with $N > 0$. We can make a sequence of degree-N monic polynomials $P_n \in \mathscr{O}_{\overline{K}}[X]$ converging to P termwise in coefficients. More specifically, for each $n \geqslant 0$ choose

$$P_n = X^N + a_{1,n} X^{N-1} + \cdots + a_{N,n} \in \mathscr{O}_{\overline{K}}[X]$$

with $P - P_n \in p^{Nn}\mathscr{O}_{\mathbf{C}_K}[X]$. By monicity, each P_n splits over $\mathscr{O}_{\overline{K}}$; let $\alpha_n \in \mathscr{O}_{\overline{K}}$ be a root of P_n.

Since $P_{n+1} - P_n \in p^{Nn}\mathscr{O}_{\mathbf{C}_K}[X]$, we have $P_{n+1}(\alpha_n) \in p^{Nn}\mathscr{O}_{\mathbf{C}_K}$ for all n. Expanding P_{n+1} as $\prod_{i=0}^N (X - \rho_{i,n+1})$ with roots $\rho_{i,n+1} \in \mathscr{O}_{\overline{K}}$, the product of the N differences $\alpha_n - \rho_{i,n+1}$ is divisible by p^{Nn}, so for some root α_{n+1} of P_{n+1} we must have that $\alpha_{n+1} - \alpha_n$ is divisible by p^n. In this way, proceeding by induction on n we have constructed a Cauchy sequence $\{\alpha_n\}$ in $\mathscr{O}_{\overline{K}}$ such that $P_n(\alpha_n) = 0$ for all n. Hence, if $\alpha \in \mathscr{O}_{\mathbf{C}_K}$ is the limit of the α_n's then $P(\alpha) = 0$ by continuity (since $P_n \to P$ coefficient-wise). ■

Since $G_K = \mathrm{Gal}(\overline{K}/K)$ acts on $\overline{K}$ by isometries, this action uniquely extends to an action on the field $\mathbf{C}_K$ by isometries, and so identifies G_K with the isometric automorphism group of $\mathbf{C}_K$ over K. It is then natural to ask if there is a kind of "completed" Galois theory: how does $\mathbf{C}_K^H$ compare with $\overline{K}^H$ for a closed subgroup $H \subseteq \mathbf{G}_K$? Since G_K acts by isometries, $\mathbf{C}_K^H$ is a closed subfield of $\mathbf{C}_K$, so it contains the closure of $\overline{K}^H$. Is it any bigger? For example, taking $H = G_K$, is $\mathbf{C}_K^{G_K}$ larger than K? By Galois theory we have $\mathbf{C}_K^{G_K} \cap \overline{K} = K$, so another way to put the question is: are there transcendental invariants? The following proposition shows that there are none:

Proposition 2.1.2. *Let H be a closed subgroup of G_K. Then $\mathbf{C}_K^H$ is the completion $\widehat{L}$ of $L = \overline{K}^H$ for the valuation v. In particular, if H is an open subgroup of G_K then $\mathbf{C}_K^H$ is the finite extension $\overline{K}^H$ of K, and $\widehat{L} \cap \overline{K} = L$.*

Proof. Choose $x \in \mathbf{C}_K^H$, so we want to show x is a limit of points in $\overline{K}^H$. To do this, we approximation x by algebraic elements and then try to modify the approximating sequence by using that assumed H-invariance of x. Pick a sequence $\{x_n\}_{n\geqslant 0}$ in $\overline{K}$ with $x_n \to x$; more specifically, arrange that $v(x - x_n) \geqslant n$ for all n. For $g \in H$ we have

$$v(g(x_n) - x_n) = v(g(x_n - x) - (x_n - x)) \geqslant \min(v(g(x_n - x))), v(x_n - x)) = v(x_n - x) \geqslant n.$$

Since x_n in $\overline{K}$ is close to its entire H-orbit (as made precise above), it is natural to guess that this may be explained by x being essentially as close to an algebraic H-invariant element. This is indeed true: by [2, Prop. 1], for each n there exists $y_n \in \overline{K}^H$ such that $v(x_n - y_n) \geqslant n - p/(p-1)^2$. But $x_n \to x$, so we conclude that likewise $y_n \to x$. That is, x is a limit of points in $\overline{K}^H$, as desired. ■

2.2. **Theorems of Tate–Sen and Faltings.** Let X be a smooth proper scheme over a p-adic field K. Tate discovered in special cases (abelian varieties with good reduction) that although the p-adic representation spaces $\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ for G_K are mysterious, they become *much simpler* after we apply the drastic operation

$$V \rightsquigarrow \mathbf{C}_K \otimes_{\mathbf{Q}_p} V,$$

with the G_K-action on $\mathbf{C}_K \otimes_{\mathbf{Q}_p} V$ defined by $g(c \otimes v) = g(c) \otimes g(v)$ for $c \in \mathbf{C}_K$ and $v \in V$. Before we examine this operation in detail, we introduce the category in which its output lives.

Definition 2.2.1. A $\mathbf{C}_K$*-representation of* G_K is a finite-dimensional $\mathbf{C}_K$-vector space W equipped with a continuous G_K-action map $G_K \times W \to W$ that is semilinear (i.e., $g(cw) = g(c)g(w)$ for all $c \in \mathbf{C}_K$ and $w \in W$). The category of such objects (using $\mathbf{C}_K$-linear G_K-equivariant morphisms) is denoted $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$.

This is a p-adic analogue of the notion of a complex vector space endowed with a conjugate-linear automorphism. In concrete terms, if we choose a $\mathbf{C}_K$-basis $\{w_1, \dots, w_n\}$ of W then we may uniquely write $g(w_j) = \sum_i a_{ij}(g) w_i$ for all j, and $\mu : G_K \to \mathrm{Mat}_{n\times n}(\mathbf{C}_K)$ defined by $g \mapsto (a_{ij}(g))$ is a continuous map that satisfies $\mu(1) = \mathrm{id}$ and $\mu(gh) = \mu(g) \cdot g(\mu(h))$ for all $g, h \in G_K$. In particular, μ takes its values in $\mathrm{GL}_n(\mathbf{C}_K)$ (with $g(\mu(g^{-1}))$ as inverse to

$\mu(g)$) but beware that μ is *not* a homomorphism in general (due to the semilinearity of the G_K-action).

Example 2.2.2. If $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ then $W := \mathbf{C}_K \otimes_{\mathbf{Q}_p} V$ is an object in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$. We will be most interested in W that arise in this way, but it clarifies matters at the outset to work with general W as above.

The category $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ is an abelian category with evident notions of tensor product, direct sum, and exact sequence. If we are attentive to the semilinearity then we can also define a reasonable notion of duality: for any W in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, the *dual* $W^\vee$ is the usual $\mathbf{C}_K$-linear dual on which G_K acts according to the formula $(g.\ell)(w) = g(\ell(g^{-1}(w)))$ for all $w \in W$, $\ell \in W^\vee$, and $g \in G_K$. This formula is rigged to ensure that $g.\ell : W \to \mathbf{C}_K$ is $\mathbf{C}_K$-linear (even though the action of g^{-1} on W is generally not $\mathbf{C}_K$-linear). Since G_K acts continuously on W and on $\mathbf{C}_K$, this action on $W^\vee$ is continuous. In concrete terms, if we choose a basis $\{w_i\}$ of W and describe the G_K-action on W via a continuous function $\mu : G_K \to \mathrm{GL}_n(\mathbf{C}_K)$ as above Example 2.2.2 then $W^\vee$ endowed with the dual basis is described by the function $g \mapsto g(\mu(g^{-1})^t)$ that is visibly continuous. Habitual constructions from linear algebra such as the isomorphisms $W \simeq W^{\vee\vee}$ and $W^\vee \otimes W'^\vee \simeq (W \otimes W')^\vee$ as well as the evaluation morphism $W \otimes W^\vee \to \mathbf{C}_K$ are seen to be morphisms in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$.

The following deep result of Faltings answers a question of Tate.

Theorem 2.2.3 (Faltings)**.** *Let K be a p-adic field. For smooth proper K-schemes X, there is a canonical isomorphism*

$$\mathbf{C}_K \otimes_{\mathbf{Q}_p} \mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \simeq \bigoplus_q (\mathbf{C}_K(-q) \otimes_K \mathrm{H}^{n-q}(X, \Omega^q_{X/K})) \tag{2.2.1}$$

in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, where the G_K-action on the right side is defined through the action on each $\mathbf{C}_K(-q) = \mathbf{C}_K \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(-q)$. In particular, non-canonically

$$\mathbf{C}_K \otimes_{\mathbf{Q}_p} \mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \simeq \bigoplus_q \mathbf{C}_K(-q)^{h^{n-q,q}}$$

in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, with $h^{p,q} = \dim_K \mathrm{H}^p(X, \Omega^q_{X/K})$.

This is a remarkable theorem for two reasons: it says that $\mathbf{C}_K \otimes_{\mathbf{Q}_p} \mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ as a $\mathbf{C}_K$-representation space of G_K is a direct sum of extremely simple pieces (the $\mathbf{C}_K(-q)$'s with suitable multiplicity), and we will see that this isomorphism enables us to recover the K-vector spaces $\mathrm{H}^{n-q}(X, \Omega^q_{X/K})$ from $\mathbf{C}_K \otimes_{\mathbf{Q}_p} \mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ by means of operations that make sense on all objects in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$. This is a basic example of a *comparison isomorphism* that relates one p-adic cohomology theory to another. (Faltings established a version of his result without requiring X to be smooth or proper, but then the Hodge cohomology terms must be replaced with something else.) It is extremely important to keep in mind (as we shall soon see) that we *cannot* recover the p-adic representation space $\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ from the Hodge cohomologies $\mathrm{H}^{n-q}(X, \Omega^q_{X/K})$ in (2.2.1). In general, $\mathbf{C}_K \otimes_{\mathbf{Q}_p} V$ loses a lot of information about V. This fact is very fundamental in motivating many of the basic constructions in p-adic Hodge theory, and it is best illustrated by the following example.

Example 2.2.4. Let E be an elliptic curve over K with split multiplicative reduction, and consider the representation space $V_p(E) = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathrm{T}_p(E) \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$. The theory of Tate curves provides an exact sequence

$$0 \to \mathbf{Q}_p(1) \to V_p(E) \to \mathbf{Q}_p \to 0 \tag{2.2.2}$$

that is non-split in $\mathrm{Rep}_{\mathbf{Q}_p}(G_{K'})$ for all finite extensions K'/K inside of $\overline{K}$.

If we apply $\overline{K} \otimes_{\mathbf{Q}_p} (\cdot)$ to (2.2.2) then we get an exact sequence

$$0 \to \overline{K}(1) \to \overline{K} \otimes_{\mathbf{Q}_p} V_p(E) \to \overline{K} \to 0$$

in the category $\mathrm{Rep}_{\overline{K}}(G_K)$ of semilinear representations of G_K on $\overline{K}$-vector spaces. We claim that this sequence cannot be split in $\mathrm{Rep}_{\overline{K}}(G_K)$. Assume it is split. Since $\overline{K}$ is the directed union of finite subextensions K'/K, there would then exist such a K' over which the splitting occurs. That is, applying $K' \otimes_{\mathbf{Q}_p} (\cdot)$ to (2.2.2) would give an exact sequence admitting a G_K-equivariant K'-linear splitting. Viewing this as a split sequence of $K'[G_{K'}]$-modules, we could apply a $\mathbf{Q}_p$-linear projection $K' \to \mathbf{Q}_p$ that restricts to the identity on $\mathbf{Q}_p \subseteq K'$ so as to recover (2.2.2) equipped with a $\mathbf{Q}_p[G_{K'}]$-linear splitting. But (2.2.2) has no splitting in $\mathrm{Rep}_{\mathbf{Q}_p}(G_{K'})$, so we have a contradiction. Hence, applying $\overline{K} \otimes_{\mathbf{Q}_p} (\cdot)$ to (2.2.2) gives a non-split sequence in $\mathrm{Rep}_{\overline{K}}(G_K)$, as claimed.

This non-splitting over $\overline{K}$ makes it all the more remarkable that if we instead apply $\mathbf{C}_K \otimes_{\mathbf{Q}_p} (\cdot)$ to (2.2.2) then the resulting sequence in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ does (uniquely) split! This is a special case of the second part of the following fundamental result that pervades all that follows. It rests on a deep study of the ramification theory of local fields.

Definition 2.2.5. Let Γ be a topological group, and M a topological G-module. The *continuous cohomology* group $\mathrm{H}^1_{\mathrm{cont}}(G, M)$ (often just denoted $\mathrm{H}^1(G, M)$ by abuse of notation) is defined using continuous 1-cochains.

Imposing the continuity condition on cycles really does affect the H^1, and in many interesting cases (such as with profinite G and discrete G-module M) the associated group cohomology defined without continuity conditions is of no real interest. Exercise 2.5.2 illustrates this. The justification that $\mathrm{H}^1_{\mathrm{cont}}(G, M)$ is the right concept for the consideration of exactness properties of G-invariants in topological settings is explained in Exercise 2.5.3.

Example 2.2.6. Let $\eta : G_K \to \mathbf{Z}_p^\times$ be a continuous character. We identify $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbf{C}_K(\eta))$ with the set of isomorphism classes of extensions

$$0 \to \mathbf{C}_K(\eta) \to W \to \mathbf{C}_K \to 0 \tag{2.2.3}$$

in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ as follows: using the matrix description

$$\begin{pmatrix} \eta & * \\ 0 & 1 \end{pmatrix}$$

of such a W, the homomorphism property for the G_K-action on W says that the upper right entry function is a 1-cocycle on G_K with values in $\mathbf{C}_K(\eta)$, and changing the choice of $\mathbf{C}_K$-linear splitting changes this function by a 1-coboundary. The continuity of the 1-cocycle says exactly that the G_K-action on W is continuous. Changing the choice of $\mathbf{C}_K$-basis of W that is compatible with the filtration in (2.2.3) changes the 1-cocycle by a 1-coboundary.

In this way we get a well-defined continuous cohomology class, and the procedure can be reversed (up to isomorphism of the extension structure (2.2.3) in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$).

Theorem 2.2.7 (Tate–Sen). *For any p-adic field K we have $K = \mathbf{C}_K^{G_K}$ (i.e., there are no transcendental invariants) and $\mathbf{C}_K(r)^{G_K} = 0$ for $r \neq 0$ (i.e., if $x \in \mathbf{C}_K$ and $g(x) = \chi(g)^{-r}x$ for all $g \in G_K$ and some $r \neq 0$ then $x = 0$). Also, $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbf{C}_K(r)) = 0$ if $r \neq 0$ and $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbf{C}_K)$ is 1-dimensional over K.*

More generally, if $\eta : G_K \to \mathscr{O}_K^\times$ is a continuous character such that $\eta(G_K)$ is a commutative p-adic Lie group of dimension at most 1 (i.e., $\eta(G_K)$ is finite or contains $\mathbf{Z}_p$ as an open subgroup) and if $\mathbf{C}_K(\eta)$ denotes $\mathbf{C}_K$ with the twisted G_K-action $g.c = \eta(g)g(c)$ then $\mathrm{H}^i_{\mathrm{cont}}(G_K, \mathbf{C}_K(\eta)) = 0$ for $i = 0, 1$ when $\eta(I_K)$ is infinite and these cohomologies are 1-dimensional over K when $\eta(I_K)$ is finite (i.e., when the splitting field of η over K is finitely ramified).

Theorem 2.2.7 is proved in §14 via a "Tate–Sen formalism", as we record in Theorem 14.3.4. (There is no circular reasoning; §14 is entirely self-contained.) This result implies that all exact sequences (2.2.3) are split when $\eta(I_K)$ is infinite. Moreover, in such cases the splitting is *unique*. Indeed, any two splittings $\mathbf{C}_K \rightrightarrows W$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ differ by an element of $\mathrm{Hom}_{\mathrm{Rep}_{\mathbf{C}_K}(G_K)}(\mathbf{C}_K, \mathbf{C}_K(\eta))$, and by chasing the image of $1 \in \mathbf{C}_K$ this Hom-set is identified with $\mathbf{C}_K(\eta)^{G_K}$. But by the Tate–Sen theorem this vanishes when $\eta(I_K)$ is infinite.

The real importance of Theorem 2.2.7 is revealed when we consider an arbitrary $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ admitting an isomorphism as in Faltings' Theorem 2.2.3:

$$W \simeq \bigoplus_q \mathbf{C}_K(-q)^{h_q}. \tag{2.2.4}$$

Although such a direct sum decomposition is non-canonical in general (in the sense that the individual lines $\mathbf{C}_K(-q)$ appearing in the direct sum decomposition are generally not uniquely determined within W when $h_q > 1$), we shall see that for any such W there is a *canonical* decomposition $W \simeq \oplus_q(\mathbf{C}_K(-q) \otimes_K W\{q\})$ for a canonically associated K-vector space $W\{q\}$ with dimension h_q.

Keep in mind that although the G_K-action on any $\mathbf{Q}_p(r)$ factors through G_K^{ab}, the action on $\mathbf{C}_K(r)$ does not since the G_K-action is not $\mathbf{C}_K$-linear but rather is $\mathbf{C}_K$-semilinear. In particular, for nonzero W as in (2.2.4) the G_K-action on W does not factor through G_K^{ab}.

Example 2.2.8. In (2.2.4) we have $W^{G_K} \simeq \oplus_q(\mathbf{C}_K(-q)^{G_K})^{h_q} \simeq K^{h_0}$ by the Tate–Sen theorem, so $h_0 = \dim_K W^{G_K}$. A priori it is not clear that $\dim_K W^{G_K}$ should be finite for typical $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$. Such finiteness holds in much greater generality, as we shall see, and the W that arise as in (2.2.4) will be intrinsically characterized in terms of such finiteness properties.

2.3. **Hodge–Tate decomposition.** The companion to Theorem 2.2.7 that gets p-adic Hodge theory off the ground is a certain lemma of Serre and Tate that we now state. For $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ and $q \in \mathbf{Z}$, consider the K-vector space

$$W\{q\} := W(q)^{G_K} \simeq \{w \in W \mid g(w) = \chi(g)^{-q}w \text{ for all } g \in G_K\}, \tag{2.3.1}$$

where the isomorphism rests on a choice of basis of $\mathbf{Z}_p(1)$. In particular, this isomorphism is not canonical when $q \neq 0$ and $W\{q\} \neq 0$, so $W\{q\}$ is canonically a K-subspace of $W(q)$ but it is only non-canonically a K-subspace of W when $q \neq 0$ and $W\{q\} \neq 0$. More importantly, $W\{q\}$ is not a $\mathbf{C}_K$-subspace of $W(q)$ when it is nonzero. In fact, $W\{q\}$ contains no $\mathbf{C}_K$-lines, for if $x \in W\{q\}$ is nonzero and cx lies in $W\{q\}$ for all $c \in \mathbf{C}_K$ then $g(c) = c$ for all $c \in \mathbf{C}_K$ and all $g \in G_K$, which is absurd since $\overline{K} \subseteq \mathbf{C}_K$.

We have a natural G_K-equivariant K-linear multiplication map

$$K(-q) \otimes_K W\{q\} \hookrightarrow K(-q) \otimes_K W(q) \simeq W,$$

so extending scalars defines maps

$$\mathbf{C}_K(-q) \otimes_K W\{q\} \to W$$

in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ for all $q \in \mathbf{Z}$.

Lemma 2.3.1 (Serre–Tate). *For $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$, the natural $\mathbf{C}_K$-linear G_K-equivariant map*

$$\xi_W : \bigoplus_q (\mathbf{C}_K(-q) \otimes_K W\{q\}) \to W$$

is injective. In particular, $W\{q\} = 0$ for all but finitely many q and $\dim_K W\{q\} < \infty$ for all q, with $\sum_q \dim_K W\{q\} \leqslant \dim_{\mathbf{C}_K} W$; equality holds here if and only if ξ_W is an isomorphism.

Proof. The idea is to consider a hypothetical nonzero element in $\ker \xi_W$ with "shortest length" in terms of elementary tensors and to use that $\ker \xi_W$ is a $\mathbf{C}_K$-subspace yet each $W\{q\}$ contains no $\mathbf{C}_K$-lines. To carry out this strategy, consider a nonzero $v = (v_q)_q \in \ker \xi_W$. We choose such v with minimal length, where the length $\ell(x)$ for

$$x = (x_q) \in \oplus_q (\mathbf{C}_K(-q) \otimes_K W\{q\})$$

is defined as follows. For an element x_q of $\mathbf{C}_K \otimes_K W\{q\}$ we define $\ell(x_q)$ to be the least integer $n_q \geqslant 0$ such that x_q is a sum of n_q elementary tensors, and for a general $x = (x_q)$ we define $\ell(x) = \sum \ell(x_q)$ (which makes sense since $\ell(x_q) = 0$ for all but finitely many q). Observe that $\mathbf{C}_K^\times$-scaling preserves length.

It suffices to prove that $\ell(v) = 1$. Indeed, this forces $v = c \otimes w$ for some $c \in \mathbf{C}_K^\times$ and nonzero $w \in W\{q_0\}$ (with some $q_0 \in \mathbf{Z}$), which is a contradiction since $\xi_W(v) = cw \neq 0$ in W. To prove $\ell(v) = 1$, first observe that there is some q_0 such that v_{q_0} is nonzero. By applying a $\mathbf{C}_K^\times$-scaling we can arrange that v_0 has a minimal-length expression $v_{q_0} = \sum_j c_j \otimes y_j$ with $c_j \in \mathbf{C}_K^\times$, $y_j \in W\{q_0\} = W(q_0)^{G_K}$, and some nonzero $c_{j_0} = \mathbf{Q}_p(q_0)$.

Pick $g \in G_K$, so $g(v) \in \ker \xi_W$ and hence $g(v) - \chi(g)^{-q_0} v \in \ker \xi_W$. For each $q \in \mathbf{Z}$ the qth component of $g(v) - \chi(g)^{-q_0} v$ is $g(v_q) - \chi(g)^{-q_0} v_q$. If $\sum c_{j,q} \otimes y_{j,q}$ is a minimal-length expression for v_q then since

$$g(v_q) - \chi(g)^{-q_0} v_q = \sum (\chi(g)^{-q} g(c_{j,q}) - \chi(q)^{-q_0} c_{j,q}) \otimes y_{j,q},$$

we see that $\ell(g(v_q) - v_q) \leqslant \ell(v_q)$. Hence, $g(v) - \chi(g)^{-q_0} v$ has length at most $\ell(v)$. But $g(v_{q_0}) - \chi(g)^{-q_0} v_{q_0} = \sum_j (\chi(g)^{q_0} g(c_j) - \chi(g)^{-q_0} c_j) \otimes y_j$ since $g(y_j) = \chi(g)^{q_0} y_j$ for all j (as $y_j \in W\{q_0\}$), and this has strictly smaller length than v_{q_0} because $c_{j_0} \in \mathbf{Q}_p(q_0)$. Hence, the point $g(v) - \chi(g)^{-q_0} v \in \ker \xi_W$ has strictly smaller length than v, so it vanishes. Thus,

$\chi(g)^{q_0}g(v) = v$ for all $g \in G_K$. In other words, $v \in \mathbf{Q}_p(-q_0) \otimes_{\mathbf{Q}_p} W\{q_0\}$. But all elements of this space are elementary tensors, so $\ell(v) = 1$ (as $v \neq 0$). ■

Remark 2.3.2. An alternative formulation of the Serre–Tate lemma can be given in terms of the K-subspaces

$$W[q] := \{w \in W \mid g(w) = \chi(g)^{-q}w \text{ for all } g \in G_K\} \subseteq W$$

instead of the K-subspaces $W\{q\} \subseteq W(q)$ from (2.3.1) for all $q \in \mathbf{Z}$. Since $W[q] = \mathbf{Q}_p(-q) \otimes_{\mathbf{Q}_p} W\{q\}$, the Serre–Tate lemma says exactly that the $W[q]$'s are finite-dimensional over K, vanish for all but finitely many q, and are mutually $\mathbf{C}_K$-linearly independent within W in the sense that the natural map $\oplus(\mathbf{C}_K \otimes_K W[q]) \to W$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ is injective.

In the special case $W = \mathbf{C}_K \otimes_{\mathbf{Q}_p} \mathrm{H}^n(X_{\overline{K}}, \mathbf{Q}_p)$ for a smooth proper scheme X over K, Faltings' Theorem 2.2.3 says that ξ_W is an isomorphism and $W\{q\}$ (rather than $W[q]$!) is canonically K-isomorphic to $\mathrm{H}^{n-q}(X, \Omega^q_{X/K})$ for all $q \in \mathbf{Z}$.

Example 2.3.3. Let $W = \mathbf{C}_K(\eta)$ for a continuous character $\eta : G_K \to \mathbf{Z}_p^\times$. By the Tate–Sen theorem, $W\{q\} = \mathbf{C}_K(\eta\chi^{-q})^{G_K}$ is 1-dimensional over K if $\eta\chi^{-q}|_{I_K}$ has finite order (equivalently, if $\eta = \chi^q\psi$ for a finitely ramified character $\psi : G_K \to \mathbf{Z}_p^\times$) and $W\{q\}$ vanishes otherwise. In particular, there is at most one q for which $W\{q\}$ can be nonzero, since if $W\{q\}, W\{q'\} \neq 0$ with $q \neq q'$ then $\eta = \chi^q\psi$ and $\eta = \chi^{q'}\psi'$ with finitely ramified $\psi, \psi' : G_K \rightrightarrows \mathbf{Z}_p^\times$, so $\chi^r|_{I_K}$ has finite image for $r = q - q' \neq 0$, which is absurd (use Example 1.1.5).

An interesting special case of Example 2.3.3 is when K contains $\mathbf{Q}_p(\mu_p)$, so $\chi(G_K)$ is contained in the pro-p group $1 + p\mathbf{Z}_p$. Hence, $\eta = \chi^s$ makes sense for all $s \in \mathbf{Z}_p$, and for $W = \mathbf{C}_K(\eta)$ with such η the space $W\{q\}$ vanishes for all q when $s \notin \mathbf{Z}$ whereas $W\{-s\}$ is 1-dimensional over K if $s \in \mathbf{Z}$. Thus for $s \in \mathbf{Z}_p$ the map $\xi_{\mathbf{C}_K(\chi^s)}$ vanishes if $s \notin \mathbf{Z}$ and it is an isomorphism if $s \in \mathbf{Z}$. The case $s \notin \mathbf{Z}$ is of "non-algebraic" nature, and this property situation is detected by the map $\xi_{\mathbf{C}_K(\chi^s)}$.

Definition 2.3.4. A representation W in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ is *Hodge–Tate* if ξ_W is an isomorphism. We say that V in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ is *Hodge–Tate* if $\mathbf{C}_K \otimes_{\mathbf{Q}_p} V \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ is Hodge–Tate.

Example 2.3.5. If W is Hodge–Tate then by virtue of ξ_W being an isomorphism we have a non-canonical isomorphism $W \simeq \oplus \mathbf{C}_K(-q)^{h_q}$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ with $h_q = \dim_K W\{q\}$. Conversely, consider an object $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ admitting a finite direct sum decomposition $W \simeq \oplus \mathbf{C}_K(-q)^{h_q}$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ with $h_q \geqslant 0$ for all q and $h_q = 0$ for all but finitely many q. The Tate–Sen theorem gives that $W\{q\}$ has dimension h_q for all q, so $\sum_q \dim_K W\{q\} = \sum_q h_q = \dim_{\mathbf{C}_K} W$ and hence W is Hodge–Tate. In other words, the intrinsic property of being Hodge–Tate is equivalent to the concrete property of being isomorphic to a finite direct sum of various objects $\mathbf{C}_K(r_i)$ (with multiplicity permitted).

For any Hodge–Tate object W in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ we define the *Hodge–Tate weights* of W to be those $q \in \mathbf{Z}$ such that $W\{q\} := (\mathbf{C}_K(q) \otimes_{\mathbf{C}_K} W)^{G_K}$ is nonzero, and then we call $h_q := \dim_K W\{q\} \geqslant 1$ the *multiplicity* of q as a Hodge–Tate weight of W. Beware that, according to this definition, $q \in \mathbf{Z}$ is a Hodge–Tate weight of W precisely when there is an injection $\mathbf{C}_K(-q) \hookrightarrow W$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, as opposed to when there is an injection $\mathbf{C}_K(q) \hookrightarrow W$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$. For example, $\mathbf{C}_K(q)$ has $-q$ as its unique Hodge–Tate weight.

Obviously (by Example 2.3.5) if W is Hodge–Tate then so is $W^\vee$, with negated Hodge–Tate weights (compatibly with multiplicities), so it is harmless to change the definition of "Hodge–Tate weight" by a sign. In terms of p-adic Hodge theory, this confusion about signs comes down to later choosing to use covariant or contravariant functors when passing between p-adic representations and semilinear algebra objects (as replacing a representation space with its dual will be the mechanism by which we pass between covariant and contravariant versions of various functors on categories of representations).

2.4. **Formalism of Hodge–Tate representations.** We saw via Example 2.3.5 that for any W in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, W is Hodge–Tate if and only if its dual $W^\vee$ is Hodge–Tate. By the same reasoning, since

$$\left(\oplus_q \mathbf{C}_K(-q)^{h_q}\right) \otimes_{\mathbf{C}_K} \left(\oplus_{q'} \mathbf{C}_K(-q')^{h'_{q'}}\right) \simeq \oplus_r \mathbf{C}_K(-r)^{\sum_i h_i h'_{r-i}}$$

in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ we see that if W and W' are Hodge–Tate then so is $W \otimes W'$ (with Hodge–Tate weights that are suitable sums of products of those of W and W'); we also have in such cases that $W \oplus W'$ is also Hodge–Tate. To most elegantly express how the Hodge–Tate property interacts with tensorial and other operations, it is useful to introduce some terminology.

Definition 2.4.1. A ($\mathbf{Z}$-)*graded vector space* over a field F is an F-vector space D equipped with direct sum decomposition $\oplus_{q \in \mathbf{Z}} D_q$ for F-subspaces $D_q \subseteq D$ (and we define the *qth graded piece* of D to be $\mathrm{gr}^q(D) := D_q$). Morphisms $T : D' \to D$ between graded F-vector spaces are F-linear maps that respect the grading (i.e., $T(D'_q) \subseteq D_q$ for all q). The category of these is denoted Gr_F; we let $\mathrm{Gr}_{F,f}$ denote the full subcategory of D for which $\dim_F D$ is finite.

For any field F, Gr_F is an abelian category with the evident notions of kernel, cokernel, and exact sequence (working in separate degrees). We write $F\langle r \rangle$ for $r \in \mathbf{Z}$ to denote the F-vector space F endowed with the grading for which the unique non-vanishing graded piece is in degree r. For $D, D' \in \mathrm{Gr}_F$ we define the *tensor product* $D \otimes D'$ to have underlying F-vector space $D \otimes_F D'$ and to have qth graded piece $\oplus_{i+j=q}(D_i \otimes_F D'_j)$. Likewise, if $D \in \mathrm{Gr}_{F,f}$ then the *dual* $D^\vee$ has underlying F-vector space given by the F-linear dual and its qth graded piece is $D^\vee_{-q}$.

With these definitions, $F\langle r \rangle \otimes F\langle r' \rangle = F\langle r + r' \rangle$, $F\langle r \rangle^\vee = F\langle -r \rangle$, and the natural evaluation mapping $D \otimes D^\vee \to F\langle 0 \rangle$ and double duality isomorphism $D \simeq (D^\vee)^\vee$ on F-vector spaces for D in $\mathrm{Gr}_{F,f}$ are morphisms in Gr_F. Observe also that a map in Gr_F is an isomorphism if and only if it is a linear isomorphism in each separate degree.

Definition 2.4.2. The covariant functor $\underline{D} = \underline{D}_K : \mathrm{Rep}_{\mathbf{C}_K}(G_K) \to \mathrm{Gr}_K$ is

$$\underline{D}(W) = \oplus_q W\{q\} = \oplus_q (\mathbf{C}_K(q) \otimes_{\mathbf{C}_K} W)^{G_K}.$$

This functor is visibly left-exact.

Remark 2.4.3. Many functors valued in linear algebra categories are denoted with the letter "D". This stands for Dieudonné, who introduced the theory of Dieudonné modules that provides a categorical equivalence between certain categories of group schemes and certain categories of structures in (semi-)linear algebra.

In general, the Serre–Tate lemma says that $\underline{D}$ takes values in $\mathrm{Gr}_{K,f}$ and more specifically that $\dim_K \underline{D}(W) \leqslant \dim_{\mathbf{C}_K} W$ with equality if and only if W is Hodge–Tate. As a simple example, the Tate–Sen theorem gives that $\underline{D}(\mathbf{C}_K(r)) = K\langle -r\rangle$ for all $r \in \mathbf{Z}$. The functor $\underline{D}$ satisfies a useful exactness property on Hodge–Tate objects, as follows.

Proposition 2.4.4. *If* $0 \to W' \to W \to W'' \to 0$ *is a short exact sequence in* $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ *and* W *is Hodge–Tate then so are* W' *and* W''*, in which case the sequence*

$$0 \to \underline{D}(W') \to \underline{D}(W) \to \underline{D}(W'') \to 0$$

in $\mathrm{Gr}_{K,f}$ *is short exact* (*so the multiplicities for each Hodge–Tate weight are additive in short exact sequences of Hodge–Tate representations*)*.*

Proof. We have a left-exact sequence

$$0 \to \underline{D}(W') \to \underline{D}(W) \to \underline{D}(W'') \tag{2.4.1}$$

with $\dim_K \underline{D}(W') \leqslant \dim_{\mathbf{C}_K}(W')$ and similarly for W and W''. But equality holds for W by the Hodge–Tate property, so

$$\begin{aligned} \dim_{\mathbf{C}_K} W = \dim_K \underline{D}(W) &\leqslant \dim_K \underline{D}(W') + \dim_K \underline{D}(W'') \\ &\leqslant \dim_{\mathbf{C}_K} W' + \dim_{\mathbf{C}_K} W'' \\ &= \dim_{\mathbf{C}_K} W, \end{aligned}$$

forcing equality throughout. In particular, W' and W'' are Hodge–Tate and so for K-dimension reasons the left-exact sequence (2.4.1) is right-exact too. ■

Example 2.4.5. Although Proposition 2.4.4 says that any subrepresentation or quotient representation of a Hodge–Tate representation is again Hodge–Tate, the converse is false in the sense that if W' and W'' are Hodge–Tate then W can fail to have this property. To give a counterexample, we recall that $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbf{C}_K) \neq 0$ by Theorem 2.2.7. This gives a non-split exact sequence

$$0 \to \mathbf{C}_K \to W \to \mathbf{C}_K \to 0 \tag{2.4.2}$$

in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, and we claim that such a W cannot be Hodge–Tate. To see this, applying the left-exact functor $\underline{D}$ to the exact sequence above gives a left exact sequence

$$0 \to K\langle 0\rangle \to \underline{D}(W) \to K\langle 0\rangle$$

of graded K-vector spaces, so in particular $\underline{D}(W) = W\{0\} = W^{G_K}$. If W were Hodge–Tate then by Proposition 2.4.4 this left exact sequence of graded K-vector spaces would be short exact, so there would exist some $w \in W^{G_K}$ with nonzero image in $K\langle 0\rangle$. We would then get a $\mathbf{C}_K$-linear G_K-equivariant section $\mathbf{C}_K \to W$ via $c \mapsto cw$. This splits (2.4.2) in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, contradicting the non-split property of (2.4.2). Hence, W cannot be Hodge–Tate.

The functor $\underline{D} = \underline{D}_K$ is useful when studying how the Hodge–Tate property interacts with basic operations such as a finite scalar extension on K, tensor products, duality, and replacing K with $\widehat{K^{\mathrm{un}}}$ (i.e., replacing G_K with I_K), as we now explain.

Theorem 2.4.6. *For any $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$, the natural map $K' \otimes_K \underline{D}_K(W) \to \underline{D}_{K'}(W)$ in $\mathrm{Gr}_{K',f}$ is an isomorphism for all finite extensions K'/K contained in $\overline{K} \subseteq \mathbf{C}_K$. Likewise, the natural map $\widehat{K^{\mathrm{un}}} \otimes_K \underline{D}_K(W) \to \underline{D}_{\widehat{K^{\mathrm{un}}}}(W)$ in $\mathrm{Gr}_{\widehat{K^{\mathrm{un}}},f}$ is an isomorphism.*

In particular, for any finite extension K'/K inside of $\overline{K}$, an object W in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ is Hodge–Tate if and only iff it is Hodge–Tate when viewed in $\mathrm{Rep}_{\mathbf{C}_K}(G_{K'})$, and similarly W is Hodge–Tate in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ if and only if it is Hodge–Tate when viewed in $\mathrm{Rep}_{\mathbf{C}_K}(G_{\widehat{K^{\mathrm{un}}}}) = \mathrm{Rep}_{\mathbf{C}_K}(I_K)$.

This theorem says that the Hodge–Tate property is insensitive to replacing K with a finite extension or restricting to the inertia group (i.e., replacing K with $\widehat{K^{\mathrm{un}}}$). This is a prototype for a class of results that will arise in several later contexts (with properties that refine the Hodge–Tate property). The insensitivity to inertial restriction is a good feature of the Hodge–Tate property, but the insensitivity to finite (possibly ramified) extensions is a bad feature, indicating that the Hodge–Tate property is not sufficiently fine (e.g., to distinguish between good reduction and potentially good reduction for elliptic curves).

Proof. By a transitivity argument, the case of finite extensions is reduced to the case when K'/K is Galois. We first treat the finite Galois case, and then will need to do some work to adapt the method to handle the extension $\widehat{K^{\mathrm{un}}}/K$ that is generally not algebraic but should be thought of as being approximately algebraic (with Galois group $G_K/I_K = G_k$). Observe that $\mathrm{Gal}(K'/K)$ naturally acts semilinearly on the finite-dimensional K'-vector space $\underline{D}_{K'}(W)$ with invariant subspace $\underline{D}_K(W)$ over K, and likewise $G_K/I_K = G_k$ naturally acts semilinearly on the finite-dimensional $\widehat{K^{\mathrm{un}}}$-vector space $\underline{D}_{\widehat{K^{\mathrm{un}}}}(W)$ with invariant subspace $\underline{D}_K(W)$ over K.

Hence, for the case of finite (Galois) extensions our problem is a special case of *classical Galois descent for vector spaces*: if F'/F is a finite Galois extension of fields and D' is a finite-dimensional F'-vector space endowed with a semilinear action by $\mathrm{Gal}(F'/F)$ then the natural map

$$F' \otimes_F (D'^{\mathrm{Gal}(F'/F)}) \to D' \tag{2.4.3}$$

is an isomorphism. (See [47, Ch. II, Lemma 5.8.1] for a proof, resting on the non-vanishing of discriminants for finite Galois extensions.) This has a generalization to arbitrary Galois extensions F'/F with possibly infinite degree: we just need to impose the additional "discreteness" hypothesis that each element of D' has an open stabilizer in $\mathrm{Gal}(F'/F)$ (so upon choosing an F'-basis of D' there is an open normal subgroup $\mathrm{Gal}(F'/F_1)$ that fixes the basis vectors and hence reduces our problem to the finite case via the semilinear $\mathrm{Gal}(F_1/F)$-action on the F_1-span of the chosen F'-basis of D').

For the case of $\widehat{K^{\mathrm{un}}}$, we have to modify the preceding argument since $\widehat{K^{\mathrm{un}}}/K$ is generally not algebraic and the group of isometric automorphisms $\mathrm{Aut}(\widehat{K^{\mathrm{un}}}/K) = \mathrm{Gal}(K^{\mathrm{un}}/K) = G_K/I_K = G_k$ generally acts on the space of I_K-invariants in W with stabilizer groups that are closed but not open. Hence, we require a variant of the Galois descent isomorphism (2.4.3) subject to a (necessary) auxiliary continuity hypothesis.

First we check that the natural semilinear action on $D' := D_{\widehat{K^{\mathrm{un}}}}(W)$ by the profinite group $G_k = G_K/I_K$ is continuous relative to the natural topology on D' as a finite-dimensional

$\widehat{K^{\mathrm{un}}}$-vector space. It suffices to check such continuity on the finitely many nonzero graded pieces D'_q separately, and $\mathbf{C}_K(-q) \otimes_{\widehat{K^{\mathrm{un}}}} D'_q$ with its G_K-action is naturally embedded in W (by the Serre–Tate injection ξ_W). Since G_K acts continuously on W by hypothesis and the natural topology on D'_q coincides with its subspace topology from naturally sitting in the $\mathbf{C}_K$-vector space $\mathbf{C}_K(-q) \otimes_{\widehat{K^{\mathrm{un}}}} D'_q$, we get the asserted continuity property for the action of $G_k = G_K/I_K$ on D'_q.

Although G_k acts $\widehat{K^{\mathrm{un}}}$-semilinearly rather than $\widehat{K^{\mathrm{un}}}$-linearly on D', since $\widehat{K^{\mathrm{un}}}$ is the fraction field of a complete discrete valuation ring $\mathscr{O} := \mathscr{O}_{\widehat{K^{\mathrm{un}}}}$ the proof of Lemma 1.2.6 adapts (using continuity of the semilinear G_k-action on D') to construct a G_k-stable $\mathscr{O}$-lattice $\Lambda \subseteq D'$. Consider the natural $\mathscr{O}$-linear G_k-equivariant map

$$\mathscr{O} \otimes_{\mathscr{O}_K} \Lambda^{G_k} \to \Lambda. \tag{2.4.4}$$

We shall prove that this is an isomorphism with Λ^{G_k} a finite free $\mathscr{O}_K$-module. Once this is proved, inverting p on both sides will give the desired isomorphism $\widehat{K^{\mathrm{un}}} \otimes_K \underline{D}_K(W) \simeq D' = \underline{D}_{\widehat{K^{\mathrm{un}}}}(W)$.

To verify the isomorphism property for (2.4.4), we shall argue via successive approximation by lifting from the residue field $\overline{k}$ of $\widehat{K^{\mathrm{un}}}$. Let $\pi \in \mathscr{O}_K$ be a uniformizer, so it is also a uniformizer of $\mathscr{O} = \mathscr{O}_{\widehat{K^{\mathrm{un}}}}$ and G_k acts trivially on π. The quotient $\Lambda/\pi\Lambda$ is a vector space over $\overline{k}$ with dimension equal to $d = \mathrm{rank}_{\mathscr{O}}\Lambda = \dim_{\widehat{K^{\mathrm{un}}}} D'$ and it is endowed with a natural semilinear action by $G_k = \mathrm{Gal}(\overline{k}/k)$ that has open stabilizers for all vectors (due to the continuity of the G_k-action on D' and the fact that Λ gets the π-adic topology as its subspace topology from D'). Hence, classical Galois descent in (2.4.3) (applied to $\overline{k}/k$) gives that $\Lambda/\pi\Lambda = \overline{k} \otimes_k \Delta$ in $\mathrm{Rep}_{\overline{k}}(G_k)$ for the d-dimensional k-vector space $\Delta = (\Lambda/\pi\Lambda)^{G_k}$. In particular, $\Lambda/\pi\Lambda \simeq \overline{k}^d$ compatibly with G_k-actions, so $\mathrm{H}^1(G_k, \Lambda/\pi\Lambda)$ vanishes since $\mathrm{H}^1(G_k, \overline{k}) = 0$. Since π is G_k-invariant, a successive approximation argument with continuous 1-cocycles (see [42, §1.2, Lemma 3], applied successively to increasing finite quotients of G_k) then gives that $\mathrm{H}^1_{\mathrm{cont}}(G_k, \Lambda) = 0$. Hence, passing to G_k-invariants on the exact sequence

$$0 \to \Lambda \xrightarrow{\pi} \Lambda \to \Lambda/\pi\Lambda \to 0$$

gives an exact sequence

$$0 \to \Lambda^{G_k} \xrightarrow{\pi} \Lambda^{G_k} \to (\Lambda/\pi\Lambda)^{G_k} \to 0.$$

That is, we have $\Lambda^{G_k}/\pi \cdot \Lambda^{G_k} \simeq (\Lambda/\pi\Lambda)^{G_k}$ as k-vector spaces.

Since Λ^{G_k} is a closed $\mathscr{O}_K$-submodule of the finite free $\mathscr{O}_{\widehat{K^{\mathrm{un}}}}$-module Λ of rank d and we have just proved that $\Lambda^{G_k}/\pi\Lambda^{G_k}$ is finite-dimensional of dimension d over $k = \mathscr{O}_K/(\pi)$, a simple approximation argument gives that any lift of a k-basis of $\Lambda^{G_k}/\pi\Lambda^{G_k}$ to a subset of Λ^{G_k} is an $\mathscr{O}_K$-spanning set of Λ^{G_k} of size d. Thus, Λ^{G_k} is a finitely generated torsion-free $\mathscr{O}_K$-module, so it is free of rank d since its reduction modulo π is d-dimensional over k. Our argument shows that the map (2.4.4) is a map between finite free $\mathscr{O}$-modules of the same rank and that this map becomes an isomorphism modulo π, so it is an isomorphism. ■

Further properties of $\underline{D}$ are best expressed by recasting the definition of $\underline{D}$ in terms of a "period ring" formalism. This rests on the following innocuous-looking definition whose mathematical (as opposed to linguistic) importance will only be appreciated after some later developments.

Definition 2.4.7. The *Hodge–Tate ring* of K is the $\mathbf{C}_K$-algebra $B_{\mathrm{HT}} = \oplus_{q\in\mathbf{Z}}\mathbf{C}_K(q)$ in which multiplication is defined via the natural maps $\mathbf{C}_K(q)\otimes_{\mathbf{C}_K}\mathbf{C}_K(q') \simeq \mathbf{C}_K(q+q')$.

Remark 2.4.8. We will encounter many rings denoted with the letter "B". This stands for Barsotti, who was one of the pioneers in using large ring-theoretic constructions to study group schemes and related structures.

Observe that B_{HT} is a graded $\mathbf{C}_K$-algebra in the sense that its graded pieces are $\mathbf{C}_K$-subspaces with respect to which multiplication is additive in the degrees, and that the natural G_K-action respects the gradings and the ring structure (and is semilinear over $\mathbf{C}_K$). Concretely, if we choose a basis t of $\mathbf{Z}_p(1)$ then we can identify B_{HT} with the Laurent polynomial ring $\mathbf{C}_K[t,t^{-1}]$ with the evident grading (by monomials in t) and G_K-action (via $g(t^i) = \chi(g)^i t^i$ for $i\in\mathbf{Z}$ and $g\in G_K$).

By the Tate–Sen theorem, we have $B_{\mathrm{HT}}^{G_K} = K$. For any $W\in\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, we have

$$\underline{D}(W) = \oplus_q(\mathbf{C}_K(q)\otimes_{\mathbf{C}_K} W)^{G_K} = (B_{\mathrm{HT}}\otimes_{\mathbf{C}_K} W)^{G_K}$$

in Gr_K, where the grading is induced from the one on B_{HT}. Since B_{HT} compatibly admits all three structures of interest ($\mathbf{C}_K$-vector space structure, G_K-action, grading), we can go in the reverse direction (from graded K-vector spaces to $\mathbf{C}_K$-representations of G_K) as follows.

Let D be in $\mathrm{Gr}_{K,f}$, so $B_{\mathrm{HT}}\otimes_K D$ is a graded $\mathbf{C}_K$-vector space with typically infinite $\mathbf{C}_K$-dimension:

$$\mathrm{gr}^n(B_{\mathrm{HT}}\otimes_K D) = \oplus_q\,\mathrm{gr}^q(B_{\mathrm{HT}})\otimes_K D_{n-q} = \oplus_q\mathbf{C}_K(q)\otimes_K D_{n-q}.$$

Moreover, the G_K-action on $B_{\mathrm{HT}}\otimes_K D$ arising from that on B_{HT} respects the grading since such compatibility holds in B_{HT}, so we get the object

$$\underline{V}(D) := \mathrm{gr}^0(B_{\mathrm{HT}}\otimes_K D) = \oplus_q\mathbf{C}_K(-q)\otimes_K D_q \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$$

since D_q vanishes for all but finitely many q and is finite-dimensional over K for all q (as $D\in\mathrm{Gr}_{K,f}$). By inspection $\underline{V}(D)$ is a Hodge–Tate representation, and $\underline{V}:\mathrm{Gr}_{K,f}\to\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ is a covariant exact functor.

Example 2.4.9. For each $r\in\mathbf{Z}$, recall that $K\langle r\rangle$ denotes the 1-dimensional K-vector space K endowed with unique nontrivial graded piece in degree r. One checks that $\underline{V}(K\langle r\rangle) = \mathbf{C}_K(-r)$. In particular, $\underline{V}(K\langle 0\rangle) = \mathbf{C}_K$.

For any W in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, the multiplicative structure on B_{HT} defines a natural B_{HT}-linear composite *comparison morphism*

$$\gamma_W : B_{\mathrm{HT}}\otimes_K \underline{D}(W) \hookrightarrow B_{\mathrm{HT}}\otimes_K (B_{\mathrm{HT}}\otimes_{\mathbf{C}_K} W) \to B_{\mathrm{HT}}\otimes_{\mathbf{C}_K} W \qquad (2.4.5)$$

that respects the G_K-actions (from B_{HT} on both sides and from W) and the gradings (from B_{HT} on both sides and from $\underline{D}(W)$) since the second step in (2.4.5) rests on the multiplication in B_{HT} which is G_K-equivariant and respects the grading of B_{HT}. The Serre–Tate lemma admits the following powerful reformulation:

Lemma 2.4.10. *For W in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, the comparison morphism γ_W is injective. It is an isomorphism if and only if W is Hodge–Tate, in which case there is a natural isomorphism*

$$\underline{V}(\underline{D}(W)) = \mathrm{gr}^0(B_{\mathrm{HT}} \otimes_K \underline{D}(W)) \overset{\gamma_W}{\simeq} \mathrm{gr}^0(B_{\mathrm{HT}} \otimes_{\mathbf{C}_K} W) = \mathrm{gr}^0(B_{\mathrm{HT}}) \otimes_{\mathbf{C}_K} W = W$$

in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$.

Proof. The map γ_W on gr^n's is the $\mathbf{Q}_p(n)$-twist of ξ_W. ■

We have seen above that if D is an object in $\mathrm{Gr}_{K,f}$ then $\underline{V}(D)$ is a Hodge–Tate object in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, so by Lemma 2.4.10 we obtain a B_{HT}-linear comparison isomorphism

$$\gamma_{\underline{V}(D)} : B_{\mathrm{HT}} \otimes_K \underline{D}(\underline{V}(D)) \simeq B_{\mathrm{HT}} \otimes_{\mathbf{C}_K} \underline{V}(D)$$

respecting G_K-actions and gradings. Since $B_{\mathrm{HT}}^{G_K} = K$ and the G_K-action on the target of $\gamma_{\underline{V}(D)}$ respects the grading induced by $B_{\mathrm{HT}} = \oplus \mathbf{C}_K(r)$, by passing to G_K-invariants on the source and target of $\gamma_{\underline{V}(D)}$ we get an isomorphism

$$\underline{D}(\underline{V}(D)) \simeq \oplus_r (\underline{V}(D)(r))^{G_K}$$

in Gr_K with $\underline{V}(D)(r) \simeq \oplus_q \mathbf{C}_K(r-q) \otimes_K D_q$. Hence, $(\underline{V}(D)(r))^{G_K} = D_r$ by the Tate–Sen theorem, so we get an isomorphism

$$\underline{D}(\underline{V}(D)) \simeq \oplus_r D_r = D$$

in Gr_K. This proves the first part of:

Theorem 2.4.11. *The covariant functors $\underline{D}$ and $\underline{V}$ between the categories of Hodge–Tate representations in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ and finite-dimensional objects in Gr_K are quasi-inverse equivalences.*

For any W, W' in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ the natural map

$$\underline{D}(W) \otimes \underline{D}(W') \to \underline{D}(W \otimes W')$$

in Gr_K induced by the G_K-equivariant map

$$(B_{\mathrm{HT}} \otimes_{\mathbf{C}_K} W) \otimes_{\mathbf{C}_K} (B_{\mathrm{HT}} \otimes_{\mathbf{C}_K} W') \to B_{\mathrm{HT}} \otimes_{\mathbf{C}_K} (W \otimes_{\mathbf{C}_K} W')$$

defined by multiplication in B_{HT} is an isomorphism when W and W' are Hodge–Tate. Likewise, if W is Hodge–Tate then the natural map

$$\underline{D}(W) \otimes_K \underline{D}(W^\vee) \to \underline{D}(W \otimes W^\vee) \to \underline{D}(\mathbf{C}_K) = K\langle 0 \rangle$$

in Gr_K is a perfect duality (between $W\{q\}$ and $W^\vee\{-q\}$ for all q), so the induced map $\underline{D}(W^\vee) \to \underline{D}(W)^\vee$ is an isomorphism in $\mathrm{Gr}_{K,f}$. In other words, $\underline{D}$ is compatible with tensor products and duality on Hodge–Tate objects.

Similar compatibilities hold for $\underline{V}$ with respect to tensor products and duality.

Proof. For the tensor product and duality claims for $\underline{D}$, one first checks that both sides have compatible evident functorial behavior with respect to direct sums in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$. Hence, we immediately reduce to the special case $W = \mathbf{C}_K(q)$ and $W' = \mathbf{C}_K(q')$ for some $q, q' \in \mathbf{Z}$, and this case is a straightforward calculation. Likewise, to analyze the natural map $\underline{V}(D) \otimes_{\mathbf{C}_K} \underline{V}(D') \to \underline{V}(D \otimes D')$ we can reduce to the special case of the graded objects $D = K\langle r \rangle$ and $D' = K\langle r' \rangle$ for $r, r' \in \mathbf{Z}$; the case of duality goes similarly. ■

Definition 2.4.12. Let $\mathrm{Rep}_{\mathrm{HT}}(G_K) \subseteq \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ be the full subcategory of objects V that are Hodge–Tate (i.e., $\mathbf{C}_K \otimes_{\mathbf{Q}_p} V$ is Hodge–Tate in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$), and define the functor $D_{\mathrm{HT}} : \mathrm{Rep}_{\mathbf{Q}_p}(G_K) \to \mathrm{Gr}_{K,f}$ by

$$D_{\mathrm{HT}}(V) = \underline{D}_K(\mathbf{C}_K \otimes_{\mathbf{Q}_p} V) = (B_{\mathrm{HT}} \otimes_{\mathbf{Q}_p} V)^{G_K}$$

with grading induced by that on B_{HT}.

Our results in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ show that $\mathrm{Rep}_{\mathrm{HT}}(G_K)$ is stable under tensor product, duality, subrepresentations, and quotients (but not extensions) in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, and that the formation of D_{HT} naturally commutes with finite extension on K as well as with scalar extension to $\widehat{K^{\mathrm{un}}}$. Also, our preceding results show that on $\mathrm{Rep}_{\mathrm{HT}}(G_K)$ the functor D_{HT} is exact and is compatible with tensor products and duality. The comparison morphism

$$\gamma_V : B_{\mathrm{HT}} \otimes_K D_{\mathrm{HT}}(V) \to B_{\mathrm{HT}} \otimes_{\mathbf{Q}_p} V$$

for $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ is an isomorphism precisely when V is Hodge–Tate (apply Lemma 2.4.10 to $W = \mathbf{C}_K \otimes_{\mathbf{Q}_p} V$), and hence $D_{\mathrm{HT}} : \mathrm{Rep}_{\mathrm{HT}}(G_K) \to \mathrm{Gr}_{K,f}$ is a faithful functor.

Example 2.4.13. Theorem 2.2.3 can now be written in the following more appealing form: if X is a smooth proper K-scheme then for $n \geqslant 0$ the representation $V := \mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ is in $\mathrm{Rep}_{\mathrm{HT}}(G_K)$ with $D_{\mathrm{HT}}(V) \simeq \mathrm{H}^n_{\mathrm{Hodge}}(X/K) := \oplus_q \mathrm{H}^{n-q}(X, \Omega^q_{X/K})$. Thus, the comparison morphism γ_V takes the form of a B_{HT}-linear G_K-equivariant isomorphism

$$B_{\mathrm{HT}} \otimes_K \mathrm{H}^n_{\mathrm{Hodge}}(X/K) \simeq B_{\mathrm{HT}} \otimes_{\mathbf{Q}_p} \mathrm{H}^n(X_{\overline{K}}, \mathbf{Q}_p) \tag{2.4.6}$$

in Gr_K.

This is reminiscent of the de Rham isomorphism

$$\mathrm{H}^n_{\mathrm{dR}}(M) \simeq \mathbf{R} \otimes_{\mathbf{Q}} \mathrm{H}_n(M, \mathbf{Q})^\vee$$

for smooth manifolds M, which in the case of finite-dimensional cohomology is described by the matrix $(\int_{\sigma_j} \omega_i)$ for an $\mathbf{R}$-basis $\{\omega_i\}$ of $\mathrm{H}^n_{\mathrm{dR}}(M)$ and a $\mathbf{Q}$-basis $\{\sigma_j\}$ of $\mathrm{H}_n(M, \mathbf{Q})$. The numbers $\int_\sigma \omega$ are classically called *periods* of M, and to define the de Rham isomorphism relating de Rham cohomology to topological cohomology we must use the coefficient ring $\mathbf{R}$ on the topological side. For this reason, the ring B_{HT} that serves as a coefficient ring for Faltings' comparison isomorphism (2.4.6) between Hodge and étale cohomologies is called a *period ring*. Likewise, the more sophisticated variants on B_{HT} introduced by Fontaine as a means of passing between other pairs of p-adic cohomology theories are all called period rings.

Whereas $\underline{D}$ on the category of Hodge–Tate objects in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ is fully faithful into $\mathrm{Gr}_{K,f}$, D_{HT} on the category $\mathrm{Rep}_{\mathrm{HT}}(G_K)$ of Hodge–Tate representations of G_K over $\mathbf{Q}_p$ is *not* fully faithful. For example, if $\eta : G_K \to \mathbf{Z}_p^\times$ has finite order then $D_{\mathrm{HT}}(\mathbf{Q}_p(\eta)) \simeq K\langle 0\rangle = D_{\mathrm{HT}}(\mathbf{Q}_p)$ by the Tate–Sen theorem, but $\mathbf{Q}_p(\eta)$ and $\mathbf{Q}_p$ have no nonzero maps between them when $\eta \neq 1$. This lack of full faithfulness is one reason that the functor $\mathrm{Rep}_{\mathrm{HT}}(G_K) \to \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ given by $V \rightsquigarrow \mathbf{C}_K \otimes_{\mathbf{Q}_p} V$ is a drastic operation and needs to be replaced by something more sophisticated.

To improve on D_{HT} so as to get a *fully faithful* functor from a nice category of p-adic representations of G_K into a category of semilinear algebra objects, we need to do two things: we must refine B_{HT} to a ring with more structure (going beyond a mere grading with a compatible G_K-action) and we need to introduce a target semilinear algebra category that is richer than $\mathrm{Gr}_{K,f}$. As a warm-up, we will next turn to the category of étale φ-modules. This involves a digression away from studying p-adic representations of G_K (it really involves representations of the closed subgroup G_{K_∞} for certain infinitely ramified algebraic extensions K_∞/K inside of $\overline{K}$), but it will naturally motivate some of the objects of semilinear algebra that have to be considered in any reasonable attempt to refine the theory of Hodge–Tate representations.

2.5. **Exercises.**

Exercise 2.5.1. Let K be a p-adic field, and $\overline{K}/K$ an algebraic closure. There are plenty of elements of $\mathbf{C}_K$ not in $\overline{K}$. That is, $\overline{K}$ is never complete. Indeed, since $[\overline{K}:K]$ is infinite (as follows by ramification considerations, for example), the non-completeness follows from [8, 3.4.3/1].

Nonetheless, prove that if L/K is a subextension of $\overline{K}/K$ then the subfield $\widehat{L} \subseteq \mathbf{C}_K$ determines L. More specifically, prove that $\widehat{L} \cap \overline{K} = L$.

Exercise 2.5.2. Let M be a topological module for a topological group G. If M has trivial G-action then prove $\mathrm{H}^1_{\mathrm{cont}}(G, M) = \mathrm{Hom}_{\mathrm{cont}}(G, M)$. Show by example with $M = \mathbf{Z}/2\mathbf{Z}$ and $G = G_{\mathbf{Q}}$ that dropping the continuity condition here makes this much larger, and in particular gives rise to many nontrivial cohomology classes that are everywhere locally trivial (i.e., have trivial restriction to the corresponding cohomology for the $G_{\mathbf{Q}_v}$'s, say again without the continuity condition).

Exercise 2.5.3. Let G be a topological group, and $0 \to M' \to M \to M'' \to 0$ a short exact sequence of G-modules such that M is a topological G-module and M' (resp. M'') is given the subspace topology (resp. quotient topology).

(1) Verify that M' and M'' are then topological G-modules (so we say that the given short exact sequence is *topologically exact*). We write $\mathrm{H}^1(G, M)$ to denote $\mathrm{H}^1_{\mathrm{cont}}(G, M)$ and $\mathrm{H}^1_{\mathrm{alg}}(G, M)$ to denote the usual algebraic G-cohomology of M ignoring topologies.

(2) Explain why $\mathrm{H}^1(G, M)$ naturally sits inside of $\mathrm{H}^1_{\mathrm{alg}}(G, M)$, and show that the usual 6-term exact sequence

$$0 \to M'^G \to M^G \to M''^G \xrightarrow{\delta} \mathrm{H}^1_{\mathrm{alg}}(G, M') \to \mathrm{H}^1_{\mathrm{alg}}(G, M) \to \mathrm{H}^1_{\mathrm{alg}}(G, M'')$$

restricts to a 6-term exact sequence

$$0 \to M'^G \to M^G \to M''^G \xrightarrow{\delta} \mathrm{H}^1(G, M') \to \mathrm{H}^1(G, M) \to \mathrm{H}^1(G, M'').$$

In particular, for any $m'' \in M''^G$, the obstruction to lifting it to M^G lies not only in $\mathrm{H}^1_{\mathrm{alg}}(G, M')$ but even in $\mathrm{H}^1(G, M')$.

(3) Consider a G-module M' that is a topological G-module relative to two topologies τ'_1 and τ'_2 such that $\mathrm{H}^1_{\tau'_1}(G, M') = 0$ but $\mathrm{H}^1_{\tau'_2}(G, M) \neq 0$. (For example, $G = \mathbf{Z}_p$ with the usual topology, and $M' = \mathbf{Z}_p$ endowed with the trivial G-action, and τ'_1 is the discrete topology whereas τ'_2 is the p-adic topology.) Construct a topologically exact sequence

$$0 \to M' \to M \to M'' \to 0$$

relative to τ_2' with $M'' = \mathbf{Z}$ (having discrete topology and trivial G-action) such that $M^G \to M''^G$ is *not* surjective (and so the nontrivial obstructions do not lie in $\mathrm{H}^1_{\tau_1'}(G, M')$). Thus, it is rather important to keep track of subspace topologies when trying to use a cohomological vanishing result for M' to deduce surjectivity for $M^G \to M''^G$!

Exercise 2.5.4. This exercise pushes Example 2.2.6 a bit further. Choose $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ and consider an exact sequence

$$0 \to W \to W' \to \mathbf{C}_K \to 0$$

of $\mathbf{C}_K$-vector spaces equipped with compatible $\mathbf{C}_K$-semilinear G_K-actions (usual action on $\mathbf{C}_K$ and given one on W).

(1) By choosing a $\mathbf{C}_K$-linear splitting, show that under the resulting identification $W \oplus \mathbf{C}_K$ the G_K-action is given by

$$g(w, c) = (g.w + g(c)\tau(g), g(c))$$

where $\tau : G_K \to W$ is a function satisfying $\tau(g'g) = g'(\tau(g)) + \tau(g)$; i.e., τ is a 1-coboundary valued in the G_K-module W.

(2) Prove that changing the $\mathbf{C}_K$-linear splitting corresponds exactly to changing τ by a 1-coboundary, and that τ is continuous if and only if the $\mathbf{C}_K$-semilinear G_K-action on W' is continuous.

(3) Now assume $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ (i.e., the G_K-action is continuous). Show that the cohomology class $[\tau] \in \mathrm{H}^1_{\mathrm{cont}}(G_K, W)$ only depends on the isomorphism class of the given exact sequence in $\mathrm{Ext}^1_{\mathbf{C}_K[G_K]}(\mathbf{C}_K, W)$, and that this procedure defines a bijection $\mathrm{Ext}^1_{\mathbf{C}_K[G_K]}(\mathbf{C}_K, W) \to \mathrm{H}^1_{\mathrm{cont}}(G_K, W)$.

(4) Prove that the bijection in (3) is $\mathbf{C}_K$-linear. (Hint: use the description of the $\mathbf{C}_K$-vector space structure on the left side via pushout and pullback operations),

3. Étale φ-modules

We now switch themes to describe p-adic representations of G_E for arbitrary fields E of characteristic $p > 0$; later this will be applied with $E = k((u))$ for a perfect field k of characteristic p, so in particular E must be allowed to be imperfect. The reason such Galois groups will be of interest to us was sketched in Example 1.3.3. In contrast with the case of Hodge–Tate representations in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, for which there was an equivalence with the relatively simple category $\mathrm{Gr}_{K,f}$ of finite-dimensional graded K-vector spaces, in our new setting we will construct an equivalence between various categories of representations of G_E and some interesting categories of modules equipped with an endomorphism that is semilinear over a "Frobenius" operator on the coefficient ring.

We shall work our way up to $\mathbf{Q}_p$-representation spaces for G_E by first studying $\mathbf{F}_p$-representation spaces for G_E, then general torsion $\mathbf{Z}_p$-representation spaces for G_E, and finally $\mathbf{Z}_p$-lattice representations of G_E (from which the $\mathbf{Q}_p$-case will be analyzed via Lemma 1.2.6).

Throughout this section we work with a fixed field E that is arbitrary with $\mathrm{char}(E) = p > 0$ and we fix a separable closure E_s. We let $G_E = \mathrm{Gal}(E_s/E)$. We emphasize that E is *not* assumed to be perfect, so the p-power endomorphisms of E and E_s are generally not surjective.

3.1. **p-torsion representations.** We are first interested in the category $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$ of continuous representations of G_E on finite-dimensional $\mathbf{F}_p$-vector spaces V_0 (so continuity means that the G_E-action on V_0 factors through an action by $\mathrm{Gal}(E'/E)$ for some finite Galois extension E'/E contained in E_s that may depend on V_0). The role of the ring B_{HT} in §2.4 will now be played by E_s, and the relevant structures that this ring admits are twofold: a G_E-action and the p-power endomorphism $\varphi_{E_s} : E_s \to E_s$ (i.e., $x \mapsto x^p$). These two structures on E_s respectively play roles analogous to the G_K-action on B_{HT} and the grading on B_{HT}, and the properties $B_{\mathrm{HT}}^{G_K} = K$ and $\mathrm{gr}^0(B_{\mathrm{HT}}) = \mathbf{C}_K$ have as their respective analogues the identities $E_s^{G_E} = E$ and $E_s^{\varphi_{E_s}=1} = \mathbf{F}_p$. The compatibility of the G_K-action and grading on B_{HT} has as its analogue the evident fact that the G_E-action on E_s commutes with the endomorphism $\varphi_{E_s} : x \mapsto x^p$ (i.e., $g(x^p) = g(x)^p$ for all $x \in E_s$ and $g \in G_E$). We write $\varphi_E : E \to E$ to denote the p-power endomorphism of E, so $\varphi_{E_s}|_E = \varphi_E$.

Whereas in Theorem 2.4.11 we used B_{HT} to set up inverse equivalences $\underline{D}$ and $\underline{V}$ between the category of Hodge–Tate objects in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ and the category $\mathrm{Gr}_{K,f}$ of graded K-vector spaces, now we will use E_s to set up an equivalence between the category $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$ and a certain category of finite-dimensional E-vector spaces equipped with a suitable Frobenius semilinear endomorphism.

The following category of semilinear algebra objects to later be identified with $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$ looks complicated at first, but we will soon see that it is not too bad. Below, we write $\varphi_E^*(M_0)$ for an E-vector space M_0 to denote the scalar extension $E \otimes_{\varphi_E, E} M_0$ with its natural E-vector space structure via the left tensor factor.

Definition 3.1.1. An *φ-module over E* is a pair (M_0, φ_{M_0}) where M_0 is a finite-dimensional E-vector space and $\varphi_{M_0} : M_0 \to M_0$ is a φ_E-semilinear endomorphism. A φ-module (M_0, φ_{M_0}) is *étale* if the E-linearization $\varphi_E^*(M_0) \to M_0$ of φ_{M_0} (i.e., the E-linear map $c \otimes m \mapsto c\varphi_{M_0}(m)$) is an isomorphism. Equivalently, $\varphi_{M_0}(M_0)$ spans M_0 over E, or in other words the "matrix" of φ_{M_0} relative to a choice of E-basis of M_0 is invertible). The notion of morphism between étale φ-modules over E is defined in the evident manner, and the category of étale φ-modules over E is denoted $\Phi M_E^{\text{ét}}$.

Remark 3.1.2. The reason for the word "étale" in the terminology is that a scheme X locally of finite type over a field k of characteristic $p > 0$ is étale if and only if the relative Frobenius map $F_{X/k} : X \to X^{(p)} = k \otimes_{\varphi_k, k} X$ over k is an isomorphism.

We often write M_0 rather than (M_0, φ_{M_0}) to denote an object in the category $\Phi M_E^{\text{ét}}$. The simplest interesting example of an object in $\Phi M_E^{\text{ét}}$ is $M_0 = E$ endowed with $\varphi_{M_0} = \varphi_E$; this object will simply be denoted as E. It may not be immediately evident how to make more interesting objects in $\Phi M_E^{\text{ét}}$, but shortly we will associate such an object to every object in $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$.

We now give some basic constructions for making new objects out of old ones. There is an evident notion of tensor product in $\Phi M_E^{\text{ét}}$. The notion of duality is defined as follows. For $M_0 \in \Phi M_E^{\text{ét}}$, the dual $M_0^\vee$ has as its underlying E-vector space the usual E-linear dual of M_0, and $\varphi_{M_0^\vee} : M_0^\vee \to M_0^\vee$ carries an E-linear functional $\ell : M_0 \to E$ to the composite of the

E-linear pullback functional $\varphi_E^*(\ell) : \varphi_E^*(M_0) \to E$ (i.e., $c \otimes m \mapsto c \cdot \ell(m)^p = c \cdot \varphi_E(\ell(m))$) and the inverse $M_0 \simeq \varphi_E^*(M_0)$ of the E-linearized isomorphism $\varphi_E^*(M_0) \simeq M_0$ induced by φ_{M_0}. To show that this is an étale Frobenius structure, the problem is to check that $\varphi_{M_0^\vee}$ linearizes to an isomorphism. A slick method to establish this is via the alternative description of $\varphi_{M_0^\vee}$ that is provided in Exercise 3.4.2.

In concrete terms, if we choose an E-basis for M_0 and use the dual basis for $M_0^\vee$, then the resulting "matrices" that describe the φ_E-semilinear endomorphisms φ_{M_0} and $\varphi_{M_0^\vee}$ are transpose inverse to each other. The notions of tensor product and duality as defined in $\Phi M_E^{\text{ét}}$ satisfy the usual relations (e.g., the natural double duality isomorphism $M_0 \simeq M_0^{\vee\vee}$ is an isomorphism in $\Phi M_E^{\text{ét}}$, and the evaluation pairing $M_0 \otimes M_0^\vee \to E$ is a morphism in $\Phi M_E^{\text{ét}}$).

Lemma 3.1.3. *The category $\Phi M_E^{\text{ét}}$ is abelian. More specifically, if $h : M' \to M$ is a morphism in $\Phi M_E^{\text{ét}}$ then the induced Frobenius endomorphisms of* $\ker h$, $\operatorname{im} h$, *and* $\operatorname{coker} h$ *are étale (i.e., have E-linearization that is an isomorphism).*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \varphi_E^*(M') & \xrightarrow{\varphi_E^*(h)} & \varphi_E^*(M) \\ \simeq\downarrow & & \downarrow\simeq \\ M' & \xrightarrow[h]{} & M \end{array}$$

This induces isomorphisms between kernels, cokernels and images formed in the horizontal directions, and the formation of kernels, cokernels, and images of linear maps commutes with arbitrary ground field extension (such as $\varphi_E : E \to E$). Hence, the desired étaleness properties are obtained. ∎

We now use E_s equipped with its *compatible* G_E-action and φ_E-semilinear endomorphism φ_{E_s} to define covariant functors D_E and V_E between $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$ and $\Phi M_E^{\text{ét}}$.

Definition 3.1.4. For any $V_0 \in \mathrm{Rep}_{\mathbf{F}_p}(G_E)$, define $\mathrm{D}_E(V_0) = (E_s \otimes_{\mathbf{F}_p} V_0)^{G_E}$ as an E-vector space equipped with the φ_E-semilinear endomorphism $\varphi_{\mathrm{D}_E(V_0)}$ induced by $\varphi_{E_s} \otimes 1$. (This makes sense since φ_{E_s} commutes with the G_E-action on E_s.)

For any $M_0 \in \Phi M_E^{\text{ét}}$ we define $\mathrm{V}_E(M_0)$ to be the $\mathbf{F}_p$-vector space $(E_s \otimes_E M_0)^{\varphi=1}$ with its evident G_E-action induced by the G_E-action on E_s; here, $\varphi = \varphi_{E_s} \otimes \varphi_{M_0}$.

Some work is needed to check that D_E takes values in $\Phi M_E^{\text{ét}}$ and that V_E takes values in $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$. Indeed, it is not at all obvious that $\mathrm{D}_E(V_0)$ is finite-dimensional over E in general, nor that that E-linearization of $\varphi_{\mathrm{D}_E(V_0)}$ is an isomorphism (so $\mathrm{D}_E(V_0) \in \Phi M_E^{\text{ét}}$). Likewise, it is not obvious that $\mathrm{V}_E(M_0)$ is finite-dimensional over $\mathbf{F}_p$, though it follows from the definition that each element of $\mathrm{V}_E(M_0)$ has an open stabilizer in G_E (since such an element is a finite sum of elementary tensors in $E_s \otimes_E M_0$, and a finite intersection of open subgroups is open). Thus, once finite-dimensionality over $\mathbf{F}_p$ is established then $\mathrm{V}_E(M_0)$ will be an object in $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$.

Example 3.1.5. There are two trivial examples that can be worked out by hand. We have $\mathrm{D}_E(\mathbf{F}_p) = E$ with Frobenius endomorphism φ_E and $\mathrm{V}_E(E) = \mathbf{F}_p$ with trivial G_E-action.

Remark 3.1.6. It is sometimes convenient to use contravariant versions D_E^* and V_E^* of the functors D_E and V_E. These may be initially defined in an *ad hoc* way via

$$\mathrm{D}_E^*(V_0) = \mathrm{D}_E(V_0^\vee), \;\; \mathrm{V}_E^*(M_0) = \mathrm{V}_E(M_0^\vee)$$

but the real usefulness is due to an alternative formulation: since $E_s \otimes_{\mathbf{F}_p} V_0^* \simeq \mathrm{Hom}_{\mathbf{F}_p}(V_0, E_s)$ compatibly with the φ_{E_s}-actions and the G_E-actions (defined in the evident way on the Hom-space, namely $(g.\ell)(v) = g(\ell(g^{-1}v)))$, by passing to G_E-invariants we naturally get $\mathrm{D}_E^*(V_0) \simeq \mathrm{Hom}_{\mathbf{F}_p[G_E]}(V_0, E_s)$ as E-vector spaces equipped with a φ_E-semilinear endomorphism. Likewise, we naturally have an $\mathbf{F}_p[G_E]$-linear identification $\mathrm{V}_E^*(M_0) \simeq \mathrm{Hom}_{E,\varphi}(M_0, E_s)$ onto the space of E-linear Frobenius-compatible maps from M_0 into E_s.

Let us begin our study of D_E and V_E by checking that they take values in the expected target categories.

Lemma 3.1.7. *For any $V_0 \in \mathrm{Rep}_{\mathbf{F}_p}(G_E)$, the E-vector space $\mathrm{D}_E(V_0)$ is finite-dimensional with dimension equal to $\dim_{\mathbf{F}_p} V_0$, and the E-linearization of $\varphi_{\mathrm{D}_E(V_0)}$ is an isomorphism. In particular, $\mathrm{D}_E(V_0)$ lies in $\Phi M_E^{\text{ét}}$ with E-rank equal to the $\mathbf{F}_p$-rank of V_0.*

For any $M_0 \in \Phi M_E^{\text{ét}}$, the $\mathbf{F}_p$-vector space $\mathrm{V}_E(M_0)$ is finite-dimensional with dimension at most $\dim_E M_0$. In particular, $\mathrm{V}_E(M_0)$ lies in $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$ with $\mathbf{F}_p$-rank at most $\dim_E M_0$.

The upper bound for $\dim_{\mathbf{F}_p} \mathrm{V}_E(M_0)$ in this lemma will be proved to be an equality in Theorem 3.1.8, but for now it is simpler (and sufficient) to just prove the upper bound.

Proof. Observe that $E_s \otimes_{\mathbf{F}_p} V_0$ equipped with its diagonal G_E-action is a finite-dimensional E_s-vector space equipped with a semilinear G_E-action that is continuous for the discrete topology in the sense that each element has an open stabilizer (as this is true for each element of E_s and V_0, and hence for finite sums of elementary tensors). Thus, the classical theorem on Galois descent for vector spaces in (2.4.3) (applied to E_s/E) implies that $E_s \otimes_{\mathbf{F}_p} V_0$ is naturally identified with the scalar extension to E_s of its E-vector subspace of G_E-invariant vectors. That is, the natural E_s-linear G_E-equivariant map

$$E_s \otimes_E \mathrm{D}_E(V_0) = E_s \otimes_E (E_s \otimes_{\mathbf{F}_p} V_0)^{G_E} \to E_s \otimes_{\mathbf{F}_p} V_0 \tag{3.1.1}$$

induced by multiplication in E_s is an isomorphism. In particular, $\mathrm{D}_E(V_0)$ has finite E-dimension equal to $\dim_{\mathbf{F}_p} V_0$. (This isomorphism is an analogue of the comparison morphism γ_W in (2.4.5) defined via multiplication in B_{HT} in our study of Hodge–Tate representations.)

A crucial observation is that (3.1.1) satisfies a further compatibility property beyond the E_s-linearity and G_E-actions, namely that it respects the natural Frobenius endomorphisms of both sides (as follows from the definition). To exploit this, we first recall that for any vector space D over any field F of characteristic $p > 0$, if $\varphi_D : D \to D$ is a φ_F-semilinear endomorphism (with $\varphi_F : F \to F$ denoting $x \mapsto x^p$) then the F-linearization $\varphi_F^*(D) \to D$ of φ_D is compatible with arbitrary extension of the ground field $j : F \to F'$ (as the reader may check, ultimately because φ_F and $\varphi_{F'}$ are compatible via j). Applying this to the field extension $E \to E_s$, we see that the E-linearization of $\varphi_{\mathrm{D}_E(V_0)}$ is an isomorphism if and only if the E_s-linearization of $\varphi_{E_s} \otimes \varphi_{\mathrm{D}_E(V_0)}$ is an isomorphism. But Frobenius-compatibility of the E_s-linear isomorphism (3.1.1) renders this property equivalent to the assertion that for any finite-dimensional $\mathbf{F}_p$-vector space V_0 the E_s-linearization of the Frobenius endomorphism

$\varphi_{E_s} \otimes 1$ of $E_s \otimes_{\mathbf{F}_p} V_0$ is an isomorphism. By unravelling definitions we see that this E_s-linearization is naturally identified with the identity map of $E_s \otimes_{\mathbf{F}_p} V_0$, so it is indeed an isomorphism. Hence, we have proved the claims concerning $\mathrm{D}_E(V_0)$.

Now we turn to the task of proving that $\mathrm{V}_E(M_0)$ has finite $\mathbf{F}_p$-dimension at most $\dim_E M_0$ (and in particular, it is finite). To do this, we will prove that the natural E_s-linear G_E-compatible and Frobenius-compatible map

$$E_s \otimes_{\mathbf{F}_p} \mathrm{V}_E(M_0) = E_s \otimes_{\mathbf{F}_p} (E_s \otimes_E M_0)^{\varphi=1} \to E_s \otimes_E M_0 \tag{3.1.2}$$

induced by multiplication in E_s is injective. (This map is another analogue of the comparison morphism for Hodge–Tate representations.) Such injectivity will give $\dim_{\mathbf{F}_p} \mathrm{V}_E(M_0) \leqslant \dim_E M_0$ as desired.

Since any element in the left side of (3.1.2) is a finite sum of elementary tensors, even though $\mathrm{V}_E(M_0)$ is not yet known to be finite-dimensional over $\mathbf{F}_p$ it suffices to prove that if $v_1, \dots, v_r \in \mathrm{V}_E(M_0) = (E_s \otimes_E M_0)^{\varphi=1}$ are $\mathbf{F}_p$-linearly independent then in $E_s \otimes_E M_0$ they are E_s-linearly independent. We assume to the contrary and choose a least $r \geqslant 1$ for which there is a counterexample, say $\sum a_i v_i = 0$ with $a_i \in E_s$ not all zero. By minimality we have $a_i \neq 0$ for all i, and we may therefore apply $E_s^\times$-scaling to arrange that $a_1 = 1$. Hence, $v_1 = -\sum_{i>1} a_i v_i$. But $v_1 = \varphi(v_1)$ since $v_1 \in \mathrm{V}_E(M_0)$, so

$$v_1 = -\sum_{i>1} \varphi_{E_s}(a_i)\varphi(v_i) = -\sum_{i>1} \varphi_{E_s}(a_i) v_i$$

since $v_i \in \mathrm{V}_E(M_0)$ for all $i > 1$. Hence,

$$\sum_{i>1} (a_i - \varphi_{E_s}(a_i)) v_i = 0.$$

By minimality of r we must have $a_i = \varphi_{E_s}(a_i)$ for all $i > 1$, so $a_i \in E_s^{\varphi_{E_s}=1} = \mathbf{F}_p$ for all $i > 1$. Thus, the identity $v_1 = -\sum_{i>1} a_i v_i$ has coefficients in $\mathbf{F}_p$, so we have contradicted the assumption that the v_j's are $\mathbf{F}_p$-linearly independent. ∎

By Lemma 3.1.7, we have covariant functors $\mathrm{D}_E : \mathrm{Rep}_{\mathbf{F}_p}(G_E) \to \Phi M_E^{\text{ét}}$ and $\mathrm{V}_E : \Phi M_E^{\text{ét}} \to \mathrm{Rep}_{\mathbf{F}_p}(G_E)$, and D_E is rank-preserving. Also, since

$$(E_s \otimes_{\mathbf{F}_p} V_0)^{\varphi=1} = E_s^{\varphi=1} \otimes_{\mathbf{F}_p} V_0 = V_0, \;\; (E_s \otimes_E M_0)^{G_E} = E_s^{G_E} \otimes_E M_0 = M_0$$

(use an $\mathbf{F}_p$-basis of V_0 and an E-basis of M_0 respectively), passing to Frobenius-invariants on the isomorphism (3.1.1) defines an isomorphism $\mathrm{V}_E(\mathrm{D}_E(V_0)) \to V_0$ in $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$ and passing to G_E-invariants on the injection (3.1.2) defines an injection $\mathrm{D}_E(\mathrm{V}_E(M_0)) \hookrightarrow M_0$ in $\Phi M_E^{\text{ét}}$.

Theorem 3.1.8. *Via the natural map* $\mathrm{V}_E \circ \mathrm{D}_E \simeq \mathrm{id}$ *and the inclusion* $\mathrm{D}_E \circ \mathrm{V}_E \hookrightarrow \mathrm{id}$*, the covariant functors* D_E *and* V_E *are exact rank-preserving quasi-inverse equivalences of categories, and each functor is naturally compatible with tensor products and duality.*

Proof. The isomorphism (3.1.1) implies that D_E is an exact functor (as it becomes exact after scalar extension from E to E_s). For any two objects V_0 and V_0' in $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$, the natural map

$$\mathrm{D}_E(V_0) \otimes_E \mathrm{D}_E(V_0') \to \mathrm{D}_E(V_0 \otimes_{\mathbf{F}_p} V_0')$$

induced by the Frobenius-compatible and G_E-equivariant map

$$(E_s \otimes_{\mathbf{F}_p} V_0) \otimes_E (E_s \otimes_{\mathbf{F}_p} V_0') \to E_s \otimes_E (V_0 \otimes V_0')$$

arising from multiplication on E_s is a map in $\Phi M_E^{\text{ét}}$. This map is an isomorphism (and so D_E is naturally compatible with the formation of tensor products) because we may apply scalar extension from E to E_s and use the isomorphism (3.1.1) to convert this into the elementary claim that the natural map

$$(E_s \otimes_{\mathbf{F}_p} V_0) \otimes_{E_s} (E_s \otimes_{\mathbf{F}_p} V_0') \to E_s \otimes_{\mathbf{F}_p} (V_0 \otimes_{\mathbf{F}_p} V_0')$$

is an isomorphism.

Similarly we get that D_E is compatible with the formation of duals: we claim that the natural map

$$\mathrm{D}_E(V_0) \otimes_E \mathrm{D}_E(V_0^\vee) \simeq \mathrm{D}_E(V_0 \otimes_{\mathbf{F}_p} V_0^\vee) \to \mathrm{D}_E(\mathbf{F}_p) = E \tag{3.1.3}$$

(with second step induced by functoriality of D_E relative to the evaluation morphism $V_0 \otimes V_0^\vee \to \mathbf{F}_p$ in $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$) is a perfect E-bilinear duality between $\mathrm{D}_E(V_0)$ and $\mathrm{D}_E(V_0^\vee)$, or equivalently the induced morphism $\mathrm{D}_E(V_0^\vee) \to \mathrm{D}_E(V_0)^\vee$ that is checked to be a morphism in $\Phi M_E^{\text{ét}}$ is an isomorphism. To verify this perfect duality claim it suffices to check it after scalar extension from E to E_s, in which case via (3.1.1) the pairing map is identified with the natural map

$$(E_s \otimes_{\mathbf{F}_p} V_0) \otimes_{E_s} (E_s \otimes_{\mathbf{F}_p} V_0^\vee) \simeq E_s \otimes_{\mathbf{F}_p} (V_0 \otimes_{\mathbf{F}_p} V_0^\vee) \to E_s$$

that is a perfect E_s-bilinear duality pairing.

To carry out our analysis of V_E and $\mathrm{D}_E \circ \mathrm{V}_E$, the key point is to check that V_E is rank-preserving. That is, we have to show that if $\dim_E M_0 = d$ then $\dim_{\mathbf{F}_p} \mathscr{V}(M_0) = d$. Once this is proved, the injective map (3.1.2) is an isomorphism for E_s-dimension reasons and so passing to G_E-invariants on this isomorphism gives that $\mathrm{D}_E \circ \mathrm{V}_E \to \mathrm{id}$ is an isomorphism. The compatibility of V_E with respect to tensor products and duality can then be verified exactly as we did for D_E by replacing (3.1.1) with (3.1.2) and using $\mathrm{V}_E(E) = \mathbf{F}_p$ to replace the above use of the identification $\mathrm{D}_E(\mathbf{F}_p) = E$.

Our problem is now really one of counting: we must prove that the inequality $\# \mathrm{V}_E(M_0) \leqslant p^d$ for $d := \dim_E M_0$ is an equality. Arguing as in Remark 3.1.6 with $M_0^\vee$ in the role of M_0 and using double duality gives $\mathrm{V}_E(M_0) \simeq \mathrm{Hom}_{E,\varphi}(M_0^\vee, E_s)$. The key idea is to interpret this set of maps in terms of a system of étale polynomial equations in d variables. Choose a basis $\{m_1, \dots, m_d\}$ of M_0, so $M_0^\vee$ has a dual basis $\{m_i^\vee\}$ and $\varphi_{M_0^\vee}(m_j^\vee) = \sum_i c_{ij} m_i^\vee$ with $(c_{ij}) \in \mathrm{Mat}_{d\times d}(E)$ an invertible matrix. A general E-linear map $M_0^\vee \to E_s$ is given by $m_i^\vee \mapsto x_i \in E_s$ for each i, and Frobenius-compatibility for this map amounts to the system of equations $x_j^p = \sum_i c_{ij} x_i$ for all j. Hence, we have the identification $\mathrm{V}_E(M_0) = \mathrm{Hom}_{E\text{-alg}}(A, E_s)$, where

$$A = E[X_1, \dots, X_d]/(X_j^p - \sum_i c_{ij} X_i)_{1 \leqslant j \leqslant d}.$$

Clearly A is a finite E-algebra with rank p^d, and we wish to prove that its set of E_s-valued points has size equal to $p^d = \dim_E A$. In other words, we claim that A is an étale E-algebra in the sense of commutative algebra. This property amounts to the vanishing of $\Omega^1_{A/E}$, and by direct calculation

$$\Omega^1_{A/E} = (\oplus A\mathrm{d}X_i)/(\sum_j c_{ij}\mathrm{d}X_j)_{1\leqslant j\leqslant d}.$$

Since $\det(c_{ij}) \in E^\times \subseteq A^\times$, the vanishing follows. ■

3.2. **Torsion and lattice representations.** We wish to improve on the results in §3.1 by describing the entire category $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ of continuous G_E-representations on finitely generated (not necessarily free) $\mathbf{Z}_p$-modules, and then passing to $\mathrm{Rep}_{\mathbf{Q}_p}(G_E)$ by a suitable localization process. The basic strategy is to first handle torsion objects using $\mathbf{Z}_p$-length induction (and using the settled p-torsion case from §3.1 to get inductive arguments off the ground), and then pass to the inverse limit to handle general objects in $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ (especially those that are finite free as $\mathbf{Z}_p$-modules). One difficulty at the outset is that since we are lifting our coefficients from $\mathbf{F}_p$ to $\mathbf{Z}_p$ on the G_E-representation side, we need to lift the E-coefficients in characteristic p on the semilinear algebra side to some ring of characteristic 0 admitting a natural endomorphism lifting φ_E (as well as an analogue for E_s so as to get a suitable lifted "period ring"). Since E is generally not perfect, we cannot work with the Witt ring $\mathrm{W}(E)$ (which is generally quite bad if E is imperfect).

Thus, we impose the following hypothesis involving additional auxiliary data that will be fixed for the remainder of the present discussion: we assume that we are given a complete discrete valuation ring $\mathscr{O}_{\mathscr{E}}$ with characteristic 0, uniformizer p, and residue field E, and we assume moreover that there is specified an endomorphism $\varphi : \mathscr{O}_{\mathscr{E}} \to \mathscr{O}_{\mathscr{E}}$ lifting φ_E on the residue field E. We write $\mathscr{E}$ to denote the fraction field $\mathscr{O}_{\mathscr{E}}[1/p]$ of $\mathscr{O}_{\mathscr{E}}$. Abstract commutative algebra (the theory of Cohen rings [35, Thm. 29.1, 29.2]) ensures that if we drop the Frobenius-lifting hypothesis then there is such an $\mathscr{O}_{\mathscr{E}}$ and it is unique up to non-canonical isomorphism. It can also be proved [35, Thm. 29.2] that the lift φ always exists. The present discussion is generally only applied with a special class of fields E for which we can write down an explicit such pair $(\mathscr{O}_{\mathscr{E}}, \varphi)$. We shall now construct such a pair in a special case, and then for the remainder of this section we return to the general case and assume that such an abstract pair $(\mathscr{O}_{\mathscr{E}}, \varphi)$ has been given to us.

Example 3.2.1. Assume that $E = k(\!(u)\!)$ with k perfect of characteristic $p > 0$. Let $\mathrm{W}(k)$ denote the ring of Witt vectors of k. This is the unique absolutely unramified complete discrete valuation ring with mixed characteristic $(0, p)$ and residue field k; see §4.2. (If k is finite, $\mathrm{W}(k)$ is the valuation ring of the corresponding finite unramified extension of $\mathbf{Q}_p$.) In this case an explicit pair $(\mathscr{O}_{\mathscr{E}}, \varphi)$ satisfying the above axioms can be constructed as follows.

Let $\mathfrak{S} = \mathrm{W}(k)[\![u]\!]$; this is a 2-dimensional regular local ring in which (p) is a prime ideal at which the residue field is $k(\!(u)\!) = E$. Since the localization $\mathfrak{S}_{(p)}$ at the prime ideal (p) is a 1-dimensional regular local ring, it is a discrete valuation ring with uniformizer p. But u is a unit in this localized ring (since $u \notin (p)$ in $\mathfrak{S}$), so $\mathfrak{S}_{(p)}$ is identified with the localization of the Dedekind domain $\mathfrak{S}[1/u]$ at the prime ideal generated by p. Hence, the completion $\mathfrak{S}^\wedge_{(p)}$ of this discrete valuation ring is identified with the p-adic completion of the Laurent-series ring $\mathfrak{S}[1/u]$ over $\mathrm{W}(k)$. In other words, this completion is a ring of Laurent series over $\mathrm{W}(k)$

with a decay condition on coefficients in the negative direction:

$$\mathfrak{S}_{(p)}^{\wedge} \simeq \left\{ \sum_{n \in \mathbf{Z}} a_n u^n \mid a_n \in \mathrm{W}(k) \text{ and } a_n \to 0 \text{ as } n \to -\infty \right\}.$$

We define $\mathscr{O}_{\mathscr{E}} = \mathfrak{S}_{(p)}^{\wedge}$. The endomorphism $\sum a_n u^n \mapsto \sum \sigma(a_n) u^{np}$ of $\mathfrak{S}$ (with σ the unique Frobenius-lift on $\mathrm{W}(k)$) uniquely extends to a local endomorphism of $\mathfrak{S}_{(p)}$ and hence to a local endomorphism φ of the completion $\mathscr{O}_{\mathscr{E}}$.

Fix a choice of a pair $(\mathscr{O}_{\mathscr{E}}, \varphi)$ as required above. Since $\mathscr{O}_{\mathscr{E}}$ is a complete discrete valuation ring with residue field E and we have fixed a separable closure E_s of E, the maximal unramified extension (i.e., strict henselization) $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ of $\mathscr{O}_{\mathscr{E}}$ with residue field E_s makes sense and is unique up to unique isomorphism. It is a strictly henselian (generally not complete) discrete valuation ring with uniformizer p, so its fraction field $\mathscr{E}^{\mathrm{un}}$ is $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}[1/p]$. By the universal property of the maximal unramified extension (or rather, of the strict henselization), if $f : \mathscr{O}_{\mathscr{E}} \to \mathscr{O}_{\mathscr{E}}$ is a local map (such as φ or the identity) whose reduction $\overline{f} : E \to E$ is endowed with a specified lifting $\overline{f}' : E_s \to E_s$ then there is a unique local map $f' : \mathscr{O}_{\mathscr{E}}^{\mathrm{un}} \to \mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ over f lifting $\overline{f}'$. By uniqueness, the formation of such an f' is compatible with composition.

By taking $f = \varphi$ and $\overline{f}' = \varphi_{E_s}$, we get a unique local endomorphism of $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ again denoted φ that extends the given abstract endomorphism φ of $\mathscr{O}_{\mathscr{E}}$ and lifts the p-power map on E_s. Additionally, by taking f to be the identity map and considering varying $\overline{f}' \in G_E = \mathrm{Gal}(E_s/E)$ we get an induced action of G_E on $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ that is simply the classical identification of $\mathrm{Aut}_{\mathscr{O}_{\mathscr{E}}}(\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}) = \mathrm{Gal}(\mathscr{E}^{\mathrm{un}}/\mathscr{E})$ with the Galois group G_E of the residue field. Moreover, this G_E-action on $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ is continuous and it commutes with φ on $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ because the uniqueness of our lifting procedure reduces this to the compatibility of the G_E-action and Frobenius endomorphism on both $\mathscr{O}_{\mathscr{E}}$ and E_s. In particular, the induced G_E-action on $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$ is continuous and commutes with the induced Frobenius endomorphism.

Definition 3.2.2. The category $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$ of *étale φ-modules over $\mathscr{O}_{\mathscr{E}}$* consists of pairs $(\mathscr{M}, \varphi_{\mathscr{M}})$ where $\mathscr{M}$ is a finitely generated $\mathscr{O}_{\mathscr{E}}$-module and $\varphi_{\mathscr{M}}$ is a φ-semilinear endomorphism of $\mathscr{M}$ whose $\mathscr{O}_{\mathscr{E}}$-linearization $\varphi^*(\mathscr{M}) \to \mathscr{M}$ is an isomorphism.

Obviously $\Phi M_E^{\text{ét}}$ is the full subcategory of p-torsion objects in $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$. Note that in the preceding definition we do not require $\mathscr{M}$ to be a finite free module over $\mathscr{O}_{\mathscr{E}}$ or over one of its artinian quotients $\mathscr{O}_{\mathscr{E}}/(p^n)$; this generality is essential for the category $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$ to have nice stability properties. In particular, the étaleness condition in Definition 3.2.2 that $\varphi_{\mathscr{M}}$ linearize to an isomorphism cannot generally be described by a matrix condition. Since $\varphi^*(\mathscr{M})$ and $\mathscr{M}$ have the same $\mathscr{O}_{\mathscr{E}}$-rank and the same invariant factors (due to the uniformizer p being fixed by φ), the linearization of $\varphi_{\mathscr{M}}$ is a linear map between two abstractly isomorphic finitely generated $\mathscr{O}_{\mathscr{E}}$-modules, whence it is an isomorphism if and only if it is surjective. But surjectivity can be checked modulo p, so we conclude that the étaleness property on $\varphi_{\mathscr{M}}$ can be checked by working with the finite-dimensional vector space $\mathscr{M}/p\mathscr{M}$ over $\mathscr{O}_{\mathscr{E}}/(p) = E$.

The category $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ has a good notion of tensor product, as well as duality functors $\mathrm{Hom}_{\mathbf{Z}_p}(\cdot, \mathbf{Q}_p/\mathbf{Z}_p)$ and $\mathrm{Hom}_{\mathbf{Z}_p}(\cdot, \mathbf{Z}_p)$ on the respective full subcategories of objects that are of finite $\mathbf{Z}_p$-length and finite free over $\mathbf{Z}_p$.

There are similar tensor and duality functors in the category $\Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$. Indeed, tensor products $\mathscr{M} \otimes \mathscr{M}'$ are defined in the evident manner using the $\mathscr{O}_{\mathscr{E}}$-module tensor product $\mathscr{M} \otimes_{\mathscr{O}_{\mathscr{E}}} \mathscr{M}'$ and the Frobenius endomorphism $\varphi_{\mathscr{M}} \otimes \varphi_{\mathscr{M}'}$, and this really is an étale φ-module; i.e., the $\mathscr{O}_{\mathscr{E}}$-linearization of the tensor product Frobenius endomorphism is an isomorphism (since this $\mathscr{O}_{\mathscr{E}}$-linearization is identified with the tensor product of the $\mathscr{O}_{\mathscr{E}}$-linearizations of $\varphi_{\mathscr{M}}$ and $\varphi_{\mathscr{M}'}$). For duality, we use the functor $\mathrm{Hom}_{\mathscr{O}_{\mathscr{E}}}(\cdot, \mathscr{O}_{\mathscr{E}})$ on objects that are finite free over $\mathscr{O}_{\mathscr{E}}$ and the Frobenius endomorphism of this linear dual is defined similarly to the p-torsion case over E. That is, for $\ell \in \mathscr{M}^{\vee} = \mathrm{Hom}_{\mathscr{O}_{\mathscr{E}}}(\mathscr{M}, \mathscr{O}_{\mathscr{E}})$ the element $\varphi_{\mathscr{M}^{\vee}}(\ell) \in \mathscr{M}^{\vee}$ is the composite of the $\mathscr{O}_{\mathscr{E}}$-linear pullback functional $\varphi^*(\ell) : \varphi^*(\mathscr{M}) \to \mathscr{O}_{\mathscr{E}}$ and the inverse $\mathscr{M} \simeq \varphi^*(\mathscr{M})$ of the $\mathscr{O}_{\mathscr{E}}$-linearization of $\varphi_{\mathscr{M}}$. To verify that this Frobenius structure is étale (i.e., it linearizes to an isomorphism $\varphi^*(\mathscr{M}) \simeq \mathscr{M}$) one can establish an alternative description of $\varphi_{\mathscr{M}^{\vee}}$ exactly as in Exercise 3.4.2.

Likewise, on the full subcategory $\Phi M^{\text{ét,tor}}_{\mathscr{O}_{\mathscr{E}}}$ of objects of finite $\mathscr{O}_{\mathscr{E}}$-length we use the duality functor $\mathrm{Hom}_{\mathscr{O}_{\mathscr{E}}}(\cdot, \mathscr{E}/\mathscr{O}_{\mathscr{E}})$ on which we define a φ-semilinear endomorphism akin to the finite free case, now using the natural Frobenius structure on $\mathscr{E}/\mathscr{O}_{\mathscr{E}}$ to identify $\mathscr{E}/\mathscr{O}_{\mathscr{E}}$ with its own scalar extension by $\varphi : \mathscr{O}_{\mathscr{E}} \to \mathscr{O}_{\mathscr{E}}$. To see that this is really an étale Frobenius structure one again works out an alternative description akin to Exercise 3.4.2, but now it is necessary to give some thought (left to the reader) to justifying that scalar extension by $\varphi : \mathscr{O}_{\mathscr{E}} \to \mathscr{O}_{\mathscr{E}}$ commutes with the formation of the $\mathscr{E}/\mathscr{O}_{\mathscr{E}}$-valued dual (hint: the scalar extension φ is flat since it is a local map between discrete valuation rings with a common uniformizer).

Lemma 3.2.3. *The category $\Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$ is abelian.*

Proof. The content of this verification is to check the étaleness property for the linearized Frobenius maps between kernels, cokernels, and images. Since the formation of cokernels is right exact (and so commutes with reduction modulo p), the case of cokernels follows from Lemma 3.1.3 and the observed sufficiency of checking the étaleness property modulo p. Thus, if $f : \mathscr{M}' \to \mathscr{M}$ is a map in $\Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$ then $\operatorname{coker} f$ has an étale Frobenius endomorphism.

Since $\varphi : \mathscr{O}_{\mathscr{E}} \to \mathscr{O}_{\mathscr{E}}$ is flat, the formation of $\operatorname{im} f$ and $\ker f$ commutes with scalar extension by φ. That is, $\operatorname{im} \varphi^*(f) \simeq \varphi^*(\operatorname{im} f)$ and similarly for kernels. Since the image of a linear map in a "module category" is naturally identified with the kernel of projection to the cokernel, the known isomorphism property for the linearizations of the Frobenius endomorphisms of $\mathscr{M}$ and $\operatorname{coker} f$ thereby implies the same for $\operatorname{im} f$. Repeating the same trick gives the result for $\ker f$ due to the étaleness property for $\mathscr{M}'$ and $\operatorname{im} f$. ∎

Fontaine discovered that by using the completion $\widehat{\mathscr{O}^{\mathrm{un}}_{\mathscr{E}}}$ as a "period ring", one can define inverse equivalences of categories between $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ and $\Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$ recovering the inverse equivalences D_E and V_E between p-torsion subcategories in Theorem 3.1.8. To make sense of this, we first require a replacement for the basic identities $E_s^{G_E} = E$ and $E_s^{\varphi_{E_s}=1} = \mathbf{F}_p$ that lay at the bottom of our work in the p-torsion case in §3.1.

Lemma 3.2.4. *The natural inclusions $\mathscr{O}_{\mathscr{E}} \to \widehat{\mathscr{O}^{\mathrm{un}}_{\mathscr{E}}}^{G_E}$ and $\mathscr{E} \to (\widehat{\mathscr{E}^{\mathrm{un}}})^{G_E}$ are equalities, and likewise $\mathbf{Z}_p = (\widehat{\mathscr{O}^{\mathrm{un}}_{\mathscr{E}}})^{\varphi=1}$ and $\mathbf{Q}_p = (\widehat{\mathscr{E}^{\mathrm{un}}})^{\varphi=1}$.*

The successive approximation method used to prove this lemma will arise again later, but for now we hold off on axiomatizing it to a wider context.

Proof. Since G_E and φ fix p, and $\widehat{\mathscr{E}^{\mathrm{un}}} = \widehat{\mathscr{O}_{\mathscr{E}}}[1/p]$, the integral claims imply the field claims. Hence, we focus on the integral claims. The evident inclusions $\mathscr{O}_{\mathscr{E}} \to \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}^{G_E}$ and $\mathbf{Z}_p \to (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{\varphi=1}$ are local maps between p-adically separated and complete rings, so it suffices to prove surjectivity modulo p^n for all $n \geqslant 1$. We shall verify this by induction on n, so we first check the base case $n = 1$.

By left-exactness of the formation of G_E-invariants, the exact sequence

$$0 \to \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \xrightarrow{p} \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \to E_s \to 0$$

of $\mathscr{O}_{\mathscr{E}}$-modules gives a linear injection $(\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{G_E}/(p) \hookrightarrow E_s^{G_E} = E$ of nonzero modules over $\mathscr{O}_{\mathscr{E}}/(p) = E$, so this injection is bijective for E-dimension reasons. In particular, the natural map $\mathscr{O}_{\mathscr{E}} \to (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{G_E}/(p)$ is surjective. Since $E_s^{\varphi_{E_s}=1} = \mathbf{F}_p = \mathbf{Z}_p/(p)$, a similar argument gives that $\mathbf{Z}_p \to (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{\varphi=1}/(p)$ is surjective. This settles the case $n = 1$.

Now consider $n > 1$ and assume that $\mathscr{O}_{\mathscr{E}} \to (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{G_E}/(p^{n-1})$ is surjective. Choose any $\xi \in (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{G_E}$; we seek $x \in \mathscr{O}_{\mathscr{E}}$ such that $\xi \equiv x \bmod p^n$. We can choose $c \in \mathscr{O}_{\mathscr{E}}$ such that $\xi \equiv c \bmod p^{n-1}$, so $\xi - c = p^{n-1}\xi'$ with $\xi' \in (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{G_E}$. By the settled case $n = 1$ there exists $c' \in \mathscr{O}_{\mathscr{E}}$ such that $\xi' \equiv c' \bmod p$, so $\xi \equiv c + p^{n-1}c' \bmod p^n$ with $c + p^{n-1}c' \in \mathscr{O}_{\mathscr{E}}$. The case of φ-invariants goes similarly. ∎

Theorem 3.2.5 (Fontaine)**.** *There are covariant naturally quasi-inverse equivalences of abelian categories*

$$\mathrm{D}_{\mathscr{E}} : \mathrm{Rep}_{\mathbf{Z}_p}(G_E) \to \Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}, \ \ \mathrm{V}_{\mathscr{E}} : \Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}} \to \mathrm{Rep}_{\mathbf{Z}_p}(G_E)$$

defined by

$$\mathrm{D}_{\mathscr{E}}(V) = (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} V)^{G_E}, \ \ \mathrm{V}_{\mathscr{E}}(M) = (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} M)^{\varphi=1}.$$

(The operator $\varphi_{\mathrm{D}_{\mathscr{E}}(V)}$ is induced by φ on $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$.) These functors preserve rank and invariant factors over $\mathscr{O}_{\mathscr{E}}$ and $\mathbf{Z}_p$ (in particular, they are length-preserving over $\mathscr{O}_{\mathscr{E}}$ and $\mathbf{Z}_p$ for torsion objects and preserve the property of being finite free modules over $\mathscr{O}_{\mathscr{E}}$ and $\mathbf{Z}_p$), and are compatible with tensor products.

The functors $\mathrm{D}_{\mathscr{E}}$ and $\mathrm{V}_{\mathscr{E}}$ are each naturally compatible with the formation of the duality functors $\mathrm{Hom}_{\mathscr{O}_{\mathscr{E}}}(\cdot, \mathscr{E}/\mathscr{O}_{\mathscr{E}})$ and $\mathrm{Hom}_{\mathbf{Z}_p}(\cdot, \mathbf{Q}_p/\mathbf{Z}_p)$ on torsion objects, as well as with the formation of the duality functors $\mathrm{Hom}_{\mathscr{O}_{\mathscr{E}}}(\cdot, \mathscr{O}_{\mathscr{E}})$ and $\mathrm{Hom}_{\mathbf{Z}_p}(\cdot, \mathbf{Z}_p)$ on finite free module objects.

We emphasize that it is not evident from the definitions that $\mathrm{D}_{\mathscr{E}}(V)$ is finitely generated over $\mathscr{O}_{\mathscr{E}}$ for every V in $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$, let alone that its Frobenius endomorphism (induced by the Frobenius of $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$) is étale. Likewise, it is not evident that $\mathrm{V}_{\mathscr{E}}(M)$ is finitely generated over $\mathbf{Z}_p$ for every M in $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$, nor that the G_E-action on this (arising from the G_E-action on $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$) is continuous for the p-adic topology. These properties will be established in the course of proving Theorem 3.2.5.

Before we prove Theorem 3.2.5 we dispose of the problem of $\mathscr{O}_{\mathscr{E}}$-module finiteness of $\mathrm{D}_{\mathscr{E}}(V)$ for $V \in \mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ via the following lemma that is a generalization of the completed unramified descent for finite free modules that was established in the course of proving Theorem 2.4.6.

Lemma 3.2.6. *Let R be a complete discrete valuation ring with residue field k. Choose a separable closure k_s of k and let $R' = \widehat{R^{\mathrm{un}}}$ be the completion of the associated maximal unramified extension R^{un} of R (with residue field k_s). Let $G_k = \mathrm{Gal}(k_s/k)$ act on R' over R in the canonical manner.*

Let M be a finitely generated R'-module equipped with a semilinear G_k-action that is continuous with respect to the natural topology on M. The R-module M^{G_k} is finitely generated, and the natural map

$$\alpha_M : R' \otimes_R (M^{G_k}) \to M$$

is an isomorphism, so M^{G_k} has the same rank and invariant factors over R as M does over R'. In particular, $M \rightsquigarrow M^{G_k}$ is an exact functor and M^{G_k} is a free R-module if and only if M is a free R'-module.

This lemma goes beyond the completed unramified descent result that was established (for the special case $R = \mathscr{O}_K$ but using general methods) in the proof of Theorem 2.4.6 because we now allow M to have nonzero torsion. This requires some additional steps in the argument.

Proof. Once the isomorphism result is established, the exactness of M^{G_k} in M follows from the faithful flatness of $R \to R'$.

Let π be a uniformizer of R, so it is also a uniformizer of R' and is fixed by the G_k-action. We first treat the case when M has finite R'-length, which is to say that it is killed by π^r for some $r \geqslant 1$. We shall induct on r in this case. If $r = 1$ then M is a finite-dimensional k_s-vector space equipped with a semilinear action of G_k having open stabilizers, so classical Galois descent for vector spaces as in (2.4.3) implies that the natural map $k_s \otimes_k M^{G_k} \to M$ is an isomorphism. (In particular, M^{G_k} is a finite-dimensional k-vector space.) This is the desired result in the π-torsion case.

Now suppose $r > 1$ and that the result is known in the π^{r-1}-torsion case. Let $M' = \pi^{r-1}M$ and $M'' = M/M''$. Clearly M' is π-torsion and M'' is π^{r-1}-torsion. In particular, the settled π-torsion case gives that M' is G_k-equivariantly isomorphic to a product of finitely many copies of k_s, so $\mathrm{H}^1(G_k, M') = 0$. Hence, the left-exact sequence of R-modules

$$0 \to M'^{G_k} \to M^{G_k} \to M''^{G_k} \to 0$$

is exact. The flat extension of scalars $R \to R'$ gives exactness of the top row in the following commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & R' \otimes_R M'^{G_k} \arrow[r] \arrow[d, "\alpha_{M'}", "\simeq"'] & R' \otimes_R M^{G_k} \arrow[r] \arrow[d, "\alpha_M"] & R' \otimes_R M''^{G_k} \arrow[r] \arrow[d, "\alpha_{M''}"', "\simeq"] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

in which the outer vertical maps $\alpha_{M'}$ and $\alpha_{M''}$ are isomorphisms by the inductive hypothesis. Thus, the middle map α_M is an isomorphism. This settles the case when M is a torsion R'-module. In particular, the functor $M \rightsquigarrow M^{G_k}$ is exact in the torsion case.

In the general case we shall pass to inverse limits from the torsion case. Fix $n \geqslant 1$. For all $m \geqslant n$ we have an R'-linear G_k-equivariant right exact sequence

$$M/(\pi^m) \xrightarrow{\pi^n} M/(\pi^m) \to M/(\pi^n) \to 0 \tag{3.2.1}$$

of torsion objects, so applying the exact functor of G_k-invariants gives a right-exact sequence of finite-length R-modules. But $M^{G_k} \simeq \varprojlim (M/(\pi^m))^{G_k}$ since $M = \varprojlim (M/(\pi^m))$, and passage to inverse limits is exact on sequences of finite-length R-modules, so passing to the inverse limit (over m) on the right-exact sequence of G_k-invariants of (3.2.1) gives the right-exact sequence

$$M^{G_k} \xrightarrow{\pi^n} M^{G_k} \to (M/(\pi^n))^{G_k} \to 0$$

for all $n \geqslant 1$. In other words, the natural R-module map $M^{G_k}/(\pi^n) \to (M/(\pi^n))^{G_k}$ is an isomorphism for all $n \geqslant 1$.

In the special case $n = 1$, we have just shown that $M^{G_k}/(\pi) \simeq (M/(\pi))^{G_k}$, and our results in the π-torsion case ensure that $(M/(\pi))^{G_k}$ is finite-dimensional over k. Hence, $M^{G_k}/(\pi)$ is finite-dimensional over $k = R/(\pi)$ in general. Since M^{G_k} is a closed R-submodule of the finitely generated R'-module M, the R-module M^{G_k} is π-adically separated and complete. Thus, if we choose elements of M^{G_k} lifting a finite k-basis of $M^{G_k}/(\pi)$ then a π-adic successive approximation argument shows that such lifts span M^{G_k} over R. In particular, M^{G_k} is a finitely generated R-module in general.

Now consider the natural map $\alpha_M : R' \otimes_R M^{G_k} \to M$. This is a map between finitely generated R'-modules, so to show that it is an isomorphism it suffices to prove that the reduction modulo π^n is an isomorphism for all $n \geqslant 1$. But $\alpha_M \bmod \pi^n$ is identified with $\alpha_{M/(\pi^n)}$ due to the established isomorphism $M^{G_k}/(\pi^n) \simeq (M/(\pi^n))^{G_k}$. Hence, the settled isomorphism result in the general torsion case completes the argument. ■

Now we are ready to take up the proof of Theorem 3.2.5.

Proof. For $V \in \mathrm{Rep}_{\mathbf{Z}_p}(G_E)$, consider the natural $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$-linear "comparison morphism"

$$\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} \mathrm{D}_{\mathscr{E}}(V) = \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} V)^{G_E} \to \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} V. \tag{3.2.2}$$

This is compatible with the natural G_E-action and Frobenius endomorphism on both sides. Setting $M = \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} V$, the semilinear action of G_E on M is continuous (due to the hypothesis that G_E acts continuously on V and the evident continuity of its action on $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$). Thus, we can apply Lemma 3.2.6 with $R = \mathscr{O}_{\mathscr{E}}$ to deduce that $\mathrm{D}_{\mathscr{E}}(V) = M^{G_E}$ is a finitely generated $\mathscr{O}_{\mathscr{E}}$-module and that (3.2.2) is an isomorphism.

We immediately obtain some nice consequences. First of all, the Frobenius structure on $\mathrm{D}_{\mathscr{E}}(V)$ is étale (i.e., its $\mathscr{O}_{\mathscr{E}}$-linearization is an isomorphism) because it suffices to check this after the faithfully flat Frobenius-compatible scalar extension $\mathscr{O}_{\mathscr{E}} \to \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$, whereupon the isomorphism (3.2.2) reduces this étaleness claim to the fact that the Frobenius endomorphism $\varphi \otimes 1$ on the target of (3.2.2) linearizes to an isomorphism. Hence, we have shown that $\mathrm{D}_{\mathscr{E}}$ does indeed take values in the category $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$. As such, we claim that $\mathrm{D}_{\mathscr{E}}$ is an exact functor that preserves rank and invariant factors (of $\mathbf{Z}_p$-modules and $\mathscr{O}_{\mathscr{E}}$-modules) and is naturally compatible with tensor products (in a manner analogous to the tensor compatibility that we have already established in the p-torsion case in Theorem 3.1.8). It suffices to check these properties after faithfully flat scalar extension to $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$, and after applying such a scalar extension we may use (3.2.2) to transfer the claims to their analogues for the functor $V \rightsquigarrow \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} V$, all of which are obvious.

Now we can establish half of the claim concerning inverse functors: for any V in $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ we claim that $\mathrm{V}_{\mathscr{E}}(\mathrm{D}_{\mathscr{E}}(V))$ is naturally $\mathbf{Z}_p[G_E]$-linearly isomorphic to V (but we have not yet proved that $\mathrm{V}_{\mathscr{E}}$ carries general étale φ-modules over $\mathscr{O}_{\mathscr{E}}$ into $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$!). By passing to φ-invariants on the isomorphism (3.2.2) we get a natural $\mathbf{Z}_p[G_E]$-linear isomorphism

$$\mathrm{V}_{\mathscr{E}}(\mathrm{D}_{\mathscr{E}}(V)) \simeq (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} V)^{\varphi=1},$$

so we just have to show that the natural $\mathbf{Z}_p[G_E]$-linear map

$$V \to (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} V)^{\varphi=1}$$

defined by $v \mapsto 1 \otimes v$ is an isomorphism. To justify this, it suffices to show that the diagram

$$0 \to \mathbf{Z}_p \to \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \stackrel{\varphi-1}{\to} \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \to 0$$

is an exact sequence, since the rightmost term is $\mathbf{Z}_p$-flat (so applying $V \otimes_{\mathbf{Z}_p} (\cdot)$ then yields an exact sequence, giving the desired identification of V with a space of φ-invariants).

The identification of $\mathbf{Z}_p$ with $\ker(\varphi - 1)$ in $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$ follows from Lemma 3.2.4, so we just have to show that $\varphi - 1$ is surjective as a $\mathbf{Z}_p$-linear endomorphism of $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$. By p-adic completeness and separatedness of $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$, along with the fact that $\varphi - 1$ commutes with multiplication by p, we can use successive approximation to reduce to checking the surjectivity on $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}/(p) = E_s$. But on E_s the self-map $\varphi - 1$ becomes $x \mapsto x^p - x$, which is surjective since E_s is separably closed.

We now turn our attention to properties of $\mathrm{V}_{\mathscr{E}}$, the first order of business being to show that it takes values in the category $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$. Our analysis of $\mathrm{V}_{\mathscr{E}}$ rests on an analogue of Lemma 3.2.6:

Lemma 3.2.7. *For any M in $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$, the $\mathbf{Z}_p$-module $\mathrm{V}_{\mathscr{E}}(M)$ is finitely generated and the natural $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$-linear G_E-equivariant Frobenius-compatible map*

$$\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} \mathrm{V}_{\mathscr{E}}(M) = \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} M)^{\varphi=1} \to \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} M$$

is an isomorphism. In particular, $\mathrm{V}_{\mathscr{E}}(M)$ is exact in M, it has the same rank and invariant factors as M, and its formation is naturally compatible with tensor products.

Proof. We will handle the case when M is a torsion object, and then the general case is deduced from this by passage to inverse limits as in the proof of Lemma 3.2.6. Hence, we assume that M is killed by p^r for some $r \geqslant 1$, and we shall induct on r. The case $r = 1$ is the known case of étale φ-modules over E that we worked out in the proof of Theorem 3.1.8. To carry out the induction, consider $r > 1$ such that the desired isomorphism result is known in the general p^{r-1}-torsion case. Letting $M' = p^{r-1}M$ and $M'' = M/M'$, we have an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

in $\Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$ with M' killed by p and M'' killed by p^{r-1}. Applying the flat scalar extension $\mathscr{O}_{\mathscr{E}} \to \widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}}$ gives an exact sequence, and we just need to check that the resulting left-exact sequence

$$0 \to \mathrm{V}_{\mathscr{E}}(M') \to \mathrm{V}_{\mathscr{E}}(M) \to \mathrm{V}_{\mathscr{E}}(M'')$$

of φ-invariants is actually surjective on the right, for then we can do a diagram chase to infer the desired isomorphism property for M from the settled cases of M' and M'' much like in the proof of Lemma 3.2.6.

Consider the commutative diagram of exact sequences of $\mathbf{Z}_p$-modules

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}} \otimes_{\mathscr{O}_{\mathscr{E}}} M' \arrow[r] \arrow[d, "\varphi-1"'] & \widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}} \otimes_{\mathscr{O}_{\mathscr{E}}} M \arrow[r] \arrow[d, "\varphi-1"] & \widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}} \otimes_{\mathscr{O}_{\mathscr{E}}} M'' \arrow[r] \arrow[d, "\varphi-1"] & 0 \\
0 \arrow[r] & \widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}} \otimes_{\mathscr{O}_{\mathscr{E}}} M' \arrow[r] & \widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}} \otimes_{\mathscr{O}_{\mathscr{E}}} M \arrow[r] & \widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}} \otimes_{\mathscr{O}_{\mathscr{E}}} M'' \arrow[r] & 0
\end{tikzcd}$$

The kernels of the maps $\varphi - 1$ are the submodules of φ-invariants, so the induced diagram of kernels is the left-exact sequence that we wish to prove is short exact. Hence, by the snake lemma it suffices to show that the cokernel along the left side vanishes. Since M' is p-torsion, the left vertical map is the self-map $\varphi - 1$ of $E_s \otimes_E M'$, and we just need to show that this self-map is surjective. But M' is an étale φ-module over E, so our work in the p-torsion case (see (3.1.1)) gives the Frobenius-compatible $\mathbf{F}_p[G_E]$-linear comparison isomorphism

$$E_s \otimes_E M' \simeq E_s \otimes_{\mathbf{F}_p} V'$$

with $V' = \mathrm{V}_E(M') \in \mathrm{Rep}_{\mathbf{F}_p}(G_E)$. Hence, the surjectivity problem is reduced to the surjectivity of $\varphi_{E_s} - 1 : x \mapsto x^p - x$ on E_s, which holds since E_s is separably closed. ■

Returning to the proof of Theorem 3.2.5, as an immediate application of Lemma 3.2.7 we can prove that the G_E-action on the finitely generated $\mathbf{Z}_p$-module $\mathrm{V}_{\mathscr{E}}(M)$ is continuous (for the p-adic topology). It just has to be shown that the action is discrete (i.e., has open stabilizers) modulo p^n for all $n \geqslant 1$, but the exactness in Lemma 3.2.7 gives $\mathrm{V}_{\mathscr{E}}(M)/(p^n) \simeq \mathrm{V}_{\mathscr{E}}(M/(p^n))$, so it suffices to treat the case when M is p^n-torsion for some $n \geqslant 1$. In this case $\mathrm{V}_{\mathscr{E}}(M)$ is the space of φ-invariants in $\widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}} \otimes_{\mathscr{O}_{\mathscr{E}}} M = \mathscr{O}^{\text{un}}_{\mathscr{E}}/(p^n) \otimes_{\mathscr{O}_{\mathscr{E}}/(p^n)} M$, so it suffices to prove that the G_E-action on $\mathscr{O}^{\text{un}}_{\mathscr{E}}/(p^n)$ has open stabilizers. Even the action on $\mathscr{O}^{\text{un}}_{\mathscr{E}}$ has open stabilizers, since $\mathscr{O}^{\text{un}}_{\mathscr{E}}$ is the rising union of finite étale extensions $\mathscr{O}_{\mathscr{E}} \to \mathscr{O}'_{\mathscr{E}}$ corresponding to finite separable extensions E'/E inside of E_s (with $\mathscr{O}'_{\mathscr{E}}/(p) = E'$) and such a finite étale extension is invariant by the action of the open subgroup $G_{E'} \subseteq G_E$ (as can be checked by inspecting actions on the residue field). Thus, we have shown that $\mathrm{V}_{\mathscr{E}}$ takes values in the expected category $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$.

If we pass to G_E-invariants on the isomorphism in Lemma 3.2.7 then we get an $\mathscr{O}_{\mathscr{E}}$-linear Frobenius-compatible isomorphism

$$\mathrm{D}_{\mathscr{E}}(\mathrm{V}_{\mathscr{E}}(M)) \simeq (\widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}} \otimes_{\mathscr{O}_{\mathscr{E}}} M)^{G_E}$$

for any $M \in \Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$. Let us now check that the target of this isomorphism is naturally isomorphic to M via the $\mathscr{O}_{\mathscr{E}}$-linear Frobenius-compatible map $h : M \to (\widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}} \otimes_{\mathscr{O}_{\mathscr{E}}} M)^{G_E}$ defined by $d \mapsto 1 \otimes d$. It suffices to check the isomorphism property after the faithfully flat scalar extension $\mathscr{O}_{\mathscr{E}} \to \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$. By Lemma 3.2.6 applied to $\mathscr{M} := \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} M$ with $\mathscr{O}_{\mathscr{E}}$ in the role of R there, the $\mathscr{O}_{\mathscr{E}}$-module $\mathscr{M}^{G_E}$ is finitely generated and the natural map $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} \mathscr{M}^{G_E} \to \mathscr{M}$ is an isomorphism. But this isomorphism carries the scalar extension $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} h$ of h over to the identity map $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} M = \mathscr{M}$. Hence, the scalar extension of h is an isomorphism, so h is as well. This completes the verification that $\mathrm{V}_{\mathscr{E}}$ and $\mathrm{D}_{\mathscr{E}}$ are naturally quasi-inverse functors.

It remains to check the behavior of $\mathrm{D}_{\mathscr{E}}$ and $\mathrm{V}_{\mathscr{E}}$ with respect to duality functors. First consider the full subcategories $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)^{\mathrm{tor}}$ and $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét},\mathrm{tor}}$ of torsion objects, on which we use the respective duality functors $V^\vee = \mathrm{Hom}_{\mathbf{Z}_p}(V, \mathbf{Q}_p/\mathbf{Z}_p)$ and $M^\vee = \mathrm{Hom}_{\mathscr{O}_{\mathscr{E}}}(M, \mathscr{E}/\mathscr{O}_{\mathscr{E}})$. In this torsion case the already established tensor compatibility of $\mathrm{D}_{\mathscr{E}}$ gives a natural $\mathscr{O}_{\mathscr{E}}$-linear Frobenius-compatible map

$$\mathrm{D}_{\mathscr{E}}(V) \otimes \mathrm{D}_{\mathscr{E}}(V^\vee) \simeq \mathrm{D}_{\mathscr{E}}(V \otimes V^\vee) \to \mathrm{D}_{\mathscr{E}}(\mathbf{Q}_p/\mathbf{Z}_p),$$

where (i) we use the evaluation mapping $V \otimes V^\vee \to \mathbf{Q}_p/\mathbf{Z}_p$ in the category of $\mathbf{Z}_p[G_E]$-modules and (ii) for any $\mathbf{Z}_p[G_E]$-module W (such as $\mathbf{Q}_p/\mathbf{Z}_p$) we define $\mathrm{D}_{\mathscr{E}}(W) = (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} W)^{G_E}$ as an $\mathscr{O}_{\mathscr{E}}$-module endowed with a φ-semilinear Frobenius endomorphism via the G_E-equivariant Frobenius endomorphism of $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$. Clearly $\mathrm{D}_{\mathscr{E}}(\mathbf{Q}_p/\mathbf{Z}_p) = (\widehat{\mathscr{E}^{\mathrm{un}}}/\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{G_E} = (\mathscr{E}^{\mathrm{un}}/\mathscr{O}_{\mathscr{E}}^{\mathrm{un}})^{G_E}$, and the following lemma identifies this space of G_E-invariants.

Lemma 3.2.8. *The natural Frobenius-compatible map $\mathscr{E}/\mathscr{O}_{\mathscr{E}} \to (\mathscr{E}^{\mathrm{un}}/\mathscr{O}_{\mathscr{E}}^{\mathrm{un}})^{G_E}$ is an isomorphism.*

Proof. If we express $\mathscr{E}^{\mathrm{un}}/\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ as the direct limit of its p^n-torsion levels $(\mathscr{O}_{\mathscr{E}}^{\mathrm{un}} \cdot p^{-n})/\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ for $n \to \infty$, it suffices to prove the analogous claim for the p^n-torsion level for each $n \geqslant 1$, and using multiplication by p^n converts this into the claim that $\mathscr{O}_{\mathscr{E}}/(p^n) \to (\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}/(p^n))^{G_E}$ is an isomorphism for all $n \geqslant 1$. The injectivity is straightfoward, and the surjectivity was shown in the proof of Lemma 3.2.4. ■

By Lemma 3.2.8, we get a natural $\mathscr{O}_{\mathscr{E}}$-linear Frobenius compatible map

$$\mathrm{D}_{\mathscr{E}}(V) \otimes \mathrm{D}_{\mathscr{E}}(V^\vee) \to \mathscr{E}/\mathscr{O}_{\mathscr{E}} \tag{3.2.3}$$

for $V \in \mathrm{Rep}_{\mathbf{Z}_p}(G_E)^{\mathrm{tor}}$, so this in turn defines a natural $\mathscr{O}_{\mathscr{E}}$-linear Frobenius-compatible *duality comparison morphism*

$$\mathrm{D}_{\mathscr{E}}(V^\vee) \to \mathrm{D}_{\mathscr{E}}(V)^\vee.$$

We claim that this latter map in $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$ is an isomorphism (or equivalently the $\mathscr{O}_{\mathscr{E}}$-bilinear $\mathscr{E}/\mathscr{O}_{\mathscr{E}}$-valued duality pairing (3.2.3) is a perfect pairing), thereby expressing the natural compatibility of $\mathrm{D}_{\mathscr{E}}$ with respect to duality functors on torsion objects. To establish this isomorphism property for torsion V, we observe that both sides of the duality comparison morphism are exact functors in V, whence we can reduce the isomorphism problem to the p-torsion case. But in this case our duality pairing is precisely the one constructed for D_E in our study of étale φ-modules over E in the proof of Theorem 3.1.8 (using the natural Frobenius-compatible E-linear identification of $(\mathscr{E}/\mathscr{O}_{\mathscr{E}})[p]$ with E via the basis $1/p$), and in that earlier work we already established the perfectness of the duality pairing.

In a similar manner we can establish the compatibility of $\mathrm{V}_{\mathscr{E}}$ with duality functors on torsion objects, by considering the functor

$$\mathrm{V}_{\mathscr{E}} : M \rightsquigarrow (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} M)^{\varphi=1}$$

from the category of $\mathscr{O}_{\mathscr{E}}$-modules endowed with a φ-semilinear endomorphism to the category of $\mathbf{Z}_p[G_E]$-modules and verifying that

$$\mathrm{V}_{\mathscr{E}}(\mathscr{E}/\mathscr{O}_{\mathscr{E}}) = (\widehat{\mathscr{E}^{\mathrm{un}}}/\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{\varphi=1} \simeq \mathbf{Q}_p/\mathbf{Z}_p$$

via an analogue of Lemma 3.2.8. The details are left to the reader.

Finally, we consider the behavior with respect to duality on objects with finite free module structures over $\mathbf{Z}_p$ and $\mathscr{O}_{\mathscr{E}}$. In this case we use the duality functors $V^{\vee} = \mathrm{Hom}_{\mathbf{Z}_p}(V, \mathbf{Z}_p)$ and $M^{\vee} = \mathrm{Hom}_{\mathscr{O}_{\mathscr{E}}}(M, \mathscr{O}_{\mathscr{E}})$ (endowed with the evident G_E and Frobenius structures), and the tensor compatibility enables us to define duality pairings similarly to the torsion case, now resting on the identifications $\mathrm{D}_{\mathscr{E}}(\mathbf{Z}_p) = \mathscr{O}_{\mathscr{E}}$ and $\mathrm{V}_{\mathscr{E}}(\mathscr{O}_{\mathscr{E}}) = \mathbf{Z}_p$ from Lemma 3.2.4. We then get morphisms

$$\mathrm{D}_{\mathscr{E}}(V^{\vee}) \to \mathrm{D}_{\mathscr{E}}(V)^{\vee}, \;\; \mathrm{V}_{\mathscr{E}}(M^{\vee}) \to \mathrm{V}_{\mathscr{E}}(M)^{\vee}$$

in $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$ and $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ respectively which we want to prove are isomorphisms. In view of the finite freeness of the underlying module structures it suffices to check that these are isomorphisms modulo p, and the exactness of $\mathrm{V}_{\mathscr{E}}$ and $\mathrm{D}_{\mathscr{E}}$ identifies these mod-p reductions with the corresponding duality comparison morphisms from the p-torsion theory for $V/pV \in \mathrm{Rep}_{\mathbf{F}_p}(G_E)$ and $M/pM \in \Phi M_E^{\text{ét}}$. But we proved in our study of p-torsion objects that such p-torsion duality comparison morphisms are isomorphisms. ■

3.3. $\mathbf{Q}_p$-representations of G_E.

We conclude our study of p-adic representations of G_E by using our results for $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ to describe the category $\mathrm{Rep}_{\mathbf{Q}_p}(G_E)$ in a similar Frobenius-semilinear manner. Inspired by Lemma 1.2.6, the idea is that we should use finite-dimensional $\mathscr{E}$-vector spaces (equipped with suitable Frobenius semilinear automorphisms) rather than finite free $\mathscr{O}_{\mathscr{E}}$-modules. However, we will see that there is a subtlety, namely that we need to impose some integrality requirements on the Frobenius structure (whereas in the Galois case the analogous integrality condition, the existence of a Galois-stable $\mathbf{Z}_p$-lattice, is always automatically satisfied: Lemma 1.2.6). For clarity, we now write $\varphi_{\mathscr{O}_{\mathscr{E}}}$ to denote the Frobenius endomorphism of $\mathscr{O}_{\mathscr{E}}$ and $\varphi_{\mathscr{E}}$ to denote the induced endomorphism of its fraction field $\mathscr{E} = \mathscr{O}_{\mathscr{E}}[1/p]$.

To motivate the correct definition of an étale φ-module over $\mathscr{E}$, consider $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_E)$ and define the $\mathscr{E}$-vector space

$$\mathrm{D}_{\mathscr{E}}(V) = (\widehat{\mathscr{E}^{\mathrm{un}}} \otimes_{\mathbf{Q}_p} V)^{G_E}$$

equipped with the $\varphi_{\mathscr{E}}$-semilinear endomorphism $\varphi_{\mathrm{D}_{\mathscr{E}}(V)}$ induced by the G_E-equivariant Frobenius endomorphism of $\widehat{\mathscr{E}^{\mathrm{un}}}$. It may not be immediately evident if $\mathrm{D}_{\mathscr{E}}(V)$ is finite-dimensional over $\mathscr{E}$ or if its Frobenius structure $\mathscr{E}$-linearizes to an isomorphism, but by Lemma 1.2.6 both of these properties and more can be readily deduced from our work in the integral case:

Proposition 3.3.1. *For $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_E)$ $D := \mathrm{D}_{\mathscr{E}}(V)$ has finite $\mathscr{E}$-dimension $\dim_{\mathscr{E}} D = \dim_{\mathbf{Q}_p} V$, and the $\mathscr{E}$-linearization $\varphi_{\mathscr{E}}^*(D) \to D$ of φ_D is an isomorphism. Moreover, there is a φ_D-stable $\mathscr{O}_{\mathscr{E}}$-lattice $L \subseteq D$ such that the $\mathscr{O}_{\mathscr{E}}$-linearization $\varphi_{\mathscr{O}_{\mathscr{E}}}^*(L) \to L$ is an isomorphism.*

Proof. By Lemma 1.2.6, we have $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda$ for $\Lambda \in \mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ that is finite free as a $\mathbf{Z}_p$-module. Thus, from the definition we have

$$\mathrm{D}_{\mathscr{E}}(V) = \mathrm{D}_{\mathscr{E}}(\Lambda)[1/p] \simeq \mathscr{E} \otimes_{\mathscr{O}_{\mathscr{E}}} \mathrm{D}_{\mathscr{E}}(\Lambda)$$

as $\mathscr{E}$-vector spaces endowed with a $\varphi_{\mathscr{E}}$-semilinear endomorphism. Since $\mathrm{D}_{\mathscr{E}}(\Lambda) \in \Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$ and this is finite free as an $\mathscr{O}_{\mathscr{E}}$-module with rank equal to $\mathrm{rank}_{\mathbf{Z}_p}(\Lambda) = \dim_{\mathbf{Q}_p}(V)$, we are done (take $L = \mathrm{D}_{\mathscr{E}}(\Lambda)$). ■

Proposition 3.3.1 motivates the following definition.

Definition 3.3.2. An *étale φ-module over $\mathscr{E}$* is a finite-dimensional $\mathscr{E}$-vector space D equipped with a $\varphi_{\mathscr{E}}$-semilinear endomorphism $\varphi_D : D \to D$ whose linearization $\varphi^*_{\mathscr{E}}(D) \to D$ is an isomorphism and which admits a φ_D-stable $\mathscr{O}_{\mathscr{E}}$-lattice $L \subseteq D$ such that $(L, \varphi_D|_L) \in \Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$ (i.e., the linearization $\varphi^*_{\mathscr{O}_{\mathscr{E}}}(L) \to L$ induced by φ_D is an isomorphism). The category of such pairs (D, φ_D) is denoted $\Phi M^{\text{ét}}_{\mathscr{E}}$.

The lattice L in Definition 3.3.2 is auxiliary data and is not at all canonical. In Definition 3.3.2 the existence of the φ_D-stable $L \in \Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$ forces φ_D to $\mathscr{E}$-linearize to an isomorphism, but it seems more elegant to impose this latter étaleness property on φ_D before we mention the hypothesis concerning the existence of the non-canonical L. Such $\mathscr{O}_{\mathscr{E}}$-lattices L are analogous to Galois-stable $\mathbf{Z}_p$-lattices in an object of $\mathrm{Rep}_{\mathbf{Q}_p}(\Gamma)$ for a profinite group Γ: their existence is a useful device in proofs, but they are not part of the intrinsic structure of immediate interest.

Example 3.3.3. The naive definition one may have initially imagined for an étale φ-module over $\mathscr{E}$ is a finite-dimensional $\mathscr{E}$-vector space D equipped with a $\varphi_{\mathscr{E}}$-semilinear endomorphism φ_D whose $\mathscr{E}$-linearization is an isomorphism. However, this is insufficient for getting an equivalence with $\mathrm{Rep}_{\mathbf{Q}_p}(G_E)$ because such objects (D, φ_D) can fail to admit a Frobenius-stable (let alone étale) $\mathscr{O}_{\mathscr{E}}$-lattice L as in Proposition 3.3.1. The problem is that the Frobenius endomorphism φ_D can lack good integrality properties; there is no analogue of Lemma 1.2.6 on the Frobenius-semilinear module side.

To give a concrete example, let $D = \mathscr{E}$ and define $\varphi_D = p^{-1} \cdot \varphi_{\mathscr{E}}$. In this case for any nonzero $x \in D$ we have

$$\varphi_D(x) = p^{-1} \cdot \varphi_{\mathscr{E}}(x) = p^{-1} \cdot \frac{\varphi_{\mathscr{E}}(x)}{x} \cdot x.$$

Since the multiplier $\varphi_{\mathscr{E}}(x)/x$ lies in $\mathscr{O}_{\mathscr{E}}^{\times}$, the additional factor of $1/p$ prevents $\varphi_D(x)$ from being an $\mathscr{O}_{\mathscr{E}}$-multiple of x. The $\mathscr{O}_{\mathscr{E}}$-lattices in $\mathscr{E}$ are precisely the $\mathscr{O}_{\mathscr{E}}$-modules $\mathscr{O}_{\mathscr{E}} \cdot x$ for $x \in \mathscr{E}^{\times}$, so we conclude that there is no φ_D-stable $\mathscr{O}_{\mathscr{E}}$-lattice L in D (let alone one whose Frobenius endomorphism linearizes to a lattice isomorphism).

There is an evident functor $\Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}} \to \Phi M^{\text{ét}}_{\mathscr{E}}$ given by $L \rightsquigarrow L[1/p] = \mathscr{E} \otimes_{\mathscr{O}_{\mathscr{E}}} L$, and

$$\mathrm{Hom}_{\Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}}(L, L')[1/p] = \mathrm{Hom}_{\Phi M^{\text{ét}}_{\mathscr{E}}}(L[1/p], L'[1/p]),$$

so $\Phi M^{\text{ét}}_{\mathscr{E}}$ is identified with the "isogeny category" of $\Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$. In particular, $\Phi M^{\text{ét}}_{\mathscr{E}}$ is abelian.

Theorem 3.3.4. *The functors* $\mathrm{D}_{\mathscr{E}}(V) := (\widehat{\mathscr{E}^{\mathrm{un}}} \otimes_{\mathbf{Q}_p} V)^{\varphi=1}$ *and* $\mathrm{V}_{\mathscr{E}}(D) := (\widehat{\mathscr{E}^{\mathrm{un}}} \otimes_{\mathscr{E}} D)^{G_E}$ *are rank-preserving exact quasi-inverse equivalences between* $\mathrm{Rep}_{\mathbf{Q}_p}(G_E)$ *and* $\Phi M_{\mathscr{E}}^{\text{ét}}$ *that naturally commute with the formation of tensor products and duals.*

Proof. If Λ is a G_E-stable $\mathbf{Z}_p$-lattice in V then we have seen that $\mathrm{D}_{\mathscr{E}}(V) = \mathrm{D}_{\mathscr{E}}(\Lambda)[1/p]$, and likewise if we choose (as we may by definition) an étale φ-module L that is a Frobenius-stable $\mathscr{O}_{\mathscr{E}}$-lattice in a chosen $D \in \Phi M_{\mathscr{E}}^{\text{ét}}$ then $\mathrm{V}_{\mathscr{E}}(D) = \mathrm{V}_{\mathscr{E}}(L)[1/p]$. Thus, everything is immediately obtained by p-localization on our results comparing $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ and $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$ (using the full subcategories of objects with finite free module structures over $\mathbf{Z}_p$ and $\mathscr{O}_{\mathscr{E}}$). ■

3.4. **Exercises.**

Exercise 3.4.1. It is crucial to recognize that in the semilinear setting, matrices describing maps have slightly twisted transformation laws (and so concepts like eigenvalue and characteristic polynomial no longer make sense, though are useful for inspiration).

Let R be a ring equipped with an endomorphism $\phi : R \to R$. For any R-module M, let $\phi^*(M) = R \otimes_{\phi, R} M$ be an R-module via the left tensor factor.

1. A *ϕ-semilinear map* $T : M' \to M$ between two R-modules is an additive map such that $T(cm') = \phi(c)T(m')$ for all $m' \in M'$ and $c \in R$. Let $\mathrm{Hom}_R^{\phi}(M', M)$ denote the set of these. Give it a natural R-module structure in two ways.
2. Associated to any $T \in \mathrm{Hom}_R^{\phi}(M', M)$ is the R-linear map $T_{\mathrm{lin}} : \phi^*(M') \to M$ defined by $c \otimes m' \mapsto cT(m')$, called the *linearization* of T. Show that linearization defines an additive bijection $\mathrm{Hom}_R^{\phi}(M', M) \simeq \mathrm{Hom}_R(\phi^*(M'), M)$. The natural R-module structure on the target Hom-set matches one of the two on the source Hom-set. Which one?
3. Now suppose M' and M are free R-modules of respective ranks $d', d \geqslant 1$, Fix bases $\mathbf{e} = \{e_1, \dots, d_d\}$ of M and $\mathbf{e}' = \{e_1', \cdots e_{d'}'\}$ of M'. For any $T \in \mathrm{Hom}_R^{\phi}(M', M)$, the *associated matrix* ${}_{\mathbf{e}'}[T]_{\mathbf{e}} \in \mathrm{Mat}_{d \times d'}(R)$ is the matrix (c_{ij}) defined via the conditions $T(e_j') = \sum_i c_{ij} e_i$. Show this defines a bijection $\mathrm{Hom}_R^{\phi}(M', M) \simeq \mathrm{Mat}_{d \times d'}(R)$. The R-module structure on matrices matches one of the two on $\mathrm{Hom}_R^{\phi}(M', M)$. Which one? Also translate composition of ϕ-semilinear maps into the language of matrices (there is a ϕ-twist in the formula).
4. Continuing with the same notation, let $\phi^*(\mathbf{e}')$ be the R-basis $\{1 \otimes e_i\}$ of $\phi^*(M)$. Show that ${}_{\mathbf{e}'}[T]_{\mathbf{e}}$ is exactly the usual matrix ${}_{\phi^*(\mathbf{e}')}[T_{\mathrm{lin}}]_{\mathbf{e}}$ associated to the linearization (relative to the corresponding bases).
5. Let $\mathbf{f} = \{f_1, \dots, f_d\}$ and $\mathbf{f}' = \{f_1', \dots, f_d'\}$ be other choices of bases, and let $A = {}_{\mathbf{f}}[\mathrm{id}_M]_{\mathbf{e}}$ and $A' = {}_{\mathbf{f}'}[\mathrm{id}_{M'}]_{\mathbf{e}'}$. Double check that these are the "change of basis matrices" converting $\mathbf{e}$-coordinates into $\mathbf{f}$-coordinates and $\mathbf{e}'$-coordinates into $\mathbf{f}'$-coordinates (not the other way around).

   Prove the following twisted version of the usual transformation law:
   $${}_{\mathbf{f}}[T]_{\mathbf{f}'} = A_{\mathbf{e}}[T]_{\mathbf{e}'} \phi(A')^{-1},$$
   where by $\phi(A')$ we mean the matrix obtained by applying ϕ to all matrix entries. Explain by pure thought why the ϕ appears where it does in this formula. In the special case $M' = M$, $\mathbf{e}' = \mathbf{e}$, and $\mathbf{f}' = \mathbf{f}$, explain why concepts such as characteristic

polynomial, trace, and determinant generally make no sense when $\phi \neq \mathrm{id}_R$. (Example 8.1.3 gives an especially nice example in characteristic 0.) Show that the element $\det({}_{\mathbf{e}}[T]_{\mathbf{e}}) \in R$ is well-defined up to multiplication by $u/\phi(u)$ for $u \in R^\times$.

(6) Now take $R = E = k(\!(u)\!)$ with k a perfect field of characteristic $p > 0$ (so $R^\times = E^\times \neq k[\![u]\!]^\times$), and let (M, φ_M) be a φ-module of dimension d over E. Using suitable u-power multiplications on an initial choice of basis, show that there is always a basis such that the associated matrix of φ_M lies in $\mathrm{Mat}_d(k[\![u]\!])$; that is, there is *always* a φ-stable $k[\![u]\!]$-lattice in M. Assuming $\varphi_M \neq 0$, find another basis for which the matrix does *not* lie in $\mathrm{Mat}_d(k[\![u]\!])$.

Exercise 3.4.2. Let E be a field of characteristic $p > 0$, and M_0 an étale φ-module over E.

Prove that $\varphi_{M_0^\vee} : M_0^\vee \to M_0^\vee$ is the φ_E-semilinear map whose E-linearization is the isomorphism

$$\varphi_E^*(M_0^\vee) \simeq (\varphi_E^*(M_0))^\vee \simeq M_0^\vee$$

with the final isomorphism defined to be inverse to the linear dual of the E-linear isomorphism $\varphi_E^*(M_0) \simeq M_0$ induced by linearization of φ_{M_0}.

Exercise 3.4.3. Let E be a field of characteristic $p > 0$ and fix an associated pair $(\mathscr{O}_{\mathscr{E}}, \varphi)$ consisting of a Cohen ring for E and a Frobenius lift. Consider Fontaine's equivalence $\mathrm{Rep}_{\mathbf{Z}_p}(G_E) \simeq \Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$ between p-adic representations of G_E over $\mathbf{Z}_p$ and étale φ-modules over $\mathscr{O}_{\mathscr{E}}$. Let E'/E be a separable algebraic extension inside of E_s, and let $(\mathscr{O}_{\mathscr{E}'}, \varphi')$ be the canonically associated pair over $(\mathscr{O}_{\mathscr{E}}, \varphi)$.

(1) The restriction functor $\mathrm{Res}_{G_{E'}}^{G_E} : \mathrm{Rep}_{\mathbf{Z}_p}(G_E) \to \mathrm{Rep}_{\mathbf{Z}_p}(G_{E'})$ translates into a functor $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}} \to \Phi M_{\mathscr{O}_{\mathscr{E}'}}^{\text{ét}}$. Prove it is completed extension of scalars: $D \rightsquigarrow \mathscr{O}_{\mathscr{E}'} \otimes_{\mathscr{O}_{\mathscr{E}}} D$ with the associated diagonal Frobenius operator (which *is* still étale).

(2) Assume E'/E is finite. The *induction* functor $\mathrm{Rep}_{\mathbf{Z}_p}(G_{E'}) \to \mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ is defined as follows: $\mathrm{Ind}_{G_{E'}}^{G_E}(V')$ is the set of functions $f : G_E \to V'$ which "transform according to the $G_{E'}$-action on V'; that is, $f(g'g) = g'.f(g)$ for all $g \in G_E$ and $g' \in G_{E'}$. The G_E-action is defined to be $(g.f)(x) = f(xg)$ (so indeed $g.f \in \mathrm{Ind}_{G_{E'}}^{G_E}(V')$ and $f \mapsto g.f$ is a left G_E-action). There is a natural $\mathbf{Z}_p[G_{E'}]$-module map $\eta_{V'} : \mathrm{Res}_{G_{E'}}^{G_E}\mathrm{Ind}_{G_{E'}}^{G_E}(V') \to V'$ via $f \mapsto f(1)$. Prove that the resulting composite

$$\mathrm{Hom}_{\mathbf{Z}_p[G_E]}(V, \mathrm{Ind}_{G_{E'}}^{G_E}(V')) \to \mathrm{Hom}_{\mathbf{Z}_p[G_{E'}]}(\mathrm{Res}_{G_{E'}}^{G_E}(V), V')$$

via $T \mapsto \eta_{V'} \circ \mathrm{Res}_{G_{E'}}^{G_E}(T)$ is bijective, so induction is right adjoint to restriction.

Interpret induction in terms of étale φ-modules. Watch out for the variances of the functors! What if we work with the alternative construction $V' \rightsquigarrow \mathbf{Z}_p[G_E] \otimes_{\mathbf{Z}_p[G_{E'}]} V'$ (in the spirit of compact induction)?

Exercise 3.4.4. Why is Example 3.3.3 not inconsistent with the existence of φ-stable lattices in the setting of Exercise 3.4.1(6)? (That is, why does the solution of that exercise not apply to Example 3.3.3?)

4. Better ring-theoretic constructions

4.1. **From gradings to filtrations.** The ring B_{HT} provides a convenient mechanism for working with Hodge–Tate representations, but the Hodge–Tate condition on a p-adic representation of the Galois group G_K of a p-adic field K is too weak to be really useful. What we seek is a class of p-adic representations that is broad enough to include the representations arising from algebraic geometry but also small enough to permit the existence of an equivalence of categories with (or at least a fully faithful exact tensor functor to) a category of semilinear algebra objects. Based on our experience with Hodge–Tate representations and étale φ-modules, we can expect that on the semilinear algebra side we will need to work with modules admitting some kind of structures like Frobenius endomorphisms and gradings (or filtrations). We also want the functor relating our "good" p-adic representations of G_K to semilinear algebra to be defined by a period ring that is "better" than B_{HT} and allows us to recover B_{HT} (i.e., whatever class of good representations we study should at least be of Hodge–Tate type).

The ring $B_{\mathrm{HT}} = \oplus_q \mathbf{C}_K(q)$ is a graded $\mathbf{C}_K$-algebra endowed with a compatible semilinear G_K-action. In view of the isomorphism (2.4.6) in Gr_K, the grading on B_{HT} is closely related to the grading on the Hodge cohomology $\mathrm{H}^n_{\mathrm{Hodge}}(X) = \oplus_{p+q=n} \mathrm{H}^p(X, \Omega^q_{X/K})$ for smooth proper K-schemes X. To motivate how we should refine B_{HT}, we can get a clue from the refinement of $\mathrm{H}^n_{\mathrm{Hodge}}(X)$ given by the algebraic de Rham cohomology $\mathrm{H}^n_{\mathrm{dR}}(X/K)$. This is not the place to enter into the definition of algebraic de Rham cohomology, but it is instructive to record some of its properties.

For any proper scheme X over any field k whatsoever, the algebraic de Rham cohomologies $\mathrm{H}^n(X) = \mathrm{H}^n_{\mathrm{dR}}(X/k)$ are finite-dimensional k-vector spaces endowed with a natural decreasing (Hodge) filtration

$$\mathrm{H}^n(X) = \mathrm{Fil}^0(\mathrm{H}^n(X)) \supseteq \mathrm{Fil}^1(\mathrm{H}^n(X)) \supseteq \cdots \supseteq \mathrm{Fil}^{n+1}(\mathrm{H}^n(X)) = 0$$

by k-subspaces and $\mathrm{Fil}^q(\mathrm{H}^n(X))/\mathrm{Fil}^{q+1}(\mathrm{H}^n(X))$ is naturally a subquotient of $\mathrm{H}^{n-q}(X, \Omega^q_{X/k})$, with a natural equality

$$\mathrm{Fil}^q(\mathrm{H}^n(X))/\mathrm{Fil}^{q+1}(\mathrm{H}^n(X)) = \mathrm{H}^{n-q}(X, \Omega^q_{X/k})$$

if $\mathrm{char}(k) = 0$.

Definition 4.1.1. A *filtered module* over a commutative ring R is an R-module M endowed with a collection $\{\mathrm{Fil}^i M\}_{i \in \mathbf{Z}}$ of submodules that is decreasing in the sense that $\mathrm{Fil}^{i+1}(M) \subseteq \mathrm{Fil}^i(M)$ for all $i \in \mathbf{Z}$. If $\cup \mathrm{Fil}^i(M) = M$ then the filtration is *exhaustive* and if $\cap \mathrm{Fil}^i(M) = 0$ then the filtration is *separated*. For any filtered R-module M, the *associated graded module* is $\mathrm{gr}^\bullet(M) = \oplus_i (\mathrm{Fil}^i(M)/\mathrm{Fil}^{i+1}(M))$.

A *filtered ring* is a ring R equipped with an exhaustive and separated filtration $\{R^i\}$ by additive subgroups such that $1 \in R^0$ and $R^i \cdot R^j \subseteq R^{i+j}$ for all $i, j \in \mathbf{Z}$. (In particular, R^0 is a subring of R and each R^i is an R^0-submodule of R.) The *associated graded ring* is $\mathrm{gr}^\bullet(R) = \oplus_i R^i/R^{i+1}$. If k is a ring then a *filtered k-algebra* is a k-algebra A equipped with a structure of filtered ring such that the filtered pieces A^i are k-submodules of A, and the *associated graded k-algebra* is $\mathrm{gr}^\bullet(A) = \oplus_i A^i/A^{i+1}$.

Example 4.1.2. Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and residue field k, and let $A = \mathrm{Frac}(R)$. There is a natural structure of filtered ring on A via $A^i = \mathfrak{m}^i$ for $i \in \mathbf{Z}$. In this case the associated graded ring $\mathrm{gr}^\bullet(A)$ is a k-algebra that is non-canonically isomorphic to a Laurent polynomial ring $k[t, 1/t]$ upon choosing a k-basis of $\mathfrak{m}/\mathfrak{m}^2$. Note that canonically $\mathrm{gr}^\bullet(A) = \mathrm{gr}^\bullet(\widehat{A})$, where $\widehat{A}$ denotes the fraction field of the completion $\widehat{R}$ of R.

For a smooth proper $\mathbf{C}$-scheme X, Grothendieck constructed a natural $\mathbf{C}$-linear isomorphism $\mathrm{H}^n_{\mathrm{dR}}(X/\mathbf{C}) \simeq \mathbf{C} \otimes_{\mathbf{Q}} \mathrm{H}^n_{\mathrm{top}}(X(\mathbf{C}), \mathbf{Q})$. Complex conjugation on the left tensor factor of the target defines a conjugate-linear automorphism $v \mapsto \overline{v}$ of $\mathrm{H}^n_{\mathrm{dR}}(X/\mathbf{C})$, and by Hodge theory this determines a canonical splitting of the Hodge filtration on $\mathrm{H}^n_{\mathrm{dR}}(X/\mathbf{C})$ via the $\mathbf{C}$-subspaces $H^{n-q,q} := \overline{F^{n-q}} \cap F^q$ where $F^j = \mathrm{Fil}^j(\mathrm{H}^n_{\mathrm{dR}}(X/\mathbf{C}))$; i.e., $H^{n-q,q} \simeq F^q/F^{q+1}$ for all q, so $F^j = \oplus_{q \geqslant j} H^{n-q,q}$. Moreover, in Hodge theory one constructs a natural isomorphism $H^{n-q,q} \simeq \mathrm{H}^{n-q}(X, \Omega^q_{X/\mathbf{C}})$. In particular, complex conjugation gives rise to a canonical splitting of the Hodge filtration when the ground field is $\mathbf{C}$. This splitting rests on algebraic topology and complex conjugation on $\mathbf{C}$.

In the general algebraic case over an arbitrary field k of characteristic 0, the best one has canonically is that for any smooth proper k-scheme X, the k-vector space $\mathrm{H}^n_{\mathrm{dR}}(X/k)$ is naturally endowed with an exhaustive and separated filtration whose associated graded vector space

$$\mathrm{gr}^\bullet(\mathrm{H}^n_{\mathrm{dR}}(X/k)) := \bigoplus_q \mathrm{Fil}^q(\mathrm{H}^n_{\mathrm{dR}}(X/k))/\mathrm{Fil}^{q+1}(\mathrm{H}^n_{\mathrm{dR}}(X/k))$$

is the Hodge cohomology $\oplus_q \mathrm{H}^{n-q}(X, \Omega^q_{X/k})$ of X. This filtration generally does not admit a functorial splitting.

A natural idea for improving Faltings' comparison isomorphism (2.4.6) between p-adic étale and graded Hodge cohomology via B_{HT} is to replace the graded $\mathbf{C}_K$-algebra B_{HT} with a filtered K-algebra B_{dR} endowed with a G_K-action respecting the filtration such that (i) B_{dR} is $(\mathbf{Q}_p, G_K)$-regular, with $B_{\mathrm{dR}}^{G_K} = K$, (ii) $\mathrm{Fil}^0(B_{\mathrm{dR}})/\mathrm{Fil}^1(B_{\mathrm{dR}}) \simeq \mathbf{C}_K$ as rings with G_K-action, and (iii) there is a canonical G_K-equivariant isomorphism $\mathrm{gr}^\bullet(B_{\mathrm{dR}}) \simeq B_{\mathrm{HT}}$ as graded $\mathbf{C}_K$-algebras. Given such a B_{dR}, consider the associated functor $D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K}$ on $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ with values in finite-dimensional K-vector spaces. This has a functorial filtration via $\mathrm{Fil}^i(D_{\mathrm{dR}}(V)) = (\mathrm{Fil}^i(B_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V)^{G_K}$, and it is exhaustive and separated since the same holds for the filtration on B_{dR} (by the definition of a filtered ring). By left-exactness of $(\cdot)^{G_K}$, there is an evident natural injective map

$$\mathrm{gr}^\bullet(D_{\mathrm{dR}}(V)) \hookrightarrow (\mathrm{gr}^\bullet(B_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V)^{G_K} = (B_{\mathrm{HT}} \otimes_{\mathbf{Q}_p} V)^{G_K} = D_{\mathrm{HT}}(V)$$

of graded K-vector spaces, so if V is B_{dR}-admissible then

$$\dim_{\mathbf{Q}_p} V = \dim_K D_{\mathrm{dR}}(V) = \dim_K \mathrm{gr}^\bullet(D_{\mathrm{dR}}(V)) \leqslant \dim_K D_{\mathrm{HT}}(V) \leqslant \dim_{\mathbf{Q}_p} V,$$

so V is necessarily Hodge–Tate. In this sense, any such D_{dR} is a finer invariant than D_{HT}.

A serious test of a good definition for B_{dR} is that it should lead to a refinement of Faltings' comparison theorem between p-adic étale and Hodge cohomology, by using de Rham cohomology instead. That is, for smooth proper X over K the p-adic representations $\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ should be B_{dR}-admissible with a natural isomorphism $D_{\mathrm{dR}}(\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)) \simeq \mathrm{H}^n_{\mathrm{dR}}(X/K)$

whose induced isomorphism between associated graded K-vector spaces is Faltings' comparison isomorphism between p-adic étale and Hodge cohomologies.

Inspired by Example 4.1.2 and the description $B_{\mathrm{HT}} \simeq \mathbf{C}_K[T, T^{-1}]$, we are led to seek a complete discrete valuation ring B_{dR}^+ over K (with maximal ideal denoted $\mathfrak{m}$) endowed with a G_K-action such that the residue field is naturally G_K-equivariantly isomorphic to $\mathbf{C}_K$ and the Zariski cotangent space $\mathfrak{m}/\mathfrak{m}^2$ is naturally isomorphic to $\mathbf{C}_K(1)$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$. Since there is a canonical isomorphism $\mathfrak{m}^i/\mathfrak{m}^{i+1} \simeq (\mathfrak{m}/\mathfrak{m}^2)^{\otimes i}$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ for all $i \in \mathbf{Z}$, for the fraction field B_{dR} of such a ring B_{dR}^+ we would then canonically have $\mathrm{gr}^\bullet(B_{\mathrm{dR}}) \simeq B_{\mathrm{HT}}$ as graded $\mathbf{C}_K$-algebras with G_K-action.

Example 4.1.3. A naive guess is to take $B_{\mathrm{dR}}^+ = \prod_{q \geqslant 0} \mathbf{C}_K(q) \simeq \mathbf{C}_K[\![t]\!]$ with G_K-action given by $g(\sum a_n t^n) = \sum g(a_n)\chi(g)^n t^n$. This does not lead to new concepts refining the theory of Hodge–Tate representations since the product decomposition canonically defines a G_K-equivariant splitting of the filtration on $\mathfrak{m}^i/\mathfrak{m}^j$ for any $i, j \in \mathbf{Z}$ with $j > i$. In other words, for such a choice of complete discrete valuation ring the filtration is too closely related to a grading to give anything interesting (beyond what we already get from the Hodge–Tate theory). More specifically, with such a definition we would get $D_{\mathrm{dR}} = D_{\mathrm{HT}}$ (with canonically split filtration), so there could not be any comparison isomorphism obtained in this way between p-adic étale and de Rham cohomolgies, as the filtration on the latter is not functorially split.

A more promising idea is to imitate the procedure in commutative algebra whereby for perfect fields k of characteristic $p > 0$ there is a functorially associated complete discrete valuation ring $\mathrm{W}(k)$ (of Witt vectors) that has uniformizer p and residue field k. (See §4.2.) A big difference is that now we want to functorially build a complete discrete valuation ring with residue field $\mathbf{C}_K$ of characteristic 0 (and we will not expect to have a canonical uniformizer). Thus, we cannot use a naive Witt construction (as in §4.2). Nonetheless, we shall see that an artful application of Witt-style ideas will give rise to the right equicharacteristic-0 complete discrete valuation ring B_{dR}^+ for our purposes (and though any complete discrete valuation ring with residue field F of characteristic 0 is abstractly isomorphic to $F[\![t]\!]$ by commutative algebra, such a structure will not exist for B_{dR}^+ in a G_K-*equivariant* manner).

Remark 4.1.4. We should emphasize at the outset that B_{dR}^+ will differ from $\prod_{q \geqslant 0} \mathbf{C}_K(q)$ (as complete discrete valuation rings with G_K-action and residue field $\mathbf{C}_K$) in at least two key respects. First, as we just noted, there will be no G_K-equivariant ring-theoretic section to the reduction map from B_{dR}^+ onto its residue field $\mathbf{C}_K$. Second, even the quotient $B_{\mathrm{dR}}^+/\mathfrak{m}^2$ as an extension of $\mathbf{C}_K$ by $\mathbf{C}_K(1)$ will have no G_K-equivariant additive splitting. (This is not inconsistent with Example 2.2.6 because $B_{\mathrm{dR}}^+/\mathfrak{m}^2$ does not admit a G_K-equivariant $\mathbf{C}_K$-structure as required there.)

Roughly speaking, the idea behind the construction of B_{dR}^+ is as follows. Rather than try to directly make a canonical complete discrete valuation ring with residue field $\mathbf{C}_K$, we observe that $\mathbf{C}_K = \mathscr{O}_{\mathbf{C}_K}[1/p]$ with $\mathscr{O}_{\mathbf{C}_K} = \varprojlim \mathscr{O}_{\mathbf{C}_K}/(p^n) = \varprojlim \mathscr{O}_{\overline{K}}/(p^n)$ closely related to p-power torsion rings. Hence, it is more promising to try to adapt Witt-style constructions for $\mathscr{O}_{\mathbf{C}_K}$ than for $\mathbf{C}_K$. We will make a certain height-1 valuation ring R of equicharacteristic p whose fraction field $\mathrm{Frac}(R)$ is algebraically closed (hence perfect) such that there is a natural G_K-action on R and a natural surjective G_K-equivariant map

$$\theta : \mathrm{W}(R) \twoheadrightarrow \mathscr{O}_{\mathbf{C}_K}.$$

(Note that $\mathrm{W}(R) \subseteq \mathrm{W}(\mathrm{Frac}(R))$, so $\mathrm{W}(R)$ is a domain of characteristic 0.) We would then get a surjective G_K-equivariant map $\theta_{\mathbf{Q}} : \mathrm{W}(R)[1/p] \twoheadrightarrow \mathscr{O}_{\mathbf{C}_K}[1/p] = \mathbf{C}_K$. Since R is like a 1-dimensional ring, $\mathrm{W}(R)$ is like a 2-dimensional ring and so $\mathrm{W}(R)[1/p]$ is like a 1-dimensional ring. The ring structure of $\mathrm{W}(A)$ is generally pretty bad if A is not a perfect field of characteristic p, but as long as the maximal ideal $\ker \theta_{\mathbf{Q}}$ is principal and nonzero we can replace $\mathrm{W}(R)[1/p]$ with its $\ker \theta_{\mathbf{Q}}$-adic completion to obtain a canonical complete discrete valuation ring B_{dR}^+ having residue field $\mathbf{C}_K$ (and it will satisfy all of the other properties that we shall require).

4.2. **Witt vectors and universal Witt constructions.** Let k be a finite field of characteristic p and let A be the valuation ring of the finite unramified extension of $\mathbf{Z}_p$ with residue field k. Let $[\cdot] : k \to A$ be the multiplicative Teichmüller lifting (carrying 0 to 0 and sending $k^\times$ isomorphically onto $\mu_{q-1}(A)$ with $q = \#k$), so every element $a \in A$ admits a unique expansion $a = \sum_{n \geqslant 0} [c_n] p^n$ with $c_n \in k$. For any such $a \in A$ and $a' = \sum_{n \geqslant 0} [c'_n] p^n \in A$, it is natural to ask if we can compute the Teichmüller expansions of $a + a'$ and aa' by "universal formulas" (independent of k beyond the specification of the characteristic as p) involving only algebraic operations over $\mathbf{F}_p$ on the sequences $\{c_n\}$ and $\{c'_n\}$ in k. Since A is functorially determined by k it is not unreasonable to seek this kind of reconstruction of A in such a direct manner in terms of k.

One can work out such formulas in some low-degree Teichmüller coefficients, and then it becomes apparent that what really matters about k is not its finiteness but rather its perfectness. Rather than give a self-contained complete development of Witt vectors from scratch, we refer the reader to [44, Ch. II, §4–§6] for such a development. Some aspects of this theory will be reviewed below as necessary, but we assume that the reader has some previous experience with the ring of Witt vectors $\mathrm{W}(A)$ for an arbitrary commutative ring A (not just for $\mathbf{F}_p$-algebras A).

Let A be a perfect $\mathbf{F}_p$-algebra (i.e., an $\mathbf{F}_p$-algebra for which $a \mapsto a^p$ is an automorphism of A). Observe that the additive multiplication map $p : \mathrm{W}(A) \to \mathrm{W}(A)$ is given by $(a_i) \mapsto (0, a_0^p, a_1^p, \dots)$, so it is injective and the subset $p^n \mathrm{W}(A) \subseteq \mathrm{W}(A)$ consists of Witt vectors (a_i) such that $a_0 = \dots = a_{n-1} = 0$ since A is perfect, so we naturally have $\mathrm{W}(A)/(p^n) \simeq \mathrm{W}_n(A)$ by projection to the first n Witt components. Hence, the natural map $\mathrm{W}(A) \to \varprojlim \mathrm{W}(A)/(p^n)$ is an isomorphism. Thus, $\mathrm{W}(A)$ for perfect $\mathbf{F}_p$-algebras A is a strict p-ring in the sense of the following definition.

Definition 4.2.1. A *p-ring* is a ring B that is separated and complete for the topology defined by a specified decreasing collection of ideals $\mathfrak{b}_1 \supseteq \mathfrak{b}_2 \supseteq \dots$ such that $\mathfrak{b}_n \mathfrak{b}_m \subseteq \mathfrak{b}_{n+m}$ for all $n, m \geqslant 1$ and $B/\mathfrak{b}_1$ is a perfect $\mathbf{F}_p$-algebra (so $p \in \mathfrak{b}_1$).

We say that B is a *strict p-ring* if moreover $\mathfrak{b}_i = p^i B$ for all $i \geqslant 1$ (i.e., B is p-adically separated and complete with B/pB a perfect $\mathbf{F}_p$-algebra) and $p : B \to B$ is injective.

In addition to $\mathrm{W}(A)$ being a strict p-ring for perfect $\mathbf{F}_p$-algebras A, a wide class of (generally non-strict) p-rings is given by complete local noetherian rings with a perfect residue field of characteristic $p > 0$ (taking $\mathfrak{b}_i$ to be the ith power of the maximal ideal).

Lemma 4.2.2. *Let B be a p-ring. There is a unique set-theoretic section $r_B : B/\mathfrak{b}_1 \to B$ to the reduction map such that $r_B(x^p) = r_B(x)^p$ for all $x \in B/\mathfrak{b}_1$. Moreover, r_B is multiplicative and $r_B(1) = 1$.*

Proof. This proceeds by the same method as used in the development of the theory of Witt vectors, as follows. By perfectness of the $\mathbf{F}_p$-algebra $B/\mathfrak{b}_1$, we can make sense of $x^{p^{-n}}$ for all $x \in B/\mathfrak{b}_1$ and all $n \geqslant 1$. For any choice of lift $\widehat{x^{p^{-n}}} \in B$ of $x^{p^{-n}}$, the sequence of powers $\widehat{x^{p^{-n}}}^{p^n}$ is Cauchy for the $\mathfrak{b}_1$-adic topology. Indeed, for $n' \geqslant n$ we have $\widehat{x^{p^{-n'}}}^{p^{n'-n}} \equiv \widehat{x^{p^{-n}}} \bmod \mathfrak{b}_1$, so raising to the p^n-power gives $\widehat{x^{p^{-n'}}}^{p^{n'}} \equiv \widehat{x^{p^{-n}}}^{p^n} \bmod (p\mathfrak{b}_1^n, \mathfrak{b}_1^{p^n})$ since in general if $y \equiv y' \bmod J$ for an ideal J in a ring R with $p \in J$ (such as $J = \mathfrak{b}_1$ in $R = B$) then $y^{p^n} \equiv y'^{p^n} \bmod (pJ^n, J^{p^n})$ for all $n \geqslant 1$. Since $\mathfrak{b}_1^i \subseteq \mathfrak{b}_i$ for all $i \geqslant 1$ and B is assumed to be separated and complete for the topology defined by the $\mathfrak{b}_i$'s, there is a well-defined limit

$$r_B(x) = \lim_{n\to\infty} \widehat{x^{p^{-n}}}^{p^n} \in B$$

relative to this topology. Obviously $r_B(x^p) = r_B(x)^p$. If we make another choice of lifting $\widetilde{x^{p^{-n}}}$ then the congruence $\widehat{x^{p^{-n}}} \equiv \widetilde{x^{p^{-n}}} \bmod \mathfrak{b}_1$ implies $\widehat{x^{p^{-n}}}^{p^n} \equiv \widetilde{x^{p^{-n}}}^{p^n} \bmod (p\mathfrak{b}_1^n, \mathfrak{b}_1^{p^n})$ for all $n \geqslant 1$, whence the limit $\widetilde{r}_B(x)$ constructed using these other liftings satisfies $\widetilde{r}_B(x) \equiv r_B(x) \bmod \mathfrak{b}_n$ for all $n \geqslant 1$, so $\widetilde{r}_B(x) = r_B(x)$. In other words, $r_B(x)$ is independent of the choice of liftings $\widehat{x^{p^{-n}}}$.

In particular, if ρ_B is a p-power compatible section as in the statement of the lemma then we could choose $\widehat{x^{p^{-n}}} = \rho_B(x^{p^{-n}})$ for all $n \geqslant 1$ in the construction of $r_B(x)$, so

$$\widehat{x^{p^{-n}}}^{p^n} = \rho_B((x^{p^{-n}})^{p^n}) = \rho_B(x).$$

Passing to the limit gives $r_B(x) = \rho_B(x)$. This proves the uniqueness in the lemma, so it remains to check that r_B is multiplicative and $r_B(1) = 1$. The latter condition follows from the construction, and for the multiplicativity we observe that $\widehat{(xy)^{p^{-n}}}$ can be chosen to be $r_B(x^{p^{-n}})r_B(y^{p^{-n}})$ in the construction of $r_B(xy)$, so passing to p^n-powers and then to the limit gives $r_B(x)r_B(y) = r_B(xy)$. ∎

An immediate consequence of this lemma is that in a strict p-ring B endowed with the p-adic topology (relative to which it is separated and complete), each element $b \in B$ has the unique form $b = \sum_{n\geqslant 0} r_B(b_n)p^n$ with $b_n \in B/\mathfrak{b}_1 = B/pB$. This leads to the following useful universal property of certain Witt rings.

Proposition 4.2.3. *If A is a perfect $\mathbf{F}_p$-algebra and B is a p-ring, then the natural "reduction" map $\mathrm{Hom}(\mathrm{W}(A), B) \to \mathrm{Hom}(A, B/\mathfrak{b}_1)$ (which makes sense since $A = \mathrm{W}(A)/(p)$ and $p \in \mathfrak{b}_1$) is bijective. More generally, for any strict p-ring $\mathscr{B}$, the natural map*

$$\mathrm{Hom}(\mathscr{B}, B) \to \mathrm{Hom}(\mathscr{B}/(p), B/\mathfrak{b}_1)$$

is bijective for every p-ring B.

In particular, since $\mathscr{B}$ and $\mathrm{W}(\mathscr{B}/(p))$ satisfy the same universal property in the category of p-rings for any strict p-ring $\mathscr{B}$, strict p-rings are precisely the rings of the form $\mathrm{W}(A)$ for perfect $\mathbf{F}_p$-algebras A.

Proof. Elements $\beta \in \mathscr{B}$ have the unique form $\beta = \sum_n r_{\mathscr{B}}(\beta_n)p^n$ for $\beta_n \in \mathscr{B}/(p)$. By construction, the multiplicative sections r_B and $r_{\mathscr{B}}$ are functorial with respect to any ring map $h : \mathscr{B} \to B$ and the associated reduction $\overline{h} : \mathscr{B}/(p) \to B/\mathfrak{b}_1$, so

$$h(\beta) = \sum h(r_{\mathscr{B}}(\beta_n))p^n = \sum r_B(\overline{h}(\beta_n))p^n,$$

whence h is uniquely determined by $\overline{h}$. To go in reverse and lift ring maps, we have to show that if $\overline{h} : \mathscr{B}/(p) \to B/\mathfrak{b}_1$ is a given ring map then the map of sets $\mathscr{B} \to B$ defined by

$$\beta = \sum r_{\mathscr{B}}(\beta_n)p^n \mapsto \sum r_B(\overline{h}(\beta_n))p^n$$

is a ring map. This map respects multiplicative identity elements, so we have to check additivity and multiplicativity. For this it suffices to prove quite generally that in an arbitrary p-ring C, the ring structure on a pair of elements $c = \sum r_C(c_n^{p^{-n}})p^n$ and $c' = \sum r_C({c'_n}^{p^{-n}})p^n$ with sequences $\{c_n\}$ and $\{c'_n\}$ in $C/\mathfrak{c}_1$ is given by formulas

$$c + c' = \sum r_C(S_n(c_0, \dots, c_n; c'_0, \dots, c'_n)^{p^{-n}})p^n, \quad cc' = \sum r_C(P_n(c_0, \dots, c_n; c'_0, \dots, c'_n)^{p^{-n}})p^n$$

for universal polynomials $S_n, P_n \in \mathbf{Z}[X_0, \dots, X_n; Y_0, \dots, Y_n]$. In fact, we can take S_n and P_n to be the universal nth Witt addition and multiplication polynomials in the theory of Witt vectors. The validity of such universal formulas is proved by the same arguments as in the proof of uniqueness of such Witt polynomials. ■

Let us give two important applications of Proposition 4.2.3. First of all, for a p-adic field K with (perfect) residue field k we recover the theory of its maximal unramified subextension. Indeed, since $\mathscr{O}_K$ endowed with the filtration by powers $\{\mathfrak{m}^i\}_{i \geqslant 1}$ of its maximal ideal $\mathfrak{m}$ is a p-ring, there is a unique map of rings $\mathrm{W}(k) \to \mathscr{O}_K$ lifting the identification $\mathrm{W}(k)/(p) = k = \mathscr{O}_K/\mathfrak{m}$. Since p has nonzero image in the maximal ideal $\mathfrak{m}$ of the domain $\mathscr{O}_K$, this map $\mathrm{W}(k) \to \mathscr{O}_K$ is local and injective. Moreover, $\mathscr{O}_K/(p)$ is thereby a vector space over $\mathrm{W}(k)/(p) = k$ with basis $\{1, \pi, \dots, \pi^{e-1}\}$ for a uniformizer π and $e = \mathrm{ord}_K(p)$, so by successive approximation and p-adic completeness and separatedness of $\mathscr{O}_K$ it follows that $\{\pi^i\}_{0 \leqslant i < e}$ is a $\mathrm{W}(k)$-basis of $\mathscr{O}_K$. In particular, $\mathscr{O}_K$ is a finite free module over $\mathrm{W}(k)$ of rank e, so likewise $K = \mathscr{O}_K[1/p]$ is a finite extension of $K_0 = \mathrm{W}(k)[1/p]$ of degree e, and it must be totally ramified as such since the residue fields coincide. We call K_0 the *maximal unramified subfield* of K, and for finite k this coincides with the classical notion that goes by the same name.

Remark 4.2.4. Let $\overline{k}$ denote the algebraic closure of k given by the residue field of $\mathscr{O}_{\overline{K}}$. Although $\mathscr{O}_{\overline{K}}$ is not p-adically complete – so we cannot generally embed $\mathrm{W}(\overline{k})$ into $\mathscr{O}_{\overline{K}}$ – the (non-noetherian) valuation ring $\mathscr{O}_{\mathbf{C}_K}$ is p-adically separated and complete and there is a canonical local embedding $\mathrm{W}(\overline{k}) \to \mathscr{O}_{\mathbf{C}_K}$. However, this is not directly constructed by the general formalism of p-rings since no quotient of $\mathscr{O}_{\mathbf{C}_K}$ modulo a proper ideal containing p is a perfect $\mathbf{F}_p$-algebra. Rather, since $K_0 \subseteq K$ with $[K : K_0] < \infty$, we have $\mathbf{C}_K = \mathbf{C}_{K_0}$ and $\mathrm{W}(\overline{k})$ is the valuation ring of the completion $\widehat{K_0^{\mathrm{un}}}$ of the maximal unramified extension of K_0 (with residue field $\overline{k}$). In particular, $\mathscr{O}_{\overline{K}}/(p) = \mathscr{O}_{\mathbf{C}_K}/(p)$ is not only an algebra over $\mathrm{W}(k)/(p) = k$ in a canonical manner, but also over $\mathrm{W}(\overline{k})/(p) = \overline{k}$ (as can also be proved by other methods, such as Hensel's lemma).

For a second application of Proposition 4.2.3, we require some preparations. If A is any $\mathbf{F}_p$-algebra whatsoever (e.g., $A = \mathscr{O}_{\overline{K}}/(p)$) then we can construct a canonically associated perfect $\mathbf{F}_p$-algebra $\underline{R}(A)$ as follows:

$$\underline{R}(A) = \varprojlim_{x \mapsto x^p} A = \{(x_0, x_1, \dots) \in \prod_{n \geqslant 0} A \mid x_{i+1}^p = x_i \text{ for all } i\} \tag{4.2.1}$$

with the product ring structure. This is perfect because the additive pth power map on $\underline{R}(A)$ is surjective by construction and is injective since if $(x_i) \in \underline{R}(A)$ satisfies $(x_i)^p = (0)$ then $x_{i-1} = x_i^p = 0$ for all $i \geqslant 1$ (so $(x_i) = 0$). In terms of universal properties, observe that the map $\underline{R}(A) \to A$ defined by $(x_i) \mapsto x_0$ is a map to A from a perfect $\mathbf{F}_p$-algebra, and this is final among all maps to A from perfect $\mathbf{F}_p$-algebras. The functoriality of $\underline{R}(A)$ in A is exhibited in the evident manner in terms of compatible p-power sequences.

Example 4.2.5. If A is a perfect $\mathbf{F}_p$-algebra then the canonical map $\underline{R}(A) \to A$ is an isomorphism (as follows by inspection in such cases), and the inverse map is explicitly given by $a \mapsto (a, a^{1/p}, a^{1/p^2}, \dots)$.

If F is any field of characteristic p, then $\underline{R}(F)$ is the largest perfect subfield of F. For example, $\underline{R}(\mathbf{F}_p(x)) = \mathbf{F}_p$.

We will be particularly interested in the perfect $\mathbf{F}_p$-algebra

$$R := \underline{R}(\mathscr{O}_{\overline{K}}/(p)) = \underline{R}(\mathscr{O}_{\mathbf{C}_K}/(p))$$

endowed with its natural G_K-action via functoriality. Since $\mathscr{O}_{\overline{K}}/(p)$ is canonically an algebra over the perfect field $\overline{k}$, likewise by functoriality we have a ring map

$$\overline{k} = \underline{R}(\overline{k}) \to \underline{R}(\mathscr{O}_{\overline{K}}/(p)) = R \tag{4.2.2}$$

described concretely by

$$c \mapsto (j(c), j(c^{1/p}), j(c^{1/p^2}), \dots) \tag{4.2.3}$$

where $j : \overline{k} \to \mathscr{O}_{\overline{K}}/(p)$ is the canonical (even unique) k-algebra section to the reduction map $\mathscr{O}_{\overline{K}}/(p) \twoheadrightarrow \overline{k}$. Although $\mathscr{O}_{\mathbf{C}_K}$ is p-adically separated and complete, $\mathscr{O}_{\mathbf{C}_K}/(p)$ is not perfect. If we ignore this for a moment, then the canonical G_K-equivariant map $R \to \mathscr{O}_{\mathbf{C}_K}/(p)$ defined by $(x_n) \mapsto x_0$ would uniquely lift to a ring map

$$\theta : \mathrm{W}(R) \to \mathscr{O}_{\mathbf{C}_K}$$

due to the universal property of $\mathrm{W}(R)$ in Proposition 4.2.3. It will later be shown how to actually construct a canonical such G_K-equivariant surjection θ despite the fact that we actually cannot apply Proposition 4.2.3 in this way (due to $\mathscr{O}_{\mathbf{C}_K}/(p)$ not being perfect). The induced G_K-equivariant surjection $\mathrm{W}(R)[1/p] \to \mathbf{C}_K$ via θ then solves our original motivating problem of expressing $\mathbf{C}_K$ as a G_K-equivariant quotient of a "one-dimensional" ring, and further work will enable us to replace $\mathrm{W}(R)[1/p]$ with a canonical complete discrete valuation ring.

To proceed further (e.g., to prove that R is a valuation ring with algebraically closed fraction field and to actually construct θ as above), it is necessary to investigate the properties of the ring R. This is taken up in the next section.

4.3. **Properties of R.** Although $R = \underline{R}(\mathscr{O}_{\mathbf{C}_K}/(p))$ for a p-adic field K is defined ring-theoretically in characteristic p as a ring of p-power compatible sequences, it is important that such sequences can be uniquely lifted to p-power compatible sequences in $\mathscr{O}_{\mathbf{C}_K}$ (but possibly not in $\mathscr{O}_{\overline{K}}$). This lifting process behaves well with respect to multiplication in R, but it expresses the additive structure of R in a slightly complicated manner. To explain how this lifting works, it is convenient to work more generally with any p-adically separated and complete ring (e.g., $\mathscr{O}_{\mathbf{C}_K}$ but not $\mathscr{O}_{\overline{K}}$).

Proposition 4.3.1. *Let $\mathscr{O}$ be a p-adically separated and complete ring, and let $\mathfrak{a} \subseteq \mathscr{O}$ be an ideal containing $p\mathscr{O}$ such that $\mathfrak{a}^N \subseteq p\mathscr{O}$ for some $N \gg 0$ (i.e., the $\mathfrak{a}$-adic and p-adic topologies on $\mathscr{O}$ coincide). The multiplicative map of sets*

$$\varprojlim_{x \mapsto x^p} \mathscr{O} \to \underline{R}(\mathscr{O}/\mathfrak{a}) \tag{4.3.1}$$

defined by $(x^{(n)})_{n \geqslant 0} \mapsto (x^{(n)} \bmod \mathfrak{a})$ is bijective. Also, for any $x = (x_n) \in \underline{R}(\mathscr{O}/\mathfrak{a})$ and arbitrary lifts $\widehat{x_r} \in \mathscr{O}$ of $x_r \in \mathscr{O}/\mathfrak{a}$ for all $r \geqslant 0$, the limit $\ell_n(x) = \lim_{m\to\infty} \widehat{x_{n+m}}^{p^m}$ exists in $\mathscr{O}$ for all $n \geqslant 0$ and is independent of the choice of lifts $\widehat{x_r}$. Moreover, the inverse to (4.3.1) is given by $x \mapsto (\ell_n(x))$.

In particular, $\underline{R}(\mathscr{O}/p\mathscr{O}) \to \underline{R}(\mathscr{O}/\mathfrak{a})$ is an isomorphism, and this common ring is a domain if $\mathscr{O}$ is a domain.

Proof. The given map of sets $\varprojlim \mathscr{O} \to \underline{R}(\mathscr{O}/\mathfrak{a})$ makes sense and is multiplicative, and to make sense of the proposed inverse map we observe that for each $n \geqslant 0$ and $m' \geqslant m \geqslant 0$ we have

$$\widehat{x_{n+m'}}^{p^{m'-m}} \equiv \widehat{x_{n+m}} \bmod p\mathscr{O},$$

so $\widehat{x_{n+m'}}^{p^{m'}} \equiv \widehat{x_{n+m}}^{p^m} \bmod p^{m+1}\mathscr{O}$. Hence, the limit $\ell_n(x)$ makes sense for each $n \geqslant 0$, and the same argument as in the proof of Lemma 4.2.2 shows that $\ell_n(x)$ is independent of the choice of liftings $\widehat{x_r}$. The proposed inverse map $x \mapsto (\ell_n(x))$ is therefore well-defined, and in view of it being independent of the liftings we see that it is indeed an inverse map. ∎

In what follows, for any $x \in \underline{R}(\mathscr{O}/\mathfrak{a}) = \underline{R}(\mathscr{O}/p\mathscr{O})$ as in Proposition 4.3.1 we write $x^{(n)} \in \mathscr{O}$ to denote the limit $\ell_n(x) = \lim_{m\to\infty} \widehat{x_{n+m}}^{p^m}$ for all $n \geqslant 0$.

Remark 4.3.2. The bijection in Proposition 4.3.1 allows us to transfer the natural $\mathbf{F}_p$-algebra structure on $\underline{R}(\mathscr{O}/\mathfrak{a})$ over to such a structure on the inverse limit set $\varprojlim \mathscr{O}$ of p-power compatible sequences $x = (x^{(n)})_{n \geqslant 0}$ in $\mathscr{O}$. The multiplicative structure translate through this bijection as $(xy)^{(n)} = x^{(n)}y^{(n)}$. For addition, the proof of the proposition gives

$$(x+y)^{(n)} = \lim_{m\to\infty} (x^{(n+m)} + y^{(n+m)})^{p^m}.$$

Also, if p is odd then $(-1)^p = -1$ in $\mathscr{O}$, so $(-x^{(n)})$ is a p-power compatible sequence for any x. Hence, from the description of the additive structure we see that $(-x)^{(n)} = -x^{(n)}$ for all

$n \geqslant 0$ and all x when $p \neq 2$. This argument fails to work if $p = 2$, but then $(-x)^{(n)} = x^{(n)}$ for all $n \geqslant 0$ since $-x = x$ in such cases (as $\underline{R}(\mathscr{O}/\mathfrak{a})$ is an $\mathbf{F}_2$-algebra if $p = 2$).

We now fix a p-adic field K and let R denote the perfect domain $\underline{R}(\mathscr{O}_{\overline{K}}/(p)) = \underline{R}(\mathscr{O}_{\mathbf{C}_K}/(p))$ of characteristic p. An element $x \in R$ will be denoted $(x_n)_{n\geqslant 0}$ when we wish to view its p-power compatible components as elements of $\mathscr{O}_{\mathbf{C}_K}/(p)$ and we use the notation $(x^{(n)})_{n\geqslant 0}$ to denote its unique representation using a p-power compatible sequence of elements $x^{(n)} \in \mathscr{O}_{\mathbf{C}_K}$. An element $x \in R$ is a unit if and only if the component $x_0 \in \mathscr{O}_{\overline{K}}/(p)$ is a unit, so R is a local ring. Also, since every element of $\mathscr{O}_{\overline{K}}$ is a square, it follows (e.g., via Proposition 4.3.1) that the nonzero maximal ideal $\mathfrak{m}$ of R satisfies $\mathfrak{m} = \mathfrak{m}^2$. In particular, R is not noetherian. The ring R has several non-obvious properties which are used throughout the development of p-adic Hodge theory, and the remainder of this section is devoted to stating and proving these properties.

Lemma 4.3.3. *Let $|\cdot|_p : \mathbf{C}_K \twoheadrightarrow p^{\mathbf{Q}} \cup \{0\}$ be the normalized absolute value satisfying $|p|_p = 1/p$. The map $|\cdot|_R : R \to p^{\mathbf{Q}} \cup \{0\}$ defined by $x = (x^{(n)}) \mapsto |x^{(0)}|_p$ is a G_K-equivariant absolute value on R that makes R the valuation ring for the unique valuation v_R on* $\mathrm{Frac}(R)$ *extending $-\log_p|\cdot|_R$ on R (and having value group $\mathbf{Q}$).*

Also, R is v_R-adically separated and complete, and the subfield $\overline{k}$ of R maps isomorphically onto the residue field of R.

Proof. Obviously $x^{(0)} = 0$ if and only if $x = 0$, and $|xy|_R = |x|_R|y|_R$ since $(xy)^{(0)} = x^{(0)}y^{(0)}$.

To show that $|x+y|_R \leqslant \max(|x|_R, |y|_R)$ for all $x, y \in R$, we may assume $x, y \neq 0$, so $x^{(0)}, y^{(0)} \neq 0$. By symmetry we may assume $|x^{(0)}|_p \leqslant |y^{(0)}|_p$, so for all $n \geqslant 0$ we have

$$|x^{(n)}|_p = |x^{(0)}|_p^{p^{-n}} \leqslant |y^{(0)}|_p^{p^{-n}} = |y^{(n)}|_p.$$

The ratios $x^{(n)}/y^{(n)}$ therefore lie in $\mathscr{O}_{\mathbf{C}_K}$ for $n \geqslant 0$ and form a p-power compatible sequence. This sequence is therefore an element $z \in R$, and $yz = x$ in R so $y|x$ in R. Hence,

$$|x+y|_R = |y(z+1)|_R = |y|_R|z+1|_R \leqslant |y|_R \leqslant \max(|x|_R, |y|_R).$$

The same argument shows that R is the valuation ring of v_R on $\mathrm{Frac}(R)$.

To prove $|\cdot|_R$-completeness of R, first note that if we let $v = -\log_p|\cdot|_p$ on $\mathbf{C}_K$ then $v_R(x) = v(x^{(0)}) = p^n v(x^{(n)})$ for $n \geqslant 0$. Thus, $v_R(x) \geqslant p^n$ if and only if $v(x^{(n)}) \geqslant 1$ if and only if $x^{(n)} \bmod p = 0$. Hence, if we let

$$\theta_n : R \to \mathscr{O}_{\mathbf{C}_K}/(p)$$

denote the ring homomorphism $x = (x_m)_{m\geqslant 0} \mapsto x_n$ then $\{x \in R \,|\, v_R(x) \geqslant p^n\} = \ker\theta_n$. In view of how the inverse limit R sits within the product space $\prod_{m\geqslant 0}(\mathscr{O}_{\mathbf{C}_K}/(p))$, or more specifically since $x_n = 0$ implies $x_m = 0$ for all $m \leqslant n$, we conclude that the v_R-adic topology on R coincides with its subspace topology within $\prod_{m\geqslant 0}(\mathscr{O}_{\mathbf{C}_K}/(p))$ where the factors are given the discrete topology, so the v_R-adic completeness follows (as R is closed in this product space due to the definition of $R = \underline{R}(\mathscr{O}_{\mathbf{C}_K}/(p))$).

Finally, the definition of the k-embedding of $\overline{k}$ into R in (4.2.2) implies that $\theta_0 : R \twoheadrightarrow \mathscr{O}_{\mathbf{C}_K}/(p)$ is a $\overline{k}$-algebra map, but θ_0 is local and so induces an injection on residue fields. Since $\overline{k} \to \mathscr{O}_{\mathbf{C}_K}/(p)$ induces an isomorphism on residue fields, we are done. ∎

For $x = (x^{(n)})$ and $y = (y^{(n)})$ in R, we have $x^{(n)} \equiv y^{(n)} \bmod p$ if and only if $x^{(i)} \equiv y^{(i)} \bmod p^{n-i+1}$ for all $0 \leqslant i \leqslant n$, so the v_R-adic topology on R also coincides with its closed subspace topology from sitting as a multiplicative inverse limit within $\prod_{n \geqslant 0} \mathscr{O}_{\mathbf{C}_K}$ where each factor is given the p-adic topology. This gives an alternative way of seeing the v_R-adic completeness of R.

Example 4.3.4. An important example of an element of R is

$$\varepsilon = (\varepsilon^{(n)})_{n \geqslant 0} = (1, \zeta_p, \zeta_{p^2}, \dots)$$

with $\varepsilon^{(0)} = 1$ but $\varepsilon^{(1)} \neq 1$ (so $\varepsilon^{(1)} = \zeta_p$ is a primitive pth root of unity and hence $\varepsilon^{(n)}$ is a primitive p^nth root of unity for all $n \geqslant 0$). Any two such elements are $\mathbf{Z}_p^\times$-powers of each other. For any such choice of element we claim that

$$v_R(\varepsilon - 1) = \frac{p}{p-1}.$$

To see this, by definition we have $v_R(\varepsilon - 1) = v((\varepsilon - 1)^{(0)})$ where $v = \mathrm{ord}_p = -\log_p |\cdot|_p$, so we need to describe $(\varepsilon - 1)^{(0)} \in \mathscr{O}_{\mathbf{C}_K}$. By Remark 4.3.2, in $\mathscr{O}_{\mathbf{C}_K}$ we have

$$(\varepsilon - 1)^{(0)} = \lim_{n \to \infty} (\varepsilon^{(n)} + (-1)^{(n)})^{p^n},$$

with $\varepsilon^{(n)} = \zeta_{p^n}$ a primitive p^nth root of unity in $\overline{K}$ and $(-1)^{(n)} = -1$ if $p \neq 2$ whereas $(-1)^{(n)} = 1$ if $p = 2$. We shall separately treat the cases of odd p and $p = 2$.

If p is odd then

$$v_R(\varepsilon - 1) = \lim_{n \to \infty} p^n \,\mathrm{ord}_p(\zeta_{p^n} - 1) = \lim_{n \to \infty} \frac{p^n}{p^{n-1}(p-1)} = \frac{p}{p-1}.$$

If $p = 2$ then

$$v_R(\varepsilon - 1) = \lim_{n \to \infty} 2^n \,\mathrm{ord}_2(\zeta_{2^n} + 1) = \lim_{n \to \infty} 2^n \,\mathrm{ord}_2((\zeta_{2^n} - 1) + 2).$$

Since $\mathrm{ord}_2(\zeta_{2^n} - 1) = 1/2^{n-1} < \mathrm{ord}_2(2)$ for $n > 1$, we have $\mathrm{ord}_2((\zeta_{2^n} - 1) + 2) = \mathrm{ord}_2(\zeta_{2^n} - 1)$ for $n > 1$, so we may conclude as for odd p.

Theorem 4.3.5. *The field* $\mathrm{Frac}(R)$ *of characteristic* p *is algebraically closed.*

Proof. Since R is a valuation ring, it suffices to construct a root in R for any monic polynomial $f \in R[X]$ with $d = \deg P > 0$. We may and do assume $d \geqslant 2$.

For each $m \geqslant 1$, consider the ring map $\theta_m : R \to \mathscr{O}_{\mathbf{C}_K}/(p)$ defined by $x = (x_i) \mapsto x_m$. Let $f_m = \theta_m(f) \in (\mathscr{O}_{\mathbf{C}_K}/(p))[X]$ (apply θ_m to coefficients). This is a monic polynomial of degree d, so it lifts to a monic polynomial $\widetilde{f}_m \in \mathscr{O}_{\mathbf{C}_K}[X]$ of degree d. Since $\mathscr{O}_{\mathbf{C}_K}$ is the valuation ring of the field $\mathbf{C}_K$ that is algebraically closed (Proposition 2.1.1), $\widetilde{f}_m$ admits a set of d roots (with multiplicity) $\{\rho_{1,m}, \dots, \rho_{d,m}\}$ in $\mathscr{O}_{\mathbf{C}_K}$. The reductions $\overline{\rho}_{i,m}$ of these modulo p are roots of $f_m = \theta_m(f)$, and if we could arrange a p-power compatible sequence of these as $m \to \infty$ we could get the desired root of f in $R = \varprojlim \mathscr{O}_{\mathbf{C}_K}/(p)$. Since the p-power map on $\mathscr{O}_{\mathbf{C}_K}/(p)$ is a ring homomorphism carrying the map θ_{m+1} to the map θ_m (by the definition of R), the p-powers $\overline{\rho}_{i,m+1}^p$ are roots of f_m. The problem is that $\mathscr{O}_{\mathbf{C}_K}/(p)$ is not a domain, and so f_m always has infinitely many roots. In particular, the $\overline{\rho}_{i,m}$'s are not the only roots of f_m, so we cannot conclude that every $\overline{\rho}_{i,m+1}^p$ is equal to some $\overline{\rho}_{i',m}$. If we did have such a conclusion then the sets $\{\overline{\rho}_{i,m}\}_{1\leqslant i\leqslant d}$ would form an inverse system of non-empty finite sets via the p-power map, so there would have to be a compatible system via the pigeonhole principle and hence we would get the root we seek.

To circumvent the infinitude of roots, we use a nice trick observed by Coleman. The key point is that the $\overline{\rho}_{i,m}$'s are not merely roots of f_m but are actually reductions of roots from $\mathscr{O}_{\mathbf{C}_K}$, where finiteness for the set of roots *does* hold. We will exploit a big p-power mapping to transfer this property to characteristic p, as follows. Since $f_m(\overline{\rho}_{i,m+1}^p) = f_{m+1}(\overline{\rho}_{i,m+1})^p = 0$ in $\mathscr{O}_{\mathbf{C}_K}/(p)$, we have $\widetilde{f}_m(\rho_{i,m+1}^p) \in p\mathscr{O}_{\mathbf{C}_K}$ for each $1 \leqslant i \leqslant d$. But $\widetilde{f}_m = \prod_j (X - \rho_{j,m})$, so for each i we have $\prod_j(\rho_{i,m+1}^p - \rho_{j,m}) \in p\mathscr{O}_{\mathbf{C}_K}$. There are d terms in the product, so at least one of them, say $\rho_{i,m+1}^p - \rho_{j(i),m}$, lies in $p^{1/d}\mathscr{O}_{\mathbf{C}_K}$. In other words, for each $1 \leqslant i \leqslant d$ there exists $1 \leqslant j(i) \leqslant d$ such that $\rho_{i,m+1}^p \equiv \rho_{j(i),m} \bmod p^{1/d}\mathscr{O}_{\mathbf{C}_K}$. By a straightforward calculation, any congruence $a \equiv b \bmod p^{r/d}\mathscr{O}_{\mathbf{C}_K}$ with $1 \leqslant r < d$ implies $a^p \equiv b^p \bmod p^{(r+1)/d}\mathscr{O}_{\mathbf{C}_K}$. Applying this repeatedly, we conclude that $\rho_{i,m+1}^{p^d} \equiv \rho_{j(i),m}^{p^{d-1}} \bmod p\mathscr{O}_{\mathbf{C}_K}$. In other words, $\overline{\rho}_{i,m+1}^{p^d} = \overline{\rho}_{j(i),m}^{p^{d-1}}$. Hence, the finite sets

$$\{\overline{\rho}_{1,m+1}^{p^{d-1}}, \dots, \overline{\rho}_{d,m+1}^{p^{d-1}}\}$$

do form a compatible system under the p-power maps. These p^{d-1}-powers of roots of f_{m+1} are roots of $f_{m+1-(d-1)} = f_{m-d+2}$. In other words, if we define $x_{i,m} = \overline{\rho}_{i,m+d-2}^{p^{d-1}} \in \mathscr{O}_{\mathbf{C}_K}/(p)$ for $m \geqslant 0$ and $1 \leqslant i \leqslant d$ (which makes sense even for $m = 0$ since we arranged that $d \geqslant 2$), then the non-empty finite sets $\{x_{1,m}, \dots, x_{d,m}\}$ for $m \geqslant 0$ form an inverse system under the p-power mapping. By the pigeonhole principle we may therefore select $x_{i(m),m}$ for each $m \geqslant 0$ such that $x_{i(m+1),m+1}^p = x_{i(m),m}$ for all $m \geqslant 0$. Hence, $x = (x_{i(m),m}) \in R$ and $\rho_m(f(x)) = f_m(x_{i(m),m}) = 0$ for all m, so $f(x) = 0$ in R. ∎

Consider an element $\varepsilon \in R$ as in Example 4.3.4 (so $\varepsilon^{(0)} = 1$ and $\varepsilon^{(1)} \neq 1$). Thus, $\theta_0(\varepsilon) = 1 \in \mathscr{O}_{\mathbf{C}_K}/(p)$, so the image of ε in the residue field $\overline{k}$ of R is 1. Hence, $\varepsilon - 1$ lies in the maximal ideal $\mathfrak{m}_R$ of R, which we knew anyway from Example 4.3.4 since there we proved $v_R(\varepsilon-1) = p/(p-1) > 0$. By the completeness of R, we get a unique local k-algebra map $k[\![u]\!] \to R$ satisfying $u \mapsto \varepsilon - 1 \neq 0$. This map depends on the choice of ε, but its image does not:

Lemma 4.3.6. *The image of $k[\![u]\!]$ in R is independent of ε.*

Proof. Consider a second choice ε', so $\varepsilon' = \varepsilon^a$ for some $a \in \mathbf{Z}_p^\times$. (Note that ε lies in the multiplicative group $1 + \mathfrak{m}_R$ whose "strict" neighborhoods of 1 are $1 + c\mathfrak{m}_R$ with $v_R(c) > 0$, and these are p-adically separated and complete, so $\mathbf{Z}_p$-exponentiation on $1 + \mathfrak{m}_R$ makes sense.) Letting $x = \varepsilon - 1$ and $x' = \varepsilon' - 1$ in $\mathfrak{m}_R$, we can compute formally

$$x' = \varepsilon^a - 1 = (1+x)^a - 1 = ax + \dots$$

in R. Rigorously, the unique local k-algebra self-map of $k[\![u]\!]$ satisfying $u \mapsto (1+u)^a - 1$ carries the map $k[\![u]\!] \to R$ resting on ε to the one resting on ε'. But this self-map is an automorphism since $(1+u)^a - 1 = au + \dots$ with $a \in \mathbf{Z}_p^\times$. ∎

In view of the lemma, we may define the canonical subfield $E \subseteq \mathrm{Frac}(R)$ to be the fraction field of the canonical image of $k[\![u]\!]$ in R for any choice of ε as in Lemma 4.3.6. By Theorem

4.3.5, the separable closure E_s of E within $\mathrm{Frac}(R)$ is a separable closure of E. The action of the Galois group G_K on R extends uniquely to an action on $\mathrm{Frac}(R)$ (preserving v_R), and this does not fix the image $\varepsilon - 1$ of u. However, for the extension $K_\infty = K(\mu_{p^\infty})$ generated by the components $\varepsilon^{(n)}$ of ε (for all choices of ε) we see that the subgroup $G_{K_\infty} \subseteq G_K$ is the isotropy group of $\varepsilon - 1 \in R$ and so is the isotropy group of the intrinsic subfield $E \subseteq \mathrm{Frac}(R)$. Hence, G_{K_∞} preserves the separable closure $E_s \subseteq \mathrm{Frac}(R)$, so we get a group homomorphism

$$G_{K_\infty} \to \mathrm{Aut}(E_s/E) = G_E.$$

Lemma 4.3.7. *The map of Galois groups $G_{K_\infty} \to G_E$ is continuous.*

Proof. Fix a finite Galois extension E' of E inside of $E_s \subseteq \mathrm{Frac}(R)$. We may choose a primitive element $x \in E'^\times$ for E' over E. By replacing x with $1/x$ if necessary, we can arrange that $x \in R$. The algebraicity of x over E implies that the G_{K_∞}-orbit of x is finite, say $\{x = x_1, \dots, x_n\}$, with all $x_i \in R$. To find an open subgroup of G_{K_∞} that has trivial image in $\mathrm{Gal}(E'/E)$, or equivalently lands in $G_{E'} \subseteq G_E$, we just need to show that if $g \in G_{K_\infty}$ is sufficiently close to 1 then $g(x)$ is distinct from the finitely many elements $x_2, \dots, x_n$ that are distinct from x (forcing $g(x) = x$). The existence of such a neighborhood of the identity is immediate from the continuity of the action of G_K on the Hausdorff space R. ■

A much deeper fact that is best understood as part of the theory of norm fields is that the continuous map in Lemma 4.3.7 is in fact bijective and so is a topological isomorphism. Even better, there is a functorial equivalence between the categories of finite separable extensions of K_∞ and of E. This is a concrete realization of a special case of the general isomorphism in (1.3.1), and it will be proved in §13.4 (see Theorem 13.4.3).

4.4. **The field of p-adic periods B_{dR}.** We have now assembled enough work to carry out the first important refinement on the graded ring B_{HT}, namely the construction of the field of p-adic periods B_{dR} as promised in the discussion following Example 4.1.2. Inspired by the universal property of Witt vectors in Proposition 4.2.3 and the perfectness of the $\mathbf{F}_p$-algebra R, we seek to lift the G_K-equivariant surjective ring map $\theta_0 : R \to \mathscr{O}_{\mathbf{C}_K}/(p)$ defined by $(x_i) \mapsto x_0$ to a G_K-equivariant surjective ring map $\theta : \mathrm{W}(R) \to \mathscr{O}_{\mathbf{C}_K}$. As we have already observed, although $\mathscr{O}_{\mathbf{C}_K}$ is p-adically separated and complete, we cannot use Proposition 4.2.3 because $\mathscr{O}_{\mathbf{C}_K}/(p)$ is not perfect. Nonetheless, we will construct such a θ in a canonical (in particular, G_K-equivariant) manner.

Our definition for θ as a set-theoretic map is simple and explicit:

$$\theta(\sum [c_n]p^n) = \sum c_n^{(0)} p^n.$$

(Recall that $\mathrm{W}(R)$ is a strict p-ring with $\mathrm{W}(R)/(p) = R$, so each of its elements has the unique form $\sum [c_n]p^n$ with $c_n \in R$.) This is very much in the spirit of the proof of Proposition 4.2.3 since $c^{(0)} = \lim_{m\to\infty} \widehat{c_m}^{p^m}$ for any $c \in R$ using any choice of lift $\widehat{c_m} \in \mathscr{O}_{\mathbf{C}_K}$ of $c_m \in \mathscr{O}_{\mathbf{C}_K}/(p)$ (with $\{c_m\}$ a compatible sequence of p-power roots of $c_0 \in \mathscr{O}_{\mathbf{C}_K}/(p)$). In terms of the Witt coordinatization $(r_0, r_1, \dots) = \sum p^n [r_n^{p^{-n}}]$ this says $\theta : (r_0, r_1, \dots) \mapsto \sum (r_n^{p^{-n}})^{(0)} p^n$, but for any $r \in R$ we have $(r^{p^{-n}})^{(0)} = ((r^{p^{-n}})^{(n)})^{p^n} = r^{(n)}$ in $\mathscr{O}_{\mathbf{C}_K}$ since $r \mapsto r^{(n)}$ is multiplicative. Hence, we have the formula $\theta : (r_0, r_1, \dots) \mapsto \sum r_n^{(n)} p^n$. By definition θ is G_K-equivariant, and the only real issue is to check that it is a ring map:

Lemma 4.4.1. *The map* $\theta : \mathrm{W}(R) \to \mathscr{O}_{\mathbf{C}_K}$ *is a ring homomorphism.*

Proof. It suffices to prove that

$$\theta_n = \theta \bmod p^n : \mathrm{W}_n(R) = \mathrm{W}(R)/p^n\,\mathrm{W}(R) \to \mathscr{O}_{\mathbf{C}_K}/p^n\mathscr{O}_{\mathbf{C}_K} = \mathscr{O}_{\overline{K}}/p^n\mathscr{O}_{\overline{K}}$$

is a ring map for all $n \geqslant 1$. Once additivity is established, both side of the multiplicativity identity $\theta_n(ww') = \theta_n(w)\theta_n(w')$ depend $\mathbf{Z}$-bilinearly on (w, w') and so via Teichmüller expansions the verification of this identity is reduced to the case $w = [r]$ and $w' = [r']$:

$$\theta([r][r']) = \theta([rr']) = (rr')^{(0)} = r^{(0)}r'^{(0)} = \theta([r])\theta([r']).$$

Hence, we just have to check that each θ_n is additive.

Writing $w = (x_0, \ldots, x_{n-1})$ with $x_i \in R$, by definition

$$\theta_n(w) = \sum_{i=0}^{n-1} p^i x_i^{(i)} = \sum_{i=0}^{n-i} p^i (x_i^{(n)})^{p^{n-1}} = \Phi_n(x_0^{(n)} \bmod p^n, \ldots, x_{n-1}^{(n)} \bmod p^n)$$

where $\Phi_n : \mathrm{W}_n(\mathscr{O}_{\overline{K}}/p^n\mathscr{O}_{\overline{K}}) \to \mathscr{O}_{\overline{K}}/p^n\mathscr{O}_{\overline{K}}$ is the "nth ghost component" map defined by

$$(z_0, \ldots, z_{n-1}) \mapsto \sum_{i=0}^{n-1} p^i z_i^{p^{n-i}}.$$

By the very definition of the additive structure on $\mathrm{W}_n(A)$ for any ring A, the map Φ_n is additive. But $\Phi_n(z_0, \ldots, z_{n-1})$ only depends on the $z_i \bmod p$ since if $a \equiv b \bmod p$ then $a^{p^{n-i}} \equiv b^{p^{n-i}} \bmod p^{n+1-i}$ and so $p^i a^{p^{n-i}} \equiv p^i b^{p^{n-i}} \bmod p^n$. In other words, Φ_n factors as $\overline{\Phi}_n \circ \pi_n$ where $\pi_n : \mathrm{W}_n(\mathscr{O}_{\overline{K}}/p^n\mathscr{O}_{\overline{K}}) \twoheadrightarrow \mathrm{W}_n(\mathscr{O}_{\overline{K}}/p\mathscr{O}_{\overline{K}})$ is the natural quotient map and $\overline{\Phi}_n : \mathrm{W}_n(\mathscr{O}_{\overline{K}}/p\mathscr{O}_{\overline{K}}) \to \mathscr{O}_{\overline{K}}/p^n\mathscr{O}_{\overline{K}}$ is the map of sets $(\overline{z}_0, \ldots, \overline{z}_{n-1}) \mapsto \sum_{i=0}^{n-1} p^i z_i^{p^{n-i}}$ where $z_i \in \mathscr{O}_{\overline{K}}/p^n\mathscr{O}_{\overline{K}}$ is a lift of $\overline{z}_i$.

Since π_n is surjective and additive (by functoriality of the additive structure on W_n) and Φ_n is additive, $\overline{\Phi}_n$ is also additive. Letting $f_n : R \to \mathscr{O}_{\overline{K}}/p\mathscr{O}_{\overline{K}}$ denote the projection $r \mapsto r^{(n)} \bmod p$ to the nth member of the p-power compatible system that "is" r, we have

$$\theta_n = \overline{\Phi}_n \circ \mathrm{W}_n(f_n).$$

The map $\mathrm{W}_n(f_n)$ is additive since f_n is a ring homomorphism and the additive structure on W_n is functorial in ring homomorphisms, and we have just seen that $\overline{\Phi}_n$ is additive. We conclude that θ_n is additive as well. ■

This explicit definition of θ makes it evident that θ is surjective (since $R \to \mathscr{O}_{\mathbf{C}_K}/(p)$ via $r \mapsto r^{(n)} \bmod p$ is surjective for each $n \geqslant 0$). In concrete terms, the formula shows that θ fits into the following family of commutative diagrams:

$$\begin{tikzcd}
\mathrm{W}(R) \arrow[r, "\theta"] \arrow[d] & \mathscr{O}_{\mathbf{C}_K} \arrow[dr] & \\
\mathrm{W}_n(R) \arrow[r, "\Theta_n"'] & \mathrm{W}_n(\mathscr{O}_{\mathbf{C}_K}/(p)) \arrow[r, "\psi_n"'] & \mathscr{O}_{\mathbf{C}_K}/(p^n)
\end{tikzcd}$$

Proposition 4.4.2. *The continuous surjective G_K-equivariant map $\theta : \mathrm{W}(R) \to \mathscr{O}_{\mathbf{C}_K}$ constructed above is open. Also, using the canonical k-algebra map $j : \overline{k} \to R$ to make $\mathrm{W}(R)$ into a $\mathrm{W}(\overline{k})$-algebra via $\mathrm{W}(j)$, θ is a $\mathrm{W}(\overline{k})$-algebra map via the natural $\mathrm{W}(\overline{k})$-algebra structure on $\mathscr{O}_{\mathbf{C}_K}$.*

Proof. To prove openness, using the product of the valuation topology from R on $\mathrm{W}(R)$ and the p-adic topology on $\mathscr{O}_{\mathbf{C}_K}$, we just have to show that if J is an open ideal in R then the image under θ of the additive subgroup of vectors (r_i) with $r_0, \dots, r_n \in J$ (for fixed n) is open in $\mathscr{O}_{\mathbf{C}_K}$. This image is $J^{(0)} + pJ^{(1)} + \cdots + p^{n-1}J^{(n-1)}$, where $J^{(m)}$ is the image of J under the map of sets $R \to \mathscr{O}_{\mathbf{C}_K}$ defined by $r \mapsto r^{(m)}$. Since $\mathscr{O}_{\mathbf{C}_K}$ has the p-adic topology, it suffices to show that $J^{(m)}$ is open in $\mathscr{O}_{\mathbf{C}_K}$ for each $m \geqslant 0$. But $J^{(m)} = (J^{p^m})^{(0)}$, so to prove that θ is open we just have to show that if J is an open ideal in R then $J^{(0)}$ is open in $\mathscr{O}_{\mathbf{C}_K}$. It is enough to work with J's running through a base of open ideals, so we take $J = \{r \in R \,|\, v_R(r) \geqslant c\}$ with $c \in \mathbf{Q}$. Since $v_R(r) = v(r^{(0)})$ and the map $r \mapsto r^{(0)}$ is a surjection from R onto $\mathscr{O}_{\mathbf{C}_K}$, for such J we have that $J^{(0)} = \{t \in \mathscr{O}_{\mathbf{C}_K} \,|\, v(t) \geqslant c\}$, which is certainly open in $\mathscr{O}_{\mathbf{C}_K}$. This concludes the proof that θ is an open map.

Next, consider the claim that θ is a map of $\mathrm{W}(\overline{k})$-algebras. Recall that $\mathscr{O}_{\mathbf{C}_K}$ is made into a $\mathrm{W}(\overline{k})$-algebra via the unique continuous $\mathrm{W}(k)$-algebra map $h : \mathrm{W}(\overline{k}) \to \mathscr{O}_{\mathbf{C}_K}$ lifting the identity map on $\overline{k}$ at the level of residue fields. (By such continuity and the p-adic separatedness and completeness of $\mathscr{O}_{\mathbf{C}_K}$, the existence and uniqueness of such an h is reduced to the case when $\overline{k}$ is replaced with a finite extension k'/k, and the unique $\mathrm{W}(k)$-algebra map $\mathrm{W}(k') \to \mathscr{O}_{\mathbf{C}_K}$ lifting the inclusion $k' \to \overline{k}$ is built as follows: by $\mathrm{W}(k)$-finiteness it must land in the valuation ring of a *finite* extension of K if it exists, so we can pass to the case when the target is a complete discrete valuation ring, whence the universal property of $\mathrm{W}(k')$ can be used. Concretely, $\mathrm{W}(k')$ is just a finite unramified extension of $\mathrm{W}(k)$ within $\overline{K}$, the point being that the map on residue fields uniquely determines the map in characteristic 0.) Using p-adic continuity, it is enough to chase Teichmüller digits.

Our problem is now to show that for each $c \in \overline{k}$ the image $h([c])$ is equal to $\theta([j(c)])$, where $j : \overline{k} \to R$ is the canonical k-algebra map defined by $c \mapsto (c^{1/p^m})_{m \geqslant 0} \in \underline{R}(\mathscr{O}_{\overline{K}}/(p)) = R$ and we view $\mathscr{O}_{\overline{K}}/(p)$ as a $\overline{k}$-algebra over its k-algebra structure via Hensel's Lemma. The key point is that c viewed in $\mathscr{O}_{\overline{K}}/(p) = \mathscr{O}_{\mathbf{C}_K}/(p)$ is just $h([c]) \bmod p$ (check!), so $j(c) = (h([c^{1/p^m}]) \bmod p) \in R$. Since the sequence of elements $h([c^{1/p^m}])$ in $\mathscr{O}_{\mathbf{C}_K}$ is p-power compatible, $j(c)^{(0)} = h([c])$. Thus, $\theta([j(c)]) = j(c)^{(0)} = h([c])$. ■

We now have a G_K-equivariant surjective ring homomorphism

$$\theta_{\mathbf{Q}} : \mathrm{W}(R)[1/p] \twoheadrightarrow \mathscr{O}_{\mathbf{C}_K}[1/p] = \mathbf{C}_K,$$

but the source ring is not a complete discrete valuation ring. We shall replace $\mathrm{W}(R)[1/p]$ with its $\ker \theta_{\mathbf{Q}}$-adic completion, and the reason this works is that $\ker \theta_{\mathbf{Q}} = (\ker \theta)[1/p]$ turns out to be a principal ideal. We now record some facts about $\ker \theta$.

Proposition 4.4.3. *Choose $\widetilde{p} \in R$ such that $\widetilde{p}^{(0)} = p$ (i.e., $\widetilde{p} = (p, p^{1/p}, p^{1/p^2}, \dots) \in \varprojlim_{x \mapsto x^p} \mathscr{O}_{\mathbf{C}_K} = R$, so $v_R(\widetilde{p}) = 1$) and let $\xi = \xi_{\widetilde{p}} = [\widetilde{p}] - p = (\widetilde{p}, -1, \dots) \in \mathrm{W}(R)$.*

(1) *The ideal $\ker \theta \subseteq \mathrm{W}(R)$ is the principal ideal generated by ξ.*
(2) *An element $w = (r_0, r_1, \dots) \in \ker \theta$ is a generator of $\ker \theta$ if and only if $r_1 \in R^\times$.*

A defect of ξ, despite its explicitness, is that G_K does not act on ξ in a nice way (but it does preserve $\xi \cdot \mathrm{W}(R) = \ker\theta$). This will be remedied after replacing $\mathrm{W}(R)[1/p]$ with its $\ker\theta_{\mathbf{Q}}$-adic completion.

Proof. Clearly $\theta(\xi) = \theta([\widetilde{p}]) - p = \widetilde{p}^{(0)} - p = 0$ and $\ker\theta \cap p^n\,\mathrm{W}(R) = p^n \cdot \ker\theta$ since $\mathrm{W}(R)/(\ker\theta) = \mathscr{O}_{\mathbf{C}_K}$ has no nonzero p-torsion. Since $\mathrm{W}(R)$ is p-adically separated and complete (as R is a perfect domain, so the p-adic topology on $\mathrm{W}(R)$ is just the product topology on $\mathrm{W}(R)$ using the discrete topology of R), to prove that ξ is a principal generator of $\ker\theta$ it therefore suffices to show $\ker\theta \subseteq (\xi, p) = ([\widetilde{p}], p)$. But if $w = (r_0, r_1, \dots) \in \ker\theta$ then $r_0^{(0)} \equiv 0 \bmod p$, so $v_R(r_0) = \mathrm{ord}_p(r_0^{(0)}) \geqslant 1 = v_R(\widetilde{p})$ and hence $r_0 \in \widetilde{p}R$. We conclude that $w \in ([r_0], p) \subseteq ([\widetilde{p}], p)$, as desired.

A general element $w = (r_0, r_1, \dots) \in \ker\theta$ has the form

$$w = \xi \cdot (r_0', r_1', \dots) = (\widetilde{p}, -1, \dots)(r_0', r_1', \dots) = (\widetilde{p}r_0', \widetilde{p}^p r_1' - r_0'^{\,p}, \dots),$$

so $r_1 = \widetilde{p}^p r_1' - r_0'^{\,p}$. Hence, $r_1 \in R^\times$ if and only if $r_0' \in R^\times$, and this final unit condition is equivalent to the multiplier $(r_0', r_1', \dots)$ being a unit in $\mathrm{W}(R)$, which amounts to w being a principal generator of $\ker\theta$ (since $\mathrm{W}(R)$ is a domain). ■

Example 4.4.4. Using the criterion in Proposition 4.4.3(2), prove that the element $\varepsilon - 1 \in \ker\theta$ is a generator when $p = 2$. Then prove this is *false* whenever $p > 2$ (hint: $v_R(\varepsilon) = p/(p-1)$ for all p).

Corollary 4.4.5. *For all $j \geqslant 1$,*

$$\mathrm{W}(R) \cap (\ker\theta_{\mathbf{Q}})^j = (\ker\theta)^j.$$

Also, $\cap(\ker\theta)^j = \cap(\ker\theta_{\mathbf{Q}})^j = 0$.

Proof. By a simple induction on j and chasing multiples of ξ, to prove the displayed equality it suffices to check the case $j = 1$. This case holds since $\mathrm{W}(R)/(\ker\theta) = \mathscr{O}_{\mathbf{C}_K}$ has no nonzero p-torsion.

Since any element of $\mathrm{W}(R)[1/p]$ admits a p-power multiple in $\mathrm{W}(R)$, we conclude that

$$\cap(\ker\theta_{\mathbf{Q}})^j = (\cap(\ker\theta)^j)[1/p].$$

To prove this vanishes, it suffices to consider an arbitrary $w = (r_0, r_1, \dots) \in \mathrm{W}(R)$ lying in $\cap(\ker\theta)^j$. Thus, w is divisible by arbitrarily high powers of $\xi = [\widetilde{p}] - p = (\widetilde{p}, -1, \dots)$, so r_0 is divisible by arbitrarily high powers of $\widetilde{p}$ in R. But $v_R(\widetilde{p}) = 1 > 0$, so by v_R-adic separatedness of R we see that $r_0 = 0$. This says that $w = pw'$ for some $w' \in \mathrm{W}(R)$ since R is a perfect $\mathbf{F}_p$-algebra. Hence, $w' \in (\cap(\ker\theta)^j)[1/p] = \cap(\ker\theta_{\mathbf{Q}})^j$. Thus, $w' \in \mathrm{W}(R) \cap (\ker\theta_{\mathbf{Q}})^j = (\ker\theta)^j$ for all j. This shows that each element of $\cap(\ker\theta)^j$ in $\mathrm{W}(R)$ lies in $\cap p^n\,\mathrm{W}(R)$, and this vanishes since $\mathrm{W}(R)$ is a strict p-ring. ■

We conclude that $\mathrm{W}(R)[1/p]$ injects into the inverse limit

$$B_{\mathrm{dR}}^+ := \varprojlim_j \mathrm{W}(R)[1/p]/(\ker\theta_{\mathbf{Q}})^j \tag{4.4.1}$$

whose transition maps are G_K-equivariant, so B_{dR}^+ has a natural G_K-action that is compatible with the action on its subring $\mathrm{W}(R)[1/p]$. (Beware that in (4.4.1) we cannot move the p-localization outside of the inverse limit: algebraic localization and inverse limit do not generally commute with each other, as is most easily seen when comparing the t-adic completion $\mathbf{Q}_p[\![t]\!]$ of $\mathbf{Q}_p[t] = \mathbf{Z}_p[t][1/p]$ with its subring $\mathbf{Z}_p[\![t]\!][1/p]$ of power series with "bounded denominators".) The inverse limit B^+_{dR} maps G_K-equivariantly onto each quotient $\mathrm{W}(R)[1/p]/(\ker\theta_{\mathbf{Q}})^j$ via the evident natural map, and in particular for $j = 1$ the map $\theta_{\mathbf{Q}}$ induces a natural G_K-equivariant surjective map $\theta^+_{\mathrm{dR}} : B^+_{\mathrm{dR}} \twoheadrightarrow \mathbf{C}_K$. From the definitions $\ker\theta^+_{\mathrm{dR}} \cap \mathrm{W}(R) = \ker\theta$, and $\ker\theta^+_{\mathrm{dR}} \cap \mathrm{W}(R)[1/p] = \ker\theta_{\mathbf{Q}}$ since θ^+_{dR} restricts to $\theta_{\mathbf{Q}}$ on the subring $\mathrm{W}(R)[1/p]$.

Proposition 4.4.6. *The ring B^+_{dR} is a complete discrete valuation ring with residue field $\mathbf{C}_K$, and any generator of $\ker\theta_{\mathbf{Q}}$ in $\mathrm{W}(R)[1/p]$ is a uniformizer of B^+_{dR}. The natural map $B^+_{\mathrm{dR}} \to \mathrm{W}(R)[1/p]/(\ker\theta_{\mathbf{Q}})^j$ is identified with the projection to the quotient modulo the jth power of the maximal ideal for all $j \geqslant 1$.*

Proof. Since $\ker\theta_{\mathbf{Q}}$ is a nonzero principal maximal ideal (with residue field $\mathbf{C}_K$) in the domain $\mathrm{W}(R)[1/p]$, for $j \geqslant 1$ we see that $\mathrm{W}(R)[1/p]/(\ker\theta_{\mathbf{Q}})^j$ is an artin local ring whose only ideals are $(\ker\theta_{\mathbf{Q}})^i/(\ker\theta_{\mathbf{Q}})^j$ for $0 \leqslant i \leqslant j$. In particular, an element of B^+_{dR} is a unit if and only if it has nonzero image under θ^+_{dR}. In other words, the maximal ideal $\ker\theta^+_{\mathrm{dR}}$ consists of precisely the non-units, so B^+_{dR} is a local ring.

Consider a non-unit $b \in B^+_{\mathrm{dR}}$, so its image in each $\mathrm{W}(R)[1/p]/(\ker\theta_{\mathbf{Q}})^j$ has the form $b_j\xi$ with b_j uniquely determined modulo $(\ker\theta_{\mathbf{Q}})^{j-1}$ (with ξ as above). In particular, the residue classes $b_j \bmod (\ker\theta_{\mathbf{Q}})^{j-1}$ are a compatible sequence and so define an element $b' \in B^+_{\mathrm{dR}}$ with $b = \xi b'$. The construction of b' shows that it is unique. Hence, the maximal ideal of B^+_{dR} has the principal generator ξ, and ξ is not a zero divisor in B^+_{dR}.

It now follows that for each $j \geqslant 1$ the multiples of ξ^j in B^+_{dR} are the elements killed by the surjective projection to $\mathrm{W}(R)[1/p]/(\ker\theta_{\mathbf{Q}})^j$. In particular, B^+_{dR} is ξ-adically separated, so it is a discrete valuation ring with uniformizer ξ. We have identified the construction of B^+_{dR} as the inverse limit of its artinian quotients, so it is a complete discrete valuation ring. ■

The Frobenius automorphism φ of $\mathrm{W}(R)[1/p]$ does not naturally extend to B^+_{dR} since it does not preserve $\ker\theta_{\mathbf{Q}}$; for example, $\varphi(\xi) = [\widetilde{p}^p] - p \notin \ker\theta_{\mathbf{Q}}$. There is no natural Frobenius structure on B^+_{dR}. Nonetheless, we do have a filtration via powers of the maximal ideal, and this is a G_K-stable filtration. We get the same on the fraction field:

Definition 4.4.7. The *field of p-adic periods* (or the *de Rham period ring*) is $B_{\mathrm{dR}} := \mathrm{Frac}(B^+_{\mathrm{dR}})$ equipped with its natural G_K-action and G_K-stable filtration via the $\mathbf{Z}$-powers of the maximal ideal of B^+_{dR}.

To show that the filtered field B_{dR} is an appropriate refinement of B_{HT}, we wish to prove that the associated graded algebra $\mathrm{gr}^\bullet(B_{\mathrm{dR}})$ over the residue field $\mathbf{C}_K$ of B^+_{dR} (see Example 4.1.2) is G_K-equivariantly identified with the graded $\mathbf{C}_K$-algebra B_{HT}. This amounts to proving that the Zariski cotangent space of B^+_{dR}, which is 1-dimensional over the residue field $\mathbf{C}_K$, admits a canonical copy of $\mathbf{Z}_p(1)$; this would be a canonical $\mathbf{Z}_p$-line on which G_K acts by the p-adic cyclotomic character, and identifies the Zariski cotangent space with $\mathbf{C}_K(1)$ as required.

We will do better: we shall prove that B^+_{dR} admits a uniformizer t, canonical up to $\mathbf{Z}_p^\times$-multiple, on which G_K acts by the cyclotomic character, and that the set of such t's is naturally $\mathbf{Z}_p^\times$-equivariantly bijective with the set of $\mathbf{Z}_p$-bases of $\mathbf{Z}_p(1) = \varprojlim \mu_{p^n}(\overline{K})$. (Such elements t do not live in $\mathrm{W}(R)[1/p]$, so it is essential to have passed to the completion B_{dR}^+ to find such a uniformizer on which there is such a nice G_K-action.) The construction of t rests on elements $\varepsilon \in R$ from Example 4.3.4 as follows.

Choose $\varepsilon \in R$ with $\varepsilon^{(0)} = 1$ and $\varepsilon^{(1)} \neq 1$, so $\theta([\varepsilon] - 1) = \varepsilon^{(0)} - 1 = 0$. Hence, $[\varepsilon] - 1 \in \ker\theta \subseteq \ker\theta_{\mathrm{dR}}^+$, so $[\varepsilon] = 1 + ([\varepsilon] - 1)$ is a 1-unit in the complete discrete valuation ring B_{dR}^+ over K. We can therefore make sense of the logarithm

$$t := \log([\varepsilon]) = \log(1 + ([\varepsilon] - 1)) = \sum_{n\geqslant 1} (-1)^{n+1} \frac{([\varepsilon] - 1)^n}{n} \in B_{\mathrm{dR}}^+.$$

This lies in the maximal ideal of B_{dR}^+. Note that if we make another choice ε' then $\varepsilon' = \varepsilon^a$ for a unique $a \in \mathbf{Z}_p^\times$ using the natural $\mathbf{Z}_p$-module structure on 1-units in R. Hence, by continuity of the Teichmüller map $R \to \mathrm{W}(R)$ relative to the v_R-adic topology of R we have $[\varepsilon'] = [\varepsilon]^a$ in $\mathrm{W}(R)$. Thus, $t' = \log([\varepsilon']) = \log([\varepsilon]^a)$.

We wish to claim that $\log([\varepsilon]^a) = a \cdot \log([\varepsilon])$, but this requires an argument because the logarithm is defined as a convergent sum relative to a topology on B_{dR}^+ that "ignores" the v_R-adic topology of R whereas the exponentiation procedure $[\varepsilon]^a$ involves the v_R-adic topology of R in an essential manner. A good way to deal with this is to introduce a topological ring structure on B_{dR}^+ that is finer than its discrete valuation topology and relative to which the natural map $\mathrm{W}(R) \to B_{\mathrm{dR}}^+$ is continuous. We leave this to the reader in the form of the multi-part Exercise 4.5.3.

The reader is *strongly encouraged* to read over the statements in Exercise 4.5.3, and to try to solve some of the parts, as this exercise will play an essential role in numerous later arguments and constructions. It is the key to ensuring that the constructions of p-adic Hodge theory retain the right kind of continuity conditions without which proofs would break down. (For example, in Theorem 2.2.7, it is essential that we work with continuous cohomology.)

We now use Exercise 4.5.3(5). Let $U_R \subseteq 1 + \mathfrak{m}_R$ be the subgroup of elements x such that $x^{(0)} = 1$ (such as any choice of ε). We claim that the logarithm $\log([x]) \in B_{\mathrm{dR}}^+$ formed as a convergent sum for the discrete valuation topology is continuous in x relative to the v_R-adic topology of the topological group $U_R \subseteq 1 + \mathfrak{m}_R$ and the topological ring structure just constructed on B_{dR}^+. Since $x \mapsto \log([x])$ is an abstract homomorphism $U_R \to B_{\mathrm{dR}}^+$ between topological groups, it suffices to check continuity at the identity. If $\mathfrak{a} \subseteq R$ is an ideal and $x \in (1 + \mathfrak{a}) \cap U_R$ then working in $\mathrm{W}(R/\mathfrak{a})$ shows that $[x] - 1 \in \mathrm{W}(\mathfrak{a})$, so $([x] - 1)^n/n \in p^{-j}\,\mathrm{W}(\mathfrak{a}^{p^j})$ with $j = \mathrm{ord}_p(n)$ for all $n \geqslant 1$. This gives the required continuity, in view of how the topology on B_{dR}^+ is defined in Exercise 4.5.3.

For any $a \in \mathbf{Z}_p$ and $x \in U_R$ we have $x^a \in U_R$ by continuous extension from the case $a \in \mathbf{Z}^+$ via the tautological continuity of the map $x \mapsto x^{(0)}$ from R to $\mathscr{O}_{\mathbf{C}_K}$. Likewise, by continuity of $\log : U_R \to B_{\mathrm{dR}}^+$, for any $a \in \mathbf{Z}_p$ and $x \in U_R$ we have $\log([x^a]) = a\log([x])$ by continuous extension from the case $a \in \mathbf{Z}^+$. Hence, for $\varepsilon' = \varepsilon^a$ with $a \in \mathbf{Z}_p^\times$ we have $t' := \log([\varepsilon']) = a\log([\varepsilon]) = at$.

In other words, the line $\mathbf{Z}_p t$ in the maximal ideal of B_{dR}^+ is intrinsic (i.e., independent of the choice of ε) and making a choice of $\mathbf{Z}_p$-basis of this line is the same as making a choice of ε. Also, choosing ε is literally a choice of $\mathbf{Z}_p$-basis of $\mathbf{Z}_p(1) = \varprojlim \mu_{p^n}(\overline{K})$. For any $g \in G_K$

we have $g(\varepsilon) = \varepsilon^{\chi(g)}$ in R since $g(\varepsilon^{(n)}) = (\varepsilon^{(n)})^{\chi(g)}$ for the primitive p^nth roots of unity $\varepsilon^{(n)} \in \mathscr{O}_{\overline{K}}$ for all $n \geqslant 0$. Thus, by the G_K-equivariance of the logarithm on 1-units of B_{dR}^+,

$$g(t) = \log(g([\varepsilon])) = \log([g(\varepsilon)]) = \log([\varepsilon^{\chi(g)}]) = \log([\varepsilon]^{\chi(g)}) = \chi(g)t.$$

We conclude that $\mathbf{Z}_p t$ is a canonical copy of $\mathbf{Z}_p(1)$ as a G_K-stable line in B_{dR}^+. Intuitively, this line is viewed as an analogue of the $\mathbf{Z}$-line $\mathbf{Z}(1) := \ker(\exp) \subseteq \mathbf{C}$, and in particular the choice of a $\mathbf{Z}_p$-basis element t is analogous to a choice of $2\pi i$ in complex analysis.

The key fact concerning such elements t is that they are uniformizers of B_{dR}^+, and hence we get a canonical isomorphism $\mathrm{gr}^\bullet(B_{\mathrm{dR}}) \simeq B_{\mathrm{HT}}$. We now prove this uniformizer property.

Proposition 4.4.8. *The element $t = \log([\varepsilon])$ in B_{dR}^+ is a uniformizer.*

Proof. By construction of t, $\theta_{\mathrm{dR}}^+(t) = 0$. Hence, t is a non-unit. We have to prove that t is not in the square of the maximal ideal. In view of its definition as an infinite series in powers $([\varepsilon]-1)^n/n$ with $[\varepsilon]-1$ in the maximal ideal, all such terms with $n \geqslant 2$ can be ignored. Thus, we just have to check that $[\varepsilon]-1$ is not in the square of the maximal ideal. But the projection from B_{dR}^+ onto the quotient modulo the square of its maximal ideal is the same as the natural map onto $\mathrm{W}(R)[1/p]/(\ker\theta_{\mathbf{Q}})^2$, so we have to prove that $[\varepsilon]-1$ is not contained in $(\ker\theta_{\mathbf{Q}})^2$, or equivalently is not contained in $\mathrm{W}(R) \cap (\ker\theta_{\mathbf{Q}})^2 = (\ker\theta)^2 = \xi^2\,\mathrm{W}(R)$ with $\xi = [\widetilde{p}] - p$ for $\widetilde{p} \in R$ defined by a compatible sequence of p-power roots of p.

To show that $[\varepsilon]-1$ is not a $\mathrm{W}(R)$-multiple of ξ^2, it suffices to project into the 0th component of $\mathrm{W}(R)$ and show that $\varepsilon - 1$ is not an R-multiple of $\widetilde{p}^2$. That is, it suffices to prove $v_R(\varepsilon-1) < v_R(\widetilde{p}^2) = 2$. But $v_R(\varepsilon - 1) = p/(p-1)$ by Example 4.3.4, so for $p > 2$ we have a contradiction. Now suppose $p = 2$. In this case we will work in $\mathrm{W}_2(R)$. Since $\xi^2 = [\widetilde{p}^2] - 2[\widetilde{p}] + 4 = (\widetilde{p}^2, 0, \dots)$ in $\mathrm{W}(R)$, for any $w = (r_0, r_1, \dots) \in \mathrm{W}(R)$ we compute $\xi^2 w = (r_0\widetilde{p}^2, r_1\widetilde{p}^4, \dots)$. However, for $p = 2$ we have $-1 = (1, 1, \dots)$ in $\mathbf{Z}_2 = \mathrm{W}(\mathbf{F}_2)$ since $-1 = 1 + 2\cdot 1 \bmod 4$, so $[\varepsilon] - 1 = (\varepsilon - 1, \varepsilon - 1, \dots)$ in $\mathrm{W}(R)$. Thus, if $[\varepsilon]-1$ were a $\mathrm{W}(R)$-multiple of ξ^2 for $p = 2$ then $\varepsilon - 1 = r_1\widetilde{p}^4$ for some $r_1 \in R$. This says $v_R(\varepsilon - 1) \geqslant v_R(\widetilde{p}^4) = 4$, a contradiction since $v_R(\varepsilon - 1) = p/(p-1) = 2$. ■

Remark 4.4.9. Note that the construction of B_{dR}^+ only involves the field K through its completed algebraic closure $\mathbf{C}_K$. More specifically, if $K' \subseteq \mathbf{C}_K$ is a complete discretely-valued subfield (so it is a p-adic field, as its residue field k' is perfect due to sitting between k and $\overline{k}$) then we get the same ring B_{dR}^+ whether we use K or K'. The actions of G_K and $G_{K'}$ on this common ring are related in the evident manner, namely via the inclusion $G_{K'} \hookrightarrow G_K$ as subgroups of the isometric automorphism group of $\mathbf{C}_K$. For example, replacing K with $\widehat{K^{\mathrm{un}}}$ does not change B_{dR}^+ but replaces the G_K-action with the underlying I_K-action. Likewise, the ring B_{dR}^+ is unaffected by replacing K with a finite extension within $\overline{K}$.

We end our preliminary discussion of B_{dR}^+ by recording some important properties that are not easily seen from its explicit construction. First of all, whereas $\mathrm{W}(R)[1/p]$ does not contain any nontrivial finite totally ramified extension of $K_0 = \mathrm{W}(k)[1/p]$ (as it lies inside of the absolutely unramified p-adic field $\mathrm{W}(\mathrm{Frac}(R))[1/p]$), the situation is quite different for B_{dR}^+:

Lemma 4.4.10. *The K_0-algebra B_{dR}^+ contains a unique copy of $\overline{K}$ as a subfield over K_0, and this lifting from the residue field is compatible with the action of G_{K_0}.*

Moreover, any extension K'/K_0 inside of $\overline{K}$ with finite ramification index gets its valuation topology as the subspace topology from B^+_{dR}. In particular, K' is closed in B^+_{dR} if it is complete.

Proof. Since B^+_{dR} is a complete discrete valuation ring over K_0, and $\overline{K}$ is a subfield of the residue field $\mathbf{C}_K$ that is separable algebraic over K_0, it follows from Hensel's Lemma that $\overline{K}$ uniquely lifts to a subfield over K_0 in B^+_{dR}. The uniqueness of the lifting ensures that this is a G_K-equivariant lifting.

Now pick an algebraic extension K'/K_0 with finite ramification index. To check that K' gets its valuation topology as the subspace topology, first recall that B^+_{dR} only depends on $\mathbf{C}_K$, so we can construct it from the view of the completion $\widehat{K}_0 = W(\overline{k})[1/p]$. In particular, B^+_{dR} contains $\widehat{K}_0K'$ over K', and by Exercise 4.5.3(3) the induced topology on $\widehat{K}_0$ is the usual one. Hence, to check that the topology on K' is as expected it suffices to replace K' with the subfield $K'\widehat{K}_0$ which we may then rename as K (upon replacing k with $\overline{k}$). In other words, we just have to check that K gets the expected subspace topology.

Since B^+_{dR} is a topological K_0-algebra and the valuation topology on K is its product topology for a K_0-basis, if we give K its valuation topology then the natural map $K \to B^+_{\mathrm{dR}}$ is continuous. To see that this is an embedding it suffices to compare convergent sequences. By continuity of the map $\theta_{\mathbf{Q}} : B^+_{\mathrm{dR}} \to \mathbf{C}_K$ onto $\mathbf{C}_K$ with its valuation topology, we are done. ■

Remark 4.4.11. Beware that the subspace topology on $\overline{K}$ from B^+_{dR} is *not* its valuation topology, nor is the inclusion $\overline{K} \to B^+_{\mathrm{dR}}$ even continuous! Indeed, iof it were continuous then by completeness of the topology on B^+_{dR} we would get a unique continuous extension $\mathbf{C}_K = \widehat{\overline{K}} \to B^+_{\mathrm{dR}}$. By uniqueness, this would have to be a G_K-equivariant section to the projection to the residue field, so the filtration on B_{dR} would be canonically split and B_{dR} would be isomorphic to B_{HT} as graded rings equipped with a G_K-action. In particular, all Hodge–Tate representations would be de Rham. But this is false; we will give simple examples in Example 6.3.5 (but the *proof* that these simple examples are not de Rham is very far from elementary).

It turns out that relative to the topology on B^+_{dR}, the subfield $\overline{K}$ is *dense*. This is proved by Colmez in [21, §A2], where he gives a direct description of this subspace topology on $\overline{K}$.

The following innocuous-looking further topological result seems to be less elementary to prove than one might expect. We need it in the proofs of some important facts (Proposition 6.3.8 and Corollary 15.3.10).

Lemma 4.4.12. *Any finite-dimensional K-subspace W in B^+_{dR} or any $B^+_{\mathrm{dR}}/(t^m)$ is closed and acquires its natural K-linear topology as its subspace topology.*

Proof. Since B^+_{dR} is a topological K-algebra (see Exercise 4.5.3) and it is Hausdorff with a countable base of opens around the origin, the subspace topology on W is a Hausdorff topological vector space structure and closedness of W in B^+_{dR} can be checked using sequences. Consider any point $b \in B^+_{\mathrm{dR}}$ lying in the closure of W, so $b = lim w_m$ for a sequence $\{w_0, w_1, \dots\}$ in W. Then $\{w_m\}$ is a Cauchy sequence for the subspace topology of W, so if this topology is the usual (complete) one then $\{w_m\}$ has a limit $w \in W$. The Hausdorff property of the topology on B^+_{dR} would then force $b = w \in W$.

It remains to prove that the subspace topology on W is the linear topology defined using a finite K-basis. By [7, I, §3, Thm. 2], a finite-dimensional vector space over a field complete with respect to a nontrivial absolute value has a unique structure of Hausdorff topological vector space, namely the one defined using a finite basis. ■

The canonical $\overline{K}$-structure on B_{dR}^+ (and hence on its fraction field B_{dR}) plays an important role in the study of finer period rings; it can be shown that there is no G_K-equivariant lifting of the *entire* residue field $\mathbf{C}_K$ into B_{dR}^+ (whereas such an abstract lifting exists by commutative algebra and is not useful).

Another property of B_{dR} that is hard to see directly from the construction is the determination of its subfield of G_K-invariants. As we have just seen, there is a canonical G_K-equivariant embedding $\overline{K} \hookrightarrow B_{\mathrm{dR}}^+$, whence $K \subseteq B_{\mathrm{dR}}^{G_K}$. (Nothing like this holds for $\mathrm{W}(R)[1/p]$ if $K \neq K_0$.) This inclusion is an equality, due to the Tate–Sen theorem:

Theorem 4.4.13. *The inclusion $K \subseteq B_{\mathrm{dR}}^{G_K}$ is an equality.*

Proof. Since the G_K-actions respect the (exhaustive and separated) filtration, the field extension $B_{\mathrm{dR}}^{G_K}$ of K with the subspace filtration has associated graded K-algebra that injects into $(\mathrm{gr}^\bullet(B_{\mathrm{dR}}))^{G_K} = B_{\mathrm{HT}}^{G_K}$. But by the Tate–Sen theorem this latter space of invariants is K. We conclude that $\mathrm{gr}^\bullet(B_{\mathrm{dR}}^{G_K})$ is 1-dimensional over K, so the same holds for $B_{\mathrm{dR}}^{G_K}$. ■

The final property of B_{dR} that we record is its dependence on K. An inspection of the construction shows that B_{dR}^+ depends solely on $\mathscr{O}_{\mathbf{C}_K}$ and not on the particular p-adic field $K \subseteq \mathscr{O}_{\mathbf{C}_K}[1/p] = \mathbf{C}_K$ whose algebraic closure is dense in $\mathbf{C}_K$. More specifically, B_{dR}^+ depends *functorially* on $\mathscr{O}_{\mathbf{C}_K}$ (this requires reviewing the construction of R and θ), and the action of $\mathrm{Aut}(\mathscr{O}_{\mathbf{C}_K})$ on B_{dR}^+ via functoriality induces the action of G_K (via the natural inclusion of G_K into $\mathrm{Aut}(\mathscr{O}_{\mathbf{C}_K})$). Hence, if $K \to K'$ is a map of p-adic fields and we pick a compatible embedding $\overline{K} \to \overline{K'}$ of algebraic closures then the induced map $\mathscr{O}_{\mathbf{C}_K} \to \mathscr{O}_{\mathbf{C}_{K'}}$ induces a map $B_{\mathrm{dR},K}^+ \to B_{\mathrm{dR},K'}^+$ that is equivariant relative to the corresponding map of Galois groups $G_{K'} \to G_K$. In particular, if the induced map $\mathbf{C}_K \to \mathbf{C}_{K'}$ is an isomorphism then we have $B_{\mathrm{dR},K}^+ = B_{\mathrm{dR},K'}^+$ (compatibly with the inclusion $G_{K'} \hookrightarrow G_K$) and likewise for the fraction fields. This applies in two important cases: K'/K a finite extension and $K' = \widehat{K^{\mathrm{un}}}$. In other words, B_{dR}^+ and B_{dR} are naturally insensitive to replacing K with a finite extension or with a completed maximal unramified extension. The invariance of B_{dR}^+ and B_{dR} under these two kinds of changes in K is important in practice when replacing G_K with an open subgroup or with I_K in the context of studying de Rham representations in §6. We will return to this issue in more detail in Proposition 6.3.8 and the discussion immediately preceding it.

4.5. **Exercises.**

Exercise 4.5.1. Let K be a p-adic field and K_0 its maximal unramified subfield (as defined above Remark 4.2.4). Prove that the natural map $\widehat{K_0^{\mathrm{un}}} \otimes_{K_0} K \to \widehat{K^{\mathrm{un}}}$ is an isomorphism.

Exercise 4.5.2. Let K be a p-adic field and $R = \underline{R}(\mathscr{O}_{\overline{K}}/(p))$ the associated perfect valuation ring. Prove that a subset $\Sigma \subseteq R$ is dense for the v_R-adic topology if and only if the maps $\theta_n : R \to \mathscr{O}_{\mathbf{C}_K}/(p)$ have surjective restriction to Σ for all $n \geqslant 0$. Consider a nonzero $x = (x_n)_{n \geqslant 0} \in R$. Prove that $x_n \neq 0$ for all sufficiently large n, and show that if $\widehat{x}_n \in \mathscr{O}_{\mathbf{C}_K}$ is a lift of x_n then $|\widehat{x}_n| \to 1$ as $n \to \infty$. In other words, the x_n's are "almost units" in $\mathscr{O}_{\mathbf{C}_K}/(p)$ for large n. This is a very useful fact.

Exercise 4.5.3. Let K be a p-adic field (with residue field k) and let $R = \underline{R}(\mathscr{O}_{\overline{K}}/(p))$ be the associated valuation ring of an algebraically closed field of characteristic $p > 0$.

This crucial exercise introduces a topological ring structure on $\mathrm{W}(R)[1/p]$ that induces the natural v_R-adic product topology on the subring $\mathrm{W}(R)$ and extends it to a natural topological ring structure on B^+_{dR} whose induced quotient topology on the residue field $\mathbf{C}_K$ is the natural valuation topology. Roughly speaking, for $\mathrm{W}(R)[1/p]$ the idea is to impose a topology using controlled decay of coefficients of Laurent series in p. The situation is fundamentally different from topologizing $\mathbf{Q}_p = \mathbf{Z}_p[1/p]$ from the topology on $\mathbf{Z}_p$ because $p\,\mathrm{W}(R)$ is *not* open in $\mathrm{W}(R)$ (in contrast with $p\mathbf{Z}_p \subseteq \mathbf{Z}_p$) when R is given its v_R-adic (rather than its discrete) topology. Since this exercise has many parts, you may prefer to just do a few parts now and come back to the rest as you see them used later.

(1) For any open ideal $\mathfrak{a} \subseteq R$ and $N \geqslant 0$, let

$$U_{N,\mathfrak{a}} = \bigcup_{j>-N} (p^{-j}\,\mathrm{W}(\mathfrak{a}^{p^j}) + p^N\,\mathrm{W}(R)) \subseteq \mathrm{W}(R)[1/p],$$

where $\mathrm{W}(J)$ for an ideal J of R means the ideal of Witt vectors in $\mathrm{W}(R)$ whose components all lie in J. Prove that $U_{N,\mathfrak{a}}$ is a G_K-stable $\mathrm{W}(R)$-submodule of $\mathrm{W}(R)[1/p]$.

(2) Prove $U_{N+M,\mathfrak{a}\cap\mathfrak{b}} \subseteq U_{N,\mathfrak{a}} \cap U_{M,\mathfrak{b}}$ and that $U_{N,\mathfrak{a}} \cdot U_{N,\mathfrak{a}} \subseteq U_{N,\mathfrak{a}}$. Deduce that $\mathrm{W}(R)[1/p]$ has a unique structure of topological ring with the $U_{N,\mathfrak{a}}$'s a base of open neighborhoods of 0, and that the G_K-action on $\mathrm{W}(R)[1/p]$ is continuous.

(3) Prove that $U_{N,\mathfrak{a}} \cap \mathrm{W}(R) = \mathrm{W}(\mathfrak{a}) + p^N\,\mathrm{W}(R)$, and deduce that $\mathrm{W}(R)$ endowed with its product topology using the v_R-adic topology on R is a closed topological subring of $\mathrm{W}(R)[1/p]$. Conclude that $K_0 = \mathrm{W}(k)[1/p] \subseteq \mathrm{W}(R)[1/p]$ is a closed subfield with its usual p-adic topology (hint: k is a discrete subring of R).

(4) For each $N \geqslant 0$, prove that $p^N \mathscr{O}_{\mathbf{C}_K} \subseteq \theta_{\mathbf{Q}}(U_{N,\mathfrak{a}})$ and show that this containment gets arbitrarily close to an equality for the p-adic topology (i.e., $\theta_{\mathbf{Q}}(U_{N,\mathfrak{a}})$ is contained in $p^{N+a}\mathscr{O}_{\mathbf{C}_K}$ for arbitrarily small $a > 0$) by taking $\mathfrak{a}$ to be sufficiently small. In particular, deduce that $\theta_{\mathbf{Q}} : \mathrm{W}(R)[1/p] \to \mathbf{C}_K$ is a continuous open map.

(5) Prove that the multiplication map $\xi : \mathrm{W}(R)[1/p] \to \mathrm{W}(R)[1/p]$ is a closed embedding, so all ideals $(\ker\theta_{\mathbf{Q}})^j = \xi^j\,\mathrm{W}(R)[1/p]$ are closed. Conclude that with the quotient topology on each $\mathrm{W}(R)[1/p]/(\ker\theta_{\mathbf{Q}})^j$, the inverse limit topology on B^+_{dR} makes it a Hausdorff topological ring relative to which the powers of the maximal ideal are closed, $\mathrm{W}(R)$ is a closed subring (with its natural topology as subspace topology), the G_K-action is continuous, the multiplication map by ξ on B^+_{dR} (and hence by *any* uniformizer!) is a closed embedding, and the residue field $\mathbf{C}_K$ inherits its *valuation topology* as the quotient topology. (It is not clear if $\mathrm{W}(R)[1/p]$ recovers its initial topology as the subspace topology from B^+_{dR}, but this is never needed. On the other hand, it is elementary from the construction that the map from B^+_{dR} onto each of its artinian quotients is an open mapping, so these quotients recover their initial topology as their quotient topology from B^+_{dR}. Also, (3) implies that B^+_{dR} is a topological K-algebra, by working with a K_0-basis of K.)

(6) Prove that this topology on $B_{\rm dR}^+$ is *complete*. That is, Cauchy sequences (defined in an evident manner) converge, or equivalently $\sum b_n$ converges in $B_{\rm dR}^+$ whenever $b_n \to 0$. (Hint: Really prove that each $\mathrm{W}(R)[1/p]/(\ker\theta_{\mathbf{Q}})^j$ is complete. For this you'll need to use how the $U_{N,\mathfrak{a}}$'s were defined in order to prove, akin to the familiar case of $\mathbf{C}_K$, that a sequence converging to 0 in such a quotient is represented by one in $\mathrm{W}(R)[1/p]$ that is contained in $p^{-N}\,\mathrm{W}(R)$ for a single N; this boundedness on denominators is *not* in the definition of the topology, so it really must be proved.)

(7) Recall that the adele ring of a global field is a natural example of a topological rings whose subset of units is not a topological group (inversion is not continuous). Prove that the subset of units is open. Can you determine if inversion is continuous relative to the subspace topology? (This is never needed.)

5. Formalism of admissible representations

Now that we have developed some experience with various functors between Galois representations and semilinear algebra categories via suitable rings with structure, we wish to axiomatize this kind of situation for constructing and analyzing functors defined via "period rings" in order that we do not have to repeat the same kinds of arguments every time we introduce a new period ring. In §6 we shall use the following formalism.

5.1. **Definitions and examples.** Let F be a field and G be a group. Let B be an F-algebra domain equipped with a G-action (as an F-algebra), and assume that the invariant F-subalgebra $E = B^G$ is a field. We do not impose any topological structure on B or F or G. Our goal is to use B to construct an interesting functor from finite-dimensional F-linear representations of G to finite-dimensional E-vector spaces (endowed with extra structure, depending on B).

We let $C = \mathrm{Frac}(B)$, and observe that G also acts on C in a natural way.

Definition 5.1.1. We say B is (F,G)-*regular* if $C^G = B^G$ and if every nonzero $b \in B$ whose F-linear span Fb is G-stable is a unit in B.

Note that if B is a field then the conditions in the definition are obviously satisfied. The cases of most interest will be rather far from fields. We now show how the Tate–Sen theorem (Theorem 2.2.7) provides two interesting examples of (F,G)-regular domains.

Example 5.1.2. Let K be a p-adic field with a fixed algebraic closure $\overline{K}$, and let $\mathbf{C}_K$ denote the completion of $\overline{K}$. Let $G = G_K = \mathrm{Gal}(\overline{K}/K)$. Let $B = B_{\rm HT} = \oplus_{n\in\mathbf{Z}}\mathbf{C}_K(n)$ endowed with its natural G-action. Non-canonically, $B = \mathbf{C}_K[T,1/T]$ with G acting through the p-adic cyclotomic character $\chi: G_K \to \mathbf{Z}_p^\times$ via $g(\sum a_nT^n) = \sum g(a_n)\chi(g)^nT^n$. Obviously in this case $C = \mathbf{C}_K(T)$. We claim that B is $(\mathbf{Q}_p, G)$-regular (with $B^G = K$).

By the Tate–Sen theorem, $B^G = \oplus\mathbf{C}_K(n)^G = K$. To compute that C^G is also equal to K, consider the G_K-equivariant inclusion of $C = \mathbf{C}_K(T)$ into the formal Laurent series field $\mathbf{C}_K(\!(T)\!)$ equipped with its evident G-action. It suffices to show that $\mathbf{C}_K(\!(T)\!)^G = K$. The action of $g \in G$ on a formal Laurent series $\sum c_nT^n$ is given by $\sum c_nT^n \mapsto \sum g(c_n)\chi(g)^nT^n$, so G-invariance amounts to the condition $c_n \in \mathbf{C}_K(n)^G$ for all $n \in \mathbf{Z}$. Hence, by the Tate–Sen theorem we get $c_n = 0$ for $n \neq 0$ and $c_0 \in K$, as desired.

Verifying the second property in $(\mathbf{Q}_p, G_K)$-regularity goes by a similar method, as follows: if $b \in B - \{0\}$ spans a G_K-stable $\mathbf{Q}_p$-line then G_K acts on the line $\mathbf{Q}_p b$ by some character $\psi : G_K \to \mathbf{Q}_p^\times$. It is a crucial fact (immediate from the continuity of the G_K-action on each direct summand $\mathbf{C}_K(n)$ of $B = B_{\mathrm{HT}}$) that ψ must be continuous (so it takes values in $\mathbf{Z}_p^\times$). Writing the Laurent polynomial b as $b = \sum c_j T^j$, we have $\psi(g)b = g(b) = \sum g(c_j)\chi(g)^j T^j$, so for each j we have $(\psi^{-1}\chi^j)(g) \cdot g(c_j) = c_j$ for all $g \in G_K$. That is, each c_j is G_K-invariant in $\mathbf{C}_K(\psi^{-1}\chi^j)$. But by the Tate–Sen theorem, for a $\mathbf{Z}_p^\times$-valued continuous character η of G_K, if $\mathbf{C}_K(\eta)$ has a nonzero G_K-invariant element then $\eta|_{I_K}$ has finite order. Hence, $(\psi^{-1}\chi^j)|_{I_K}$ has finite order whenever $c_j \neq 0$. It follows that we cannot have $c_j, c_{j'} \neq 0$ for some $j \neq j'$, for otherwise taking the ratio of the associated finite-order characters would give that $\chi^{j-j'}|_{I_K}$ has finite order, so $\chi|_{I_K}$ has finite order (as $j - j' \neq 0$), but this is a contradiction since χ cuts out an infinitely ramified extension of K. It follows that there is at most one j such that $c_j \neq 0$, and there is a nonzero c_j since $b \neq 0$. Hence, $b = cT^j$ for some j and some $c \in \mathbf{C}_K^\times$, so $b \in B^\times$.

Example 5.1.3. Consider $B = B_{\mathrm{dR}}^+$ equipped with its natural action by $G = G_K$. This is a complete discrete valuation ring with uniformizer t on which G acts through χ and with fraction field $C = B_{\mathrm{dR}} = B[1/t]$. We have seen in Theorem 4.4.13 (using that the associated graded ring to B_{dR} is B_{HT}) that $C^G = K$, so $B^G = K$ too. Since B_{dR} is a field, it follows trivially that B_{dR} is $(\mathbf{Q}_p, G)$-regular. Let us consider whether $B = B_{\mathrm{dR}}^+$ is also $(\mathbf{Q}_p, G)$-regular. The first requirement in the definition of $(\mathbf{Q}_p, G)$-regularity for B is satisfied in this case, as we have just seen. But the second requirement in $(\mathbf{Q}_p, G)$-regularity fails: $t \in B$ spans a G-stable $\mathbf{Q}_p$-line but $t \notin B^\times$.

The most interesting examples of $(\mathbf{Q}_p, G_K)$-regular rings are Fontaine's rings B_{cris} and B_{st} (certain subrings of B_{dR} with "more structure"), which turn out (ultimately by reducing to the study of B_{HT}) to be $(\mathbf{Q}_p, G_K)$-regular with subring of G_K-invariants equal to $K_0 = \mathrm{Frac}(\mathrm{W}(k)) = \mathrm{W}(k)[1/p]$ and K respectively.

In the general axiomatic setting, if B is an (F, G)-regular domain and E denotes the field $C^G = B^G$ then for any object V in the category $\mathrm{Rep}_F(G)$ of finite-dimensional F-linear representations of G we define

$$D_B(V) = (B \otimes_F V)^G,$$

so $D_B(V)$ is an E-vector space equipped with a canonical map

$$\alpha_V : B \otimes_E D_B(V) \to B \otimes_E (B \otimes_F V) = (B \otimes_E B) \otimes_F V \to B \otimes_F V.$$

This is a B-linear G-equivariant map (where G acts trivially on $D_B(V)$ in the right tensor factor of the source), by inspection.

As a simple example, for $V = F$ with trivial G-action we have $D_B(F) = B^G = E$ and the map $\alpha_V : B = B \otimes_E E \to B \otimes_F F = B$ is the identity map. It is not a priori obvious if $D_B(V)$ always lies in the category Vec_E of finite-dimensional vector spaces over E, but we shall now see that this and much more is true.

5.2. **Properties of admissible representations.** The aim of this section is to prove the following theorem which shows (among other things) that $\dim_E D_B(V) \leqslant \dim_F V$; in case equality holds we call V a *B-admissible* representation. For example, $V = F$ is always B-admissible. In case we fix a p-adic field K and let $F = \mathbf{Q}_p$ and $G = G_K$ then for $B = B_{\mathrm{HT}}$ this coincides with the concept of being a *Hodge-Tate* representation. For the ring B_{dR} and Fontaine's finer period rings B_{cris}, and B_{st} the corresponding notions are called being a *de Rham*, *crystalline*, and *semi-stable* representation respectively.

Theorem 5.2.1. *Fix V as above.*

1. *The map α_V is always injective and $\dim_E D_B(V) \leqslant \dim_F V$, with equality if and only if α_V is an isomorphism.*
2. *Let $\mathrm{Rep}_F^B(G) \subseteq \mathrm{Rep}_F(G)$ be the full subcategory of B-admissible representations. The covariant functor $D_B : \mathrm{Rep}_F^B(G) \to \mathrm{Vec}_E$ to the category of finite-dimensional E-vector spaces is exact and faithful, and any subrepresentation or quotient of a B-admissible representation is B-admissible.*
3. *If $V_1, V_2 \in \mathrm{Rep}_F^B(G)$ then there is a natural isomorphism*
$$D_B(V_1) \otimes_E D_B(V_2) \simeq D_B(V_1 \otimes_F V_2),$$
so $V_1 \otimes_F V_2 \in \mathrm{Rep}_F^B(G)$. If $V \in \mathrm{Rep}_F^B(G)$ then its dual representation $V^\vee$ lies in $\mathrm{Rep}_F^B(G)$ and the natural map
$$D_B(V) \otimes_E D_B(V^\vee) \simeq D_B(V \otimes_F V^\vee) \to D_B(F) = E$$
is a perfect duality between $D_B(V)$ and $D_B(V^\vee)$.
In particular, $\mathrm{Rep}_F^B(G)$ is stable under the formation of duals and tensor products in $\mathrm{Rep}_F(G)$, and D_B naturally commutes with the formation of these constructions in $\mathrm{Rep}_F^B(G)$ and in Vec_E.
Moreover, B-admissibility is preserved under the formation of exterior and symmetric powers, and D_B naturally commutes with both such constructions.

Before proving the theorem, we make some remarks.

Remark 5.2.2. In practice $F = \mathbf{Q}_p$, $G = G_K$ for a p-adic field K, and $E = K$ or $E = K_0$ (the maximal unramified subfield, $\mathrm{W}(k)[1/p]$), and the ring B has more structure (related to a Frobenius operator, filtration, monodromy operator, etc.). Corresponding to this extra structure on B, the functor D_B takes values in a category of finite-dimensional E-vector spaces equipped with "more structure", with morphisms being those E-linear maps which "respect the extra structure".

By viewing D_B with values in such a category, it can fail to be fully faithful (such as for $B = B_{\mathrm{HT}}$ or $B = B_{\mathrm{dR}}$ using categories of graded or filtered vector spaces respectively), but for more subtle period rings such as B_{cris} and B_{st} one does get full faithfulness into a suitably enriched category of linear algebra objects. One of the key results in recent years in p-adic Hodge theory is a purely linear algebraic description of the essential image of the fully faithful functor D_B for such better period rings (with the D_B viewed as taking values in a suitably enriched subcategory of Vec_E).

Remark 5.2.3. Once the theorem is proved, there is an alternative description of the B-admissibility condition on V: it says that $B \otimes_F V$ with its B-module structure and G-action is isomorphic to a direct sum $B^{\oplus r}$ (for some r) respecting the B-structure and G-action.

Indeed, since α_V is G-equivariant and B-linear, we get the necessity of this alternative description by choosing an E-basis of $D_B(V)$. As for sufficiency, if $B \otimes_F V \simeq B^{\oplus r}$ as B-modules and respecting the G-action then necessarily $r = d := \dim_F V$ (as $B \otimes_F V$ is finite free of rank d over B), and taking G-invariants gives $D_B(V) \simeq (B^G)^{\oplus d} = E^{\oplus d}$ as modules over $B^G = E$. This says $\dim_E D_B(V) = d = \dim_F V$, which is the dimension equality definition of B-admissibility.

Proof. First we prove (1). Granting for a moment that α_V is injective, let us show the rest of (1). Extending scalars from B to $C := \mathrm{Frac}(B)$ preserves injectivity (by flatness of C over B), so $C \otimes_E D_B(V)$ is a C-subspace of $C \otimes_F V$. Comparing C-dimensions then gives $\dim_E D_B(V) \leqslant \dim_F V$. Let us show that in case of equality of dimensions, say with common dimension d, the map α_V is an isomorphism (the converse now being obvious). Let $\{e_j\}$ be an E-basis of $D_B(V)$ and let $\{v_i\}$ be an F-basis of V, so relative to these bases we can express α_V using a $d \times d$ matrix (b_{ij}) over B (thanks to the assumed dimension equality). In other words, $e_j = \sum b_{ij} \otimes v_i$. The determinant $\det(\alpha_V) := \det(b_{ij}) \in B$ is nonzero due to the isomorphism property over $C = \mathrm{Frac}(B)$ (as $C \otimes_B \alpha_V$ is a C-linear injection between C-vector spaces with the same finite dimension d, so it must be an isomorphism). We want $\det(\alpha_V) \in B^\times$, so then α_V is an isomorphism over B. Since B is an (F, G)-regular ring, to show the nonzero $\det(\alpha_V) \in B$ is a unit it suffices to show that it spans a G-stable F-line in B.

The vectors $e_j = \sum b_{ij} \otimes v_i \in D_B(V) \subseteq B \otimes_F V$ are G-invariant, so passing to dth exterior powers on α_V gives that

$$\wedge^d(\alpha_V)(e_1 \wedge \cdots \wedge e_d) = \det(b_{ij}) v_1 \wedge \cdots \wedge v_d$$

is a G-invariant vector in $B \otimes_F \wedge^d(V)$. But G acts on $v_1 \wedge \cdots \wedge v_d$ by some character $\eta : G \to F^\times$ (just the determinant of the given F-linear G-representation on V), so G must act on $\det(b_{ij}) \in B - \{0\}$ through the $F^\times$-valued η^{-1}.

This completes the reduction of (1) to the claim that α_V is injective. Since B is (F, G)-regular, we have that $E = B^G$ is equal to C^G. For $D_C(V) := (C \otimes_F V)^G$ we also have a commutative diagram

$$\begin{tikzcd} B \otimes_E D_B(V) \arrow[r, "\alpha_V"] \arrow[d] & B \otimes_F V \arrow[d] \\ C \otimes_E D_C(V) \arrow[r] & C \otimes_F V \end{tikzcd}$$

in which the sides are injective. To prove injectivity of the top it suffices to prove it for the bottom. Hence, we can replace B with C so as to reduce to the case when B is a field. In this case the injectivity amounts to the claim that α_V carries an E-basis of $D_B(V)$ to a B-linearly independent set in $B \otimes_F V$, so it suffices to show that if $x_1, \ldots, x_r \in B \otimes_F V$ are E-linearly independent *and* G-invariant then they are B-linearly independent. Assuming to the contrary that there is a nontrivial B-linear dependence relation among the x_i's, consider such a relation of minimal length. We may assume it to have the form

$$x_r = \sum_{i<r} b_i \cdot x_i$$

for some $r \geqslant 2$ since B is a field and all x_i are nonzero. Applying $g \in G$ gives

$$x_r = g(x_r) = \sum_{i<r} g(b_i) \cdot g(x_i) = \sum_{i<r} g(b_i) \cdot x_i.$$

Thus, minimal length for the relation forces equality of coefficients: $b_i = g(b_i)$ for all $i < r$, so $b_i \in B^G = E$ for all i. Hence, we have a nontrivial E-linear dependence relation among $x_1, \ldots, x_r$, a contradiction.

Now we prove (2). For any B-admissible V we have a natural isomorphism $B \otimes_E D_B(V) \simeq B \otimes_F V$, so D_B is exact and faithful on the category of B-admissible V's (since a sequence of E-vector spaces is exact if and only if it becomes so after applying $B \otimes_E (\cdot)$, and similarly from F to B). To show that subrepresentations and quotients of a B-admissible V are B-admissible, consider a short exact sequence

$$0 \to V' \to V \to V'' \to 0$$

of $F[G]$-modules with B-admissible V. We have to show that V' and V'' are B-admissible. From the definition D_B is left-exact without any B-admissibility hypothesis, so we have a left-exact sequence of E-vector spaces

$$0 \to D_B(V') \to D_B(V) \to D_B(V'')$$

with $\dim_E D_B(V) = d$ by B-admissibility of V, so $d \leqslant \dim_E D_B(V') + \dim_E D_B(V'')$. By (1) we also know that the outer terms have respective E-dimensions at most $d' = \dim_F V'$ and $d'' = \dim_F V''$. But $d = d' + d''$ from the given short exact sequence of $F[G]$-modules, so these various inequalities are forced to be equalities, and in particular V' and V'' are B-admissible.

Finally, we consider (3). For B-admissible V_1 and V_2, say with $d_i = \dim_F V_i$, there is an evident natural map

$$D_B(V_1) \otimes_E D_B(V_2) \to (B \otimes_F V_1) \otimes_E (B \otimes_F V_2) \to B \otimes_F (V_1 \otimes V_2)$$

that is seen to be invariant under the G-action on the target, so we obtain a natural E-linear map

$$t_{V_1,V_2} : D_B(V_1) \otimes_E D_B(V_2) \to D_B(V_1 \otimes_F V_2),$$

with source having E-dimension $d_1 d_2$ (by B-admissibility of the V_i's) and target having E-dimension at most $\dim_F(V_1 \otimes_F V_2) = d_1 d_2$ by applying (1) to $V_1 \otimes_F V_2$. Hence, as long as this map is an injection then it is forced to be an isomorphism and so $V_1 \otimes_F V_2$ is forced to be B-admissible. To show that t_{V_1,V_2} is injective it suffices to check injectivity after composing with the inclusion of $D_B(V_1 \otimes_F V_2)$ into $B \otimes_F (V_1 \otimes_F V_2)$, and by construction this composite is seen to coincide with the composition of the injective map

$$D_B(V_1) \otimes_E D_B(V_2) \to B \otimes_E (D_B(V_1) \otimes_E D_B(V_2)) = (B \otimes_E D_B(V_1)) \otimes_B (B \otimes_E D_B(V_2))$$

and the isomorphism $\alpha_{V_1} \otimes_B \alpha_{V_2}$ (using again that the V_i are B-admissible).

Having shown that B-admissibility is preserved under tensor products and that D_B naturally commutes with the formation of tensor products, as a special case we see that if V is B-admissible then so is $V^{\otimes r}$ for any $r \geqslant 1$, with $D_B(V)^{\otimes r} \simeq D_B(V^{\otimes r})$. The quotient

$\wedge^r(V)$ of $V^{\otimes r}$ is also B-admissible (since $V^{\otimes r}$ is B-admissible), and there is an analogous map $\wedge^r(D_B(V)) \to D_B(\wedge^r V)$ that fits into a commutative diagram

$$\begin{tikzcd} D_B(V)^{\otimes r} \arrow[r, "\simeq"] \arrow[d] & D_B(V^{\otimes r}) \arrow[d] \\ \wedge^r(D_B(V)) \arrow[r] & D_B(\wedge^r V) \end{tikzcd} \tag{5.2.1}$$

in which the left side is the canonical surjection and the right side is surjective because it is D_B applied to a surjection between B-admissible representations. Thus, the bottom side is surjective. But the left and right terms on the bottom have the same dimension (since V and $\wedge^r V$ are B-admissible, with $\dim_F V = \dim_E D_B(V)$), so the bottom side is an isomorphism!

The same method works with symmetric powers in place of exterior powers. Note that the diagram (5.2.1) without an isomorphism across the top can be constructed for any $V \in \mathrm{Rep}_F(G)$, so for any such V there are natural E-linear maps $\wedge^r(D_B(V)) \to D_B(\wedge^r V)$ and likewise for rth symmetric powers, just as we have for tensor powers (and in the B-admissible case these are isomorphisms).

The case of duality is more subtle. Let V be a B-admissible representation of G over F. To show that $V^\vee$ is B-admissible and that the resulting natural pairing between $D_B(V)$ and $D_B(V^\vee)$ is perfect, we use a trick with tensor algebra. For any finite-dimensional vector space W over a field with $\dim W = d \geqslant 1$ there is a natural isomorphism

$$\det(W^\vee) \otimes \wedge^{d-1}(W) \simeq W^\vee$$

defined by

$$(\ell_1 \wedge \cdots \wedge \ell_d) \otimes (w_2 \wedge \cdots \wedge w_d) \mapsto (w_1 \mapsto \det(\ell_i(w_j))),$$

and this is equivariant for the naturally induced group actions in case W is a linear representation space for a group. Hence, to show that $V^\vee$ is a B-admissible F-linear representation space for G we are reduced to proving B-admissibility for $\det(V^\vee) = (\det V)^\vee$ (as then its tensor product against the B-admissible $\wedge^{d-1}(V)$ is B-admissible, as required). Since $\det V$ is B-admissible, we are reduced to the 1-dimensional case (for proving preservation of B-admissibility under duality).

Now assume the B-admissible V satisfies $\dim_F V = 1$, and let v_0 be an F-basis of V, so B-admissibility gives that $D_B(V)$ is 1-dimensional (rather than 0). Hence, $D_B(V) = E(b \otimes v_0)$ for some nonzero $b \in B$. The isomorphism $\alpha_V : B \otimes_E D_B(V) \simeq B \otimes_F V = B(1 \otimes v_0)$ between free B-modules of rank 1 carries the B-basis $b \otimes v_0$ of the left side to $b \otimes v_0 = b \cdot (1 \otimes v_0)$ on the right side, so $b \in B^\times$. The G-invariance of $b \otimes v_0$ says $g(b) \otimes g(v_0) = b \otimes v_0$, and we have $g(v_0) = \eta(g)v_0$ for some $\eta(g) \in F^\times$ (as V is a 1-dimensional representation space of G over F, say with character η), so $\eta(g)g(b) = b$. Thus, $b/g(b) = \eta(g) \in F^\times$. Letting $v_0^\vee$ be the dual basis of $V^\vee$, one then computes that $D_B(V^\vee)$ contains the nonzero vector $b^{-1} \otimes v_0^\vee$, so it is a nonzero space. The 1-dimensional $V^\vee$ is therefore B-admissible, as required.

Now that we know duality preserves B-admissibility in general, we fix a B-admissible V and aim to prove the perfectness of the pairing defined by

$$\langle \cdot, \cdot \rangle_V : D_B(V) \otimes_E D_B(V^\vee) \simeq D_B(V \otimes_F V^\vee) \to D_B(F) = E.$$

For $\dim_F V = 1$ this is immediate from the above explicitly computed descriptions of $D_B(V)$ and $D_B(V^\vee)$ in terms of a basis of V and the corresponding dual basis of $V^\vee$. In the general case, since V and $V^\vee$ are both B-admissible, for any $r \geqslant 1$ we have natural isomorphisms $\wedge^r(D_B(V)) \simeq D_B(\wedge^r(V))$ and $\wedge^r(D_B(V^\vee)) \simeq D_B(\wedge^r(V^\vee)) \simeq D_B((\wedge^r V)^\vee)$ with respect to which the pairing

$$\wedge_E^r(D_B(V)) \otimes_E \wedge_E^r(D_B(V^\vee)) \to E$$

induced by $\langle \cdot, \cdot \rangle_V$ on rth exterior powers is identified with $\langle \cdot, \cdot \rangle_{\wedge^r V}$. Since perfectness of a bilinear pairing between finite-dimensional vector spaces of the same dimension is equivalent to perfectness of the induced bilinear pairing between their top exterior powers, by taking $r = \dim_F V$ we see that the perfectness of the pairing $\langle \cdot, \cdot \rangle_V$ for the B-admissible V is equivalent to perfectness of the pairing associated to the B-admissible 1-dimensional $\det V$. But the 1-dimensional case is settled, so we are done. ■

5.3. **Exercises.**

Exercise 5.3.1. Verify the unchecked linear-algebra compatibility assertions in the proofs in §5.

Part II. Period rings and functors

6. de Rham representations

6.1. **Basic definitions.** Since B_{dR} is $(\mathbf{Q}_p, G_K)$-regular with $B_{\mathrm{dR}}^{G_K} = K$, the general formalism of admissible representations provides a good class of p-adic representations: the B_{dR}-admissible ones. More precisely, we define the covariant functor $D_{\mathrm{dR}} : \mathrm{Rep}_{\mathbf{Q}_p}(\mathbf{G}_K) \to \mathrm{Vec}_K$ valued in the category Vec_K of finite-dimensional K-vector spaces by

$$D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K},$$

so $\dim_K D_{\mathrm{dR}}(V) \leqslant \dim_{\mathbf{Q}_p} V$. In case this inequality is an equality we say that V is a *de Rham representation* (i.e., V is B_{dR}-admissible). Let $\mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K) \subseteq \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ denote the full subcategory of de Rham representations.

By the general formalism from §5, for $V \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K)$ we have a B_{dR}-linear G_K-compatible comparison isomorphism

$$\alpha_V : B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V) \to B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$$

and the subcategory $\mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K) \subseteq \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ is stable under passage to subquotients, tensor products, and duals (and so also exterior and symmetric powers), and moreover the functor $D_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K) \to \mathrm{Vec}_K$ is faithful and exact and commutes with the formation of duals and tensor powers (and hence exterior and symmetric powers).

Since duality does not affect whether or not the de Rham property holds, working with D_{dR} is equivalent to working with the contravariant functor

$$D^*_{\mathrm{dR}}(V) := D_{\mathrm{dR}}(V^\vee) \simeq \mathrm{Hom}_{\mathbf{Q}_p[G_K]}(V, B_{\mathrm{dR}});$$

this alternative functor can be very useful. In general $D^*_{\mathrm{dR}}(V)$ is a finite-dimensional K-vector space, and its elements correspond to $\mathbf{Q}_p[G_K]$-linear maps from V into B_{dR}. In particular, for any $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ the collection of all such maps spans a finite-dimensional K-subspace of B_{dR}, generally called the space of *p-adic periods of* V (or of $V^\vee$, depending on one's point of view). This space of periods for V is the only piece of B_{dR} that is relevant in the formation of $D^*_{\mathrm{dR}}(V)$. As an example, if V is an irreducible $\mathbf{Q}_p[G_K]$-module then any nonzero map from V to B_{dR} is injective and so $D^*_{\mathrm{dR}}(V) \neq 0$ precisely when V occurs as a subrepresentation of B_{dR}. In general $\dim_K D^*_{\mathrm{dR}}(V) \leqslant \dim_{\mathbf{Q}_p}(V)$, so an irreducible V appears in B_{dR} with finite multiplicity at most $\dim_{\mathbf{Q}_p}(V)$, and this maximal multiplicity is attained precisely when V is de Rham (as this is equivalent to $V^\vee$ being de Rham).

Example 6.1.1. For $n \in \mathbf{Z}$, $D_{\mathrm{dR}}(\mathbf{Q}_p(n)) = Kt^{-n}$ if we view $\mathbf{Q}_p(n)$ as $\mathbf{Q}_p$ with G_K-action by χ^n. This is 1-dimensional over K, so $\mathbf{Q}_p(n)$ is de Rham for all n.

The output of the functor D_{dR} has extra K-linear structure (arising from additional structure on the K-algebra B_{dR}), namely a K-linear filtration arising from the canonical K-linear filtration on the fraction field B_{dR} of the complete discrete valuation ring B^+_{dR} over K. Before we explain this in §6.3 and axiomatize the resulting finer target category of D_{dR} (as a subcategory of Vec_K), in §6.2 we review some terminology from linear algebra.

6.2. **Filtered vector spaces.** Let F be a field, and let Vec_F be the category of finite-dimensional F-vector spaces. In Definition 4.1.1 we defined the notion of a filtered vector space over F. In the finite-dimensional setting, if $(D, \{\mathrm{Fil}^i(D)\})$ is a filtered vector space over F with $\dim_F D < \infty$ then the filtration is exhaustive if and only if $\mathrm{Fil}^i(D) = D$ for $i \ll 0$ and it is separated if and only if $\mathrm{Fil}^i(D) = 0$ for $i \gg 0$. We let Fil_F denote the category of finite-dimensional filtered vector spaces $(D, \{\mathrm{Fil}^i(D)\})$ over F equipped with an exhaustive and separated filtration, where a *morphism* between such objects is a linear map $T : D' \to D$ that is filtration-compatible in the sense that $T(\mathrm{Fil}^i(D')) \subseteq \mathrm{Fil}^i(D)$ for all i.

In the category Fil_F there are good functorial notions of kernel and cokernel of a map $T : D' \to D$ between objects, namely the usual F-linear kernel and cokernel endowed respectively with the *subspace filtration*

$$\mathrm{Fil}^i(\ker T) := \ker(T) \cap \mathrm{Fil}^i(D') \subseteq \ker T$$

and the *quotient filtration*

$$\mathrm{Fil}^i(\operatorname{coker} T) := (\mathrm{Fil}^i(D) + T(D'))/T(D') \subseteq \operatorname{coker}(T).$$

These have the expected universal properties (for linear maps $D'_0 \to D'$ killed by T and linear maps $D \to D_0$ composing with T to give the zero map respectively), but beware that Fil_F is *not* abelian!!

More specifically, it can happen that $\ker T = \operatorname{coker} T = 0$ (i.e., T is an F-linear isomorphism) but T is not an isomorphism in Fil_F. The problem is that the even if T is an isomorphism when viewed in Vec_F, the filtration on D may be "finer" than on D' and so although $T(\mathrm{Fil}^i(D')) \subseteq \mathrm{Fil}^i(D))$ for all i, such inclusions may not always be equalities (so the linear inverse is not a filtration-compatible map).

Example 6.2.1. For example, we could take $D = D'$ as vector spaces and give D' the trivial filtration $\mathrm{Fil}^i(D') = D'$ for $i \leqslant 0$ and $\mathrm{Fil}^i(D') = 0$ for $i > 0$ whereas we define $\mathrm{Fil}^i(D) = D$ for $i \leqslant 4$ and $\mathrm{Fil}^i(D) = 0$ for $i > 4$. The identity map T is then bijective but not an isomorphism in Fil_F. Thus, the forgetful functor $\mathrm{Fil}_F \to \mathrm{Vec}_F$ loses too much information (though it is a faithful functor).

Despite the absence of a good abelian category structure on Fil_F, we can still define basic notions of linear algebra in the filtered setting, as follows.

Definition 6.2.2. For $D, D' \in \mathrm{Fil}_F$, the *tensor product* $D \otimes D'$ has underlying F-vector space $D \otimes_F D'$ and filtration

$$\mathrm{Fil}^n(D \otimes D') = \sum_{p+q=n} \mathrm{Fil}^p(D) \otimes_F \mathrm{Fil}^q(D')$$

that is checked to be exhaustive and separated. The *unit object* $F[0]$ is F as a vector space with $\mathrm{Fil}^i(F[0]) = F$ for $i \leqslant 0$ and $\mathrm{Fil}^i(F[0]) = 0$ for $i > 0$. (Canonically, $D \otimes F[0] \simeq F[0] \otimes D \simeq D$ in Fil_F for all D.)

The *dual* $D^\vee$ of $D \in \mathrm{Fil}_F$ has underlying F-vector space given by the F-linear dual $\mathrm{Hom}_F(D, F)$, and has the (exhaustive and separated) filtration

$$\mathrm{Fil}^i(D^\vee) = (\mathrm{Fil}^{1-i} D)^\perp := \{\ell \in D^\vee \mid \mathrm{Fil}^{1-i}(D) \subseteq \ker \ell\}.$$

The reason we use $\mathrm{Fil}^{1-i}(D)$ rather than $\mathrm{Fil}^{-i}(D)$ is to ensure that $F[0]^\vee = F[0]$ (check this identification!).

A *short exact sequence* in Fil_F is a diagram

$$0 \to D' \to D \to D'' \to 0$$

in Fil_F that is short exact as vector spaces with $D' = \ker(D \to D'')$ (i.e., D' has the subspace filtration from D) and $D'' = \mathrm{coker}(D \to D'')$ (i.e., D'' has the quotient filtration from D). Equivalently, for all i the diagram

$$0 \to \mathrm{Fil}^i(D') \to \mathrm{Fil}^i(D) \to \mathrm{Fil}^i(D'') \to 0 \tag{6.2.1}$$

is short exact as vector spaces.

There is also a naturally induced filtration on $\mathrm{Hom}_F(D', D)$ for $D, D' \in \mathrm{Fil}_F$, and it is useful that this can be defined in two equivalent ways. This is discussed in Exercise 6.4.1.

Example 6.2.3. The unit object $F[0]$ is naturally self-dual in Fil_F, and that there is a natural isomorphism $D^\vee \otimes D'^\vee \simeq (D \otimes D')^\vee$ in Fil_F induced by the usual F-linear isomorphism. Likewise we have the usual double-duality isomorphism $D \simeq D^{\vee\vee}$ in Fil_F, and the evaluation morphism $D \otimes D^\vee \to F[0]$ is a map in Fil_F.

Example 6.2.4. There is a natural "shift" operation in Fil_F: for $D \in \mathrm{Fil}_F$ and $n \in \mathbf{Z}$, define $D[n] \in \mathrm{Fil}_F$ to have the same underlying F-vector space but $\mathrm{Fil}^i(D[n]) = \mathrm{Fil}^{i+n}(D)$ for all $i \in \mathbf{Z}$. (There seems little risk of confusion caused by the notation $F[0]$ that we use for the unit object.)

We have $D[n]^\vee \simeq D^\vee[-n]$ in Fil_F in the evident manner, and shifting can be passed through either factor of a tensor product.

Observe that if $T : D' \to D$ is a map in Fil_F there are two notions of "image" that are generally distinct in Fil_F but have the same underlying space. We define the *image* of T to be $T(D') \subseteq D$ with the subspace filtration from D. We define the *coimage* of T to be $T(D')$ with the quotient filtration from D'. Equivalently, $\mathrm{coim}\, T = D'/\ker T$ with the quotient filtration and $\mathrm{im}\, T = \ker(D \to \mathrm{coker}\, T)$ with the subspace filtration. There is a canonical map $\mathrm{coim}\, T \to \mathrm{im}\, T$ in Fil_F that is a linear bijection, and it is generally not an isomorphism in Fil_F.

Definition 6.2.5. A morphism $T : D' \to D$ in Fil_F is *strict* if the canonical map $\mathrm{coim}\, T \to \mathrm{im}\, T$ is an isomorphism, which is to say that the quotient and subspace filtrations on $T(D')$ coincide.

There is a natural functor $\mathrm{gr} = \mathrm{gr}^\bullet : \mathrm{Fil}_F \to \mathrm{Gr}_{F,f}$ to the category of finite-dimensional graded F-vector spaces via $\mathrm{gr}(D) = \oplus_i \mathrm{Fil}^i(D)/\mathrm{Fil}^{i+1}(D)$. This functor is dimension-preserving, and it is exact in the sense that if carries short exact sequences in Fil_F (see Definition 6.2.2, especially (6.2.1)) to short exact sequences in $\mathrm{Gr}_{F,f}$. By choosing bases compatible with filtrations we see that the functor gr is compatible with tensor products in the sense that there is a natural isomorphism

$$\mathrm{gr}(D) \otimes \mathrm{gr}(D') \simeq \mathrm{gr}(D \otimes D')$$

in $\mathrm{Gr}_{F,f}$ for any $D, D' \in \mathrm{Fil}_F$, using the tensor product grading on the left side and the tensor product filtration on $D \otimes D'$ on the right side.

6.3. **Filtration on D_{dR}.** For $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, the K-vector space $D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes V)^{G_K} \in \mathrm{Vec}_K$ has a natural structure of object in Fil_K: since B_{dR} has an exhaustive and separated G_K-stable K-linear filtration via $\mathrm{Fil}^i(B_{\mathrm{dR}}) = t^i B_{\mathrm{dR}}^+$, we get an evident K-linear G_K-stable filtration $\{\mathrm{Fil}^i(B_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V\}$ on $B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$, so this induces an exhaustive and separated filtration on the *finite-dimensional* K-subspace $D_{\mathrm{dR}}(V)$ of G_K-invariant elements. Explicitly,

$$\mathrm{Fil}^i(D_{\mathrm{dR}}(V)) = (t^i B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V)^{G_K}.$$

The finite-dimensionality of $D_{\mathrm{dR}}(V)$ is what ensures that this filtration fills up all of $D_{\mathrm{dR}}(V)$ for sufficiently negative filtration degrees and vanishes for sufficiently positive filtration degrees.

Example 6.3.1. For $n \in \mathbf{Z}$, $D_{\mathrm{dR}}(\mathbf{Q}_p(n))$ is 1-dimensional with its unique filtration jump in degree $-n$ (i.e., gr^{-n} is nonzero).

Proposition 6.3.2. *If V is de Rham then V is Hodge–Tate and $\mathrm{gr}(D_{\mathrm{dR}}(V)) = D_{\mathrm{HT}}(V)$ as graded K-vector spaces. In general there is an injection $\mathrm{gr}(D_{\mathrm{dR}}(V)) \hookrightarrow D_{\mathrm{HT}}(V)$ and it is an equality of $\mathbf{C}_K$-vector spaces when V is de Rham.*

The inclusion in the proposition can be an equality in some cases with V not de Rham, such as when $D_{\mathrm{HT}}(V) = 0$ and $V \neq 0$.

Proof. By left exactness of the formation of G_K-invariants, we get a natural K-linear injection

$$\mathrm{gr}(D_{\mathrm{dR}}(V)) \hookrightarrow D_{\mathrm{HT}}(V)$$

for all $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ because $\mathrm{gr}(B_{\mathrm{dR}}) = B_{\mathrm{HT}}$ as graded $\mathbf{C}_K$-algebras with G_K-action. Thus,

$$\dim_K D_{\mathrm{dR}}(V) = \dim_K \mathrm{gr}(D_{\mathrm{dR}}(V)) \leqslant \dim_K D_{\mathrm{HT}}(V) \leqslant \dim_{\mathbf{Q}_p}(V)$$

for all V. In the de Rham case the outer terms are equal, so the inequalities are all equalities. ∎

In the spirit of the Hodge–Tate case, we say that the *Hodge–Tate weights* of a de Rham representation V are those i for which the filtration on $D_{\mathrm{dR}}(V)$ "jumps" from degree i to degree $i+1$, which is to say $\mathrm{gr}^i(D_{\mathrm{dR}}(V)) \neq 0$. This says exactly that the graded vector space $\mathrm{gr}(D_{\mathrm{dR}}(V)) = D_{\mathrm{HT}}(V)$ has a nonzero term in degree i, which is the old notion of $\mathbf{C}_K \otimes_{\mathbf{Q}_p} V$ having i as a Hodge–Tate weight. The multiplicity of such an i as a Hodge–Tate weight is the K-dimension of the filtration jump, which is to say $\dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(V))$.

Since $D_{\mathrm{dR}}(\mathbf{Q}_p(n))$ is a line with nontrivial gr^{-n}, we have that $\mathbf{Q}_p(n)$ has Hodge–Tate weight $-n$ (with multiplicity 1). Thus, sometimes it is more convenient to define Hodge–Tate weights using the same filtration condition ($\mathrm{gr}^i \neq 0$) applied to the contravariant functor $D_{\mathrm{dR}}^*(V) = D_{\mathrm{dR}}(V^\vee) = \mathrm{Hom}_{\mathbf{Q}_p[G_K]}(V, B_{\mathrm{dR}})$ so as to negate things (so that $\mathbf{Q}_p(n)$ acquires Hodge–Tate weight n instead).

The general formalism of §5 tells us that D_{dR} on the full subcategory $\mathrm{Rep}_{\mathbf{Q}}^{\mathrm{dR}}(G_K)$ is exact and respects tensor products and duals when viewed with values in Vec_K, but it is a stronger property to ask if the same is true as a functor valued in Fil_K. For example, when D_{dR} on $\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}}(G_K)$ is viewed with values in Fil_K it is a faithful functor, since the forgetful functor

$\mathrm{Fil}_K \to \mathrm{Vec}_K$ is faithful and D_{dR} is faithful when valued in Vec_K. However, it is less mechanical to check if the general isomorphism

$$D_{\mathrm{dR}}(V') \otimes_K D_{\mathrm{dR}}(V) \simeq D_{\mathrm{dR}}(V' \otimes_{\mathbf{Q}_p} V)$$

in Vec_K for de Rham representations V and V' is actually an isomorphism in Fil_K (using the tensor product filtration on the left side). Fortunately, such good behavior of isomorphisms relative to filtrations does hold:

Proposition 6.3.3. *The faithful functor $D_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K) \to \mathrm{Fil}_K$ carries short exact sequences to short exact sequences and is compatible with the formation of tensor products and duals. In particular, if V is a de Rham representation and*

$$0 \to V' \to V \to V'' \to 0$$

is a short exact sequence in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ (so V' and V'' are de Rham) then $D_{\mathrm{dR}}(V') \subseteq D_{\mathrm{dR}}(V)$ has the subspace filtration and the linear quotient $D_{\mathrm{dR}}(V'')$ of $D_{\mathrm{dR}}(V)$ has the quotient filtration.

Once this proposition is proved, it follows that D_{dR} with its filtration structure is compatible with the formation of exterior and symmetric powers (endowed with their natural quotient filtrations as operations on Fil_K).

Proof. For any short exact sequence

$$0 \to V' \to V \to V'' \to 0 \tag{6.3.1}$$

in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ the sequence

$$0 \to \mathrm{Fil}^i(D_{\mathrm{dR}}(V')) \to \mathrm{Fil}^i(D_{\mathrm{dR}}(V)) \to \mathrm{Fil}^i(D_{\mathrm{dR}}(V'')) \tag{6.3.2}$$

is always left-exact, but surjectivity may fail on the right. However, when V is de Rham all terms in (6.3.1) are Hodge–Tate and so the functor D_{HT} applied to (6.3.1) yields an exact sequence. Passing to separate graded degrees gives that the sequence of $\mathrm{gr}^i(D_{\mathrm{HT}}(\cdot))$'s is short exact, but this is the same as the $\mathrm{gr}^i(D_{\mathrm{dR}}(\cdot))$'s since V', V, and V'' are de Rham (by Theorem 5.2.1(2)). Hence, adding up dimensions of gr^j's for $j \leqslant i$ gives

$$\dim_K \mathrm{Fil}^i(D_{\mathrm{dR}}(V)) = \dim_K \mathrm{Fil}^i(D_{\mathrm{dR}}(V')) + \dim_K \mathrm{Fil}^i(D_{\mathrm{dR}}(V'')),$$

so the left-exact sequence (6.3.2) is also right-exact in the de Rham case. This settles the exactness properties for the Fil_K-valued D_{dR}, as well as the subspace and quotient filtration claims.

Now consider the claims concerning the behavior of D_{dR} with respect to tensor product and dual filtrations. By the general formalism of §5 we have K-linear isomorphisms

$$D_{\mathrm{dR}}(V) \otimes_K D_{\mathrm{dR}}(V') \simeq D_{\mathrm{dR}}(V \otimes_{\mathbf{Q}_p} V'),\ \ D_{\mathrm{dR}}(V)^\vee \simeq D_{\mathrm{dR}}(V^\vee)$$

for $V, V' \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K)$. The second of these isomorphisms is induced by the mapping

$$D_{\mathrm{dR}}(V) \otimes_K D_{\mathrm{dR}}(V^\vee) \simeq D_{\mathrm{dR}}(V \otimes_{\mathbf{Q}_p} V^\vee) \to D_{\mathrm{dR}}(\mathbf{Q}_p) = K[0],$$

and so if the tensor-compatibility is settled then at least the duality comparison isomorphism in Vec_K is a morphism in Fil_K.

The construction of the tensor comparison isomorphism for the Vec_K-valued D_{dR} rests on the multiplicative structure of B_{dR}, so since B_{dR} is a filtered ring it is immediate that the tensor comparison isomorphism in Vec_K for D_{dR} is at least a morphism in Fil_K. In view of the finite-dimensionality and the exhaustiveness and separatedness of the filtrations, this morphism in Fil_K that is known to be an isomorphism in Vec_K is an isomorphism in Fil_K precisely when the induced map on associated graded spaces is an isomorphism. But $\mathrm{gr}(D_{\mathrm{dR}}) = D_{\mathrm{HT}}$ on de Rham representations and $\mathrm{gr} : \mathrm{Fil}_K \to \mathrm{Gr}_{K,f}$ is compatible with the formation of tensor products, so our problem is reduced to the Hodge–Tate tensor comparison isomorphism being an isomorphism in $\mathrm{Gr}_{K,f}$ (and not just in Vec_K). But this final assertion is part of Theorem 2.4.11. The same mechanism works for the case of dualities. ■

The following corollary is very useful, and is often invoked without comment.

Corollary 6.3.4. *For $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ and $n \in \mathbf{Z}$, V is de Rham if and only if $V(n)$ is de Rham.*

Proof. By Example 6.3.1, this follows from the tensor compatibility in Proposition 6.3.3 and the isomorphism $V \simeq (V(n))(-n)$. ■

Example 6.3.5. We now give an example of a Hodge–Tate representation that is not de Rham. Consider a *non-split* short exact sequence

$$0 \to \mathbf{Q}_p \to V \to \mathbf{Q}_p(1) \to 0 \tag{6.3.3}$$

in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$. The existence of such a non-split extension amounts to the non-vanishing of $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbf{Q}_p(-1))$, and at least when k is finite such non-vanishing is a consequence of the Euler characteristic formula for H^1's in the $\mathbf{Q}_p$-version of Tate local duality.

We now show that any such extension V is Hodge–Tate. Applying $\mathbf{C}_K \otimes_{\mathbf{Q}_p} (\cdot)$ to (6.3.3) gives an extension of $\mathbf{C}_K(1)$ by $\mathbf{C}_K$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, and $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbf{C}_K(-1)) = 0$ by the Tate–Sen theorem. Thus, our extension structure on $\mathbf{C}_K \otimes_{\mathbf{Q}_p} V$ is split in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, so implies $\mathbf{C}_K \otimes_{\mathbf{Q}_p} V \simeq \mathbf{C}_K \oplus \mathbf{C}_K(-1)$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$. The Hodge–Tate property for V therefore holds. However, we claim that such a non-split extension V is *never* de Rham!

There is no known elementary proof of this fact. The only known proof rests on very deep results, namely that de Rham representations must be *potentially* semistable in the sense of being $B_{\mathrm{st},K'}$-admissible after restriction to $G_{K'}$ for a suitable finite extension K'/K inside of $\overline{K}$, where $B_{\mathrm{st},K'} \subseteq B_{\mathrm{dR},K'} = B_{\mathrm{dR},K}$ is Fontaine's semistable period ring. It is an important fact that the category of $B_{\mathrm{st},K'}$-admissble p-adic representations of $G_{K'}$ admits a fully faithful functor $D_{\mathrm{st},K'}$ into a concrete abelian semilinear algebra category (of weakly admissible filtered (ϕ, N)-modules over K'), and that the Ext-group for $D_{\mathrm{st},K'}(\mathbf{Q}_p(1))$ by $D_{\mathrm{st},K'}(\mathbf{Q}_p)$ in this abelian category can be shown to vanish via a direct calculation in linear algebra. By full faithfulness of $D_{\mathrm{st},K'}$, this would force the original extension structure (6.3.3) on V to be $\mathbf{Q}_p[G_{K'}]$-linearly split. But the restriction map $\mathrm{H}^1(G_K, \mathbf{Q}_p(-1)) \to \mathrm{H}^1(G_{K'}, \mathbf{Q}_p(-1))$ is injective due to $[K' : K]$ being a unit in the coefficient ring $\mathbf{Q}_p$, so the original extension structure (6.3.3) on V in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ would then have to split, contrary to how V was chosen.

Example 6.3.6. To compensate for the incomplete justification (at the present time) of the preceding example, we now prove that for any extension structure

$$0 \to V' \to V \to V'' \to 0 \tag{6.3.4}$$

such that V' and V'' are de Rham (hence Hodge–Tate) and the Hodge–Tate weights of V' are all strictly larger than those of V'', the representation V is de Rham. (A basic example is $V'' = \mathbf{Q}_p$ and $V' = \mathbf{Q}_p(r)$ with $r > 0$. Likewise, by induction we see that any upper-triangular representation $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ having diagonal characters that are finitely-ramified twists of powers χ^{a_i} for which the exponents a_i are *strictly decreasing* is a de Rham representation.)

Before proving this claim, we first make a side remark that will not be used. If k is finite then by Kummer theory the space of isomorphism classes of such extensions with $V'' = \mathbf{Q}_p$ and $V' = \mathbf{Q}_p(1)$ has $\mathbf{Q}_p$-dimension $1 + [K : \mathbf{Q}_p]$, and a calculation with weakly admissible filtered (ϕ, N)-modules shows that there is only a 1-dimensional space of such extensions for which V is semistable (i.e., B_{st}-admissible), namely those V's that arise from "Tate curves" over K. Hence, these examples exhibit the difference between the de Rham property and the much finer admissibility property with respect to the finer period ring $B_{\mathrm{st}} \subseteq B_{\mathrm{dR}}$.

The de Rham property for V as above is the statement that $\dim_K D_{\mathrm{dR}}(V) = \dim_{\mathbf{Q}_p}(V)$. It is harmless to make a Tate twist, so we may and do arrange that the Hodge–Tate weights of V' are all $\geqslant 1$ and those of V'' are $\leqslant 0$. In particular, $D_{\mathrm{dR}}(V'') = (B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V'')^{G_K}$. We have a left exact sequence

$$0 \to D_{\mathrm{dR}}(V') \to D_{\mathrm{dR}}(V) \to D_{\mathrm{dR}}(V'')$$

in Fil_K with $\dim_K D_{\mathrm{dR}}(V') = \dim_{\mathbf{Q}_p} V'$ and $\dim_K D_{\mathrm{dR}}(V'') = \dim_{\mathbf{Q}_p} V''$. Hence, our problem is to prove surjectivity on the right, for which it suffices to prove that the natural map $(B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V)^{G_K} \to (B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V'')^{G_K} = D_{\mathrm{dR}}(V'')$ is surjective.

Applying $B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} (\cdot)$ to the initial short exact sequence (6.3.4) gives a G_K-equivariant short exact sequence of finite free B_{dR}^+-modules, so it admits a B_{dR}^+-linear splitting. The problem is to give such a splitting that is G_K-equivariant, and the obstruction is a *continuous* 1-cocycle on G_K valued in the topological module $B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V'$. This has the filtration by G_K-stable closed B_{dR}^+-submodules $t^n B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V'$ with $n \geqslant 0$. (Recall from Exercise 4.5.3 that multiplication by any uniformizer of B_{dR}^+ is a closed embedding, so $t^n B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V'$ has as its subspace topology exactly its topology as a free module of rank 1 over the topological ring B_{dR}^+.) It suffices to prove $\mathrm{H}^1_{\mathrm{cont}}(G_K, B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V') = 0$. The G_K-equivariant exact sequence

$$0 \to t^{n+1} B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V' \to t^n B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V' \to (\mathbf{C}_K \otimes_{\mathbf{Q}_p} V')(n) \to 0$$

is topologically exact for $n \geqslant 0$. Since $\mathbf{C}_K \otimes_{\mathbf{Q}_p} V' \simeq \oplus \mathbf{C}_K(m_i)$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ for some $m_i \geqslant 1$, so $m_i + n \geqslant 1$ for all $n \geqslant 0$ and all i, $\mathrm{H}^1_{\mathrm{cont}}(G_K, (\mathbf{C}_K \otimes_{\mathbf{Q}_p} V')(n)) = 0$ for all $n \geqslant 0$ by the Tate–Sen theorem. We can therefore use a successive approximation argument with continuous 1-cocycles and the topological identification $B_{\mathrm{dR}}^+ = \varprojlim B_{\mathrm{dR}}^+ / t^n B_{\mathrm{dR}}^+$ to deduce that $\mathrm{H}^1_{\mathrm{cont}}(G_K, B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V') = 0$. (Concretely, by successive approximation we exhibit each continuous 1-cocycle as a 1-coboundary.)

An important refinement of Proposition 6.3.3 is that the de Rham comparison isomorphism is also filtration-compatible:

Proposition 6.3.7. *For $V \in \mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}}(G_K)$, the G_K-equivariant B_{dR}-linear comparison isomorphism*

$$\alpha : B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V) \simeq B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$$

respects the filtrations and its inverse does too.

Proof. By construction α is filtration-compatible, so the problem is to prove that its inverse is as well. It is equivalent to show that the induced B_{HT}-linear map $\mathrm{gr}(\alpha)$ on associated graded objects is an isomorphism. On the right side the associated graded object is naturally identified with $B_{\mathrm{HT}} \otimes_{\mathbf{Q}_p} V$. For the left side, we first recall that (by a calculation with filtration-adapted bases) the formation of the associated graded space of an arbitrary filtered K-vector space (of possibly infinite dimension) is naturally compatible with the formation of tensor products (in the graded and filtered senses), so the associated graded object for the left side is naturally identified with $B_{\mathrm{HT}} \otimes_K \mathrm{gr}(D_{\mathrm{dR}}(V))$.

By Proposition 6.3.2, the de Rham representation V is Hodge–Tate and there is a natural graded isomorphism $\mathrm{gr}(D_{\mathrm{dR}}(V)) \simeq D_{\mathrm{HT}}(V)$. In this manner, $\mathrm{gr}(\alpha)$ is naturally identified with the graded comparison morphism

$$\alpha_{\mathrm{HT}} : B_{\mathrm{HT}} \otimes_K D_{\mathrm{HT}}(V) \to B_{\mathrm{HT}} \otimes_{\mathbf{Q}_p} V$$

that is a graded isomorphism because V is Hodge–Tate. ■

Recall that the construction of B_{dR}^+ as a topological ring with G_K-action only depends on $\mathscr{O}_{\mathbf{C}_K}$ endowed with its G_K-action. Thus, replacing K with a discretely-valued complete subfield $K' \subseteq \mathbf{C}_K$ has no effect on the construction (aside from replacing G_K with the closed subgroup $G_{K'}$ within the isometric automorphism group of $\mathbf{C}_K$). It therefore makes sense to ask if the property of $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ being de Rham is insensitive to replacing K with such a K', in the sense that this problem involves the same period ring B_{dR} throughout (but with action by various subgroups of the initial G_K).

For accuracy, we now write $D_{\mathrm{dR},K}(V) := (B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K}$, so for a discretely-valued complete extension K'/K inside of $\mathbf{C}_K$ we have $D_{\mathrm{dR},K'}(V) = (B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_{K'}}$. There is an evident map

$$K' \otimes_K D_{\mathrm{dR},K}(V) \to D_{\mathrm{dR},K'}(V)$$

in $\mathrm{Fil}_{K'}$ for all $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ via the canonical compatible embeddings of K and K' into the same B_{dR}^+ (determined by the embedding of $\mathrm{W}(\overline{k})[1/p]$ into B_{dR}^+ and considerations with Hensel's Lemma and the residue field $\mathbf{C}_K$).

Proposition 6.3.8. *For any complete discretely-valued extension K'/K inside of $\mathbf{C}_K$ and any $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, the natural map $K' \otimes_K D_{\mathrm{dR},K}(V) \to D_{\mathrm{dR},K'}(V)$ is an isomorphism in $\mathrm{Fil}_{K'}$. In particular, V is de Rham as a G_K-representation if and only if V is de Rham as a $G_{K'}$-representation.*

As special cases, the de Rham property for G_K can be checked on $I_K = G_{\widehat{K^{\mathrm{un}}}}$ and it is insensitive to replacing K with a finite extension inside of $\mathbf{C}_K$. It must be emphasized that the insensitivity of the de Rham condition to ramified extension on K is a "bad" feature, akin to not distinguishing between good reduction and potentially good reduction for abelian varieties. The more subtle (and important) properties of being a crystalline or semi-stable representation will exhibit sensitivity to ramified extension on K, as we will explain and illustrate in §9.3.

Proof. The fields $\widehat{K'^{\mathrm{un}}}$ and $\widehat{K^{\mathrm{un}}}$ have the same residue field $\overline{k}$, so by finiteness of the absolute ramification we see that the resulting extension $\widehat{K^{\mathrm{un}}} \to \widehat{K'^{\mathrm{un}}}$ of completed maximal unramified extensions is of finite degree. Hence, it suffices to separately treat two special cases:

K'/K of finite degree and $K' = \widehat{K^{\mathrm{un}}}$. In the case of finite-degree extensions a transitivity argument reduces us to the case when K'/K is finite Galois. It follows from the definitions that for all $i \in \mathbf{Z}$, the finite-dimensional K'-vector space $\mathrm{Fil}^i(D_{\mathrm{dR},K'}(V))$ has a natural semilinear action by $\mathrm{Gal}(K'/K)$ whose K-subspace of invariants is $\mathrm{Fil}^i(D_{\mathrm{dR},K}(V))$. Thus, classical Galois descent for vector spaces as in (2.4.3) (applied to K'/K) gives the desired isomorphism result in $\mathrm{Fil}_{K'}$ in this case.

To adapt this argument to work in the case $K' = \widehat{K^{\mathrm{un}}}$, we wish to apply the "completed unramified descent" argument for vector spaces as in the proof of Theorem 2.4.6. It is follow from the definitions that for all $i \in \mathbf{Z}$, the finite-dimensional K'-vector space $\mathrm{Fil}^i(D_{\mathrm{dR},K'}(V))$ has a natural semilinear action by $G_K/I_K = G_k$ and the K-subspace of invariants is $\mathrm{Fil}^i(D_{\mathrm{dR},K}(V))$. Hence, to apply the completed unramified descent result we just have to check that the G_k-action on each $\mathrm{Fil}^i(D_{\mathrm{dR},K'}(V))$ is continuous for the natural topology on this finite-dimensional K'-vector space. More generally, consider the G_K-action on $t^i B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V$. We view this as a free module of finite rank over the topological ring B_{dR}^+ (using the topology from Exercise 4.5.3). It suffices to prove two things: (i) the G_K-action on $t^i B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V$ relative to the finite free module topology is continuous, and (ii) any finite-dimensional K'-subspace of $t^i B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V$ inherits as its subspace topology the natural topology as such a finite-dimensional vector space (over the p-adic field K'). Note that for the proof of (ii) we may rename K' as K since this does not affect the formation of B_{dR}^+, so it suffices for both claims to consider a common but arbitrary p-adic field K.

For (i), we can use multiplication by t^{-i} and replacement of V by $V(i)$ to reduce to checking continuity of the G_K-action on $B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V$ for any $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$. Continuity of the G_K-action on V and on B_{dR}^+ then gives the continuity of the G_K-action on $B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V$ by computing relative to a $\mathbf{Q}_p$-basis of V. To prove (ii) with $K' = K$, we may again replace V with $V(i)$ to reduce to the case $i = 0$. It is harmless to replace the given finite-dimensional K-subspace of $B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V$ with a larger one, so by considering elementary tensor expansions relative to a choice of $\mathbf{Q}_p$-basis of V we reduce to the case when the given finite-dimensional K-vector space has the form $W \otimes_{\mathbf{Q}_p} V$ for a finite-dimensional K-subspace of B_{dR}^+. We may therefore immediately reduce to showing that if $W \subseteq B_{\mathrm{dR}}^+$ is a finite-dimensional K-subspace then its subspace topology from B_{dR}^+ is its natural topology as a finite-dimensional K-vector space. This is part of Lemma 4.4.12. ■

Example 6.3.9. In the 1-dimensional case, the Hodge–Tate and de Rham properties are equivalent. Indeed, we have seen in general that de Rham representations are always Hodge–Tate (in any dimension), and for the converse suppose that V is a 1-dimensional Hodge–Tate representation. Thus, it has some Hodge–Tate weight i, so if we replace V with $V(-i)$ (as we may without loss of generality since every $\mathbf{Q}_p(n)$ is de Rham) we may reduce to the case when the continuous character $\psi : G_K \to \mathbf{Z}_p^\times$ of V is Hodge–Tate with Hodge–Tate weight 0. Hence, $\mathbf{C}_K(\psi)^{G_K} \neq 0$, so by the Tate–Sen theorem $\psi(I_K)$ is finite. By choosing a sufficiently ramified finite extension K'/K we can thereby arrange that $\psi(I_{K'}) = 1$. Since the de Rham property is insensitive to replacing K with $\widehat{K'^{\mathrm{un}}}$, we thereby reduce to the case of the trivial character, which is de Rham.

The same argument shows that any finite-dimensional p-adic representation W of G_K with open kernel on I_K is de Rham with 0 as the only Hodge–Tate weight, and that $D_{\mathrm{dR}}(W)$ is then a direct sum of copies of the unit object $K[0]$ in Fil_K.

Example 6.3.9 shows that the exact faithful tensor functor $D_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K) \to \mathrm{Fil}_K$ is not fully faithful. Indeed, in Example 6.3.9 we saw that if $\rho : G_K \to \mathrm{GL}(W)$ is a p-adic representation with *finite* image on I_K then W is necessarily de Rham and $D_{\mathrm{dR}}(W) \in \mathrm{Fil}_K$ is a direct sum of copies of the unit object $K[0]$ in Fil_K; this has lost all information about W beyond $\dim_{\mathbf{Q}_p} W$. In particular, the functor $D_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K) \to \mathrm{Fil}_K$ really is not fully faithful. In effect, D_{dR} is insensitive to finite ramification information. This is a serious deficiency, akin to losing the distinction between good reduction and potentially good reduction.

To improve on the situation we need to do two things. First, we have to replace B_{dR} with a period ring having "finer structure", in the same spirit as how B_{dR} has finer structure than B_{HT} (a filtration rather than just a grading) and is a kind of refinement of B_{HT} (i.e., $\mathrm{gr}(B_{\mathrm{dR}}) \simeq B_{\mathrm{HT}}$). More specifically, we will find a G_K-stable K_0-subalgebra $B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$ that admits a finer structure than subspace filtration. The second thing we have to do is to study properties of the functor $D_{\mathrm{cris}} := D_{B_{\mathrm{cris}}}$ with values in a richer linear algebra category than filtered vector spaces. The extra linear algebra structure that we seek is a synthesis of filtrations and Frobenius operators, and in §7 we will look at a number of examples arising from algebraic geometry which point the way to the right kind of synthesis which leads to a good generalization of the de Rham condition on p-adic representations of G_K.

6.4. **Exercises.**

Exercise 6.4.1. Let F be a field. For $D, D' \in \mathrm{Fil}_F$ we can naturally endow $\mathrm{Hom}_F(D', D)$ with a structure in Fil_F (denoted $\mathrm{Hom}(D', D)$). This can be done in two equivalent ways. First of all, the usual linear isomorphism $D \otimes_F D'^{\vee} \simeq \mathrm{Hom}_F(D', D)$ imposes a Fil_F-structure by using the dual filtration on $D'^{\vee}$ and the tensor product filtration on $D \otimes_F D'^{\vee}$.

This is too *ad hoc* to be useful by itself, so the usefulness rests on the ability to also describe this filtration in more direct terms in the language of Hom's: prove that this *ad hoc* definition yields

$$\mathrm{Fil}^i(\mathrm{Hom}_F(D', D)) = \{T \in \mathrm{Hom}_F(D', D) \,|\, T(\mathrm{Fil}^j(D')) \subseteq \mathrm{Fil}^{j+i}(D) \text{ for all } j\}.$$

In other words, $\mathrm{Fil}^i(\mathrm{Hom}_F(D', D)) = \mathrm{Hom}_{\mathrm{Fil}_F}(D', D[i])$ for all $i \in \mathbf{Z}$. (Hint: Compute using bases of D and D' adapted to the filtrations on these spaces.) What does this say for $i = 0$? Or $D = F[0]$?

Exercise 6.4.2. Verify the successive approximation argument at the end of Example 6.3.6.

Exercise 6.4.3. Although p-adic Hodge theory addresses finite-dimensional $\mathbf{Q}_p$-linear representations, in practice one often has to work with continuous linear representations of G_K on finite-dimensional vector spaces V over a finite extension $E/\mathbf{Q}_p$. In such cases one may want a variant of the basic formalism, adapted to E-linear structures. At least for the de Rham and Hodge–Tate properties (for which replacing K with a finite extension is harmless), it is generally harmless to suppose that K is large enough to contain a Galois closure of E over

$\mathbf{Q}_p$, and then it makes sense to try to redo the theory with E replacing $\mathbf{Q}_p$ in some of the basic constructions.

In this exercise we take up the first interesting question in this direction: does Example 6.3.9 carry over to 1-dimensional representations over a finite extension $E/\mathbf{Q}_p$? Let $\psi : G_K \to \mathscr{O}_E^\times$ be a continuous character. This is an object V_ψ in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ with $\mathbf{Q}_p$-dimension $[E : \mathbf{Q}_p]$. We aim to prove that V_ψ is de Rham if (and only if) it is Hodge–Tate, so we may and do assume that K contains a Galois closure of E over $\mathbf{Q}_p$. The method of solution of Example 6.3.9 cannot be used, since $\mathscr{O}_E^\times$ near the identity is generally not $\mathbf{Z}_p$.

(1) Prove that $\mathbf{C}_K \otimes_{\mathbf{Q}_p} V_\psi \simeq \prod_\sigma \mathbf{C}_K(\psi^\sigma)$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, where σ ranges through all $\mathbf{Q}_p$-embeddings $E \hookrightarrow K$ (and $\psi^\sigma = \sigma \circ \psi$ is viewed as a $K^\times$-valued character of G_K). Deduce that V_ψ is Hodge–Tate if and only if there are integers n_σ such that $\mathbf{C}_K(\psi^\sigma) \simeq \mathbf{C}_K(\chi^{n_\sigma})$ for all σ. (Beware that the n_σ's really can be different from each other. This already occurs for elliptic curves with complex multiplication: if L is an imaginary quadratic field that is inert at p and if E is an elliptic curve over L with complex multiplication by L then for $K = L_p$ the representation $V_p(E_K)$ is a 1-dimensional object in $\mathrm{Rep}_K(G_K)$, say with G_K-action given by a continuous character $\psi : G_K \to K^\times$, and as an object in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ is it Hodge-Tate with Hodge–Tate weights $\{0, 1\}$. Hence, $V_\psi \simeq V_p(E_K)$ has two distinct Hodge–Tate weights.)

(2) For $W \in \mathrm{Rep}_K(G_K)$, define $D_{\mathrm{dR},K}(W) = (B_{\mathrm{dR}} \otimes_K W)^{G_K} \in \mathrm{Fil}_K$. Prove that for $W \in \mathrm{Rep}_E(G_K)$, $D_{\mathrm{dR}}(W) = \prod_\sigma D_{\mathrm{dR},K}(W^\sigma)$ in Fil_K, where σ ranges through all $\mathbf{Q}_p$-embeddings $E \to K$ and $W^\sigma = K \otimes_{\sigma,E} W \in \mathrm{Rep}_K(G_K)$. Deduce that each $D_{\mathrm{dR},K}(\psi^\sigma)$ is either 0 or 1-dimensional over K, and that V_ψ if de Rham if and only if $D_{\mathrm{dR},K}(\psi^\sigma) \neq 0$ for all σ.

(3) Using a suitable Tate twist, reduce to showing that if $\psi : G_K \to \mathscr{O}_K^\times$ is continuous and $\mathbf{C}_K(\psi) \simeq \mathbf{C}_K$ then $D_{\mathrm{dR},K}(\psi) \neq 0$.

(4) Assuming $\mathbf{C}_K(\psi) \simeq \mathbf{C}_K$, prove that $\mathrm{H}^1(G_K, \mathbf{C}_K(\psi\chi^n)) = 0$ for all $n \geqslant 1$, and deduce that $\mathrm{H}^1(G_K, tB_{\mathrm{dR}}^+(\psi)) = 0$. Finally, conclude that $D_{\mathrm{dR},K}(\psi)$ maps onto $\mathrm{H}^0(G_K, \mathbf{C}_K(\psi)) = K$, so $D_{\mathrm{dR},K}(\psi) \neq 0$, as desired.

7. Why filtered isocrystals?

To motivate how to generalize Fil_K to classify "good" p-adic representations, we shall now study good reduction for an abelian variety A over a p-adic field K. By "good reduction" we mean that A is the K-fiber of an *abelian scheme* (i.e., smooth proper group scheme over $\mathrm{Spec}\, \mathscr{O}_K$ with connected geometric fibers). For example, when A is an elliptic curve this amounts to the existence of a Weierstrass model over $\mathscr{O}_K$ with smooth reduction. Below we will provide a summary of some basic facts from the theory of group schemes, assuming the reader has some prior experience with the concept of a group variety, and knows how to think functorially with schemes (i.e., Yoneda's Lemma). A nice introduction to the subject of group schemes is given in [36, Ch. 3].

(In [37, Ch. 6] there is given a self-contained development of the basic theory of abelian schemes; the main point to keep in mind is that they are always commutative, just like abelian varieties, and they exhibit many of the familiar features of abelian varieties. One should think of an abelian scheme $A \to S$ as a family of abelian varieties parameterized by S. The most "concrete" examples of abelian schemes beyond Weierstrass models with unit discriminant are the relative Jacobians $\mathrm{Pic}^0_{X/S}$ for a smooth proper curve $X \to S$ with connected geometric fibers.)

For any prime ℓ, the ℓ-adic representation space

$$V_\ell(A) = \mathbf{Q}_\ell \otimes_{\mathbf{Z}_\ell} \varprojlim A[\ell^n](\overline{K})$$

for G_K encodes information about A. However, the case $\ell \neq p$ is far simpler to understand, and we will begin with this case in order to get some intuition. The most basic interesting fact is the *Néron-Ogg-Shafarevich criterion* [9, 1.2/8, 7.4/5]: A has good reduction if and only if $V_\ell(A)$ is unramified. This is a striking result: a Galois-theoretic property (unramifiedness) implies a geometric property (existence of a nice integral model over $\mathscr{O}_K$). Of course, its proof uses an a-priori theory of best possible integral models over $\mathscr{O}_K$, namely the theory of Néron models.

The Néron-Ogg-Shafarevich criterion is extremely useful in the study of abelian varieties, but it requires $\ell \neq p$. It is natural to wonder if there is a property $\mathbf{P}$ of general p-adic representations of G_K such that good reduction for A is equivalent to $V_p(A)$ satisfying property $\mathbf{P}$. We already saw in Question 1.1.3 and the discussion following it that the most naive guess (i.e., unramifiedness) no longer works.

The right answer to the problem of formulating a Néron-Ogg-Shafarevich criterion "at p" will lead us to some new kinds of objects in linear algebra, merging filtered objects from Fil_K and K_0-vector spaces equipped with a bijective Frobenius endomorphism in the spirit of §3 (with the field $\mathscr{E}$ there replaced by the much simpler field $K_0 = \mathrm{W}(k)[1/p]$ equipped with its canonical Frobenius automorphism). These combined structures will be called *filtered isocrystals*, and they are ubiquitous throughout p-adic Hodge theory. A study of abelian varieties will lead us to many interesting examples of filtered isocrystals.

7.1. **Finite flat group schemes.** Grothendieck gave a good analogue of the Néron-Ogg-Shafarevich criterion for $\ell = p$. He recognized that the criterion can be formulated in a language different from Galois representations (using group schemes), as we will present in Remark 7.1.12, and he then used the theory of Néron models to prove that his modified formulation works even when $\ell = p$. Ultimately p-adic Hodge theory does provide a purely Galois-theoretic criterion for good reduction: a necessary and sufficient condition for A to have good reduction is that $V_p(A)$ is a *crystalline* G_K-representation, which is to say that it is B_{cris}-admissible for a certain "crytalline period ring" $B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$ to be introduced in §9.1. This does not obviate the need for Grothendieck's criterion! In fact, the way good reduction is connected to p-adic Hodge theory is via Grothendieck's criterion: Grothendieck established an equivalence between good reduction (for abelian varieties) and a property involving group schemes, and work of Fontaine [19, 5.5, 6.2], Breuil [10, Thm. 1.4], and Kisin [30, Cor. 2.2.6] establishes an equivalence between Grothendieck's group scheme criterion and the crystalline condition over any p-adic field K.

Definition 7.1.1. Let S be a scheme. A group scheme over S (or an S-*group*) is a group object $G \to S$ in the category of S-schemes. That is, there are given maps $m : G \times_S G \to G$, $i : G \to G$, and $e : S \to G$ over S satisfying the commutative diagrams that characterize a group law with inversion and identity. Equivalently (by Yoneda's Lemma), for any S-scheme T the set $G(T) = \mathrm{Hom}_S(T, G)$ is endowed with a group structure and this structure is functorial in T.

A *homomorphism* of S-group schemes is a map $f : G \to H$ that is compatible with the multiplication and inversion morphisms, and with the identity sections. Equivalently, for all S-schemes T, the induced map of sets $G(T) \to H(T)$ is a group homomorphism.

The (scheme-theoretic) *kernel* of an S-group homomorphism f is the S-group scheme $\ker f = G \times_{H,e_H} S$ obtained by pullback of f along the identity section $e_H : S \to H$; functorially, $(\ker f)(T) = \ker(G(T) \to H(T))$ for every S-scheme T.

We are most interested in Definition 7.1.1 for $S = \mathrm{Spec}\, K$, $\mathrm{Spec}\, \mathscr{O}_K$, and $\mathrm{Spec}\, k$. However, considerations over more general S clarify the initial development of the theory (and are necessary to give valid arguments via Yoneda's Lemma).

Example 7.1.2. If E is an elliptic curve over a field k and n is a positive integer, then the k-group $[n] : E \to E$ is finite flat of degree n^2. Hence, the scheme-theoretic kernel $E[n] := [n]^{-1}(0)$ is a finite k-group scheme of degree n^2, even if $\mathrm{char}(k)|n$. For example, if E is a supersingular elliptic curve and $\mathrm{char}(k) = p$ then $E[p]$ is a finite k-group scheme of degree p^2 but it has only the trivial geometric point (so it is an infinitesimal k-scheme).

This example is the reason why group schemes are essential for a good understanding of torsion in abelian varieties in positive characteristic. (The scheme-theoretic kernel of an isogeny is another good example: it could be nontrivial but have only the trivial geometric point, such as the Frobenius isogeny of an elliptic curve.)

Example 7.1.3. The $\mathbf{Z}$-group schemes $\mathbf{G}_a$ and $\mathbf{G}_m$ correspond to the additive structure on the affine line and the multiplicative structure on the complement of the origin. The corresponding functors associate to any scheme T the additive group of global functions on T and the multiplicative group of global units on T respective. See Exercise 7.4.1.

Observe that if G is an S-group scheme and $S' \to S$ is any map of schemes then the base change $G' = G \times_S S'$ has a natural structure of S'-group scheme. As an illustration of the utility of base change with group schemes, we note that many classical "matrix groups" are really group schemes over $\mathbf{Z}$, and base change to fields recover the classical viewpoint of such matrix groups as group varieties defined over a field via "universal" matrix equations that are independent of the base field. For example, GL_n, SL_n, and Sp_{2g} are really affine group schemes over $\mathbf{Z}$ (with GL_n having coordinate ring $\mathbf{Z}[x_{ij}]_{\det(x_{ij})}$, etc.), and $\det : \mathrm{GL}_n \to \mathrm{GL}_1$ is a $\mathbf{Z}$-group homomorphism whose kernel is SL_n. This viewpoint gives a precise way of saying that certain group-theoretic considerations with matrices are "universal" (i.e., independent of a base field) since they are really statements about group schemes over $\mathbf{Z}$.

An important example of a group scheme homomorphism in characteristic $p > 0$ is the relative Frobenius morphism. For example, if E is an elliptic curve over a field k of characteristic $p > 0$, the relative Frobenius morphism is the degree-p isogeny $E \to E^{(p)}$ described by $(x, y) \mapsto (x^p, y^p)$ in Weierstrass coordinates. Its scheme-theoretic kernel if an infinitesimal k-subgroup scheme of E of degree p. We now explain how to generalize this to all group schemes in characteristic $p > 0$.

Consider a base scheme S over $\operatorname{Spec} \mathbf{F}_p$. Let $F_S : S \to S$ denote the absolute Frobenius endomorphism (which corresponds to the p-power map on the coordinate ring of any affine open; it is the identity map on underlying topological spaces). This is functorial in S since the p-power map commutes with all homomorphisms of $\mathbf{F}_p$-algebras. If X is an S-scheme then we let $X^{(p)}$ denote the base change $S \times_{F_S, S} X$ viewed as an S-scheme via pr_1; loosely speaking, $X^{(p)}$ is the scheme obtained from X by replacing the coefficients of the defining equations over S by their pth powers (but leaving the indeterminates alone).

Definition 7.1.4. The *relative Frobenius morphism* $F_{X/S} : X \to X^{(p)}$ is the unique S-map fitting into the commutative diagram

$$\begin{tikzcd} X \arrow[r, "F_{X/S}"'] \arrow[rr, bend left, "F_X"] \arrow[dr, bend right] & X^{(p)} \arrow[r, "\mathrm{pr}_2"'] \arrow[d] & X \arrow[d] \\ & S \arrow[r, "F_S"'] & S \end{tikzcd}$$

in which the square is cartesian. Loosely speaking, $F_{X/S}$ is the map given by pth powers on coordinates over S.

We emphasize that $F_{X/S}$ is a map of S-schemes (unlike F_X, which lies over the map F_S). From the definition one checks:

Lemma 7.1.5. *Let $X \to S$ be a map of $\mathbf{F}_p$-schemes. The formation of the relative Frobenius map $F_{X/S}$ is functorial in X, compatible with arbitrary base change on S (i.e., if $X' = S' \times_S X$ then $S' \times_S X^{(p)}$ is naturally identified with $X'^{(p)}$ over S' carrying $1_{S'} \times F_{X/S}$ over to $F_{X'/S'}$), and compatible with residue to products over S. In particular, $X = G$ is an S-group scheme then $F_{G/S} : G \to G^{(p)}$ is an S-group homomorphism.*

The notion of a commutative group scheme is defined in an evident manner, and if $G \to S$ is such a group scheme then we write $[n]_G : G \to G$ to denote the multiplication-by-n map and

$$G[n] := \ker([n]_G)$$

to denote its kernel. For example, an abelian scheme is always commutative, and we know from the theory over a field that its torsion subgroup schemes in that case are a powerful tool in the investigation of the structure of the abelian variety. This example generalizes to the case of an arbitrary base scheme, as follows.

Let $A \to S$ be an abelian scheme with fibers of constant dimension $g > 0$, and choose $n \in \mathbf{Z}_{>0}$. We are especially interested in the case $S = \operatorname{Spec} \mathscr{O}_K$. Consider the multiplication map $[n]_A : A \to A$. By the theory of abelian varieties, on geometric fibers over S this map becomes finite flat with constant degree n^{2g}. It follows by general fibral techniques (in case S is not the spectrum of a field) that the map $[n]_A : A \to A$ is finite flat and the pushforward $[n]_{A*}(\mathscr{O}_A)$ is locally free of constant rank n^{2g}. These properties are inherited by any base change of $[n]_A$ over its target, so using base change by $e_A : S \to A$ gives that the (scheme-theoretic) n-torsion $A[n] := \ker([n]_A)$ is a commutative finite flat S-group scheme whose geometric fibers all have rank n^{2g}.

As we know from elliptic curves in characteristic $p > 0$, the number of geometric points in the fibers of $A[n] \to S$ can be considerably less than n^{2g} when $p|n$. The n-torsion group schemes $A[n]$ motivate the interest in the following definition.

Definition 7.1.6. Let S be a scheme. A *finite flat group scheme* over S is a commutative group scheme $f : G \to S$ whose structural morphism to S is finite and flat with $f_*(\mathscr{O}_G)$ a locally free $\mathscr{O}_S$-module of some constant rank $r > 0$. We call r the *order* of the group scheme.

This definition can certainly be given without requiring commutativity, but it is of no interest to us in such cases. As an example, if $A \to S$ is an abelian scheme with fibers of constant dimension $g > 0$ then for any integer $n \geqslant 1$ the S-group $A[n]$ is a finite flat group scheme with order n^{2g}. Here are some more examples illustrating the appropriateness of the concept of *order* as defined above.

Example 7.1.7. Let S be a scheme.

(1) If Γ is a finite abelian group of size n then the disjoint union $\Gamma_S = \coprod_{\gamma \in \Gamma} S$ of copies of S indexed by Γ has a natural S-group scheme structure via the identitification
$$\Gamma_S \times_S \Gamma_S = \coprod_{(\gamma,\gamma') \in \Gamma \times \Gamma} S$$
and the group law on Γ. (Keep in mind that S may be disconnected!) More concretely, in terms of functors, for any S-scheme T we see that $\Gamma_S(T)$ is the set of locally constant maps $T \to \Gamma$ endowed with the evident pointwise group structure. From the definition, $f_*(\mathscr{O}_{\Gamma_S}) = \prod_{\gamma \in \Gamma} \mathscr{O}_S$ as an $\mathscr{O}_S$-algebra, so Γ_S is a finite flat group scheme of order n. We call Γ_S the *constant S-group* associated to Γ.

(2) The kernel μ_n of the nth-power map $t^n : \mathrm{GL}_1 \to \mathrm{GL}_1$ is a finite flat group scheme of order n. This is the *scheme of nth roots of unity*, and its "coordinate ring" over $\mathscr{O}_S$ is $\mathscr{O}_S[T]/(T^n - 1)$.

(3) Similarly, over an $\mathbf{F}_p$-scheme S, the kernel α_{p^m} of the p^mth-power map $t^{p^m} : \mathbf{G}_a \to \mathbf{G}_a$ is a finite flat group scheme of order p^m. This is the *scheme of p^mth roots of* 0, and its "coordinate ring" over $\mathscr{O}_S$ is $\mathscr{O}_S[T]/(T^{p^m})$. Although α_p and μ_p are isomorphic as S-groups, they are not isomorphic as S-groups when S is non-empty. (Try to write down a homomorphism between them over a geometric point!)

(4) A finite group scheme G over a field k is connected if and only if $G_{\mathrm{red}} = \mathrm{Spec}\, k$ (so G is infinitesimal), since $G(k)$ is always non-empty. Such G arise a lot when $\mathrm{char}(k) > 0$.

(5) Let E be an elliptic curve over a field k of characteristic $p > 0$. Then $E[p]$ is a finite flat k-group scheme of order p^2, and $E[p](\overline{k})$ has size 1 or p. In particular, $E[p]$ is *never* étale, and $E[p]$ is infinitesimal if and only if E is supersingular.

Remark 7.1.8. If $f : G \to H$ is a homomorphism between finite flat group schemes over a base S, then the kernel $\ker f$ is often not flat over S (unless S is the spectrum of a field). For example, if $S = \mathrm{Spec}\, \mathbf{Z}[\zeta_p]$ and $f : (\mathbf{Z}/p\mathbf{Z})_S \to \mu_p$ is the S-group map sending j to ζ_p^j for all $j \in \mathbf{Z}/p\mathbf{Z}$ then $\ker f$ has generic fiber $\{0\}$ of rank 1 but special fiber $(\mathbf{Z}/p\mathbf{Z})_{\mathbf{F}_p}$ of rank p, so $\ker f$ is not S-flat due to rank-jumping in the fibers.

It is a general theorem of Deligne [38, §1] that a finite flat (commutative!) group scheme is killed by its order; that is, if $G \to S$ has order n then the multiplication map $[n]_G : G \to G$ is the zero map (i.e., it is the composite of the structure map $G \to S$ and the identity section $e_G : S \to S$). If we drop the commutativity requirement on the group scheme then the analogous assertion is an unsolved problem. Deligne's theorem has the following nice consequence (which can be proved in other ways):

Lemma 7.1.9. *Let $f : G \to S$ be a finite flat group scheme whose order n is a unit on S. Then $G \to S$ is an étale map. In particular, if $S = \mathrm{Spec}(F)$ for a field F and $\mathrm{char}(F) \nmid n$ then $G(F_s)$ is a finite group of size n.*

The final part of the lemma is due to the fact that an étale scheme over a separably closed field is a disjoint union of rational points. In view of this lemma, the most interesting aspects of the theory of finite flat group schemes involve cases when n is not a unit on S, such as p-power torsion in an abelian scheme over $\mathrm{Spec}\, \mathscr{O}_K$.

Proof. Since $f_*(\mathscr{O}_G)$ is a finite locally free $\mathscr{O}_S$-module, it follows from the general theory of étale maps that f is étale if and only if it is so on geometric fibers over S. Hence, we can assume that $S = \mathrm{Spec}(F)$ for an algebraically closed field F. In this case étaleness amounts to being a disjoint union of rational points. Since all physical points of G have residue field F (as F is algebraically closed), to prove that the artinian coordinate ring of G is a product of copies of F we can use translation in $G(F)$ to reduce to showing that the artin local ring at the identity point has vanishing maximal ideal $\mathfrak{m}$. But $\mathfrak{m}/\mathfrak{m}^2$ is dual to the tangent space $T_0(G)$, so it suffices to prove $T_0(G)$ vanishes.

By Deligne's theorem, the multiplication map $[n]_G : G \to G$ is the zero map, so it factors as

$$G \to \mathrm{Spec}\, F \overset{e_G}{\to} G.$$

Applying the Chain Rule, $\mathrm{d}[n]_G(0) : T_0(G) \to T_0(G)$ factors through 0 and hence vanishes. But we have another way to compute $\mathrm{d}[n]_G(0)$! From the definition of a group scheme it can be checked (as in [36, Ch. II, §4] for group varieties) that $m : G \times G \to G$ has differential

$$T_0(G) \oplus T_0(G) \simeq T_{(0,0)}(G \times G) \to T_0(G)$$

that is ordinary addition. Hence, by the Chain Rule, $\mathrm{d}(f + h)(0) = \mathrm{d}f(0) + \mathrm{d}h(0)$ for any two F-group maps $f, h : G \rightrightarrows G$. By induction, it follows that $\mathrm{d}[n]_G(0) : T_0(G) \to T_0(G)$ is multiplication by $n \in F$ for any $n \in \mathbf{Z}$. Since $\mathrm{d}[n]_G(0)$ has been shown to vanish, we conclude that n kills the F-vector space $T_0(G)$. But by hypothesis $\mathrm{char}(F) \nmid n$, so the only possibility is $T_0(G) = 0$. ■

Using terminology from Example 1.2.4, we have a construction in the opposite direction:

Lemma 7.1.10. *Let F be a field and $G_F = \mathrm{Gal}(F_s/F)$. The functor $X \rightsquigarrow X(F_s)$ from finite étale F-schemes to finite G_F-sets is an equivalence of categories and is compatible with fiber products. In particular, $G \rightsquigarrow G(F_s)$ is an equivalence between the category of finite étale F-groups and the category of finite G_F-modules.*

Proof. Since fiber products are characterized by categorical means, the compatibility with fiber products is a formal consequence of the categorical equivalence (once it is proved).

Also, since group objects are characterized in categorical terms involving products, it will be automatic that the equivalence will restrict to one between group objects, as well as commutative group objects.

Every finite étale F-scheme becomes totally split (i.e., a disjoint union of rational points) over some finite Galois extension of F, so it suffices to prove the more precise statement that for a finite Galois extension F'/F, the functor $X \rightsquigarrow X(F')$ from finite étale F-schemes split by F' to finite $\mathrm{Gal}(F'/F)$-sets is an equivalence (compatibly with change in F').

The classical Galois descent isomorphism for vector spaces in (2.4.3) is compatible with tensor products over the base field and so restricts to an equivalence between the category of finite commutative F-algebras and the category of finite commutative F'-algebras equipped with an action by $\mathrm{Gal}(F'/F)$. Under this latter equivalence, those F-algebras A that are finite étale and split by F are exactly the ones for which $A' = F' \otimes_F A$ is a finite product of copies of F'. Hence, we are reduced to the elementary fact that the category of F'-algebras that are finite products of copies of F' is equivalent to the category of finite sets via the formation of F'-rational points. ■

Lemma 7.1.9 and Lemma 7.1.10 show that over a field F of characteristic 0, the theory of finite flat group schemes is the same as the theory of finite G_F-modules. But this reformulation is not merely linguistic: it enables us to define a good theory of "integral models" for Galois modules when $F = K$ is a p-adic field: if M is a finite discrete G_K-module then an *integral model* for M is a finite flat group scheme $\mathscr{M}$ over $\mathscr{O}_K$ whose generic fiber is the finite étale K-group scheme associated to M; i.e., $M \simeq \mathscr{M}(\overline{K})$ as G_K-modules. (In particular, $\mathscr{M}$ has order equal to the size of M.) This concept provides good insight into the role of unramifiedness in the study of finite G_K-modules with order prime to p:

Proposition 7.1.11. *The functor $\mathscr{X} \rightsquigarrow \mathscr{X}_K$ from finite étale $\mathscr{O}_K$-schemes to finite étale K-schemes is an equivalence onto the category of finite étale K-schemes X for which $X(\overline{K})$ has unramified G_K-action.*

In particular, a finite G_K-module with size not divisible by p admits an integral model if and only if it is unramified, in which case such a model is finite étale and unique.

Proof. The second part follows from the first due to Lemma 7.1.9. As for the first part, since a finite étale $\mathscr{O}_K$-algebra always has unramified K-fiber (essentially by definition of étale) we see that the functor lands where we expect. The full faithfulness follows by using normalization to construct an inverse functor. That is, any finite étale $\mathscr{O}_K$-algebra may be reconstructed from its unramified K-fiber as the integral closure of $\mathscr{O}_K$ in the coordinate ring of this K-fiber.

Finally, we have to check that any finite étale K-scheme with unramified G_K-action on its $\overline{K}$-points arises as the K-fiber of a finite étale $\mathscr{O}_K$-scheme. Passing to physical points, this says that if K'/K is a finite unramified extension then $K' = K \otimes_{\mathscr{O}_K} A'$ for a finite étale $\mathscr{O}_K$-algebra A'. Simply take $A' = \mathscr{O}_{K'}$! ■

Remark 7.1.12. Proposition 7.1.11 lets us reformulate the Néron-Ogg-Shafarevich criterion in a new form: if A is an abelian variety over K and $\ell \neq p$ is a prime then A has good reduction if and only if $A[\ell^n]$ admits an integral model $\mathscr{G}_n$ for all $n \geqslant 1$, in which case $\mathscr{G}_n = \mathscr{G}_{n+1}[\ell^n]$ for all $n \geqslant 1$.

In view of Remark 7.1.12, Grothendieck's version [26, Exp. IX, Thm. 5.13] allowing $\ell = p$ is very natural:

Theorem 7.1.13 (Grothendieck)**.** *Let A be an abelian variety over K and let ℓ be an arbitrary prime. Then A has good reduction if and only if $A[\ell^n]$ admits an integral model $\mathscr{G}_n$ for all $n \geqslant 1$ with $\mathscr{G}_n = \mathscr{G}_{n+1}[\ell^n]$ (respecting the K-fiber identification $A[\ell^n] = (A[\ell^{n+1}][\ell^n])$ for all $n \geqslant 1$. In such cases, if $\mathscr{A}$ is the abelian scheme over $\mathscr{O}_K$ with K-fiber A then necessarily $\mathscr{G}_n = \mathscr{A}[\ell^n]$ for all $n \geqslant 1$.*

In this result, each $\mathscr{G}_{n+1}$ has p^n-torsion that is equal to the $\mathscr{O}_K$-group $\mathscr{G}_n$ which is *flat* over $\mathscr{O}_K$. Such flatness for the p^n-torsion of $\mathscr{G}_{n+1}$ is a nontrivial condition, in view of Remark 7.1.8. Beware that when $\ell = p$ it is generally the case that there can be more than one integral model for a given finite flat K-group scheme (if there is any such model at all!), but in [39] Raynaud proved that for $\ell = p$ the integral model is unique if it exists when $e(K) < p - 1$, and remarkably without any restriction on $e(K)$ he showed that in Theorem 7.1.13 with $\ell = p$ it is actually not necessary to assume $\mathscr{G}_n = \mathscr{G}_{n+1}[\ell^n]$ for all $n \geqslant 1$. (More specifically, he showed that if *some* integral model exists for $A[p^n]$ for all $n \geqslant 1$ then models can be found satisfying the compatibility requirement in Theorem 7.1.13.)

Using the language of ℓ-divisible groups (to be introduced in §7.2) rather than the language of Galois representations, Theorem 7.1.13 admits the following linguistic reformulation: for any prime ℓ (especially $\ell = p$), an abelian variety A over K has good reduction if and only if the ℓ-divisible group of A extends to an ℓ-divisible group over $\mathscr{O}_K$. The utility of this point of view when $\ell = p$ is that p-divisible groups over $\mathscr{O}_K$ and k admit a very rich structure theory.

7.2. **p-divisible groups and Dieudonné modules.** The proof of Theorem 7.1.13 for $\ell = p$ makes essential use of some results of Tate involving p-divisible groups, and the theory of p-divisible groups is also a useful "testing ground" for many concepts in p-adic Hodge theory. Thus, we now explain some basic aspects of the theory, with an eye on later applications. (It must be stressed that the significance of p-divisible groups goes far beyond the contexts that we will consider.) The basic idea behind the theory of p-divisible groups is to make a scheme-theoretic substitute for Tate modules when working with group schemes that are not finite étale (and so cannot be studied as Galois modules via Proposition 7.1.11). It is instructive to first consider Tate modules from another point of view.

For an abelian variety A of dimension $g > 0$ over a field F, if $\ell \neq \mathrm{char}(F)$ is a prime then the directed system $\{A[\ell^n]\}$ of finite étale F-groups of ℓ-power order can be analyzed in terms of the associated directed system of Galois modules $\{A[\ell^n](F_s)\}$ due to Lemma 7.1.10. Due to the basic exact sequences

$$0 \to A[\ell](F_s) \to A[\ell^{n+1}](F_s) \to A[\ell^n](F_s) \to 0$$

(using ℓ-multiplication to construct the right map), we can repackage this data in terms of the limit

$$T_\ell(A) = \varprojlim A[\ell^n](F_s)$$

that is a finite free $\mathbf{Z}_\ell$-module equipped with a continuous G_F-action.

But can we work directly with the directed system $\{A[\ell^n]\}$? One reason to try to avoid the inverse limit step is that the directed system makes sense even when $\ell = \mathrm{char}(F)$ (in which case the Galois-theoretic framework cannot be used). More generally, if $A \to S$ is an abelian scheme with fibers of constant dimension $g > 0$ and p is a prime, then $G_n := A[p^n]$ is a finite flat p^n-torsion group scheme of order p^{2gn} for all $n \geqslant 1$ and $\{G_n\}$ is a directed system via isomorphisms $G_n \simeq G_{n+1}[p^n]$ for all $n \geqslant 1$. This structure motivates the following definition.

Definition 7.2.1. Let p be a prime and $h \geqslant 0$ an integer. A *p-divisible group* (or *Barsotti-Tate group*) of height h over a scheme S is a directed system $G = \{G_n\}$ of finite flat group schemes over S such that each G_n is p^n-torsion of order p^{nh} and the transition map $i_n : G_n \to G_{n+1}$ is an isomorphism of G_n onto $G_{n+1}[p^n]$ for all $n \geqslant 1$.

A *morphism* $f : G \to H$ between p-divisible groups is a compatible system of S-group maps $f_n : G_n \to H_n$ for all $n \geqslant 1$.

If $S' \to S$ is a map of schemes then $G \times_S S' := \{G_n \times_S S'\}$ is the p-divisible group of height h over S' obtained by *base change*.

If $G = \{G_n\}$ is a p-divisible group, we will often write $G[p^n]$ as convenient shorthand for G_n. For an abelian scheme $A \to S$ with fibers of constant dimension $g > 0$, we sometimes write $A[p^\infty]$ to denote the associated p-divisible group $\{A[p^n]\}$ over S.

Let us see how p-divisible groups generalize Tate modules. Suppose $S = \operatorname{Spec} F$ for a field F, and let p be a prime distinct from $\operatorname{char}(F)$. A p-divisible group $\Gamma = \{\Gamma_n\}$ over F has each Γ_n necessarily finite étale over F, by Lemma 7.1.9, so the axioms for a p-divisible group can be formulated entirely in the language of the Galois modules $M_n = \Gamma_n(F_s)$: each M_n is a discrete G_F-module of size p^{nh} that is killed by p^n and $M_{n+1}[p^n] = M_n$. In particular, each M_n has p-torsion M_1 of size p^h, so necessarily M_n is finite free of rank h over $\mathbf{Z}/(p^n)$ for all n.

We can form two kinds of limits: (i) the direct limit $M_\infty = \varinjlim M_n$ is $(\mathbf{Q}_p/\mathbf{Z}_p)^h$ with a continuous G_F-action for the discrete topology, and (ii) multiplication by p on M_{n+1} provides a quotient map $M_{n+1} \twoheadrightarrow M_n$ of discrete G_F-modules yielding an inverse limit $T_p(\Gamma) = \varprojlim M_n$ that is a finite free $\mathbf{Z}_p$-module of rank h equipped with a continuous action of G_F for the p-adic topology.

We can recover the directed system of M_n's from both limits, namely $M_n = M_\infty[p^n]$ and $M_n = T_p(\Gamma)/(p^n)$. The viewpoint of M_∞ explains the "p-divisible" aspect of the situation (since multiplication by p is surjective on $(\mathbf{Q}_p/\mathbf{Z}_p)^h$), whereas $T_p(\Gamma)$ has a nicer $\mathbf{Z}_p$-module structure. This proves:

Proposition 7.2.2. *If F is a field with $p \neq \operatorname{char}(F)$, then the functor $\Gamma \rightsquigarrow T_p(\Gamma)$ is an equivalence from the category of p-divisible groups over $\operatorname{Spec} F$ to the category of p-adic representations of G_F on finite free $\mathbf{Z}_p$-modules, with $T_p(\Gamma)$ having $\mathbf{Z}_p$-rank equal to the height of Γ.*

For example, if $\Gamma = A[p^\infty]$ is the p-divisible group associated to an abelian variety A over F then $T_p(\Gamma)$ is the p-adic Tate module representation $T_p(A)$ of G_F. Doing the same with A replaced by GL_1 yields $\Gamma = \{\mu_{p^n}\}$ with the evident transition maps, for which we have $T_p(\Gamma) = \mathbf{Z}_p(1)$. When working over a base S on which p is not a unit, such as $S = \operatorname{Spec} \mathscr{O}_K$, such Galois-theoretic considerations do not apply.

Although Tate modules are not available when studying p-divisible groups over base schemes on which p is not a unit, over some special base schemes there is a very good replacement. The two simplest cases for which a variant is available is $S = \operatorname{Spec} k$ for a perfect field k of characteristic $p > 0$ and $S = \operatorname{Spec} \mathrm{W}(k)$ for such a k. (The case $S = \operatorname{Spec} \mathscr{O}_K$ is rather more subtle when $e(K) > 1$, and we postpone it to §12 where we use integral p-adic Hodge theory.) The theory over $\operatorname{Spec} k$ is due to Dieudonné, as part of his theory of Dieudonné modules, and the theory over $\operatorname{Spec} \mathrm{W}(k)$ is due to Fontaine (building on earlier work of Barsotti and Honda). We now turn to a discussion of each of these cases in turn. The case $S = \operatorname{Spec} \mathrm{W}(k)$ is of more arithmetic interest, but to understand it we first need to review Dieudonné's method for working with the special fiber over k.

Definition 7.2.3. Let k be a perfect field of characteristic $p > 0$, and let $\sigma : \mathrm{W}(k) \simeq \mathrm{W}(k)$ be the Frobenius automorphism lifting the p-power map on k. The *Dieudonné ring* of k is the associative ring $\mathscr{D}_k = \mathrm{W}(k)[\mathscr{F}, \mathscr{V}]$ subject to the relations $\mathscr{F}\mathscr{V} = \mathscr{V}\mathscr{F} = p$, $\mathscr{F}c = \sigma(c)\mathscr{F}$ for $c \in \mathrm{W}(k)$, and $c\mathscr{V} = \mathscr{V}\sigma(c)$ for $c \in \mathrm{W}(k)$. (This is non-commutative when $k \neq \mathbf{F}_p$, and is $\mathbf{Z}_p[x, y]/(xy - p)$ when $k = \mathbf{F}_p$.)

Observe that a left $\mathscr{D}_k$-module is the same thing as a $\mathrm{W}(k)$-module D equipped with a σ-semilinear endomorphism $\mathscr{F} : D \to D$ and a σ^{-1}-semilinear endomorphism $\mathscr{V} : D \to D$ such that $\mathscr{F}\mathscr{V} = \mathscr{V}\mathscr{F} = [p]_D$. An excellent reference for the basic theory of Dieudonné modules and their relations with group schemes over k is [18, Ch. I–III], and some of the main results as summarized as follows.

Theorem 7.2.4. *There is an additive anti-equivalence of categories $G \rightsquigarrow \mathbf{D}(G)$ from the category of finite flat k-group schemes of p-power order to the category of left $\mathscr{D}_k$-modules of finite W-length. The following additional properties hold.*

1. *The order of G is $p^{\ell_{\mathrm{W}(k)}(\mathbf{D}(G))}$.*
2. *If k'/k is an extension of perfect fields then naturally $\mathrm{W}(k') \otimes_{\mathrm{W}(k)} \mathbf{D}(G) \simeq \mathbf{D}(G_{k'})$ as left $\mathscr{D}_{k'}$-modules. In particular, taking $k \to k'$ to be the Frobenius map $\sigma : k \simeq k$, we naturally have $\sigma^*(\mathbf{D}(G)) \simeq \mathbf{D}(G^{(p)})$ as $\mathrm{W}(k)$-modules, where $G^{(p)}$ is the base change of G along $\operatorname{Spec} \sigma$.*
3. *Let $F_{G/k} : G \to G^{(p)}$ be the relative Frobenius morphism as in Definition 7.1.4. The σ-semilinear map $\mathbf{D}(G) \to \mathbf{D}(G)$ corresponding to the $\mathrm{W}(k)$-linear map*
$$\sigma^*(\mathbf{D}(G)) \simeq \mathbf{D}(G^{(p)}) \overset{\mathbf{D}(F_{G/k})}{\to} \mathbf{D}(G)$$
is the action of $\mathscr{F}$. Moreover, G is connected if and only if $\mathscr{F}$ is nilpotent on $\mathbf{D}(G)$.
4. *The k-vector space $\mathbf{D}(G)/\mathscr{F}\mathbf{D}(G)$ is canonically identified with the linear dual t_G^* of the tangent space t_G of G at the origin. In particular, G is étale if and only if $\mathscr{F}$ is bijective on $\mathbf{D}(G)$.*

Example 7.2.5. Let k be a perfect field of characteristic $p > 0$. Finite flat p-torsion k-groups correspond to 1-dimensional k-vector spaces equipped with a compatible left $\mathscr{D}_k$-module structure. That is, there is given a pair of operators $\mathscr{F}$ and $\mathscr{V}$ with appropriate semilinearity properties such that $\mathscr{F} \circ \mathscr{V}$ and $\mathscr{V} \circ \mathscr{F}$ both vanish. In the 1-dimensional case we have $\mathbf{D}(\mu_p) = k$ with $\mathscr{F} = 0$ and $\mathscr{V} = \sigma^{-1}$, $\mathbf{D}(\mathbf{Z}/p\mathbf{Z}) = k$ with $\mathscr{V} = 0$ and $\mathscr{F} = \sigma$, and $\mathbf{D}(\alpha_p) = k$ with $\mathscr{F} = \mathscr{V} = 0$. When k is algebraically closed there are no other 1-dimensional examples, but otherwise the cases in which $\mathscr{F} \neq 0$ or $\mathscr{V} \neq 0$ admit other possibilities.

In the 2-dimensional case, an especially interesting example is $M = k^2$ with

$$\mathscr{F} = \sigma \circ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathscr{V} = \sigma^{-1} \circ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The corresponding k-group is the p-torsion of a supersingular elliptic curve over k.

The usefulness of Theorem 7.2.4 is perhaps most strikingly illustrated by applying it to describe p-divisible groups over k in a manner very much like a Tate module, except that the Galois action is replaced with a $\mathscr{D}_k$-module structure and the "coefficients" are $\mathrm{W}(k)$ rather than $\mathbf{Z}_p$:

Proposition 7.2.6. *The functor $G \rightsquigarrow \mathbf{D}(G) := \varprojlim \mathbf{D}(G_n)$ is an anti-equivalence of categories between the category of p-divisible groups over k and the category of finite free $\mathrm{W}(k)$-modules D equipped with a Frobenius semilinear endomorphism $\mathscr{F} : D \to D$ such that $pD \subseteq \mathscr{F}(D)$. The height of G is the $\mathrm{W}(k)$-rank of $\mathbf{D}(G)$, and this equivalence is compatible with any extension k'/k of perfect fields.*

The torsion-levels G_n of the p-divisible group G satisfy $\mathbf{D}(G_n) \simeq \mathbf{D}(G)/(p^n)$ compatibly with change in n, so the G_n's are connected if and only if $\mathscr{F}$ is topologically nilpotent on $\mathbf{D}(G)$ with its p-adic topology.

The point of the condition $pD \subseteq \mathscr{F}(D)$ in this proposition is to ensure that the $\mathscr{V}$ operator can also be defined on D, as is required to specify a left $\mathscr{D}_k$-module structure.

Proof. Choose a p-divisible group $G = \{G_n\}$ over k with height $h \geqslant 0$. Since G_1 is the categorical kernel of p on G_n and $\mathbf{D}$ is an additive categorical anti-equivalence, it follows that $\mathbf{D}(G_1)$ is naturally identified with $\mathbf{D}(G_n)/(p)$ for all n. Thus, each $\mathbf{D}(G_n)$ is a $\mathrm{W}(k)/(p^n)$-module of length p^{nh} such that $\mathbf{D}(G_n)/(p) \simeq \mathbf{D}(G_1)$ has k-dimension h.

It follows by the same argument used to work out the structure of torsion in abelian varieties that each $\mathbf{D}(G_n)$ is a finite free $\mathrm{W}(k)/(p^n)$-module of rank h and $G_n \hookrightarrow G_{n+1}$ induces an isomorphism $\mathbf{D}(G_{n+1})/(p^n) \simeq \mathbf{D}(G_n)$ for all $n \geqslant 1$. Hence,

$$\mathbf{D}(G) = \varprojlim \mathbf{D}(G_n)$$

is $\mathscr{D}_k$-module that is finite free over $\mathrm{W}(k)$ with rank h, and $\mathbf{D}(G)/(p^n) \simeq \mathbf{D}(G_n)$ as $\mathscr{D}_k$-modules compatibly with change in n. ■

To give another illustration of the usefulness of Dieudonné modules, recall that if A and B are abelian varieties over a field k with $\mathrm{char}(k) \neq \ell$ then

$$\mathbf{Z}_\ell \otimes_{\mathbf{Z}} \mathrm{Hom}_k(A, B) \to \mathrm{Hom}_{\mathbf{Z}_\ell[G_k]}(T_\ell(A), T_\ell(B)) \tag{7.2.1}$$

is injective. We can rewrite the target as the space of maps $\mathrm{Hom}_k(A[\ell^\infty], B[\ell^\infty])$ of ℓ-divisible groups over k, suggesting that for $p = \mathrm{char}(k) > 0$ the map

$$\mathbf{Z}_p \otimes_{\mathbf{Z}} \mathrm{Hom}_k(A, B) \to \mathrm{Hom}_k(A[p^\infty], B[p^\infty])$$

should be injective. Though k may not be perfect, to check such injectivity it suffices to do so over a perfect extension of k, in which case we can use Dieudonné theory to reformulate the assertion as:

Proposition 7.2.7. *Let A and B be abelian varieties over a perfect field k with* $\mathrm{char}(k) = p > 0$. *The natural map*

$$\mathbf{Z}_p \otimes_{\mathbf{Z}} \mathrm{Hom}_k(A, B) \to \mathrm{Hom}_{\mathscr{D}_k}(\mathbf{D}(B[p^\infty]), \mathbf{D}(A[p^\infty])) \tag{7.2.2}$$

is injective.

This proposition is proved by *exactly* the same argument as in the ℓ-adic case for $\ell \neq p$ (as in [36, §19, Thm, 3]). Although (7.2.2) looks very similar to the analogue for Tate modules (up to the intervention of contravariance that swaps A and B in the target), there are two important distinctions to keep in mind: (i) when k is algebraically closed the $\mathscr{D}_k$-module structure cannot be ignored (whereas the Galois action becomes trivial in the ℓ-adic case), and (ii) although the Dieudonné modules are finite free $\mathrm{W}(k)$-modules, if $k \neq \mathbf{F}_p$ then the target of (7.2.2) is *not* a $\mathrm{W}(k)$-module when it is nonzero because $\mathrm{W}(k)$ is not central in $\mathscr{D}_k$ (and so it does not act through $\mathscr{D}_k$-linear endomorphisms on a general $\mathscr{D}_k$-module).

It is an important result of Tate that (7.2.1) and (7.2.2) are isomorphisms when k is finite. This is essential in Tate's analysis ([48], [50]) of the precise structure (especially at p-adic places) of the endomorphism algebras of simple abelian varieties over finite fields. Tate went much further with p-divisible groups than applications over finite fields. In mixed characteristic, he showed that although p-divisible groups over $\mathscr{O}_K$ are not abelian schemes, they sometimes behave as if they were. More specifically, Tate proved (as the main result in [49]) that p-divisible groups over $\mathscr{O}_K$ are functorial in their generic fiber, exactly like abelian schemes:

Theorem 7.2.8 (Tate)**.** *If Γ and Γ' are p-divisible groups over $\mathscr{O}_K$ then* $\mathrm{Hom}(\Gamma, \Gamma') \to \mathrm{Hom}_K(\Gamma_K, \Gamma'_K)$ *is bijective.*

This theorem (and its technique of proof) marks the true beginning of p-adic Hodge theory, as it was Tate's proof of this result that led him to discover the Hodge–Tate decomposition for $\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbf{Q}_p) = V_p(A)^\vee$ for abelian varieties A over K with good reduction, and to then ask whether a similar such decomposition might hold in complete generality (for the p-adic étale cohomology of all smooth proper K-schemes).

We conclude our discussion of Dieudonné modules by describing Fontaine's classification of finite flat group schemes (of p-power order) and p-divisible groups G over $\mathrm{W}(k)$ in terms of the Dieudonné module $\mathbf{D}(G_k)$ of the special fiber and some "lifting data".

Recall that $\mathbf{D}(G_k)/\mathscr{F}\mathbf{D}(G_k)$ is the cotangent space of G_k at the origin (Theorem 7.2.4). In [18, Ch. IV], Fontaine proved via a general study of formal groups that if G_k is a p-divisible group over k then when $p > 2$ any lift G of G_k to $\mathrm{W}(k)$ provides a $\mathrm{W}(k)$-submodule $L \subseteq \mathbf{D}(G_k)$ of "logarithms" such that $L/pL \to \mathbf{D}(G_k)/\mathscr{F}\mathbf{D}(G_k)$ is an isomorphism. For $p = 2$ he obtained the same result provided that G_k (or rather, each $G_k[p^n]$) is connected. In [17] Fontaine proved an analogous theorem when G_k is a finite flat group scheme of p-power order (assumed to be connected when $p = 2$), with the additional property that $\mathscr{V}|_L : L \to \mathbf{D}(G_k)$ is injective (as is automatic when working with finite free $\mathrm{W}(k)$-modules, since $\mathscr{F} \circ \mathscr{V} = p$). This motivates the following definition.

Definition 7.2.9. Let k be a perfect field of characteristic $p > 0$. A *Honda system* over $\mathrm{W}(k)$ is a pair (M, L) consisting of a finite free $\mathrm{W}(k)$-module M, a $\mathrm{W}(k)$-submodule L, and a Frobenius-semilinear endomorphism $\mathscr{F} : M \to M$ such that

- $pM \subseteq \mathscr{F}(M)$,
- the induced map $L/pL \to M/\mathscr{F}(M)$ is an isomorphism.

If $\mathscr{F}$ is topologically nilpotent on M then (M, L) is called *connected*.

A *finite Honda system* over $\mathrm{W}(k)$ is a pair (M, L) consisting of a left $\mathscr{D}_k$-module M of finite $\mathrm{W}(k)$-length and a $\mathrm{W}(k)$-submodule L such that

- the restriction $\mathscr{V}|_L : L \to M$ is injective
- the induced map $L/pL \to M/\mathscr{F}(M)$ is an isomorphism.

If $\mathscr{F}$ is nilpotent on M then (M, L) is called *connected*.

Morphisms of Honda systems and finite Honda systems are defined in the evident manner, and there is an evident notion of base change (corresponding to extensions $k \to k'$ of perfect fields). Note that the condition $pM \subseteq \mathscr{F}(M)$ in the definition of a Honda system is just a way of encoding that M is really a left $\mathscr{D}_k$-module (in which case the map $\mathscr{V}|_L : L \to M$ is automatically injective since $\mathscr{F}\mathscr{V} = p$ and M is torsion-free over $\mathrm{W}(k)$). It is true but not obvious that if (M, L) is a Honda system over $\mathrm{W}(k)$ then $(M/p^nM, L/p^nL)$ is a finite Honda system over $\mathrm{W}(k)$. This expresses a basic compatibility property in the following result of Fontaine:

Theorem 7.2.10 (Fontaine)**.** *Let k be a perfect field of characteristic $p > 0$. If $p > 2$ then there is a natural anti-equivalence of categories $G \rightsquigarrow (\mathbf{D}(G_k), L(G))$ from the category of p-divisible groups over $\mathrm{W}(k)$ to the category of Honda systems, and the same holds for $p = 2$ if we restrict attention to connected objects on both sides.*

Likewise, there is a natural anti-equivalence of categories from the category of finite flat $\mathrm{W}(k)$-group schemes of p-power order to the category of finite Honda systems when $p > 2$, and similarly for connected objects when $p = 2$.

Both anti-equivalences respect extension of the perfect residue field, and if G is a p-divisible group over $\mathrm{W}(k)$ then $(\mathbf{D}(G_k)/(p^n), L(G)/(p^n))$ is naturally identified with the finite Honda system associated to $G[p^n]$ for all $n \geqslant 1$.

Proof. The case of p-divisible groups is treated in [18, Ch. IV] using the theory of formal groups. The case of finite group schemes was announced by Fontaine in [17], unfortunately without proofs; in [15, §1] proofs are provided. ■

Recall that K_0 denotes $\mathrm{W}(k)[1/p]$ (as in the discussion preceding Remark 4.2.4). In the spirit of p-adic Hodge theory it is natural to ask for a description of the Galois module $G(\overline{K}_0)$ associated to a finite flat group scheme G over $\mathrm{W}(k)$ in terms of its associated finite Honda system, and similarly for the p-adic representation of G_{K_0} associated to the p-adic Tate module of the generic fiber of a p-divisible group over $\mathrm{W}(k)$. Such descriptions can be given, but we will not need them and so we refer the interested reader to [15, Thm. 1.9] for further details.

7.3. **Motivation from crystalline and de Rham cohomologies.** In Proposition 7.2.6, we saw that there is some interest in studying the following kind of semilinear algebra object: a finite free $\mathrm{W}(k)$-module D equipped with a Frobenius semi-linear endomorphism $\phi : D \to D$ such that $pD \subseteq \phi(D)$. For any such D, ϕ is bijective on $D[1/p]$. Hence, $D[1/p]$ is an instance of the following kind of structure:

Definition 7.3.1. An *isocrystal over* K_0 is a finite-dimensional K_0-vector space D equipped with a bijective Frobenius-semilinear endomorphism $\phi_D : D \to D$. (The "iso" refers to isogeny, and is related to working over K_0 rather than $\mathrm{W}(k)$ and assuming that ϕ_D is bijective on D.)

The abelian category of isocrystals over K_0 is denoted $\mathrm{Mod}_{K_0}^{\phi}$, with evident notions of tensor product and dual (the latter resting on the Frobenius structure $(\phi_{D^\vee}.\ell)(d) := \sigma(\ell(\phi_D^{-1}(d)))$ for $\ell \in D^\vee$ and $d \in D$, with σ denoting the Frobenius automorphism of $K_0 = \mathrm{W}(k)[1/p]$).

Just as the isogeny category of abelian varieties over a field is much simpler than the category of abelian varieties (e.g., Poincaré complete reducibility holds up to isogeny, and endomorphism algebras are semisimple over $\mathbf{Q}$), working with an "isogeny category" of p-divisible groups over a field is sometimes a big simplification. Put another way, just as $V_\ell(A)$ can be more convenient than $T_\ell(A)$, we will likewise find that working with the isocrystal $\mathbf{D}(A[p^\infty])[1/p]$ can be simpler than working with the Dieudonné module $\mathbf{D}(A[p^\infty])$. Of course, sometimes it is necessary to keep track of the integral structure (e.g., for deformation theory of abelian varieties and Galois representations).

The following example shows that one can really "write down" isocrystals over K_0. The isocrystals given below lead to a classification of all isocrystals when $k = \overline{k}$ (Theorem 8.1.4). Nothing like this is true if we work over $\mathrm{W}(k)$ rather than over $K_0 = \mathrm{W}(k)[1/p]$, so it demonstrates one of the virtues of working in the "isogeny category" $\mathrm{Mod}_{K_0}^{\phi}$.

Example 7.3.2. Let $K_0[\phi] = \mathscr{D}_k[1/p]$ (with $\phi = \mathscr{F}$ from Definition 7.2.3) be the twisted polynomial ring satisfying $\phi c = \sigma(c)\phi$ for $c \in K_0$. (See Exercise 7.4.7.) An interesting class of isocrystals over K_0 is given by the quotients

$$D_{r,s} = K_0[\phi]/(K_0[\phi](\phi^r - p^s))$$

modulo the left ideals $K_0[\phi](\phi^r - p^s)$ in $K_0[\phi]$ for any integers r and s with $r > 0$ (but possibly $s < 0$). The Frobenius structure on $D_{r,s}$ defined by left multiplication by ϕ.

By a "division algorithm'" argument we see that $D_{r,s}$ has finite dimension r over K_0, and it is an isocrystal over K_0. Although it does not make sense to speak of eigenvalues for the ϕ-operator on $D_{r,s}$ when $k \neq \mathbf{F}_p$ (since this operator is just semilinear rather than linear), it is good to imagine that ϕ should have "eigenvalues" on $D_{r,s}$ that are integral unit multiples of $p^{s/r}$.

General isocrystals over K_0 will play an essential role in the theory of crystalline representations, and the ones arising from Dieudonné modules of p-divisible groups over k as in Proposition 7.2.6 are extremely special: in general an isocrystal need not contain a ϕ-stable $\mathrm{W}(k)$-lattice, and even when such a lattice M exists it is a very stringent condition that $pM \subseteq \phi(M)$. Moreover, when $e(K) > 1$ the appropriate integral counterpart to isocrystals is far more subtle than the $\mathrm{W}(k)$-module structures arising from p-divisible groups over k.

At this point we have seen two ways in which Frobenius structures naturally arise in the study of p-adic Galois representations: in §3 via étale φ-modules (which encode information about G_K-representations restricted to a certain closed subgroup G_{K_∞}), and in §7.2 via isocrystals over K_0 and integral counterparts related to p-divisible groups over k and especially over $\mathrm{W}(k)$ (such as arise from abelian varieties over K_0 with good reduction).

To figure out how to introduce Frobenius structures into the study of objects in Fil_K in general, we first consider an abelian variety A over K_0 with good reduction, say with $\mathscr{A}$ the corresponding abelian scheme over $\mathrm{W}(k)$. Let $\Gamma = \mathscr{A}[p^\infty]$ be the corresponding p-divisible group over $\mathrm{W}(k)$. By Theorem 7.2.8, Γ is determined up to isomorphism by the corresponding generic fiber Γ_{K_0}, or equivalently the associated $\mathbf{Z}_p[G_{K_0}]$-module $T_p(A)$. Since torsion in abelian varieties over an algebraically closed field classifies finite étale covering spaces [36, §18], this Tate module is the $\mathbf{Z}_p$-linear dual of $\mathrm{H}^1_{\text{ét}}(A_{\overline{K}_0}, \mathbf{Z}_p)$. But Theorem 7.2.10 shows (assuming connectedness when $p = 2$) that Γ is determined up to isomorphism by an entirely different kind of structure: the Dieudonné module $\mathbf{D}(\Gamma_k)$ equipped with the filtration provided by the $\mathrm{W}(k)$-submodule $L \subseteq \mathbf{D}(\Gamma_k)$ satisfying $L/pL \simeq \mathbf{D}(\Gamma_k)/\mathscr{F}\mathbf{D}(\Gamma_k)$.

Upon inverting p, we now have two kinds of structure that each determine Γ up to isogeny: the p-adic Galois representation $\mathrm{H}^1_{\text{ét}}(A_{\overline{K}_0}, \mathbf{Q}_p)$ and the isocrystal $\mathbf{D}(\Gamma_k)[1/p]$ over K_0 equipped with a 1-step filtration $\mathrm{Fil}^1 = L[1/p] \subseteq \mathbf{D}(\Gamma_k)[1/p]$. It is therefore natural to speculate whether there is a way to bypass Γ entirely and define a "mysterious functor" going directly between p-adic G_{K_0}-representations and isocrystals D over K_0 endowed with a structure from Fil_{K_0}. This suggests that to generalize Fil_{K_0} we should bring the category $\mathrm{Mod}^{\phi}_{K_0}$ of isocrystals over K_0 into the picture.

Now allowing ramification, there is a broader context in which one sees isocrystals over K_0 and filtered vector spaces over K interacting with each other (even when $e(K) > 1$, in which case Frobenius operators make no sense on K-vector spaces). Let X be a smooth proper K-scheme, and assume $X = \mathscr{X}_K$ for a smooth proper $\mathscr{O}_K$-scheme $\mathscr{X}$ (i.e., "good reduction"). Let $\mathscr{X}_0 = \mathscr{X} \otimes_{\mathscr{O}_K} k$ denote the smooth proper special fiber over the perfect field k. The theory of crystalline cohomology over k provides a finitely generated (possibly not free) $\mathrm{W}(k)$-module $\mathrm{H}^i_{\mathrm{cris}}(\mathscr{X}_0/\mathrm{W}(k))$ equipped with a Frobenius semilinear endomorphism ϕ such that the induced endomorphism of the K_0-vector space $\mathrm{H}^i_{\mathrm{cris}}(\mathscr{X}_0/\mathrm{W}(k))[1/p]$ is bijective (due to a calculation with Poincaré duality). This is an isocrystal over K_0.

The comparison isomorphism between crystalline and de Rham cohomology [6, Cor. 2.5] is a canonical K-linear isomorphism

$$\mathrm{H}^i_{\mathrm{dR}}(X/K) \simeq K \otimes_{K_0} \mathrm{H}^i_{\mathrm{cris}}(\mathscr{X}_0/\mathrm{W}(k))[1/p]. \tag{7.3.1}$$

(In general the integral structures on both sides provided by the image of $\mathrm{H}^i_{\mathrm{cris}}(\mathscr{X}_0/\mathrm{W}(k))$ and the image of $\mathrm{H}^i_{\mathrm{dR}}(\mathscr{X}/\mathscr{O}_K)$ are *not* compatible in either direction [6, Rem. 2.10]! So it is important that we have inverted p.) Thus, $D = \mathrm{H}^i_{\mathrm{cris}}(\mathscr{X}_0/\mathrm{W}(k))[1/p]$ is an isocrystal over K_0 for which the scalar extension $D_K := K \otimes_{K_0} D$ is endowed with a structure of object in Fil_K via the Hodge filtration on the right side of (7.3.1).

Remark 7.3.3. In case $\mathscr{X} = \mathscr{A}$ is an abelian scheme over $\mathrm{W}(k)$, it is natural to ask how to compare the two preceding constructions (one using Dieudonné theory, the other using crystalline cohomology), assuming when $p = 2$ that $\mathscr{A}_k$ has connected p-power torsion. By work of Berthelot–Breen–Messing [4, 2.2.9, 2.5.7, 2.5.8(ii), 3.3.7, 4.2.14, 4.2.15(ii)], for $A = \mathscr{A}_k$ there are canonical $\mathrm{W}(k)$-linear isomorphisms

$$\mathbf{D}(A[p^\infty])^{(p)} \simeq \mathrm{H}^1_{\mathrm{cris}}(A/\mathrm{W}(k)) \simeq \mathrm{H}^1_{\mathrm{dR}}(\mathscr{A}/\mathrm{W}(k))$$

in which the first one is compatible with Frobenius operators, the second coincides (by construction) with (7.3.1) upon inverting p, and the composite map converts the Hodge filtration $\mathrm{H}^0(\mathscr{A}, \Omega^1_{\mathscr{A}/\mathrm{W}(k)})$ on the right side over to the Frobenius twist of Fontaine's module of "logarithms" $L \subseteq \mathbf{D}(A[p^\infty])$ on the left side.

The preceding considerations lead us to the following concept that is a refinement of Fil_K.

Definition 7.3.4. Let K be a p-adic field. A *filtered ϕ-module over K* is a triple $(D, \phi, \mathrm{Fil}^\bullet)$ where (D, ϕ_D) is an isocrystal over K_0 and $(D_K, \mathrm{Fil}^\bullet)$ is an object in Fil_K (i.e., $\{\mathrm{Fil}^i\}$ is a decreasing exhaustive and separated filtration on D_K).

A *morphism* $D' \to D$ between two filtered ϕ-modules is a K_0-linear map $D' \to D$ that is compatible with both $\phi_{D'} : D' \to D'$ and $\phi_D : D \to D$ and has K-linear extension $D'_K \to D_K$ that is a morphism in Fil_K. The category of triples $(D, \phi, \mathrm{Fil}^\bullet)$ is denoted MF_K^ϕ.

It must be emphasized that a filtered ϕ-module over K is really a vector space D over K_0 equipped with some auxiliary structure (one part of which is a filtration on D_K), and there is *no* required relationship between the filtration structure on D_K and the Frobenius structure on D over K_0. When $K \neq K_0$ it makes no sense to speak of a Frobenius structure on D_K, so it is not obvious how we can possibly link up the filtration and the Frobenius structure. When $K = K_0$, it would be "wrong" to ask that ϕ respect the filtration. (For example, with an elliptic curve E over $\mathbf{F}_p$ the Frobenius action on the isocrystal has eigenvalues which may not lie in $\mathbf{Q}_p$, due to Exercise 7.4.8.) Later we will define a finer class of objects in MF_K^ϕ for which there is a deep connection between these two structures.

The category MF_K^ϕ of filtered ϕ-modules over K admits many of the basic "linear algebra" concepts that we have defined earlier for the category of étale φ-modules in §3 and for the category Fil_K in §6.2:

- we form the K_0-linear kernel and cokernel endowed with their induced Frobenius semilinear maps (that are bijective for the same reasons given in §3), and their scalar extensions to K are endowed with subspace and quotient filtrations respectively;
- we define the notions of image and coimage, akin to the case of Fil_K in §6.2, so MF_K^ϕ is not abelian (for the same reasons as for filtered vector spaces in Example 6.2.1) but we have the notion of a *strict morphism* as in Definition 6.2.5;
- there is an evident notion of short exact sequence, as in Fil_K;
- we define tensor product and dual by merging such notions as introduced for étale φ-modules in §3 and for filtered vector spaces in Definition 6.2.2 (since tensor products and duals commute with scalar extension from K_0 to K).

For example, the *dual* $D^\vee$ of a filtered ϕ-module D is the usual K_0-linear dual endowed with the dual Frobenius (i.e., $\phi_{D^\vee} : \ell \mapsto \sigma \circ \ell \circ \phi_D^{-1}$, where σ is the Frobenius self-map of $K_0 = \mathrm{W}(k)[1/p]$, or proceed alternatively as in Exercise 3.4.2), and the scalar extension $(D^\vee)_K \simeq (D_K)^\vee$ is given the dual filtration to the filtration on D_K (as in Definition 6.2.2). Natural linear isomorphisms such as $D^\vee \otimes_{K_0} D'^\vee \simeq (D \otimes_{K_0} D')^\vee$ are isomorphisms in MF_K^ϕ when using dual and tensor product filtration and Frobenius structures in MF_K^ϕ (exactly as for Fil_K via Exercise 6.2.3).

Since the Frobenius self-map σ of K_0 is bijective (whereas $\varphi_{\mathscr{E}}$ in §3 was not bijective in the most interesting cases, due to the residue field E being imperfect in such cases), for some constructions in MF_K^ϕ we may suppress the intervention of scalar extension by the Frobenius map of K_0. As an illustration of this, Exercise 6.4.1 can be extended to incorporate Frobenius structures; see Exercise 7.4.11.

7.4. **Exercises.**

Exercise 7.4.1. Write out the "additive" group scheme structure $\mathbf{G}_a$ on $\operatorname{Spec}\mathbf{Z}[x]$ and the "multiplicative" group scheme structure $\mathbf{G}_m$ on $\operatorname{Spec}\mathbf{Z}[x, 1/x]$ in terms of maps of rings. Do the same for $\mathrm{GL}_n = \operatorname{Spec}(\mathbf{Z}[x_{ij}][1/\det])$. How about PGL_n and SL_n?

Describe all finite subgroup schemes of $\mathbf{G}_a$ over an algebraically closed field of any characteristic. How about $\mathbf{G}_m$?

Exercise 7.4.2. Using functorial considerations, show that if $G \to S$ is a commutative group scheme killed by nm for relative prime integers $n, m \geqslant 1$ then the natural map $G[n] \times_S G[m] \to G$ is an isomorphism.

If G is finite flat over S prove the same for $G[n]$ and $G[m]$. Thus, finite flat groups admit "primary decomposition" just like finite abelian groups.

Exercise 7.4.3. Let S be a scheme, and F a field.

(1) Check the equivalences in Definition 7.1.1, via Yoneda's Lemma. In particular, deduce that if G and H are S-group schemes and $f : G \to H$ is an S-scheme map compatible with multiplication laws then f is an S-group homomorphism.
(2) Prove that there are no nontrivial group scheme homomorphisms from $\mathbf{G}_m$ to $\mathbf{G}_a$ over any ring R, and prove the same in the opposite direction over any reduced ring. But if there exists a non-zero $\epsilon \in R$ such that $\epsilon^2 = 0$ then construct a nontrivial R-group homomorphism from $\mathbf{G}_a$ to $\mathbf{G}_m$. If moreover R is an $\mathbf{F}_p$-algebra, show that $x \mapsto x + \epsilon x^p$ is an R-group automorphism of $\mathbf{G}_a$ not arising from an $R^\times$-scaling!
(3) Prove that $\det(x_{ij}) \in F[x_{ij}]$ is irreducible, and by using the structure of units in the ring $F[x_1, \dots, x_N][1/f]$ for an irreducible polynomial f show that the *only* F-group scheme maps $\mathrm{GL}_n \to \mathrm{GL}_1$ are $\det^n$ for $n \in \mathbf{Z}$. Show the same over $\operatorname{Spec}\mathbf{Z}$ by using the result over $\operatorname{Spec}\mathbf{Q}$.
(4) Let G be a smooth F-group with $\dim G > 0$. Explain why the underlying topological space of the *scheme* G cannot be given a group structure compatibly with the group law on $G(F)$ such that translations are continuous.

Exercise 7.4.4. Let R be a Dedekind domain, and X a flat R-scheme of finite type.

(1) Show that scheme-theoretic closure sets up a bijective correspondence between closed subschemes of the generic fiber and closed subschemes of X that are R-flat. Prove that this is compatible with the formation of products over $\operatorname{Spec} R$, and so deduce a similar correspondence for R-flat group schemes of finite type.
(2) Suppose R has fraction field F with characteristic 0, and let $\mathscr{G}$ be a finite flat R-group (so $\mathscr{G}_F$ is étale, and hence can be interpreted as a finite $\operatorname{Gal}(F_s/F)$-module). Construct a bijective correspondence between Galois submodules of $\mathscr{G}(F_s)$ and R-flat closed subschemes of G.

Exercise 7.4.5. Let k be an arbitrary field with characteristic $p > 0$. Using the addition law on length-n Witt vectors (with values in any k-algebra), explain how this gives affine n-space over k a structure of smooth group variety W_n (compatible with extension of the base field, so the case $k = \mathbf{F}_p$ is the most important). Describe it explicitly for $n = 2$ and any k. Can you define a concept of "commutative ring scheme" and exhibit W_n as such an example?

One interesting feature of Witt groups is that they give rise to nontrivial extension structures. For example, construct a complex of k-groups

$$0 \to \mathbf{G}_a \to W_2 \xrightarrow{\pi} \mathbf{G}_a \to 0$$

that is an "exact sequence" in the sense that the first map is the scheme-theoretic kernel of π, and π is faithfully flat (which forces it to have the universal mapping property one would want for a good notion of quotient, though do not try to prove it if you have not studied descent theory). Also prove that over this sequence is non-split: there is no k-group section to π (but there are lots of k-scheme sections!). It can be proved that in the category of smooth commutative k-groups, this is the *universal* "extension" of $\mathbf{G}_a$ by $\mathbf{G}_a$ over k. (In contrast, in characteristic 0 all such extensions are split.)

Exercise 7.4.6. Let $X \to S$ be a map of $\mathbf{F}_p$-schemes. Verify the compatibilities asserted for $F_{X/S}$ in Lemma 7.1.5, and check that if $G = \mathrm{GL}_n$ over $\mathbf{F}_p$ then $F_{G/\mathbf{F}_p}$ is the p-power map on matrix entries. Do likewise for $X = \mathbf{P}^n_{\mathbf{F}_p}$ in terms of standard homogenous coordinates. Also check that if $G = E$ is an elliptic curve over a field k of characteristic $p > 0$ then $F_{E/k}$ as in Definition 7.1.4 really is the Frobenius isogeny from the theory of elliptic curves.

Finally, show that if X is smooth over k with pure dimension $d > 0$ then $F_{X/k}$ is a finite flat map with degree p^d. If A is an abelian variety of dimension $g > 0$ over k then deduce that $F_{A/k}$ is a purely inseparable isogeny of degree p^g.

Exercise 7.4.7. Let k be a perfect field of characteristic $p > 0$. Prove that elements of the Dieudonné ring $\mathscr{D}_k$ admit unique expansions

$$c_0 + \sum_{j>0} c_j \mathscr{F}^j + \sum_{j>0} c'_j \mathscr{V}^j$$

as finite sums with $c_j, c'_j \in \mathrm{W}(k)$. Deduce that $\mathscr{D}_k$ has center $\mathrm{W}(\mathbf{F}_p) = \mathbf{Z}_p$ if k is infinite and center $\mathbf{Z}_p[\mathscr{F}^f, \mathscr{V}^f] \simeq \mathbf{Z}_p[x, y]/(xy - p^f)$ if k has finite size $q = p^f$. For any extension of perfect fields k'/k define a natural ring map $\mathrm{W}(k') \otimes_{\mathrm{W}(k)} \mathscr{D}_k \to \mathscr{D}_{k'}$, and prove it is an isomorphism.

Prove that $\mathscr{D}_k[1/p]$ has a much simpler structure than $\mathscr{D}_k$: if we let $K_0 = \mathrm{W}(k)[1/p]$ then $\mathscr{D}_k[1/p]$ is the twisted polynomial ring $K_0[\mathscr{F}]$ in a variable $\mathscr{F}$ satisfying the commutation relation $\mathscr{F} c = \sigma(c)\mathscr{F}$ for all $c \in K_0$ (where σ denotes the Frobenius automorphism of K_0).

Exercise 7.4.8. Check the proof of Proposition 7.2.7 really works. Also check that if A is an abelian variety over a finite field k and $f \in \mathrm{End}_k(A)$ is an endomorphism of $A \neq 0$ then the common characteristic polynomial $P_f \in \mathbf{Z}[T]$ of all $T_\ell(f) \in \mathrm{End}_{\mathbf{Z}_\ell}(T_\ell(A))$ with $\ell \neq \mathrm{char}(k)$ is also the characteristic polynomial of $\mathbf{D}(f) \in \mathrm{End}_{\mathrm{W}(k)}(\mathbf{D}(A[p^\infty]))$. (Hint: the proof for Tate modules [36, Thm. 4, p. 180] carries over verbatim!)

Exercise 7.4.9. As an application of Dieudonné modules, show that if G is a p-divisible group over a field k of characteristic $p > 0$ and if $\gamma \in \mathrm{Aut}_k(G)$ is a finite-order automorphism that is trivial on $G[p]$ then $\gamma = 1$ provided $p > 2$. Give a counterexample if $p = 2$.

Exercise 7.4.10. Note that the Frobenius map on $K_0 = \mathrm{W}(k)[1/p]$ is an automorphism (in contrast with the considerations with $\mathscr{E}$ in §3, whose residue field was imperfect!). Using this, prove that any Frobenius semilinear injection $D \to D$ for a *finite-dimensional* K_0-vector space D is automatically bijective. Hence, in the definition of an isocrystal over K_0 it suffices to assume that $\phi_D : D \to D$ is injective rather than bijective. (This is very useful in some constructions of isocrystals, such as in constructions involving B_{cris}.)

Exercise 7.4.11. Let K be a p-adic field. For $D, D' \in \mathrm{MF}_K^{\phi}$, give $\mathrm{Hom}_{K_0}(D, D')$ a structure of object in MF_K^{ϕ} (denoted $\mathrm{Hom}(D, D')$) by using the Hom-filtration from Exercise 6.4.1 on $\mathrm{Hom}_{K_0}(D, D')_K = \mathrm{Hom}_K(D_K, D'_K)$ and using the Frobenius structure $\phi : \mathrm{Hom}_{K_0}(D, D') \to \mathrm{Hom}_{K_0}(D, D')$ defined by $\phi(L) = \phi_{D'} \circ L \circ \phi_D^{-1}$. (This is K_0-linear because the two Frobenius semilinearities from $\phi_{D'}$ and ϕ_D cancel out.) Using this definition of the MF_K^{ϕ}-structure on $\mathrm{Hom}_{K_0}(D, D')$, prove that the usual linear isomorphism $D' \otimes_{K_0} D^{\vee} \simeq \mathrm{Hom}_{K_0}(D, D')$ is really an isomorphism $D' \otimes D^{\vee} \simeq \mathrm{Hom}(D, D')$ in MF_K^{ϕ}.

8. Filtered (ϕ, N)-modules

We now wish to take up a systematic study of some basic properties of filtered ϕ-modules, and even a more general kind of structure (filtered (ϕ, N)-modules) that is designed to deal with the p-adic representation analogue of "bad reduction". Before we dive in, it may help to orient ourselves as to how filtered ϕ-modules are going to ultimately fit into Fontaine's period ring formalism.

The category MF_K^{ϕ} of filtered ϕ-modules will naturally arise as the target category of the functor $D_{B_{\mathrm{cris}}}$ associated to a certain $(\mathbf{Q}_p, G_K)$-regular K_0-subalgebra $B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$ to be introduced in §9. This subalgebra contains t and $\mathrm{W}(R)[1/p]$ (hence it contains $\mathrm{W}(\overline{k})[1/p] = \widehat{K_0^{\mathrm{un}}}$), it admits a canonical Frobenius-semilinear and $K_0[G_K]$-algebra endomorphism $\varphi_{\mathrm{cris}} : B_{\mathrm{cris}} \to B_{\mathrm{cris}}$, and the following two crucial properties hold:

- φ_{cris} on B_{cris} is injective (though not surjective),
- the natural map $K \otimes_{K_0} B_{\mathrm{cris}} \to B_{\mathrm{dR}}$ is injective.

In particular, $K \otimes_{K_0} B_{\mathrm{cris}}$ is endowed with a G_K-stable exhaustive and *separated* K-linear subspace filtration from B_{dR}:

$$\mathrm{Fil}^i(K \otimes_{K_0} B_{\mathrm{cris}}) = (K \otimes_{K_0} B_{\mathrm{cris}}) \cap \mathrm{Fil}^i(B_{\mathrm{dR}}).$$

Note that $K \otimes_{K_0} B_{\mathrm{cris}}^{G_K} \subseteq B_{\mathrm{dR}}^{G_K} = K$, so for K_0-dimension reasons the inclusion $K_0 \subseteq B_{\mathrm{cris}}^{G_K}$ is an equality.

It will be shown in Theorem 9.1.6 that B_{cris} is a $(\mathbf{Q}_p, G_K)$-regular ring in the sense of Definition 5.1.1. Thus, by the general formalism of §5, we will get a faithful functor $D_{\mathrm{cris}} : \mathrm{Rep}_{\mathbf{Q}_p}(G_K) \to \mathrm{Vec}_{K_0}$ defined by

$$D_{\mathrm{cris}}(V) := (B_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V)^{G_K}.$$

This finite-dimensional K_0-vector space has two kinds of structure: (i) an *injective* Frobenius-semilinear endomorphism induced by the G_K-equivariant injective Frobenius φ_{cris} on B_{cris} (so this is bijective by Exercise 7.4.10!); (ii) an exhaustive and *separated* K-linear filtration on the scalar extension

$$D_{\mathrm{cris}}(V)_K = ((K \otimes_{K_0} B_{\mathrm{cris}}) \otimes_{\mathbf{Q}_p} V)^{G_K}$$

via the G_K-stable filtration on $K \otimes_{K_0} B_{\mathrm{cris}}$. (Equivalently, there is a natural injective K-linear map $D_{\mathrm{cris}}(V)_K \hookrightarrow D_{\mathrm{dR}}(V)$ making the filtration on $D_{\mathrm{cris}}(V)_K$ into the subspace filtration.)

Thus, $D_{\mathrm{cris}}(V)_K$ has a structure of object in Fil_K, so D_{cris} is a functor

$$D_{\mathrm{cris}} : \mathrm{Rep}_{\mathbf{Q}_p}(G_K) \to \mathrm{MF}_K^{\phi}.$$

Since B_{cris} is $(\mathbf{Q}_p, G_K)$-regular, this is a faithful functor on the full subcategory $\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{cris}}(G_K)$ of B_{cris}-admissible representations (to be called *crystalline representations*) since the forgetful functor $\mathrm{MF}_K^{\phi} \to \mathrm{Vec}_{K_0}$ is faithful. Somewhat deeper is the fact that $D_{\mathrm{cris}} : \mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{cris}}(G_K) \to \mathrm{MF}_K^{\phi}$ is *fully faithful*, as we shall prove later. (In Proposition 9.1.9 it will be proved that crystalline representations V are always de Rham and that $D_{\mathrm{dR}}(V) \in \mathrm{Fil}_K$ can be reconstructed from $D_{\mathrm{cris}}(V)$, so the crystalline condition really does refine the de Rham condition.)

8.1. **Newton and Hodge polygons.** A general filtered ϕ-module is not so useful, since there is no relationship between its Frobenius and filtration structures. The filtered ϕ-modules that arise from algebraic geometry (as well as the ones which will arise from crystalline representations) satisfy some additional properties that relate their Frobenius and filtration structures in a nontrivial manner. This motivates the introduction of a certain full subcategory of MF_K^{ϕ} (the *weakly admissible* filtered ϕ-modules) consisting of objects satisfying such additional properties, and remarkably this subcategory will be abelian.

To define this special class of objects in MF_K^{ϕ}, we need to introduce two important invariants of a filtered ϕ-module, its *Hodge polygon* and its *Newton polygon*. The Hodge polygon is really associated to the underlying object in Fil_K and the Newton polygon is associated to the underlying isocrystal over K_0.

Definition 8.1.1. Let F be a field and let $(D, \{D^i\})$ be a nonzero object in Fil_F. Let $\{i_0 < \cdots < i_n\}$ be the distinct i's such that $\mathrm{gr}^i(D) \neq 0$. The *Hodge polygon* $P_H = P_H(D)$ is the convex polygon in the plane that has leftmost endpoint $(0,0)$ and has $\dim_F \mathrm{gr}^{i_j}(D)$ consecutive segments with horizontal distance 1 and slope i_j for $0 \leqslant j \leqslant r$. (If $D = 0$ then define $P_H(D)$ to be the single point $(0,0)$.) See Figure 1.

The y-coordinate of the rightmost endpoint of $P_H(D)$ is the *Hodge number*

$$t_H(D) = \sum_{i \in \mathbf{Z}} i \cdot \dim_F \mathrm{gr}^i(D).$$

The horizontal length of $P_H(D)$ is $\dim D$, so the rightmost endpoint is $(\dim D, t_H(D))$. For example, if $D^{i_0} = D$ and $D^{i_0+1} = 0$ for some i_0 (i.e., if the filtration is supported in a single degree, at least for $D \neq 0$) then $P_H(D)$ is the segment joining $(0,0)$ and $(d, i_0 d)$ for $d = \dim_F D$. This covers all cases with $\dim D \leqslant 1$. For any nonzero D with dimension $d > 0$, the top exterior power $\det D$ has the natural quotient filtration from $D^{\otimes d}$, and as such we see (by considering a basis adapted to the filtration) that $t_H(D) = t_H(\det D)$.

The natural linear isomorphisms

$$\det(D^\vee) \simeq (\det D)^\vee, \ \ \det(D)^{d'} \otimes \det(D')^d \simeq \det(D \otimes D')$$

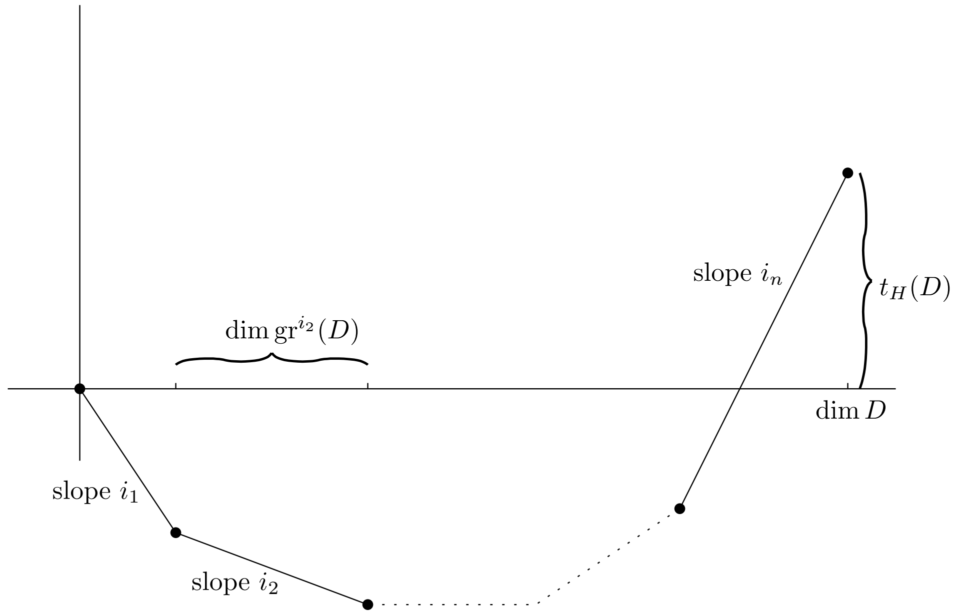


FIGURE 1. A typical Hodge polygon

are isomorphisms in Fil_F for $D, D' \in \mathrm{Fil}_F$ with respective dimensions d and d', and likewise if

$$0 \to D' \to D \to D'' \to 0$$

is a short exact sequence in Fil_F then the natural linear isomorphism $\det(D') \otimes \det(D'') \simeq \det(D)$ is an isomorphism in Fil_F. Thus, the general equality $t_H(D) = t_H(\det D)$ and direct calculations in the 1-dimensional case yield the following useful result concerning the relationship between t_H and tensorial notions in Fil_F.

Proposition 8.1.2. *For $D \in \mathrm{Fil}_F$ we have $t_H(D^\vee) = -t_H(D)$, and for $D, D' \in \mathrm{Fil}_F$ we have*

$$t_H(D \otimes D') = (\dim D)t_H(D') + (\dim D')t_H(D).$$

(In particular, $t_H(D^{\otimes r}) = r(\dim D)^{r-1}t_H(D)$ for $r \geqslant 1$.) Moreover, t_H is additive on short exact sequences in Fil_F in the sense that if $0 \to D' \to D \to D'' \to 0$ is a short exact sequence in Fil_F then

$$t_H(D) = t_H(D') + t_H(D'').$$

To define Newton polygons we switch our attention to the category $\mathrm{Mod}_{K_0}^{\phi}$ of isocrystals over K_0 rather than the category Fil_K. Recall that in p-adic analysis, one attaches a convex Newton polygon to a polynomial or power series $\prod(1 - t_i T)$ with constant term 1, and the *slopes* of this polygon are numbers $\mathrm{ord}_p(t_i)$ as t_i varies through the reciprocal zeros. An interesting example of this occurs in the study of abelian varieties over finite fields: if A is an abelian variety of dimension $g > 0$ over a finite field k of size $q = p^r$, then although the Frobenius operator ϕ on $D = \mathbf{D}(A[p^\infty])[1/p]$ is not K_0-linear (if $r \neq 1$), the iterate ϕ^r is K_0-linear. Moreover, by Theorem 7.2.4(3) this linear operator is induced by the q-Frobenius endomorphism of A. Thus, by Exercise 7.4.8 the characteristic polynomial of ϕ^r is exactly the usual one over $\mathbf{Z}$ from the ℓ-adic Tate modules, so its zeros $\lambda_1, \ldots, \lambda_{2g}$ encode the zeta function of A over k. (For example, the $\mathrm{ord}_p(\lambda_i)$'s encode information about whether A is ordinary, supersingular, or somewhere in between.)

The ratios $\mathrm{ord}_p(\lambda_i)/r = \mathrm{ord}_p(\lambda_i)/\mathrm{ord}_p(q)$ are insensitive to replacing k with a finite extension, and so it is such "normalized" p-adic ordinals that are of interest (and have a chance to be generalized to the case of infinite k, such as $\overline{\mathbf{F}}_p$).

Rather generally, for any isocrystal D over $K_0 = \mathrm{W}(k)[1/p]$ with a finite k of size $q = p^r$ we can make sense of eigenvalues for the K_0-linear map ϕ_D^r, so we can define the set of *slopes* of D (with multiplicity) to be the set of ratios $\mathrm{ord}_p(\lambda)/\mathrm{ord}_p(q)$ where λ ranges through the set of eigenvalues of ϕ_D^r (in an algebraic closure of K_0). This set of ratios is invariant under finite extension on k, but if k is not finite then this linearization trick is not available. Thus, in general we have to proceed in another way.

Let us first explain a possible approach that turns out not to work (but whose failure is instructive). Fix a basis $\{e_i\}$ of D and consider the resulting "matrix" for ϕ_D, by which we mean the (visibly invertible) matrix (a_{ij}) over K_0 with $\phi_D(e_j) = \sum a_{ij} e_i$. This matrix transforms in a semilinear-conjugation manner under a change of basis, so its set of eigenvalues $\{\lambda_i\}$ is not basis-independent in general, but it is natural to wonder if the set of p-adic ordinals $\mathrm{ord}_p(\lambda_i)$ (with multiplicity) is independent of the choice of basis. In the 1-dimensional case this holds since $\sigma(c)/c \in \mathrm{W}(k)^\times$ for any $c \in K_0^\times$, but unfortunately it fails in the 2-dimensional case, as the following example shows.

Example 8.1.3 (Katz). Let $K_0 = \mathrm{W}(\mathbf{F}_{p^2})[1/p]$ with $p \equiv 3 \bmod 4$, and let $i = \sqrt{-1} \in K_0$. Let $D = K_0 e_1 \oplus K_0 e_2$ and define $\phi_D : D \to D$ by the matrix

$$\begin{pmatrix} p-1 & (p+1)i \\ (p+1)i & -(p-1) \end{pmatrix}.$$

That is, we define $\phi_D(e_1) = (p-1)e_1 + (p+1)ie_2$ and $\phi_D(e_2) = (p+1)ie_1 - (p-1)e_2$ and we then extend ϕ_D uniquely by Frobenius-semilinearity. This matrix has characteristic polynomial $X^2 - 4p$, so its roots are $\pm 2\sqrt{p}$. These have p-adic ordinal $1/2$. However, if we pass to the basis $e_1' = e_1 + ie_2$ and $e_2' = ie_1 + e_2$ then since the Frobenius of K_0 carries i to $-i$ (as $p \equiv 3 \bmod 4$) we compute that $\phi_D(e_1') = 2pe_1'$ and $\phi_D(e_2') = 2e_2'$. So in this new basis the matrix for ϕ_D has eigenvalues 2 and $2p$ with respective p-adic ordinals 0 and 1.

In view of the preceding example, we have to use an alternative procedure to define a concept of slope for an isocrystal D over $K_0 = \mathrm{W}(k)[1/p]$ when k is a general perfect field of characteristic $p > 0$. The procedure that will work rests on the important Dieudonné–Manin classification of isocrystals when k is algebraically closed, so we now review this classification.

Let $\overline{k}$ be an algebraic closure of k. For any isocrystal D over K_0 we get an isocrystal over $\widehat{K_0^{\mathrm{un}}} = \mathrm{W}(\overline{k})[1/p]$ by scalar extension: $\widehat{D} = \widehat{K_0^{\mathrm{un}}} \otimes_{K_0} D$ endowed with the bijective semilinear tensor-product Frobenius structure $\phi_{\widehat{D}}(c \otimes d) = \sigma(c) \otimes \phi_D(d)$. The *Dieudonné–Manin classification* [34, II, §4.1] describes the possibilities for $\widehat{D}$:

Theorem 8.1.4 (Dieudonné–Manin)**.** *For an algebraically closed field k of characteristic $p > 0$, the category $\mathrm{Mod}_{K_0}^{\phi}$ of isocrystals over $K_0 = \mathrm{W}(k)[1/p]$ is semisimple* (*i.e., all objects are finite direct sums of simple objects and all short exact sequences split*)*. Moreover, the* *simple objects are given up to isomorphism (without repetition) by the isocrystals $D_{r,s}$ in Example* 7.3.2 *with* $\gcd(r,s)=1$.

This theorem says that if $k=\overline{k}$ then the isomorphism classes of simple isocrystals over K_0 are in natural bijection with $\mathbf{Q}$, where a rational number α expressed uniquely in reduced form s/r with $r>0$ corresponds to $D_{r,s}$. In view of the definition of $D_{r,s}$, where ϕ looks as if it acts with eigenvalues of p-adic ordinal s/r, the decomposition in Theorem 8.1.4 is akin to an eigenspace decomposition for a semisimple operator, and we shall write Δ_α to denote $D_{r,s}$; this is called the simple object with *pure slope* α in $\mathrm{Mod}^{\phi}_{K_0}$ (when $k=\overline{k}$).

For any perfect field k with characteristic $p>0$ and any isocrystal D over $K_0=\mathrm{W}(k)[1/p]$, the Dieudonné–Manin classification provides a unique decomposition of $\widehat{D}:=\widehat{K_0^{\mathrm{un}}}\otimes_{K_0} D$ in the form

$$\widehat{D}=\oplus_{\alpha\in\mathbf{Q}}\widehat{D}(\alpha) \tag{8.1.1}$$

for subobjects $\widehat{D}(\alpha)\simeq\Delta_\alpha^{e_\alpha}$ having "pure slope α" (and $\widehat{D}(\alpha)=0$ for all but finitely many α). For each $\alpha=s/r\in\mathbf{Q}$ in reduced form (with $r>0$), the integer $\dim_{\widehat{K_0^{\mathrm{un}}}}\widehat{D}(\alpha)=re_\alpha$ is the number (with multiplicity) of "eigenvalues" of ϕ_D with slope α.

Definition 8.1.5. The $\alpha\in\mathbf{Q}$ for which $\widehat{D}(\alpha)\neq 0$ are the *slopes* of D, and $\dim_{\widehat{K_0^{\mathrm{un}}}}\widehat{D}(\alpha)$ is called the *multiplicity* of this slope. We say that D is *isoclinic* (with slope α_0) if $D\neq 0$ and $\widehat{D}=\widehat{D}(\alpha_0)$ for some $\alpha_0\in\mathbf{Q}$ (i.e., $\widehat{D}\simeq\Delta_{\alpha_0}^{e}$ for some $e\geqslant 1$).

In Exercise 8.4.1, an interesting class of examples will be worked out in which slopes actually do correspond to p-adic ordinals of eigenvalues. For now, we illustrate the definition of slopes by revisiting D as in Example 8.1.3. It is natural to guess that it either has slopes $\{0,1\}$ or the single slope $1/2$ with multiplicity 2. Let us check that the first of these guesses is correct. Note that by using the basis $\{e_1',e_2'\}$ gives an isomorphism of D with a direct sum of two 1-dimensional objects on which Frobenius acts (relative to a suitable basis vector over K_0) via multiplication by $2p$ and 2. Letting σ denote the absolute Frobenius automorphism of $\mathrm{W}(\overline{\mathbf{F}}_p)$, a successive approximation argument shows that the self-map of $\mathrm{W}(\overline{\mathbf{F}}_p)^\times$ defined by $u\mapsto\sigma(u)/u$ is surjective. In particular, we can find $c\in\mathrm{W}(\overline{\mathbf{F}}_p)^\times$ such that $\sigma(c)/c=1/2$, and over $\mathrm{W}(\overline{\mathbf{F}}_p)[1/p]=\widehat{\mathbf{Q}_p^{\mathrm{un}}}$ we compute that ϕ fixes ce_2' and multiplies ce_1' by p. Thus, we get an isomorphism $\widehat{\mathbf{Q}_p^{\mathrm{un}}}\otimes_{\mathbf{Q}_p}D\simeq\Delta_1\oplus\Delta_0$, so the slopes are as claimed.

Remark 8.1.6. The theory of slopes for modules with a "Frobenius" endomorphism arises in many contexts far beyond the setting of isocrystals over K_0. In §10.3 we will see an illustration of the relevance of more general theories of "Frobenius slope" in p-adic Hodge theory.

A convenient visualization device for recording information about slopes and their multiplicities is the Newton polygon:

Definition 8.1.7. Let D be a nonzero isocrystal over K_0 with slopes $\{\alpha_0<\cdots<\alpha_n\}$ having multiplicities $\{\mu_0,\ldots,\mu_n\}$. The *Newton polygon* $P_N(D)$ of D is the convex polygon with leftmost endpoint $(0,0)$ and having μ_i consecutive segments of horizontal distance 1 and slope α_i. (If $D=0$ then define $P_N(D)$ to be the point $(0,0)$.) See Figure 2.

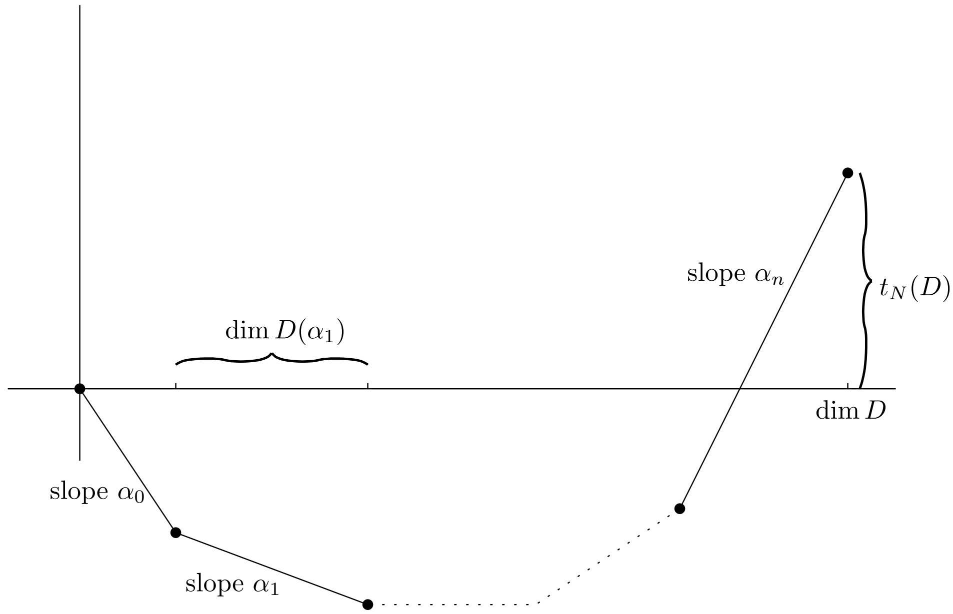


FIGURE 2. A typical Newton polygon

The y-coordinate of the rightmost endpoint of $P_N(D)$ is the *Newton number*

$$t_N(D) = \sum \alpha_i \dim \widehat{D}(\alpha_i).$$

The rightmost endpoint of $P_N(D)$ is $(\dim D, t_N(D))$, and all corners of $P_N(D)$ are in $\mathbf{Z}^2$ since $\alpha_i \mu_i \in \mathbf{Z}$ for all i. Note that a nonzero isocrystal D over K_0 is isoclinic of slope α if and only if $P_N(D)$ is a segment with slope α, which is to say that $\widehat{D}$ is isoclinic of slope α.

Lemma 8.1.8. *Let $K_0 = \mathrm{W}(k)[1/p]$ for a perfect field k with characteristic $p > 0$. If D_1 and D_2 are isocrystals over K_0 that are isoclinic with respective slopes α_1 and α_2 then $D_1 \otimes D_2$ is isoclinic with slope $\alpha_1 + \alpha_2$.*

Beware that this lemma cannot be proved by "eigenvalue" considerations over $\widehat{K_0^{\mathrm{un}}}$, due to the problems exhibited in Example 8.1.3.

Proof. By the definition of being isoclinic, we can assume k is algebraically closed and we
1 need to exhibit a decomposition of $\Delta_{\alpha_1} \otimes \Delta_{\alpha_2}$ into a direct sum of copies of $\Delta_{\alpha_1+\alpha_2}$.[1] ■

Lemma 8.1.8 yields the following analogue of Proposition 8.1.2 that is proved by the same determinantal isomorphisms as in the proof of Proposition 8.1.2.

Proposition 8.1.9. *For an isocrystal D over K_0 we have $t_N(D) = t_N(\det D)$, $t_N(D^\vee) = -t_N(D)$, and for two isocrystals D and D' over K_0 we have*

$$t_N(D \otimes D') = (\dim D) t_N(D') + (\dim D') t_N(D).$$

[1] need to insert proof!

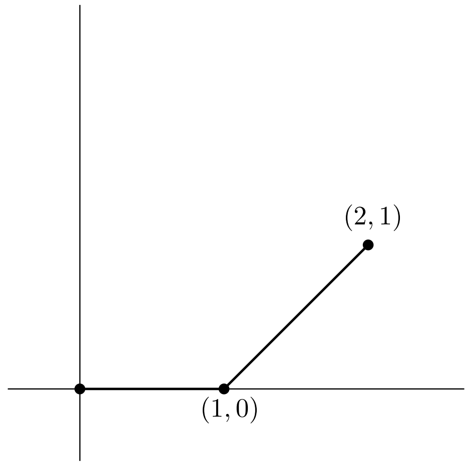


FIGURE 3. Hodge polygon of an elliptic curve.

(In particular, $t_N(D^{\otimes r}) = r(\dim D)^{r-1}t_N(D)$ for all $r \geqslant 1$.) Moreover, t_N is additive on short exact sequences of isocrystals over K_0 in the sense that if $0 \to D' \to D \to D'' \to 0$ is a short exact sequence of isocrystals over K_0 then $t_N(D) = t_N(D') + t_N(D'')$.

For any filtered ϕ-module D in MF_K^{ϕ}, there are now associated two convex polygons with leftmost endpoint $(0,0)$: the Hodge polygon $P_H(D)$ associated to $D_K \in \mathrm{Fil}_K$ and the Newton polygon $P_N(D)$ associated to the isocrystal D over K_0. One way to relate the two structures is to consider the relative positions of these two polygons in the plane. This is best understood with a concrete example, as follows.

Example 8.1.10. Let E be an elliptic curve over K with good reduction, say with $\mathscr{E}$ the unique elliptic curve over $\mathscr{O}_K$ having generic fiber E and with $\mathscr{E}_0$ denoting its special fiber. Let D be the filtered ϕ-module over K associated to E, which is to say $D = \mathrm{H}^1_{\mathrm{cris}}(\mathscr{E}_0/\mathrm{W}(k))[1/p]$ with its natural Frobenius structure and with D_K filtered by means of the comparison isomorphism $D_K \simeq \mathrm{H}^1_{\mathrm{dR}}(E/K)$. (Recall from Remark 7.3.3 that if $K = K_0$ then it is equivalent to work with Fontaine's Honda system for $\mathscr{E}[p^\infty]$, provided $\mathscr{E}_0$ is supersingular when $p = 2$.)

The object D_K in Fil_K is the same for all E, a 2-dimensional K-vector space with gr^0 and gr^1 each 1-dimensional, so the Hodge polygon $P_H(D)$ is the same for *all* E. See Figure 3.

In contrast, the structure of D as an isocrystal depends on whether the reduction $\mathscr{E}_0$ over k is ordinary or supersingular. Indeed, using the Frobenius-compatible $\mathrm{W}(k)$-linear isomorphism $\mathrm{H}^1_{\mathrm{cris}}(\mathscr{E}_0/\mathrm{W}(k)) \simeq \mathbf{D}(\mathscr{E}_0[p^\infty])^{(p)}$ from [4, 2.5.6, 2.5.7, 3.3.7, 4.2.14], we see that $P_N(D)$ looks as in Figure 4. In particular, for all E we see that $P_N(D)$ *lies on or above* $P_H(D)$ *and their right endpoints coincide.*

Although the Dieudonné–Manin classification does not extend to the case when k is not assumed to be algebraically closed, the "slope decomposition" (8.1.1) into isoclinic parts does uniquely descend:

Lemma 8.1.11. *For a nonzero isocrystal D over K_0 whose Newton polygon has slopes $\alpha_1 < \cdots < \alpha_n$, there is a unique decomposition $D = \oplus D(\alpha_i)$ into a direct sum of nonzero subobjects that are isoclinic with respective slopes $\alpha_1 < \cdots < \alpha_n$.*

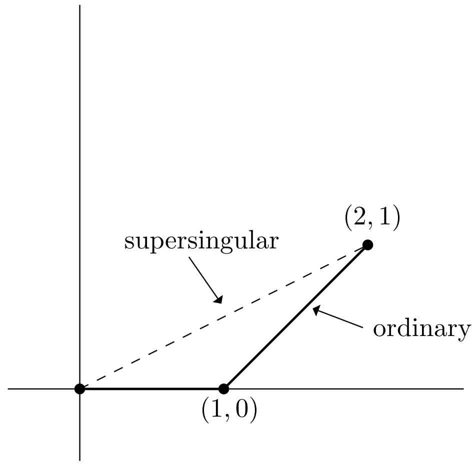


FIGURE 4. Newton polygons of elliptic curves.

The decomposition in this lemma is called the *slope decomposition* of the isocrystal D. In Kedlaya's theory of slope filtrations for modules over another kind of ring with Frobenius endomorphism, one gets not a direct sum decomposition into isoclinic parts but rather a filtration with isoclinic successive quotients; see Theorem 10.3.6.

Proof. By the Dieudonné–Manin classification, the result holds when k is algebraically closed. Thus, the only issue is to descend the isoclinic decomposition for $\widehat{D} = \widehat{K_0^{\mathrm{un}}} \otimes_{K_0} D$. The natural action of $G_k = G_K/I_K$ on $\widehat{D}$ that is semilinear over the $\widehat{K_0^{\mathrm{un}}}$-vector space structure commutes with the Frobenius structure on $\widehat{D}$. Each $\widehat{D}(\alpha_i)$ is spanned over $\widehat{K_0^{\mathrm{un}}}$ by the images of all maps $\Delta_{\alpha_i} \to \widehat{D}$ as isocrystals over $\widehat{K_0^{\mathrm{un}}}$, which is to say it is spanned over $\widehat{K_0^{\mathrm{un}}}[\phi]$ by all elements $v \in \widehat{D}$ such that $\phi^{r_i}(v) = p^{s_i}v$ with s_i/r_i the reduced form of $\alpha_i \in \mathbf{Q}$. Hence, each $\widehat{D}(\alpha_i)$ is G_k-stable.

By the completed unramified descent in Lemma 3.2.6, the ϕ-stable K_0-subspace $D(\alpha_i) := \widehat{D}(\alpha_i)^{G_k}$ of $\widehat{D}^{G_k} = D$ satisfies $\widehat{K_0^{\mathrm{un}}} \otimes_{K_0} D(\alpha_i) \simeq \widehat{D}(\alpha_i)$. Thus, each $D(\alpha_i)$ is an isocrystal over K_0 that is isoclinic of slope α_i, and $\oplus D(\alpha_i) \to D$ is an isomorphism of isocrystals over K_0. ■

A nice application of the slope decomposition is given in Exercise 8.4.1, which gives a situation in which slopes actually do correspond to p-adic ordinals of eigenvalues (with multiplicity).

Example 8.1.12. Consider the isocrysal $D = \mathbf{D}(G)[1/p]$ arising from a p-divisible group G over k. Since $p\mathbf{D}(G) \subseteq \phi(\mathbf{D}(G)) \subseteq \mathbf{D}(G)$, we see that $p \cdot \phi^{\vee}$ preserves the $\mathrm{W}(k)$-lattice $\mathbf{D}(G)^{\vee}$ in $D^{\vee}$ and hence are power-bounded in the sense of Exercise 8.4.1(4). It follows that the isocrystals (D, ϕ) and $(D^{\vee}, p\phi^{\vee})$ have power-bounded Frobenius. Each therefore has slopes $\geqslant 0$. By applying duality to a Dieudonne–Manin decomposition, we conclude that such D have all slopes in the interval $[0, 1]$.

Vastly generalizing Example 8.1.10, it was conjectured by Katz (and proved by Mazur in special cases and Berthelot–Ogus [5, 8.36] in general) that if $\mathscr{X}$ is a smooth proper $\mathrm{W}(k)$-scheme then the Newton polygon of the isocrystal $\mathrm{H}^i_{\mathrm{cris}}(\mathscr{X}_0/\mathrm{W}(k))[1/p]$ lies on or above the Hodge polygon of the filtered vector space $\mathrm{H}^i_{\mathrm{dR}}(\mathscr{X}_{K_0}/K_0)$ and that these polygons in the first quadrant have the same right endpoint. This positional condition on the two polygons partially motivates the interest in the following lemma of Fontaine.

Lemma 8.1.13 (Fontaine)**.** *Let $D \in \mathrm{MF}^{\phi}_K$ be arbitrary. The following two conditions are equivalent.*

1. *For all subobjects $D' \subseteq D$, $P_N(D')$ lies on or above $P_H(D')$.*
2. *For all subobjects $D' \subseteq D$, the rightmost endpoint of $P_N(D')$ lies on or above the one for $P_H(D')$; i.e., $t_N(D') \geqslant t_H(D')$.*

Moreover, these properties hold for D in MF^{ϕ}_K if and only if they hold for $\widehat{D} := \widehat{K_0^{\mathrm{un}}} \otimes_{K_0} D$ in $\mathrm{MF}^{\phi}_{\widehat{K^{\mathrm{un}}}}$.

Proof. The first condition certainly implies the second. For the converse, we assume that there is some subobject $D' \subseteq D$ such that $P_N(D')$ contains a point lying strictly below the point of $P_H(D')$ on the same vertical line and we seek to construct a subobject $D'' \subseteq D$ violating the second condition (i.e., $t_N(D'') < t_H(D'')$). Necessarily $D' \neq 0$. Both polygons $P_N(D')$ and $P_H(D')$ are convex with common left endpoint $(0,0)$, and by hypothesis the right endpoint of $P_N(D')$ lies on or above that of $P_H(D')$. Hence, there is some $0 < x_0 < \dim D'$ such that the line $x = x_0$ meets $P_N(D')$ and $P_H(D')$ at the respective points (x_0, y_N) and (x_0, y_H) where $y_N < y_H$.

By small deformation of x_0 and continuity considerations, we can arrange that neither of these two points on $x = x_0$ is a corner of their respective polygon, so there is a well-defined slope of the polygons at such points. Depending on which of the two slopes is larger, by convexity we can move either forwards or backwards to get to the case when (x_0, y_N) is the final point of the part of $P_N(D')$ with some slope α_0; note that we still have $0 < x_0 < \dim D$ since the left endpoints of $P_N(D)$ and $P_H(D)$ coincide with $(0,0)$ and the respective right endpoints are $(\dim D, t_N(D))$ and $(\dim D, t_H(D))$ where $t_N(D) \geqslant t_H(D)$ by hypothesis on D.

Consider the isoclinic decomposition $\widehat{D} = \oplus_{\alpha\in\mathbf{Q}}\widehat{D}(\alpha)$ of $\widehat{D} \in \mathrm{MF}^{\phi}_K$ from Lemma 8.1.11. Let $\widehat{D}' = \oplus_{\alpha\leqslant\alpha_0}\widehat{D}(\alpha)$ and give $\widehat{D}'_K$ the subspace filtration from $\widehat{D}_K$, so $\widehat{D}'$ is a subobject of $\widehat{D}$ in MF^{ϕ}_K. By construction, $P_N(\widehat{D}')$ is the subset of $P_N(\widehat{D})$ through slopes up to α_0, so its right endpoint is (x_0, y_N). Hence, $t_N(\widehat{D}') = y_N$. Since $\widehat{D}'_K$ has the subspace filtration from $\widehat{D}_K$, the filtration jumps in $\widehat{D}'_K$ stay on or ahead of those of $\widehat{D}_K$ for the first $\dim \widehat{D}'$ segments of the Hodge polygons, which is to say that $P_H(\widehat{D}')$ lies on or above $P_H(\widehat{D})$ across $0 \leqslant x \leqslant \dim \widehat{D}'$. Thus, $t_H(\widehat{D}') \geqslant y_H > y_N = t_N(\widehat{D}')$, contradicting our hypothesis about right endpoints of Hodge and Newton polygons of all subobjects of D.

Finally, it remains to check that scalar extension by $K_0 \to \widehat{K_0^{\mathrm{un}}}$ does not affect whether or not the equivalent properties (1) and (2) hold. This is not obvious because $\widehat{D}$ may have subobjects that do not arise from subobjects of D. When $\widehat{D}$ satisfies these conditions in $\mathrm{MF}^{\phi}_{\widehat{K^{\mathrm{un}}}}$ then so does D in MF^{ϕ}_K (since the formation of P_H and P_N is unchanged by the scalar extension $K_0 \to \widehat{K_0^{\mathrm{un}}}$). Conversely, suppose $\widehat{D}$ violates these conditions in $\mathrm{MF}^{\phi}_{\widehat{K^{\mathrm{un}}}}$; we seek to prove the same for D in MF^{ϕ}_K. The preceding argument produces a slope α_0 such that the subobject $\widehat{\Delta} = \oplus_{\alpha \leqslant \alpha_0} \widehat{D}(\alpha)$ of $\widehat{D}$ spanned by the isoclinic parts of $\widehat{D}$ with slope at most α_0 (with $\widehat{\Delta}_{\widehat{K^{\mathrm{un}}}}$ given the subspace filtration from $\widehat{D}_{\widehat{K^{\mathrm{un}}}}$) has Newton polygon $P_N(\widehat{\Delta})$ that does not lie on or above the Hodge polygon $P_H(\widehat{\Delta})$. But then $\Delta = \oplus_{\alpha \leqslant \alpha_0} D(\alpha)$ is a subobject of D (with Δ_K given the subspace filtration from D_K) such that $\widehat{\Delta} = \widehat{K_0^{\mathrm{un}}} \otimes_{K_0} \Delta$ as subobjects of $\widehat{D}$, so $P_N(\Delta) = P_N(\widehat{\Delta})$ does not lie on or above $P_H(\widehat{\Delta}) = P_H(\Delta)$. ■

8.2. **Weakly admissible modules.** The conditions in Lemma 8.1.13 inspire the following definition.

Definition 8.2.1. A filtered ϕ-module D over K is *weakly admissible* if $t_N(D') \geqslant t_H(D')$ for all subobjects $D' \subseteq D$ in MF^{ϕ}_K, with equality when $D' = D$. (This final condition $t_H(D) = t_N(D)$ says exactly that $P_H(D)$ and $P_N(D)$ have the same right endpoint.)

The full subcategory of MF^{ϕ}_K consisting of weakly admissible objects is denoted $\mathrm{MF}^{\phi,\mathrm{w.a.}}_K$.

Keeping in mind the Newton and Hodge polygons associated to ordinary and supersingular elliptic curves is the easiest way to remember that it is P_N that lies on or above P_H.

By Lemma 8.1.13, the property of being weakly admissible is unaffected by the scalar extension $K_0 \to \widehat{K_0^{\mathrm{un}}}$. The case of duality requires a bit of thought, since the definition is in terms of subobjects rather than quotients.

Proposition 8.2.2. *If $D \in \mathrm{MF}^{\phi}_K$ then D is weakly admissible if and only if its dual $D^{\vee}$ is weakly admissible.*

Proof. Since t_H and t_N are negated under duality, it suffices to show that in the definition of weak admissibility it is equivalent to work with the alternative condition that for all quotients $D \twoheadrightarrow D''$ we have $t_N(D'') \leqslant t_H(D'')$ with equality when $D'' = D$. For any D in MF^{ϕ}_K there is a natural bijective correspondence between subobjects $D' \subseteq D$ and quotient objects $\pi : D \twoheadrightarrow D''$ (up to isomorphism), namely $D' \mapsto D'' := D/D'$ and $D'' \mapsto \ker \pi$. Since $t_H(D') + t_H(D/D') = t_H(D)$ and $t_N(D') + t_N(D/D') = t_N(D)$ with the values $t_H(D)$ and $t_N(D)$ fixed and independent of D', we are done. ■

Remark 8.2.3. The weak admissibility property is also inherited under tensor products, but this is a very difficult fact to prove directly since it is hard to describe subobjects of $D \otimes D'$. The "right" way to understand this compatibility is by using the deeper result of Fontaine and Colmez [14, Thm. A] that says the weakly admissible filtered ϕ-modules are exactly the $D_{\mathrm{cris}}(V)$'s for crystalline representations V, in which case the compatibility with tensor products becomes a special case of the general formalism of period rings in Theorem 5.2.1(3).

It is a remarkable fact that $\mathrm{MF}^{\phi,\mathrm{w.a.}}_K$ is an abelian category (using kernels and cokernels as in the additive category MF^{ϕ}_K that is *not* abelian), and more specifically that any morphism between weakly admissible filtered ϕ-modules is strict with respect to filtrations over K in the sense of Definition 6.2.5. To avoid later duplication of effort, rather than prove these properties for $\mathrm{MF}^{\phi,\mathrm{w.a.}}_K$ now, we prefer to establish such a result for a larger category of structures beyond MF^{ϕ}_K that we will need later. Whereas MF^{ϕ}_K was inspired by the study of smooth proper K-schemes X with good reduction (i.e., $X \simeq \mathscr{X}_K$ for $\mathscr{X}$ smooth and proper over $\mathscr{O}_K$), we need to enlarge MF_K^ϕ to include linear algebra objects associated to the p-adic representations arising from more general smooth proper K-schemes X (with "bad reduction").

What additional structure(s) should we impose on the linear algebra side to capture p-adic representations arising from X with "bad reduction"? One source of motivation is an observation of Grothendieck concerning the structure of *general* ℓ-adic representations of Galois groups of finite extensions K of $\mathbf{Q}_p$ (with $\ell \neq p$), so we now explain his observation.

For simplicity, consider K that is a finite extension of $\mathbf{Q}_p$; i.e., assume the residue field k is finite of some size q. We let ℓ be a prime distinct from p and consider a continuous representation $\rho : G_K \to \mathrm{GL}(W)$ of G_K on a finite-dimensional $\mathbf{Q}_\ell$-vector space W. Since $\mathrm{GL}(W)$ contains a pro-ℓ neighborhood of the identity and the wild inertia group $P_K \subseteq I_K$ is pro-p, by continuity of ρ there is a finite extension K'/K such that the restriction $\rho|_{I_{K'}}$ is tame (i.e., kills $P_{K'}$) and even factors through the maximal pro-ℓ quotient of the abelian tame inertia group $I_{K'}^{\mathrm{t}}$.

By considering the I_K-action on root extractions of a uniformizer of K (the choice of which does not matter), one obtains [43, 1.3] a canonical isomorphism $t_K : I_K^{\mathrm{t}} \simeq \prod_{p' \neq p} \mathbf{Z}_{p'}(1)$ that is G_k-equivariant, and by [43, 1.4] we have the basic compatibility

$$t_K|_{I_{K'}^{\mathrm{t}}} = e(K'/K) t_{K'}. \tag{8.2.1}$$

By a clever argument with cyclotomic characters on the Galois group of the residue field, Grothendieck proved [46, App.]:

Lemma 8.2.4 (Grothendieck)**.** *The representation on W by the pro-ℓ $\mathbf{Z}_\ell(1)$-quotient of $I_{K'}^{\mathrm{t}}$ is unipotent if K'/K is sufficiently ramified.*

To use the lemma, we first recall a special fact about unipotent and nilpotent matrices over a field of characteristic 0: the operators $\log(U) = \log(1+(U-1))$ for unipotent U and $\exp(N)$ for nilpotent N are *finite* sums (not more than the dimension of the vector space) since $U-1$ and N are both nilpotent. Hence, these are readily checked to be inverse bijections, and they carry commutative groups to commutative groups. In particular, any commutative group of unipotent matrices (such as $\rho(I_{K'}^{\mathrm{t}})$ in Grothendieck's lemma) is uniquely expressed as the exponential of an additive group of nilpotent matrices. Hence, the representation of $\mathbf{Z}_\ell(1)$ on W via $\rho|_{G_{K'}}$ is described by exponentiating $\mathbf{Z}_\ell$-multiples of a nilpotent matrix.

That is, if K'/K is sufficiently ramified then $\rho|_{I_{K'}^{\mathrm{t}}}$ has the unique form $g \mapsto \exp(t_{K',\ell}(g) N_{K'})$ for $N_{K'} \in \mathrm{Hom}_{\mathbf{Q}_\ell}(W, W(-1))$ that is nilpotent. By (8.2.1), the modified operator $N = e(K'/K)^{-1} N_{K'}$ is *independent* of K'/K and the representation ρ on $I_{K'}^{\mathrm{t}}$ has the form $g \mapsto \exp(t_{K,\ell}(g)N)$. This is rather striking: for suitable K'/K, the representation $\rho|_{I_K}$ is encoded on an open subgroup in terms of the single nilpotent operator N!

In geometric contexts arising from étale cohomology or topology, the logarithm of a unipotent inertial action encodes the "monodromy", so N is called the *monodromy operator* for the ℓ-adic representation of G_K on W. It is important to determine how $N : W \to W(-1)$ interacts with the action of $\rho(\phi_K^{-1})$ for a choice of q-Frobenius $\phi_K \in G_K$ (with $q = \#k$). (It is the "geometric" q-Frobenius ϕ_K^{-1} whose action on ℓ-adic cohomology has good integrality properties in the smooth proper case with good reduction.) To carry this out, since N encodes most of the tame inertial action of $I_{K'}$ we should first understand how a q-Frobenius element of G_K conjugates on $I^t_{K'}$, or even I^t_K.

In general, the left action of G_K on I_K via conjugation induces a left action by the quotient group $G_k = G_K/I_K$ on the abelian quotient $I^{\rm t}_K$, and this is given by the formula

$$\widetilde{g}\tau\widetilde{g}^{-1} = \tau^{\chi_{\rm cyc}(g)} \tag{8.2.2}$$

for any $\tau \in I^{\rm t}_K$ and any $\widetilde{g} \in G_K$ lifting $g \in G_k$, where $\chi_{\rm cyc} : G_K \to \widehat{\mathbf{Z}}^\times$ denotes the total cyclotomic character (whose ℓ-adic component for any prime $\ell \neq p$ is the ℓ-adic cyclotomic character). In particular, $t_K(\widetilde{g}\tau\widetilde{g}^{-1}) = \chi_{\rm cyc}(g)t_K(\tau)$, so for $t = t_{K,\ell}$ we have

$$\begin{aligned}\exp(t(\tau)\rho(\widetilde{g})N\rho(\widetilde{g})^{-1}) = \rho(\widetilde{g})\exp(t(\tau)N)\rho(\widetilde{g})^{-1} &= \rho(\widetilde{g}\tau\widetilde{g}^{-1})\\ &= \exp(t(\widetilde{g}\tau\widetilde{g}^{-1})N)\\ &= \exp(t(\tau)\chi_\ell(g)N).\end{aligned}$$

Hence, taking logarithms of both sides and letting τ vary gives

$$\rho(\widetilde{g})N\rho(\widetilde{g})^{-1} = \chi_\ell(g)N.$$

Now choose $g \in G_k$ to be the *geometric* q-Frobenius (i.e., the inverse of $x \mapsto x^q$), and pick a uniformizer π of K. The extension field $K^{\rm un}(\pi^{1/\ell^\infty})/K$ is Galois and accounts for the entire ℓ-adic part of $I^{\rm t}_K$. Thus, it makes sense to define the lift $\widetilde{g}_\pi \in {\rm Gal}(K^{\rm un}(\pi^{1/\ell^\infty})/K)$ of g by the condition that it fixes the chosen compatible system of ℓ-power roots of π. Let $\varphi = \varphi_\pi := \rho(\widetilde{g}_\pi)$, so φ is a linear endomorphism of W depending on π and $\varphi N\varphi^{-1} = \chi_\ell(\widetilde{g}_\pi)N = q^{-1}N$. In other words, $N\varphi = q\varphi N$.

In the p-adic case ($\ell = p$) we shall now impose a similar kind of structure on the semilinear algebra side.

Definition 8.2.5. A *(ϕ, N)-module* (over K_0) is an isocrystal (D, ϕ_D) over K_0 equipped with a K_0-linear endomorphism $N_D : D \to D$ (called the *monodromy operator*) such that $N_D\phi_D = p\phi_D N_D$. The notion of morphism between such objects is the evident one. The category of these is denoted ${\rm Mod}^{\phi,N}_{K_0}$.

A *filtered (ϕ, N)-module* (over K) is a (ϕ, N)-module D over K_0 for which D_K is endowed with a structure of object in ${\rm Fil}_K$. The notion of morphism between such objects is the evident one, and the category of these is denoted ${\rm MF}^{\phi,N}_K$.

In Definition 8.2.5 we do not *assume* N_D is nilpotent; it will be deduced later (in Lemma 8.2.8). We write $K_0[0]$ to denote the 1-dimensional unit object of ${\rm MF}^\phi_K$ (i.e., $D = K_0$ with ${\rm gr}^0(D_K) \neq 0$ and ϕ equal to the Frobenius automorphism); this is a "unit object" for the tensor product. Upon endowing it with the monodromy operator $N = 0$ it likewise becomes the unit object for the tensor product in ${\rm MF}^{\phi,N}_K$. Note that in general ${\rm MF}^\phi_K$ is exactly the full subcategory of ${\rm MF}^{\phi,N}_K$ consisting of objects whose monodromy operator vanishes.

The categories ${\rm Mod}^{\phi,N}_{K_0}$ and ${\rm MF}^{\phi,N}_K$ have evident notions of short exact sequence, kernel, cokernel, image, and coimage. We also define duals and tensor products in the evident manner, and the one subtlety is how to define the monodromy operator on the tensor product and dual. To see how to define $N_{D\otimes D'}$ in terms of N_D and $N_{D'}$, and how to define $N_{D^\vee}$ in terms of N_D, we use the motivating situation of ℓ-adic representations of G_K (with $\ell \neq p$) to see what to do: if $\rho(g) = \exp(t(g)N)$ and $\rho'(g) = \exp(t(g)N')$ then

$$\begin{aligned} (\rho \otimes \rho')(g) = \rho(g) \otimes \rho'(g) &= \exp(t(g)(N \otimes 1)) \circ \exp(t(g)(1 \otimes N')) && (8.2.3)\\ &= \exp(t(g)(N \otimes 1 + 1 \otimes N')) && (8.2.4) \end{aligned}$$

$$\rho^\vee(g) = \rho(g^{-1})^\vee = \exp(-t(g)N^\vee).$$

This motivates the following definitions (which one checks do satisfy "$N\phi = p\phi N$"):

$$N_{D\otimes D'} = 1_D \otimes N_{D'} + N_D \otimes 1_{D'},\ \ N_{D^\vee} = -N_D^\vee.$$

(Note that the evaluation pairing $D \otimes D^\vee \to K_0[0]$ *is* thereby a morphism in $\mathrm{Mod}_K^{\phi,N}$!) These formulas may look familiar from the theory of Lie algebra representations, and the similarity is no coincidence since the formula $\rho(g) = \exp(t(g)N)$ essentially makes the monodromy operator like the derivative of the representation at the identity element.

We likewise define N on $\mathrm{Hom}(D, D')$ by the rule $N(L) = N_{D'} \circ L - L \circ N_D$, and in this way the natural isomorphism $D' \otimes D^\vee \simeq \mathrm{Hom}(D, D')$ in $\mathrm{Mod}_{K_0}^{\phi}$ is an isomorphism in $\mathrm{MF}_K^{\phi,N}$. By combining the procedures for tensor products and quotients, we can define exterior and symmetric power operations on $\mathrm{MF}_K^{\phi,N}$.

Remark 8.2.6. Beware that for D in $\mathrm{MF}_K^{\phi,N}$, the concept of *subobject* of D is very sensitive to the specified monodromy operator N on D since a subobject must be stable by N on D. For example, if we replace the given N with 0 then we get a new object in place of D and it has many more subobjects than the original D does in general since the monodromy-stability condition has become much weaker.

Example 8.2.7. For $D \in \mathrm{MF}_K^{\phi,N}$, consider the isoclinic decomposition $D = \oplus_{\alpha\in\mathbf{Q}} D(\alpha)$ of the underlying isocrystal. By the definition of $D(\alpha)$, its scalar extension $\widehat{D}(\alpha)$ over $\widehat{K_0^{\mathrm{un}}}$ is spanned by vectors v such that $\phi_{\widehat{D}}^r(v) = p^s v$ for s/r the reduced form of α, so

$$\phi_{\widehat{D}}^r(Nv) = p^{-r} N\phi_{\widehat{D}}^r(v) = p^{s-r}Nv.$$

But $(s-r)/r = \alpha - 1$, so $Nv \in \widehat{D}(\alpha - 1)$. Hence, by descent from $\widehat{K_0^{\mathrm{un}}}$, we get $N(D(\alpha)) \subseteq D(\alpha - 1)$. Due to this relationship between N and the $D(\alpha)$'s, we see that $\oplus_{\alpha\leqslant a} D(\alpha)$ is N-stable for any $a \in \mathbf{Q}$.

This has two applications. First, as in the motivating ℓ-adic case, nilpotence holds:

Lemma 8.2.8. *For any $D \in \mathrm{Mod}_{K_0}^{\phi,N}$, the monodromy operator N_D on D is nilpotent. In particular, if $\dim D = 1$ then $N_D = 0$.*

Proof. The intuitive idea is that $\phi^{-1} \circ N \circ \phi = pN$, so the finite nonempty set of eigenvalues of N in $\overline{K}$ is stable under p-multiplication and thus is $\{0\}$, which is to say that N is nilpotent. But since ϕ is not generally linear this argument is merely suggestive of nilpotence and is not a proof. To carry out an actual proof based on this idea we shall use the isoclinic decomposition $D = \oplus_{\alpha\in\mathbf{Q}} D(\alpha)$. We saw in Example 8.2.7 that $N(D(\alpha)) \subseteq D(\alpha - 1)$. Since $D(\alpha) = 0$ for all but finitely many α, the nilpotence of N now follows. ∎

Definition 8.2.1 now extends to incorporate a monodromy operator:

Definition 8.2.9. An object $D \in \mathrm{MF}_K^{\phi,N}$ is *weakly admissible* if for all subobjects $D' \subseteq D$ in $\mathrm{MF}_K^{\phi,N}$ (so D' is required to be N-stable in D) we have $t_N(D') \geqslant t_H(D')$ with equality for $D' = D$. Equivalently, for all quotient objects $D \twoheadrightarrow D''$ in $\mathrm{MF}_K^{\phi,N}$ we have $t_N(D'') \leqslant t_H(D'')$ with equality for $D'' = D$.

These objects constitute a full subcategory $\mathrm{MF}_K^{\phi,N,\mathrm{w.a.}}$ of $\mathrm{MF}_K^{\phi,N}$. (Clearly $\mathrm{MF}_K^{\phi,\mathrm{w.a.}}$ consists of objects in $\mathrm{MF}_K^{\phi,N,\mathrm{w.a.}}$ for which $N = 0$.)

Using Example 8.2.7, Fontaine's Lemma 8.1.13 carries over verbatim to $\mathrm{MF}_K^{\phi,N}$, and so weak admissibility can also be described in terms of Hodge and Newton polygons for subobjects or quotients of D. Weak admissibility is a very subtle link between three structures: the Frobenius, the filtration, and the monodromy operator (whose only role here is to constrain the possible subobjects in $\mathrm{MF}_K^{\phi,N}$ via the N-stability condition). Since $N_{D^\vee} = -N_D^\vee$, we see as in the case $N = 0$ that D in $\mathrm{MF}_K^{\phi,N}$ is weakly admissible if and only if $D^\vee$ is weakly admissible.

Continuing the theme of Remark 8.2.6, what happens if we simply redefine the monodromy operator to be 0? That is, for D in $\mathrm{MF}_K^{\phi,N}$, consider the object D' that is obtained by setting the monodromy operator to be 0 but leaving everything else (the underlying isocrystal over K_0 and filtration structure over K) unchanged. It can and does happen when $\dim D > 1$ that D may be weakly admissible whereas D' is not! The problem is that the N-stability condition on subobjects of D' is weaker than that for D, so D' may admit K_0-subspaces that are subobjects (i.e., Frobenius-stable) but are not subobjects of D (i.e., not stable by N_D). Some of these extra K_0-subspaces may lead to violation of the weak admissibility property for D' even if D is weakly admissible. This phenomenon already occurs in the 2-dimensional case for $K = K_0 = \mathbf{Q}_p$, as we will see in the classification of 2-dimensional objects in $\mathrm{MF}_{\mathbf{Q}_p}^{\phi,N,\mathrm{w.a.}}$ in §8.3.

The next two results in $\mathrm{MF}_K^{\phi,N}$ could have been proved much earlier in MF_K^{ϕ}, but we waited so that we could handle $\mathrm{MF}_K^{\phi,N}$ in general.

Proposition 8.2.10. *If $0 \to D' \to D \to D'' \to 0$ is a short exact sequence in $\mathrm{MF}_K^{\phi,N}$ and any two of the three terms are weakly admissible then so is the third.*

Proof. If D is weakly admissible then for any subobject D'_1 of D' we may view D'_1 as a subobject of D and hence $t_H(D'_1) \leqslant t_N(D'_1)$. If in addition D'' is weakly admissible then $t_H(D'') = t_N(D'')$, so $t_H(D') = t_H(D) - t_H(D'') = t_N(D) - t_N(D'') = t_N(D')$. Thus, D' is weakly admissible when D and D'' are so. Applying these considerations after dualizing the original exact sequence and using the general identity that t_H and t_N negate under duality, we conclude that if D and D' are weakly admissible then so is D''.

Now suppose that D' and D'' are weakly admissible. By additivity in short exact sequences we see that $t_H(D) = t_N(D)$ due to the analogous such equalities for D' and D''. It remains to prove $t_H(D_1) \leqslant t_N(D_1)$ for all subobjects $D_1 \subseteq D$. We let $D'_1 = D' \cap D_1$ and give $(D'_1)_K$ the subspace filtration from either $(D_1)_K$ or D'_K (these subspace filtrations coincide!), and let $D''_1 = D_1/D'_1$ with the quotient filtration on $(D''_1)_K$. There is a natural injective map $j : D''_1 \hookrightarrow D'' = D/D'$ in $\mathrm{MF}_K^{\phi,N}$, but a priori it may not be strict (i.e., the quotient filtration on $(D''_1)_K$ from $(D_1)_K$ may be finer than the subspace filtration from D''_K). Since D'_1 is a subobject of the weakly admissible D', $t_H(D_1') \leqslant t_N(D_1')$. Thus,

$$t_H(D_1) = t_H(D_1') + t_H(D_1'') \leqslant t_N(D_1') + t_H(D_1'')$$

and $t_N(D_1) = t_N(D_1') + t_N(D_1'')$, so it suffices to prove that $t_H(D_1'') \leqslant t_N(D_1'')$.

Let $j(D_1'')$ denote D_1'' endowed with the subspace filtration from D'', so the natural map $D_1'' \to j(D_1'')$ in $\mathrm{MF}_K^{\phi,N}$ is a linear isomorphism. We have $t_N(D_1'') = t_N(j(D_1''))$ since j is an isomorphism in the category $\mathrm{Mod}_{K_0}^{\phi}$ of isocrystals over K_0. Hence, it is enough to prove $t_H(D_1'') \leqslant t_N(j(D_1''))$. But $j(D_1'')$ is a subobject of the weakly admissible D'', so $t_H(j(D_1'')) \leqslant t_N(j(D_1''))$ and hence our problem reduces to proving the inequality $t_H(D_1'') \leqslant t_H(j(D_1''))$ between Hodge numbers for the bijective morphism $j : D_1'' \to j(D_1'')$ in $\mathrm{MF}_K^{\phi,N}$.

In general, if $h : \Delta' \to \Delta$ is a bijective morphism in Fil_K then we claim that $t_H(\Delta') \leqslant t_H(\Delta)$ with equality if and only if h is an isomorphism in Fil_K (i.e., it is a strict morphism). To prove this, first note that $t_H(\Delta) = t_H(\det \Delta)$ and $t_H(\Delta') = t_H(\det \Delta')$, and a consideration of bases adapted to filtrations shows that a bijective morphism in Fil_K is an isomorphism in Fil_K if and only if the induced map on top exterior powers is an isomorphism in Fil_K. Thus, by passing to $\det h : \det \Delta' \to \det \Delta$ we reduce to the 1-dimensional case, for which t_H is the unique i such that $\mathrm{gr}^i \neq 0$. This concludes the argument. ■

We now come to the remarkable fact that in the presence of the weak admissibility condition the filtration structures behave as in an abelian category:

Theorem 8.2.11. *Let $h : D \to D'$ be a map in $\mathrm{MF}_K^{\phi,N,\mathrm{w.a.}}$. The map h is strict (i.e., $D/\ker h \to \mathrm{im}\, h$ is an isomorphism in $\mathrm{MF}_K^{\phi,N}$), and $\ker h$ and $\mathrm{coker}\, h$ with their respective subspace and quotient filtration structures are weakly admissible. In particular, the object $\mathrm{im}\, h \simeq D/\ker h$ is weakly admissible and the category $\mathrm{MF}_K^{\phi,N}$ is abelian.*

Proof. Consider the diagram

$$\ker h \hookrightarrow D \twoheadrightarrow \mathrm{coim}\, h \to \mathrm{im}\, h \hookrightarrow D' \twoheadrightarrow \mathrm{coker}\, h$$

in $\mathrm{MF}_K^{\phi,N}$ with $\mathrm{coim}\, h := D/\ker h$ given the quotient filtration structure. Both $\ker h$ and $\mathrm{im}\, h$ have subspace filtration structures (from D and D' respectively), and $\mathrm{coker}\, h$ has the quotient filtration structure from D'. Since the map $\mathrm{coim}\, h \to \mathrm{im}\, h$ is a bijective morphism in $\mathrm{MF}_K^{\phi,N}$, the argument at the end of the proof of Proposition 8.2.10 gives $t_H(\mathrm{coim}\, h) \leqslant t_H(\mathrm{im}\, h)$ with equality if and only if h is a strict morphism. Weak admissibility of D gives the inequality $t_N(\mathrm{coim}\, h) \leqslant t_H(\mathrm{coim}\, h)$ for the quotient object $\mathrm{coim}\, h$ of D, and likewise the weak admissibility of D' gives the inequality $t_H(\mathrm{im}\, h) \leqslant t_N(\mathrm{im}\, h)$ for the subobject $\mathrm{im}\, h$ of D'.

Putting these inequalities together gives

$$t_N(\mathrm{coim}\, h) \leqslant t_H(\mathrm{coim}\, h) \leqslant t_H(\mathrm{im}\, h) \leqslant t_N(\mathrm{im}\, h),$$

but the Newton numbers given by the outer terms are equal since the map $\mathrm{coim}\, h \to \mathrm{im}\, h$ on underlying isocrystals over K_0 is an isomorphism (although we have not yet shown the induced map over K to be an isomorphism in Fil_K). Hence, equality holds throughout, so the equality in the middle gives that h is a strict morphism. That is, $\mathrm{coim}\, h \to \mathrm{im}\, h$ is an isomorphism in $\mathrm{MF}_K^{\phi,N}$.

Since $\ker h = \ker(D \to \operatorname{coim} h)$ and $\operatorname{coker} h = D'/\operatorname{im} h$, and we know that weak admissibility is inherited by the third term of any short exact sequence in $\mathrm{MF}_K^{\phi,N}$ in which two of the objects are weakly admissible, it remains to prove that the object $\Delta := \operatorname{coim} h \simeq \operatorname{im} h$ in $\mathrm{MF}_K^{\phi,N}$ is weakly admissible. It is a subobject (or quotient) of a weakly admissible object, so the only aspect requiring justification is that $t_H(\Delta) = t_N(\Delta)$. However, this equality was already proved above. ■

In §9.2 we shall define an intermediate $(\mathbf{Q}_p, G_K)$-regular $K_0[G_K]$-algebra

$$B_{\mathrm{cris}} \subseteq B_{\mathrm{st}} \subseteq B_{\mathrm{dR}}$$

such that B_{st} admits a natural *injective* Frobenius-semilinear endomorphism $\varphi : B_{\mathrm{st}} \to B_{\mathrm{st}}$ extending the one on B_{cris} and a natural K_0-linear derivation $N : B_{\mathrm{st}} \to B_{\mathrm{st}}$ such that $N\varphi = p\varphi N$ and $B_{\mathrm{cris}} = B_{\mathrm{st}}^{N=0}$ (so N is even B_{cris}-linear). Also, the natural map $K \otimes_{K_0} B_{\mathrm{st}} \to B_{\mathrm{dR}}$ will turn out to be injective, so $K_0 = B_{\mathrm{st}}^{G_K}$ and the functor $V \rightsquigarrow (B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V)^{G_K}$ is a covariant functor

$$D_{\mathrm{st}} : \mathrm{Rep}_{\mathbf{Q}_p}(G_K) \to \mathrm{MF}_K^{\phi,N}.$$

For all B_{st}-admissible representations V (to be called *semistable* representations), we will prove that $D_{\mathrm{st}}(V)$ is weakly admissible. Hence, we will get a covariant faithful tensor functor

$$D_{\mathrm{st}} : \mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{st}}(V) \to \mathrm{MF}_K^{\phi,N,\mathrm{w.a.}}.$$

Later we will show that this is *fully faithful.* This is why weak admissibility is an interesting notion for our purposes. It is a deep theorem of Fontaine and Colmez [14, Thm. A] that this functor is an equivalence of categories. Passing to objects with vanishing monodromy then yields an equivalence $D_{\mathrm{cris}} : \mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{cris}}(V) \simeq \mathrm{MF}_K^{\phi,\mathrm{w.a.}}$.

8.3. **Twisting and low-dimensional examples.** The construction of period rings is rather dry and boring, and likewise calculations in semi-linear algebra can feel quite dull without some sense of why anyone should care. This section represents a bit of a compromise: accepting on faith some properties of B_{cris} and B_{st} to be developed in §9, we wish to illustrate the theory of filtered (ϕ, N)-modules by working out some basic examples and giving Galois-theoretic interpretations to the result. The reader who is uncomfortable with this can skip this section until reading §9, but probably there will be more motivation to wade through §9 (and more appreciation of how one works with filtered (ϕ, N)-modules) if one reads this section first, at least in a superficial manner. This area of mathematics abounds in unsolved problems of pedagogy.

To get started, we introduce a twisting operation that corresponds (under suitable *contravariant* Fontaine functors $D_B^* = \mathrm{Hom}_{\mathbf{Q}_p[G_K]}(\cdot, B)$) to the operation $V \rightsquigarrow V \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(i)$ on the Galois side for $i \in \mathbf{Z}$. Suppose that $B \subseteq B_{\mathrm{dR}}$ is a $\mathbf{Q}_p[G_K]$-subalgebra containing the canonical $\mathbf{Z}_p(1)$ (of which the two most important examples are B_{cris} and B_{st}). For any basis t of $\mathbf{Z}_p(1)$, elements of $W' = \mathrm{Hom}_{\mathbf{Q}_p}(V(i), B)$ can be written as $w' = t^{-i}w$ for $w \in W := \mathrm{Hom}_{\mathbf{Q}_p}(V, B)$, so $w' \in W'$ is G_K-invariant if and only if $w \in W$ is G_K-invariant. Clearly w' lies in $\mathrm{Hom}_{\mathbf{Q}_p}(V, t^r B_{\mathrm{dR}}^+)$ if and only if $w \in \mathrm{Hom}_{\mathbf{Q}_p}(V, t^{r+i} B_{\mathrm{dR}}^+)$. The subring B_{cris} will contain t with the Frobenius of B_{cris} acting on t^{-i} as multiplication by p^{-i}, and since $t \in B_{\mathrm{cris}}$ we also will have $N(t) = 0$. Thus, we are led to the following definition.

Definition 8.3.1. For $D \in \mathrm{MF}_K^{\phi,N}$, the *$i$-fold Tate twist of* D is the object $D\langle i\rangle$ whose underlying K_0-vector space is D, monodromy operator $N_{D\langle i\rangle}$ is N_D, Frobenius operator $\phi_{D\langle i\rangle}$ is $p^{-i}\phi_D$, and filtration structure over K is $\mathrm{Fil}^r(D\langle i\rangle_K) = \mathrm{Fil}^{r+i}(D_K)$.

Beware that this definition is adapted to the use of *contravariant* Fontaine functors $D_B^*(V) = \mathrm{Hom}_{\mathbf{Q}_p[G_K]}(\cdot, B)$. If using the covariant D_B then i should be replaced with $-i$ everywhere on the output. (It is always confusing to keep track of signs for Tate twists. It is best to rederive things with $\mathbf{Q}_p(1)$ by a little calculation using t each time rather than trying to memorize formulas.)

Using the above definition, the Hodge polygon $P_H(D\langle i\rangle)$ is obtained from the Hodge polygon $P_H(D)$ by decreasing all slopes in the polygon by i, and likewise for Newton polygons. Thus, $t_H(D\langle i\rangle) = t_H(D) - i\dim D$ and likewise for t_N. Since $D' \mapsto D'\langle i\rangle$ sets up a bijection between the set of subobjects of D and the set of subobjects of $D\langle i\rangle$, we see that D is weakly admissible if and only if $D\langle i\rangle$ is weakly admissible. In terms of later *contravariant* period ring constructions, we will have $D^*_{\mathrm{st}}(V(i)) \simeq D^*_{\mathrm{st}}(V)\langle i\rangle$ for any $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ and $i \in \mathbf{Z}$, and similarly for D^*_{cris}.

Example 8.3.2. We can now parameterize all 1-dimensional objects in $\mathrm{MF}_K^{\phi,N}$ and describe the weak admissibility property in terms of the parameters. By 1-dimensionality, we have: $N_D = 0$, $D = K_0e$, and $\phi(e) = \lambda e$ for some $\lambda \in K_0^\times$. If we replace e with $e' = ce$ for some $c \in K_0^\times$ then λ is replaced with $\lambda' = (\sigma(c)/c)\lambda$, where σ is the Frobenius automorphism of $K_0 = \mathrm{W}(k)[1/p]$. In particular, $\mathrm{ord}_p(\lambda)$ is *independent* of the choice of basis; this is the unique slope of D and it is equal to $t_N(D)$.

Since $\dim D_K = 1$, there is a unique $r \in \mathbf{Z}$ such that $\mathrm{gr}^r(D_K) \neq 0$, which is to say $\mathrm{Fil}^r(D_K) = D_K$ and $\mathrm{Fil}^{r+1}(D_K) = 0$; hence, $t_H(D) = r$. By passing to $D\langle r\rangle$ if necessary, we get to the case where $\mathrm{gr}^0(D_K) \neq 0$. Then the Hodge polygon $P_H(D)$ is the horizontal segment with endpoints $(0,0)$ and $(1,0)$, and $P_N(D)$ is the segment with endpoints $(0,0)$ and $(1, \mathrm{ord}_p(\lambda))$. Hence, a 1-dimensional D with $\mathrm{gr}^0(D_K) \neq 0$ is weakly admissible if and only if $\mathrm{ord}_p(\lambda) = 0$, which is to say $\lambda \in \mathrm{W}(k)^\times$. (In general, the necessary and sufficient condition for weak admissibility is $\mathrm{ord}_p(\lambda) = r$, where $\mathrm{ord}_p(\lambda) = t_N(D)$ and $r = t_H(D)$.)

Upon specifying the discrete filtration parameter $r = t_H(D) \in \mathbf{Z}$ and the discrete slope parameter $\mu \in \mathbf{Z}$, the isomorphism class is determined by $\lambda \in K_0^\times$ with $\mathrm{ord}_p(\lambda) = \mu$ up to the equivalence relation $\lambda \sim (\sigma(c)/c)\lambda$ for $c \in K_0^\times$ (or even just $c \in \mathrm{W}(k)^\times$).

We can refine the preceding example as follows. For $n \geqslant 1$ let $F : \mathrm{W}_n^\times \to \mathrm{W}_n^\times$ be the relative Frobenius morphism of the smooth affine $\mathbf{F}_p$-group of units in the length-n Witt vectors. (See Exercise 7.4.5.) There is a short exact sequence of smooth affine $\mathbf{F}_p$-groups

$$1 \to \mathrm{W}_n(\mathbf{F}_p)^\times \to \mathrm{W}_n^\times \xrightarrow{\wp} \mathrm{W}_n^\times \to 1 \tag{8.3.1}$$

with $\wp(x) := F(x)/x$ and $\mathrm{W}_n(\mathbf{F}_p)^\times$ denoting the finite constant $\mathbf{F}_p$-group $(\mathbf{Z}/p^n\mathbf{Z})^\times$. (By "short exact" we can take the meaning that one has short exactness on geometric points and the left term is the functorial kernel of $\wp$.) Thus, passing to k-points on (8.3.1) gives rise to an isomorphism

$$\mathrm{W}_n(k)^\times/\wp(\mathrm{W}_n(k)^\times) \simeq \mathrm{H}^1(k, \mathrm{W}_n(\mathbf{F}_p)^\times) \simeq \mathrm{H}^1(G_k, (\mathbf{Z}/p^n\mathbf{Z})^\times)$$

because $\mathrm{H}^1(k, \mathrm{W}_n^\times) = 1$ (as $\mathrm{W}_n^\times$ has a finite filtration by smooth closed k-subgroups with successive quotients $\mathbf{G}_m$ and $\mathbf{G}_a$, each of which has vanishing degree-1 cohomology over k). Passing to the inverse limit over n and using successive approximation and p-adic completeness and separatedness of $\mathrm{W}(k)$ then gives a natural isomorphism

$$\mathrm{W}(k)^\times/\wp(\mathrm{W}(k)^\times) \simeq \mathrm{Hom}_{\mathrm{cont}}(G_k, \mathbf{Z}_p^\times) = \mathrm{Hom}_{\mathrm{cont}}^{\mathrm{un}}(G_K, \mathbf{Q}_p^\times) \tag{8.3.2}$$

onto the group of unramified p-adic characters of G_K.

In other words, we have parameterized such characters by integral units $\lambda \in \mathrm{W}(k)^\times$ up to the equivalence relation $\lambda \sim (\sigma(c)/c)\lambda = \wp(c)\cdot\lambda$ for $c \in \mathrm{W}(k)^\times$. But such equivalence classes have been seen in Example 8.3.2 to also parameterize isomorphism classes of 1-dimensional weakly admissible filtered (ϕ, N)-modules D over K with $t_H(D) = 0$, so to each continuous unramified character $\eta : G_K \to \mathbf{Q}_p^\times$ we can associate the isomorphism class D_η of a 1-dimensional weakly admissible filtered (ϕ, N)-module over K. This abstract conclusion can be interpreted very nicely:

Lemma 8.3.3. *The bijective correspondence $\eta \mapsto D_\eta$ from continuous unramified characters of G_K to isomorphism classes of 1-dimensional weakly admissible filtered (ϕ, N)-modules over K with $t_H = 0$ is the contravariant Fontaine functor $D_{\mathrm{cris}}^* = \mathrm{Hom}_{\mathbf{Q}_p[G_K]}(\cdot, B_{\mathrm{cris}})$. That is, $D_{\mathrm{cris}}^*(\mathbf{Q}_p(\eta))$ is in the isomorphism class D_η.*

In terms of the covariant Fontaine functor, $D_{\mathrm{cris}}(\mathbf{Q}_p(\eta)) = D_{\eta^{-1}}$.

Proof. Let $\eta : G_K \to \mathbf{Z}_p^\times$ be an unramified character. The proof of the isomorphism (8.3.2) produces a $\lambda \in \mathrm{W}(k)^\times$ such that for $w \in \mathrm{W}(\overline{k})^\times$ satisfying $\wp(w) = \lambda$ we have $g(w) = \eta(g)w$ for all $g \in G_K$. The choice of w is unique up to a $\mathbf{Z}_p^\times$-multiple, so the line $D = K_0 w \subseteq \mathrm{W}(\overline{k})$ only depends on λ. The construction of B_{cris} to be given in §9.1 realizes it as a G_K-stable K_0-subalgebra of B_{dR} containing $\mathrm{W}(R)[1/p]$ in a Frobenius-compatible manner and hence containing $\mathrm{W}(\overline{k})[1/p] = \widehat{K_0^{\mathrm{un}}}$ in a Galois-equivariant and Frobenius-compatible manner. Thus, $D_{\mathrm{cris}}^*(\mathbf{Q}_p(\eta)) = \mathrm{Hom}_{\mathbf{Q}_p[G_K]}(\mathbf{Q}_p(\eta), B_{\mathrm{cris}})$ contains a nonzero element e corresponding to the map $1 \mapsto w$. But $\dim_{K_0} D_{\mathrm{cris}}^*(\mathbf{Q}_p(\eta)) \leqslant \dim_{\mathbf{Q}_p} \mathbf{Q}_p(\eta) = 1$, so $D_{\mathrm{cris}}^*(\mathbf{Q}_p(\eta))$ is 1-dimensional over K_0 with basis e. Clearly the nontrivial gr^i is for $i = 0$ (as $w \in \widehat{K_0^{\mathrm{un}}}^\times$), and $\phi(e) = \lambda e$ because $\sigma(w) = \lambda w$ by the way we chose w. ∎

In Example 9.1.12 we will verify that $D_{\mathrm{cris}}^*(\mathbf{Q}_p(1))$ identifed with the Tate twist $(K_0[0])\langle 1\rangle$ of the unit object (if we use the covariant D_{cris} the answer would be dualized: $(K_0[0])\langle -1\rangle$). Hence, in view of the tensor compatibility of the (contravariant) Fontaine functors and the direct calculation of the filtered ϕ-module $D_{\mathrm{cris}}^*(\mathbf{Q}_p(1))$, it follows from Lemma 8.3.3 via Tate-twisting that every 1-dimensional weakly admissible filtered (ϕ, N)-module over K is D_{cris}^* applied to the Tate twist of an unramified character (all of which are crystalline, since we shall see that the crystalline property can be checked on the inertia group and is invariant under Tate twisting). Since D_{cris}^* will be shown to be fully faithful on crystalline representations, it follows there are no further crystalline characters to be found. That is, granting basic properties of B_{cris} and D_{cris} to be proved later, we have shown:

Proposition 8.3.4. *The functor D_{cris}^* is an equivalence of categories between 1-dimensional crystalline representations of G_K and 1-dimensional weakly admissible filtered (ϕ, N)-modules* *over K. The characters arising in this way are precisely the Tate twists of the $\mathbf{Z}_p^\times$-valued unramified characters of G_K.*

Our classification of 1-dimensional crystalline representations of G_K did not require knowing in advance that $D^*_{\mathrm{cris}}(V)$ is weakly admissible when V is crystalline, nor that every weakly admissible module over K arises from a semistable representation. For the 2-dimensional case it seems hopeless to give an elementary analysis of the classification problem for crystalline or semistable representations for all K (even granting elementary facts about B_{cris} and B_{st}).

The rest of §8.3 is a very long "exercise" in linear algebra: we will solve the purely algebraic problem of classifying all 2-dimensional weakly admissible filtered (ϕ, N)-modules over $K = \mathbf{Q}_p$. If we grant (as will be proved in Proposition 9.2.11, Proposition 9.2.14, Theorem 9.3.4, and Remark 11.3.4) that there is a dimension-preserving contravariant tensor equivalence between semistable representations and weakly admissible filtered (ϕ, N)-modules via an appropriate period ring B_{st}, under which crystalline representations are precisely those for which $N = 0$, we will have then classified *all* 2-dimensional semistable representations of $G_{\mathbf{Q}_p}$.

One reason that the case $K = \mathbf{Q}_p$ is much simpler to analyze on the linear algebra side than the case of general K is that in such cases ϕ is linear over $K_0 = K$ and Exercise 8.4.1 relates slopes to actual eigenvalues (i.e., roots of a characteristic polynomial).

For the 2-dimensional classification in ${}^{\mathrm{w.a.}}\,\mathrm{MF}^{\phi,N}_{\mathbf{Q}_p}$, we will encounter both irreducible and reducible cases, and within the reducible cases it is the non-semisimple ones that will be the most interesting (especially the relationship between the Hodge-Tate weights of their "diagonal characters"). We shall state the classification (Theorem 8.3.6) in terms of the contravariant Fontaine functor $V \mapsto D^*_{\mathrm{cris}}(V) = \mathrm{Hom}_{\mathbf{Q}_p[G_{\mathbf{Q}_p}]}(V, B_{\mathrm{cris}})$ because $D^*_{\mathrm{cris}}(\mathbf{Q}_p(r))$ has nonzero gr^i precisely for $i = r$ (rather than $i = -r$), but the main work in the proof involves only semilinear algebra (for which all relevant notions have already been defined already, whereas B_{cris} and D_{cris} will be defined in §9).

Let $(D, \varphi, \mathrm{Fil}^\bullet(D), N)$ be a 2-dimensional weakly admissible filtered (ϕ, N)-module over $\mathbf{Q}_p$. By applying a suitable Tate twist (in the sense of Definition 8.3.1), we may arrange that $\mathrm{Fil}^0(D) = D$ and $\mathrm{Fil}^1(D) \neq D$. To systematically treat all possibilities, we need to consider various special situations.

First consider the special case when the filtration structure is trivial: $\mathrm{Fil}^1(D) = 0$. In this case we shall drop the 2-dimensionality hypothesis and allow $n = \dim_{K_0} D \geqslant 1$ to be arbitrary. Removing the effect of the Tate twist at the start (i.e., assume $\mathrm{Fil}^r(D) = D$ and $\mathrm{Fil}^{r+1}(D) = 0$ for some r), these are the cases in which the Hodge polygon is a straight line. In this case by convexity and agreement of both endpoints it follows that $P_N(D) = P_H(D)$, so in terms of the isoclinic decomposition there is only one slope. Hence, without any hypotheses on $\dim_K D$ we must have $N = 0$ and $\varphi : D \simeq D$ with pure slope 0, so by Exercise 8.4.1 the map φ has characteristic polynomial $f_\varphi(X) \in \mathbf{Z}_p[X]$ with all roots in $\overline{\mathbf{Z}}_p^\times$.

The subobjects are the φ-stable subspaces, each of which has Hodge and Newton polygons that coincide (as segments along the x-axis). Hence, weak admissibility always holds. Also, there is always a lattice $\Lambda \subseteq D$ that is φ-stable and on which φ acts as an automorphism. Indeed, φ is 'power-bounded (in the sense of Exercise 8.4.1(4)) since $f_\varphi \in \mathbf{Z}_p[X]$ (compute after extending scalars to acquire all eigenvalues, and use generalized eigenvectors), so for any $\mathbf{Z}_p$-lattice $L \subseteq D$ the span $\Lambda = \sum_{n \geqslant 0} \varphi^n(L)$ is bounded and hence is a φ-stable lattice. But $\det \varphi \in \mathbf{Z}_p^\times$, so φ is an automorphism of Λ.

To summarize, when the filtration structure is trivial we are simply studying $\mathbf{Q}_p$-isogeny classes of pairs (Λ, T) consisting of a lattice Λ over $\mathbf{Z}_p$ and a linear automorphism T of Λ. In other words, this is the study of $\mathrm{GL}_n(\mathbf{Q}_p)$-conjugacy classes of elements of $\mathrm{GL}_n(\mathbf{Z}_p)$. These will correspond (via D^*_{st}) to unramified n-dimensional representations of $G_{\mathbf{Q}_p}$ (which may be semisimple or not, and when irreducible are never absolutely irreducible because $G_{\mathbf{F}_p}$ is abelian). Removing the effect of the Tate twist, the cases in which the Hodge polygon is a straight line correspond to cyclotomic twists of unramified representations. In particular, these have a single Hodge-Tate weight (equal to the unique i such that $\mathrm{gr}^i(D) \neq 0$ – rather than $\mathrm{gr}^{-i}(D) \neq 0!$ – when we use the *contravariant* Fontaine functors). We record our conclusions in this special case:

Proposition 8.3.5. *The n-dimensional weakly admissible filtered (ϕ, N)-modules over $\mathbf{Q}_p$ with a single Hodge–Tate weight have vanishing N, and in case of Hodge–Tate weight 0 are parameterized up to isomorphism by $\mathrm{GL}_n(\mathbf{Q}_p)$-conjugacy classes of elements of $\mathrm{GL}_n(\mathbf{Z}_p)$. In general if the Hodge–Tate weight is i then such objects naturally correspond under D^*_{cris} to χ^i-twists of n-dimensional unramified p-adic representations of $G_{\mathbf{Q}_p}$*

We now turn to the more interesting case in which there are two distinct filtration jumps, or in Galois-theoretic terms (via D^*_{st}) two distinct Hodge-Tate weights. Taking into account our initial Tate twist to get to the case $\mathrm{Fil}^0(D) = D$ and $\mathrm{Fil}^1(D) \neq D$, we must have that $L := \mathrm{Fil}^1(D)$ is a line in D. There is a discrete invariant $r \geqslant 1$: $\mathrm{Fil}^j(D) = L$ when $1 \leqslant j \leqslant r$ and $\mathrm{Fil}^{r+1}(D) = 0$. In terms of Galois representations (using D^*_{st}), the Hodge-Tate weights are 0 and r (or more invariantly, r is the gap between the Hodge-Tate weights). In particular, $t_H(D) = r$. It will be convenient to separately treat the cases when $N = 0$ and when $N \neq 0$ (i.e., crystalline representations and semistable non-crystalline representations).

First we consider the case $N = 0$, which is to say 2-dimensional *crystalline* representations with two distinct Hodge–Tate weights, Once again, we make an initial Tate twist so that the smaller such weight is 0.

Theorem 8.3.6. *The set of isomorphism classes of 2-dimensional crystalline representations V of $G_{\mathbf{Q}_p}$ that have distinct Hodge-Tate weights $\{0, r\}$ with $r > 0$ and are not a direct sum of two characters is naturally parameterized by the set of quadratic polynomials $f(X) = X^2 + aX + b \in \mathbf{Z}_p[X]$ with $\mathrm{ord}_p(b) = r$, where f is the characteristic polynomial of φ on $D = D^*_{\mathrm{cris}}(V)$.*

If f is irreducible then $D = \mathbf{Q}_p e_1 \oplus \mathbf{Q}_p e_2$ with $\mathrm{Fil}^j(D) = \mathbf{Q}_p e_1$ precisely for $1 \leqslant j \leqslant r$ and $[\varphi] = \begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}$. The crystalline Galois representation $V^*_{\mathrm{cris}}(D)$ contravariantly associated to D is irreducible.

If f is reducible with distinct roots then $D = \mathbf{Q}_p e_1 \oplus \mathbf{Q}_p e_2$ with $\mathrm{Fil}^j(D) = \mathbf{Q}_p(e_1 + e_2)$ precisely for $1 \leqslant j \leqslant r$ and each e_i an eigenvector for φ. If f is reducible with a repeated *root λ (so $r \in 2\,\mathrm{ord}_p(\lambda) \in 2\mathbf{Z}^+$) then the same description holds except that e_1 spans the λ-eigenspace and φ has the matrix $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.*

In all of these cases, φ does not act as a scalar on D. Also, the associated Galois representation $V^*_{\mathrm{cris}}(D)$ is reducible if and only if f has a unit root $\mu \in \mathbf{Z}_p^\times$ (so this never occurs when f has a repeated root), in which case the other root is $p^r\mu'$ for some $\mu' \in \mathbf{Z}_p^\times$ and $V^*_{\mathrm{cris}}(D)$ is an extension of the unramified character ψ_μ associated to μ by the r-fold Tate twist $\chi^r\psi_{\mu'}$ of the unramified character $\psi_{\mu'}$ associated to μ'.

Remark 8.3.7. This theorem does not claim that *every* 2-dimensional crystalline representation of $G_{\mathbf{Q}_p}$ with distinct Hodge-Tate weights is determined up to isomorphism by the monic quadratic characteristic polynomial of Frobenius in $\mathbf{Q}_p[X]$. Indeed, each such quadratic polynomial that is reducible can arise in two ways: from a direct sum of characters with distinct Hodge-Tate weights or from a non-split extension of the lower-weight character by the higher-weight character. Keep in mind that the slopes of this polynomial (i.e., the p-adic ordinals of its roots) are *not* the Hodge-Tate weights in general, but by Exercise 8.4.1 they are the slopes which define the Newton polygon of D.

Proof. Let $f_\varphi(X) = X^2 + aX + b \in \mathbf{Q}_p[X]$ be the characteristic polynomial of φ acting on D, so $b \neq 0$. The condition $r = t_H(D) = t_N(D) = \mathrm{ord}_p(b)$ forces $b \in p^r\mathbf{Z}_p^\times$.

Step 1 (Irreducible case). Suppose f_φ is irreducible over $\mathbf{Q}_p$, so its roots in $\overline{\mathbf{Q}}_p$ have the same valuation and hence this valuation is > 0 (as $r \geqslant 1$). Necessarily $a \in p^{\lfloor r/2 \rfloor}\mathbf{Z}_p$ in such cases. In these cases there are no nontrivial subobjects of D, and in particular $\varphi(L)$ is not contained in L. Thus, if we choose a basis vector e_1 for L then $e_2 := \varphi(e_1)$ is linearly independent from e_1 and $\{e_1, e_2\}$ is an ordered basis of D.

The matrix of φ relative to this ordered basis is $\begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}$. Since $\mathrm{Fil}^j(D) = L = \mathbf{Q}_p e_1$ for $1 \leqslant j \leqslant r$, and otherwise $\mathrm{Fil}^j(D)$ is equal to D (for $j \leqslant 0$) or vanishes (for $j \geqslant r+1$), we have classified the cases with irreducible f_φ up to isomorphism in terms of the parameters $(a, b) \in p^{\lfloor r/2 \rfloor}\mathbf{Z}_p \times p^r\mathbf{Z}_p^\times$ (subject to the constraint that $b^2 - 4a$ a nonsquare in $\mathbf{Q}_p^\times$). These are exactly the 2-dimensional crystalline representations of $G_{\mathbf{Q}_p}$ with Hodge-Tate weights 0 and r for which φ acts irreducibly on D. Removing the effect of the initial Tate twist on these examples amounts to allowing the smaller of the two distinct Hodge-Tate weights to be an arbitrary integer.

Step 2 (Reducible case with distinct eigenvalues). Assume $f_\varphi(X) = (X - \lambda_1)(X - \lambda_2)$ with $\lambda_i \in \mathbf{Q}_p^\times$ and $\mathrm{ord}_p(\lambda_1) \leqslant \mathrm{ord}_p(\lambda_2)$. These ord's are both integers. The equality $t_H(D) = t_N(D)$ from the weak admissibility requirement says $r = \mathrm{ord}_p(\lambda_1) + \mathrm{ord}_p(\lambda_2)$, so $\mathrm{ord}_p(\lambda_2) \geqslant 1$ since $r \geqslant 1$. We separately treat the cases when $\lambda_1 \neq \lambda_2$ and $\lambda_1 = \lambda_2$.

First assume that the eigenvalues are distinct, and choose an eigenvector e_i for λ_i, so $D = \mathbf{Q}_p e_1 \oplus \mathbf{Q}_p e_2$. The only nontrivial subobjects of D are the two eigenlines, with $t_N(\mathbf{Q}_p e_i) = \mathrm{ord}_p(\lambda_i)$. Weak admissibility amounts to the requirement $t_H(\mathbf{Q}_p e_i) \leqslant t_N(\mathbf{Q}_p e_i) = \mathrm{ord}_p(\lambda_i)$ for both i's. The filtration of the line $\mathbf{Q}_p e_i$ has its unique nontrivial graded subquotient in degree r (i.e., $t_H(\mathbf{Q}_p e_i) = r$) if this line is equal to L and in degree 0 (i.e., $t_H(\mathbf{Q}_p e_i) = 0$) otherwise. In particular, $t_H(\mathbf{Q}_p e_i) \geqslant 0$ for both i's, so each λ_i is integral. But $\mathrm{ord}_p(\lambda_2) \geqslant \mathrm{ord}_p(\lambda_1) \geqslant 0$

with $\mathrm{ord}_p(\lambda_1) + \mathrm{ord}_p(\lambda_2) = r \geqslant 1$, so $\mathrm{ord}_p(\lambda_1) < r$. Hence, necessarily $L \neq \mathbf{Q}_p e_1$, so $t_H(\mathbf{Q}_p e_1) = 0$. We separately consider the possibilities that $L = \mathbf{Q}_p e_2$ or not.

The case $L = \mathbf{Q}_p e_2$ can only occur if $\mathrm{ord}_p(\lambda_1) = 0$ and $\mathrm{ord}_p(\lambda_2) = r$, in which case it corresponds to D that is a direct sum of the 1-dimensional objects $\mathbf{Q}_p e_1$ (the eigenline for the smaller slope) and $L = \mathbf{Q}_p e_2$, with these subobjects having respective filtration jumps in degrees 0 and r. In contravariant Galois-theoretic terms, by Lemma 8.3.3, these are the direct sums $\psi_1 \oplus \psi_2(r)$ with each ψ_i unramified (and the integral units λ_1 and λ_2/p^r encode the Frobenius action for ψ_i); removing the effect of the Tate twist makes this into the reducible decomposable crystalline case with distinct Hodge-Tate weights.

Now suppose $L \neq \mathbf{Q}_p e_2$ (and still $\lambda_1 \neq \lambda_2$), so by scaling the e_i's we can arrange that $L = \mathbf{Q}_p(e_1 + e_2)$. In such cases $0 = t_H(\mathbf{Q}_p e_i) \leqslant t_N(\mathbf{Q}_p e_i)$ by weak admissibility. We have a pair of distinct φ-eigenvalues $\lambda_i \in \mathbf{Q}_p^\times$ with $0 \leqslant \mathrm{ord}_p(\lambda_1) \leqslant \mathrm{ord}_p(\lambda_2)$ (by weak admissibility) and $\mathrm{ord}_p(\lambda_1) + \mathrm{ord}_p(\lambda_2) = r$, $D = \mathbf{Q}_p e_1 \oplus \mathbf{Q}_p e_2$, and $\varphi(e_i) = \lambda_i e_i$. Moreover, $\mathrm{Fil}^j(D) = \mathbf{Q}_p(e_1 + e_2)$ for $1 \leqslant j \leqslant r$, and $\mathrm{Fil}^j(D) = D$ (resp. $\mathrm{Fil}^j(D) = 0$) if $j \leqslant 0$ (resp. $j \geqslant r+1$). Since $t_H(\mathbf{Q}_p e_2) = 0 < \mathrm{ord}_p(\lambda_2) = t_N(\mathbf{Q}_p e_2)$, the subobject $\mathbf{Q}_p e_2$ is *not* a weakly admissible filtered ϕ-module. Hence, the only possibility in these cases for a nontrivial weakly admissible subobject is $\mathbf{Q}_p e_1$, and this happens if and only if $\mathrm{ord}_p(\lambda_1) = 0$. Thus, we have obtained all crystalline representations of $G_{\mathbf{Q}_p}$ with distinct Hodge-Tate weights 0 and $r \geqslant 1$ (under the contravariant Fontaine functor) such that the representation is not a direct sum of two characters and the φ-action has distinct eigenvalues. These are parameterized by unordered pairs of distinct nonzero $\lambda, \lambda' \in \mathbf{Z}_p$ such that $\mathrm{ord}_p(\lambda) + \mathrm{ord}_p(\lambda') = r \geqslant 1$.

In terms of this parameterization, the reducible Galois representations are exactly those for which one (and then necessarily only one) of λ or λ' is in $\mathbf{Z}_p^\times$. Moreover, in these reducible (non-decomposable) cases the unique nontrivial weakly admissible subobject of D is the φ-eigenline for the unit eigenvalue, so in terms of the contravariant Fontaine functor the Galois representation has the nonsemisimple form

$$\begin{pmatrix} \psi'(r) & * \\ 0 & \psi \end{pmatrix}$$

with ψ and ψ' unramified characters of $G_{\mathbf{Q}_p}$ (valued in $\mathbf{Z}_p^\times$). These unramified characters correspond respectively to the units λ_1 and λ_2/p^r in the above notation, and our analysis shows that the knowledge of these eigenvalues determines the Galois representation up to isomorphism!

This gives two interesting results: for any pair of unramified characters $\psi, \psi' : G_{\mathbf{Q}_p} \rightrightarrows \mathbf{Z}_p^\times$ and any $r \geqslant 1$ there is exactly one non-semisimple crystalline representation $\rho_{\psi,\psi'}$ containing $\psi'(r)$ and admitting ψ as a quotient (in fancier language, the space

$$\mathrm{H}^1_{\mathrm{cris}}(\mathbf{Q}_p, \psi^{-1}\psi'(r)) := \mathrm{Ext}^1_{\mathrm{cris}}(\psi, \psi'(r)) \subseteq \mathrm{Ext}^1_{\mathbf{Q}_p[G_{\mathbf{Q}_p}]}(\psi, \psi'(r)) \simeq \mathrm{H}^1(\mathbf{Q}_p, \psi^{-1}\psi'(r))$$

of extension classes with underlying crystalline representation is a 1-dimensional $\mathbf{Q}_p$-subspace of $\mathrm{H}^1(\mathbf{Q}_p, \psi'\psi^{-1}(r))$), and more importantly there is *no* non-split crystalline extension of $\psi'(r)$ by ψ with $r \geqslant 1$. That is, if $\chi, \chi' : G_{\mathbf{Q}_p} \rightrightarrows \mathbf{Q}_p^\times$ are crystalline characters (i.e., Tate twists of unramified characters) with respective Hodge-Tate weights n and n', then there is *no* non-split crystalline extension of χ' by χ if $n' > n$. In other words, the Hodge-Tate weights can only "drop" as we move up a Jordan-Hölder filtration of a reducible non-split crystalline Galois representation.

Step 3 (Reducible case with equal slopes). There remains the case in which $\lambda_1 = \lambda_2 = \lambda$, so $2\,\mathrm{ord}_p(\lambda) = r$. (Hence, r is even and $\lambda \in p\mathbf{Z}_p$.) We cannot have that φ is a scalar, for otherwise L would be a subobject yet $t_H(L) = r = 2\,\mathrm{ord}_p(\lambda)$ whereas $t_N(L) = \mathrm{ord}_p(\lambda) < 2\,\mathrm{ord}_p(\lambda)$, contradicting weak admissibility. The λ-eigenspace is therefore 1-dimensional, and if we choose such an eigenvector e_1 then arguing as above allows us to choose a basis $\{e_1, e_2\}$ for D such that $L = \mathbf{Q}_p(e_1 + e_2)$ and φ has the matrix

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

We have uniquely determined the filtration on D, so these cases are parameterized by the arbitrary nonzero $\lambda \in p\mathbf{Z}_p$ (with $r = 2\,\mathrm{ord}_p(\lambda)$).

There are no nontrivial weakly admissible proper subobjects since the unique eigenline $\mathbf{Q}_p e_1$ has Hodge number 0 yet it has Newton number $\mathrm{ord}_p(\lambda) \neq 0$. The corresponding Galois representations via the contravariant Fontaine functor are the irreducible crystalline representations with Hodge-Tate weights 0 and $r \in 2\mathbf{Z}^+$ such that the φ-action has a double root (with slope $r/2$) for its characteristic polynomial. The explicit description shows that up to isomorphism such examples are completely determined by this repeated root $\lambda \in p^{r/2}\mathbf{Z}_p^\times$ (in particular, φ is non-scalar on D), so this nicely fits with the parameterization by unordered pairs $\{\lambda, \lambda'\}$ of distinct elements as given in Step 2, now filling in the cases with $\lambda = \lambda'$. Note that we can remove the effect of the initial Tate twist by allowing any $\lambda \in \mathbf{Q}_p^\times$ (in which case $\mathrm{ord}_p(\lambda) \in \mathbf{Z}$ is the average of the two distinct Hodge-Tate weights). ■

Finally, we consider 2-dimensional semistable $G_{\mathbf{Q}_p}$-representations that have a nonzero monodromy operator, which is to say that they are non-crystalline. The presence of this nonzero operator severely restricts the possible subobjects, and so correspondingly the determination of all possibilities winds up being much easier than in the crystalline case (where the subobject property came down to φ-stability).

Let D be a 2-dimensional weakly admissible filtered (ϕ, N)-module over $\mathbf{Q}_p$ with $N_D \neq 0$. This N_D is a *nonzero* nilpotent operator, so over the completed maximal unramified extension $\mathrm{W}(\overline{\mathbf{F}}_p)[1/p]$ the relation $N\phi = p\phi N$ and the Dieudonné-Manin classification force there to be exactly two distinct slopes which moreover differ by 1. Hence, $f_\varphi(X) \in \mathbf{Q}_p[X]$ has roots $\lambda_1, \lambda_2 \in \overline{\mathbf{Q}}_p^\times$ with $\mathrm{ord}_p(\lambda_1) = \mathrm{ord}_p(\lambda_2) - 1$. In particular, λ_1 and λ_2 cannot be conjugate over $\mathbf{Q}_p$, so necessarily f_φ is not irreducible over $\mathbf{Q}_p$. That is, $\lambda_1, \lambda_2 \in \mathbf{Q}_p^\times$.

Proposition 8.3.8. *The non-crystalline semistable 2-dimensional representations V of $G_{\mathbf{Q}_p}$ with smallest Hodge-Tate weight equal to 0 are parameterized as follows: there is a Hodge-Tate weight $r > 0$ of the form $r = 2m + 1$ with $m \geqslant 0$, and V is parameterized up to isomorphism by a pair (λ, c) with $\lambda \in p^m\mathbf{Z}_p^\times$ and $c \in \mathbf{Q}_p$.*

For a given (λ, c), the contravariantly associated filtered (ϕ, N)-module $D = D^*_{\mathrm{st}}(V)$ given explicitly by $D = \mathbf{Q}_p e_1 \oplus \mathbf{Q}_p e_2$ with N and φ as in (8.3.3) and $\mathrm{Fil}^j(D) = \mathbf{Q}_p(ce_1 + e_2)$ precisely for $1 \leqslant j \leqslant 2m + 1$. In these cases there is a unique nontrivial subobject D' of D, namely $D' = \mathbf{Q}_p e_1$, and $t_H(D') = 0$ and $t_N(D') = m$.

In particular, D' is weakly admissible if and only if $m = 0$, which is to say $\lambda \in \mathbf{Z}_p^\times$.

Accordingly to this parameterization, if $m > 0$ (i.e., the necessarily distinct Hodge-Tate weights 0 and $2m+1$ are not *consecutive* integers) then the semistable representation is irreducible, whereas if $m = 0$ then it is necessarily reducible and non-semisimple (as $N \neq 0$).

Proof. The condition of 0 being the smallest Hodge–Tate weight says that $\mathrm{Fil}^0(D) = D$ and $\mathrm{Fil}^1(D) \neq D$. We claim $\mathrm{Fil}^1(D) \neq 0$. If $\mathrm{Fil}^1(D) = 0$ then the Hodge polygon would be a straight line, and so by weak admissibility the Newton polygon is the same straight line. But the monodromy operator always drops the slope by 1 on the isotypic parts, so triviality of the filtration structure would force the monodromy operator to vanish, contradicting our non-crystalline hypothesis. Hence, $\mathrm{Fil}^1(D)$ is equal to a line L in D.

There is a unique $r \geqslant 1$ such that $\mathrm{Fil}^j(D) = L$ for $1 \leqslant j \leqslant r$ and $\mathrm{Fil}^{r+1}(D) = 0$. Let $m = \mathrm{ord}_p(\lambda_1) = \mathrm{ord}_p(\lambda_2) - 1$, so $r = t_H(D) = t_N(D) = 2m+1$. The line $\ker N$ is stable by φ and N, and hence it is a subobject of D. Direct analysis of the relation $N\phi = p\phi N$ shows that the eigenline $\ker N$ must support the eigenvalue of φ with the smaller slope, so $t_N(\ker N) = m$. But $t_H(\ker N) \geqslant 0$, so $m \geqslant 0$ by weak admissibility.

Since N carries the λ_2-eigenline onto the λ_1-eigenline (as $N \neq 0$) we can choose an eigenvector e_2 with $\varphi(e_2) = \lambda_2 e_2$ and define $e_1 = N(e_2)$ to get an ordered basis $\{e_1, e_2\}$ of D. This forces $\lambda_2 = p\lambda_1$ since

$$p\lambda_1 e_1 = p\varphi(e_1) = p\varphi N(e_2) = N\varphi(e_2) = \lambda_2 N(e_2) = \lambda_2 e_1.$$

To summarize, there is a parameter $\lambda \in p^m\mathbf{Z}_p^\times$ satisfying $2m+1 = r \geqslant 1$ and a linear decomposition $D = \mathbf{Q}_p e_1 \oplus \mathbf{Q}_p e_2$ relative to which N and φ have matrices

$$[N] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad [\varphi] = \begin{pmatrix} \lambda & 0 \\ 0 & p\lambda \end{pmatrix}. \tag{8.3.3}$$

It remains to determine the possibilities for the line $L \subseteq D$ satisfying $\mathrm{Fil}^j(D) = L$ precisely for $1 \leqslant j \leqslant r$.

The requirement of (ϕ, N)-stability implies that the only nontrivial subobject is the line $\mathbf{Q}_p e_1 = \ker N$. Clearly $t_N(\mathbf{Q}_p e_1) = \mathrm{ord}_p(\lambda) = m < 2m+1 = r$. This rules out the possibility $L = \mathbf{Q}_p e_1$, for in such a case we would have $t_H(\mathbf{Q}_p e_1) = r > t_N(\mathbf{Q}_p e_1)$, contradicting weak admissibility for D. Thus, $L = \mathbf{Q}_p(ce_1 + e_2)$ for some uniquely determined $c \in \mathbf{Q}_p$. (If we replace the initial choice of e_2 with a $\mathbf{Q}_p^\times$-multiple then $e_1 = N(e_2)$ is scaled in the same way and so c does not change. Thus, c is intrinsic to D.) Since $\mathbf{Q}_p e_1 \neq L$ we have $t_H(\mathbf{Q}_p e_1) = 0 \leqslant m = t_N(\mathbf{Q}_p e_1)$. The weak admissibility condition therefore imposes no requirements on c and is satisfied in all examples as just described. ■

Using Lemma 8.3.3 and the contravariant D^*_{st}, the reducible cases of Proposition 8.3.8 are non-split extensions of ψ by $\psi(1)$ for the unramified character $\psi : G_{\mathbf{Q}_p} \to \mathbf{Q}_p^\times$ classified by $\lambda \in \mathbf{Z}_p^\times$. In particular, for these reducible cases the larger Hodge-Tate weight appears on the subobject, exactly as in the crystalline reducible non-semisimple cases in Theorem 8.3.6 (but now the gap between the weights is necessarily 1). Hence, the unique unramified quotient character ψ determines the 2-dimensional representation space (though not its non-split extension structure) up to isomorphism.

Applying the unramified twist by ψ^{-1} brings us to the case $\lambda = 1$ because D^*_{st} is tensor-compatible, so since $D^*_{\mathrm{st}}(\mathbf{Q}_p) = D^*_{\mathrm{cris}}(\mathbf{Q}_p) \simeq K_0[0]$ we see that, up to unramified twisting, the

2-dimensional reducible non-crystalline semistable representations of $G_{\mathbf{Q}_p}$ are parameterized by a single parameter $c \in \mathbf{Q}_p$. Note that to choose a basis of the line $D^*_{\mathrm{cris}}(\mathbf{Q}_p)$ amounts to making a choice of $\mathbf{Q}_p$-basis of the canonical line $\mathbf{Q}_p(1) = \mathbf{Q}_p \cdot t \subseteq B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$.

Focusing on the case $\psi = 1$, we have described all of the lines in the subspace space

$$\mathrm{H}^1_{\mathrm{st}}(G_{\mathbf{Q}_p}, \mathbf{Q}_p(1)) := \mathrm{Ext}^1_{\mathrm{st}}(\mathbf{Q}_p, \mathbf{Q}_p(1)) \subseteq \mathrm{Ext}^1_{\mathbf{Q}_p[G_{\mathbf{Q}_p}]}(\mathbf{Q}_p, \mathbf{Q}_p(1)) \simeq \mathrm{H}^1(G_{\mathbf{Q}_p}, \mathbf{Q}_p(1))$$

of extension classes with underlying semistable representation. There is a distinguished line whose nonzero elements are the non-split crystalline extension classes of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$ (all of which are mutually isomorphic as representation spaces, forgetting the extension structure). The set of other lines is naturally parameterized by a parameter c as above. The nontrivial filtration step L is given by $\mathbf{Q}_p(ce_1 + e_2)$ in the non-crystalline cases (with $e_1 = N(e_2)$), and it is given by $\mathbf{Q}_p(e_1 + e_2)$ in the crystalline non-split case. In each case the pair (e_1, e_2) is uniquely determined up to a common nonzero scaling factor, and this scalar may be viewed as a parameter for the nonzero elements of a $\mathbf{Q}_p$-line in the space $\mathrm{H}^1_{\mathrm{st}}(G_{\mathbf{Q}_p}, \mathbf{Q}_p(1))$ of semistable extension classes.

By Kummer theory, $\mathrm{H}^1(G_{\mathbf{Q}_p}, \mathbf{Q}_p(1))$ is 2-dimensional when $p > 2$. There is also a concrete description of the vector space structure on this cohomology in terms of the language of extension classes (using pushouts and pullbacks). Hence, the proved existence of a line of crystalline classes and a line whose nonzero elements are semistable classes shows (via the preservation of semistability under subrepresentations, quotients, and direct sums) that when $p > 2$ all elements in $\mathrm{H}^1(G_{\mathbf{Q}_p}, \mathbf{Q}_p(1))$ correspond to semistable representations, and that there is a distinguished line consisting of the crystalline classes. Here is a vast generalization:

Lemma 8.3.9. *For any p-adic field K, each element in $\mathrm{H}^1(G_K, \mathbf{Q}_p(1))$ corresponds to a semistable G_K-representation and there is a $\mathbf{Q}_p$-hyperplane consisting of the crystalline classes.*

Proof. Kummer theory provides a concrete description of $\mathrm{H}^1(G_K, \mathbf{Q}_p(1))$ for any p-adic field K whatsoever: it is the tensor product of $\mathbf{Q}_p$ against the p-adic completion of $K^\times$. This is naturally an extension of $\mathbf{Q}_p$ by $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} (1 + \mathfrak{m}_K) \simeq K$ (isomorphism defined via the p-adic exponential map), where the $\mathbf{Q}_p$-hyperplane K parameterizes the cohomology classes arising from integral units of K. These latter classes V are 2-dimensional p-adic representations of G_K that are crystalline: see Example 9.2.8.

Since we have a hyperplane of crystalline classes, to show that in general all elements of $\mathrm{H}^1(G_K, \mathbf{Q}_p(1))$ are semistable as Galois representations it suffices to exhibit a single non-crystalline but semistable extension class. The p-adic Tate module of a single Tate curve over K does the job; see Example 9.2.9. ■

We can push the Tate curve case further when $K = \mathbf{Q}_p$. The Tate curve E_q for $q \in \mathbf{Q}_p^\times$ with $|q| < 1$ gives rise to a representation $V_p(E_q) := \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathrm{T}_p(E)$ of $G_{\mathbf{Q}_p}$ that is non-crystalline but semistable, and $V_p(E_q)$ has a canonical structure of extension of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$. Thus, $D^*_{\mathrm{st}}(V_p(E_q))$ is classified by some parameter $c_q \in \mathbf{Q}_p^\times$ in terms of our preceding description of crystalline classes in $\mathrm{H}^1(G_{\mathbf{Q}_p}, \mathbf{Q}_p(1))$, and one can ask to compute c_q explicitly. In order to do this we have to fix the embedding of B_{st} into B_{dR} (in order to define the filtration structure on $D^*_{\mathrm{st}}(V)_K$). Such an embedding will depend on a choice of G_K-equivariant logarithm

$\lambda : \overline{K}^{\times} \to \overline{K}$ extending the usual one on $\mathscr{O}_{\overline{K}}^{\times} = \overline{k}^{\times} \times (1 + \mathfrak{m}_{\mathscr{O}_{\overline{K}}})$. If one does a direct calculation with B_{st} using the contravariant functors, one finds that $c_q = -\lambda(q)$.

8.4. **Exercises.**

Exercise 8.4.1. Consider a linear automorphism $T : D \to D$ of a finite-dimensional $\mathbf{Q}_p$-vector space. Let $K = \mathrm{W}(k)[1/p]$ for a perfect field k with characteristic $p > 0$.

By extending scalars Frobenius-semilinearly, we get an isocrystal structure on the finite-dimensional K-vector space $K \otimes_{\mathbf{Q}_p} D$ via $\phi(c \otimes d) = \sigma(c) \otimes T(d)$. The following steps prove that the slopes of ϕ are exactly the $\mathrm{ord}_p(\lambda)$'s, where λ ranges through eigenvalues of T in $\overline{\mathbf{Q}}_p$, each occurring with multiplicity equal to its eigenvalue multiplicity for T.

(1) Prove that in Definition 8.1.5, it is equivalent to apply the Dieudonné-Manin classification after extending scalars to $\mathrm{W}(k')[1/p]$ for any algebraically closed extension k'/k (not necessarily an algebraic closure). In particular, deduce that it suffices to treat the case $K = \mathbf{Q}_p$.

(2) With $K = \mathbf{Q}_p$, use the slope decomposition to reduce the problem to the case when ϕ is isoclinic (i.e., the isocrystal $\widehat{\mathbf{Q}_p^{\mathrm{un}}} \otimes_{\mathbf{Q}_p} D$ has some pure slope). Let α be the slope. Show that passing to T^{-1} corresponds to negative the slope, and so reduce to the case $\alpha \geqslant 0$.

(3) Write $\alpha = s/r$ in reduced form with $r \geqslant 1$, $s \geqslant 0$. Using $\mathbf{Q}_p(p^{1/r}) \otimes_{\mathbf{Q}_p} D$ and $(p^{1/r})^{-s} \otimes T$, with $\mathbf{Q}_p(p^{1/r})$ linearly disjoint from $\widehat{\mathbf{Q}_p^{\mathrm{un}}}$ over $\mathbf{Q}_p$, reduce to the case $\alpha = 0$.

(4) Define an isocrystal (Δ, ϕ) over a p-adic field to be *power-bounded* if there is a $\mathrm{W}(k)$-lattice $\Lambda \subseteq \Delta$ such that the sequence of $\mathrm{W}(k)$-lattices $\{\phi^n(\Lambda)\}$ for $n \geqslant 0$ is "bounded" in Δ (in the sense that these lattices are contained in a common $\mathrm{W}(k)$-lattice of Δ). Prove that if such a property holds for one ϕ-stable lattice then it holds for all of them (so this concept is well-defined). Then use the Dieudonné-Manin classification to prove that a (nonzero) power-bounded isocrystal is exactly one for which the slopes are $\geqslant 0$.

(5) Using that $\alpha = 0$, deduce that for any $\mathbf{Z}_p$-lattice $L \subseteq D$, the family of lattices $\{T^n(L)\}$ for *all* $n \in \mathbf{Z}$ is bounded. By suitable extension of scalars, conclude that all eigenvalues of T are integral units in $\overline{\mathbf{Q}}_p$.

Exercise 8.4.2. The classification in §8.3 is basically one very extensive worked example. Read through it carefully in order to see how the various aspects of weak admissibility restrict possibilities on the linear algebra side.

Exercise 8.4.3. In practice it is important to consider p-adic representations "with coefficients". That is, we need to work with $\mathrm{Rep}_F(G_K)$ for a finite extension $F/\mathbf{Q}_p$. We may view $\mathrm{Rep}_F(G_K)$ as a subcategory of $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ (since $[F : \mathbf{Q}_p]$ is finite), and so for $B \in \{B_{\mathrm{HT}}, B_{\mathrm{dR}}, B_{\mathrm{cris}}, B_{\mathrm{st}}\}$ we define B-admissibility on $\mathrm{Rep}_F(G_K)$ in terms of the underlying $\mathbf{Q}_p$-linear representation space.

By functoriality, the G_K-equivariant F-action on V endows $D_B(V)$ with an action by F. In particular, for the field $E := B^G$ the E-vector space $D_B(V)$ is naturally a module over $F \otimes_{\mathbf{Q}_p} E$. Consequently, the classification of such V's (especially for $B = \{B_{\mathrm{st}}, B_{\mathrm{cris}}\}$, in which case D_B is fully faithful onto a certain subcategory of Vec_E) is influenced by the E-algebra structure of $F \otimes_{\mathbf{Q}_p} E$. It is therefore simplest to analyze things when F and E are linearly disjoint over $\mathbf{Q}_p$ (i.e., $F \otimes_{\mathbf{Q}_p} E$ is a field) or when F contains a Galois closure of E over $\mathbf{Q}_p$ (in which case $F \otimes_{\mathbf{Q}_p} E$ is a product of copies of F indexed by the $\mathbf{Q}_p$-embeddings $E \to F$). This exercise takes up low-dimensional examples of this situation.

(1) Let $F/\mathbf{Q}_p$ be a finite extension linearly disjoint from K over $\mathbf{Q}_p$ (automatic if $K = \mathbf{Q}_p$). Generalize Example 8.3.2 to classify the objects D in ${}^{\text{w.a.}}\,\mathrm{MF}_K^{\phi,N}$ corresponding to semistable representations $\rho : G_K \to F^\times$. More precisely, use F-linear functoriality to prove that (i) D is 1-dimensional over $FK_0 := F \otimes_{\mathbf{Q}_p} K_0$ with $N_D = 0$ (so D must be crystalline), (ii) $\mathrm{gr}^r(D_K) \neq 0$ for a unique r, with $t_H(D) = r[F : \mathbf{Q}_p]$, (iii) D has pure slope $\mathrm{ord}_p(\mathrm{N}_{FK_0/K_0}(\lambda))/[F : \mathbf{Q}_p] = \mathrm{ord}_{FK_0}(\lambda)/[F : F_0]$ where $\phi(e) = \lambda e$ for $\lambda \in (FK_0)^\times$ and an FK_0-basis $\{e\}$ of D (so $t_N(D) = [F_0 : \mathbf{Q}_p]\,\mathrm{ord}_{FK_0}(\lambda)$, with λ unique up to multiplication by $(1 \otimes \sigma)(c)/c$ for $c \in (FK_0)^\times$). Deduce via weak admissibility that $\mathrm{ord}_{FK_0}(\lambda) = e(F)r$ where r is the unique Hodge–Tate weight, and that such ρ are precisely the Tate twists of unramified $\mathscr{O}_F^\times$-valued characters of G_K.

(2) Let $F/\mathbf{Q}_p$ be as in (1). Prove that if $V \in \mathrm{Rep}_F(G_K)$ is semistable then the multiplicity of each Hodge–Tate weight is a multiple of $[F : \mathbf{Q}_p]$. More specifically, show that if $D \in \mathrm{MF}_K^{\phi,N}$ has an action by F then all $\mathrm{gr}^r(D_K)$'s have K-dimension that is a multiple of $[F : \mathbf{Q}_p]$. (This fails if semistable is relaxed to de Rham, as occurs already for elliptic curves over $K = \mathbf{Q}_p$ that have geometric complex multiplication by an imaginary quadratic field in which p is inert or ramified!) In particular, if $\dim_F V = 2$ and V does not have a single Hodge–Tate weight, deduce that after a Tate twist it has Hodge–Tate weights $\{0, r\}$ for some $r > 0$ with each weight having multiplicity $[F : \mathbf{Q}_p]$.

(3) Now take $K = \mathbf{Q}_p$, so (2) applies with any finite extension $F/\mathbf{Q}_p$. Let $V \in \mathrm{Rep}_F(G_{\mathbf{Q}_p})$ be 2-dimensional and semistable with Hodge–Tate weights $\{0, r\}$ with $r > 0$, and let $D = D_{\mathrm{st}}(V) \in {}^{\text{w.a.}}\,\mathrm{MF}_{\mathbf{Q}_p}^{\phi,N}$ be the corresponding 2-dimensional object over F. Let $f_\varphi(X) = X^2 + aX + b \in F[X]$ be the characteristic polynomial of the F-linear φ acting on D, so $b \neq 0$. Show that $t_H(D) = r[F : \mathbf{Q}_p]$ and $t_N(D) = [F_0 : \mathbf{Q}_p]\,\mathrm{ord}_F(b)$, and deduce that $r = \mathrm{ord}_F(b)/e(F)$ and $b \in p^r \mathscr{O}_F^\times$.

(4) By using F, $\mathscr{O}_F$, and ord_F, adapt the statement and proof of Theorem 8.3.6 so that it classifies the 2-dimensional F-linear representations of $G_{\mathbf{Q}_p}$ that are crystalline with Hodge–Tate weights $\{0, r\}$ for $r > 0$ and are not a direct sum of two $F^\times$-valued characters. In particular, show that reducibility of such representations over F is equivalent to the quadratic characteristic polynomial $f_\varphi(X) \in F[X]$ of the F-linear φ having a root in $\mathscr{O}_F^\times$. Beware that if f_φ has a repeated root $\lambda \in F^\times$ then the condition "$r = 2\,\mathrm{ord}_p(\lambda)$" for $F = \mathbf{Q}_p$ is replaced with $e(F)r = 2\,\mathrm{ord}_F(\lambda)$, so r may not be even (if $e(F)$ is even) and there are nontrivial constraints on $\mathrm{ord}_F(\lambda)$ when $e(F) > 2$.

(5) Can you likewise generalize Proposition 8.3.8 to allow coefficients in any finite extension F of $\mathbf{Q}_p$?

9. Crystalline and semistable period rings

Recall that $R = \varprojlim \mathscr{O}_{\mathbf{C}_K}/(p)$ is a perfect valuation ring in characteristic p, with each $x = (x_n) \in R$ uniquely lifting to a p-power compatible sequence $(x^{(n)})$ in $\mathscr{O}_{\mathbf{C}_K}$. We constructed a continuous open G_K-equivariant surjection $\theta : \mathrm{W}(R) \twoheadrightarrow \mathscr{O}_{\mathbf{C}_K}$ given by $\theta([x]) = x^{(0)}$ for $x \in R$, and more generally $\theta(r_0, r_1, \dots) = \sum p^n r_n^{(n)}$ for $r_0, r_1, \dots \in R$. Inverting p gave a G_K-equivariant surjection $\theta_{\mathbf{Q}} : \mathrm{W}(R)[1/p] \twoheadrightarrow \mathbf{C}_K$. By Proposition 4.4.3, $\ker\theta = (\xi_{\widetilde{p}})$ where $\xi_{\widetilde{p}} = [\widetilde{p}] - p$ for $\widetilde{p} \in R$ such that $\widetilde{p}^{(0)} = p$. The ring B_{dR}^+ was defined to be the $\ker\theta_{\mathbf{Q}}$-adic completion of $\mathrm{W}(R)[1/p]$.

One defect of B_{dR}^+ is that the Frobenius automorphism of $\mathrm{W}(R)[1/p]$ does not preserve $\ker\theta_{\mathbf{Q}}$, so there is no natural Frobenius endomorphism of $B_{\mathrm{dR}} = \mathrm{Frac}(B_{\mathrm{dR}}^+) = B_{\mathrm{dR}}^+[1/t]$. To remedy this defect we will introduce an auxiliary subring $A_{\mathrm{cris}}^0 \subseteq \mathrm{W}(R)[1/p]$ that is Frobenius-stable and gives rise to a large subring $B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$ on which there is a natural Frobenius endomorphism.

9.1. Construction and properties of B_{cris}.

We let A_{cris}^0 denote the "divided power envelope" of $\mathrm{W}(R)$ with respect to $\ker\theta$, which in concrete terms means that it is the G_K-stable $\mathrm{W}(R)$-subalgebra

$$\mathrm{W}(R)[\alpha^m/m!]_{m\geqslant 1, \alpha\in\ker\theta} = \mathrm{W}(R)[\xi^m/m!]_{m\geqslant 1} \tag{9.1.1}$$

in $\mathrm{W}(R)[1/p]$ generated by "divided powers" of all elements of $\ker\theta$ (or equivalently by the divided powers of a single generator ξ of $\ker\theta$, as $(cx)^n/n! = c^n \cdot x^n/n!$). (There is a general abstract notion of divided powers and divided power envelopes [5, §3, App. A] that can be very useful; we discuss it in §12.1.) Since A_{cris}^0 is a $\mathbf{Z}$-flat domain, if we define

$$A_{\mathrm{cris}} = \varprojlim A_{\mathrm{cris}}^0/p^n \cdot A_{\mathrm{cris}}^0$$

to be the p-adic completion of A_{cris}^0 then A_{cris} is p-adically separated and complete and the natural map $A_{\mathrm{cris}}^0/p^n \cdot A_{\mathrm{cris}}^0 \to A_{\mathrm{cris}}/p^n \cdot A_{\mathrm{cris}}^0$ is an isomorphism for all $n \geqslant 1$. In particular, it follows that A_{cris} is $\mathbf{Z}_p$-flat. However, it is not at all evident if A_{cris} is a domain or if $A_{\mathrm{cris}}^0 \to A_{\mathrm{cris}}$ is injective (i.e., A_{cris}^0 is p-adically separated); these properties will be addressed shortly.

As a $\mathrm{W}(R)$-module, A_{cris}^0 is spanned by the divided powers $\xi^m/m!$ for $m \in p\mathbf{Z}$, with ξ a generator of $\ker\theta$, but beware that A_{cris}^0 is not a free $\mathrm{W}(R)$-module! To understand some of its properties after p-adic completion, we need to be careful since this ring is rather far from being noetherian. Unfortunately, verifying basic properties of A_{cris} appears to require a lot of effort, more so than we can explain in these notes. In Exercise 9.4.1 we give some experience with this ring. (Some useful techniques for studying A_{cris} are contained in [19] and [21].)

Using Exercise 9.4.1 and a somewhat tedious amount of algebra, it can be proved that there is a way to fill in a continuous top row in a commutative diagram

$$\begin{tikzcd} A_{\mathrm{cris}} \arrow[r, "j"] & B_{\mathrm{dR}}^+ \\ A_{\mathrm{cris}}^0 \arrow[u] \arrow[r] & \mathrm{W}(R)[1/p] \arrow[u] \end{tikzcd} \tag{9.1.2}$$

using the p-adic topology on A_{cris} and the topology from Exercise 4.5.3 on B_{dR}^+. Such a continuous map j across the top in (9.1.2) is unique since A_{cris}^0 is dense in A_{cris} and B_{dR}^+ is Hausdorff. Using the uniqueness (or the construction) of j, it follows that j is G_K-equivariant. Rather more effort (which we omit) is required to prove that j is actually injective. (The existence of a diagram (9.1.2) with continuous j can be deduced from [19, Prop. 4.4.7], but its injectivity seems difficult to verify via that method.) One consequence of the injectivity of j and the commutativity of the diagram is that A_{cris} really is a domain and $A_{\mathrm{cris}}^0 \to A_{\mathrm{cris}}$ is indeed injective.

Concretely, the image of A_{cris} in B_{dR}^+ is the subring of elements

$$\left\{\sum_{n\geqslant 0} a_n \frac{\xi^n}{n!} \mid a_n \in \mathrm{W}(R), a_n \to 0 \text{ for the } p\text{-adic topology}\right\}$$

in which the infinite sums are taken with respect to the discretely-valued topology of B_{dR}^+; such sums converge since ξ lies in the maximal ideal of B_{dR}^+. In terms of this description, it can be proved that the p-adic topology on A_{cris} is characterized by uniform p-adic smallness of all a_n's in $\mathrm{W}(R)$, but beware that the divided power series expansions $\sum a_n \xi^n/n!$ for an element of A_{cris} are not unique since $a_n \in \mathrm{W}(R)$ rather than $a_n \in \mathrm{W}(\overline{k})$. We summarize our conclusions:

Proposition 9.1.1. *The abstract p-adic completion A_{cris} is a $\mathbf{Z}_p$-flat domain, and the composite map $A_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}^+ \twoheadrightarrow \mathbf{C}_K$ of $\mathrm{W}(R)$-algebras lands in $\mathscr{O}_{\mathbf{C}_K}$.*

Note that the composite map $A_{\mathrm{cris}} \to \mathscr{O}_{\mathbf{C}_K}$ is surjective since $\mathrm{W}(R)$ maps onto $\mathscr{O}_{\mathbf{C}_K}$ via θ, and it is trivially continuous relative to p-adic topologies (since $p \mapsto p$). It is natural to wonder if the G_K-action on A_{cris} is continuous for the p-adic topology. A moment's thought shows that this is really not obvious. Its proof requires a new idea:

Proposition 9.1.2. *The G_K-action on A_{cris} is continuous for the p-adic topology. Equivalently, for any $r \geqslant 1$, the G_K-action on $A_{\mathrm{cris}}/(p^r)$ has open stabilizers.*

Proof. Although $A_{\mathrm{cris}}/(p^r) = A_{\mathrm{cris}}^0/(p^r)$, and A_{cris}^0 has a "concrete" description via Exercise 9.4.1, to prove this continuity property it seems necessary to make use of a completely different description of A_{cris}, or at least of its quotients $A_{\mathrm{cris}}/(p^r)$. In [21, 5.2.7] Fontaine gives an elegant G_K-equivariant description of the $\mathrm{W}(R)$-algebra A_{cris} as a p-adically completed tensor product, and passing to the quotient modulo p^r on this description gives that $A_{\mathrm{cris}}/(p^r)$ is generated over $\mathrm{W}_r(R)$ by elements (arising from $K_0((t))$) on which G_K acts through χ mod p^r.

Passing to a finite extension of K to make $\chi \bmod p^r = 1$ therefore makes $A_{\mathrm{cris}}/(p^r)$ be generated over $\mathrm{W}_r(R)$ by G_K-invariant elements. Hence, provided that the $\mathrm{W}_r(R)$-algebra $A_{\mathrm{cris}}/(p^r)$ is a $\mathrm{W}_r(R/\mathfrak{a})$-algebra for some open ideal $\mathfrak{a}$ in R, the discreteness of the G_K-action on $R/\mathfrak{a}$ will then complete the proof. By using Teichmüller expansions (via perfectness of R) it suffices to treat the case $r = 1$, which is to say that we are reduced to proving that the map $R = \mathrm{W}(R)/(p) \to A_{\mathrm{cris}}/(p)$ has nonzero kernel. But for $\xi = [\widetilde{p}] - p \in \mathrm{W}(R)$ with $\widetilde{p}^{(0)} = p$ we have $\xi^p \in pA_{\mathrm{cris}}$, so $[\widetilde{p}^p] \in pA_{\mathrm{cris}}$. Thus, $\widetilde{p}^p \in \ker(R \to A_{\mathrm{cris}}/(p))$. ■

Define the G_K-stable $\mathrm{W}(R)[1/p]$-subalgebra

$$B^+_{\mathrm{cris}} := A_{\mathrm{cris}}[1/p] \subseteq B^+_{\mathrm{dR}}.$$

Recall the element $t = \log([\varepsilon]) = \sum_{n\geqslant 1}(-1)^{n+1}([\varepsilon]-1)^n/n \in B^+_{\mathrm{dR}}$ that is killed by θ^+_{dR}, where $\varepsilon = (\varepsilon^{(n)}) \in R$ satisfies $\varepsilon^{(0)} = 1$ and $\varepsilon^{(1)} \neq 1$ (so $\varepsilon^{(n)}$ is a primitive p^nth root of unity in $\overline{K}$ for all $n \geqslant 0$).

Proposition 9.1.3. *We have $t \in A_{\mathrm{cris}}$ and $t^{p-1} \in pA_{\mathrm{cris}}$, so $t^p/p! \in A_{\mathrm{cris}}$. In fact, $t^m/m! \in A_{\mathrm{cris}}$, and more generally for any $a \in \ker(A_{\mathrm{cris}} \twoheadrightarrow \mathscr{O}_{\mathbf{C}_K})$ we have $a^m/m! \in A_{\mathrm{cris}}$, for all $m \geqslant 1$.*

Proof. Choose a generator ξ of $\ker\theta$. Since $[\varepsilon]-1 \in \ker\theta = \xi\,\mathrm{W}(R)$, we have $[\varepsilon]-1 = w\xi$ for some $w \in \mathrm{W}(R)$. Thus, in B^+_{dR} we have

$$t = \sum_{n\geqslant 1}(-1)^{n+1}\frac{([\varepsilon]-1)^n}{n} = \sum_{n\geqslant 1}(-1)^{n+1}(n-1)!w^n\cdot\frac{\xi^n}{n!} \tag{9.1.3}$$

with $(n-1)!w^n \to 0$ in $\mathrm{W}(R)$ relative to the p-adic topology. Hence, $t \in A_{\mathrm{cris}}$ inside of B^+_{dR}.

For any $a \in A_{\mathrm{cris}}$ (such as $a = t$), whether or not $a^{p-1} \in pA_{\mathrm{cris}}$ only depends on $a \bmod p$. Thus, the infinite sum expression (9.1.3) for t allows us to check whether or not $t^{p-1} \in pA_{\mathrm{cris}}$ by replacing t with a suitable finite truncation of the sum on the right side of (9.1.3), namely dropping terms whose coefficient $(n-1)!$ is divisible by p. Hence, we can restrict to the sum over $1 \leqslant n \leqslant p$. The terms for $1 \leqslant n < p$ are A_{cris}-multiples of $[\varepsilon]-1$, and the term for $n = p$ is

$$(-1)^{p+1}\cdot\frac{([\varepsilon]-1)^{p-1}}{p}\cdot([\varepsilon]-1),$$

so $t = ([\varepsilon]-1)(a+(-1)^{p+1}([\varepsilon]-1)^{p-1}/p)$ for some $a \in A_{\mathrm{cris}}$. Hence, to prove that $t^{p-1} \in pA_{\mathrm{cris}}$ it remains to check (and apply twice) that $([\varepsilon]-1)^{p-1} \in pA_{\mathrm{cris}}$. But $p\,\mathrm{W}(R) \subseteq pA_{\mathrm{cris}}$ and

$$[\varepsilon]-1 \equiv [\varepsilon-1] \bmod p\,\mathrm{W}(R),$$

so it suffices to show $[(\varepsilon-1)^{p-1}] \in pA_{\mathrm{cris}}$.

By Example 4.3.4 we have $v_R(\varepsilon-1) = p/(p-1)$, so for $\widetilde{p} \in R$ such that $\widetilde{p}^{(0)} = p$ we have $v_R((\varepsilon-1)^{p-1}) = p = v_R(\widetilde{p}^p)$. Hence, $(\varepsilon-1)^{p-1} = \widetilde{p}^p r$ for some $r \in R^\times$, so $[(\varepsilon-1)^{p-1}]$ is a $\mathrm{W}(R)^\times$-multiple of $[\widetilde{p}]^p = (\xi_{\widetilde{p}}+p)^p \equiv \xi_{\widetilde{p}}^p \bmod pA_{\mathrm{cris}}$ with $\xi_{\widetilde{p}} = [\widetilde{p}]-p$ a generator of $\ker\theta$ in $\mathrm{W}(R)$. But $\xi_{\widetilde{p}}^p = p\cdot(\xi_{\widetilde{p}}^p/p!)\cdot(p-1)! \in pA_{\mathrm{cris}}$.

Finally, we check that if $a \in \ker(A_{\mathrm{cris}} \twoheadrightarrow \mathscr{O}_{\mathbf{C}_K})$ then $a^m/m! \in A_{\mathrm{cris}}$ for all $m \geqslant 0$. Fix a choice of m. Since a in A_{cris} is a (convergent for the p-adic topology!) sum of terms $a_n\xi^n/n!$ with $n \geqslant 1$ and coefficients $a_n \in \mathrm{W}(R)$ that tend to 0 in $\mathrm{W}(R)$ for the p-adic topology, it suffices to treat the case when this infinite sum is replaced with a finite truncation far enough out so that the tail lies in $p^N A_{\mathrm{cris}}$ with $m!|p^N$. In other words, we are reduced to the case when a is a finite sum of terms $a_n\xi^n/n!$ with $n \geqslant 1$. Letting $\gamma_N(x) = x^N/N!$ in any $\mathbf{Q}$-algebra for any $N \geqslant 1$, the binomial theorem says

$$\gamma_m(x+y) = \sum_{i=0}^{m}\gamma_i(x)\gamma_j(y).$$

Thus, to show $a^m/m! \in A_{\mathrm{cris}}$ when a is a finite sum of terms $a_n\xi^n/n!$ with $n \geqslant 1$, it suffices to treat the case when a is a single such term: $a = w\xi^n/n!$ with $w \in \mathrm{W}(R)$. But $\gamma_m(wx) = w\gamma_m(x)$, so finally we are reduced to the case $a = \xi^n/n! = \gamma_n(\xi)$ with $n \geqslant 1$, and we wish to prove that the divided power $a^m/m! = \gamma_m(a)$ lies in A_{cris}. But for all $n, m \geqslant 1$ we have the universal identity $\gamma_m(\gamma_n(x)) = C_{m,n}\gamma_{mn}(x)$ in any $\mathbf{Q}$-algebra, with $C_{m,n} = (mn)!/(m!(n!)^m)$. Since $C_{m,n} \in \mathbf{Z}$ [5, 3.1], taking $x = \xi$ gives $\gamma_m(a) \in A_{\mathrm{cris}}$ for $a = \gamma_n(\xi)$ and all $m \geqslant 1$, as required. ∎

Definition 9.1.4. The *crystalline period ring* B_{cris} for K is the G_K-stable $\mathrm{W}(R)[1/p]$-subalgebra $B^+_{\mathrm{cris}}[1/t] = A_{\mathrm{cris}}[1/t]$ inside of $B^+_{\mathrm{dR}}[1/t] = B_{\mathrm{dR}}$. (Since $t^{p-1} \in pA_{\mathrm{cris}}$, inverting t makes p become a unit, which is why $A_{\mathrm{cris}}[1/t] = B^+_{\mathrm{cris}}[1/t]$)

Observe that the definitions of B^+_{cris} and B_{cris} (with their Frobenius and Galois structures) only depend on the valued field $\mathbf{C}_K$ and not on K, just like for B^+_{dR} and B_{dR}. (The action of G_K is encoded via functoriality in $\mathbf{C}_K$ through its identification with the isometric automorphism group of $\mathbf{C}_K$.) The same holds for the embeddings of B^+_{cris} and B_{cris} into B_{dR}. Since $\mathrm{W}(k) \subseteq \mathrm{W}(R) \subseteq A_{\mathrm{cris}}$, we have $K_0 = \mathrm{W}(k)[1/p] \subseteq B_{\mathrm{cris}}$, so $K_0 \subseteq B^{G_K}_{\mathrm{cris}} \subseteq B^{G_K}_{\mathrm{dR}} = K$. We claim that $B^{G_K}_{\mathrm{cris}} = K_0$. This is immediate from the following non-obvious crucial fact.

Theorem 9.1.5. *The natural G_K-equivariant map $K \otimes_{K_0} B_{\mathrm{cris}} \to B_{\mathrm{dR}}$ is injective, and if we give $K \otimes_{K_0} B_{\mathrm{cris}}$ the subspace filtration then the induced map between the associated graded algebras is an isomorphism.*

Proof. Unfortuntately, the proof of injectivity in [21, §4.1.2–4.1.3] is incomplete when $e(K) > 1$ since the generator $\xi_{\widetilde{p}}$ of $\ker\theta$ does not generate the kernel of the associated $\mathscr{O}_K$-algebra map

$$\mathscr{O}_K \otimes_{\mathrm{W}(k)} \mathrm{W}(R) \to \mathscr{O}_{\mathbf{C}_K}.$$

To handle this, a proof can be given in the spirit of the construction of the map $j : A_{\mathrm{cris}} \to B^+_{\mathrm{dR}}$ in (9.1.2) using delicate direct calculations resting on [19, Prop. 4.7]. The computations are too tedious to be included here. As for the isomorphism property on associated graded objects, since $t \in B_{\mathrm{cris}}$ and A_{cris} map onto $\mathscr{O}_{\mathbf{C}_K}$, we get the isomorphism result since $\mathrm{gr}(B_{\mathrm{dR}}) = B_{\mathrm{HT}}$ has its graded components of dimension 1 over $\mathrm{gr}^0(B_{\mathrm{dR}}) = \mathbf{C}_K$). ∎

Since B_{dR} is a field, it follows from Theorem 9.1.5 that $K \otimes_{K_0} \mathrm{Frac}(B_{\mathrm{cris}}) \to B_{\mathrm{dR}}$ is injective. Hence, we likewise deduce that $\mathrm{Frac}(B_{\mathrm{cris}})^{G_K} = K_0$. This proves part of:

Proposition 9.1.6. *The domain B_{cris} is $(\mathbf{Q}_p, G_K)$-regular.*

Proof. It remains to show that if $b \in B_{\mathrm{cris}}$ is nonzero and $\mathbf{Q}_p b$ is G_K-stable then $b \in B^\times_{\mathrm{cris}}$. Since $t \in B^\times_{\mathrm{cris}}$, if the nonzero b has exact filtration degree i in B_{dR} then by replacing b with $t^{-i}b$ we can arrange that $b \in B^+_{\mathrm{dR}}$ and b is not in the maximal ideal. Let $\eta : G_K \to \mathbf{Q}_p^\times$ be the abstract character on the line $\mathbf{Q}_p b$. Thus, the residue class $\overline{b}$ in $\mathbf{C}_K$ spans a $\mathbf{Q}_p$-line in $\mathbf{C}_K$ with G_K-action by η. This forces η to be continuous and hence $\mathbf{Z}_p^\times$-valued, with $\mathbf{C}_K(\eta^{-1})^{G_K} \neq 0$. By Theorem 2.2.7 we conclude that $\eta(I_K)$ is *finite*. But $I_K = G_{\widehat{K^{\mathrm{un}}}}$, and replacing K with $\widehat{K^{\mathrm{un}}}$ does not affect the formation of B_{cris}, so again using Theorem 2.2.7

(for the absence of transcendental invariants, applied over a finite extension of $\widehat{K^{\mathrm{un}}}$ splitting η), we deduce that the element $\overline{b} \in \mathbf{C}_K$ is algebraic over $\widehat{K^{\mathrm{un}}} = \mathrm{W}(\overline{k})[1/p] \subseteq B_{\mathrm{dR}}^+$.

Such an element $\overline{b}$ in the residue field $\mathbf{C}_K$ of the $\widehat{K^{\mathrm{un}}}$-algebra B_{dR}^+ uniquely lifts to an element $\beta \in B_{\mathrm{dR}}^+$ that is algebraic over $\widehat{K^{\mathrm{un}}}$ by Hensel's Lemma for the complete discrete valuation ring B_{dR}^+ with residue characteristic 0, so $b - \beta \in \mathrm{Fil}^1(B_{\mathrm{dR}}^+)$. The G_K-action on B_{dR}^+ restricted to β is given by the $\mathbf{Q}_p^\times$-valued η due to the *uniqueness* of β as a lifting of $\overline{b}$ that is algebraic over $\widehat{K^{\mathrm{un}}}$. Hence, $b - \beta$ spans a G_K-stable $\mathbf{Q}_p$-line in $\mathrm{Fil}^1(B_{\mathrm{dR}})^+$ with character η if $b - \beta \neq 0$. If there is such a $\mathbf{Q}_p$-line then its nonzero elements live in some exact filtration degree $r \geqslant 1$ and so passing to the quotient by the next filtered piece would give a nonzero element in $\mathbf{C}_K(r)$ on which G_K acts through η. In other words, $\mathbf{C}_K(\chi^r \cdot \eta)$ has a nonzero G_K-invariant element. But by Theorem 2.2.7 this forces $\chi^r \eta(I_K)$ to be finite, which is a contradiction since $\eta(I_K)$ is finite and $r > 0$. We conclude that $b - \beta = 0$, so $b = \beta$ is algebraic over $\widehat{K^{\mathrm{un}}}$.

Thus, $L := \widehat{K_0^{\mathrm{un}}}(b) \subseteq B_{\mathrm{cris}}$ is a finite extension of $\widehat{K_0^{\mathrm{un}}}$, and is maximal unramified subfield L_0 must be $\widehat{K_0^{\mathrm{un}}}$. By applying Theorem 9.1.5 over the ground field L (in the role of K in that theorem) we get that the map of rings $L \otimes_{L_0} B_{\mathrm{cris}} \to B_{\mathrm{dR}}$ is injective. Hence, the subring $L \otimes_{L_0} L$ is a domain (as B_{dR} is a domain), so $L = L_0$ and therefore $b \in L_0^\times = \widehat{K_0^{\mathrm{un}}}^\times \subseteq B_{\mathrm{cris}}^\times$. ∎

By the general formalism in §5, we have a functor $D_{\mathrm{cris}} : \mathrm{Rep}_{\mathbf{Q}_p}(G_K) \to \mathrm{Vec}_{K_0}$ defined by $V \rightsquigarrow (B_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V)^{G_K}$, and there is a natural descending exhaustive and separated filtration on $K \otimes_{K_0} D_{\mathrm{cris}}(V)$ via its natural injection into $D_{\mathrm{dR}}(V)$ (using Theorem 9.1.5). By Exercise 7.4.10, we may conclude that D_{cris} is naturally valued in MF_K^ϕ once we construct an *injective* G_K-equivariant endomorphism of B_{cris} that extends the Frobenius automorphism ϕ_R of $\mathrm{W}(R)[1/p]$. We now prepare to construct such an endomorphism. (See Theorem 9.1.8.)

Fix $\widetilde{p} \in R$ such that $\widetilde{p}^{(0)} = p$, so for $\xi = [\widetilde{p}] - p \in \ker \theta$ we have that $B_{\mathrm{cris}} = A_{\mathrm{cris}}[1/t]$ with A_{cris} defined to be the p-adic completion of $A_{\mathrm{cris}}^0 = \mathrm{W}(R)[\xi^m/m!]_{m \geqslant 1}$. We now examine how ϕ_R on $\mathrm{W}(R)[1/p]$ acts on the subring A_{cris}^0. The key point is:

Lemma 9.1.7. *The* $\mathrm{W}(R)$*-subalgebra* $A_{\mathrm{cris}}^0 \subseteq \mathrm{W}(R)[1/p]$ *is* ϕ_R*-stable.*

Proof. We compute $\phi_R(\xi) = [\widetilde{p}^p] - p = [\widetilde{p}]^p - p = (\xi + p)^p - p = \xi^p + pw$ for some $w \in \mathrm{W}(R)$. Thus,

$$\phi_R(\xi) = p \cdot (w + (p-1)! \cdot (\xi^p/p!)),$$

so $\phi_R(\xi^m) = p^m(w + (p-1)! \cdot (\xi^p/p!))^m$ for all $m \geqslant 1$. But $p^m/m! \in \mathbf{Z}_p$ for all $m \geqslant 1$, so $\phi_R(\xi^m/m!) \in A_{\mathrm{cris}}^0$ for all $m \geqslant 1$. ∎

The endomorphism of A_{cris}^0 induced by ϕ_R on $\mathrm{W}(R)[1/p]$ extends uniquely to a continuous endomorphism of the p-adic completion A_{cris}, and hence an endomorphism ϕ of $B_{\mathrm{cris}}^+ = A_{\mathrm{cris}}[1/p]$ that extends the Frobenius automorphism ϕ_R of the subring $\mathrm{W}(R)[1/p]$. We claim that for $t \in A_{\mathrm{cris}}$ (inside of B_{dR}^+) we have $\phi(t) = pt$ with $p \in (B_{\mathrm{cris}}^+)^\times$, so ϕ uniquely extends to an endomorphism of $B_{\mathrm{cris}} = B_{\mathrm{cris}}^+[1/t] = B_{\mathrm{cris}}$. Intuitively the reason that $\phi(t) = pt$ is that $t = \log([\varepsilon])$ and $\phi_R([\varepsilon]) = [\varepsilon^p] = [\varepsilon]^p$ with $\log([\varepsilon]^p) = p\log([\varepsilon]) = pt$, but this is merely a plausibility argument and not a proof because (i) there is no Frobenius on the ring B_{dR} in which t was initially defined by a max-adic completion process, and (ii) ϕ on A_{cris} was defined by passing to an abstract p-adic completion on A^0_{cris} that was only embedded into B^+_{dR} after its construction (and only after this step was it shown that t lies in A_{cris}, as opposed to intrinsically constructing t in the abstract p-adic completion A_{cris}).

To rigorously prove that $\phi(t) = pt$, first recall that to prove $t \in A_{\mathrm{cris}}$ we showed that the summation $\sum_{n\geqslant 1}(-1)^{n+1}([\varepsilon]-1)^n/n$ initially defining t in B^+_{dR} actually made sense as a convergent sum in the p-adic topology of A_{cris}, with such a sum thereby defining the element of A_{cris} that "is" t via the embedding $A_{\mathrm{cris}} \hookrightarrow B^+_{\mathrm{dR}}$. Thus, we may use p-adic continuity to compute

$$\phi(t) = \sum_{n\geqslant 1}(-1)^{n+1}\frac{(\phi([\varepsilon])-1)^n}{n} = \sum_{n\geqslant 1}(-1)^{n+1}\frac{([\varepsilon^p]-1)}{n}$$

since ϕ on A_{cris} extends the usual Frobenius map on $\mathrm{W}(R)$. Thus, $\phi(t) = \log([\varepsilon^p])$ after all, and we have already seen below Example 4.5.3 that this is equal to pt. Rather more difficult is the following fundamental fact:

Theorem 9.1.8. *The Frobenius endomorphism $\phi : A_{\mathrm{cris}} \to A_{\mathrm{cris}}$ is injective. In particular, the induced Frobenius endomorphism of $B_{\mathrm{cris}} = A_{\mathrm{cris}}[1/t]$ is injective.*

Proof. Unfortunately, the proof was omitted from [21]. We do not know of a published reference. A proof will be included in the final version of these notes. ■

We conclude that $D_{\mathrm{cris}} : \mathrm{Rep}_{\mathbf{Q}_p}(G_K) \to \mathrm{Vec}_{K_0}$ is naturally promoted to a functor valued in MF^{ϕ}_K, and we shall always view it as such. Beware that the Frobenius operator on B_{cris} does *not* preserve the subspace filtration acquired via

$$B_{\mathrm{cris}} \hookrightarrow K \otimes_{K_0} B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}.$$

The basic reason for this incompatibility is that $\ker\theta$ is not stable by the Frobenius. More specifically, if we choose $\widetilde{p} \in R$ such that $\widetilde{p}^{(0)} = p$ then $\xi = [\widetilde{p}] - p$ is killed by θ whereas $\phi(\xi) = [\widetilde{p}^p] - p$ is not $(\theta(\phi(\xi)) = p^p - p \neq 0)$, so $\xi \in \mathrm{Fil}^1(B_{\mathrm{cris}})$ and $\phi(\xi) \notin \mathrm{Fil}^1(B_{\mathrm{cris}})$.

A p-adic representation of G_K is *crystalline* if it is B_{cris}-admissible, and the full subcategory of these is denoted $\mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(G_K)$. By §5 and Proposition 9.1.6, this full subcategory is stable under duality and tensor products. The same filtration arguments as used earlier for D_{dR} show that as an MF^{ϕ}_K-valued functor, the faithful covariant functor D_{cris} on $\mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(G_K)$ is exact and naturally commutes with the formation of tensor products and duals (in MF^{ϕ}_K and $\mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(G_K)$).

Proposition 9.1.9. *If $V \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(G_K)$ then the natural map $j_V : K\otimes_{K_0} D_{\mathrm{cris}}(V) \to D_{\mathrm{dR}}(V)$ in Fil_K is an isomorphism. In particular, crystalline representations are de Rham.*

Moreover, the B_{cris}-linear Frobenius-compatible G_K-equivariant crystalline comparison isomorphism

$$\alpha : B_{\mathrm{cris}} \otimes_{K_0} D_{\mathrm{cris}}(V) \simeq B_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V$$

satisfies the property that α_K is a filtered isomorphism.

Before we give the proof, we mention a nice application. Using the identifications

$$D_{\mathrm{dR}}(V^\vee) = \mathrm{Hom}_{\mathbf{Q}_p[G_K]}(V, B_{\mathrm{dR}}), \quad K \otimes_{K_0} D_{\mathrm{cris}}(V^\vee) = \mathrm{Hom}_{\mathbf{Q}_p[G_K]}(V, K \otimes_{K_0} B_{\mathrm{cris}}),$$

the isomorphism property for $j_{V^\vee}$ says that if V is a crystalline p-adic representation of G_K then every $\mathbf{Q}_p[G_K]$-linear map $V \to B_{\mathrm{dR}}$ lands in the $K[G_K]$-subalgebra $K \otimes_{K_0} B_{\mathrm{cris}}$. Loosely speaking, the "de Rham periods" of a crystalline representation are the same as its "crystalline periods" up to an extension of scalars by $K_0 \to K$.

Proof. The natural map j_V is a subobject inclusion in Fil_K by definition of the filtration structure on $D_{\mathrm{cris}}(V)_K$, so the problem is one of comparing K-dimensions. The crystalline condition says $\dim_{K_0} D_{\mathrm{cris}}(V) = \dim_{\mathbf{Q}_p}(V)$, and since $\dim_K D_{\mathrm{dR}}(V) \leqslant \dim_{\mathbf{Q}_p} V$ we must have equality, so V is de Rham. To verify that the K-linear inverse α_K^{-1} is filtration-compatible too, or in other words that the filtration-compatible α_K is a filtered isomorphism, it is equivalent to show that $\mathrm{gr}(\alpha_K)$ is an isomorphism. Since j_V is an isomorphism and $\mathrm{gr}(K \otimes_{K_0} B_{\mathrm{cris}}) = \mathrm{gr}(B_{\mathrm{dR}}) = B_{\mathrm{HT}}$ by Theorem 9.1.5, the method of proof of Proposition 6.3.7 adapts to show that $\mathrm{gr}(\alpha_K)$ is identified with the Hodge–Tate comparison isomorphism for V. ■

Give B_{cris} the subspace filtration from $K \otimes_{K_0} B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$; i.e., define

$$\mathrm{Fil}^i B_{\mathrm{cris}} = B_{\mathrm{cris}} \cap \mathrm{Fil}^i B_{\mathrm{dR}}.$$

Beware that (since there is no Frobenius on B_{dR}) this is not ϕ-stable! We require a fundamental property of the filtration on B_{cris}.

Theorem 9.1.10. *The space* $(\mathrm{Fil}^0 B_{\mathrm{cris}})^{\phi=1} = \{b \in \mathrm{Fil}^0(B_{\mathrm{cris}}) \mid \phi(b) = b\}$ *of ϕ-invariant elements in the 0th filtered piece of B_{cris} is equal to $\mathbf{Q}_p$.*

Proof. This is difficult; see [21, 5.3.7]. ■

This theorem underlies the key to the full faithfulness properties for D_{cris}. The reason for the importance of Theorem 9.1.10 is that it shows how to extract $\mathbf{Q}_p$ out of B_{cris} using only its "linear structures": the G_K-action, the Frobenius operator, and the filtration. To see how useful this is, we finally come to the key point of the story: we can recover V from $D_{\mathrm{cris}}(V)$ when V is crystalline!

Indeed, consider the crystalline comparison isomorphism

$$\alpha : B_{\mathrm{cris}} \otimes_{K_0} D_{\mathrm{cris}}(V) \simeq B_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V \tag{9.1.4}$$

for $V \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(G_K)$. We have seen that not only is α only B_{cris}-linear, G_K-equivariant, and Frobenius-compatible, but α_K is a filtered isomorphism too. Hence, by intersecting with the 0th filtered parts after scalar extension to K we get a G_K-equivariant K_0-linear isomorphism

$$\mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} D_{\mathrm{cris}}(V)) \simeq \mathrm{Fil}^0(B_{\mathrm{cris}}) \otimes_{\mathbf{Q}_p} V$$

that is compatible with the Frobenius actions on both sides (within the ambient B_{cris}-modules as in (9.1.4)). Passing to ϕ-fixed parts therefore gives a $\mathbf{Q}_p[G_K]$-linear isomorphism

$$\mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} D_{\mathrm{cris}}(V))^{\phi=1} \simeq V. \tag{9.1.5}$$

In other words, if we define the covariant functor

$$V_{\mathrm{cris}} : \mathrm{MF}_K^{\phi} \to \mathbf{Q}_p[G_K]\text{-mod}$$

by $D \rightsquigarrow \mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} D)^{\phi=1}$ then $V \simeq V_{\mathrm{cris}}(D_{\mathrm{cris}}(V))$ for crystalline representations V of G_K. Hence, modulo the issue that $V_{\mathrm{cris}}(D)$ may not be finite-dimensional over $\mathbf{Q}_p$ with continuous G_K-action for arbitrary D in MF_K^ϕ, the functor V_{cris} provides an inverse to D_{cris} (or rather, D_{cris} restricted to $\mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(G_K)$)! Most importantly, we have almost shown:

Proposition 9.1.11. *The exact tensor-functor $D_{\mathrm{cris}} : \mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(G_K) \to \mathrm{MF}_K^\phi$ is fully faithful, with inverse on its essential image given by V_{cris}. The same holds for the contravariant D^*_{cris} using the contravariant functor $V^*_{\mathrm{cris}}(D) = \mathrm{Hom}_{\mathrm{Fil},\phi}(D, B_{\mathrm{cris}})$.*

Proof. The full faithfulness needs further discussion. Suppose that V and V' are crystalline p-adic representations of G_K and let $D = D_{\mathrm{cris}}(V)$ and $D' = D_{\mathrm{cris}}(V')$ in MF_K^ϕ. If $T : D' \to D$ is a map in MF_K^ϕ then via the crystalline comparison isomorphisms as in (9.1.4) for V and V', the B_{cris}-linear extension $1 \otimes T : B_{\mathrm{cris}} \otimes_{K_0} D' \to B_{\mathrm{cris}} \otimes_{K_0} D$ of T is identified with a B_{cris}-linear, G_K- and Frobenius-compatible, and filtration-compatible isomorphism $\widetilde{T} : B_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V' \simeq B_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V$.

Explicitly, $\widetilde{T} = \alpha_{\mathrm{cris}}(V) \circ T \circ \alpha_{\mathrm{cris}}(V')^{-1}$. The map $\widetilde{T}$ respects the formation of the ϕ-fixed part in filtration degree 0, which is to say (by (9.1.5)) that this B_{cris}-linear isomorphism must carry V' into V by a G_K-equivariant map. Hence, $\widetilde{T}$ is the B_{cris}-scalar extension of some map $V' \to V$ in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, so by functoriality of the crystalline comparison isomorphism we see that this map $V' \to V$ between Galois representations induces the given map $T : D_{\mathrm{cris}}(V') = D' \to D = D_{\mathrm{cris}}(V)$. This gives full faithfulness as desired. ∎

We conclude with a basic calculation.

Example 9.1.12. Let's calculate $D^*_{\mathrm{cris}}(\mathbf{Q}_p(r)) = \mathrm{Hom}_{\mathbf{Q}_p[G_K]}(\mathbf{Q}_p(r), B_{\mathrm{cris}})$. Given any $\mathbf{Q}_p[G_K]$-linear map $\mathbf{Q}_p(r) \to B_{\mathrm{cris}}$, if we multiply it by t^{-r} then we get a $\mathbf{Q}_p[G_K]$-linear map $\mathbf{Q}_p \to B_{\mathrm{cris}}$. In other words, $D = D^*_{\mathrm{cris}}(\mathbf{Q}_p(r)) = B_{\mathrm{cris}}^{G_K} \cdot t^r = K_0 t^r$. This has Frobenius action $\phi(ct^r) = \sigma(c)(\phi t)^r = p^r\sigma(c)t^r$, and the unique filtration jump for D_K happens in degree r (i.e., $\mathrm{gr}^r(D_K) \neq 0$). In other words, $D^*_{\mathrm{cris}}(\mathbf{Q}_p(r))$ is the Tate twist $(K_0[0])\langle r\rangle$ in the sense of Definition 8.3.1.

Let's push this further and compute $V^*_{\mathrm{cris}}(D^*_{\mathrm{cris}}(\mathbf{Q}_p(r))) = V^*_{\mathrm{cris}}((K_0[0])\langle r\rangle)$. This consists of K_0-linear maps $T : K_0 \to \mathrm{Fil}^r(B_{\mathrm{cris}})$ that satisfy $\phi(T(c)) = T(p^r\sigma(c))$ for all $c \in K_0$, or in other words $\sigma(c) \cdot \phi(T(1)) = p^r\sigma(c)T(1)$ for all $c \in K_0$. This says $\phi(T(1)) = p^rT(1)$ with $T(1) \in \mathrm{Fil}^r B_{\mathrm{cris}}$, and if we write $T(1) = bt^r$ with $b \in \mathrm{Fil}^0(B_{\mathrm{cris}})$ (as we may since $t \in B_{\mathrm{cris}}^\times$) then the condition on b is exactly $b \in (\mathrm{Fil}^0 B_{\mathrm{cris}})^{\phi=1} = \mathbf{Q}_p$. Hence, $V^*_{\mathrm{cris}}(D^*_{\mathrm{cris}}(\mathbf{Q}_p(r))) = \mathbf{Q}_p t^r$ is the canonical copy of $\mathbf{Q}_p(r)$ inside of B_{cris}. This illustrates in a special (but important!) case of the general fact that V^*_{cris} is "inverse" to D^*_{cris} restricted to crystalline representations.

The next step in the development of D_{cris} is to show that it takes values in the full subcategory of *weakly admissible* filtered ϕ-modules over K. Rather than prove this result now, we shall first digress to develop the theory of another $(\mathbf{Q}_p, G_K)$-regular period ring B_{st} containing B_{cris} whose associated theory of admissible representations (to be called *semistable*) generalizes the theory of crystalline representations. The desired weak admissibility property for $D_{\mathrm{cris}}(V)$ with crystalline V will be a special case of a more general weak admissibility property that we will prove for $D_{\mathrm{st}}(V) = (B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V)^{G_K} \in \mathrm{MF}_K^{\phi,N}$ for semistable V.

9.2. **Construction of B_{st}.** The period ring B_{st} will be a canonical extension ring of B_{cris} endowed with compatible Galois and Frobenius structures, as well as a filtration on $K \otimes_{K_0} B_{\mathrm{st}}$, but there will not be a canonical injective map $B_{\mathrm{st}} \to B_{\mathrm{dR}}$ as B_{cris}-algebras with G_K-action. Instead, such a map will depend on a certain non-canonical choices, but the image of the map will be independent of the choices. (Don't forget: the map to this canonical image will not be independent of the choices!).

To motivate what is to be done, we recall that crystalline representations are meant to capture (among other things) the p-adic étale cohomology of smooth proper K-schemes X with good reduction. But what is X has "bad reduction"? By the techniques of cohomological descent, coupled with deJong's alterations theorem, it turns out that the worst case (up to finite extension of K) is essentially that of "semistable reduction". Loosely speaking, this is the case in which $X = \mathscr{X}_K$ where $\mathscr{X}$ is a proper flat $\mathscr{O}_K$-scheme whose special fiber is reduced and has singularities that look étale-locally like transverse intersections of hyperplanes in an affine space. The most basic example of such a singularity is the local equation $uv = q$ over $\mathscr{O}_K$ with q a nonzero element of the maximal ideal of $\mathscr{O}_K$ (so it has reduction $uv = 0$). The primordial example in which this singularity arise is the regular proper model for the (algebraization of the) Tate curve $E_q = \mathbf{G}_m^{\mathrm{an}}/q^{\mathbf{Z}}$ over K. Thus, before we proceed we first consider this example.

Example 9.2.1. The p-adic Tate module representation $T_p(E_q)$ has a $\mathbf{Z}_p$-basis given choices of $\varepsilon = (\zeta_{p^n}) \in R$ and $\widetilde{q} \in R$ satisfying $\widetilde{q}^{(0)} = q \in K^\times$, and the G_K-action relative to this basis is

$$\begin{pmatrix} \chi & \eta_{\widetilde{q}} \\ 0 & 1 \end{pmatrix}$$

where $g(\widetilde{q})/\widetilde{q} = \varepsilon^{\eta_{\widetilde{q}}(g)}$ for a continuous 1-cocycle $\eta_{\widetilde{q}} : G_K \to \mathbf{Z}_p$ relative to the χ-action. (This formula for $g(\widetilde{q})/\widetilde{q}$ rests crucially on the fact that $q \in K^\times$; if merely $q \in \overline{K}^\times$ then the formula is more complicated.)

To discover a copy of $V_p(E_q)$ inside of B_{dR}^+, we proceed as follows. The element $t = \log([\varepsilon]) \in B_{\mathrm{cris}}$ has G_K-action via χ, just like the first basis vector of $T_p(E_q)$ chosen above. Since $g(\widetilde{q}) = \varepsilon^{\eta_{\widetilde{q}}(g)}\widetilde{q}$, by applying Teichmüller lifts to this and imagining we can then take logarithms, we see that to match the G_K-action on the second basis vector of $T_p(E_q)$ we should define an element "$\log([\widetilde{q}])$" in B_{dR}^+, as then the identities

$$g(t) = \chi(g)t, \quad g(\log[\widetilde{q}]) = \eta_{\widetilde{q}}(g)t + \log([\widetilde{q}])$$

should hold. This would define a copy of $V_p(E_q)$ inside of B_{dR}^+.

The most optimistic idea for defining a period ring $B_{\mathrm{st}} \subseteq B_{\mathrm{dR}}$ containing the "periods" of the p-adic étale cohomology of all smooth proper K-schemes X with semistable reduction is that we should need all crystalline periods (i.e., $B_{\mathrm{cris}} \subseteq B_{\mathrm{st}}$) and the periods of the simplest semistable singularities of all, namely the ones arising from Tate curves. By thinking in terms of isogenies of Tate curves, it seems plausible that adjoining the periods of a single Tate curve should then be enough to get everything. That is, B_{st} should be generated over B_{cris} the hypothetical element $\log([\widetilde{q}])$ as in Example 9.2.1 for a single q. This will turn out to work!

Concretely, if we choose $\widetilde{q} \in \mathfrak{m}_R - \{0\}$ such that $q := \widetilde{q}^{(0)} \in \mathscr{O}_K$ (rather than just $q \in \mathscr{O}_{\mathbf{C}_K}$ or $q \in \mathscr{O}_{\overline{K}}$) then $0 < |q| < 1$ and B_{st} will be identified with $B_{\mathrm{cris}}[X]$ where G_K acts as usual on B_{cris} and by the formula $g(X) = X + \log([\varepsilon_{\widetilde{q}}(g)])$ on X, where $\varepsilon_{\widetilde{q}}(g) = g(\widetilde{q})/\widetilde{q} \in R^\times$ is a compatible sequence of (possibly non-primitive) p^nth roots of unity (so $\log([\varepsilon_{\widetilde{q}}(g)])$ lies in the canonical $\mathbf{Z}_p(1)$ in A_{cris}).

We prefer to first give an abstract construction of B_{st} unrelated to B_{dR} and to then relate it more concretely to B_{dR} by means of various choices.

Now fix a choice of q in the maximal ideal of $\mathscr{O}_K$ and pick $\widetilde{q} \in \mathfrak{m}_R - \{0\}$ such that $\widetilde{q}^{(0)} = q$. Since $\theta([\widetilde{q}]) = \widetilde{q}^{(0)} = q \neq 1$, so $[\widetilde{q}]$ is not a 1-unit in B_{dR}^+ (in contrast with $[\varepsilon]$), to make sense of $\log([\widetilde{q}])$ in B_{dR}^+ we need to generalize the B_{dR}^+-valued logarithms as constructed in the discussion following Exercise 4.5.3. We will now use the p-adic topology of A_{cris} (which has no good analogue on B_{dR}^+) to carry this out.

Lemma 9.2.2. *For $x \in 1 + \mathfrak{m}_R$, if $n \gg 0$ then the element*

$$\frac{([x]-1)^n}{n} \in \mathrm{W}(R)[1/p] \subseteq A_{\mathrm{cris}}[1/p] = B_{\mathrm{cris}}^+$$

lies in A_{cris}, and it tends to 0 for the p-adic topology of A_{cris} as $n \to \infty$. In particular, the infinite sum

$$\log_{\mathrm{cris}}([x]) = \sum_{n \geqslant 1} (-1)^{n+1} \cdot \frac{([x]-1)^n}{n} \in B_{\mathrm{cris}}^+$$

makes sense for all $x \in 1 + \mathfrak{m}_R$.

Moreover, $x \mapsto \log_{\mathrm{cris}}([x])$ is G_K-equivariant homomorphism and

$$\phi(\log_{\mathrm{cris}}([x])) = \log_{\mathrm{cris}}([x^p]) = p \log_{\mathrm{cris}}([x])$$

for all $x \in 1 + \mathfrak{m}_R$.

Proof. Since $\theta([x]-1) = x^{(0)} - 1 \in \mathfrak{m}_{\mathscr{O}_{\mathbf{C}_K}}$, for some $N \gg 0$ we have $\theta(([x]-1)^N) = (x^{(0)}-1)^N \in p\mathscr{O}_{\mathbf{C}_K}$. But $\theta : \mathrm{W}(R) \to \mathscr{O}_{\mathbf{C}_K}$ is surjective with kernel generated by some ξ, so $([x]-1)^N = pw_1 + \xi w_2$ with $w_1, w_2 \in \mathrm{W}(R)$. Both elements p and ξ in A_{cris} admit divided powers in A_{cris} (since $p^n/n! \in \mathbf{Z}_p$ for all $n \geqslant 1$), so $([x]-1)^{Nj}/j! \in A_{\mathrm{cris}}$ for all $j \geqslant 0$.

Now consider $([x]-1)^n/n$ for $n \geqslant 1$. Writing $n = Nq_n + r_n$ with $0 \leqslant r_n < N$,

$$\frac{([x]-1)^n}{n} = \frac{q_n!}{n} \cdot ([x]-1)^{r_n} \cdot \frac{([x]-1)^{Nq_n}}{q_n!}$$

with the final factor in A_{cris}. Hence, for the membership in A_{cris} (for sufficiently large n) and the p-adic convergence to 0 as $n \to \infty$ we just need that $q_n!/n \to 0$ in $\mathbf{Q}_p$ as $n \to \infty$. But for any $j \geqslant 1$ we have

$$\frac{j}{p-1} \geqslant \mathrm{ord}_p(j!) \geqslant \frac{j-1}{p-1} - \log_p(j)$$

where we use base-p logarithm, so $\mathrm{ord}_p(q_n!)$ grows at linear rate in n (since $q_n = \lfloor n/N \rfloor$) whereas $\mathrm{ord}_p(n) \leqslant \log_p(n)$. This gives the required decay toward 0.

The G_K-equivariance, homomorphism property, and Frobenius-compatibility for $x \mapsto \log_{\mathrm{cris}}([x])$ are deduced by passage to the limit on finite sum approximations (due to how the Frobenius on $B_{\mathrm{cris}}^+ = A_{\mathrm{cris}}[1/p]$ was defined). ∎

Define the G_K-equivariant "logarithm" homorphism

$$\lambda : R^\times = \overline{k}^\times \times (1 + \mathfrak{m}_R) \to B_{\mathrm{cris}}^+$$

by requiring it to be trivial on $\overline{k}^\times$ (inspired by the case of finite k, since $B_{\rm cris}^+$ is torsion-free as a $\mathbf{Z}$-module) and to be $x \mapsto \log_{\rm cris}([x])$ on 1-units. From the definitions, $\phi(\lambda(r)) = \lambda(r^p) = p\lambda(r)$ for all $r \in R^\times$. Since $B_{\rm cris}^+$ is a $\mathbf{Q}$-algebra, λ induces a canonical G_K-equivariant $\mathbf{Q}$-algebra map

$$\mathrm{Sym}_{\mathbf{Q}}(R^\times) \to B_{\rm cris}^+$$

where $\mathrm{Sym}_{\mathbf{Q}}(\Gamma)$ for an abelian group Γ means the symmetric algebra $\mathrm{Sym}_{\mathbf{Q}}(\Gamma_{\mathbf{Q}})$ on the associated $\mathbf{Q}$-vector space $\Gamma_{\mathbf{Q}} = \mathbf{Q} \otimes_{\mathbf{Z}} \Gamma$.

Consider the G_K-equivariant exact sequence of abelian groups

$$1 \to R^\times \to \mathrm{Frac}(R)^\times \stackrel{v_R}{\to} \mathbf{Q} \to 1; \tag{9.2.1}$$

this is an analogue of $1 \to \mathscr{O}_{\overline{K}}^\times \to \overline{K}^\times \to \mathbf{Q} \to 1$. This exact sequence implies that $\mathrm{Sym}_{\mathbf{Q}}(\mathrm{Frac}(R)^\times)$ is a 1-variable polynomial ring over $\mathrm{Sym}_{\mathbf{Q}}(R^\times)$, where the choice of variable rests on a choice of $y \in \mathrm{Frac}(R)^\times$ with $v_R(y) \neq 0$ (e.g., $y \in \mathfrak{m}_R - \{0\}$). Indeed, if we apply $\mathbf{Q}\otimes_{\mathbf{Z}}(\cdot)$ to (9.2.1) then we get a short exact sequence of $\mathbf{Q}$-vector spaces, and rather generally if

$$0 \to W' \to W \to W'' \to 0$$

is a short exact sequence of vector spaces over a field then the symmetric algebra $\mathrm{Sym}(W)$ is a polynomial ring over $\mathrm{Sym}(W')$ in variables given by a lift to W of a basis of W'' (since symmetric algebras of vector spaces are polynomial algebras in a basis).

Definition 9.2.3. As a $B_{\rm cris}$-algebra with G_K-action,

$$B_{\rm st}^+ := \mathrm{Sym}_{\mathbf{Q}}(\mathrm{Frac}(R)^\times) \otimes_{\mathrm{Sym}_{\mathbf{Q}}(R^\times)} B_{\rm cris}^+$$

and the canonical G_K-equivariant homomorphism $\mathrm{Frac}(R)^\times \to B_{\rm st}^+$ via $h \mapsto h \otimes 1$ is denoted $\lambda_{\rm st}^+$. Define $B_{\rm st} = B_{\rm st}^+[1/t]$ with its evident G_K-action.

Non-canonically, $B_{\rm st}^+ \simeq B_{\rm cris}^+[X]$ and $B_{\rm st} \simeq B_{\rm cris}[X]$ upon choosing $y \in \mathrm{Frac}(R)^\times$ with $y \notin R^\times$ (and setting $X = \lambda_{\rm st}^+(y)$).

Remark 9.2.4. The pair $(B_{\rm st}^+, \lambda_{\rm st}^+)$ is an initial object in the category of pairs (S, λ_S) consisting of a $B_{\rm cris}^+$-algebra S equipped with a G_K-equivariant homomorphism $\lambda_S : \mathrm{Frac}(R)^\times \to S$ extending λ.

It is natural to wonder if there is ring $A_{\rm st}$ analogous to $A_{\rm cris}$ that is an integral counterpart to $B_{\rm st}$ (in the sense that p is not a unit in $A_{\rm st}$ and $A_{\rm st}[1/t] = B_{\rm st}$). In work on comparison theorems for p-adic cohomology one needs integral versions of $B_{\rm st}$, but we will not address the issue here.

Roughly speaking, $B_{\rm st}^+$ is obtained from $B_{\rm cris}^+$ by universally adjoining $\log y$ for elements of $\mathrm{Frac}(R)^\times$ not in $R^\times$. As with the de Rham and crystalline period rings, the rings $B_{\rm st}^+$ and $B_{\rm st}$ (equipped with their Frobenius and Galois structures, as well as their $B_{\rm cris}$-algebra structure) only depend on $\mathbf{C}_K$ and not on K.

Since ϕ on $B_{\rm cris}^+$ satisfies $\phi(t) = pt$ and $\phi(\lambda(x)) = p\lambda(x)$ for $x \in R^\times$, we canonically extend the injective Frobenius ϕ on $B_{\rm cris}^+$ to a (visibly injective) Frobenius ϕ on $B_{\rm st}^+$ and $B_{\rm st}$ via the requirement $\phi(\lambda_{\rm st}^+(x)) = p\lambda_{\rm st}^+(x)$ for all $x \in \mathrm{Frac}(R)^\times$. (In terms of the non-canonical presentations $B_{\rm cris}^+[X]$ and $B_{\rm cris}[X]$ for $B_{\rm st}^+$ and $B_{\rm st}$, this amounts to the single condition $\phi(X) = pX$.)

The ring B_{st} admits an additional crucial structure, a monodromy operator N whose interaction with ϕ satisfies $N\phi = p\phi N$. We now construct this N. Loosely speaking, the idea is to define $N = \mathrm{d}/\mathrm{d}X$ on $B_{\mathrm{st}}^+ = B_{\mathrm{cris}}^+[X]$, but to proceed canonically we need to formulate the definition in a slightly different manner.

Observe that for a choice of $y_0 \in \mathfrak{m}_R - \{0\}$ with $v_0 := v_R(y_0) \in \mathbf{Q}_{>0}^\times$, $\lambda_{\mathrm{st}}^+(y_0)\lambda(R^\times)$ is the set of $\lambda_{\mathrm{st}}^+(y)$'s for $y \in v_R^{-1}(v_0)$. Thus, if we set $X = \lambda_{\mathrm{st}}^+(y_0)$ to identify B_{st}^+ with $B_{\mathrm{cris}}^+[X]$ then changing y_0 to $y_0 u$ for $u \in R^\times$ changes X to $X + \lambda(u)$ with $\lambda(u) \in (B_{\mathrm{cris}}^+)^\times$. Hence, the operator $\mathrm{d}/\mathrm{d}X$ on $B_{\mathrm{st}}^+ = B_{\mathrm{cris}}^+[X]$ is invariant under replacing y_0 with $y_0 u$ and so only depends on $v_0 = v_R(y_0)$ rather than on y_0. We define

$$N := v_0 \cdot \mathrm{d}/\mathrm{d}X$$

(which we will soon see is independent of v_0, so $v_0 = 1$ is useful for doing computations).

This operator N is a B_{cris}^+-linear derivation of $B_{\mathrm{st}}^+ = B_{\mathrm{cris}}^+[X]$ with kernel B_{cris}^+. We uniquely extend N to a B_{cris}-linear derivaton of $B_{\mathrm{st}} = B_{\mathrm{st}}^+[1/t] = B_{\mathrm{cris}}[X]$ that is also denoted by N, and $B_{\mathrm{st}}^{N=0} = B_{\mathrm{cris}}$. The identity $N\phi = p\phi N$ holds because we can check it on $X = \lambda_{\mathrm{st}}^+(y_0)$ (using that $\phi(\lambda_{\mathrm{st}}^+(y_0)) = p\lambda_{\mathrm{st}}^+(y_0)$). Similarly, we see that N on B_{st} is G_K-equivariant and only depends on $\mathbf{C}_K$ rather than on K.

Remark 9.2.5. If we change v_0 to $v_0' = cv_0$ for $c = m/n \in \mathbf{Q}_{>0}^\times$ (with $m, n \in \mathbf{Z}^+$) then correspondingly the element $y_0 \in \mathfrak{m}_R - \{0\}$ can be replaced with any $y_0' \in \mathfrak{m}_R - \{0\}$ satisfying $y_0'^n = y_0^m u$ for some $u \in R^\times$. Choose such a y_0'. Clearly $n\lambda_{\mathrm{st}}^+(y_0') = m\lambda_{\mathrm{st}}^+(y_0) + \lambda(u)$, so $X = \lambda_{\mathrm{st}}^+(y_0)$ is replaced with $X' = cX + \lambda(u)/n$, where $\lambda(u)/n \in (B_{\mathrm{cris}}^+)^\times$ (since $u \in R^\times$). Thus, $v_0' \cdot \mathrm{d}/\mathrm{d}X' = cv_0 \cdot \mathrm{d}/\mathrm{d}(cX) = v_0 \mathrm{d}/\mathrm{d}X$, so N is independent of v_0!

Another point of view that may be used is that to each $\widetilde{q} \in \mathfrak{m}_R - \{0\}$ there is associated a B_{cris}-linear derivation $N_{\widetilde{q}} = \mathrm{d}/\mathrm{d}X_{\widetilde{q}}$ for $X_{\widetilde{q}} = \lambda_{\mathrm{st}}^+(\widetilde{q})$, and $N := v_R(\widetilde{q}) \cdot N_{\widetilde{q}}$ is independent of $\widetilde{q}$.

To define a filtration on $K \otimes_{K_0} B_{\mathrm{st}}$ extending the one on $K \otimes_{K_0} B_{\mathrm{cris}}$, we seek to construct a G_K-equivariant B_{cris}-algebra embedding $B_{\mathrm{st}} \to B_{\mathrm{dR}}$ carrying B_{st}^+ into B_{dR}^+. The image of such a map will be canonical but the actual map will depend on a choice of G_K-equivariant homomorphism

$$\log_{\overline{K}} : \overline{K}^\times \to \overline{K}$$

extending the usual log on 1-units and equal to the trivial map on Teichmüller lifts $\overline{k}^\times$. We write $\log : \mathscr{O}_{\overline{K}}^\times \to \overline{K}^\times$ to denote the canonical log map on units that kills $\overline{k}^\times$ and is the usual logarithm on 1-units. This latter canonical logarithm map is G_K-equivariant.

To construct such a map $\log_{\overline{K}}$ on $\overline{K}^\times$, pick any $q \in \mathfrak{m}_K - \{0\}$ and any $c \in K$ for which we want to define $\log_{\overline{K}}(q) = c$.

Lemma 9.2.6. *There is a unique homomorphism* $\log_{\overline{K}} : \overline{K}^\times \to \overline{K}$ *extending* $\log$ *on* $\mathscr{O}_{\overline{K}}^\times$ *and satisfying* $\log_{\overline{K}}(q) = c$. *It is also G_K-equivariant, and if* $\log_{\overline{K}}'$ *corresponds to the condition* $\log_{\overline{K}}(q) = c' \in K$ *then* $\log_{\overline{K}}'(x) - \log_{\overline{K}}(x) = (\mathrm{ord}_p(x)/\mathrm{ord}_p(q))(c' - c) \in K$ *for all* $x \in \overline{K}^\times$. *In particular, if $c, c' \in K_0$ then the associated logarithms have difference valued in K_0.*

The standard convention is to take $q = p$ (and Iwasawa's convention is to also take $c = 0$).

Proof. Once uniqueness and existence are proved then G_K-equivariance follows, as $q, c \in K$ are fixed by G_K and log on units is G_K-equivariant. For a general $x \in \overline{K}^\times$ we have

$\mathrm{ord}_p(x) = (m/n)\,\mathrm{ord}_p(q)$ for some $m, n \in \mathbf{Z}$ with $n \neq 0$, so $x^n/q^m \in \mathscr{O}_{\overline{K}}^{\times}$. We therefore know what $\log(x^n/q^m)$ means, and so if $\log_{\overline{K}}$ is to be homomorphism extending log and satisfying $\log_{\overline{K}}(q) = c$ then the only choice is to define

$$\log_{\overline{K}}(x) = \frac{\log(x^n/q^m) + mc}{n}.$$

This proves uniqueness, and also exhibits the desired variation under change in c, but we have to show this formula actually works.

If we scale m and n by a common nonzero integer then proposed formula does not change, so it is a well-posed definition (i.e., it only depends on m and n through the ratio m/n). To check the homomorphism property we simply observe that if $x' \in \overline{K}^{\times}$ with $\mathrm{ord}_p(x') = (m'/n')\,\mathrm{ord}_p(q)$ then

$$\mathrm{ord}_p(xx') = (m/n + m'/n')\,\mathrm{ord}_p(q) = ((mn' + m'n)/(nn'))\,\mathrm{ord}_p(q).$$

From this the homomorphism property is a simple calculation. ∎

To define a G_K-equivariant B_{cris}^+-algebra map $B_{\mathrm{st}}^+ \to B_{\mathrm{dR}}^+$, we need to construct a G_K-equivariant homomorphism $\mathrm{Frac}(R)^{\times} \to B_{\mathrm{dR}}^+$ whose restriction to $R^{\times}$ is the G_K-equivariant homomorphism $\lambda_{\mathrm{cris}} : x \mapsto \log_{\mathrm{cris}}([x]) \in A_{\mathrm{cris}}$ from Lemma 9.2.2. The construction of such a homomorphism uses two ingredients.

First, the G_K-equivariant multiplicative Teichmüller map $R - \{0\} \to (B_{\mathrm{dR}}^+)^{\times}$ defined by $r \mapsto [r]$ uniquely extends to a multiplicative map $\mathrm{Frac}(R)^{\times} \to (B_{\mathrm{dR}}^+)^{\times}$ that we denote $y \mapsto [y]$. We similarly define $\mathrm{Frac}(R)^{\times} \to \mathbf{C}_K^{\times}$ extending the map $R - \{0\} \to \mathscr{O}_{\mathbf{C}_K} - \{0\}$ defined by $y \mapsto y^{(0)}$ (and we denote the extended map with the same notation). Both of these extended homomorphisms are G_K-equivariant.

Second, each coset in $\mathrm{Frac}(R)^{\times}/R^{\times}$ is represented by some $y \in \mathrm{Frac}(R)^{\times}$ such that $y^{(0)} \in \overline{K}^{\times}$, and even $y^{(0)} \in \overline{\mathbf{Q}}_p^{\times}$ (to make later considerations depend only on $\mathbf{C}_K$ and not on K or $\overline{K}$). Using the canonical embedding $\overline{K} \hookrightarrow B_{\mathrm{dR}}^+$ we can therefore make sense of the ratio $[y]/y^{(0)} \in (B_{\mathrm{dR}}^+)^{\times}$ for all $y \in \mathrm{Frac}(R)^{\times}$ such that $y^{(0)} \in \overline{K}^{\times}$, and this has reduction in $\mathbf{C}_K^{\times}$ that is equal to 1.

In other words, $[y]/y^{(0)}$ is a 1-unit in the complete discrete valuation ring B_{dR}^+ for all $y \in \mathrm{Frac}(R)^{\times}$ such that $y^{(0)} \in \overline{K}^{\times}$, so the usual B_{dR}^+-valued logarithm $\log_{\mathrm{dR}}$ on 1-units of B_{dR}^+ makes sense to evaluate on $[y]/y^{(0)}$. Hence,

$$\lambda(y) := \log_{\mathrm{dR}}([y]/y^{(0)}) + \log_{\overline{K}}(y^{(0)}) \in B_{\mathrm{dR}}^+$$

makes sense for the group of all $y \in \mathrm{Frac}(R)^{\times}$ such that $y^{(0)} \in \overline{K}^{\times}$, or even just $y^{(0)} \in \overline{\mathbf{Q}}_p^{\times}$ (which is sufficient for our needs and avoids dependence on $\overline{K}$), and λ is a G_K-equivariant homomorphism. In order that this λ "work" to extend λ_{cris} to $\mathrm{Frac}(R)^{\times}$, it remains to check the consistency with λ_{cris} on $R^{\times}$:

Lemma 9.2.7. *If $y \in R^{\times}$ and $y^{(0)} \in \mathscr{O}_{\overline{K}}^{\times}$ then $\log_{\mathrm{cris}}([y]) = \log_{\mathrm{dR}}([y]/y^{(0)}) + \log_{\overline{K}}(y^{(0)})$.*

Proof. By using the decomposition of y into a product of a Teichmüller lift and a 1-unit in $R^{\times}$, it suffices to separately treat the case $y \in \overline{k}^{\times}$ and $y \in 1 + \mathfrak{m}_R$. In the first case $y^{(0)} = [y]$, so $\log_{\mathrm{cris}}([y]) = 0 = \log_{\overline{K}}(y^{(0)})$ by definition for such y and $[y]/y^{(0)} = 1$, yielding the desired equality. In the second case $\log_{\mathrm{cris}}([y])$ is defined using the procedure of Lemma 9.2.2, and we need to prove

$$\log_{\mathrm{dR}}([y]/y^{(0)}) = \log_{\mathrm{cris}}([y]) - \log_{\overline{K}}(y^{(0)}) \tag{9.2.2}$$

for $y \in 1 + \mathfrak{m}_R$ such that the 1-unit $y^{(0)} \in \mathscr{O}_{\mathbf{C}_{\overline{K}}}^{\times}$ is algebraic over K.

Both sides of (9.2.2) convert products in y into sums, so since the identity takes place in a torsion-free abelian group it suffices to check the result after replacing y with y^N for sufficiently large N. Hence, we can assume $y^{(0)} \in 1 + p\mathscr{O}_{\overline{K}}$, so $y^{(0)} \in 1 + p\mathscr{O}_{K'}$ for some *finite* extension K'/K. Hence, $\log_{\overline{K}}(y^{(0)}) = \sum_{n \geqslant 1} (-1)^{n+1}(y^{(0)} - 1)^n/n$ in K' as a p-adically convergent sum, the tail of which takes place in the $\mathrm{W}(k')$-finite $\mathscr{O}_{K'}$ and so can be viewed as a p-adically convergent sum in $\mathscr{O}_{K'} \otimes_{\mathrm{W}(k')} A_{\mathrm{cris}} \subseteq K' \otimes_{K'_0} B^+_{\mathrm{cris}} \subseteq B^+_{\mathrm{dR}}$ (with k'/k the residue field of K'). By construction $\log_{\mathrm{cris}}([y]) = \sum_{n \geqslant 1}(-1)^{n+1}([y]-1)^n/n$ in $A_{\mathrm{cris}}[1/p]$ as a p-adically convergent sum, the tail of which takes place in A_{cris}.

Since $-\log_{\overline{K}}(y^{(0)}) = \log_{\overline{K}}((1/y)^{(0)})$, a p-adic approximation argument with finite sums gives that in $(\mathscr{O}_{K'} \otimes_{\mathrm{W}(k')} A_{\mathrm{cris}})[1/p]$ we have

$$\log_{\mathrm{cris}}([y]) - \log_{\overline{K}}(y^{(0)}) = \sum_{n \geqslant 1} (-1)^{n+1} \cdot \frac{([y](1/y)^{(0)} - 1)^n}{n},$$

where this sum formed in $(\mathscr{O}_{K'} \otimes_{\mathrm{W}(k')} A_{\mathrm{cris}})[1/p]$ has tail whose terms lie in $\mathscr{O}_{K'} \otimes_{\mathrm{W}(k')} A_{\mathrm{cris}}$ and is p-adically convergent in here. In B^+_{dR} we have

$$\log_{\mathrm{dR}}([y]/y^{(0)}) = \sum_{n \geqslant 1} (-1)^{n+1} \frac{([y]/y^{(0)} - 1)^n}{n} = \sum_{n \geqslant 1} (-1)^{n+1} \frac{([y](1/y)^{(0)} - 1)^n}{n}$$

relative to the discretely-valued topology of B^+_{dR}.

We may and do rename K' as K. Consider $u \in \mathscr{O}_K \otimes_{\mathrm{W}(k)} \mathrm{W}(R)$ satisfying $u \equiv 1 \bmod p$ and $\widetilde{\theta}(u) = 1$ (with $\widetilde{\theta} : \mathscr{O}_K \otimes_{\mathrm{W}(k)} \mathrm{W}(R) \to \mathscr{O}_{\mathbf{C}_K}$ induced by the usual $\mathrm{W}(k)$-algebra map θ); a basic example is $u = (1/y)^{(0)} \otimes [y]$ for y as above. It suffices to prove that for any such u, the convergent sum

$$\sum_{n \geqslant 1} (-1)^{n+1}(u-1)^n/n$$

formed in the *p-adic topology* of $\mathscr{O}_K \otimes_{\mathrm{W}(k)} \mathrm{W}(R)$ has image in B^+_{dR} that is equal to the "same" sum formed relative to the *discretely-valued topology* of B^+_{dR}. Equivalently, if $v \in \mathscr{O}_K \otimes_{\mathrm{W}(k)} \mathrm{W}(R)$ satisfies $\widetilde{\theta}(v) = 0$ then we claim that the p-adically convergent sum

$$\sum_{n \geqslant 1} (-1)^{n+1}(p^n/n)v^n \in \mathscr{O}_K \otimes_{\mathrm{W}(k)} \mathrm{W}(R)$$

has image in B^+_{dR} that is the "same" sum formed relative to the discretely-valued topology of B^+_{dR}.

In other words, for any $i \geqslant 1$ we claim that if N is sufficiently large (depending on i) then the p-adic tail $\sum_{n \geqslant N}(-1)^{n+1}(p^n/n)v^n$ in $\mathscr{O}_K \otimes_{\mathrm{W}(k)} \mathrm{W}(R)$ has image in B^+_{dR} that lies in $\mathrm{Fil}^i(B^+_{\mathrm{dR}})$. Provided that $N \geqslant i$, this image lies in $v^i B^+_{\mathrm{dR}} \subseteq \mathrm{Fil}^i(B^+_{\mathrm{dR}})$ since $v \in \mathrm{Fil}^1(B^+_{\mathrm{dR}})$

(because $\widetilde{\theta}(v) = 0$ and the K-algebra map $B_{\mathrm{dR}}^+ \twoheadrightarrow \mathbf{C}_K$ is a $\overline{K}$-algebra map due to how we *defined* the $\overline{K}$-algebra structure on B_{dR}^+ using Hensel's Lemma). ■

Example 9.2.8. Choose a unit $u \in \mathscr{O}_K^\times$ that is not a root of unity, and pick a sequence of p-power compatible roots of u, which is to say $\underline{u} \in R^\times$ with $\underline{u}^{(0)} = u$. We get the element $\log_{\mathrm{cris}}([\underline{u}]) \in B_{\mathrm{cris}}^+$, and thanks to the formula $\log([\varepsilon^a]) = at$ for all $a \in \mathbf{Z}_p$ (as we rigorously proved after Exercise 4.5.3) we can compute that for $g \in G_K$,

$$g(\log_{\mathrm{cris}}([\underline{u}])) = \log_{\mathrm{cris}}(g([\underline{u}])) = \log_{\mathrm{cris}}([g(\underline{u})/\underline{u}]) + \log_{\mathrm{cris}}([\underline{u}]) = \eta_{\underline{u}}(g)t + \log_{\mathrm{cris}}([\underline{u}])$$

where $g(\underline{u})/\underline{u} = \varepsilon^{\eta_{\underline{u}}(g)}$ for the 1-cocycle $\eta_{\underline{u}} : G_K \to \mathbf{Z}_p$ *relative to the χ-action on $\mathbf{Z}_p$) arising from the G_K-action on the chosen p-power roots of u (as in computations of p-power Kummer theory!).

We claim that the elements $t = \log([\varepsilon])$ and $\log_{\mathrm{cris}}([\underline{u}])$ in B_{cris} are $\mathbf{Q}_p$-linearly independent. Indeed, if $\log_{\mathrm{cris}}([\underline{u}]) = at$ for some $a \in \mathbf{Q}_p$ then we would have $g(at) = \eta_{\underline{u}}(g)t + at$, so $a(\chi(g) - 1) = \eta_{\underline{u}}(g)$. This would force the chosen system of compatible p-power roots of u to all lie in the cyclotomic extension $K(\mu_{p^\infty})$ that is abelian over K, so all p-power roots of u would lie in this abelian extension. Since u was assumed to not be a root of unity, this is impossible by Exercise 9.4.2 below.

The $\mathbf{Q}_p$-linear independence ensures that t and $\log_{\mathrm{cris}}([\underline{u}])$ span a 2-dimension subspace $V_u \subseteq B_{\mathrm{cris}}$ on which the G_K-action is given by the matrix form

$$\begin{pmatrix} \chi & \eta_{\underline{u}} \\ 0 & 1 \end{pmatrix}.$$

This is exactly the extension class in $\mathrm{H}^1(G_K, \mathbf{Q}_p(1))$ arising from $\underline{u}$, and its isomorphism class as an abstract $\mathbf{Q}_p[G_K]$-module only depends on u.

In addition to the $\mathbf{Q}_p[G_K]$-linear injection of V_u into B_{cris} that we have just constructed, there is a nonzero element of $\mathrm{Hom}_{\mathbf{Q}_p[G_K]}(V_u, B_{\mathrm{cris}}) = D_{\mathrm{cris}}^*(V_u)$ with a 1-dimensional kernel, namely the projection of V_u into its quotient $\mathbf{Q}_p$ that in turn naturally sits inside of B_{cris}. Hence, we have constructed two elements of $D_{\mathrm{cris}}^*(V_u)$ that are visibly linearly independent over K_0. The general inequality $\dim_{K_0} D_{\mathrm{cris}}^*(V_u) \leqslant 2$ is therefore an equality, so such V_u are crystalline.

Example 9.2.9. We can now interpret Example 9.2.1 in terms of B_{st}, somewhat extending the theme of Example 9.2.8, as follows. Choose $\widetilde{q} \in \mathfrak{m}_R - \{0\}$ with $q := \widetilde{q}^{(0)} \in \mathscr{O}_K$ (so $0 < |q| < 1$). We get an isomorphism $B_{\mathrm{st}}^+ \simeq B_{\mathrm{cris}}^+[X]$ with $X = \lambda_{\mathrm{st}}^+(\widetilde{q})$, and the B_{cris}^+-algebra map $B_{\mathrm{st}}^+ \to B_{\mathrm{dR}}^+$ carries X to the element $\log([\widetilde{q}]) := \log_{\mathrm{dR}}([\widetilde{q}]/q) + \log_{\overline{K}}(q) \in B_{\mathrm{dR}}^+$ whose image in the residue field $\mathbf{C}_K$ is $\log_{\overline{K}}(q)$ (which might vanish). Note that this definition of $\log([\widetilde{q}])$ as we vary $\widetilde{q}$ is a G_K-equivariant multiplicative map on the set of elements $[\widetilde{q}]$ with $\widetilde{q}^{(0)} \in \mathfrak{m}_K - \{0\}$.

Choose a $\mathbf{Z}_p$-basis ε of $\mathbf{Z}_p(1)$, so (using that $q \in K^\times$) we have $g(\widetilde{q}) = \widetilde{q}\varepsilon^{\eta_{\widetilde{q}}(g)}$ for a unique $\eta_{\widetilde{q}}(g) \in \mathbf{Z}_p$. Letting $t = \log([\varepsilon])$, the G_K-action is given on X by $g(X) = X + \eta_{\widetilde{q}}(g)t$ because in B_{dR}^+ we have (by G_K-equivariance of $\log_{\mathrm{dR}}$)

$$g(\log([\widetilde{q}])) = \log([g(\widetilde{q})]) = \log([\widetilde{q}][\varepsilon^{\eta_{\widetilde{q}}(g)}]) = \log([\widetilde{q}]) + \log([\varepsilon^{\eta_{\widetilde{q}}(g)}])$$

and (as we saw after Exercise 4.5.3) $\log([\varepsilon^a]) = a\log([\varepsilon])$ for all $a \in \mathbf{Z}_p$. Hence, Example 9.2.1 is now rigorously completed: for the Tate curve E_q we have explicitly realized the 2-dimensional p-adic representation $V_p(E_q)$ inside of $B_{\rm st}$. Moreover, the same linear independent argument as in Example 9.2.8 shows that $D_{\rm st}^*(V_p(E_q))$ is at least 2-dimensional. In Proposition 9.2.11 we will show that $B_{\rm st}$ is $(\mathbf{Q}_p, G_K)$-regular, whence by the formalism of §5 it follows that $D_{\rm st}^*(V_p(E_q))$ is exactly 2-dimensional and thus $V_p(E_q)$ is $B_{\rm st}$-admissible (i.e., $V_p(E_q)$ is a "semistable" representation).

Now we are ready to impose a filtration on $K \otimes_{K_0} B_{\rm st}$, depending on a choice of $\log_{\overline{K}}$, and to make this work out nicely we impose the requirement

$$\log_{\overline{K}}(p) \in K_0.$$

The reason for this condition is that if we change the choice of $\log_{\overline{K}}(p) \in K_0$ then by Lemma 9.2.6 the image of $\log([\widetilde{q}])$ changes in $B_{\rm dR}^+$ by additive translation by some element of $K_0 \subseteq B_{\rm cris}^+$. Hence, the $B_{\rm cris}^+$-subalgebra *image* of $B_{\rm st}^+$ in $B_{\rm dR}^+$ is independent of the choice of $\log_{\overline{K}}(p) \in K_0$, so we have a canonical image for the map $B_{\rm st}^+ \to B_{\rm dR}^+$ (and likewise after inverting t) even though the actual map is not canonical.

Theorem 9.2.10. *Choosing $\log_{\overline{K}}(p) \in K_0$, the resulting G_K-equivariant $K \otimes_{K_0} B_{\rm cris}^+$-algebra map $K \otimes_{K_0} B_{\rm st}^+ \to B_{\rm dR}^+$ is injective. In particular, $K \otimes_{K_0} B_{\rm st}$ injects into $B_{\rm dR}$ as a $K[G_K]$-algebra, so the inclusion $K_0 \subseteq B_{\rm st}^{G_K}$ is an equality.*

Proof. We just sketch the idea, giving a reference for the details. Upon choosing $\widetilde{q} \in \mathfrak{m}_R - \{0\}$ with $q := \widetilde{q}^{(0)} \in \mathscr{O}_K$, the problem is to prove that $X := \log_{\rm dR}([\widetilde{q}]/q) + \log_{\overline{K}}(q)$ is transcendental over $\mathrm{Frac}(B_{\rm cris}^+) = \mathrm{Frac}(B_{\rm cris})$ inside of $B_{\rm dR}$. It suffices to treat a single choice of $\widetilde{q}$, and to show that $\log_{\rm dR}([\widetilde{q}]/q)$ is transcendental over $\mathrm{Frac}(B_{\rm cris})$ (since $\log_{\overline{K}}(q) \in K$). We choose $q = p$, so $\xi = [\widetilde{p}] - p$ generates $\ker\theta \subseteq \mathrm{W}(R)$ and

$$\log_{\rm dR}([\pi]/p) = \sum_{n\geqslant 1} (-1)^{n+1} \cdot \frac{\xi^n}{np^n}$$

in the discrete valuation topology of $B_{\rm dR}^+$ (with ξ lying in the maximal ideal). It must be shown that this element of $B_{\rm dR}$ is not algebraic over $\mathrm{Frac}(B_{\rm cris})$. The proof of this is given in [21, §4.3.2–§4.3.3].

The key to the proof of such non-algebraicity is to show that this sum does not lie in $\mathrm{Frac}(B_{\rm cris})$, which rests on a delicate analysis of p-adic series and especially on proving that the (non-noetherian) subring of $B_{\rm dR}^+$ consisting of sums $\sum_{n\geqslant 0} w_n(\xi/p)^n$ with $w_n \in \mathrm{W}(R)$ is p-adically separated. Once this is shown, if $\log_{\rm dR}([\pi]/p)$ is not transcendental over $\mathrm{Frac}(B_{\rm cris})$ then its minimal polynomial has degree $d \geqslant 2$ and then applying the G_K-action to this minimal polynomial it follows from the equality $B_{\rm dR}^{G_{K_0}} = K_0 \subseteq B_{\rm cris}$ and the condition $d \geqslant 2$ that one gets a contradiction. ∎

Theorem 9.2.10 gives an exhaustive and separated filtration to $K \otimes_{K_0} B_{\rm st}$ via a $B_{\rm cris}$-algebra injection in $B_{\rm dR}$, but this injection rests on a choice of $\log_{\overline{K}}(p) \in K_0$, and so likewise the filtration on $D_{\rm st}(V)_K$ depends on this choice. We will now use Iwasawa's convention $\log_{\overline{K}}(p) = 0$ to eliminate non-canonicity in the filtration structure. Moreover, with this choice the embedding of B_{st} into B_{dR} is not only intrinsic but depends only on $\mathbf{C}_K$ rather than on K.

Proposition 9.2.11. *The ring B_{st} is $(\mathbf{Q}_p, G_K)$-regular.*

Proof. It remains to prove that if $b \in B_{\mathrm{st}}$ is nonzero and $\mathbf{Q}_p b$ is G_K-stable then $b \in B_{\mathrm{st}}^\times$. It is harmless for this purpose to replace K with $\widehat{K^{\mathrm{un}}}$, which is to say that k is algebraically closed. We shall use the concrete description $B_{\mathrm{st}} = B_{\mathrm{cris}}[X]$ with $g(X) = X + \eta(g)t$ where $t = \log([\varepsilon])$ is a fixed choice and the continuous $\eta : G_K \to \mathbf{Z}_p$ is defined by $g(\pi) = \pi\varepsilon^{\eta(g)}$ for a fixed $\pi \in R$ such that $\pi^{(0)} = p$. Let $\psi : G_K \to \mathbf{Q}_p^\times$ be the character on the line $\mathbf{Q}_p b$ in $B_{\mathrm{st}} = B_{\mathrm{cris}}[X]$. We may write $b = b_0 + \cdots + b_r X^r$ with $b_i \in B_{\mathrm{cris}}$ and $b_r \neq 0$. Our goal is to show $r = 0$, as then $b = b_0$ spans a G_K-stable $\mathbf{Q}_p$-line in B_{cris}, whence $b \in B_{\mathrm{cris}}^\times = B_{\mathrm{st}}^\times$ due to the known $(\mathbf{Q}_p, G_K)$-regularity of B_{cris}.

Consider the identity

$$\psi(g)b = g(b) = g(b_0) + g(b_1)(X + \eta(g)t) + \cdots + g(b_r)(X + \eta(g)t)^r$$

in B_{st} for $g \in G_K$. Comparing top-degree terms in X gives $\psi(g)b_r = g(b_r)$, so b_r spans a G_K-stable $\mathbf{Q}_p$-line in B_{cris}. The character ψ is continuous, by the same trick with $t^{\mathbf{Z}}$-scaling and projection into $\mathbf{C}_K$ as in the proof of $(\mathbf{Q}_p, G_K)$-regularity of B_{cris} in Proposition 9.1.6. Hence, ψ is a continuous character that appears in B_{cris}, so it is a crystalline character of G_K. As such ψ is Hodge–Tate, so it has some Hodge–Tate weight $n \in \mathbf{Z}$. Thus, $\chi^{-n}\psi$ is a crystalline character with Hodge–Tate weight 0. The proof of Proposition 8.3.4 relied on properties of B_{cris} and D_{cris} that have been established in the preceding developments, and so its conclusion may be applied: $\chi^{-n}\psi$ is a Tate twist of an unramified character of G_K. But $G_K = I_K$ since now k is algebraically closed, and so the vanishing of the Hodge–Tate weight means that there is no Tate twist at all: $\chi^{-n}\psi = 1$.

We may now replace b with $t^{-n}b$ (as $t \in B_{\mathrm{cris}}^\times$) to reduce to the case $n = 0$, so $\psi = 1$. In particular, $g(b_r) = \psi(g)b_r = b_r$ for all $g \in G_K$, so $b_r \in (B_{\mathrm{cris}}^\times)^{G_K} = K_0^\times$. Assuming $r > 0$, we seek a contradiction. Consideration of terms in X-degree $r - 1$ in our formula for $\psi(g)b$ gives

$$b_{r-1} = \psi(g)b_{r-1} = g(b_{r-1}) + g(b_r)r\eta(g)t = g(b_{r-1}) + b_r r\eta(g)t.$$

Thus, $g(b_{r-1}) - b_{r-1} = -rb_r\eta(g)t$ with $c := -rb_r \in K_0^\times$ and any $g \in G_K$. Hence,

$$g(b_{r-1}/c) - b_{r-1}/c = \eta(g)t = g(X) - X,$$

so $X - b_{r-1}/c \in B_{\mathrm{st}}^{G_K} = K_0 \subseteq B_{\mathrm{cris}}$. But $b_{r-1} \in B_{\mathrm{cris}}$ and $X \notin B_{\mathrm{cris}}$, so we have a contradiction. ∎

We may now apply the formalism of §5 to the functor $D_{\mathrm{st}} : \mathrm{Rep}_{\mathbf{Q}_p}(G_K) \to \mathrm{Vec}_{K_0}$ defined by

$$D_{\mathrm{st}}(V) = (B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V)^{G_K},$$

so $\dim_{K_0} D_{\mathrm{st}}(V) \leqslant \dim_{\mathbf{Q}_p}(V)$ for all V and equality holds precisely when V is B_{st}-admissible. A *semistable* p-adic representation of G_K is one that is B_{st}-admissible; the full subcategory of these is denoted $\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{st}}(G_K)$. By using the additional structures on B_{st} (including the subspace filtration on $K \otimes_{K_0} B_{\mathrm{st}}$ from B_{dR} via Theorem 9.2.10), we see that D_{st} is naturally valued in $\mathrm{MF}_K^{\phi,N}$.

Much like in our analysis of D_{cris}, we also see that the faithful functor

$$D_{\mathrm{st}} : \mathrm{Rep}^{\mathrm{st}}_{\mathbf{Q}_p}(G_K) \to \mathrm{MF}^{\phi,N}_K$$

is an exact functor compatible with tensor products and duals (endowed with their natural filtrations). Likewise, the B_{st}-linear G_K-equivariant Frobenius-compatible and N-compatible semistable comparison isomorphism

$$\alpha : B_{\mathrm{st}} \otimes_{K_0} D_{\mathrm{st}}(V) \simeq B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V$$

is seen to be an isomorphism with respect to the filtration structures after scalar extension to K (i.e., α_K and α_K^{-1} are filtration-compatible).

Lemma 9.2.12. *Crystalline representations are semistable, and $D_{\mathrm{cris}}(V) = D_{\mathrm{st}}(V)$ in $\mathrm{MF}^{\phi,N}_K$ for all V. If V is semistable and $D_{\mathrm{st}}(V)$ has vanishing monodromy operator then V is crystalline.*

Proof. Since $B_{\mathrm{st}}^{N=0} = B_{\mathrm{cris}}$, we see that $D_{\mathrm{st}}(V)^{N=0} = D_{\mathrm{cris}}(V)$ in MF^{ϕ}_K for every $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$. In particular, if V is crystalline then for dimension reasons the K_0-linear inclusion $D_{\mathrm{cris}}(V) \subseteq D_{\mathrm{st}}(V)$ is an isomorphism in $\mathrm{MF}^{\phi,N}_K$. Thus, crystalline representations are semistable.

If V is semistable but $D_{\mathrm{st}}(V)$ has vanishing monodromy operator then $D_{\mathrm{cris}}(V) = D_{\mathrm{st}}(V)$ and this has K_0-dimension $\dim_{\mathbf{Q}_p}(V)$, so V is crystalline. ■

It follows from this lemma that by working in the generality of semistable representations we can keep track of crystalline objects simply by observing whether or not N vanishes.

Lemma 9.2.13. *Semistable representations are de Rham, and if V is semistable then the natural injective map $K \otimes_{K_0} D_{\mathrm{st}}(V) \to D_{\mathrm{dR}}(V)$ is an isomorphism in* Fil_K.

Proof. If V is semistable then the natural injective map $K \otimes_{K_0} D_{\mathrm{st}}(V) \to D_{\mathrm{dR}}(V)$ has source with K-dimension $\dim_{\mathbf{Q}_p}(V)$ that is an upper bound on the K-dimension of the target, so it is a K-linear isomorphism. In particular, V is de Rham. By the definition of the filtration structure on $K \otimes_{K_0} B_{\mathrm{st}}$), this natural injective map is always a subobject inclusion in Fil_K, so when it is an isomorphism as K-vector spaces it must be an isomorphism in Fil_K. ■

To summarize:

$$\text{crystalline} \Rightarrow \text{semistable} \Rightarrow \text{de Rham} \Rightarrow \text{Hodge-Tate.}$$

As with crystalline representations in Proposition 9.1.11, there is a full faithfulness result for D_{st} on semistable representations and we can write down an inverse functor on the essential image of D_{st} on semistable representations, as follows. The equality $\mathrm{Fil}^0(B_{\mathrm{st}})^{N=0,\phi=1} = \mathrm{Fil}^0(B_{\mathrm{cris}})^{\phi=1} = \mathbf{Q}_p$ implies that functor

$$V_{\mathrm{st}} : \mathrm{MF}^{\phi,N}_K \to \mathbf{Q}_p[G_K]\text{-mod}$$

defined by

$$V_{\mathrm{st}}(D) = \mathrm{Fil}^0(B_{\mathrm{st}} \otimes_{K_0} D)^{N=0,\phi=1} \tag{9.2.3}$$

$$:= \ker(\delta(D) : (B_{\mathrm{st}} \otimes_{K_0} D)^{N=0,\phi=1} \to (B_{\mathrm{dR}} \otimes_K D_K)/\,\mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_K D_K)) \tag{9.2.4}$$

provides an inverse to the functor $D_{\rm st}$ on semistable representations: there is a natural $\mathbf{Q}_p[G_K]$-linear isomorphism $V \simeq V_{\rm st}(D_{\rm st}(V))$ for all $V \in {\rm Rep}^{\rm st}_{\mathbf{Q}_p}(G_K)$. (If we use the contravariant functor $D^*_{\rm st}(V) = {\rm Hom}_{\mathbf{Q}_p[G_K]}(V, B_{\rm st})$ then the inverse is given by the contravariant functor $V^*_{\rm st}(D) = {\rm Hom}_{{\rm Fil},\phi,N}(D, B_{\rm st})$.) In particular, as in the crystalline case in Proposition 9.1.11, we deduce via the comparison isomorphism:

Proposition 9.2.14. *The functor $D_{\rm st} : {\rm Rep}^{\rm st}_{\mathbf{Q}_p}(G_K) \to {\rm MF}^{\phi,N}_K$ is fully faithful, with quasi-inverse on its essential image given by $V_{\rm st}$.*

Note also that if $D \in {\rm MF}^{\phi,N}_K$ with $N_D = 0$ (i.e., $M \in {\rm MF}^{\phi}_K$) then $V_{\rm st}(D) = V_{\rm cris}(D)$ because $B^{N=0}_{\rm st} = B_{\rm cris}$.

9.3. **Finer properties of crystalline and semistable representations.** Now that we have constructed $B_{\rm cris}$ and $B_{\rm st}$ and worked out some basic properties of their respective associated functors $D_{\rm cris}$ and $D_{\rm st}$ (especially full faithfulness from ${\rm Rep}^{\rm cris}_{\mathbf{Q}_p}(G_K)$ and ${\rm Rep}^{\rm st}_{\mathbf{Q}_p}(G_K)$ into the respective target categories), we want to address some deeper properties. The most important property concerns an intrinsic characterization of the essential images of these fully faithful functors.

We first dispose of a more elementary fact: insensitivity to inertial restriction. Unlike the de Rham case (see Proposition 6.3.8), if K'/K is a finite ramified extension it is not true that $V \in {\rm Rep}_{\mathbf{Q}_p}(G_K)$ is crystalline (resp. semistable) if its $G_{K'}$-restriction is so. This was already apparent in Proposition 8.3.4 (which rests on properties of $D_{\rm cris}$ that have now been established), where we found that a 1-dimensional crystalline representation is exactly a Tate twist of an unramified character; this leaves no room for twisting by a finite-order ramified character without ruining the crystalline property.

Likewise, if K'/K is a finite ramified Galois extension then the induction ${\rm Ind}^{G_K}_{G_{K'}}(\mathbf{Q}_p) = \mathbf{Q}_p[{\rm Gal}(K'/K)]$ has trivial $G_{K'}$-action (so it is crystalline as a $G_{K'}$-representation) but has ramified G_K-action and hence is non-crystalline by Corollary 9.3.2 below. Thus, to prove good behavior for the crystalline property with respect to the restriction functor ${\rm Rep}_{\mathbf{Q}_p}(G_K) \to {\rm Rep}_{\mathbf{Q}_p}(G_{K'})$ we must restrict our attention to the case when K'/K satisfies $e(K'/K) = 1$. In other words, the essential case is $K' = \widehat{K^{\rm un}}$ (which is to say, inertial restriction):

Proposition 9.3.1. *Let $K' = \widehat{K^{\rm un}}$. The natural map $K'_0 \otimes_{K_0} D_{{\rm st},K}(V) \to D_{{\rm st},K'}(V)$ in ${\rm MF}^{\phi,N}_{K'}$ is an isomorphism for* all *$V \in {\rm Rep}_{\mathbf{Q}_p}(G_K)$, and likewise for the functor $D_{{\rm cris},K'}$ that is valued in ${\rm MF}^{\phi}_{K'}$. In particular, V is semistable as a G_K-representation if and only if it is semistable as a representation of $G_{K'} = I_K$, and likewise for the crystalline property.*

Proof. The crystalline case will follow from the semistable case since $D_{\rm cris}(V) = D_{\rm st}(V)^{N=0}$ for all $V \in {\rm Rep}_{\mathbf{Q}_p}(G_K)$. The map $K'_0 \otimes_{K_0} D_{{\rm st},K}(V) \to D_{{\rm st},K'}(V)$ is visibly a morphism in ${\rm MF}^{\phi,N}_{K'}$, so it suffices to show that its scalar extension to K' is an isomorphism in ${\rm Fil}_{K'}$. This goes via completed unramified descent as in the proof of Proposition 6.3.8. ∎

Corollary 9.3.2. *If $\rho : G_K \to {\rm GL}(V)$ is a p-adic representation with open kernel then ρ is semistable if and only if it is crystalline if and only if it is unramified.*

Also, a continuous character $\eta : G_K \to \mathbf{Q}_p^\times$ if semistable if and only if it is crystalline if and only if it is a Tate twist of an unramified character.

Proof. By Proposition 9.3.1, we may replace K with $\widehat{K^{\mathrm{un}}}$ so that k is algebraically closed. The problem for the first part of the corollary is then to show that if ρ is semistable with $\ker \rho$ open in G_K then ρ is a trivial action on V. Let L/K be the finite Galois extension corresponding to $\ker \rho$, so V is a representation space for $\mathrm{Gal}(L/K)$ and it is semistable as a G_K-representation space. Our goal is to prove that $V^{\mathrm{Gal}(L/K)} = V$.

Since k is algebraically closed we have $L_0 = K_0$. Hence, $B_{\mathrm{st}}^{G_L} = L_0 = K_0$, so

$$\begin{aligned} D_{\mathrm{st},K}(V) = (D_{\mathrm{st},L}(V))^{\mathrm{Gal}(L/K)} = (B_{\mathrm{st}}^{G_L} \otimes_{\mathbf{Q}_p} V)^{\mathrm{Gal}(L/K)} &= (K_0 \otimes_{\mathbf{Q}_p} V)^{\mathrm{Gal}(L/K)} \\ &= K_0 \otimes_{\mathbf{Q}_p} V^{\mathrm{Gal}(L/K)}. \end{aligned}$$

But $\dim_{K_0} D_{\mathrm{st},K}(V) = \dim_{\mathbf{Q}_p} V$ by semistability of V as a G_K-representation, whence $\dim_{\mathbf{Q}_p} V^{\mathrm{Gal}(L/K)} = \dim_{\mathbf{Q}_p} V$ by K_0-dimension reasons. This gives $V = V^{\mathrm{Gal}(L/K)}$ as desired.

For the claim concerning semistable characters η, since semistable representations are Hodge–Tate there is a Hodge–Tate weight $n \in \mathbf{Z}$ for η. It is harmless to twist by the crystalline (hence semistable) representation $\mathbf{Q}_p(-n)$, so we may assume that η has Hodge–Tate weight 0. Thus, by Theorem 2.2.7 we see that $\eta(G_K)$ is finite (as $G_K = I_K$). That is, $\ker \eta$ is open. By the first part of the corollary, it follows that $\eta = 1$. ∎

Lemma 9.3.3. *Let k be an algebraically closed field with characteristic $p > 0$. The map $\mathrm{W}(k)^\times \to \mathrm{W}(k)^\times$ defined by $w \mapsto \sigma(w)/w$ is surjective, where σ is the Frobenius automorphism of $\mathrm{W}(k)$.*

Proof. We shall argue by successive approximation. More specifically, we will prove that if $u \in \mathrm{W}(k)^\times$ with $u \equiv 1 \bmod p^n$ (for $n \geqslant 0$) then we can find $w \in \mathrm{W}(k)^\times$ with $w \equiv 1 \bmod p^n$ such that $\sigma(w)/w = u$. By a simple limit process with infinite products, it suffices to find w such that $\sigma(w)/w \equiv u \bmod p^{n+1}$. In case $n = 0$ this says that the map $x \mapsto x^{p-1}$ is a surjection from $k^\times$ to itself, which holds since k is algebraically closed.

For $n \geqslant 1$ we can write $u = 1 + p^n u_n$ for some $u_n \in \mathrm{W}(k)$, and the hypothetical w must have the form $w = 1 + p^n w_n$ with $w_n \in \mathrm{W}(k)$ such that $\sigma(w_n)/w_n \equiv 1 + p^n u_n \bmod p^{n+1}$. But $\sigma(w)/w \equiv 1 + p^n(\sigma(w_n) - w_n) \bmod p^{n+1}$ for any $w = 1 + p^n w_n$, so we just need that $u_n \bmod p \in k$ has the form $x^p - x$ for some $x \in k$. This holds since k is algebraically closed. ∎

Now we come to a key theorem that explains the interest in weak admissibility.

Theorem 9.3.4. *If $V \in \mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{st}}(G_K)$ then $D_{\mathrm{st}}(V) \in \mathrm{MF}_K^{\phi,N}$ is weakly admissible. In particular, if V is crystalline then $D_{\mathrm{cris}}(V) \in \mathrm{MF}_K^{\phi}$ is weakly admissible.*

Proof. Since weak admissibility is insensitive to the scalar extension $K_0 \to \widehat{K_0^{\mathrm{un}}}$, by Proposition 9.3.1 we may assume that k is algebraically closed. We let $D = D_{\mathrm{st}}(V)$ and let $D' \subseteq D$ be a subobject. We need to prove that $t_H(D') \leqslant t_N(D')$ with equality in case $D' = D$. We may assume $D' \neq 0$, so $d' = \dim D' > 0$. As a first step, we use determinant arguments to reduce to the case $d' = 1$ (so D' can be described in concrete terms). Note that $\wedge^{d'}(V)$ is semistable (being a quotient of $V^{\otimes d'}$), so $\wedge^{d'}(D)$ is naturally identified with

$D_{\rm st}(\wedge^{d'}(V))$. Also, $\det D' = \wedge^{d'}(D')$ is naturally a 1-dimensional subobject of $\wedge^{d'}(D)$. Since $t_H(D') = t_H(\det D')$ and $t_N(D') = t_N(\det D')$, we may therefore pass to $\wedge^{d'}(V)$ to reduce to the case $\dim D' = 1$.

In case $D' = D$ we have $\dim V = 1$, so $V = \mathbf{Q}_p(n)$ for some $n \in \mathbf{Z}$ (as k is algebraically closed). In this case we see with the help of $t^{-n} \in B_{\rm cris}^\times$ that $t_H(D) = t_N(D) = -n$ (as we are using the *covariant* Fontaine functors $D_{\rm st}$ and $D_{\rm cris}$). Thus, it remains to show that in general $t_H(D') \leqslant t_N(D')$. Let $e' \in D'$ be a K_0-basis, so $\phi(e') = \lambda e'$ for some $\lambda \in K_0^\times$ and $t_N(D') = {\rm ord}_p(\lambda)$. Also, $N(e') = 0$ since $N_{D'} = N_D|_{D'}$ is a nilpotent operator on a 1-dimensional space. Let $s = t_H(D')$, so $e' \in {\rm Fil}^s(B_{\rm dR} \otimes_{\mathbf{Q}_p} V) = {\rm Fil}^s(B_{\rm dR}) \otimes_{\mathbf{Q}_p} V$ but $e' \notin {\rm Fil}^{s+1}(B_{\rm dR}) \otimes_{\mathbf{Q}_p} V$.

Pick a $\mathbf{Q}_p$-basis $\{v_1, \dots, v_n\}$ of V, so the inclusion $D' \subseteq D = (B_{\rm st} \otimes_{\mathbf{Q}_p} V)^{G_K}$ gives a unique expansion $e' = \sum b_i \otimes v_i$ for $b_i \in B_{\rm st}$. The equality $\lambda e' = \phi(e') = \sum \phi(b_i) \otimes v_i$ gives $\phi(b_i) = \lambda b_i$ for all i, and the vanishing of $N(e') = \sum N(b_i) \otimes v_i$ gives $N(b_i) = 0$ for all i. In particular, $b_i \in B_{\rm st}^{N=0} = B_{\rm cris}$ for all i. Since $e' \in {\rm Fil}^s(B_{\rm dR}) \otimes_{\mathbf{Q}_p} V$ but $e' \notin {\rm Fil}^{s+1}(B_{\rm dR}) \otimes_{\mathbf{Q}_p} V$, we conclude that $b_i \in {\rm Fil}^s(B_{\rm cris})$ for all i but $b_{i_0} \notin {\rm Fil}^{s+1}(B_{\rm cris})$ for some i_0. Focusing on b_{i_0}, it suffices to show generally that if $b \in B_{\rm cris}$ lies in ${\rm Fil}^s(B_{\rm cris})$ but not in ${\rm Fil}^{s+1}(B_{\rm cris})$ (so $b \neq 0$) and $\phi(b) = \lambda b$ for $\lambda \in K_0^\times$ then $s \leqslant {\rm ord}_p(\lambda)$.

We assume to the contrary, so $s \geqslant {\rm ord}_p(\lambda) + 1$. Let $n = {\rm ord}_p(\lambda)$, so $b \in {\rm Fil}^s(B_{\rm cris}) \subseteq {\rm Fil}^{n+1}(B_{\rm cris})$. To get a contradiction, it suffices to show that the only $b \in {\rm Fil}^{n+1}(B_{\rm cris})$ such that $\phi(b) = \lambda b$ with $n = {\rm ord}_p(\lambda)$ is $b = 0$. We may replace b with b/t^n to reduce to the case $n = 0$. Hence, $b \in {\rm Fil}^1(B_{\rm cris})$ and $\phi(b) = ub$ with $u \in {\rm W}(k)^\times$. But k is algebraically closed, so $u = \sigma(u')/u'$ for some $u' \in {\rm W}(k)^\times$ (Lemma 9.3.3). Thus, $b/u' \in {\rm Fil}^1(B_{\rm cris})^{\phi=1}$. But ${\rm Fil}^0(B_{\rm cris})^{\phi=1} = \mathbf{Q}_p$ by Theorem 9.1.10, and this meets ${\rm Fil}^1(B_{\rm cris})$ in 0. ■

In Example 6.3.9 we saw that $D_{\rm dR}$ is not fully faithful, due to the de Rham property being insensitive to replacing G_K with $G_{K'}$ for a finite extension K'/K. This is best explained by a fundamental result independently due to Berger and André–Kedlaya–Mebkhout that relates p-adic differential equations with de Rham representations to prove Fontaine's potential semistability conjecture:

Theorem 9.3.5. *A p-adic representation V of G_K is de Rham if and only if it it potentially semistable in the sense that V is a semistable $G_{K'}$-representation for some finite extension K'/K.*

This theorem implies that although we cannot invert the functor $D_{\rm dR}$, the gap between de Rham representations and semistable representations amounts to an insensitivity to finite extensions of K. However, keep in mind that $D_{\rm dR}(V)$ contains too little information even to recover V as a $G_{K'}$-representation for some unknown finite extension of K', as we see by considering $V = \mathbf{Q}_p(\eta)$ for an unramified $\eta : G_K \to \mathbf{Q}_p^\times$ with infinite image (in which case $D_{\rm dR}(V) = K[0] = D_{\rm dR}(\mathbf{Q}_p)$).

A fundamental result of Colmez and Fontaine [14, Thm. A] is that the fully faithful and exact tensor functor $D_{\rm st} : {\rm Rep}_{\mathbf{Q}_p}^{\rm st}(G_K) \to {\rm MF}_K^{\phi,N,{\rm w.a.}}$ is an *equivalence*. That is, every weakly admissible filtered (ϕ, N)-module D over K is isomorphic as such to $D_{\rm st}(V)$ for a semistable p-adic representation V of G_K. In principle we know what V has to be: necessarily $V \simeq V_{\rm st}(D)$. But it is not a priori obvious that for every weakly admissible D we at least have that $V_{\mathrm{st}}(D) \in \mathrm{Rep}^{\mathrm{st}}_{\mathbf{Q}_p}(G_K)$ (in particular, it is not obvious if $V_{\mathrm{st}}(D)$ is *finite-dimensional* with continuous G_K-action), nor is it obvious that $D \simeq D_{\mathrm{st}}(V_{\mathrm{st}}(D))$ for weakly admissible D. A nice preliminary observation of Colmez and Fontaine that will be proved below is that $V_{\mathrm{st}}(D)$ is *always* in $\mathrm{Rep}^{\mathrm{st}}_{\mathbf{Q}_p}(G_K)$ for any weakly admissible $D \in \mathrm{MF}_K^{\phi,N}$, and that as long as $\dim_{\mathbf{Q}_p} V_{\mathrm{st}}(D) \geqslant \dim_{K_0}(D)$ it is automatic that $D \simeq D_{\mathrm{st}}(V_{\mathrm{st}}(D))$. In other words, the real problem is to prove that $V_{\mathrm{st}}(D)$ is "big" (and in particular nonzero when $D \neq 0$). Kisin's alternative proof [30, Prop. 2.1.5] of the Colmez–Fontaine theorem via integral p-adic Hodge theory also uses this bigness criterion for a weakly admissible module to arise from a semistable representation; we will sketch Kisin's proof in §11 (especially §11.3).

In the remainder of this section, we take up the proof that $V_{\mathrm{st}}(D)$ is always in $\mathrm{Rep}^{\mathrm{st}}_{\mathbf{Q}_p}(G_K)$ for any weakly admissible D (in particular, it is in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$), and we also prove the Colmez–Fontaine lemma that $\dim_{\mathbf{Q}_p} V_{\mathrm{st}}(D) \leqslant \dim_{K_0} D$ for all weakly admissible D, with equality if and only if $D \simeq D_{\mathrm{st}}(V_{\mathrm{st}}(D))$ in $\mathrm{MF}_K^{\phi,N}$. By considering D for which $N_D = 0$ the analogous conclusions for D_{cris}, V_{cris}, and crystalline representations are obtained, so we will say nothing further about the crystalline case.

To get started, we first consider an arbitrary $D \in \mathrm{MF}_K^{\phi,N}$ without a weak admissibility hypothesis. The $\mathbf{Q}_p[G_K]$-module $V_{\mathrm{st}}(D)$ might be infinite-dimensional, but we claim that if it is finite-dimensional then its natural G_K-action is automatically continuous for the natural topology as such a vector space. More generally, we claim that any G_K-stable finite-dimensional $\mathbf{Q}_p$-subspace of $V_{\mathrm{st}}(D)$ has continuous G_K-action (i.e., lies in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$). Since the rings B_{st} and B_{cris} do not have a useful natural topology, this continuity claim requires some thought. By definition $V_{\mathrm{st}}(D) \subseteq B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} D$ with the G_K-action doing nothing to D, so it suffices to prove:

Proposition 9.3.6. *For any $n \geqslant 1$, any $\mathbf{Q}_p[G_K]$-submodule V of B_{st}^n with finite $\mathbf{Q}_p$-dimension has continuous G_K-action relative to its natural p-adic topology.*

Proof. Consider the usual non-canonical presentation $B_{\mathrm{st}} \simeq B_{\mathrm{cris}}[X]$ (resting on a choice of $\widetilde{q} \in \mathfrak{m}_R - \{0\}$ with $\widetilde{q}^{(0)} \in \mathscr{O}_K$). The B_{cris}-submodule $B_{\mathrm{cris}}[X]_{<d}$ of polynomials with degree below a given bound $d \geqslant 1$ is G_K-stable because $g(X) = X + \eta(g)t$ for a suitable continuous $\eta : G_K \to \mathbf{Z}_p$ depending on a choice of $t = \log([\varepsilon])$. The finite-dimensional $\mathbf{Q}_p$-subspace $V \subseteq B_{\mathrm{st}}^n$ is contained in the finite free B_{cris}-submodule $B_{\mathrm{cris}}[X]_{<d}^n$ for some $d \geqslant 1$, but beware that $B_{\mathrm{cris}}[X]_{<d}^n$ is not G_K-equivariantly identified with B_{cris}^{nd} via the basis of vectors in standard monomials when $d > 1$.

Since $B_{\mathrm{cris}} = A_{\mathrm{cris}}[1/t]$ and $\dim_{\mathbf{Q}_p} V$ is finite, the t-denominators needed to describe V are bounded: for some $M \gg 0$ we have $V \subseteq \mathbf{Q}_p \cdot t^{-M} A_{\mathrm{cris}}[X]_{<d}^n$. The action by G_K on t is through the $\mathbf{Z}_p^\times$-valued continuous χ, so we can replace V with $t^M V$ for some $M \gg 0$ to arrange that V is generated over $\mathbf{Q}_p$ by the G_K-stable

$$\Lambda := V \cap A_{\mathrm{cris}}[X]_{<d}^n.$$

This $\mathbf{Z}_p$-submodule of V contains no infinitely p-divisible elements because A_{cris} is p-adically separated, so it follows that Λ must be *finitely generated* over $\mathbf{Z}_p$ and hence is a $\mathbf{Z}_p$-lattice in V. Thus, it suffices to prove that the G_K-action on Λ is continuous for the p-adic topology of Λ.

Let $\Lambda_r = \Lambda \cap (p^r A_{\mathrm{cris}}[X]^n_{<d})$, so $p^r\Lambda \subseteq \Lambda_r \subseteq \Lambda$ and Λ_r is G_K-stable. Since A_{cris} is p-adically separated we have $\cap_r \Lambda_r = 0$, so by a result of Chevalley [35, Exer. 8.7] it follows that the Λ_r's cut out the p-adic topology of Λ. Thus, our problem is reduced to showing that for each $r \geqslant 1$ the G_K-action on each finite quotient Λ/Λ_r is discrete, which is to say that points have open stabilizers. Fix such an r. The finite quotient Λ/Λ_r is naturally contained in $(A_{\mathrm{cris}}/(p^r))[X]^n_{<d}$ (with $g(X) = X + \eta(g)t$ for $g \in G_K$), so we are reduced to proving that if an element of $(A_{\mathrm{cris}}/(p^r))[X]^n_{<d}$ has a *finite* G_K-orbit then it has an open stabilizer. We will show that all orbits are finite with open stabilizer. By projection to factors of this direct sum of truncated polynomial modules, we can assume $n = 1$.

We may replace K with the finite Galois extension corresponding to $\ker(\eta \bmod p^r)$, which is to say that we can assume that the additive character $\eta \bmod p^r$ vanishes. Hence, the G_K-action on $X \bmod p^r$ has now been eliminated, so we can project to monomial coefficients in each separate X-degree less than d, which is to say that we are reduced to proving that every G_K-orbit in $A_{\mathrm{cris}}/(p^r)$ has an open stabilizer (and hence is finite) for each $r \geqslant 1$. This is Proposition 9.1.2. ∎

Now we turn to the task of analyzing $V_{\mathrm{st}}(D)$ when D is weakly admissible. The case $\dim_{K_0} D = 1$ will be analyzed first, both as a warm-up to the general case and because it is used in the treatment of the general case.

Lemma 9.3.7. *If D is an arbitrary filtered (ϕ, N)-module over K with $\dim_{K_0}(D) = 1$ then $V_{\mathrm{st}}(D)$ is 1-dimensional when D is weakly admissible (i.e., $t_H(D) = t_N(D)$), it vanishes when $t_H(D) < t_N(D)$, and it is infinite-dimensional when $t_H(D) > t_N(D)$.*

Proof. We have $D = K_0 d$ with $\phi(d) = \lambda d$ for some $\lambda \in K_0^\times$. The monodromy operator vanishes on D since it is nilpotent and $\dim_{K_0} D = 1$. By definition $t_N(D) = \mathrm{ord}_p(\lambda) \in \mathbf{Z}$ and $\mathrm{Fil}^{t_H(D)}(D_K) = D_K$, $\mathrm{Fil}^{t_H(D)+1}(D_K) = 0$. Since $\dim_{K_0}(D) = 1$, we have that D is weakly admissible if and only if $t_H(D) = t_N(D)$. We wish to relate the (possibly infinite) $\mathbf{Q}_p$-dimension of $V_{\mathrm{st}}(D)$ to the nature of the difference $t_H(D) - t_N(D)$.

Let us compute $V_{\mathrm{st}}(D)$ in general, using the K_0-basis $\{d\}$ of D. Elements of this space are elements $x \in B_{\mathrm{st}} \otimes_{K_0} D$ such that $\phi(x) = x$, $N(x) = 0$, and

$$x \in \mathrm{Fil}^0(B_{\mathrm{st}} \otimes_{K_0} D) = \mathrm{Fil}^{-t_H(D)}(B_{\mathrm{st}}) \otimes_{K_0} D.$$

In particular, $x \in B_{\mathrm{cris}} \otimes_{K_0} D$, so $x = b \otimes d$ for a unique $b \in \mathrm{Fil}^{-t_H(D)}(B_{\mathrm{cris}})$ such that $\phi(b) = b/\lambda$. We can write $\lambda = p^m u$ for $m = t_N(D)$ and $u \in \mathscr{O}_{K_0}^\times = \mathrm{W}(k)^\times$. Letting $b' = t^{t_H(D)} b \in B_{\mathrm{cris}}$, the conditions are that $b' \in \mathrm{Fil}^0(B_{\mathrm{cris}})$ with $\phi(b') = p^{t_H(D) - t_N(D)}(b'/u)$.

By Lemma 9.3.3, we may choose $w \in \mathrm{W}(\overline{k})^\times$ such that $\sigma(w)/w = u$. Replace b' with $b'' = wb'$, so $V_{\mathrm{st}}(D)$ as a $\mathbf{Q}_p$-vector space is identified the set of elements $b'' \in \mathrm{Fil}^0(B_{\mathrm{cris}})$ such that $\phi(b'') = p^{t_H(D)-t_N(D)} b''$. Thus, in the weakly admissible case (i.e., $t_H(D) = t_N(D)$) the condition on b'' says exactly that $b'' \in \mathrm{Fil}^0(B_{\mathrm{cris}})^{\phi=1} = \mathbf{Q}_p$, so $\dim V_{\mathrm{st}}(D) = 1$ in such cases. In general, if $r := t_H(D) - t_N(D)$ then $\phi(b''/t^r) = b''/t^r$, so if $r < 0$ then $b''/t^r \in \mathrm{Fil}^{-r}(B_{\mathrm{cris}}) \subseteq \mathrm{Fil}^1(B_{\mathrm{cris}})$ is a ϕ-invariant vector and thus vanishes (as the only ϕ-invariant elements of $\mathrm{Fil}^0(B_{\mathrm{cris}})$ are elements of $\mathbf{Q}_p$, none of which lie in $\mathrm{Fil}^1(B_{\mathrm{cris}})$ except for the element 0). Hence, b'' vanishes when $r < 0$. The remaining case is when $r > 0$, in which case $b''/t^r \in \mathrm{Fil}^{-r}(B_{\mathrm{cris}})$ is a ϕ-invariant vector, and the space of these is infinite-dimensional due to the so-called *fundamental exact sequence*

$$0 \to \mathbf{Q}_p \to \mathrm{Fil}^{-r}(B_{\mathrm{cris}})^{\phi=1} \to \mathrm{Fil}^{-r}(B_{\mathrm{dR}})/B_{\mathrm{dR}}^{+} \to 0 \tag{9.3.1}$$

that is valid for all $r \geqslant 0$. A proof of this exactness can be found in [14, Prop. 1.3(v)]; the essential content of its proof is already needed to handle the case $r = 0$ (i.e., to prove that $\mathrm{Fil}^0(B_{\mathrm{cris}})^{\phi=1} = \mathbf{Q}_p$). ■

Definition 9.3.8. An object $D \in \mathrm{MF}_K^{\phi,N}$ is *admissible* if $D \simeq D_{\mathrm{st}}(V)$ for some $V \in \mathrm{Rep}_{\mathrm{st}}(G_K)$.

By Theorem 9.3.4, admissible objects in $\mathrm{MF}_K^{\phi,N}$ are weakly admissible. The following preliminary result generalizing Lemma 9.3.7 is a small piece of the proof of the general result (due originally to Colmez and Fontaine) that weakly admissible filtered (ϕ, N)-modules are always admissible.

Proposition 9.3.9 (Colmez–Fontaine)**.** *Let $D \in \mathrm{MF}_K^{\phi,N}$ be weakly admissible. The vector space $V_{\mathrm{st}}(D)$ is finite-dimensional over $\mathbf{Q}_p$ with dimension at most $\dim_{K_0}(D)$, and it is semistable as a p-adic representation of G_K. Moreover, $D' := D_{\mathrm{st}}(V_{\mathrm{st}}(D))$ is naturally identified with a subobject of D, and D is admissible if and only if $\dim_{\mathbf{Q}_p}(V_{\mathrm{st}}(D)) = \dim_{K_0}(D)$, or equivalently $D' = D$.*

In such cases, the natural map

$$\delta(D) : (B_{\mathrm{st}} \otimes_{K_0} D)^{N=0,\phi=1} \to (B_{\mathrm{dR}} \otimes_K D_K)/\,\mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_K D_K)$$

from (9.2.3) *is surjective.* [2]

Proof. Let C_{st} denote the fraction field of the domain B_{st} and let $V = V_{\mathrm{st}}(D)$. We do not yet know if V has finite $\mathbf{Q}_p$-dimension. The key idea is to work with vector spaces over C_{st} rather than just with modules over B_{st}. Within the C_{st}-vector space $C_{\mathrm{st}} \otimes_{K_0} D$ of dimension $s := \dim_{K_0} D$, the $\mathbf{Q}_p$-subspace V generates a C_{st}-subspace V' of some dimension $r \leqslant s$. It is trivial to handle the case $r = 0$ (i.e., $V_{\mathrm{st}}(D) = 0$), so we now may and do assume $r > 0$. (Strictly speaking, in what follows the case $r = 0$ goes through without a problem.)

The action by G_K on $C_{\mathrm{st}} \otimes_{K_0} D$ preserves the C_{st}-subspace V'. View V' as a C_{st}-valued point of a Grassmannian variety $\mathbf{G}_r(D)$ over K_0 parameterizing r-dimensional subspaces of D. This point is invariant by G_K and so it descends to a $C_{\mathrm{st}}^{G_K}$-valued point. By the $(\mathbf{Q}_p, G_K)$-regularity of B_{st}, $C_{\mathrm{st}}^{G_K} = B_{\mathrm{st}}^{G_K} = K_0$, so V' corresponds to a K_0-valued point of $\mathbf{G}_r(D)$, which is to say that $V' = C_{\mathrm{st}} \otimes_{K_0} D'$ for a K_0-subspace $D' \subseteq D$ with dimension r. (Rather than appealing to the general theory of Grassmannians one can do explicit basis calculations by imitating how Grassmannians are constructed in order to see this descent from C_{st} down to K_0 by more direct means.) Thus,

$$V \subseteq V' \cap (B_{\mathrm{st}} \otimes_{K_0} D) = B_{\mathrm{st}} \otimes_{K_0} D'.$$

The K_0-subspace D' in D is stable by ϕ and N since this holds after scalar extension from K_0 to C_{st}. Using the subspace filtration on $D'_K \subseteq D_K$, we thereby make D' into a filtered

[2] Should include proof of converse: if $V_{\mathrm{st}}(D)$ is finite-dimensional then it is semistable using [14, Prop. 4.5], and if also $\delta(D)$ is surjective then D is weakly admissible (using proof of [14, Prop. 5.7]).

(ϕ, N)-module over K that is a subobject of D. Since $V = V_{\mathrm{st}}(D) = \mathrm{Fil}^0(B_{\mathrm{st}} \otimes_{K_0} D)^{\phi=1,N=0}$ and $V \subseteq B_{\mathrm{st}} \otimes_{K_0} D'$, we have $V \subseteq V_{\mathrm{st}}(D') \subseteq V_{\mathrm{st}}(D) = V$, so $V = V_{\mathrm{st}}(D')$.

By definition, V' is spanned over C_{st} by V, so we can find a C_{st}-basis $\{v_1, \dots, v_r\}$ for V' consisting of elements of V; the v_i's are a maximal C_{st}-linearly independent subset of V. Thus, the map $\wedge^r_{\mathbf{Q}_p}(V) \to \wedge^r_{C_{\mathrm{st}}}(V')$ carries $v_1 \wedge \cdots \wedge v_r$ to a nonzero element, and $\wedge^r_{C_{\mathrm{st}}}(V')$ is a C_{st}-subspace of $C_{\mathrm{st}} \otimes_{K_0} \wedge^r_{K_0}(D')$, so $v_1 \wedge \cdots \wedge v_r$ has nonzero image in $C_{\mathrm{st}} \otimes_{K_0} \wedge^r(D')$. In other words, if we choose a K_0-basis $\{d_1, \dots, d_r\}$ of D' and write $v_j = \sum_i b_{ij} d_i$ with $b_{ij} \in B_{\mathrm{st}}$ (recall $V = V_{\mathrm{st}}(D') \subseteq B_{\mathrm{st}} \otimes_{K_0} D'$) then $b := \det(b_{ij}) \in B_{\mathrm{st}}$ lies in $C_{\mathrm{st}}^\times$; that is, $b \neq 0$ in B_{st}. Thus, the element

$$v_1 \wedge \cdots \wedge v_r = b d_1 \wedge \cdots \wedge d_r \in B_{\mathrm{st}} \otimes_{K_0} \wedge^r(D') \tag{9.3.2}$$

lies in the 0th filtered piece and is killed by N and fixed by ϕ since each v_j lies in $V = V_{\mathrm{st}}(D')$. Hence, we have produced a nonzero element of $V_{\mathrm{st}}(\wedge^r(D'))$. But $\wedge^r(D')$ is a 1-dimensional filtered (ϕ, N)-module over K. Since we have exhibited a nonzero element of $V_{\mathrm{st}}(\wedge^r(D'))$, by Lemma 9.3.7 we cannot have $t_H(\wedge^r(D')) < t_N(\wedge^r(D'))$, or in other words the case $t_H(D') < t_N(D')$ cannot occur. The weak admissibility hypothesis on D implies $t_H(D') \leqslant t_N(D')$ for the subobject $D' \subseteq D$, so $t_H(D') = t_N(D')$. Hence, D' is weakly admissible (as D is) and $V_{\mathrm{st}}(\wedge^r(D'))$ must be exactly 1-dimensional over $\mathbf{Q}_p$.

Any r-fold wedge product of elements of $V = V_{\mathrm{st}}(D) = V_{\mathrm{st}}(D')$ is naturally an element of $V_{\mathrm{st}}(\wedge^r(D'))$, and so is a unique $\mathbf{Q}_p$-*multiple* of $v_1 \wedge \cdots \wedge v_r$. But we can view this wedge product as being formed over B_{st} within $B_{\mathrm{st}} \otimes_{K_0} \wedge^r(D')$, so if an element $v \in V \subseteq V'$ is arbitrary and we write (as we may) $v = \sum c_i v_i$ with unique $c_i \in C_{\mathrm{st}}$ then

$$v_1 \wedge \cdots \wedge v_{i-1} \wedge v \wedge v_{i+1} \wedge \cdots \wedge v_r = c_i(v_1 \wedge \cdots \wedge v_r).$$

Hence, $c_i \in \mathbf{Q}_p$ for all i. This shows that the v_i's span V over $\mathbf{Q}_p$, so they are a basis for V (as they are even linearly independent over C_{st}). In other words, V has *finite* $\mathbf{Q}_p$-dimension that is equal to $r = \dim_{K_0}(D') \leqslant \dim_{K_0}(D)$, and V must then have continuous G_K-action by Proposition 9.3.6.

The identity (9.3.2) now implies that G_K acts on b through a $\mathbf{Q}_p^\times$-valued character, so $\mathbf{Q}_p b \subseteq B_{\mathrm{st}}$ is a G_K-stable line. Hence, by $(\mathbf{Q}_p, G_K)$-regularity of B_{st} we must have that $b \in B_{\mathrm{st}}^\times$. It therefore follows from (9.3.2) that the $\mathbf{Q}_p$-basis $\{v_1, \dots, v_r\}$ for $V = V_{\mathrm{st}}(D')$ is also a B_{st}-basis of $B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} D'$, so the B_{st}-linear map $B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V \to B_{\mathrm{st}} \otimes_{K_0} D'$ induced by the identification $V = V_{\mathrm{st}}(D')$ is actually a linear isomorphism. By G_K-compatibility, we deduce that as K_0-vector spaces

$$D_{\mathrm{st}}(V) \simeq (B_{\mathrm{st}} \otimes_{K_0} D')^{G_K} = B_{\mathrm{st}}^{G_K} \otimes_{K_0} D' = D'. \tag{9.3.3}$$

This shows that $D_{\mathrm{st}}(V)$ has K_0-dimension equal to $\dim_{K_0}(D') = r = \dim_{\mathbf{Q}_p}(V)$, so V is a semistable p-adic representation of G_K with dimension $r \leqslant \dim_{K_0}(D')$.

The identification $D_{\mathrm{st}}(V) = D'$ in (9.3.3) visibly respects the Frobenius and monodromy operators, and carries $\mathrm{Fil}^j(D_{\mathrm{st}}(V))$ into $\mathrm{Fil}^j(D')$ for all j (since the B_{st}-linear isomorphism $B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V \simeq B_{\mathrm{st}} \otimes_{K_0} D'$ carries the jth filtered piece into the jth filtered piece for all j, due to the identification $V = V_{\mathrm{st}}(D')$ within $\mathrm{Fil}^0(B_{\mathrm{st}} \otimes_{K_0} D')$). But D' is weakly admissible, and $D_{\mathrm{st}}(V)$ is also weakly admissible since we have proved that D_{st} carries any semistable p-adic representation to a weakly admissible filtered (ϕ, N)-module! Any morphism of *weakly* *admissible* filtered (ϕ, N)-modules that is a linear isomorphism on K_0-vector spaces is automatically an isomorphism in $\mathrm{MF}_K^{\phi,N}$ (i.e., it is compatible with filtrations in both directions), by Theorem 8.2.11, so $D' \simeq D_{\mathrm{st}}(V)$ as filtered (ϕ, N)-modules. We conclude that $D_{\mathrm{st}}(V)$ is naturally a subobject of D, with K_0-dimension $\dim_{\mathbf{Q}_p}(V)$. Hence, $\dim_{\mathbf{Q}_p}(V) = \dim_{K_0}(D)$ if and only if the subobject $D_{\mathrm{st}}(V) \subseteq D$ has full K_0-dimension, in which case D is admissible (arising from V). Conversely, if D is admissible, say $D \simeq D_{\mathrm{st}}(V_1)$ for a semistable p-adic representation V_1 of G_K, then $V = V_{\mathrm{st}}(D) \simeq V_{\mathrm{st}}(D_{\mathrm{st}}(V_1)) \simeq V_1$ (the final isomorphism due to the semistability of V_1). Hence, in such cases $\dim_{\mathbf{Q}_p}(V) = \dim_{\mathbf{Q}_p}(V_1) = \dim_{K_0}(D)$.

Finally, we suppose we are in the case that D is admissible, so $D = D_{\mathrm{st}}(V)$ for some semistable p-adic representation V of G_K, and we wish to prove that $\delta(D)$ is surjective. Using the deRham and semistable comparison morphisms for V, the map $\delta(D)$ is identified with the natural map

$$B_{\mathrm{st}}^{N=0,\phi=1} \otimes_{\mathbf{Q}_p} V \to (B_{\mathrm{dR}}/B_{\mathrm{dR}}^+) \otimes_{\mathbf{Q}_p} V.$$

Hence, the surjectivity is reduced to the surjectivity of the natural map $B_{\mathrm{cris}}^{\phi=1} \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$. This latter surjectivity follows by passage to the direct limit over $r \to \infty$ on (9.3.1). ■

We conclude by recording an interesting observation made by Colmez and Fontaine [14, Cor. 4.7]. Suppose that in the abelian category of weakly admissible filtered (ϕ, N)-modules, the object D is a simple object. (In particular, $D \neq 0$.) If $V := V_{\mathrm{st}}(D) \neq 0$ then the above proof realizes $D_{\mathrm{st}}(V)$ as a nonzero subobject of D, in which case it must equal D by simplicity. Hence, a weakly admissible D that is simple in $\mathrm{MF}_K^{\phi,N}$ is admissible if and only if $V_{\mathrm{st}}(D) \neq 0$!

9.4. **Exercises.**

Exercise 9.4.1. Let $\theta : \mathrm{W}(R) \twoheadrightarrow \mathscr{O}_{\mathbf{C}_K}$ be the surjection as in Proposition 4.4.2. Let $\widetilde{p} \in R = \underline{R}(\mathscr{O}_{\overline{K}}/(p))$ be a choice of compatible p-power roots of p (i.e., $\underline{p}^{(0)} = p$). Consider the explicit choice of generator $\xi = \xi_{\widetilde{p}} = [\widetilde{p}] - p$ of $\ker\theta$.

(1) Using that $\xi \cdot \mathrm{W}(R)[1/p] \cap \mathrm{W}(R) = \ker\theta = \xi \cdot \mathrm{W}(R)$ (from Proposition 4.4.3), prove rigorously that the $\mathrm{W}(R)$-module sequence

$$0 \to \mathrm{W}(R)[X^n/n!]_{n\geqslant 1} \stackrel{X-\xi}{\to} \mathrm{W}(R)[X^n/n!]_{n\geqslant 1} \to A_{\mathrm{cris}}^0 \to 0$$

is exact. This gives a concrete description of A_{cris}^0.

(2) Prove that this sequence is also exact relative to p-adic topologies (i.e., when the middle term is given the p-adic topologies then the subspace and quotient topologies on the outer terms are the p-adic topologies.) Deduce that this sequence remains exact after passing to p-adic completions. This can be useful for some studying some properties of the p-adic completion A_{cris} of A_{cris}^0 (but there are more effective methods as well, such as by introducing various auxiliary rings which sit between others and are easier to work with for making p-adic estimates).

(3) Prove that the exact sequence in (1) remains exact after reduction modulo p, and prove rigorously that for any flat $\mathbf{Z}_{(p)}$-algebra W we have

$$W[X^n/n!]_{n\geqslant 1} \simeq W[Y_0, Y_1, \dots]/(Y_j^p - c_{p,j}Y_{j+1})_{j\geqslant 0}$$

where $c_{p,j} = (p^{j+1})!/(p^j!)^p \in p\mathbf{Z}_{(p)}^\times$. Deduce the important description

$$A_{\mathrm{cris}}/pA_{\mathrm{cris}} \simeq A^0_{\mathrm{cris}}/pA^0_{\mathrm{cris}} \simeq (R/(\widetilde{p}^p))[Y_0, Y_1, \dots]/(Y_0^p, Y_1^p, \dots).$$

Exercise 9.4.2. Rigorously prove the following fact that was used in Example 9.2.8: for a unit $u \in \mathscr{O}_K^\times$ that is not a root of unity, the field generated by its p-power roots is not abelian over K. (Hint: reduce to the case when $u \in 1 + \mathfrak{m}_K$ and pass to the inverse limit on the Kummer theory isomorphisms $K^\times/(K^\times)^p \simeq \mathrm{H}^1(G_K, \mathbf{Z}/(p^n))$. Then invert p and interpret the meaning of this isomorphism in terms of extension classes. Make sure your proof actually uses that u is not a root of unity, and keep in mind that K may contain some nontrivial p-power roots of unity.)

Part III. Integral p-adic Hodge theory

For many purposes (such as in Galois deformation theory with artinian coefficients) it is useful to have a finer theory in which p-adic vector spaces are replaced with lattices or torsion modules. Fontaine and Laffaille gave such a theory in the early 1980's under stringent restrictions on the Hodge-Tate weights and absolute ramification in K. The aim of Part III is to explain the more recent theory of integral p-adic Hodge theory, largely due to Breuil and Kisin, that has no ramification or weight restrictions. This is essentially a survey of [30], to which the reader should turn for more details.

For the entirety of Part III we fix a choice of uniformizer π of K, and let $E \in W[u]$ be the minimal polynomial of π over K_0. Finally, Δ denotes the rigid-analytic open unit disc over K_0 (not over K, when $e(K) > 1$), so the points of Δ are identified with the orbits of $\mathrm{Gal}(\overline{K}/K_0)$ acting on the set

$$\{x \in \overline{K} \mid |x| < 1\}.$$

The reader who is not familiar with rigid-analytic geometry should regard it as analytic analogue of working with an algebraic scheme over a field that is not algebraically closed: there are many non-rational points. The actual rigid-analytic spaces we will use are rather concrete: certain open and closed discs and annuli inside of the open unit disc. It is rings of convergent series on such discs that are of most relevance to us, and the convergence conditions can be described by explicit growth and decay properties of coefficients of formal power series or formal Laurent series over K_0. However, the geometric viewpoint is more helpful than a purely algebraic one; Using discs consisting solely of K_0-rational points or K-rational points will be insufficient.

10. Categories of linear algebra data

Our first main goal is to imbue the category $\mathrm{MF}_K^{\phi,N}$ of filtered (ϕ, N)-modules with a beautiful geometric interpretation. Following an idea originally due to Berger, we shall introduce a certain category of vector bundles (with extra structure, depending on the uniformizer $\pi \in \mathscr{O}_K$) over the rigid-analytic open unit disc Δ over K_0, and sketch the proof of the equivalence of this category with the category $\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0}$ of filtered (ϕ, N)-modules over K whose filtration is *effective* (i.e., $\mathrm{Fil}^0(M) = M$, or equivalently the associated graded module over K has its grading supported in degrees $\geqslant 0$).

Roughly speaking, the idea behind the construction of this equivalence is to show that any (effective) filtered (ϕ, N)-module D can be naturally "promoted" to a vector bundle $\mathscr{M}$ over the open unit disc Δ over K_0, with D recovered as the "fiber of $\mathscr{M}$ to the origin." See Theorem 10.2.1 for a precise statement. Using Kedlaya's theory of slopes [29] (as a black box), we then explain how to translate the condition that D be weakly admissible into a certain condition ("slope zero") on $\mathscr{M}$. This description will motivate the introducion another category of "integral" linear algebra data that enables us to study broad classes of interesting p-adic Galois representations in §11.

10.1. **Modules with φ and connection.** Fix a choice of coordinate u on Δ and let $\mathscr{O} \subseteq K_0[\![u]\!]$ be the K_0-algebra of power series that converge on Δ. For $0 < r < 1$ (and r always understood to lie in the value group $p^{\mathbf{Q}} = |\overline{K}^{\times}|$), the ring $\mathscr{O}_r$ of power series converging on the rigid-analytic closed disc $\Delta_r := \{|u| \leqslant r\}$ is equipped with the supremum norm

$$\|f\|_r := \sup_{x \in \Delta_r} |f(x)| < \infty.$$

These norms make $\mathscr{O}$ into a Fréchet space (i.e. we topologize $\mathscr{O}$ by uniform convergence on the Δ_r's for $r \to 1^-$). Concretely, $\mathscr{O}$ is the K_0-subalgebra of $K_0[\![u]\!]$ consisting of power series that converge on every closed subdisc of Δ with radius $r < 1$.

If we denote by $\varphi : \mathrm{W}(k) \to \mathrm{W}(k)$ the Frobenius automorphism of $\mathrm{W}(k)$ (lifting the Frobenius automorphism $\alpha \mapsto \alpha^p$ of the perfect field k), then φ naturally extends to an endomorphism $\varphi_{\mathscr{O}} : \mathscr{O} \to \mathscr{O}$ over φ by

$$\varphi_{\mathscr{O}}\left(\sum_{n \geqslant 0} a_n u^n\right) = \sum_{n \geqslant 0} \varphi(a_n) u^{np}.$$

Note that $\varphi_{\mathscr{O}}$ is finite and faithfully flat with degree p.

We will denote by λ the infinite product

$$\lambda := \prod_{n \geqslant 0} \varphi_{\mathscr{O}}^n \left(\frac{E(u)}{E(0)}\right), \tag{10.1.1}$$

which converges (uniformly on closed subdiscs) on Δ. (In fact, if $s(u) \in \mathrm{W}(k)[\![u]\!]\left[\frac{1}{p}\right] \subseteq \mathscr{O}$ has constant term 1, then the product $\prod_{n \geqslant 0} \varphi_{\mathscr{O}}^n(s)$ converges in $\mathscr{O}$ [28, Rem. 4.5].) Note that λ depends on the choice of uniformizer π, and that the zeroes of λ in the closed unit disc are precisely the p^nth roots of the zeroes of $E^{(\varphi^n)}$ for all $n \geqslant 0$, where $h^{(\varphi^n)}(u) = \sum_{m \geqslant 0} \varphi^n(c_m) u^m$ for $h = \sum c_m u^m \in \mathscr{O}$. We calculate

$$\varphi_{\mathscr{O}}(\lambda) = \prod_{n \geqslant 0} \varphi_{\mathscr{O}}^{n+1}\left(\frac{E(u)}{E(0)}\right) = \left(\frac{E(0)}{E(u)}\right)\lambda, \tag{10.1.2}$$

so in particular $\varphi_{\mathscr{O}}(1/\lambda) = \frac{E(u)/E(0)}{\lambda}$ and hence $\varphi_{\mathscr{O}}$ naturally acts on the ring $\mathscr{O}\left[\frac{1}{\lambda}\right]$.

The function λ should be viewed as a replacement for the p-adic logarithm; see Exercise 10.5.2 for why this is so.

Definition 10.1.1. Define the differential operator $N_{\nabla} : \mathscr{O} \to u\mathscr{O} \subseteq \mathscr{O}$ by $N_{\nabla} := -\lambda u \frac{\mathrm{d}}{\mathrm{d}u}$.

The minus sign in this definition is due to the fact that $\lambda(0) = 1$, and we cannot say more to justify this sign intervention at the outset other than that it makes certain calculations later in the theory (for semistable non-crystalline representations) work out well, such as [30, Prop. 1.7.8]. A straightforward calculation (using (10.1.2)) shows that the relation

$$N_{\nabla} \circ \varphi_{\mathscr{O}} = p\frac{E(u)}{E(0)}\varphi_{\mathscr{O}} \circ N_{\nabla} \tag{10.1.3}$$

holds, which at $u = 0$ recovers the familiar relation "$N\varphi = p\varphi N$" between Frobenius and monodromy operators in p-adic Hodge theory. Thus, we may think of the operators N_{∇} and $\varphi_{\mathscr{O}}$ as deformations of the usual N and φ.

Since K is discretely-valued, every invertible sheaf on Δ is trivial. (Indeed, for $c \in K^\times$ with $0 < |c| < 1$, the Dedekind coordinate ring of each of the exhausting discs $\{|t| \leqslant |c|^{1/n}\}$ is a UFD and hence has trivial Picard group. A line bundle on Δ therefore admits *compatible* trivializations on the Δ_r's, and hence is globally trivial, via an infinite product trick used in the proof of [11, 1.3.3]. The discreteness of $|K^\times|$ implies the exponentially decaying coefficient-estimates which ensure the convergence of the intervening infinite products.) In particular, every effective divisor on Δ is the divisor of an analytic function (which is false for more general K [25, Ex. 2.7.8]), so $\mathscr{O}$ is a *Bezout domain*; i.e. every finitely generated ideal is principal.

In general $\mathscr{O}$ is not noetherian. For example, let $\{x_n\}$ be a collection of K-points of Δ with $|x_n| \to 1$ and let the nonzero $f_r \in \mathscr{O}$ have divisor $\sum_{n\leqslant r}[x_n] + \sum_{n>r} 2[x_n]$. If the ideal $(f_r)_{r\geqslant 1}$ is finitely generated then by the Bezout property it must be principal, say (g), and g must have divisor $\sum_n [x_n]$. But such a g does not lie in the ideal $(f_r)_{r\geqslant 1}$, so we get the non-noetherian claim for $\mathscr{O}$. Nonetheless, the Bezout property for $\mathscr{O}$ ensures that finite free $\mathscr{O}$-modules behave much as if they were modules over a principal ideal domain:

Lemma 10.1.2. *Let $\mathscr{M}$ be free $\mathscr{O}$-module of finite rank, and $\mathscr{N} \subseteq \mathscr{M}$ an arbitrary submodule. The following are equivalent:*

1. *$\mathscr{N}$ is a closed submodule of $\mathscr{M}$,*
2. *$\mathscr{N}$ is finitely generated as an $\mathscr{O}$-module,*
3. *$\mathscr{N}$ is a free $\mathscr{O}$-module of finite rank.*

Proof. See [30, Lemma 1.1.4]. ∎

We remark that the implication (1) $\implies$ (3) will be especially useful for our purposes. With these preliminaries out of the way, we can now define the first category of "linear algebra data" over $\mathscr{O}$ that we shall consider.

Definition 10.1.3. Let $\mathrm{Mod}^{\varphi}_{/\mathscr{O}}$ be the category whose objects are pairs $(\mathscr{M}, \varphi_{\mathscr{M}})$ consisting of a finite free $\mathscr{O}$-module $\mathscr{M}$ and an endomorphism $\varphi_{\mathscr{M}}$ of $\mathscr{M}$ satisfying the following two conditions:

1. The map $\varphi_{\mathscr{M}} : \mathscr{M} \to \mathscr{M}$ is $\varphi_{\mathscr{O}}$-semilinear and injective.
2. The cokernel $\mathrm{coker}(1 \otimes \varphi_{\mathscr{M}})$ of the $\mathscr{O}$-linearization of $\varphi_{\mathscr{M}}$ is killed by a power E^h for some integer $h \geqslant 0$.

Morphisms in $\mathrm{Mod}^{\varphi}_{/\mathscr{O}}$ are $\mathscr{O}$-module homomorphisms that are φ-equivariant. We will abbreviate condition (2) by saying that the pair $(\mathscr{M}, \varphi_{\mathscr{M}})$ has *finite E-height*. The least integer h that works in (2) is the *E-height of $\mathscr{M}$*.

Observe that a $\varphi_{\mathscr{O}}$-semilinear operator $\varphi_{\mathscr{M}} : \mathscr{M} \to \mathscr{M}$ is injective if its $\mathscr{O}$-linearization

$$1 \otimes \varphi_{\mathscr{M}} : \varphi_{\mathscr{O}}^* \mathscr{M} = \mathscr{O} \otimes_{\mathscr{O}, \varphi_{\mathscr{O}}} \mathscr{M} \to \mathscr{M}$$

is injective, and this latter injectivity is equivalent to $\mathrm{coker}(1 \otimes \varphi_{\mathscr{M}})$ having nonzero $\mathscr{O}$-annihilator. If condition (1) in Definition 10.1.3 is satisfied and $\mathrm{ann}_{\mathscr{O}}(\mathrm{coker}(1 \otimes \varphi_{\mathscr{M}})) \neq 0$ then by arguing in terms of vector bundles we see that the cokernel of $1 \otimes \varphi_{\mathscr{M}}$ (which corresponds to a coherent sheaf on Δ that is killed by a nonzero element of $\mathscr{O}$) has discrete support in Δ. Geometrically, condition (2) in Definition 10.1.3 says that the cokernel of

$1 \otimes \varphi_{\mathscr{M}}$ is supported in the single point $\pi \in \Delta$ (recall that points of Δ correspond to $\mathrm{Gal}(\overline{K}/K_0)$-orbits of points $x \in \overline{K}$ with $|x| < 1$).

We can enhance the category $\mathrm{Mod}^{\varphi}_{/\mathscr{O}}$ by equipping a module in $\mathrm{Mod}^{\varphi}_{/\mathscr{O}}$ with the data of a *monodromy operator* over the differential operator $N_{\nabla} : \mathscr{O} \to \mathscr{O}$. This gives rise to the following category:

Definition 10.1.4. Let $\mathrm{Mod}^{\phi, N_{\nabla}}_{/\mathscr{O}}$ be the category whose objects are triples $(\mathscr{M}, \varphi_{\mathscr{M}}, N^{\mathscr{M}}_{\nabla})$ where

(1) the pair $(\mathscr{M}, \varphi_{\mathscr{M}})$ is an object of $\mathrm{Mod}^{\varphi}_{/\mathscr{O}}$,
(2) $N^{\mathscr{M}}_{\nabla} : \mathscr{M} \to \mathscr{M}$ is a K_0-linear endomorphism of $\mathscr{M}$ satisfying the relations:
   (a) for every $f \in \mathscr{O}$ and $m \in \mathscr{M}$,
$$N^{\mathscr{M}}_{\nabla}(fm) = N_{\nabla}(f)m + fN^{\mathscr{M}}_{\nabla}(m),$$
   (b)
$$N^{\mathscr{M}}_{\nabla} \circ \varphi_{\mathscr{M}} = p\frac{E(u)}{E(0)}\varphi_{\mathscr{M}} \circ N^{\mathscr{M}}_{\nabla},$$

and whose morphisms are $\mathscr{O}$-module homomorphisms that are compatible with the additional structures.

Remark 10.1.5. Given $N^{\mathscr{M}}_{\nabla} : \mathscr{M} \to \mathscr{M}$, we obtain a map
$$\nabla : \mathscr{M}\left[\frac{1}{\lambda u}\right] \to \mathscr{M}\left[\frac{1}{\lambda u}\right] \otimes_{\mathscr{O}} \Omega^1_{\Delta/K_0}$$
by defining
$$\nabla(m) := -\frac{1}{\lambda}N^{\mathscr{M}}_{\nabla}(m)\frac{\mathrm{d}u}{u},$$
where the sign is due to the appearance of the sign in the definition of the operator N_{∇} on $\mathscr{O}$. The condition (2a) in Definition 10.1.4 ensures that ∇ satisfies the Leibniz rule, and so is a meromorphic connection on $\mathscr{M}$ with at most simple poles supported in the zero locus of λu, and a straightforward calculation shows that the condition (2b) in Definition 10.1.4 guarantees that ∇ is compatible with evident actions of $\varphi_{\mathscr{M}}$. Moreover, we can reverse this construction, and associate a monodromy operator $N^{\mathscr{M}}_{\nabla}$ on $\mathscr{M}$ to any $\varphi_{\mathscr{M}}$-compatible meromorphic connection on $\mathscr{M}$ with at most simple poles supported in the zero locus of λu. Note that at $u = 0$, the relation (2b) in Definition 10.1.4 recovers the familiar relation "$N\varphi = p\varphi N$" between Frobenius and monodromy operators in p-adic Hodge theory.

Observe that both the categories $\mathrm{Mod}^{\varphi}_{/\mathscr{O}}$ and $\mathrm{Mod}^{\phi, N_{\nabla}}_{/\mathscr{O}}$ have evident notions of exactness and tensor product, and the forgetful functor from the second of these two categories to the first is neither fully faithful nor essentially surjective (but in Lemma 10.4.2 we will establish full faithfulness on the full subcategory of triples $(\mathscr{M}, \varphi_{\mathscr{M}}, N^{\mathscr{M}}_{\nabla})$ such that $N^{\mathscr{M}}_{\nabla}(\mathscr{M}) \subseteq u\mathscr{M}$). Also, neither $\mathrm{Mod}^{\varphi}_{/\mathscr{O}}$ nor $\mathrm{Mod}^{\phi, N_{\nabla}}_{/\mathscr{O}}$ is an abelian category, as the cokernel of a morphism of finite free $\mathscr{O}$-modules need not be free.

10.2. **The equivalence of categories.** In this subsection, we will discuss some ideas related to the following remarkable result:

Theorem 10.2.1. *There are exact tensor-compatible functors*

$$\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0} \underset{\underline{D}}{\overset{\underline{\mathscr{M}}}{\rightleftarrows}} \mathrm{Mod}_{/\mathscr{O}}^{\phi,N_\nabla}$$

and natural isomorphisms of functors

$$\underline{\mathscr{M}} \circ \underline{D} \xrightarrow{\simeq} \mathrm{id} \quad \textit{and} \quad \underline{D} \circ \underline{\mathscr{M}} \xrightarrow{\simeq} \mathrm{id}.$$

Remarks 10.2.2. Recall that each object of $\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0}$ is equipped with a descending, exhaustive, and separated filtration by K-subspaces. The notion of exactness in this category includes the filtration data (in the sense that an *exact sequence* of finite-dimensional filtered vector spaces is an exact sequence of vector spaces such that the natural subspace and quotient filtrations on the common kernel and image at each stage coincide). Hence, $\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0}$ is not an abelian category since maps with vanishing kernel and cokernel may fail to be filtration-compatible in the reverse direction.

The definitions of $\underline{\mathscr{M}}$ and $\underline{D}$ as module-valued functors, as well as the construction of the natural transformations as in Theorem 10.2.1, will not use N_∇. For example, the definition of $\underline{D}(\mathscr{M})$ as a K_0-vector space does not use the data of $N_\nabla^{\mathscr{M}}$ and the definition of $\underline{\mathscr{M}}(D)$ in $\mathrm{Mod}_{/\mathscr{O}}^{\varphi}$ comes before its N_∇-structure is defined. Moreover, once $\underline{\mathscr{M}}$ and $\underline{D}$ have been defined, it turns out to be straightforward to show that for any $\mathscr{M} \in \mathrm{Mod}_{/\mathscr{O}}^{\varphi}$ there is a natural map of vector bundles over Δ

$$\underline{\mathscr{M}} \circ \underline{D}(\mathscr{M}) \to \mathscr{M}$$

that is an isomorphism away from the point $\pi \in \Delta$. That this latter map is an isomorphism on π-stalks (and hence is an isomorphism) crucially uses the operator $N_\nabla^{\mathscr{M}}$.

Rather than give the proof of Theorem 10.2.1, we will content ourselves with giving the definitions of $\underline{\mathscr{M}}$ and $\underline{D}$. Moreover, we will only define $\underline{\mathscr{M}}$ on objects D with $N_D = 0$, as this simplifies the exposition. For a complete discussion, see [30, Thm. 1.2.15].

Let D be an object of $\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0}$ and denote by $\mathrm{Fil}^j D_K$ the jth filtered piece of $D_K = K \otimes_{K_0} D$. As we just noted above, to simplify the exposition of the construction of $\underline{\mathscr{M}}(D)$, we shall assume $N_D = 0$. We will define $\underline{\mathscr{M}}(D)$ as a certain $\mathscr{O}$-submodule of $\mathscr{O}\left[\frac{1}{\lambda}\right] \otimes_{K_0} D$ by imposing "polar conditions" at specific points in Δ. Roughly, we can think of elements of $\mathscr{O}\left[\frac{1}{\lambda}\right] \otimes_{K_0} D$ as certain meromorphic D-valued functions on Δ with poles supported in the divisor of λ, and we will use the additional data on D (Frobenius and filtration) to restrict the order of poles that we allow for elements of $\underline{\mathscr{M}}(D)$.

For each integer $n \geqslant 0$, let x_n be the point of Δ corresponding to the (irreducible) Eisenstein polynomial $E(u^{p^n}) \in K_0[u]$ (so x_n corresponds to the $\mathrm{Gal}(\overline{K}/K_0)$-conjugacy class of a choice of $\pi_n := \sqrt[p^n]{\pi} \in \overline{K}$). If $\mathscr{O}_{\Delta,x_n}^\wedge$ denotes the complete local ring of Δ at x_n, then the specialization map

$$\mathscr{O}_{\Delta,x_n}^\wedge \to K_0(\pi_n)$$

sending a function to its value at x_n realizes $K_0(\pi_n)$ as the residue field of $\mathscr{O}^{\wedge}_{\Delta,x_n}$. It follows that $\mathscr{O}^{\wedge}_{\Delta,x_n}$ is a complete equicharacteristic discrete valuation ring with maximal ideal $(u-\pi_n)\mathscr{O}^{\wedge}_{\Delta,x_n}$; i.e. we have a K_0-algebra isomorphism

$$\mathscr{O}^{\wedge}_{\Delta,x_n} \simeq K_0(\pi_n)[\![u-\pi_n]\!]$$

of $\mathscr{O}^{\wedge}_{\Delta,x_n}$ with the ring of x_n-centered power series over $K_0(\pi_n)$. Since

$$K_0(\pi_n) \supseteq K_0(\pi_0) = K,$$

we see that $\mathscr{O}^{\wedge}_{\Delta,x_n}$ uniquely contains K over K_0.

Denote by $\varphi_W : \mathscr{O} \to \mathscr{O}$ the "Frobenius operator" given by acting only on coefficients:

$$\varphi_W\left(\sum_{n\geqslant 0} a_n u^n\right) := \sum_{n\geqslant 0} \varphi(a_n) u^n,$$

so in particular φ_W is bijective and $\varphi_{\mathscr{O}}$ is the composition of φ_W with the pth power map $u \mapsto u^p$. From this description and the product formula (10.1.1) defining λ, we see that $\varphi_W^{-n}(\lambda)$ has a simple zero at each zero of $\varphi_W^{-n} \circ \varphi_{\mathscr{O}}^n(E(u)/E(0)) = E(u^{p^n})/E(0)$ in $\overline{K}$, and so as a function on Δ it has a simple zero at $x_n \in \Delta$. We conclude that that under the natural localization map

$$\mathscr{O} \to \mathscr{O}^{\wedge}_{\Delta,x_n} \tag{10.2.1}$$

the element $\varphi_W^{-n}(\lambda) \in \mathscr{O}$ maps to a uniformizer. Recalling that $\varphi_D : D \to D$ is bijective, the composite map

$$\mathscr{O} \otimes_{K_0} D \xrightarrow[\varphi_W^{-n}\otimes\varphi_D^{-n}]{\simeq} \mathscr{O} \otimes_{K_0} D \xrightarrow[(10.2.1)\otimes 1]{} \mathscr{O}^{\wedge}_{\Delta,x_n} \otimes_{K_0} D = \mathscr{O}^{\wedge}_{\Delta,x_n} \otimes_K D_K$$

thus induces a map

$$\iota_n : \mathscr{O}\left[\tfrac{1}{\lambda}\right] \otimes_{K_0} D \longrightarrow \mathscr{O}^{\wedge}_{\Delta,x_n}\left[\tfrac{1}{u-\pi_n}\right] \otimes_K D_K.$$

Concretely, up to the intervention of the isomorphism $\varphi_W^{-n} \otimes \varphi_D^{-n}$, the map ι_n is nothing more than the map sending a D-valued meromorphic function on Δ to its Laurent expansion at $x_n \in \Delta$.

Define

$$\underline{\mathscr{M}}(D) := \left\{ \delta \in \mathscr{O}\left[\frac{1}{\lambda}\right] \otimes_{K_0} D \;\middle|\; \iota_n(\delta) \in \sum_{j\in\mathbf{Z}} (u-\pi_n)^{-j}\,\mathrm{Fil}^j D_K \text{ for all } n \geqslant 0 \right\}.$$

Observe that the sum occurring in the definition of $\underline{\mathscr{M}}(D)$ is a *finite* sum, as $\mathrm{Fil}^j D_K = D_K$ for all $j < 0$ (D is an object of $\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0}$) and $\mathrm{Fil}^j D_K = 0$ for all j sufficiently large (the filtration on D_K is separated). Thus, this sum really makes sense as a "finite" condition on the polar part of δ at x_n.

Remark 10.2.3. Let A be any ring and let N_1 and N_2 be A-modules endowed with decreasing filtrations. Suppose that the filtration on N_2 is finite, exhaustive, and separated in the sense that $\mathrm{Fil}^j N_2 = N_2$ for $j \ll 0$ and $\mathrm{Fil}^j N_2 = 0$ for $j \gg 0$. The tensor product $N_1 \otimes_A N_2$ has a natural filtration given by

$$\mathrm{Fil}^j(N_1 \otimes_A N_2) := \sum_{m+n=j} \mathrm{image}((\mathrm{Fil}^m N_1) \otimes_A (\mathrm{Fil}^n N_2) \to N_1 \otimes_A N_2),$$

and this sum is finite because of the hypotheses on the filtration on N_2 and the fact that the filtration on N_1 is decreasing. We apply this with $A = K$, $N_2 = D_K$, and N_1 equal to the fraction field $\mathscr{O}^\wedge_{\Delta,x_n}\left[\frac{1}{u-\pi_n}\right]$ of the complete local ring $\mathscr{O}^\wedge_{\Delta,x_n}$ endowed with its natural $(u-\pi_n)$-adic filtration. The sum occurring in the definition of $\underline{\mathscr{M}}(D)$ is the $\mathscr{O}^\wedge_{\Delta,x_n}\left[\frac{1}{u-\pi_n}\right]$-module

$$\mathrm{Fil}^0\left(\mathscr{O}^\wedge_{\Delta,x_n}\left[\frac{1}{u-\pi_n}\right] \otimes_K D_K\right).$$

If $h \geqslant 0$ is any integer with $\mathrm{Fil}^{h+1} D_K = 0$, then we have $(u-\pi_n)^h \underline{\mathscr{M}}(D) \subseteq \mathscr{O} \otimes_{K_0} D$, and so (since $\iota_n(\lambda)$ is $(u-\pi_n)$ times a unit in $\mathscr{O}^\wedge_{\Delta,x_n}$)

$$\underline{\mathscr{M}}(D) \subseteq \lambda^{-h}\mathscr{O} \otimes_{K_0} D.$$

Moreover, one readily checks from consideration of finite-tailed Laurent expansions that $\underline{\mathscr{M}}(D)$ is a closed submodule of $\lambda^{-h}\mathscr{O} \otimes_{K_0} D$ (because the membership condition at each x_n in the definition of $\underline{\mathscr{M}}(D)$ is a closed condition on $\lambda^{-h}\mathscr{O} \otimes_{K_0} D$). Thus, by Lemma 10.1.2, we conclude that $\underline{\mathscr{M}}(D)$ is a finite free $\mathscr{O}$-module.

From the computation (10.1.2) we have seen that $\varphi_{\mathscr{O}}$ acts on $\mathscr{O}\left[\frac{1}{\lambda}\right]$, so a simple calculation shows that N_∇ (see Definition 10.1.1) also acts on $\mathscr{O}\left[\frac{1}{\lambda}\right]$. We define operators $\varphi_{\underline{\mathscr{M}}(D)}$ and $N_\nabla^{\underline{\mathscr{M}}(D)}$ on $\mathscr{O}\left[\frac{1}{\lambda}\right] \otimes_{K_0} D$ by the formulae

$$\varphi_{\underline{\mathscr{M}}(D)} := \varphi_{\mathscr{O}} \otimes \varphi_D \quad \text{and} \quad N_\nabla^{\underline{\mathscr{M}}(D)} := N_\nabla \otimes 1.$$

The relation (10.1.3) ensures that $\varphi_{\underline{\mathscr{M}}(D)}$ and $N_\nabla^{\underline{\mathscr{M}}(D)}$ satisfy the desired "deformation" (Definition 10.1.4(2b)) of the usual Frobenius and monodromy relation $N\varphi = p\varphi N$, and one calculates using Definition 10.1.1 that $N_\nabla^{\underline{\mathscr{M}}(D)}$ satisfies the Leibniz rule in Definition 10.1.4(2a). We remark that the above constructions can be generalized to allow for $N_D \neq 0$. (Beware that the definition of $\underline{\mathscr{M}}(D)$ must be changed if $N_D \neq 0$.) The following lemma makes no assumptions on N_D (although we have only explained the definition of $\underline{\mathscr{M}}(D)$ when $N_D = 0$).

Lemma 10.2.4. *The operators $\varphi_{\underline{\mathscr{M}}(D)}$ and $N_\nabla^{\underline{\mathscr{M}}(D)}$ preserve $\underline{\mathscr{M}}(D) \subseteq \mathscr{O}\left[\frac{1}{\lambda}\right] \otimes_{K_0} D$. Moreover, the $\mathscr{O}$-linear map*

$$1 \otimes \varphi_{\underline{\mathscr{M}}(D)} : \varphi^*_{\underline{\mathscr{M}}(D)}\underline{\mathscr{M}}(D) \to \underline{\mathscr{M}}(D)$$

is injective, and has cokernel isomorphic to

$$\bigoplus_{i \geqslant 0} \left(\mathscr{O}/E(u)^i\mathscr{O}\right)^{h_i},$$

where $h_i = \dim_K \mathrm{gr}^i D_K$ (so $\underline{\mathscr{M}}(D) \neq 0$ if $D \neq 0$).

Proof. This is essentially [30, Lemma 1.2.2]. ∎

It follows at once from Lemma 10.2.4 that $\varphi_{\underline{\mathscr{M}}(D)}$ and $N_{\nabla}^{\underline{\mathscr{M}}(D)}$ make $\underline{\mathscr{M}}(D)$ into an object of $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N_\nabla}$ and that if $D \neq 0$ then the E-height of $\underline{\mathscr{M}}(D)$ is bounded above by the largest i for which h_i is nonzero. We have thus defined the functor $\underline{\mathscr{M}}$ on objects. To define $\underline{\mathscr{M}}$ on morphisms (assuming the vanishing of the N_D's, which is the only case in which we have explained how to define $\underline{\mathscr{M}}(D)$), one must check that for any morphism $\alpha : D \to D'$ of effective filtered (ϕ, N)-modules such that $N_D = 0$ and $N_{D'} = 0$, the map

$$1 \otimes \alpha : \mathscr{O}\left[\frac{1}{\lambda}\right] \otimes_{K_0} D \to \mathscr{O}\left[\frac{1}{\lambda}\right] \otimes_{K_0} D$$

restricts to a morphism $\underline{\mathscr{M}}(\alpha) : \underline{\mathscr{M}}(D) \to \underline{\mathscr{M}}(D)$ of (ϕ, N_∇)-modules over $\mathscr{O}$; see Exercise 10.5.3.

We will define

$$\underline{D} : \mathrm{Mod}_{/\mathscr{O}}^{\phi,N_\nabla} \to \mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0}$$

by sending a (ϕ, N_∇)-module given by the data $(\mathscr{M}, \varphi_{\mathscr{M}}, N_\nabla^{\mathscr{M}})$ to its fiber at the origin of the disc:

$$\underline{D}(\mathscr{M}) := \mathscr{M}/u\mathscr{M}.$$

(Similarly, the functor $\underline{D}$ takes a morphism to its specialization at $u = 0$.) We equip $\underline{D}(\mathscr{M})$ with Frobenius and monodromy operators

$$\varphi := \varphi_{\mathscr{M}} \mod u \quad \text{and} \quad N := N_\nabla^{\mathscr{M}} \mod u.$$

Observe that $\mathscr{M}/u\mathscr{M}$ is a finite-dimensional K_0-vector space, and that $N\varphi = p\varphi N$ thanks to Definition 10.1.4(2b).

In order to show that $\underline{D}(\mathscr{M}) := \mathscr{M}/u\mathscr{M}$ is an object of $\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0}$, we must equip the K-vector space $\underline{D}(\mathscr{M})_K$ with an effective filtration. To do this, we proceed as follows. Recall that we have normalized $|\cdot|$ on $\overline{K}$ by $|p| = 1/p$. For any $r \in (|\pi|, 1)$ that is in the value group $p^{\mathbf{Q}}$ of the absolute value on $\overline{K}^\times$, "specialization at π" defines a map

$$\underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O}_r \to \underline{D}(\mathscr{M}) \otimes_{K_0} (\mathscr{O}_r/E(u)\mathscr{O}_r) = \underline{D}(\mathscr{M}) \otimes_{K_0} K = \underline{D}(\mathscr{M})_K \tag{10.2.2}$$

(recall from §10.1 that $\mathscr{O}_r$ is the ring of power series over K_0 converging on the closed rigid-analytic disc Δ_r of radius r over K_0 centered at the origin).

For any $r \in p^{\mathbf{Q}}$ with $r < 1$ we write $(\cdot)|_{\Delta_r}$ to denote the functor $(\cdot) \otimes_{\mathscr{O}} \mathscr{O}_r$ from $\mathscr{O}$-modules to $\mathscr{O}_r$-modules. If $|\pi| < r < |\pi|^{1/p}$ then we will define an infinite descending filtration on the left side of (10.2.2) by $\mathscr{O}_r$-submodules. The K-linear pushforward of this filtration will be the desired filtration on $\underline{D}(\mathscr{M})_K$; it is independent of the choice of such r. The definition of this filtration of $\underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O}_r$ by $\mathscr{O}_r$-submodules requires:

Lemma 10.2.5. *Let $\mathscr{M}$ be any object of $\mathrm{Mod}_{/\mathscr{O}}^{\varphi}$ with E-height h.*

(1) *There exists a unique $\mathscr{O}$-linear and φ-compatible map $\xi = \xi_{\mathscr{M}}$*

$$\begin{array}{ccc} \underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O} & \xrightarrow{\xi} & \mathscr{M} \\ \circlearrowleft_{\varphi_{\underline{D}(\mathscr{M})} \otimes \varphi_{\mathscr{O}}} & & \circlearrowleft_{\varphi_{\mathscr{M}}} \end{array}$$

with the property that

$$\xi \mod u = \mathrm{id}_{\underline{D}(\mathscr{M})}.$$

(2) *The map* ξ *is injective, and* $\mathrm{coker}(\xi)$ *is killed by* λ^h.
(3) *If* $r \in (|\pi|, |\pi|^{1/p})$ *is in the value group of* $\overline{K}^{\times}$, *then* $\xi\big|_{\Delta_r}$ *has the same image in* $\mathscr{M}\big|_{\Delta_r}$ *as does the linearization*

$$1 \otimes \varphi_{\mathscr{M}} : \varphi_{\mathscr{O}}^{*}\mathscr{M} \to \mathscr{M}$$

over Δ_r.

Before sketching the proof of Lemma 10.2.5, let us apply it to define a filtration on $\underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O}_r$. It follows at once from (2) that $\xi\left[\frac{1}{\lambda}\right]$ is an isomorphism. Moreover, (3) readily implies that for r as in the Lemma, $\xi\big|_{\Delta_r}$ is an isomorphism away from every $\pi_n \in \Delta_r$ and that $\xi\big|_{\Delta_r}$ induces an isomorphism

$$\underline{D}(\mathscr{M}) \otimes \mathscr{O}_r \xrightarrow{\simeq} (1 \otimes \varphi_{\mathscr{M}})(\varphi_{\mathscr{O}}^{*}\mathscr{M})\big|_{\Delta_r}. \tag{10.2.3}$$

The right side of (10.2.3) is naturally filtered by its intersections with the $E^i\mathscr{M}\big|_{\Delta_r}$. That is, we define

$$\mathrm{Fil}^i(1 \otimes \varphi_{\mathscr{M}})(\varphi_{\mathscr{O}}^{*}\mathscr{M})\big|_{\Delta_r} := (1 \otimes \varphi_{\mathscr{M}})(\varphi_{\mathscr{O}}^{*}\mathscr{M})\big|_{\Delta_r} \cap E^i\mathscr{M}\big|_{\Delta_r}.$$

Since $\mathscr{O}_r$ is Dedekind, each Fil^i is a finite free $\mathscr{O}_r$-module. Via (10.2.3), we get a filtration on $\underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O}_r$; the image of this filtration under (10.2.2) is the desired K-linear filtration on $\underline{D}(\mathscr{M})_K$. Obviously this filtration is independent of r.

Remark 10.2.6. Note that the definition of E-height implies that

$$\mathrm{Fil}^i(1 \otimes \varphi_{\mathscr{M}})(\varphi_{\mathscr{O}}^{*}\mathscr{M})\big|_{\Delta_r} = E^i\mathscr{M}\big|_{\Delta_r}$$

for $i \geqslant h$; in particular, for $i \geqslant h$ the map $\xi\big|_{\Delta_r}$ induces an isomorphism

$$\mathrm{Fil}^i(\underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O}_r) \simeq \underline{D}(\mathscr{M}) \otimes_{K_0} E^{i-h}\mathscr{O}_r.$$

Specializing at π shows that $\mathrm{Fil}^i(\underline{D}(\mathscr{M})_K) = 0$ for all $i \geqslant h+1$.

Proof of Lemma 10.2.5. The proof of the uniqueness goes via a standard trick in the theory of "Frobenius modules"; see Exercise 10.5.4. For the existence of ξ, note that the data of ξ is equivalent to a K_0-linear section

$$s : \underline{D}(\mathscr{M}) \to \mathscr{M}$$

to the natural surjection such that

$$\varphi_{\mathscr{M}} \circ s = s \circ \varphi_{\underline{D}(\mathscr{M})}.$$

Begin by choosing *any* K_0-section $s_0 : \underline{D}(\mathscr{M}) \to \mathscr{M}$. We would like to define

$$s = \lim_{n\to\infty} \varphi_{\mathscr{M}}^{n} s_0 \varphi_{\underline{D}(\mathscr{M})}^{-n}$$

pointwise on $\underline{D}(\mathscr{M})$. To see that this limit does indeed converge pointwise, one works on a fixed lattice $\mathrm{L} \subseteq \underline{D}(\mathscr{M})$ and makes p-adic estimates in L and in $\mathscr{M}\big|_{\Delta_\rho}$ for all $\rho \in (0,1) \cap p^{\mathbf{Q}}$. By construction s is φ-compatible, so ξ exists, establishing (1).

To prove (2) and (3), we fix $r \in (|\pi|, |\pi|^{1/p}) \cap p^{\mathbf{Q}}$ and proceed as follows. Since $\xi \mod u$ is an isomorphism, it follows that $\xi\big|_{\Delta_{r^{p^i}}}$ is an isomorphism for some (possibly large) $i \geqslant 1$. By devissage, we will get to the case $i = 1$. If $i > 1$, then consider the following diagram of finite $\mathscr{O}$-modules (which we think of as coherent sheaves over Δ):

(10.2.4)
$$\begin{tikzcd} \varphi_{\mathscr{O}}^* \left(\underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O}\right) \arrow[r, "\varphi_{\mathscr{O}}^*(\xi)"] \arrow[d, "\simeq"'] & \varphi_{\mathscr{O}}^*(\mathscr{M}) \arrow[d, hook, "1\otimes\varphi_{\mathscr{M}}"] \\ \underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O} \arrow[r, "\xi"'] & \mathscr{M} \end{tikzcd}$$

Due to the fact that ξ is φ-compatible, this diagram commutes. Moreover, since the cokernel of the right vertical map $1 \otimes \varphi_{\mathscr{M}}$ is killed by E^h, where h is the E-height of $\mathscr{M}$, we see that this map is an isomorphism away from the point $\pi \in \Delta$; in particular, it is an isomorphism over $\Delta_{r^p} \supseteq \Delta_{r^{p^{i-1}}}$ since $|\pi| > r^p$.

Since $\varphi_{\mathscr{O}}^{-1}(\Delta_{r^{p^i}}) = \Delta_{r^{p^{i-1}}}$ and ξ is an isomorphism over $\Delta_{r^{p^i}}$ by hypothesis, we deduce that the top arrow $\varphi_{\mathscr{O}}^*(\xi)$ is an isomorphism over $\Delta_{r^{p^{i-1}}}$. The right vertical map is also an isomorphism over $\Delta_{r^{p^{i-1}}}$, so the bottom arrow ξ must be an isomorphism over $\Delta_{r^{p^{i-1}}}$ as well. It follows by descending induction that ξ is an isomorphism over Δ_{r^p}, and hence $\varphi_{\mathscr{O}}^*(\xi)$ is an isomorphism over Δ_r. Thus, $\xi\big|_{\Delta_r}$ is injective, so ξ is injective by analytic continuation (any element of the kernel of the $\mathscr{O}$-module map ξ must vanish over Δ_r, and therefore vanishes identically on Δ). The diagram (10.2.4) also shows that $\xi\big|_{\Delta_r}$ and $(1 \otimes \varphi_{\mathscr{M}})\big|_{\Delta_r}$ have the same image. Finally, coker $\left(\xi\big|_{\Delta_r}\right)$ is killed by E^h, as this is true of $\operatorname{coker}(1 \otimes \varphi_{\mathscr{M}})$ (by definition), so running the above analysis of the diagram (10.2.4) in reverse shows that $\varphi_{\mathscr{O}}^n(E^h)$ kills $\operatorname{coker}(\xi\big|_{\Delta_{r^{1/p^n}}})$ for all $n \geqslant 0$, and hence λ^h kills $\operatorname{coker}(\xi)$. ■

10.3. **Slopes and weak admissibility.** We now recall Kedlaya's theory of slopes [29] and apply it to translate weak admissibility across the equivalence of categories in Theorem 10.2.1. Kedlaya's theory works over a certain extension of $\mathscr{O}$, the *Robba ring*:

$$\mathscr{R} := \varinjlim_{r \to 1^-} \mathscr{O}_{\{r<|u|<1\}},$$

where $\mathscr{O}_{\{r<|u|<1\}}$ denotes the ring functions on the rigid-analytic (open) annulus $\{r < |u| < 1\}$ over K_0. Observe that the transition maps in the direct limit are injective, thanks to analytic continuation, and it follows in particular that $\mathscr{O}$ is naturally a subring of $\mathscr{R}$. We identify $\mathscr{R}$ with a certain set of formal Laurent series over K_0. The ring $\mathscr{R}$ is equipped with a Frobenius endomorphism

$$\varphi_{\mathscr{R}} : \mathscr{R} \to \mathscr{R}$$

restricting to $\varphi_{\mathscr{O}}$ on $\mathscr{O}$; the map $\varphi_{\mathscr{R}}$ is faithfully flat.

The *bounded Robba ring* is the ring

$$\mathscr{R}^{\mathrm{b}} := \varinjlim_{r \to 1^-} \mathscr{O}^{\mathrm{bnd}}_{\{r<|u|<1\}},$$

where $\mathscr{O}^{\mathrm{bnd}}_{\{r<|u|<1\}}$ denotes the subring of $\mathscr{O}_{\{r<|u|<1\}}$ consisting of those functions which are bounded. We also define

$$\mathscr{R}^{\mathrm{int}} := \left\{ \sum_{n \in \mathbf{Z}} a_n u^n \in \mathscr{R} \middle| a_n \in \mathrm{W}(k) \text{ for all } n \in \mathbf{Z} \right\}; \tag{10.3.1}$$

this is a henselian discrete valuation ring with uniformizer p.

Observe that $\mathscr{R}^{\mathrm{b}} = \mathrm{Frac}(\mathscr{R}^{\mathrm{int}})$, so in particular $\mathscr{R}^{\mathrm{b}}$ is a field. In fact, the nonzero elements of $\mathscr{R}^{\mathrm{b}}$ are *precisely* the units of $\mathscr{R}$. Moreover, since $\mathscr{R}^{\mathrm{int}}$ is henselian, roots of polynomials with coefficients in $\mathscr{R}^{\mathrm{b}}$ have canonical p-adic ordinals.

Example 10.3.1. As E is a polynomial in u with $\mathrm{W}(k)$-coefficients, we have $E \in \mathscr{R}^{\mathrm{int}} \subseteq \mathscr{R}^{\mathrm{b}}$. Since the leading coefficient of E is a unit in $\mathrm{W}(k)$, we see that $\frac{1}{p}E \in \mathscr{R}^{\mathrm{b}}$ is *not* in $\mathscr{R}^{\mathrm{int}}$. It follows that the p-adic ordinal of E is 0, so $E \in (\mathscr{R}^{\mathrm{int}})^{\times}$.

Definition 10.3.2. Let $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$ be the category whose objects are pairs (M, φ_M) with M a finite free $\mathscr{R}$-module and

$$\varphi_M : M \to M$$

a $\varphi_{\mathscr{R}}$-semilinear endomorphism whose $\mathscr{R}$-linearization $1 \otimes \varphi_M : \varphi^*_{\mathscr{R}} M \to M$ is an isomorphism. Morphisms in $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$ are φ-compatible morphisms of $\mathscr{R}$-modules. We define the category $\mathrm{Mod}^{\varphi}_{/\mathscr{R}^{\mathrm{b}}}$ similarly.

Beware that although the natural inclusion map $\mathscr{R}^{\mathrm{b}} \hookrightarrow \mathscr{R}$ allows us to view any $\mathscr{R}$-module as an $\mathscr{R}^{\mathrm{b}}$-module, $\mathscr{R}$ is *not* finitely generated as an $\mathscr{R}^{\mathrm{b}}$-module (since $\mathscr{R}^{b}$ is a field but the domain $\mathscr{R}$ is not). Hence, the induced restriction functor from the category of $\mathscr{R}$-modules to $\mathscr{R}^{\mathrm{b}}$-modules does *not* restrict to a functor from $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$ to $\mathrm{Mod}^{\varphi}_{/\mathscr{R}^{\mathrm{b}}}$.

The following example will play a crucial role in what follows:

Example 10.3.3. Let $(\mathscr{M}, \varphi_{\mathscr{M}}) \in \mathrm{Mod}^{\varphi}_{/\mathscr{O}}$. We claim that the $\mathscr{R}$-module $\mathscr{M}_{\mathscr{R}} := \mathscr{M} \otimes_{\mathscr{O}} \mathscr{R}$ equipped with $\varphi_{\mathscr{M}_{\mathscr{R}}} := \varphi_{\mathscr{M}} \otimes \varphi_{\mathscr{R}}$ is an object of $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$. Obviously the $\mathscr{R}$-module $\mathscr{M}_{\mathscr{R}}$ is free. Since $\mathscr{O} \to \mathscr{R}$ is flat and

$$1 \otimes \varphi_{\mathscr{M}} : \varphi^*_{\mathscr{O}} \mathscr{M} \to \mathscr{M}$$

is injective with cokernel killed by a power of E, we see that the $\mathscr{R}$-linearization of $\varphi_{\mathscr{M}_{\mathscr{R}}}$ is an isomorphism, as E is a unit in $\mathscr{R}$ (even in $\mathscr{R}^{\mathrm{int}}$) by Example 10.3.1.

Definition 10.3.4. Let (M, φ_M) be a nonzero object of $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$. We say that (M, φ_M) is *pure of slope zero* if it descends to an object of $\mathrm{Mod}^{\varphi}_{/\mathscr{R}^{\mathrm{b}}}$ such that the matrix of φ on the descent has all eigenvalues with p-adic ordinal 0. By a suitable twisting procedure [29, Def. 1.6.1] we define *pure of slope s* similarly, for any $s \in \mathbf{Q}$. If $(\mathscr{M}, \varphi_{\mathscr{M}})$ is a nonzero object of $\mathrm{Mod}^{\varphi}_{/\mathscr{O}}$, we say that $(\mathscr{M}, \varphi_{\mathscr{M}})$ is *pure of slope s* if $(\mathscr{M}_{\mathscr{R}}, \varphi_{\mathscr{M}_{\mathscr{R}}}) \in \mathrm{Mod}^{\varphi}_{/\mathscr{R}}$ is.

Remarks 10.3.5. (1) The notion of "pure of some slope s" is well-behaved with respect to tensor and exterior products; see [29, Cor. 1.6.4] (whose proof also applies to exterior products).

(2) The condition "pure of slope zero" is equivalent to the existence of a φ_M-stable $\mathscr{R}^{\mathrm{int}}$-lattice $L \subseteq M$ with the property that the matrix of φ_M acting on L is invertible. This follows by a lattice-saturation argument with the linearization of φ_M viewed over a sufficiently large finite extension of the fraction field $\mathscr{R}^b$ of the henselian discrete valuation ring $\mathscr{R}^{\mathrm{int}}$ (where "sufficiently large" means large enough to contain certain eigenvalues); cf. Exercise 8.4.1.

(3) Since $(\mathscr{R}^{\mathrm{b}})^\times = \mathscr{R}^\times$, a linear map $M' \to M$ of finite free $\mathscr{R}^{\mathrm{b}}$-modules is a direct summand (respectively surjective) if and only if the scalar extension $M'_{\mathscr{R}} \to M_{\mathscr{R}}$ to $\mathscr{R}$ is a direct summand (respectively surjective).

The following important theorem of Kedlaya [29, Thm. 1.7.1] elucidates the structure of $\mathscr{R}$-modules.

Theorem 10.3.6 (Kedlaya)**.** *For any nonzero object (M, φ_M) of $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$, there exists a unique filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$$

in $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$ such that each successive quotient M_i/M_{i-1} is a nonzero object of $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$ that is pure of slope s_i, with the rational numbers s_i satisfying

$$s_1 < s_2 < \cdots < s_r.$$

The filtration on a nonzero object (M, φ_M) guaranteed by Kedlaya's theorem is called the *slope filtration* of M. Observe that in contrast with the slope decomposition in the Dieudonné-Manin classification of isocrystals over $\widehat{K_0^{\mathrm{un}}}$ (Theorem 8.1.4), here we have just a filtration rather than a direct sum decomposition.

Given a nonzero object $\mathscr{M}$ of $\mathrm{Mod}^{\phi,N_\nabla}_{/\mathscr{O}}$, we know that $(\mathscr{M}_{\mathscr{R}}, \varphi_{\mathscr{M}_{\mathscr{R}}})$ is a nonzero object of $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$, and it is natural to ask if the slope filtration on $(\mathscr{M}_{\mathscr{R}}, \varphi_{\mathscr{M}_{\mathscr{R}}})$ has an interpretation purely in terms of $\mathscr{M}$ in the category $\mathrm{Mod}^{\phi,N_\nabla}_{/\mathscr{O}}$. This is indeed the case, as it is possible to use φ and N_∇ to obtain the following refinement of Theorem 10.3.6:

Theorem 10.3.7. *Let $\mathscr{M}$ be a nonzero object of $\mathrm{Mod}^{\phi,N_\nabla}_{/\mathscr{O}}$. There exists a unique filtration*

$$0 = \mathscr{M}_0 \subseteq \mathscr{M}_1 \subseteq \cdots \subseteq \mathscr{M}_r = \mathscr{M} \tag{10.3.2}$$

in $\mathrm{Mod}^{\phi,N_\nabla}_{/\mathscr{O}}$ whose successive quotients $\mathscr{M}_i/\mathscr{M}_{i-1}$ are nonzero objects of $\mathrm{Mod}^{\phi,N_\nabla}_{/\mathscr{O}}$ such that (10.3.2) *descends the slope filtration of $\mathscr{M}_{\mathscr{R}}$.*

We are now able to translate the condition of weak admissibility for a filtered $(\phi,\ N)$-module across the equivalence of categories in Theorem 10.2.1.

Theorem 10.3.8. *A nonzero object D of $\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0}$ is weakly admissible if and only if the nonzero $\underline{\mathscr{M}}(D)$ is pure of slope zero.*

Proof. Since $\underline{\mathscr{M}}$ is an exact covariant tensor-compatible functor, we have

$$\wedge^i \underline{\mathscr{M}}(D) \simeq \underline{\mathscr{M}}(\wedge^i D)$$

for all $i \geqslant 0$. But $\underline{\mathscr{M}}$ is an equivalence, so therefore it preserves rank (using the characterization of rank in terms of exterior algebra). It follows that

$$\det \underline{\mathscr{M}}(D) \simeq \underline{\mathscr{M}}(\det D).$$

Recalling that

$$t_N(D) = t_N(\det D) \text{ and } t_H(D) = t_H(\det D), \tag{10.3.3}$$

we are motivated to first treat the case that $\dim_{K_0} D = 1$.

If $\dim_{K_0} D = 1$ then since N is nilpotent, we must have $N = 0$. Setting $h := t_H(D)$, by the definition of $t_H(D)$ we have $\mathrm{Fil}^j D_K = D_K$ for all $j \leqslant h$ and $\mathrm{Fil}^j D_K = 0$ for all $j \geqslant h+1$. Thus, from the definition of $\underline{\mathscr{M}}(D)$ (given in §10.2) we see that

$$\underline{\mathscr{M}}(D) = \lambda^{-h}(\mathscr{O} \otimes_{K_0} D).$$

If we select a K_0-basis $\mathbf{e}$ of D, then $\varphi_D(\mathbf{e}) = \alpha\mathbf{e}$ for some $\alpha \in K_0^\times$; by the definition of $t_N(D)$, we have $\mathrm{ord}_p(\alpha) = t_N(D)$. Viewing $\mathbf{e}$ as a $\mathscr{O}$-basis of $\mathscr{O} \otimes_{K_0} D$, we calculate (using (10.1.2))

$$\varphi_{\underline{\mathscr{M}}(D)}(\lambda^{-h}e) = \varphi_{\mathscr{O}}(\lambda)^{-h}\alpha\mathbf{e} = \left(\frac{E(u)}{E(0)}\right)^h \alpha(\lambda^{-h}\mathbf{e}). \tag{10.3.4}$$

Now $E(u) \in (\mathscr{R}^{\mathrm{int}})^\times$ by Example 10.3.1, and $E(0) \in p\cdot \mathrm{W}(k)^\times \subseteq p\cdot(\mathscr{R}^{\mathrm{int}})^\times$ so $(E(u)/E(0))^h \in p^{-t_H(D)} \cdot (\mathscr{R}^{\mathrm{int}})^\times$ by the definition of h. Since $\alpha \in p^{t_N(D)} \cdot (\mathscr{R}^{\mathrm{int}})^\times$, we conclude from (10.3.4) that $\underline{\mathscr{M}}(D)$ is pure of slope $t_N(D) - t_H(D)$ (by the definition of "pure slope"). This settles the case that D has rank 1.

It now follows formally from the properties of $\underline{\mathscr{M}}$ (such as det-compatibility) and of slopes, and the identities (10.3.3), that a nonzero D is weakly admissible when $\underline{\mathscr{M}}(D)$ is pure of slope zero.

For the converse, suppose that D is nonzero and weakly admissible. By Theorem 10.3.7, the slope filtration of $\underline{\mathscr{M}}(D)_{\mathscr{R}}$ descends to

$$0 = \mathscr{M}_0 \subseteq \mathscr{M}_1 \subseteq \cdots \subseteq \mathscr{M}_r = \underline{\mathscr{M}}(D) \tag{10.3.5}$$

in $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N_\nabla}$ with nonzero $\mathscr{M}_i/\mathscr{M}_{i-1} \in \mathrm{Mod}_{/\mathscr{O}}^{\phi,N_\nabla}$ pure of slope $s_i \in \mathbf{Q}$ such that

$$s_1 < s_2 < \cdots < s_r.$$

Our goal is to show that $r = 1$ and $s_1 = 0$.

Set $d_i := \mathrm{rk}_{\mathscr{O}}\, \mathscr{M}_i/\mathscr{M}_{i-1}$ and note that $d_i \geqslant 1$. Since $\wedge^{d_i}(\mathscr{M}_i/\mathscr{M}_{i-1})$ is pure of slope $s_i d_i$ by the proof of [29, Cor. 1.6.4], it follows that $\det \underline{\mathscr{M}}(D) \simeq \otimes \det(\mathscr{M}_i/\mathscr{M}_{i-1})$ is pure of slope $\sum_i s_i d_i$. On the other hand, we deduce from our calculations in the rank-1 case that $\underline{\mathscr{M}}(\det(D))$ is pure of slope

$$t_N(\det D) - t_H(\det D) = t_N(D) - t_H(D) = 0$$

by (10.3.3) and the weak admissibility hypothesis. Since $\det \underline{\mathscr{M}}(D) = \underline{\mathscr{M}}(\det D)$ as observed before, we conclude that

$$\sum_i s_i d_i = 0. \tag{10.3.6}$$

As $s_1 < s_2 < \cdots < s_r$, in order to show what we want ($r = 1$ and $s_1 = 0$) it is therefore enough to show that $s_1 \geqslant 0$.

Since $\underline{\mathscr{M}}$ is an equivalence of categories by Theorem 10.2.1, corresponding to the nonzero subobject $\mathscr{M}_1$ of $\underline{\mathscr{M}}(D)$ (in $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N_\nabla}$) is a nonzero subobject D_1 of D (in $\mathrm{MF}_K^{\phi,N}$) with

$$\mathscr{M}_1 = \underline{\mathscr{M}}(D_1).$$

We have calculated that $\det \mathscr{M}_1$ is pure of slope $s_1 d_1$, so since $\det \mathscr{M}_1 = \underline{\mathscr{M}}(\det D_1)$, which is pure of slope $t_N(D_1) - t_H(D_1)$ (again by the rank-1 case), we conclude that

$$s_1 d_1 = t_N(D_1) - t_H(D_1) \geqslant 0$$

as D_1 is a nonzero subobject of the weakly admissible filtered (ϕ, N)-module D (and therefore $t_N(D_1) - t_H(D_1) \geqslant 0$ by the *definition* of weak-admissibility). This gives $s_1 \geqslant 0$, as required. ■

10.4. **Integral theory.** We now describe a certain "integral theory" that will be used in §11 to study semi-stable Galois representations. To motivate this theory, we first define a new category of linear algebra data.

Definition 10.4.1. Let $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N}$ be the category whose objects are triples $(\mathscr{M}, \varphi_{\mathscr{M}}, N)$ where

(1) the pair $(\mathscr{M}, \varphi_{\mathscr{M}})$ is an object of $\mathrm{Mod}_{/\mathscr{O}}^{\varphi}$,
(2) $N : \mathscr{M}/u\mathscr{M} \to \mathscr{M}/u\mathscr{M}$ is a K_0-linear endomorphism satisfying

$$N\varphi = p\varphi N,$$

where $\varphi := \varphi_{\mathscr{M}} \mod u$.

Morphisms in $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N}$ are $\mathscr{O}$-module homomorphisms compatible with $\varphi_{\mathscr{M}}$ and N.

Note that $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N}$ is defined exactly like $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N_\nabla}$, except that we only impose a monodromy operator "at the origin." Denote by $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N_\nabla,0}$ and $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N,0}$ the full subcategories of $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N_\nabla}$ and $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N}$, respectively, consisting of those objects that are 0 or of pure slope zero (where $\mathscr{M}$ is said to be pure of slope zero if $\mathscr{M} \otimes_{\mathscr{O}} \mathscr{R}$ is; cf. Definition 10.3.4). There is a natural "forgetful" functor

$$\mathrm{Mod}_{/\mathscr{O}}^{\phi,N_\nabla} \to \mathrm{Mod}_{/\mathscr{O}}^{\phi,N} \tag{10.4.1}$$

defined by sending the triple $(\mathscr{M}, \varphi_{\mathscr{M}}, N_\nabla)$ to the triple $(\mathscr{M}, \varphi_{\mathscr{M}}, N_\nabla \mod u)$. Using the quasi-inverse equivalences of categories $\underline{\mathscr{M}}$ and $\underline{D}$, one proves (see [30, Lemma 1.3.10(2)]):

Lemma 10.4.2. *The functor* (10.4.1) *is fully faithful.*

By Theorems 10.2.1 and 10.3.8, we obtain an exact, fully faithful tensor-functor

$$^{\mathrm{w.a}}\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0} \xrightarrow[\underline{\mathscr{M}}]{\simeq} \mathrm{Mod}_{/\mathscr{O}}^{\phi,N_\nabla,0} \xhookrightarrow[(10.4.1)]{} \mathrm{Mod}_{/\mathscr{O}}^{\phi,N,0}\ . \tag{10.4.2}$$

The purpose of the "integral" theory that we will introduce is to describe the category $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N,0}$ and the essential image of (10.4.2) in more useful terms. Before we embark on this task, let us remark that by the exactness of $\underline{D}$, the "inverse" to (10.4.2) on its essential image is also exact.

Let $\mathfrak{S} := \mathrm{W}(k)[\![u]\!]$, and denote by $\varphi_{\mathfrak{S}}$ the unique semi-linear extension of the Frobenius endomorphism of $\mathrm{W}(k)$ to $\mathfrak{S}$ that satisfies $\varphi_{\mathfrak{S}}(u) = u^p$. We now define analogues of $\mathrm{Mod}^{\varphi}_{/\mathscr{O}}$ and $\mathrm{Mod}^{\phi,N}_{/\mathscr{O}}$ using $\mathfrak{S}$-modules.

Definition 10.4.3. Let $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ be the category whose objects are pairs $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ where:

(1) $\mathfrak{M}$ is a finite free $\mathfrak{S}$-module and $\varphi_{\mathfrak{M}}$ is a $\varphi_{\mathfrak{S}}$-semilinear endomorphism,
(2) $\mathfrak{M}$ is of finite E-height in the sense that the cokernel of the $\mathfrak{S}$-linearization

$$1 \otimes \varphi_{\mathfrak{M}} : \varphi_{\mathfrak{S}}^{*}\mathfrak{M} \to \mathfrak{M}$$

is killed by some power E^h of E (so $1 \otimes \varphi_{\mathfrak{S}}$ is injective, and hence so is $\varphi_{\mathfrak{M}}$).

Morphisms in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ are φ-equivariant morphisms of $\mathfrak{S}$-modules.

As usual, we enhance the category $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ by adding a "monodromy operator":

Definition 10.4.4. Let $\mathrm{Mod}^{\phi,N}_{/\mathfrak{S}}$ be the category whose objects are triples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, N_{\mathfrak{M}})$ where:

(1) the pair $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is an object of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$,
(2) $N_{\mathfrak{M}}$ is a K_0-linear endomorphism of $(\mathfrak{M}/u\mathfrak{M}) \otimes_{\mathrm{W}(k)} K_0$ which satisfies

$$N_{\mathfrak{M}} \circ \overline{\varphi}_{\mathfrak{M}} = p\overline{\varphi}_{\mathfrak{M}} \circ N_{\mathfrak{M}}$$

(with $\overline{\varphi}_{\mathfrak{M}} := \varphi_{\mathfrak{M}} \mod u$).

Morphisms in $\mathrm{Mod}^{\phi,N}_{/\mathfrak{S}}$ are morphisms in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ compatible with N mod u.

Remark 10.4.5. Note that the definition of $\mathrm{Mod}^{\phi,N}_{/\mathfrak{S}}$ parallels that of $\mathrm{Mod}^{\phi,N}_{/\mathscr{O}}$, except that we only impose $N_{\mathfrak{M}}$ on $(\mathfrak{M}/u\mathfrak{M}) \otimes_{\mathrm{W}(k)} K_0$ and *not* on $\mathfrak{M}/u\mathfrak{M}$. (This lack of integrality conditions on $N_{\mathfrak{M}}$ is solely because it is unclear if Lemma 10.4.7 is true with an integrality requirement on $N_{\mathfrak{M}}$.) Further, observe that $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ embeds as a full subcategory of $\mathrm{Mod}^{\phi,N}_{/\mathfrak{S}}$ by taking $N_{\mathfrak{M}} = 0$. We will not need the category $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ until §11.

Remark 10.4.6. By Exercise 10.5.5, we have that $\mathfrak{S}\left[\frac{1}{p}\right] = \mathscr{O}^{\mathrm{bnd}}$ (the ring of rigid-analytic functions on the open unit disc that are bounded) and that the natural inclusion

$$\mathfrak{S}\left[\frac{1}{p}\right] \to \mathscr{O}$$

is faithfully flat. Moreover, it follows at once from the definition (10.3.1) of $\mathscr{R}^{\mathrm{int}}$ that we have a natural inclusion

$$\mathfrak{S}_{(p)} \to \mathscr{R}^{\mathrm{int}}$$

which is moreover faithfully flat, as it is a local extension of discrete valuation rings.

For the convenience of the reader, we summarize the relationships between the various rings considered above in the following diagram:

$$
\begin{tikzcd}
\mathfrak{S} := \mathrm{W}(k)[\![u]\!] \arrow[r, hook] \arrow[d, hook] & \mathfrak{S}\left[\frac{1}{p}\right] \arrow[r, equal] & \mathscr{O}^{\mathrm{bnd}} \arrow[r, hook] \arrow[d, hook] & \mathscr{O} \arrow[d, hook] \\
\mathfrak{S}_{(p)} \arrow[r, hook] & \mathscr{R}^{\mathrm{int}} \arrow[r, hook] & \mathrm{Frac}(\mathscr{R}^{\mathrm{int}}) = \mathscr{R}^{\mathrm{b}} \arrow[r, hook] & \mathscr{R} \\
 & & \mathscr{R}^{\mathrm{b}} - \{0\} \arrow[r, equal] \arrow[u, hook] & \mathscr{R}^{\times} \arrow[u, hook]
\end{tikzcd}
\tag{10.4.3}
$$

Let $\mathfrak{M}$ be any object of $\mathrm{Mod}_{/\mathfrak{S}}^{\phi,N}$. Then $\mathscr{M} := \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}$ is an object of $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N}$. In fact, since the natural inclusion $\mathfrak{S} \hookrightarrow \mathscr{O} \hookrightarrow \mathscr{R}$ has image in $\mathscr{R}^{\mathrm{int}}$ and $E \in (\mathscr{R}^{\mathrm{int}})^{\times}$, it follows from Remark 10.3.5(2) that $\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}$ is pure of slope zero if $\mathfrak{M} \neq 0$. Since p is invertible in $\mathscr{O}$, the resulting functor $\mathrm{Mod}_{/\mathfrak{S}}^{\phi,N} \to \mathrm{Mod}_{/\mathscr{O}}^{\phi,N}$ factors through the p-isogeny category, so we obtain a functor

$$
\Theta : \mathrm{Mod}_{/\mathfrak{S}}^{\phi,N} \otimes \mathbf{Q}_p \to \mathrm{Mod}_{/\mathscr{O}}^{\phi,N,0} \qquad \mathfrak{M} \mapsto \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}. \tag{10.4.4}
$$

that respects tensor products and is exact.

Lemma 10.4.7. *The functor* Θ *of* (10.4.4) *is an equivalence of categories.*

Proof. We just explain how to functorially (up to p-isogeny) equip any object $\mathscr{M}$ of $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N,0}$ with a $\mathfrak{S}$-structure, and refer the reader to the proof of [30, Lemma 1.3.13] for the complete argument. The key algebraic inputs are:

$$
\mathscr{R}^{\mathrm{b}} \cap \mathscr{O} = \mathscr{O}^{\mathrm{bnd}} = \mathfrak{S}\left[\frac{1}{p}\right] \quad \text{and} \quad \mathscr{R}^{\mathrm{int}} \cap \mathscr{O} = \mathfrak{S},
$$

where both intersections are taken inside of the Robba ring $\mathscr{R}$; see (10.4.3). The idea to exploit this is the following: by definition of pure slope zero (Definition 10.3.4 and Remark 10.3.5(2)), there is a descent of $\mathscr{M}_{\mathscr{R}} := \mathscr{M} \otimes_{\mathscr{O}} \mathscr{R} \in \mathrm{Mod}_{/\mathscr{R}}^{\varphi}$ to an object $\mathscr{M}_{\mathscr{R}^{\mathrm{b}}}$ of $\mathrm{Mod}_{/\mathscr{R}^{\mathrm{b}}}^{\varphi}$ with a φ-stable $\mathscr{R}^{\mathrm{int}}$-lattice $\mathrm{L} \subseteq \mathscr{M}_{\mathscr{R}^{\mathrm{b}}}$. We "glue" the $\mathscr{O}$-module $\mathscr{M}$ and the $\mathscr{R}^{\mathrm{int}}$-lattice L to get a module $\mathfrak{M}$ over $\mathscr{O} \cap \mathscr{R}^{\mathrm{int}} = \mathfrak{S}$.

To be more precise, $\mathscr{M}_{\mathscr{R}^{\mathrm{b}}}$ is functorial in $\mathscr{M}_{\mathscr{R}}$ [29, Prop. 1.5.5], and under the isomorphism

$$
\mathscr{M}_{\mathscr{R}^{\mathrm{b}}} \otimes_{\mathscr{R}^{\mathrm{b}}} \mathscr{R} \simeq \mathscr{M}_{\mathscr{R}}
$$

there exists a subset of $\mathscr{M}_{\mathscr{R}}$ that is both an $\mathscr{O}$-basis of $\mathscr{M}$ and an $\mathscr{R}^{\mathrm{b}}$-basis of $\mathscr{M}_{\mathscr{R}^{\mathrm{b}}}$. Indeed, if we choose an $\mathscr{O}$-basis $\{v_i\}$ of $\mathscr{M}$ and an $\mathscr{R}^{\mathrm{b}}$-basis $\{w_j\}$ of $\mathscr{M}_{\mathscr{R}^{\mathrm{b}}}$ then each is an $\mathscr{R}$-basis of $\mathscr{M}_{\mathscr{R}}$, so there is an invertible matrix A over $\mathscr{R}$ carrying $\{v_i\}$ to $\{w_j\}$. By the first part of [27, Prop. 6.5], we can express A as a product (in either order) of an invertible matrix over $\mathscr{O}$ and an invertible matrix over $\mathscr{R}^{\mathrm{b}}$, so by using such factor matrices to change the respective choices of $\{v_i\}$ over $\mathscr{O}$ and $\{w_j\}$ over $\mathscr{R}^{\mathrm{b}}$ we get the asserted "common basis". It follows that

$$
\mathscr{M}^{\mathrm{b}} := \mathscr{M} \cap \mathscr{M}_{\mathscr{R}^{\mathrm{b}}} \subseteq \mathscr{M}_{\mathscr{R}}
$$

is a φ-stable, finite free $\mathscr{R}^{\mathrm{b}} \cap \mathscr{O} = \mathfrak{S}\left[\frac{1}{p}\right]$-module descending $(\mathscr{M}, \varphi_{\mathscr{M}})$. This shows that Θ is fully faithful, since for any object $\mathfrak{M}$ of $\mathrm{Mod}_{/\mathfrak{S}}^{\phi,N}$, the object $\mathscr{M} := \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}$ satisfies

$$\mathscr{M}^{\mathrm{b}} = \mathfrak{M}\left[\frac{1}{p}\right],$$

so we recover $\mathfrak{M}$ up to p-isogeny from $\mathscr{M}$.

Now for any object $\mathscr{M}$ of $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N}$, the $\mathscr{R}^{\mathrm{int}}$-lattice L inside $\mathscr{M}_{\mathscr{R}^{\mathrm{b}}}$ allows one to equip $\mathscr{M}^{\mathrm{b}} = \mathscr{M} \cap \mathscr{M}_{\mathscr{R}^{\mathrm{b}}}$ with the desired $\mathfrak{S}$-structure (up to p-isogeny); see the proof of [30, Lemma 1.3.13] for the details. ■

Using the fully faithful functor (10.4.2) and the "inverse" of Θ in (10.4.4), we have:

Corollary 10.4.8. *There exists an exact and fully faithful tensor functor*

$$\widetilde{\Theta} : {}^{\mathrm{w.a}}\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0} \hookrightarrow \mathrm{Mod}_{/\mathfrak{S}}^{\phi,N} \otimes \mathbf{Q}_p. \tag{10.4.5}$$

Thus, for any object D of ${}^{\mathrm{w.a}}\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0}$, there is a canonical $\mathfrak{S}$-structure on $\underline{\mathscr{M}}(D)$ up to p-isogeny. For example, in the next section, we will be particularly interested in the case that

$$D = D_{\mathrm{st}}(V) := \left(B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V\right)^{G_K}$$

for some object V of $\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{st}}(G_K)$.

We now wish to describe the essential image of $\widetilde{\Theta}$. To do this, we must answer the following question: for which objects $\mathfrak{M}$ of $\mathrm{Mod}_{/\mathfrak{S}}^{\phi,N}$ does the object $\mathscr{M} := \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}$ of $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N,0}$ admit an operator $N_{\nabla}^{\mathscr{M}}$ as in Definition 10.1.4(2) that lifts $N_{\mathfrak{M}} \otimes 1$ on $(\mathfrak{M}/u\mathfrak{M}) \otimes_{\mathrm{W}(k)} K_0 = \mathscr{M}/u\mathscr{M}$ and makes the triple $(\mathscr{M}, \varphi_{\mathscr{M}}, N_{\nabla}^{\mathscr{M}})$ into an object of $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N_\nabla}$?

Thanks to Lemma 10.2.5, for *any* object $\mathscr{M}$ of $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N}$ we have an injective map of finite free $\mathscr{O}$-modules

$$\xi : \underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O} \hookrightarrow \mathscr{M}$$

with cokernel killed by λ^h (where h is the E-height of $\mathscr{M}$), so in particular ξ is an isomorphism after inverting λ. Therefore, there exists a unique connection

$$\nabla_{\mathscr{M}} : \mathscr{M}\left[\frac{1}{\lambda u}\right] \to \mathscr{M}\left[\frac{1}{\lambda u}\right] \otimes_{\mathscr{O}} \Omega^1_{\Delta/K_0} \tag{10.4.6}$$

satisfying $\nabla_{\mathscr{M}}(d) = -N(d)\frac{\mathrm{d}u}{u}$ for all $d \in \underline{D}(\mathscr{M})$. Moreover, $\nabla_{\mathscr{M}}$ commutes with $\varphi_{\mathscr{M}}$ and has poles of order at most h supported on the zeroes of λ, and at worst a simple pole at $u = 0$.

Defining $N_{\nabla}^{\mathscr{M}} : \mathscr{M}[1/\lambda u] \to \mathscr{M}[1/\lambda u]$ by the relation

$$\nabla_{\mathscr{M}}(m) = -\frac{1}{\lambda} N_{\nabla}^{\mathscr{M}}(m) \frac{\mathrm{d}u}{u} \tag{10.4.7}$$

for all $m \in \mathscr{M}$, as in Remark 10.1.5, gives the only *possible* $N_{\nabla}^{\mathscr{M}}$ for $\mathscr{M} = \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}$ as above. In case $\mathscr{M}$ has $\mathscr{O}$-rank 1, it follows from a calculation (see the proof of [30, 1.3.10(3)]) that $\nabla_{\mathscr{M}}$ has at worst simple poles; that is, $N_{\nabla}^{\mathscr{M}}$ carries $\mathscr{M}$ into itself in the rank-1 case. Thus:

Corollary 10.4.9. *Let $\mathfrak{M}$ be an object of $\mathrm{Mod}_{/\mathfrak{S}}^{\phi,N} \otimes \mathbf{Q}_p$ and let $\mathscr{M} := \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}$ be the corresponding object of $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N,0}$. Then $\mathfrak{M}$ is in the image of $\widetilde{\Theta}$ if and only if the connection $\nabla_{\mathscr{M}}$ as in* (10.4.6) *has at worst simple poles* (*equivalently, if and only if the operator $N_{\nabla}^{\mathscr{M}}$ defined by* (10.4.7) *is holomorphic*)*. In particular, any such $\mathfrak{M}$ with $\mathfrak{S}$-rank* 1 *is in the image of $\widetilde{\Theta}$.*

10.5. **Exercises.**

Exercise 10.5.1. Prove that the infinite product in (10.1.1) does converge uniformly on closed subdiscs of Δ.

Exercise 10.5.2. This exercise shows how λ in (10.1.1) is an analogue of the p-adic logarithm. If we work with $E(u) = (u+1)^p - 1 = \Phi_p(u+1)$ and take $\pi = \zeta_p - 1$, then instead of working with a compatible system of p-power roots of π (which is not "Galois" over K_0) we might instead prefer to work with the system of values $\zeta_{p^n} - 1$ where $\{\zeta_{p^n}\}$ is a compatible system of p-power roots of unity. This leads to the following considerations.

Change the definition of $\varphi_{\mathscr{O}}$ by requiring $\varphi_{\mathscr{O}}(u) := (u+1)^p - 1$ (rather than $u \mapsto u^p$). We claim that with these choices, the analogous definition of λ akin to (10.1.1) will give $\lambda = \frac{\log(1+u)}{u}$. Indeed, check that with the modified definition of $\varphi_{\mathscr{O}}$ as just made, the formula in (10.1.1) works out as follows:

$$\lambda = \lim_{N\to\infty} \prod_{n=0}^{N} \frac{\Phi_{p^n}(u+1)}{p} = \frac{1}{u} \cdot \lim_{N\to\infty} \frac{(u+1)^{p^N} - 1}{p^N} = \frac{\log(1+u)}{u},$$

where you should use the binomial theorem and simple p-adic estimates on the explicit binomial coefficients to justify the final equality and the uniform convergence of the product on each Δ_r.

Exercise 10.5.3. Let $\alpha : D \to D'$ be a map of filtered ϕ-modules (so vanishing monodromy) which have effective filtrations (i.e., their nonzero gr's only occur in degrees $\geqslant 0$). Show that the map

$$1 \otimes \alpha : \mathscr{O}\left[\frac{1}{\lambda}\right] \otimes_{K_0} D \to \mathscr{O}\left[\frac{1}{\lambda}\right] \otimes_{K_0} D$$

restricts to a morphism $\underline{\mathscr{M}}(\alpha) : \underline{\mathscr{M}}(D) \to \underline{\mathscr{M}}(D)$ of (ϕ, N_∇)-modules over $\mathscr{O}$. You will need to use the definition of $\underline{\mathscr{M}}$ and the fact that α respects the filtrations.

Exercise 10.5.4. Prove the uniqueness of ξ as in Lemma 10.2.5(1). Rather generally, by taking differences, the problem is to prove that if $\xi : \mathscr{M}' \to \mathscr{M}$ is a morphism in $\mathrm{Mod}_{/\mathscr{O}}^{\varphi}$ such that $\xi(\mathscr{M}') \subseteq u\mathscr{M}$ then prove that $\xi = 0$. (Hint: if $\xi \neq 0$, show there is a maximal $n \geqslant 1$ such that $\xi = u^n h$ for an $\mathscr{O}$-linear map $h : \mathscr{M}' \to \mathscr{M}$; beware that h is not φ-compatible! Use the φ-compatibility of ξ and that $n \geqslant 1$ to get a contradiction.)

Exercise 10.5.5. Let A be a complete discrete valuation ring, and F its fraction field. Let $f = \sum c_n u^n \in F[\![u]\!]$ be a formal power series over F.

(1) Prove that f converges on the open unit disc over $\overline{F}$ if and only if $|c_n| r^n \to 0$ for each $0 < r < 1$. Give a counterexample if one only works with F-rational points of the

open unit disk. Give an example of such a convergent series with $A = \mathbf{Z}_p$ for which the $|c_n|$'s are unbounded.

(2) Assuming that f does converge on the open unit disc over $\overline{F}$, prove that it is bounded on this disc if and only if the $|c_n|$'s are bounded. In other words, prove that $A[\![u]\!]\otimes_A F$ is the F-algebra $\mathscr{O}^{\mathrm{bnd}}$ of bounded functions on the rigid-analytic open unit disk over F.

(3) Deduce that $\mathscr{O}^{\mathrm{bnd}}$ is a Dedekind domain, and prove that $\mathscr{O}^{\mathrm{bnd}} \to \mathscr{O}$ is faithfully flat.

11. $\mathfrak{S}$-modules and applications

We now turn to the task of introducing the category of $\mathfrak{S}$-modules, roughly an integral version of the category of vector bundles with connection from §10, and we set up a fully faithful functor from the category of effective weakly admissible filtered (ϕ, N)-modules to the isogeny category of $\mathfrak{S}$-modules (and we describe the essential image). In the reverse direction we construct a fully faithful functor from the category of $\mathfrak{S}$-modules into the category of G_{K_∞}-stable lattices in semistable G_K-representations, where K_∞/K is generated by compatible p-power roots of a uniformizer π of $\mathscr{O}_K$.

As applications, we obtain a proof of the conjecture of Fontaine that the natural fully faithful functor from semistable representations to weakly admissible modules is an equivalence, and we obtain a proof of the conjecture of Breuil that restriction from crystalline G_K-modules to underlying G_{K_∞}-modules is fully faithful. We also use $\mathfrak{S}$-modules to describe the category of all G_{K_∞}-stable lattices in crystalline representations of G_K.

We begin by using the fully faithful tensor-compatible functor

$$\begin{array}{ccc} {}^{\text{w.a.}}\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0} & \xhookrightarrow{\widetilde{\Theta}} & \mathrm{Mod}_{/\mathfrak{S}}^{\phi,N}\otimes\mathbf{Q}_p \\ & & \\ D & \longmapsto & \text{“}\mathfrak{S}\text{-structures on } \underline{\mathscr{M}}(D)\text{”} \end{array}$$

to study $\mathrm{Rep}^{\mathrm{st}}_{\mathbf{Q}_p} G_K$. For any profinite group Γ, we define:

$\mathrm{Rep}^{\mathrm{tor}}(\Gamma)$	$=$	category of continuous Γ-representations on finite discrete abelian p-groups,
$\mathrm{Rep}_{\mathbf{Z}_p}(\Gamma)$	$=$	category of continuous linear Γ-representations on finitely generated $\mathbf{Z}_p$-modules,
$\mathrm{Rep}^{\mathrm{free}}_{\mathbf{Z}_p}(\Gamma)$	$=$	category of continuous linear Γ-representations on finite free $\mathbf{Z}_p$-modules,
$\mathrm{Rep}_{\mathbf{Q}_p}(\Gamma)$	$=$	category of continuous linear Γ-representations on finite-dimensional $\mathbf{Q}_p$-vector spaces.

Morphisms in each category are the obvious ones. Observe that $\mathrm{Rep}_{\mathbf{Q}_p}(\Gamma)$ is the p-isogeny category of $\mathrm{Rep}^{\mathrm{free}}_{\mathbf{Z}_p}(\Gamma)$.

Recall that K/K_0 is a totally ramified extension of $K_0 = \mathrm{Frac}(\mathrm{W}(k))$ with uniformizer $\pi \in \mathscr{O}_K$. Choose a compatible sequence of p-power roots of π,

$$\pi_n := \sqrt[p^n]{\pi} \in \overline{K} \quad (\pi_0 = \pi),$$

and set $K_\infty := \cup K_0(\pi_n) \subseteq \overline{K}$ and $G_{K_\infty} := \mathrm{Gal}(\overline{K}/K_\infty) \subseteq G_K$.

The main goals of this section are:

(1) Show that weakly admissible implies admissible; i.e. that if D is a nonzero object of ${}^{\mathrm{w.a.}}\mathrm{MF}_K^{\phi,N}$ then

$$D = D_{\mathrm{st}}(V) := \left(B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V\right)^{G_K}$$

for some object V of $\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{st}}(G_K)$.

(2) Show that the restriction of the natural functor

$$\mathrm{Rep}_{\mathbf{Q}_p}(G_K) \to \mathrm{Rep}_{\mathbf{Q}_p}(G_{K_\infty})$$

to the subcategory of crystalline representations $\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{cris}}(G_K) \subseteq \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ is fully faithful, and describe G_{K_∞}-stable $\mathbf{Z}_p$-lattices in crystalline p-adic representations of G_K using $\mathrm{Mod}_{/\mathfrak{S}}^{\varphi}$ (recall from Definition 10.4.3 that this is "the category $\mathrm{Mod}_{/\mathfrak{S}}^{\phi,N}$ with $N = 0$"). Beware that the restriction functor $\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{st}}(G_K) \to \mathrm{Rep}_{\mathbf{Q}_p}(G_{K_\infty})$ is not fully faithful, so the crystalline condition in (2) above is essential. See Exercise 11.4.4.

11.1. **Étale φ-modules revisited.** In §3 we developed the theory of étale φ-modules, and now we wish to reinterpret some aspects of that theory by using the ring $R = \varprojlim \mathscr{O}_{\overline{K}}/(p) = \varprojlim \mathscr{O}_{\mathbf{C}_K}/(p)$ in a new way. Recall that in our earlier work with R we chose an element $\widetilde{p} \in R$ with $\widetilde{p}^{(0)} = p$ (which amounts to choosing a compatible system of p-power roots of p in $\mathscr{O}_{\overline{K}}$), and we used it to do several things (e.g., $[\widetilde{p}] - p$ generates $\ker\theta$). Now we shall use a variant of this element that is adapted to our particular field K. Using our fixed choice of compatible system $\{\pi_n\}$ of p^nth roots of π in $\mathscr{O}_{\overline{K}}$ $(n \geqslant 0)$, we define

$$\widetilde{\pi} := (\pi_n)_{n\geqslant 0} \in R.$$

Observe that by its definition, the isotropy subgroup of $\widetilde{\pi}$ in G_K is G_{K_∞}. There is a natural map

$$\begin{array}{ccc} \mathfrak{S} = \mathrm{W}(k)[\![u]\!] & \hookrightarrow & \mathrm{W}(R) \\ \circlearrowright \varphi_{\mathfrak{S}} & & \circlearrowright \mathrm{Frob}_R \\ \sum_{n\geqslant 0} a_n u^n & \longmapsto & \sum_{n\geqslant 0} a_n [\widetilde{\pi}]^n \end{array} \tag{11.1.1}$$

which is G_{K_∞}-invariant (by definition of $\widetilde{\pi}$) and φ-compatible (as $[\widetilde{\pi}^p] = [\widetilde{\pi}]^p$). Since $\widetilde{\pi} \in \mathrm{Frac}\, R$ is nonzero we have $[\widetilde{\pi}] \in \mathrm{W}(\mathrm{Frac}\, R)^\times$, so (11.1.1) extends to a map

$$\mathfrak{S}\left[\tfrac{1}{u}\right] \hookrightarrow \mathrm{W}(\mathrm{Frac}\, R)\,. \tag{11.1.2}$$

The source of this map is a Dedekind domain in which (p) is a prime ideal and the target is a complete discrete valuation ring with uniformizer p, so (11.1.2) gives a map

$$j : \mathfrak{S}\left[\tfrac{1}{u}\right]^\wedge_{(p)} \hookrightarrow \mathrm{W}(\mathrm{Frac}\, R)$$

that fits into a commutative diagram

$$\begin{tikzcd} \mathscr{O}_{\mathscr{E}} := \mathfrak{S}\left[\frac{1}{u}\right]^{\wedge}_{(p)} \arrow[r, hook, "j"] \arrow[d, two heads] & \mathrm{W}(\mathrm{Frac}\, R) \arrow[d, two heads] \\ k(\!(u)\!) \arrow[r] & \mathrm{Frac}\, R \end{tikzcd} \tag{11.1.3}$$

(with both horizontal maps defined by sending u to $[\widetilde{\pi}]$, and the bottom map over k). Since $\mathrm{Frac}(R)$ is algebraically closed, the bottom side of the diagram provides a separable closure of $k(\!(u)\!)$ in $\mathrm{Frac}(R)$.

The ring $\mathscr{O}_{\mathscr{E}}$ in (11.1.3) is a complete discrete valuation ring with uniformizer p, and it has a "Frobenius endomorphism" $\varphi_{\mathscr{O}_{\mathscr{E}}}$ induced by $\varphi_{\mathfrak{S}}$; due to the φ-equivariance of (11.1.1), the horizontal maps in (11.1.3) are φ-compatible. Let $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}/\mathscr{O}_{\mathscr{E}}$ be the maximal unramified extension of $\mathscr{O}_{\mathscr{E}}$ with respect to the separable closure $k(\!(u)\!)_{\mathrm{sep}} \subseteq \mathrm{Frac}(R)$ of $k(\!(u)\!)$. We define

$$\mathscr{E} := \mathrm{Frac}(\mathscr{O}_{\mathscr{E}}) \quad \text{and} \quad \mathscr{E}^{\mathrm{un}} := \mathrm{Frac}(\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}).$$

By the universal property of the strict henselization $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ of $\mathscr{O}_{\mathscr{E}}$, there exists a unique map

$$\widetilde{j} : \mathscr{O}_{\mathscr{E}}^{\mathrm{un}} \hookrightarrow \mathrm{W}(\mathrm{Frac}\, R)$$

over j which lifts the inclusion $k(\!(u)\!)_{\mathrm{sep}} \hookrightarrow \mathrm{Frac}\, R$ on residue fields. We thus obtain a commutative diagram

$$\begin{tikzcd} \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}[1/p] = \widehat{\mathscr{E}^{\mathrm{un}}} \arrow[rr, hook] & & \mathrm{W}(\mathrm{Frac}\, R)[1/p] \\ \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \arrow[u, hook] \arrow[drr, hook] & & \\ \mathscr{O}_{\mathscr{E}}^{\mathrm{un}} \arrow[u, hook] \arrow[rr, hook, "\widetilde{j}"] & & \mathrm{W}(\mathrm{Frac}\, R) \arrow[uu, hook] \\ \mathscr{O}_{\mathscr{E}} \arrow[u, hook] \arrow[urr, hook, "j"'] & & \end{tikzcd}$$

The unicity of $\widetilde{j}$ implies that the G_{K_∞}-action on $\mathrm{W}(\mathrm{Frac}\, R)$ over $\mathscr{O}_{\mathscr{E}}$ preserves the subring $\widetilde{j}(\mathscr{O}_{\mathscr{E}}^{\mathrm{un}})$.

Remark 11.1.1. The natural map $\mathfrak{S} \to \mathscr{O}_{\mathscr{E}}$ is flat since it factors through an injection from the Dedekind localization $\mathfrak{S}[1/p]$, and the natural map $\mathfrak{S}_{(p)} \to \mathscr{O}_{\mathscr{E}}$ is faithfully flat (as it is a local extension of discrete valuation rings). Moreover, since $E \equiv u^e \mod p$ ($e = [K : K_0]$), we see that $E \in \mathscr{O}_{\mathscr{E}}^{\times}$ because $\mathscr{O}_{\mathscr{E}}$ is a discrete valuation ring having residue field $k(\!(u)\!)$.

The following important theorem is a special case of the general isomorphism (1.3.1) from the theory of norm fields, and as we saw in §3 it allows us to study G_{K_∞}-representations via characteristic-p methods:

Theorem 11.1.2. *The natural action of G_{K_∞} on $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ via the inclusion $\widetilde{j}$ induces an isomorphism of topological groups*

$$G_{K_\infty} \xrightarrow{\simeq} \mathrm{Aut}(\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}/\mathscr{O}_{\mathscr{E}}) \simeq G_{k((u))}.$$

We will not prove this theorem in these notes; the analogous case of $K(\mu_{p^\infty})$ will be proved in §13.4. To handle cases beyond the cyclotomic case it seems best to argue by using the entire general theory of norm fields as in [51]. In what follows, we shall apply the theory from §3 with the axiomatic ring $\mathscr{O}_{\mathscr{E}}$ there taken to be $\mathscr{O}_{\mathscr{E}}$ as just defined above (using $\widetilde{\pi}$). In particular, we note that via Theorem 11.1.2, Fontaine's functors $\mathrm{D}_{\mathscr{E}}$ and $\mathrm{V}_{\mathscr{E}}$ from Theorem 3.2.5 and Theorem 3.3.4 provide equivalences between the category $\mathrm{Rep}_{\mathbf{Z}_p}(G_{K_\infty})$ and the category $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$ of étale φ-modules, as well as between the corresponding isogeny categories upon inverting p, all compatibly with the operations of linear algebra.

Example 11.1.3. It follows from Remark 11.1.1 that for any object $\mathfrak{M}$ of $\mathrm{Mod}_{/\mathfrak{S}}^{\varphi}$, the scalar extension $M := \mathscr{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ is an étale φ-module.

We have seen in our development of p-adic Hodge theory that contravariant functors are sometimes more convenient than covariant functors. In what follows it correspondingly turns out that contravariant versions of the functors from §3 will be more useful than the covariant ones which were studied there. in view of how the duality functors were defined in §3, we are led to define the following categories and contravariant functors between them.

Definition 11.1.4. Define $\mathrm{Mod}_{/\mathscr{O}_{\mathscr{E}}}^{\varphi}$ to be the category of étale φ-modules over $\mathscr{O}_{\mathscr{E}}$ whose underlying $\mathscr{O}_{\mathscr{E}}$-modules is finite free, and $\mathrm{Mod}_{/\mathscr{O}_{\mathscr{E}}}^{\varphi,\mathrm{tor}}$ to be the category of étale φ-modules over $\mathscr{O}_{\mathscr{E}}$ whose underlying $\mathscr{O}_{\mathscr{E}}$-module is torsion.

Define the contravariant functors

$$\underline{V}^*_{\mathscr{O}_{\mathscr{E}}} : \mathrm{Mod}_{/\mathscr{O}_{\mathscr{E}}}^{\varphi} \to \mathrm{Rep}_{\mathbf{Z}_p}^{\mathrm{free}}(G_{K_\infty}),\ \ \underline{D}^*_{\mathscr{O}_{\mathscr{E}}} : \mathrm{Rep}_{\mathbf{Z}_p}^{\mathrm{free}}(G_{K_\infty}) \to \mathrm{Mod}_{/\mathscr{O}_{\mathscr{E}}}^{\varphi}$$

by

$$\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}(M) := \mathrm{Hom}_{\mathscr{O}_{\mathscr{E}},\varphi}\left(M, \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}\right),\ \ \underline{D}^*_{\mathscr{O}_{\mathscr{E}}}(V) = \mathrm{Hom}_{\mathbf{Z}_p[G_{K_\infty}]}(V, \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}),$$

and similarly on torsion categories using $\mathscr{E}^{\mathrm{un}}/\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ in place of $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$.

In Theorem 3.2.5 we considered covariant equivalences denoted $\mathrm{D}_{\mathscr{E}}$ and $\mathrm{V}_{\mathscr{E}}$ between the bigger categories $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$ and $\mathrm{Rep}_{\mathbf{Z}_p}(G_{K_\infty})$ (allowing p-power torsion). The relation with the above contravariant functors on "finite free" objects is that for $M \in \mathrm{Mod}_{/\mathscr{O}_{\mathscr{E}}}^{\varphi}$ and $T \in \mathrm{Rep}_{\mathbf{Z}_p}^{\mathrm{free}}(G_{K_\infty})$ we have natural isomorphisms

$$\mathrm{D}_{\mathscr{E}}(T) := (T \otimes_{\mathbf{Z}_p} \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{G_{K_\infty}} \simeq \underline{D}^*_{\mathscr{O}_{\mathscr{E}}}(T^\vee)^\vee$$

and

$$\mathrm{V}_{\mathscr{E}}(M) := (M \otimes_{\mathbf{Z}_p} \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{\varphi=\mathrm{id}} \simeq \underline{V}^*_{\mathscr{O}_{\mathscr{E}}}(M^\vee)^\vee$$

in $\mathrm{Rep}_{\mathbf{Z}_p}^{\mathrm{free}}(G_{K_\infty})$ and $\mathrm{Mod}_{/\mathscr{O}_{\mathscr{E}}}^{\varphi}$ respectively, using usual linear duality on finite free modules (i.e., $\mathrm{Hom}_{\mathscr{O}_{\mathscr{E}},\varphi}(\cdot, \mathscr{O}_{\mathscr{E}})$ and $\mathrm{Hom}_{\mathbf{Z}_p[G_{K_\infty}]}(\cdot, \mathbf{Z}_p)$). We have the same formulas in the torsion cases, except that the duality functors must be defined using maps into $\mathscr{E}^{\mathrm{un}}/\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ and $\mathbf{Q}_p/\mathbf{Z}_p$.

In view of these formulas, Theorem 3.2.5 immediately gives that $\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}$ and $\underline{D}^*_{\mathscr{O}_{\mathscr{E}}}$ are quasi-inverse equivalences between the categories of finite free objects, as well as between the categories of torsion objects.

Our aim is to adapt the theory of étale φ-modules to study $\mathfrak{S}$-modules (rather than $\mathscr{O}_{\mathscr{E}}$-modules), and to find an analogue of Theorem 3.2.5 describing the essential image of $\mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(G_K)$ in $\mathrm{Rep}_{\mathbf{Q}_p}(G_{K_\infty})$ in terms of $\mathfrak{S}$-modules and certain linear algebra data on them. To do this, we will replace $\mathscr{O}^{\mathrm{un}}_{\mathscr{E}}$ and $\widehat{\mathscr{O}^{\mathrm{un}}_{\mathscr{E}}}$ with

$$\mathfrak{S}^{\mathrm{un}} := \mathscr{O}_{\mathscr{E}^{\mathrm{un}}} \cap \mathrm{W}(R) \subseteq \mathrm{W}(\operatorname{Frac} R),$$
$$\widehat{\mathfrak{S}^{\mathrm{un}}} := \widehat{\mathscr{O}_{\mathscr{E}^{\mathrm{un}}}} \cap \mathrm{W}(R) \subseteq \mathrm{W}(\operatorname{Frac} R).$$

Note that $\mathfrak{S} \subseteq \mathscr{O}_{\mathscr{E}} \cap \mathrm{W}(R)$ (so $\mathfrak{S}^{\mathrm{un}}$ is a flat $\mathfrak{S}$-module). In Exercise 11.4.1 you will show that $\widehat{\mathfrak{S}^{\mathrm{un}}}$ is isomorphic to the p-adic completion of $\mathfrak{S}^{\mathrm{un}}$, thereby justifying the notation.

Beware that, unlike the case of modules over the discrete valuation ring $\mathscr{O}_{\mathscr{E}}$, finitely generated p-power torsion $\mathfrak{S}$-modules need *not* be isomorphic to a direct sum of modules of the form $\mathfrak{S}/p^n\mathfrak{S}$. (For example, let $I \subseteq \mathfrak{S}$ be the ideal $I = (p^2 - u, u^2)$, and consider the module $\mathfrak{S}/I$.) The correct analogue of "finitely generated p-power torsion $\mathscr{O}_{\mathscr{E}}$-module" in this context turns out to be a finite p-power torsion $\mathfrak{S}$-module of projective dimension at most 1. (Over a discrete valuation ring, *all* finitely generated modules have projective dimension at most 1.)

11.2. **$\mathfrak{S}$-modules and G_{K_∞}-representations.** Recall the definition (Definition 10.4.3) of the category $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$. We will treat this category as an analogue of the category $\mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}}$ of étale φ-modules over $\mathscr{O}_{\mathscr{E}}$ with finite free underlying $\mathscr{O}_{\mathscr{E}}$-module. We now define the $\mathfrak{S}$-module analogue of the category of torsion étale φ-modules over $\mathscr{O}_{\mathscr{E}}$:

Definition 11.2.1. Let $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{/\mathfrak{S}}$ be the category whose objects are finite $\mathfrak{S}$-modules $\mathfrak{M}$ such that:

(1) $\mathfrak{M}$ is killed by some power of p and $\operatorname{projdim} \mathfrak{M} \leqslant 1$,
(2) there is a $\varphi_{\mathfrak{S}}$-semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that the $\mathfrak{S}$-linearization
$$1 \otimes \varphi_{\mathfrak{M}} : \varphi^*_{\mathfrak{S}}\mathfrak{M} \to \mathfrak{M}$$
is injective and has cokernel killed by some power E^h of E (so $\varphi_{\mathfrak{M}}$ is injective).

Morphisms in $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{/\mathfrak{S}}$ are φ-compatible maps of $\mathfrak{S}$-modules.

Observe that if $\mathfrak{M}$ is a direct sum of $\mathfrak{S}$-modules of the type $\mathfrak{S}/p^n\mathfrak{S}$ then any $\varphi_{\mathfrak{S}}$-semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ has $\mathfrak{S}$-linearization that is automatically injective since the image of E in $(\mathfrak{S}/p\mathfrak{S})[1/u] = k(\!(u)\!)$ is a unit. Although not every object $\mathfrak{M}$ of $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{/\mathfrak{S}}$ is a direct sum of objects of the form $\mathfrak{S}/p^n\mathfrak{S}$, we do have:

Lemma 11.2.2. *Every object $\mathfrak{M}$ of $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{/\mathfrak{S}}$ is a successive extension of objects that are free over $\mathfrak{S}/p\mathfrak{S}$.*

Proof. See the proof of [30, Lemma 2.3.2], and note that although that proof assumes that the cokernel of $1 \otimes \varphi_{\mathfrak{M}} : \varphi^*_{\mathfrak{S}}\mathfrak{M} \to \mathfrak{M}$ is killed by E, the same argument works with any power E^h of E. ∎

Using that
$$\mathfrak{S}/p\mathfrak{S} = k[\![u]\!] \subseteq k(\!(u)\!) = \mathscr{O}_{\mathscr{E}}/p\mathscr{O}_{\mathscr{E}},$$
some nontrivial calculations of Fontaine give:

Lemma 11.2.3. *Let $\mathfrak{M}$ be any object of $\mathrm{Mod}_{/\mathfrak{S}}^{\varphi,\mathrm{tor}}$. Then there is a natural isomorphism of $\mathbf{Z}_p[G_{K_\infty}]$-modules*
$$\underline{V}^*_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}\left(\mathfrak{M}, \mathfrak{S}^{\mathrm{un}}\left[\tfrac{1}{p}\right]/\mathfrak{S}^{\mathrm{un}}\right) \xrightarrow{\simeq} \underline{V}^*_{\mathscr{O}_{\mathscr{E}}}(\mathscr{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{M})\,.$$

It follows immediately from this lemma and Remark 11.1.1 that $\underline{V}^*_{\mathfrak{S}}$ is exact, commutes with tensor products, and
$$\text{if}\quad \mathfrak{M} \simeq \bigoplus_i \mathfrak{S}/p^{n_i}\mathfrak{S} \quad\text{then}\quad \underline{V}^*_{\mathfrak{S}}(\mathfrak{M}) \simeq \bigoplus_i \mathbf{Z}/p^{n_i}\mathbf{Z}.$$
Passing to inverse limits gives:

Corollary 11.2.4. *We have:*

(1) *Let $\mathfrak{M}$ be any object of $\mathrm{Mod}_{/\mathfrak{S}}^{\varphi}$. Then*
$$\underline{V}^*_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}\left(\mathfrak{M}, \widehat{\mathfrak{S}^{\mathrm{un}}}\right)$$
is a finite free $\mathbf{Z}_p$-module of rank equal to $\mathrm{rk}_{\mathfrak{S}}(\mathfrak{M})$, and the natural map of $\mathbf{Z}_p[G_{K_\infty}]$-modules
$$\underline{V}^*_{\mathfrak{S}}(\mathfrak{M}) \longrightarrow \underline{V}^*_{\mathscr{O}_{\mathscr{E}}}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathscr{E}})$$
obtained by extending scalars to $\mathscr{O}_{\mathscr{E}}$ is an isomorphism.

(2) *Let $\underline{V}^*_{\mathfrak{S}} : \mathrm{Mod}_{/\mathfrak{S}}^{\varphi} \to \mathrm{Rep}_{\mathbf{Z}_p}^{\mathrm{free}}(G_{K_\infty})$ be the functor defined in (1). For all $n \geqslant 1$, there are natural isomorphisms*
$$\underline{V}^*_{\mathfrak{S}}(\mathfrak{M})/(p^n) \xrightarrow{\simeq} \mathrm{Hom}_{\mathfrak{S},\varphi}\left(\mathfrak{M}/p^n\mathfrak{M}, \mathfrak{S}^{\mathrm{un}}/p^n\mathfrak{S}^{\mathrm{un}}\right) \xrightarrow{\simeq} \underline{V}^*_{\mathfrak{S}}(\mathfrak{M}/p^n\mathfrak{M}).$$

Thus, the functor $\underline{V}^*_{\mathfrak{S}}$ on the category $\mathrm{Mod}_{/\mathfrak{S}}^{\varphi}$ is exact and commutes with tensor products.

Remark 11.2.5. For any object $\mathfrak{M}$ of $\mathrm{Mod}_{/\mathfrak{S}}^{\varphi}$, the $\mathbf{Z}_p[G_{K_\infty}]$-module
$$\underline{V}_{\mathfrak{S}*}(\mathfrak{M}) := \left(\mathfrak{M} \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}^{\mathrm{un}}}\right)^{\varphi=1}$$
satisfies
$$\underline{V}_{\mathfrak{S}*}(\mathfrak{M})^\vee \simeq \underline{V}^*_{\mathfrak{S}}\left(\mathfrak{M}^\vee\right).$$

Just as the functor $\underline{D}^*_{\mathscr{O}_{\mathscr{E}}}$ provides a quasi-inverse to $\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}$, we have the following $\mathfrak{S}$-module analogue:

Lemma 11.2.6. *Let $\mathfrak{M}$ be an object of $\mathrm{Mod}_{/\mathfrak{S}}^{\varphi}$ with $\mathfrak{S}$-rank equal to d, and define*
$$\mathfrak{M}' := \mathrm{Hom}_{\mathbf{Z}_p[G_{K_\infty}]}\left(\underline{V}^*_{\mathfrak{S}}(\mathfrak{M}), \widehat{\mathfrak{S}^{\mathrm{un}}}\right).$$
Then $\mathfrak{M}'$ is a finite-free $\mathfrak{S}$-module of rank d, and the natural map $\mathfrak{M} \to \mathfrak{M}'$ is injective.

Using Corollary 11.2.4 and Lemma 11.2.6, one can prove:

Proposition 11.2.7. *The functor* $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \to \mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}}$ *given by*

$$\mathfrak{M} \mapsto \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathscr{E}} \tag{11.2.1}$$

(*see Example* 11.1.3) *is fully faithful.*

The proof of Proposition 11.2.7 is a straightforward adaptation of the "gluing argument" in the proof of Lemma 10.4.7, replacing $\mathscr{R}$ and $\mathscr{O}$ with $\mathscr{O}_{\mathscr{E}}$ and $\mathfrak{S}$ respectively. However, it requires one extra ingredient:

Lemma 11.2.8. *Let* $h : \mathfrak{M}' \to \mathfrak{M}$ *be a morphism in* $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$*. If*

$$h \otimes 1 : \mathfrak{M}' \otimes_{\mathfrak{S}} \mathscr{O}_{\mathscr{E}} \to \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathscr{E}}$$

is an isomorphism, then so is h*.*

Proof. Since $\mathfrak{M}'$ and $\mathfrak{M}$ must have the same $\mathfrak{S}$-rank, we can pass to determinants to reduce to the case that $\mathfrak{M}'$ and $\mathfrak{M}$ both have rank 1. Let $\mathscr{M}' := \Theta(\mathfrak{M}') = \mathfrak{M}' \otimes_{\mathfrak{S}} \mathscr{O}$ and $\mathscr{M} := \Theta(\mathfrak{M}) = \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}$ be the corresponding objects of $\mathrm{Mod}^{\phi,N,0}_{/\mathscr{O}}$ under the equivalence Θ of (10.4.4). The map h thus induces a nonzero map between rank-1 $\mathscr{O}$-modules

$$h \otimes 1 : \mathscr{M}' \to \mathscr{M}. \tag{11.2.2}$$

By the equivalence of categories of Theorem 10.2.1, this map corresponds to a nonzero map

$$\underline{D}(h \otimes 1) : \underline{D}(\mathscr{M}') \to \underline{D}(\mathscr{M})$$

of rank-1 objects of $\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0}$. By the final part of Corollary 10.4.9, these filtered (ϕ, N)-modules are weakly admissible.

A 1-dimensional weakly admissible filtered (ϕ, N)-module has its unique filtration jump determined by its slope, so any nonzero map between such rank-1 objects is not only a K_0-linear φ-compatible isomorphism, but also respects the filtrations in *both* directions. (This is not true without the weak admissibility property!) Hence, $\underline{D}(h \otimes 1)$ is an isomorphism. Since $\underline{D}$ is an equivalence, it follows that (11.2.2) is an isomorphism. But $\mathfrak{S}\left[\frac{1}{p}\right] \to \mathscr{O}$ is faithfully flat by Remark 10.4.6, so the map

$$h\left[\frac{1}{p}\right] : \mathfrak{M}'\left[\frac{1}{p}\right] \to \mathfrak{M}\left[\frac{1}{p}\right]$$

is an isomorphism as well. To conclude that h itself is an isomorphism, it remains to show that it is an isomorphism over $\mathfrak{S}_{(p)}$ since $\mathfrak{S}$ is a normal noetherian domain. (See Exercise 11.4.3.) But $\mathfrak{S}_{(p)} \to \mathscr{O}_{\mathscr{E}}$ is faithfully flat by Remark 11.1.1, so the isomorphism claim follows. ■

Recall from Proposition 9.1.11 that the functor

$$D_{\mathrm{cris}} : \mathrm{Rep}^{\mathrm{cris},\leqslant 0}_{\mathbf{Q}_p}(G_K) \longrightarrow {}^{\mathrm{w.a.}}\mathrm{MF}_K^{\varphi,\mathrm{Fil}\geqslant 0}$$

$$V \longmapsto \left(B_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V\right)^{G_K}$$

is fully faithful, with inverse given by the restriction of V_{cris} to the image of D_{cris}. Combining this with Corollary 10.4.8, we obtain a fully faithful functor

$$\mathrm{Rep}^{\mathrm{cris},\leqslant 0}_{\mathbf{Q}_p}(G_K) \underset{D_{\mathrm{cris}}}{\hookrightarrow} {}^{\mathrm{w.a}}\mathrm{MF}_K^{\varphi,\mathrm{Fil}\geqslant 0} \underset{\widetilde{\Theta}}{\hookrightarrow} \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbf{Q}_p. \tag{11.2.3}$$

On the other hand, by Proposition 11.2.7 we have a fully faithful functor

$$\mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbf{Q}_p \underset{(11.2.1)}{\hookrightarrow} \mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}} \otimes \mathbf{Q}_p \xrightarrow[\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}]{\simeq} \left(\mathrm{Rep}^{\mathrm{free}}_{\mathbf{Z}_p}(G_{K_\infty})\right) \otimes \mathbf{Q}_p \simeq \mathrm{Rep}_{\mathbf{Q}_p}(G_{K_\infty}) \tag{11.2.4}$$

(which coincides with the functor $\underline{V}^*_{\mathfrak{S}}$ on p-isogeny categories thanks to Corollary 11.2.4(1)).

Definition 11.2.9. An object in the essential image of (11.2.4) is called a *p-adic G_{K_∞}-representation with finite E-height.*

We will see later that D_{cris} is an equivalence of categories (i.e. weakly admissible implies admissible) and that the composite functor

$$\mathrm{Rep}^{\mathrm{cris},\leqslant 0}_{\mathbf{Q}_p}(G_K) \underset{(11.2.3)}{\hookrightarrow} \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbf{Q}_p \underset{(11.2.4)}{\hookrightarrow} \mathrm{Rep}_{\mathbf{Q}_p}(G_{K_\infty})$$

coincides with the "restriction functor" $\mathrm{Rep}_{\mathbf{Q}_p}(G_k) \to \mathrm{Rep}_{\mathbf{Q}_p}(G_{K_\infty})$ evaluated on crystalline representations.

Using $\mathfrak{S}$-modules, we now describe G_{K_∞}-stable $\mathbf{Z}_p$-lattices in p-adic G_{K_∞}-representations of finite E-height:

Lemma 11.2.10. *Fix an object $\mathfrak{M}$ of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ with $\mathfrak{S}$-rank at most d, let $V := \underline{V}^*_{\mathfrak{S}}(\mathfrak{M}) \otimes \mathbf{Q}_p$ be the corresponding d-dimensional object of $\mathrm{Rep}_{\mathbf{Q}_p}(G_{K_\infty})$, and set $\mathscr{M}_{\mathscr{E}} := \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{E}$. Then the functor*

$$\underline{V}^*_{\mathfrak{S}} : \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \longrightarrow \mathrm{Rep}^{\mathrm{free}}_{\mathbf{Z}_p}(G_{K_\infty})$$

restricts to a bijection between objects $\mathfrak{N}$ of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ that are contained in $\mathscr{M}_{\mathscr{E}}$ and have $\mathfrak{S}$-rank d, and G_{K_∞}-stable $\mathbf{Z}_p$-lattices $L \subseteq V$ with rank d.

The proof will show that the E-height of $\mathfrak{N}$ as in the lemma is independent of $\mathfrak{N}$ (and is equal to the E-height of $\mathfrak{M}$).

Idea of proof. By Theorem 3.2.5, for any G_{K_∞}-stable $\mathbf{Z}_p$-lattice $L \subseteq V$ there is a unique object $\mathscr{N}$ of $\mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}}$ that is contained in $\mathscr{M}_{\mathscr{E}}$ with full $\mathscr{O}_{\mathscr{E}}$-rank and satisfies

$$L = \underline{V}^*_{\mathscr{O}_{\mathscr{E}}}(\mathscr{N})$$

(recall $\mathscr{N}$ is given explicitly by $\mathscr{N} = \mathrm{Hom}_{\mathbf{Z}_p[G_{K_\infty}]}\left(L, \widehat{\mathscr{O}_{\mathscr{E}^{\mathrm{un}}}}\right)$). The key idea is to adapt the gluing method used in the proof of Lemma 10.4.7, using Corollary 11.2.4 and Proposition 11.2.7 to show that

$$\mathfrak{N} := \mathscr{N} \cap \mathfrak{M}\left[\frac{1}{p}\right] \subseteq \mathscr{M}_{\mathscr{E}}$$

is an object of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ (e.g., it is finite and free over $\mathfrak{S}$) and that the natural map

$$\mathfrak{N} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathscr{E}} \to \mathscr{N}$$

is an isomorphism. See the proof of [30, Lemma 2.1.15] for the details. ■

11.3. **Applications to semistable and crystalline representations.** Recall that the ring of p-adic periods $B_{\mathrm{st}} = B_{\mathrm{st},K}$ is intrinsic but its map to B_{dR} depends on a choice of G_K-equivariant $\log_{\overline{K}} : \overline{K}^{\times} \to \overline{K}$ with $\log_{\overline{K}}(p) \in K_0$; we made the convention $\log_{\overline{K}}(p) = 0$. The filtration on $D_{\mathrm{st}}(V)_K$ depends on this choice. The functor

$$\begin{aligned} D_{\mathrm{st}} : \mathrm{Rep}_{\mathbf{Q}_p}(G_k) &\longrightarrow \mathrm{MF}_K^{\phi,N} \\ V &\longmapsto \left(V \otimes_{\mathbf{Q}_p} B_{\mathrm{st}}\right)^{G_K} \end{aligned}$$

has restriction to $\mathrm{Rep}^{\mathrm{st}}_{\mathbf{Q}_p}(G_k)$ that is fully faithful and has image in the subcategory ${}^{\mathrm{w.a.}}\mathrm{MF}_K^{\phi,N}$ of weakly admissible filtered (ϕ, N)-modules (Proposition 9.2.14 and Theorem 9.3.4).

On the full subcategory $\mathrm{Rep}^{\mathrm{st},\leqslant 0}_{\mathbf{Q}_p}(G_K)$ of representations having Hodge-Tate weights $\leqslant 0$, the functor D_{st} has image contained in the subcategory ${}^{\mathrm{w.a.}}\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0}$. We have the following diagram of categories, in which all sub-diagrams commute, except possibly the large rectangle near the bottom:

$$\begin{tikzcd}
\mathrm{Rep}^{\mathrm{st},\leqslant 0}_{\mathbf{Q}_p}(G_K) \arrow[r, hook, "D_{\mathrm{st}}"] & {}^{\mathrm{w.a.}}\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0} \arrow[r, "\simeq", "\mathrm{Thm.10.3.8}"'] & \mathrm{Mod}_{/\mathscr{O}}^{\phi,N_{\nabla},0} \arrow[r, hook, "(10.4.1)"] \arrow[rrd, bend left] & \mathrm{Mod}_{/\mathscr{O}}^{\phi,N,0} & \\
\mathrm{Rep}^{\mathrm{cris},\leqslant 0}_{\mathbf{Q}_p}(G_K) \arrow[u, hook] \arrow[r, hook, "D_{\mathrm{cris}}"] \arrow[d, "\mathrm{res}"'] & {}^{\mathrm{w.a.}}\mathrm{MF}_K^{\varphi,\mathrm{Fil}\geqslant 0} \arrow[u, hook] \arrow[r, "\simeq", "\mathrm{Thm.10.3.8}"'] & \mathrm{Mod}_{/\mathscr{O}}^{\phi,N_{\nabla},0,N=0} \arrow[u, hook] \arrow[r, hook] & \mathrm{Mod}_{/\mathscr{O}}^{\varphi,0} \arrow[u, hook, "(\dagger)"] & \mathrm{Mod}_{/\mathfrak{S}}^{\phi,N} \otimes \mathbf{Q}_p \arrow[ul, "\simeq"', "(10.4.4)"] \\
\mathrm{Rep}_{\mathbf{Q}_p}(G_{K_\infty}) & \mathrm{Mod}^{\varphi}_{\mathscr{O}_{\mathscr{E}}} \otimes \mathbf{Q}_p \arrow[l, "\simeq"', "\underline{V}_{\mathscr{O}_{\mathscr{E}}*}"] & & \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbf{Q}_p \arrow[ll, hook', "(11.2.7)"'] \arrow[u, "(10.4.7)", "\simeq"'] \arrow[ur, hook, "(\dagger)"'] \arrow[lll, bend left, "\underline{V}_{\mathfrak{S}*}"] &
\end{tikzcd}$$

Note that if we start at $\mathrm{Rep}^{\mathrm{cris},\leqslant 0}_{\mathbf{Q}_p}(G_K)$ and move around the large rectangle in the bottom of the diagram in the clockwise direction, then we obtain a fully faithful embedding $\mathrm{Rep}^{\mathrm{cris},\leqslant 0}_{\mathbf{Q}_p}(G_K) \hookrightarrow \mathrm{Rep}_{\mathbf{Q}_p}(G_{K_\infty})$. If we know that this rectangle commutes, we obtain a proof of a conjecture of Breuil:

Corollary 11.3.1. *The natural restriction map*

$$\mathrm{res} : \mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(G_K) \to \mathrm{Rep}_{\mathbf{Q}_p}(G_{K_\infty})$$

is fully faithful.

Remarks 11.3.2. Before sketching the proof of Breuil's conjecture, we make the following remarks concerning the preceding large diagram:

(1) Recall that the essential image of the curving map in the upper right corner of the diagram is described by Corollary 10.4.9.

(2) The two maps labeled (†) in the diagram are *not* essentially surjective. This prevents us from generalizing Corollary 11.3.1 to the case of semistable representations (and rightly so, as is shown in Example 11.4.4.

To prove Corollary 11.3.1, we first observe that after twisting by $\mathbf{Q}_p(-n)$ for large enough n, it is enough to show that the restriction map

$$\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{cris},\leqslant 0}(G_K) \to \mathrm{Rep}_{\mathbf{Q}_p}(G_{K_\infty})$$

is fully faithful. As noted above, this follows if we can show that the large rectangle in the diagram commutes. Such commutativity follows once we know that the entire outside edge of the diagram commutes. Using the fact that

$$V_{\mathrm{st}} : {}^{\mathrm{w.a.}}\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0} \to \mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{st},\leqslant 0}(G_K)$$

is quasi-inverse to D_{st} on the essential image of D_{st}, it therefore suffices to prove the commutativity of

(11.3.1)
$$\begin{tikzcd} {}^{\mathrm{w.a.}}\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0} \arrow[r, hook, "\widetilde{\Theta}"] \arrow[d, "V_{\mathrm{st}}"] & \mathrm{Mod}_{/\mathfrak{S}}^{\phi,N} \otimes \mathbf{Q}_p \arrow[d, "(\underline{V}_{\mathfrak{S}})_* \otimes \mathbf{Q}_p"] \\ \mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{st},\leqslant 0}(G_K) \arrow[r, "\mathrm{res}"'] & \mathrm{Rep}_{\mathbf{Q}_p}(G_{K_\infty}) \end{tikzcd}$$

where the right side "forgets N" and the top horizontal arrow is (10.4.5). Note that we do not yet know that the left side is an equivalence, since we have not yet proved "weakly admissible implies admissible".

To show that (11.3.1) commutes, let D be any object of ${}^{\mathrm{w.a.}}\mathrm{MF}_K^{\phi,N,\mathrm{Fil}\geqslant 0}$. Let $\mathscr{M} := \underline{\mathscr{M}}(D)$ be the corresponding object of $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N_\nabla,0}$ (via Theorems 10.2.1 and 10.3.8), and choose an object $\mathfrak{M}$ of $\mathrm{Mod}_{/\mathfrak{S}}^{\phi,N}$ such that $\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O} \simeq \mathscr{M}$ in $\mathrm{Mod}_{/\mathscr{O}}^{\phi,N}$ (via the equivalence of Lemma 10.4.7), so $\mathfrak{M} = \widetilde{\Theta}(D)$. Recall that $\mathfrak{M}$ is functorial in D, up to p-isogeny. The commutativity of (11.3.1) follows immediately from the following statement by dualizing in $\mathrm{Rep}_{\mathbf{Q}_p}(G_{K_\infty})$.

Proposition 11.3.3. *With the notation above, there is a natural $\mathbf{Q}_p[G_{K_\infty}]$-linear isomorphism*

$$\begin{tikzcd} \mathrm{Hom}_{\mathfrak{S},\varphi}\left(\mathfrak{M}, \widehat{\mathfrak{S}^{\mathrm{un}}}\right) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \arrow[r, "\simeq"] \arrow[d, equal] & \mathrm{Hom}_{\mathrm{Fil},\phi,N}\left(D, B_{\mathrm{st}}^+\right) \arrow[d, equal] \\ \underline{V}_{\mathfrak{S}}^*(\mathfrak{M}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p & \mathrm{Hom}_{\mathrm{Fil},\phi,N}\left(D, B_{\mathrm{st}}\right) \arrow[d, equal] \\ & V_{\mathrm{st}}^*(D) \end{tikzcd}$$

Before we explain the proof of this proposition, note that by Corollary 11.2.4(1) we have

$$\dim_{\mathbf{Q}_p} V_{\mathfrak{S}}^*(\mathfrak{M}) \otimes \mathbf{Q}_p = \mathrm{rk}_{\mathfrak{S}}(\mathfrak{M}) = \dim_{K_0} \mathscr{M}/u\mathscr{M} = \dim_{K_0} D,$$

so by Proposition 11.3.3 (which gives $\dim_{\mathbf{Q}_p} V^*_{\rm st}(D) = \dim_{\mathbf{Q}_p} V^*_{\mathfrak{S}}(\mathfrak{M}) \otimes \mathbf{Q}_p$) we deduce that $\dim_{\mathbf{Q}_p} V^*_{\rm st}(D) = \dim_{K_0} D$. Thus, by weak admissibility of D and Proposition 9.3.9, the natural map

$$B_{\rm st} \otimes_{\mathbf{Q}_p} V^*_{\rm st}(D) \to B_{\rm st} \otimes_{K_0} D$$

is an isomorphism. Hence, D is admissible by [22, Prop. 5.3.6].

Remark 11.3.4. Since D above was any object in ${}^{\rm w.a}{\rm MF}_K^{\phi,N,{\rm Fil}\geqslant 0}$, this shows that "weakly admissible implies admissible" in full generality, as we can always shift the filtration to be effective.

Proof of Proposition 11.3.3. By Proposition 9.3.9, if D is admissible if and only if it is weakly admissible and $\dim_{\mathbf{Q}_p} V^*_{\rm st}(D) \geqslant \dim_{K_0} D$. Thus, it suffices to construct a natural $\mathbf{Q}_p[G_{K_\infty}]$-linear *injection*

$$\underline{V}^*_{\mathfrak{S}}(\mathfrak{M}) \otimes \mathbf{Q}_p \hookrightarrow V^*_{\rm st}(D) = {\rm Hom}_{\varphi,{\rm Fil},N}(D, B_{\rm st}). \tag{11.3.2}$$

We will just do this in the case that $N_D = 0$, as it contains the essential ideas for the general case (see the proof of [30, Prop. 2.1.5] for the details in the general case).

Recall that $B_{\rm cris} \otimes_{K_0} K$ is equipped with a filtration via its inclusion into the discretely-valued field $B_{\rm dR}$, and that a K_0-linear map $D \to B_{\rm cris}$ is *compatible with filtrations* if the extension of scalars $D_K \to B_{\rm cris} \otimes K$ respects filtrations (i.e. if the composite $D_K \to B_{\rm dR}$ is compatible with filtrations). Since $B_{\rm st}^{N=0} = B_{\rm cris}$ and $N_D =$ we have $V^*_{\rm cris}(D) = V^*_{\rm st}(D)$, so our aim is to construct a natural $\mathbf{Q}_p[G_{K_\infty}]$-linear injection

$$\underline{V}^*_{\mathfrak{S}}(\mathfrak{M}) \otimes \mathbf{Q}_p = {\rm Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \widehat{\mathfrak{S}^{\rm un}}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \hookrightarrow {\rm Hom}_{\varphi,{\rm Fil}}(D, B^+_{\rm cris}) = V^*_{\rm cris}(D) \tag{11.3.3}$$

(the final equality using the effectivity of the filtration on D_K).

An element of $\mathscr{O}$ has a Taylor expansion $\sum c_m u^m$ with $c_m \in K_0 = {\rm W}(k)[1/p]$ and $|c_m| r^m \to 0$ for all $0 < r < 1$. For $p^{-1/(p-1)} < r_0 < 1$ we have $|m!|/r_0^m \to 0$, so

$$|m! c_m| = (|c_m| r_0^m) \cdot \frac{|m!|}{r_0^m} \to 0.$$

Thus, by [21, 4.1.3] there is a natural map of K_0-algebras

$$\mathscr{O} \to B^+_{\rm cris} \tag{11.3.4}$$

extending the natural map $\mathfrak{S} \hookrightarrow {\rm W}(R) \subseteq B^+_{\rm cris}$ defined by "evaluation at $[\widetilde{\pi}]$" (i.e. $u \mapsto [\widetilde{\pi}]$). Using the the natural topologies on $\mathscr{O}$ and $B^+_{\rm cris}$, one checks that this map is moreover continuous, and since $K_0[u]$ is dense in $\mathscr{O}$ it is the unique such continuous K_0-algebra map. Since $\mathfrak{S}\left[\frac{1}{p}\right]$ is dense in $\mathscr{O}$, the map (11.3.4) is also φ-compatible, as this is true of

$$\mathfrak{S}\left[\frac{1}{p}\right] \hookrightarrow {\rm W}(R)\left[\frac{1}{p}\right]$$

(thanks to the relation $[\widetilde{\pi}^p] = [\widetilde{\pi}]^p$).

We will define the map (11.3.3) as the composite of two maps. First, recalling that $\mathscr{M} := \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}$, consider the map

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \widehat{\mathfrak{S}^{\rm un}}) \longrightarrow {\rm Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, B^+_{\rm cris}) = {\rm Hom}_{\mathscr{O},\varphi}(\mathscr{M}, B^+_{\rm cris}) \tag{11.3.5}$$

defined by composition with the natural map $\widehat{\mathfrak{S}^{\mathrm{un}}} \hookrightarrow \mathrm{W}(R) \subseteq B_{\mathrm{cris}}^+$. This map is injective.

Second, we consider the map

$$\mathrm{Hom}_{\mathscr{O},\varphi}(\mathscr{M}, B_{\mathrm{cris}}^+) \longrightarrow \mathrm{Hom}_{\mathscr{O},\varphi}(D \otimes_{K_0} \mathscr{O}, B_{\mathrm{cris}}^+) = \mathrm{Hom}_{\varphi}(D, B_{\mathrm{cris}}^+)\ . \tag{11.3.6}$$

given by composition with the φ-compatible $\mathscr{O}$-linear map

$$\xi : D \otimes_{K_0} \mathscr{O} = \underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O} \hookrightarrow \mathscr{M}$$

of Lemma 10.2.5. We claim that (11.3.6) is injective with image contained in

$$\mathrm{Hom}_{\varphi,\mathrm{Fil}}(D, B_{\mathrm{cris}}^+) \subseteq \mathrm{Hom}_{\varphi}(D, B_{\mathrm{cris}}^+).$$

To verify these claims, we proceed as follows.

Obviously $E(u) = (u - \pi)G(u)$ in $K[u]$, for some $G(u) \in K[u]$ with $G(\pi) \neq 0$. It follows that the map

$$\mathfrak{S} \to B_{\mathrm{cris}}^+ \subseteq B_{\mathrm{dR}}^+ \tag{11.3.7}$$

(defined by $u \mapsto [\widetilde{\pi}]$) carries E to

$$E([\underline{\pi}]) = ([\widetilde{\pi}] - \pi) \cdot G([\widetilde{\pi}]).$$

As $[\widetilde{\pi}] - \pi$ is a uniformizer of B_{dR}^+ and $G(\pi) \neq 0$, we see that $G([\widetilde{\pi}]) \in (B_{\mathrm{dR}}^+)^\times$ and hence that $E([\widetilde{\pi}])$ is a uniformizer of B_{dR}^+. Therefore, (11.3.7) induces a local map of local K_0-algebras

$$\mathfrak{S}\left[\frac{1}{p}\right]_{(E)} \to B_{\mathrm{dR}}^+.$$

Passing to completions (and recalling that B_{dR}^+ is a complete discrete valuation ring) we see that the map (11.3.4) extends to a K_0-algebra map

$$\mathscr{O}_{\Delta,\pi}^{\wedge} = \mathfrak{S}\left[\frac{1}{p}\right]_{(E)}^{\wedge} \to B_{\mathrm{dR}}^+$$

which is even a map of K-algebras, as can be seen (via Hensel's Lemma) by examining the induced map on residue fields.

Thus, given an $\mathscr{O}$-linear map $\mathscr{M} \to B_{\mathrm{cris}}^+ \subseteq B_{\mathrm{dR}}^+$, the map

$$(1 \otimes \varphi_{\mathscr{M}})(\varphi_{\mathscr{O}}^* \mathscr{M}) \to B_{\mathrm{dR}}^+$$

induced by restriction carries $(1 \otimes \varphi_{\mathscr{M}})(\varphi_{\mathscr{O}}^*\mathscr{M}) \cap E^i \mathscr{M}$ into $\mathrm{Fil}^i B_{\mathrm{dR}}^+$, and hence is compatible with these filtrations. Moreover, $\xi : \underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O} \hookrightarrow \mathscr{M}$ is φ-compatible and so has image landing in

$$\varphi_{\mathscr{M}}(\mathscr{M}) \subseteq (1 \otimes \varphi_{\mathscr{M}})(\varphi_{\mathscr{O}}^* \mathscr{M}).$$

But [30, 1.2.12(4)] gives that the induced map

$$\mathscr{O}_{\Delta,\pi}^{\wedge} \otimes_K \underline{D}(\mathscr{M})_K = \mathscr{O}_{\Delta,\pi}^{\wedge} \otimes_{K_0} \underline{D}(\mathscr{M}) \longrightarrow \mathscr{O}_{\Delta,\pi}^{\wedge} \otimes_{\mathscr{O}} (1 \otimes \varphi_{\mathscr{M}})(\varphi_{\mathscr{O}}^* \mathscr{M})$$

is a filtration-compatible isomorphism (where the filtrations are the usual tensor-filtrations; cf. Remark 10.2.3). It follows at once that (11.3.6) has image contained in $\mathrm{Hom}_{\varphi,\mathrm{Fil}}(D, B_{\mathrm{cris}}^+)$; moreover the resulting map

$$\mathrm{Hom}_{\mathscr{O},\varphi}(\mathscr{M}, B_{\mathrm{cris}}^+) \longrightarrow \mathrm{Hom}_{\varphi,\mathrm{Fil}}(D, B_{\mathrm{cris}}^+) \tag{11.3.8}$$

is injective since the injective map

$$(1 \otimes \varphi_{\mathscr{M}})(\varphi_{\mathscr{O}}^{*}\mathscr{M}) \hookrightarrow \mathscr{M}$$

has cokernel killed by some E^h and $E([\widetilde{\pi}]) \in B_{\mathrm{dR}}^{+}$ is a nonzero element of a domain.

Composing the injective maps (11.3.8) and (11.3.5) gives a $\mathbf{Q}_p$-linear injective map

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \widehat{\mathfrak{S}^{\mathrm{un}}}) \hookrightarrow \mathrm{Hom}_{\varphi,\mathrm{Fil}}(D, B_{\mathrm{cris}}^{+}) \ .$$

This map is moreover $\mathbf{Q}_p[G_{K_\infty}]$-linear because the action of G_{K_∞} on B_{cris}^{+} leaves the map $\mathscr{O} \to B_{\mathrm{cris}}^{+}$ invariant, as this holds on $\mathfrak{S} \subseteq \mathscr{O}$ due to the fact that $[\widetilde{\pi}]$ is G_{K_∞}-invariant (by definition of $\widetilde{\pi}$). This gives the desired map (11.3.3). ■

11.4. **Exercises.**

Exercise 11.4.1. Recall the definitions $\mathfrak{S}^{\mathrm{un}} := \mathscr{O}_{\mathscr{E}^{\mathrm{un}}} \cap \mathrm{W}(R) \subseteq \mathrm{W}(\mathrm{Frac}\, R)$ and $\widehat{\mathfrak{S}^{\mathrm{un}}} := \widehat{\mathscr{O}_{\mathscr{E}^{\mathrm{un}}}} \cap \mathrm{W}(R) \subseteq \mathrm{W}(\mathrm{Frac}\, R)$ at the end of §11.1. Use that $\mathrm{W}(R)$ is p-adically separated and complete to prove that the inclusion $\mathfrak{S}^{\mathrm{un}} \to \widehat{\mathfrak{S}^{\mathrm{un}}}$ identifies $\widehat{\mathfrak{S}^{\mathrm{un}}}$ with the p-adic completion of $\mathfrak{S}^{\mathrm{un}}$.

Exercise 11.4.2. Carry out the arguments with inverse limits needed to deduce Corollary 11.2.4 from Lemma 11.2.3. In particular, explain why it is essential that we work with $\widehat{\mathfrak{S}^{\mathrm{un}}}$ and not $\mathfrak{S}^{\mathrm{un}}$ in the description of $\underline{V}_{\mathfrak{S}}^{*}(\mathfrak{M})$.

Exercise 11.4.3. At the end of the proof of Lemma 11.2.8, the following fact was used (over a 2-dimensional base ring $\mathfrak{S}$): if A is a normal noetherian domain and $h : M' \to M$ is a map between finite projective A-modules such that the localizations $h_{\mathfrak{p}}$ are isomorphisms for all primes $\mathfrak{p}$ with height at most 1, then h is an isomorphism.

1. Prove this fact. (Hint: reduce to the case when A is local, so the modules become free.)
2. If F is the fraction field of A, prove that an F-linear map $M'_F \to M_F$ carries M' into M if and only if it carries $M'_{\mathfrak{p}}$ into $M_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ of A with height at most 1. (Geometrically, a rational map between vector bundles over $\mathrm{Spec}\, A$ is defined everywhere if it is defined in codimension at most 1, at least when A is normal.)
3. Show that if we we relax "projective" to "torsion-free' for M and M', then the conclusions of each of the preceding parts can fail when $\dim A > 1$ (i.e., give counterexamples).
4. Is (1) true if A is a non-normal noetherian domain? How about (2)?

Exercise 11.4.4. Consider the Tate curve E_π over K. The representation $V_p(E_\pi)$ is reducible and semistable (Example 9.2.1, Example 9.2.9), but upon restriction to G_{K_∞} the extension structure split. as $\mathbf{Q}_p \oplus \mathbf{Q}_p(1)$.

1. Prove that $V_p(E_\pi)$ has non-abelian splitting field over K.
2. Prove that $V_p(E_\pi)$ and $\mathbf{Q}_p \oplus \mathbf{Q}_p(1)$ are non-isomorphic as $\mathbf{Q}_p[G_K]$-modules. Deduce that restriction from G_K to G_{K_∞} is not fully faithful on the category of semistable p-adic representations of G_K.

12. Applications to p-divisible groups and finite group schemes

We now apply the theory of $\mathfrak{S}$-modules developed in §11 to the study of p-divisible groups and finite flat group schemes over $\mathscr{O}_K$. We will also discuss applications to torsion and lattice representations of G_K in the context of earlier work of Fontaine and Laffaille, and we study the restriction from G_K to G_{K_∞} for representations arising from finite flat group schemes over $\mathscr{O}_K$. This builds on ideas and results of Breuil.

We will have to use some background related to finite flat group schemes, such as the concepts of Cartier duality and short exact sequence for finite flat group schemes; see Exercise 12.5.3 for an introduction to Cartier duality.

12.1. **Divided powers and Grothendieck-Messing theory.**

Recall from §7.2 that classical Dieudonné theory classifies p-divisible groups over the perfect field k, and that Fontaine developed a variant applicable to p-divisible groups over $\mathrm{W}(k)$ (subject to a connectedness hypothesis when $p = 2$). We wish to allow ramification, which is to say we seek a classification over $\mathscr{O}_K$. To do this, we will use Grothendieck-Messing theory as our starting point, and to review this we begin with the concept of a divided power structure on an ideal in a ring.

Definition 12.1.1. Let I be an ideal in a commutative ring A. A *PD-structure* on I is a collection of maps

$$\gamma_n : I \to I \qquad n \geqslant 0$$

such that the γ_n satisfy the "obvious" properties of $t^n/n!$ in characteristic zero:

(1) For all $x \in I$, we have $\gamma_0(x) = 1$, $\gamma_1(x) = x$, and $\gamma_n(x) \in I$ for $n \geqslant 1$.
(2) For all $x, y \in I$ and all $n \geqslant 0$,

$$\gamma_n(x+y) = \sum_{i+j=n} \gamma_i(x)\gamma_j(y).$$

(3) If $a \in A$ and $x \in I$ then $\gamma_n(ax) = a^n\gamma_n(x)$ for all $n \geqslant 0$.
(4) For all $x \in I$ and all $m, n \geqslant 0$,

$$\gamma_m(x)\gamma_n(x) = \frac{(m+n)!}{m!n!}\gamma_{m+n}(x).$$

(5) For all $x \in I$ and all $m, n \geqslant 0$,

$$\gamma_n(\gamma_m(x)) = \frac{(mn)!}{(m!)^n n!}\gamma_{mn}(x)$$

Remark 12.1.2. The "PD" standard for *puissances-divisée*–literally "divided powers." Note that the combinatorial coefficients appearing in (4) and (5) are in fact integers and hence can be viewed in a unique way as elements of A.

If $I \subseteq A$ is an ideal in a commutative ring that is equipped with a PD-structure $\{\gamma_n\}$ then for $x \in I$ we will often write $x^{[n]}$ for $\gamma_n(x)$.

Example 12.1.3. It follows from (4) and (1) that $n!\gamma_n(x) = x^n$ for all n and all $x \in I$, so when A is $\mathbf{Z}$-flat there is at most one PD-structure on any ideal I of A: $\gamma_n(x) = x^n/n!$. At the other extreme, if A is a $\mathbf{Z}/N\mathbf{Z}$-algebra for some $N \geqslant 1$ and $I \subseteq A$ admits a PD-structure then $x^N = 0$ for all $x \in I$.

We say that a PD-structure $\{\gamma_n\}$ on I is *PD-nilpotent* if the ideal $I^{[n]}$ generated by all products $x_1^{[i_1]} \cdots x_r^{[i_r]}$ with $\sum i_r \geqslant n$ is zero for some (and hence all) sufficiently large n. This forces $I^n = 0$.

For $I \subseteq \mathscr{O}_K$ the maximal ideal, a PD-structure exists on I if and only if the absolute ramification index e satisfies $e \leqslant p-1$. On the other hand, the ideal $p\mathscr{O}_K$ *always* has a PD-structure, as $\gamma_n(py) = (p^n/n!) \cdot y^n$ with $p^n/n! \in p\mathbf{Z}_p$.

In general, there can be many choices of PD-structure $\{\gamma_n\}$ on an ideal I.

Example 12.1.4. Recall that A_{cris} is $\mathbf{Z}_p$-flat and comes equipped with a canonical surjection $A_{\mathrm{cris}} \twoheadrightarrow \mathscr{O}_{\mathbf{C}_K}$. The kernel of $\mathrm{Fil}^1 A_{\mathrm{cris}}$ of this surjection has a (necessarily unique) PD-structure. Indeed, the analogous claim holds for A^0_{cris}, and then passing to the p-adic completion A_{cris} provides the desired PD-structure. (See Exercise 12.5.2.)

Theorem 12.1.5 (Grothendieck-Messing)**.** *Let A_0 be a ring in which p is nilpotent and let G_0 be a p-divisible group over A_0. For any surjection $h : A \twoheadrightarrow A_0$ such that $I := \ker h$ is endowed with a PD-structure $\{\gamma_n\}$ and some power I^N vanishes, there is attached a finite locally free A-module*

$$\underline{\mathbf{D}}(G_0)(A) = \underline{\mathbf{D}}(G_0)\left(A \twoheadrightarrow A_0, \{\gamma_n\}\right)$$

with $\mathrm{rk}_A(\underline{\mathbf{D}}(G_0)(A)) = \mathrm{ht}\, G_0$. This association is contravariant in G_0 and commutes with PD-base change in A (i.e. base change that respects h and the divided power structure on $\ker h$).

The locally free A_0-module $\mathrm{Lie}(G_0)$ is naturally a quotient of $\underline{\mathbf{D}}(G_0)(A_0)$, and if $\{\gamma_n\}$ is PD-nilpotent then there is an equivalence of categories between the category of deformations of G_0 to A and the category of locally free quotients $\underline{\mathbf{D}}(G_0)(A) \twoheadrightarrow \mathscr{E}$ lifting $\underline{\mathbf{D}}(G_0)(A_0) \twoheadrightarrow \mathrm{Lie}(G_0)$.

Remarks 12.1.6. The classification of deformations at the end of the theorem can also be formulated in terms of subbundles rather than quotients. Colloquially speaking, we may say that in order to lift G_0 through a nilpotent divided power thickening A of A_0, it is equivalent to lift its Lie algebra to a locally free quotient of $\underline{\mathbf{D}}(G_0)(A)$.

1. The equivalence of categories at the end of the theorem associates to any deformation G of G_0 to A the module $\mathrm{Lie}(G)$, which is naturally a quotient of $\underline{\mathbf{D}}(G_0)(A)$.
2. This equivalence also works for deforming maps of p-divisible groups $G_0 \to H_0$. That is, a map $f_0 : G_0 \to H_0$ has at most one lift to a map $f : G \to H$, and f exists if and only if $\underline{\mathbf{D}}(f_0) : \underline{\mathbf{D}}(G_0)(A) \to \underline{\mathbf{D}}(H_0)(A)$ is compatible with the quotients associated to the liftings G and H of G_0 and H_0 respectively.
3. If $p > 2$ then $p^n/n! \to 0$ in $\mathbf{Z}_p$, whereas $2^{2^j}/(2^j)! \in 2\mathbf{Z}_2^\times$ for all $j \geqslant 0$. It follows that the (unique) PD-structure on the ideal (p) in $\mathbf{Z}_p$ is topologically PD-nilpotent for $p > 2$ but not for $p = 2$. This is one reason why the case $p = 2$ is such a headache in the crystalline theory. See Exercise 12.5.1.
4. The right way to state Theorem 12.1.5 is to use the language of *crystals*. In this terminology, $\underline{\mathbf{D}}$ is a contravariant functor from the category of p-divisible groups over

a base scheme S on which p is locally nilpotent to the category of crystals in locally free $\mathscr{O}_S$-modules.

(5) By taking projective limits, the Grothendieck-Messing Theorem has an analogue for A_0 merely p-adically separated and complete (for example, $A_0 = \mathscr{O}_K$) and $A \twoheadrightarrow A_0$ any surjection of p-adically separated and complete rings with kernel $I \subseteq A$ that is topologically nilpotent (resp. endowed with topologically PD-nilpotent divided powers).

12.2. **S-modules.** In work of Breuil on finite flat group schemes and p-divisible groups over $\mathscr{O}_K$, a certain ring S plays a vital role. Breuil's method began by studying finite flat group schemes over $\mathscr{O}_K$ in terms of S-modules, and then gave a theory for p-divisible groups by passage to the limit. Kisin provided an approach in the other direction, using Grothendieck–Messing theory to derive Breuil's description of p-divisible groups via S-modules without any preliminary work at finite level, and then used this to deduce a classification for p-divisible groups and finite flat group schemes with $\mathfrak{S}$-modules rather than S-modules. The possibility of getting a classification with the simpler ring $\mathfrak{S}$ in place of S had been conjectured in a precise form by Breuil.

We now introduce Breuil's ring S. Let $\mathrm{W}(k)[u]\left[\frac{E(u)^i}{i!}\right]_{i\geqslant 1}$ be the subring of $K_0[u]$ generated over $\mathrm{W}(k)[u]$ by the set $\{E^i/i!\}_{i\geqslant 1}$ (this is the *divided power envelope* of $\mathrm{W}(k)[u]$ with respect to the ideal $E(u)\cdot \mathrm{W}(k)[u]$). Clearly this ring is $\mathrm{W}(k)$-flat. Further, there is an evident surjective map

$$\mathrm{W}(k)[u]\left[\frac{E(u)^i}{i!}\right]_{i\geqslant 1} \twoheadrightarrow \mathscr{O}_K \tag{12.2.1}$$

defined via $u \mapsto \pi$. with kernel generated by all $E^i/i!$. Let S be the p-adic completion of $\mathrm{W}(k)[u]\left[\frac{E(u)^i}{i!}\right]_{i\geqslant 1}$ and let $\mathrm{Fil}^1 S \subseteq S$ be the ideal that is (topologically) generated by all $E^i/i!$. We view S as a topological ring via its (separated and complete) p-adic topology. The ring S is local and $\mathrm{W}(k)$-flat (but *not* noetherian), and the map (12.2.1) induces an isomorphism

$$S/\operatorname{Fil}^1 S \xrightarrow{\simeq} \mathscr{O}_K\ .$$

Moreover, there is a unique continuous map $\varphi_S : S \to S$ restricting to the Frobenius endomorphism of $\mathrm{W}(k)$ and satisfying $\varphi_S(u) = u^p$. Note that $\varphi_S(\mathrm{Fil}^1 S) \subseteq pS$ and $\varphi_S \bmod pS = \mathrm{Frob}_{S/pS}$.

The ideal $\mathrm{Fil}^1 S$ admits (topologically PD-nilpotent) divided powers, so for any p-divisible group G over $\mathscr{O}_K$ with Cartier dual G^* we get a finite free (as S is local) S-module

$$\begin{aligned}\underline{\mathscr{M}}(G) &:= \underline{\mathbf{D}}(G^*)(S \twoheadrightarrow \mathscr{O}_K)\\ &= \varprojlim_N \underline{\mathbf{D}}(G^* \bmod p^N)(S/p^N S \twoheadrightarrow \mathscr{O}_K/p^N\mathscr{O}_K)\end{aligned}$$

with $\mathrm{rk}_S\,\underline{\mathscr{M}}(G) = \mathrm{ht}(G)$. Here, the kernel of $S/p^N S \twoheadrightarrow \mathscr{O}_K/p^N\mathscr{O}_K$ is given the PD-structure induced by that on $\mathrm{Fil}^1 S$, and $\underline{\mathscr{M}}(G)$ is contravariant in G.

Since the ideal
$$\mathrm{Fil}^1 S + pS = \ker(S \twoheadrightarrow \mathscr{O}_K/p\mathscr{O}_K)$$
is also equipped with topologically PD-nilpotent divided powers if $p > 2$, and the formation of $\underline{\mathbf{D}}$ is compatible with base change (i.e., it is a crystal), by setting $G_0 = G \bmod p$ we also have
$$\underline{\mathscr{M}}(G) := \underline{\mathbf{D}}(G_0^*)(S \twoheadrightarrow \mathscr{O}_K/p\mathscr{O}_K)$$
if $p > 2$. This shows, in particular, that $\underline{\mathscr{M}}(G)$ depends contravariantly functorially on G_0 if $p > 2$. With some more work (see [30, Lemma A.2]), for all p the S-module $\underline{\mathscr{M}}(G)$ can naturally be made into an object of the following category that was introduced by Breuil.

Definition 12.2.1. Let $\mathrm{BT}^{\varphi}_{/S}$ be the category of finite free S-modules $\mathscr{M}$ that are equipped with an S-submodule $\mathrm{Fil}^1 \mathscr{M} \subseteq \mathscr{M}$ and a φ_S-semilinear map $\varphi_{\mathscr{M}} : \mathrm{Fil}^1 \mathscr{M} \to \mathscr{M}$ such that

1. $(\mathrm{Fil}^1 S) \cdot \mathscr{M} \subseteq \mathrm{Fil}^1 \mathscr{M}$,
2. the finitely generated $S/\mathrm{Fil}^1 S \simeq \mathscr{O}_K$-module $\mathscr{M}/\mathrm{Fil}^1 \mathscr{M}$ is free,
3. the subset $\varphi_{\mathscr{M}}(\mathrm{Fil}^1 \mathscr{M})$ spans $\mathscr{M}$ over S.

Morphisms are S-module homomorphisms that are compatible with $\varphi_{\mathscr{M}}$ and Fil^1. A three-term sequence of objects of $\mathrm{BT}^{\varphi}_{/S}$ is said to be a *short exact sequence* if the sequences of S-modules and Fil^1's are both short exact.

Example 12.2.2. We give the two "canonical" examples of S-modules arising from p-divisible groups over $\mathscr{O}_K$ via the functor $\underline{\mathscr{M}}$. Both examples follow from unraveling definitions (including the construction of the crystal $\underline{\mathbf{D}}$ in terms of a universal vector extension).

For $G = \mathbf{G}_m[p^\infty] = \varinjlim_n \mathbf{G}_m[p^n]$ we have
$$\underline{\mathscr{M}}(G) = S, \quad \mathrm{Fil}^1 \underline{\mathscr{M}}(G) = \mathrm{Fil}^1 S, \quad \text{and} \quad \varphi_{\underline{\mathscr{M}}(G)} = \frac{\varphi_S}{p} : \mathrm{Fil}^1 S \to S.$$
Meanwhile, for $G = \mathbf{Q}_p/\mathbf{Z}_p = \varinjlim \frac{1}{p^n}\mathbf{Z}/\mathbf{Z}$ we have
$$\underline{\mathscr{M}}(G) = S, \quad \mathrm{Fil}^1 \underline{\mathscr{M}}(G) = S, \quad \text{and} \quad \varphi_{\underline{\mathscr{M}}(G)} = \varphi_S : S \to S.$$

Example 12.2.3. The classical contravariant Dieudonné module of $G_0 = G \mod \pi$ (equipped with its $\mathscr{F}$ and $\mathscr{V}$ operators) can be recovered from $\underline{\mathscr{M}}(G)$; for example, its underlying $\mathrm{W}(k)$-module is the scalar extension of $\underline{\mathscr{M}}(G)$ along the map $S \twoheadrightarrow S/uS = \mathrm{W}(k)$ followed by scalar extension by the inverse of the Frobenius automorphism of $\mathrm{W}(k)$. In particular, G is connected if and only if $m \mapsto \varphi_{\underline{\mathscr{M}}(G)}(E(u)m)$ viewed on $\underline{\mathscr{M}}(G)/u\underline{\mathscr{M}}(G)$ is topologically nilpotent for the p-adic topology. (This evaluation of $\varphi_{\underline{\mathscr{M}}(G)}$ makes sense since $E(u)m \in (\mathrm{Fil}^1 S) \cdot \mathscr{M} \subseteq \mathrm{Fil}^1 \mathscr{M}$ for any $\mathscr{M}$ in $\mathrm{BT}^{\varphi}_{/S}$.) Thus, we say $\mathscr{M}$ in $\mathrm{BT}^{\varphi}_{/S}$ is *connected* if $m \mapsto \varphi_{\mathscr{M}}(E(u)m)$ on $\mathscr{M}/u\mathscr{M}$ is topologically nilpotent for the p-adic topology.

Using results for p-torsion groups, Breuil proved (for $p > 2$) the following theorem classifying p-divisible groups over $\mathscr{O}_K$.

Proposition 12.2.4. *If $p > 2$ then the contravariant functor $\underline{\mathscr{M}}$ from the category of p-divisible groups over $\mathscr{O}_K$ to the category $\mathrm{BT}^{\varphi}_{/S}$ is an exact equivalence of categories with exact quasi-inverse. The same statement holds for $p = 2$ working only with connected objects.*

Proof. For $p > 2$, one uses Grothendieck-Messing theory (Theorem 12.1.5) to "lift" from $\mathscr{O}_K/\pi^i\mathscr{O}_K$ to $\mathscr{O}_K/\pi^{i+1}\mathscr{O}_K$, beginning with the analogous statement for p-divisible groups over k as furnished by classical Dieudonné theory. For $p = 2$, one must adapt this method using Zink's theory of windows [32]. ■

Lemma 12.2.5. *If $p > 2$ then $\mathrm{Hom}_{S,\varphi,\mathrm{Fil}}(\underline{\mathscr{M}}(G), A_{\mathrm{cris}})$ is a finite free $\mathbf{Z}_p$-module, and there is a natural $\mathbf{Z}_p[G_{K_\infty}]$-linear isomorphism*

$$T_pG \xrightarrow{\simeq} \mathrm{Hom}_{S,\varphi,\mathrm{Fil}}(\underline{\mathscr{M}}(G), A_{\mathrm{cris}})\ .$$

The same holds for $p = 2$ provided that G is connected.

Proof. We only address the case $p > 2$. There is a unique map of $\mathrm{W}(k)$-algebras $S \to A_{\mathrm{cris}}$ such that $u \mapsto [\widetilde{\pi}]$ (and hence $E^i/i! \in S$ maps to $E([\widetilde{\pi}])^i/i!$). Since G_{K_∞} acts trivially on S and is equal to the isotropy subgroup of $\widetilde{\pi} \in R$ (by definition of $\widetilde{\pi}$), this map is G_{K_∞}-equivariant. Furthermore, the diagram

$$\begin{tikzcd} S \arrow[r, two heads] \arrow[d] & \mathscr{O}_K \arrow[d, hook] \\ A_{\mathrm{cris}} \arrow[r, two heads, "\theta"'] & \mathscr{O}_{\mathbf{C}_K} \end{tikzcd}$$

commutes, so by the "crystal" condition we have a natural isomorphism

$$\underline{\mathbf{D}}(G^*_{\mathscr{O}_{\mathbf{C}_K}})(A_{\mathrm{cris}}) \simeq \underline{\mathbf{D}}(G^*)(S) \otimes_S A_{\mathrm{cris}} \tag{12.2.2}$$

$$= \underline{\mathscr{M}}(G) \otimes_S A_{\mathrm{cris}}. \tag{12.2.3}$$

Thus, since $\underline{\mathbf{D}}(G)$ is covariant in G, we get a $\mathbf{Z}_p$-linear map

$$\begin{aligned} T_pG := \mathrm{Hom}_{\mathbf{C}_K}(\mathbf{Q}_p/\mathbf{Z}_p, G_{\mathbf{C}_K}) &= \mathrm{Hom}_{\mathscr{O}_{\mathbf{C}_K}}(\mathbf{Q}_p/\mathbf{Z}_p, G_{\mathscr{O}_{\mathbf{C}_K}}) \\ &\xrightarrow{\underline{\mathbf{D}}((\cdot)^*)} \mathrm{Hom}_{A_{\mathrm{cris}}}\left(\underline{\mathbf{D}}(G^*_{\mathscr{O}_{\mathbf{C}_K}})(A_{\mathrm{cris}}), \underline{\mathbf{D}}(\mathbf{G}_m[p^\infty])(A_{\mathrm{cris}})\right) \\ &= \mathrm{Hom}_S(\underline{\mathscr{M}}(G), A_{\mathrm{cris}}) \end{aligned}$$

and one checks that this map lands in the submodule $\mathrm{Hom}_{S,\varphi,\mathrm{Fil}}(\underline{\mathscr{M}}(G), A_{\mathrm{cris}})$. Here, the last equality above uses both the identification $\underline{\mathbf{D}}(\mathbf{G}_m[p^\infty])(A_{\mathrm{cris}}) \simeq A_{\mathrm{cris}}$ of Example 12.2.2 and the isomorphism (12.2.3).

Since $S \to A_{\mathrm{cris}}$ is G_{K_∞}-equivariant, the map

$$T_pG \longrightarrow \mathrm{Hom}_{S,\varphi,\mathrm{Fil}}(\underline{\mathscr{M}}(G), A_{\mathrm{cris}}) \tag{12.2.4}$$

thus obtained is $\mathbf{Z}_p[G_{K_\infty}]$-linear. When $G = \mathbf{G}_m[p^\infty]$ and $p > 2$, the map (12.2.4) is seen to be an isomorphism by direct calculation, using Example 12.2.2 and the fact that $A_{\mathrm{cris}}^{\varphi=1,\mathrm{Fil}\geqslant 0} = \mathbf{Z}_p$; this case of (12.2.4) is not an isomorphism if $p = 2$. Provided $p > 2$, combining this special isomorphism with the Cartier duality between G and G^* yields that (12.2.4) is an isomorphism for any G when $p > 2$. (See Exercise 12.5.4 for a concrete manifestation of this idea for using duality to establish an isomorphism.) The isomorphism claim for $p = 2$ requires more work. ■

12.3. **From S to $\mathfrak{S}$.** Let $\mathfrak{S} = \mathrm{W}(k)[\![u]\!]$ be as in §10.4. We have a unique $\mathrm{W}(k)[u]$-algebra map $\mathfrak{S} \to S$, and the diagram

$$\begin{tikzcd} & \mathrm{W}(R) \subseteq A_{\mathrm{cris}} \\ \mathfrak{S} \arrow[ur, hook] \arrow[r] & S \arrow[u] \end{tikzcd}$$

commutes. Denote by $\varphi : \mathfrak{S} \to S$ the composite map

$$\mathfrak{S} \longrightarrow S \xrightarrow{\varphi_S} S$$

and for any object $\mathfrak{M}$ of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ define

$$\mathscr{M} := S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}.$$

If E-ht$(\mathfrak{M}) \leqslant 1$ then (see [30, 2.2.3]) $\mathscr{M}$ can be naturally made into an object of $\mathrm{BT}^{\varphi}_{/S}$ for $p > 2$ (and one has an analogue using Zink's theory of windows when $p = 2$ if $\mathfrak{M}$ is *connected* in the sense that $\varphi_{\mathfrak{M}}$ on $\mathfrak{M}$ is topologically nilpotent). This motivates the following definition.

Definition 12.3.1. Denote by $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ the full subcategory of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ consisting of those $\mathfrak{S}$-modules $\mathfrak{M}$ that have E-height at most 1.

For $p > 2$ Breuil showed that the functor

$$\mathrm{BT}^{\varphi}_{/\mathfrak{S}} \longrightarrow \mathrm{BT}^{\varphi}_{/S}$$

$$\mathfrak{M} \longmapsto S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$$

is exact and fully faithful; as before, when $p = 2$ one has an analogue for connected objects $\mathfrak{M}$ of $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$. By Proposition 12.2.4, we have an anti-equivalence of categories

$$\mathrm{BT}^{\varphi}_{/S} \xleftarrow{\simeq} \{p\text{-divisible groups over } \mathscr{O}_K\}$$

that is exact with exact quasi-inverse for $p > 2$ (and a similar result for $p = 2$ using connected objects), so we get a contravariant and fully faithful functor

$$\underline{G} : \mathrm{BT}^{\varphi}_{/\mathfrak{S}} \longrightarrow \{p\text{-divisible groups over } \mathscr{O}_K\}$$

when $p > 2$ (and a similar functor working with connected objects when $p = 2$). Using Dieudonné theory over k, one shows that a 3-term complex in $\mathrm{BT}^{\varphi}_{\mathfrak{S}}$ is short exact if and only if its image under $\underline{G}$ is short exact.

Theorem 12.3.2. *For $p > 2$, the functor $\underline{G}$ is an equivalence of categories. The same statement holds for $p = 2$ if we work with connected objects.*

Proof. We only discuss the case $p > 2$ (and the case $p = 2$ is treated in [32]). We will construct a contravariant functor

$$\underline{\mathfrak{M}} : \{p\text{-divisible groups over } \mathscr{O}_K\} \longrightarrow \mathrm{BT}^{\varphi}_{/\mathfrak{S}}$$

for any p, and will show that this functor is quasi-inverse to $\underline{G}$ when $p > 2$.

Since the Tate module $V_p(G) := T_p(G) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is an object of $\mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(G_K)$ with Hodge-Tate weights in $\{0,1\}$, it is in the image of the functor

$$V^*_{\mathrm{cris}} : {}^{\mathrm{w.a.}}\mathrm{MF}_K^{\varphi,\mathrm{Fil}\geqslant 0} \longrightarrow \mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(G_K)\ .$$

Thus, using the fully faithful functor

$$\widetilde{\Theta} : {}^{\mathrm{w.a.}}\mathrm{MF}_K^{\varphi,\mathrm{Fil}\geqslant 0} \hookrightarrow \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbf{Q}_p$$

of §11, corresponding to the representation $V_p(G)$ is an $\mathfrak{S}$-module $\mathfrak{M}$, uniquely determined by and functorial in G up to p-isogeny. Moreover, we have $E\text{-ht}(\mathfrak{M}) \leqslant 1$. If h is the height of G then by Lemma 11.2.10, the functor

$$\underline{V}^*_{\mathfrak{S}} : \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \longrightarrow \mathrm{Rep}^{\mathrm{free}}_{\mathbf{Z}_p}(G_{K_\infty})$$

induces a one to one correspondence between G_{K_∞}-stable lattices $L \subseteq V_p(G)$ with rank h and objects $\mathfrak{N}$ of $\mathrm{Mod}^{\varphi}_{\mathfrak{S}}$ that are contained in $\mathscr{M}_{\mathscr{E}} := \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{E}$ and have $\mathfrak{S}$-rank h. Furthermore, since the functor

$$\mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \longrightarrow \mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}} \simeq \mathrm{Rep}_{\mathbf{Q}_p}(G_{K_\infty})$$

is fully faithful by Proposition 11.2.7, we see that $\mathfrak{N}$ is functorial in and uniquely determined by L. Taking $L = T_p(G)$ thus gives an object $\mathfrak{N}$ of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ with $\underline{V}^*_{\mathfrak{S}}(\mathfrak{N}) = T_p(G)$; by our discussion $\mathfrak{N}$ is contravariant in G and we define

$$\underline{\mathfrak{M}}(G) := \mathfrak{N}.$$

To show that $\underline{\mathfrak{M}} \circ \underline{G} \simeq \mathrm{id}$ for $p > 2$, one uses Lemma 12.2.5 to reduce to comparing divisibility by p in $\widehat{\mathfrak{S}^{\mathrm{un}}}$ and A_{cris}; this comparison works if $p > 2$.

To show that $\underline{G} \circ \underline{\mathfrak{M}} \simeq \mathrm{id}$ for $p > 2$, one must construct an isomorphism of p-divisible groups. Using Tate's Theorem 7.2.8, the crystalline property of the representations arising from p-divisible groups, and the full-faithfulness of $\mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(G_K) \to \mathrm{Rep}_{\mathbf{Q}_p}(G_{K_\infty})$ (Corollary 11.3.1), one reduces this to a problem with $\mathbf{Z}_p[G_{K_\infty}]$-modules, again solved by Lemma 12.2.5. ∎

12.4. **Finite flat group schemes and strongly divisible lattices.** For an isogeny $f : \Gamma_1 \to \Gamma_2$ between p-divisible groups over $\mathscr{O}_K$, $\ker f$ is a finite flat group scheme. Conversely, Oort showed that every finite flat group scheme G over $\mathscr{O}_K$ arises in this way. (Raynaud proved a stronger result, using abelian schemes instead of p-divisible groups [4, 3.1.1].) If the Cartier dual G^* is connected then we may arrange that Γ_1^* and Γ_2^* are connected as well.

The anti-equivalence of categories

$$\mathrm{BT}^{\varphi}_{/\mathfrak{S}} \simeq \{p\text{-divisible groups over } \mathscr{O}_K\}$$

of Theorem 12.3.2 for $p > 2$ and its "connected" analogue for $p = 2$ motivate the following definition due to Breuil.

Definition 12.4.1. Let $(\mathrm{Mod}/\mathfrak{S})$ be the category of pairs $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ in $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{/\mathfrak{S}}$ such that $E\text{-ht}(\mathfrak{M}) \leqslant 1$.

We can also define the full subcategory of "connected" objects by requiring $\varphi_{\mathfrak{M}}$ to be nilpotent.

Example 12.4.2. If $\mathfrak{M}$ is an object of $(\mathrm{Mod}\,/\mathfrak{S})$, then $\mathscr{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ is an object of $\mathrm{Mod}^{\mathrm{tor}}_{/\mathscr{O}_{\mathscr{E}}}$ (Definition 3.1.1); indeed, the image of $E \in \mathfrak{S}$ under the natural map $\mathfrak{S} \to \mathscr{O}_{\mathscr{E}}$ lands in $\mathscr{O}_{\mathscr{E}}^{\times}$ by Remark 11.1.1.

Objects in $(\mathrm{Mod}\,/\mathfrak{S})$ are precisely cokernels of maps in $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ that are isomorphisms in the isogeny category, so (with more work for $p = 2$) we get the following result, which was conjectured by Breuil and proved by him in some cases.

Theorem 12.4.3. *If $p > 2$ then there is an anti-equivalence of categories between* $(\mathrm{Mod}\,/\mathfrak{S})$ *and the category of finite flat group schemes over $\mathscr{O}_K$. For $p = 2$, one has such an equivalence working with connected objects in* $(\mathrm{Mod}\,/\mathfrak{S})$ *and connected finite flat group schemes.*

Remark 12.4.4. These equivalences are compatible with the ones for p-divisible groups. Thus, if the finite flat group scheme G over $\mathscr{O}_K$ corresponds to the object $\mathfrak{M}$ of $(\mathrm{Mod}\,/\mathfrak{S})$, then we have an isomorphism of G_{K_∞}-modules $G(\overline{K}) \simeq \underline{V}^*_{\mathfrak{S}}(\mathfrak{M})$, since the analogous statement holds for p-divisible groups (as one sees via the proof of Theorem 12.3.2).

Definition 12.4.5. We say that an object T of $\mathrm{Rep}^{\mathrm{tor}}(G_K)$ is *flat* (resp. *connected*) if $T \simeq G(\overline{K})$ for some finite flat (resp. finite flat and connected) group scheme G over $\mathscr{O}_K$.

Corollary 12.4.6. *The natural restriction functor*

$$\mathrm{Rep}^{\mathrm{tor}}(G_K) \longrightarrow \mathrm{Rep}^{\mathrm{tor}}(G_{K_\infty})$$

is fully faithful on flat (respectively connected) representations if $p > 2$ (respectively $p = 2$).

Proof. This proof is due to Breuil. We only treat the cases $p > 2$. Using the equivalence of categories $\mathrm{Rep}^{\mathrm{tor}}(G_{K_\infty}) \simeq \mathrm{Mod}^{\varphi,\mathrm{tor}}_{\mathscr{O}_{\mathscr{E}}}$ via $\underline{D}^*_{\mathscr{O}_{\mathscr{E}}}$ and $\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}$, and the fact that the diagram

$$\begin{tikzcd}
\mathrm{Rep}^{\mathrm{tor}}(G_K) \arrow[rr] & & \mathrm{Rep}^{\mathrm{tor}}(G_{K_\infty}) \\
\left\{\begin{matrix}\text{finite flat group}\\ \text{schemes over } \mathscr{O}_K\end{matrix}\right\} \arrow[u, "G(\overline{K})"] \arrow[r, "\simeq", "(12.4.3)"'] & (\mathrm{Mod}\,/\mathfrak{S}) \arrow[r, "(\cdot)\otimes_{\mathfrak{S}}\mathscr{O}_{\mathscr{E}}"'] & \mathrm{Mod}^{\varphi,\mathrm{tor}}_{\mathscr{O}_{\mathscr{E}}} \arrow[u, "\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}", "\simeq"']
\end{tikzcd}$$

commutes (due to Lemma 11.2.3 and Remark 12.4.4), it suffices to prove the following statement. Let T_1 and T_2 be flat representations and let G_1 and G_2 be the corresponding finite flat group schemes over $\mathscr{O}_K$, so $T_1 \simeq G_1(\overline{K})$ and $T_2 \simeq G_2(\overline{K})$ (Definition 12.4.5). Denote by $\mathfrak{M}_1$ and $\mathfrak{M}_2$ the objects of $(\mathrm{Mod}\,/\mathfrak{S})$ corresponding to G_1, G_2 via Theorem 12.4.3 and let $\mathscr{M}_i = \mathscr{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_i$ for $i = 1, 2$ be the corresponding objects of $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{\mathscr{O}_{\mathscr{E}}}$. If $h : \mathscr{M}_1 \to \mathscr{M}_2$ is a morphism in $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{\mathscr{O}_{\mathscr{E}}}$ then, after possibly modifying the $\mathfrak{M}_i$ without changing the generic fibers $(G_i)_K$ (so the Galois representations $G_i(\overline{K})$ remain unaffected), there is a morphism $\mathfrak{M}_1 \to \mathfrak{M}_2$ inducing h after extending scalars to $\mathscr{O}_{\mathscr{E}}$.

Due to Lemma 11.2.2, every object of $(\mathrm{Mod}\,/\mathfrak{S})$ has a filtration with successive quotients that are isomorphic to $\oplus_j \mathfrak{S}/p\mathfrak{S}$, so by a standard devissage we may restrict to the case that each $\mathfrak{M}_i$ is killed by p. In this situation, the natural map

$$\mathfrak{M}_i \longrightarrow \mathscr{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_i = k((u)) \otimes_{k[\![u]\!]} \mathfrak{M}_i$$

is injective, so

$$\mathfrak{M}_2' := \mathfrak{M}_2 + h(\mathfrak{M}_2) \subseteq \mathscr{M}_2$$

makes sense and is a φ-stable (as h is φ-equivariant) $\mathfrak{S}$-submodule of $\mathscr{M}_2$. Moreover, $\mathfrak{M}_2$ is an object of $(\mathrm{Mod}\,/\mathfrak{S})$ and so corresponds to a finite flat group scheme G_2' over $\mathscr{O}_K$. Since $\mathfrak{M}_2$ and $\mathfrak{M}_2'$ are both φ-stable lattices in $\mathscr{M}_2$, one shows that $(G_2)_K \simeq (G_2')_K$. The map h then restricts to a map $h' : \mathfrak{M}_1 \to \mathfrak{M}_2'$ that induces h after extending scalars to $\mathscr{O}_{\mathscr{E}}$; this is the desired map. ■

Now we turn to strongly divisible lattices and Fontaine-Laffaille modules. For the remainder of this section, we assume that $K = K_0$ and we take $\pi = p$, so $E = u - p$.

Definition 12.4.7. Let D be an object of ${}^{\mathrm{w.a}}\mathrm{MF}_K^{\varphi,\mathrm{Fil}\geqslant 0}$ with $K = K_0$. A *strongly divisible lattice* in D is a $\mathrm{W}(k)$-lattice $L \subseteq D$ such that

(1) $\varphi_D(L \cap \mathrm{Fil}^i D) \subseteq p^i D$ for all $i \geqslant 0$ (so $\varphi_D(L) \subseteq L$ by taking $i = 0$),
(2) $\sum_{i\geqslant 0} p^{-i}\varphi_D(L \cap \mathrm{Fil}^i D) = L$.

We set $\mathrm{Fil}^i L = L \cap \mathrm{Fil}^i D$, and we say that L is *connected* if $\varphi_D : L \to L$ is topologically nilpotent for the p-adic topology.

Theorem 12.4.8. *There are exact quasi-inverse anti-equivalences between the category of strongly divisible lattices L with $\mathrm{Fil}^p L = 0$ and the category of $\mathbf{Z}_p[G_K]$-lattices Λ in crystalline G_K-representations with Hodge-Tate weights in the set $\{0, \dots, p-1\}$.*

Proof. Let V be a crystalline G_K-representation with Hodge-Tate weights in $\{0, \dots, p-1\}$ and let $\Lambda \subseteq V$ be a G_K-stable $\mathbf{Z}_p$-lattice. Because of Corollary 11.3.1 and Lemma 11.2.10, the lattice Λ corresponds to a unique object $\mathfrak{M}$ of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ such that $\underline{V}^*_{\mathfrak{S}}(\mathfrak{M}) \simeq \Lambda$ (as G_{K_∞}-representations); moreover, $\mathfrak{M}$ is functorial in Λ. Letting $D := D^*_{\mathrm{cris}}(V)$, we have that $\mathrm{Fil}^0 D = D$ and $\mathrm{Fil}^p D = 0$ due to the condition on the Hodge-Tate weights of the crystalline representation V, and there is a natural injection

$$D \hookrightarrow \mathscr{O} \otimes_{K_0} D \xhookrightarrow{\xi} \mathscr{O} \otimes_{\mathfrak{S}} \mathfrak{M} \hookrightarrow S\left[\tfrac{1}{p}\right] \otimes_{\mathfrak{S}} \mathfrak{M}\ ,$$

so twisting by Frobenius defines a natural injection

$$D \xleftarrow[1\otimes\varphi_D]{\simeq} \varphi^*(D) \hookrightarrow S\left[\tfrac{1}{p}\right] \otimes_{\mathfrak{S},\varphi} \mathfrak{M} = S\left[\tfrac{1}{p}\right] \otimes_{\mathfrak{S}} \varphi^*_{\mathfrak{S}}\mathfrak{M}\ .$$

Viewing D as a K_0-submodule of $S\left[\frac{1}{p}\right] \otimes_{\mathfrak{S},\varphi} \mathfrak{M}$ in this way, we define

$$L := D \cap (S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}) \subseteq S\left[\frac{1}{p}\right] \otimes_{\mathfrak{S},\varphi} \mathfrak{M}.$$

Clearly L is a φ-stable $\mathrm{W}(k)$-lattice in D. We claim that L is strongly divisible. Indeed, this follows from the fact that $p | \varphi_S(E)$ in S and $D = D^*_{\mathrm{cris}}(V)$ with V crystalline. [3]

[3] need to say more

Furthermore, one shows that the association $\Lambda \rightsquigarrow L$ is exact by using the filtration bounds (in $\{0, \ldots, p-1\}$) to deduce that ξ as above induces an isomorphism $L \simeq \mathfrak{M}/u\mathfrak{M}$. [4]

Conversely, let L be any strongly divisible lattice in an object D of ${}^{\mathrm{w.a.}}\mathrm{MF}_K^{\varphi,\mathrm{Fil}\geqslant 0}$ with $\mathrm{Fil}^p L := L \cap \mathrm{Fil}^p D = 0$ (so $\mathrm{Fil}^p D = 0$). Note that $\mathrm{Fil}^0 L = L$ since $\mathrm{Fil}^0 D = D$. We set

$$\Lambda := \mathrm{Hom}_{\varphi,\mathrm{Fil}}(L, A_{\mathrm{cris}}).$$

This is a G_K-stable lattice in the crystalline representation $V^*_{\mathrm{cris}}(D)$ with $D = L[1/p]$. One shows that $L \rightsquigarrow \Lambda$ is exact and quasi-inverse to the other functor built above. [5]

One shows that these associations are quasi-inverse. ■

We now wish to apply this theory to torsion representations. In order to do this, we need a torsion replacement for strongly divisible lattices:

Definition 12.4.9. A *Fontaine-Laffaille module* over $\mathrm{W}(k)$ is a finite length $\mathrm{W}(k)$-module M equipped with a finite and separated decreasing filtration $\{\mathrm{Fil}^i M\}$ and φ-semilinear endomorphisms $\varphi^i_M : \mathrm{Fil}^i M \to M$ such that

1. the map $p\varphi^{i+1}_M : \mathrm{Fil}^{i+1} M \to M$ coincides with the restriction of φ^i_M to $\mathrm{Fil}^{i+1}_M \subseteq \mathrm{Fil}^i_M$,
2. $\sum_i \varphi^i_M(\mathrm{Fil}^i M) = M$,
3. $\mathrm{Fil}^0 M = M$.

We say M is *connected* if $\varphi^0_M : M \to M$ is nilpotent.

Example 12.4.10. If L is a strongly divisible lattice, then for each $n > 0$ we obtain a Fontaine-Laffaille module M by setting $M = L/p^n L$, taking $\mathrm{Fil}^i M$ to be the image of $\mathrm{Fil}^i L$ under the natural quotient map, and letting $\varphi^i_M := p^{-i}\varphi_L$. This is connected if and only if L is connected.

More generally, if $L' \to L$ is an isogeny of strongly divisible lattices, then L/L' has a natural structure of Fontaine-Laffaille module (and it is connected if L is connected).

Lemma 12.4.11. *Let M be any Fontaine-Laffaille module with a one-step filtration (i.e. there is some $i_0 \geqslant 0$ such that $\mathrm{Fil}^i M = M$ for all $i \leqslant i_0$ and $\mathrm{Fil}^i M = 0$ for all $i > i_0$). Then there exists an isogeny of strongly divisible lattices $L' \to L$ with cokernel M.* [6]

By using such presentations and the functoriality and exactness properties of strongly divisible lattices, we get:

Theorem 12.4.12. *Consider the contravariant functor*

$$M \rightsquigarrow \mathrm{Hom}_{\mathrm{Fil},\varphi}(M, A_{\mathrm{cris}} \otimes (\mathbf{Q}_p/\mathbf{Z}_p))$$

from the category of Fontaine-Laffaille modules M with one-step filtration that satisfies $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^p M = 0$ to the category of p-power torsion discrete G_K-modules. If $p > 2$ this is an exact and fully faithful functor into the category $\mathrm{Rep}^{\mathrm{tor}}_{G_K}$ (i.e., image objects are finite abelian groups). If $p = 2$, the same statement holds if one restricts to connected Fontaine-Laffaille modules. [7]

[4] need to say more
[5] need to say more
[6] should sketch proof or give reference
[7] should sketch proof or give reference

12.5. **Exercises.**

Exercise 12.5.1. Verify the following assertions made about any PD-structure $\{\gamma_n\}$ on an ideal I in a ring A.

1. Prove that $n!\gamma_n(x) = x^n$ for all $n \geqslant 1$.
2. Prove that if A is a discrete valuation ring of mixed characteristic $(0, p)$ and absolute ramification index e (i.e., e is the normalized order of p), then its maximal ideal has a PD-structure if and only if $e \leqslant p-1$, and this is PD-nilpotent if and only if $e < p-1$. (This is why $e = 1$ and $p = 2$ is such a problem in the crystalline theory.)
3. Give an example with $I^3 = 0$ for which there is more than one PD-structure on I, but show that if $I^2 = 0$ then defining $\gamma_n = 0$ for all $n \geqslant 2$ defines a PD-structure on I.

Exercise 12.5.2. Let A be a $\mathbf{Z}_p$-flat ring and I an ideal which admits a divided power structure (i.e., $x^n \in n!I$ for all $x \in I$ and $n \geqslant 1$). Let $\widehat{A}$ denote the p-adic completion of A.

1. Prove that $\widehat{A}$ is $\mathbf{Z}_p$-flat, and give an example for which the map $A \to \widehat{A}$ is not injective.
2. Prove that $I \cdot \widehat{A}$ admits a divided power structure. (There are general theorems in the theory of divided powers that allow one to efficiently handle the interaction of divided power structure with respect to extension rings, sums of ideals, etc. These are needed beyond the $\mathbf{Z}$-flat case.)

Exercise 12.5.3. Let $G \to S$ be a finite flat group schemes. In this exercise we develop the important concept of *Cartier dual* which is akin to duality for finite abelian groups. To ease the notation we will assume $S = \operatorname{Spec} R$ is affine and so $G = \operatorname{Spec} A$. The interested reader can work with quasi-coherent sheaves of $\mathscr{O}_S$-algebras to avoid this hypothesis.

Define the functor $G^\vee$ on R-algebras by $G^\vee(R') = \operatorname{Hom}_{R'}(G_{R'}, (\mathbf{G}_m)_{R'})$ (the group of R'-group homomorphisms from $G_{R'}$ to $(\mathbf{G}_m)_{R'}$. We aim to prove this is representable by a finite flat R-group. Beware that $G^\vee(R')$ is *not* the group $\operatorname{Hom}(G(R'), R'^{\times})$ in general!

1. In case $G = (\mathbf{Z}/n\mathbf{Z})_R$, show that $G^\vee$ is represented by the R-group μ_n.
2. If $R = \mathbf{F}_p$ and $G = \alpha_p$, show that $G^\vee(R') = 0$ for any reduced ring R' (e.g., any field) but show that $G^\vee(\mathbf{F}_p[t]/(t^2)) \neq 0$.
3. The R-group structure on A is encoded in a triple of R-linear maps $e^* : A \to R$, $m^* : A \to A \otimes_R A$, and $i^* : A \to A$. Passing to R-linear duals (recall A is a locally free R-module of constant finite rank), we get maps $R \to A^\vee$, $A^\vee \otimes_R A^\vee \to A^\vee$, and $A^\vee \to A^\vee$. Using that G is a commutative R-group, check that the first two of these maps define a commutative R-algebra structure on $A^\vee$, with identity element given by the image of 1 under $R \to A^\vee$. Hence, now $\operatorname{Spec}(A^\vee)$ makes sense.
4. Continuing with the previous part, show that dualizing the R-algebra structure maps $R \to A$ (sending 1 to 1) and $A \otimes_R A \to A$ (multiplication) imposes exactly what is needed to define a commutative R-group structure on $\operatorname{Spec}(A^\vee)$ (with the linear dual of i^* as the inversion).
5. Describe concrete what an R'-group map $G_{R'} \to (\mathbf{G}_m)_{R'}$ means, and identify this with the group of R'-points of the R-group you just constructed on $\operatorname{Spec}(A^\vee)$. This shows that $G^\vee$ is represented by a finite flat R-group.

(6) Show that to give a bi-additive pairing of R-groups $G \times H \to \mathbf{G}_m$ is the same as to give an R-group map $H \to G^\vee$. Interpret this when $H = G^\vee$, and contemplate "double duality".

Exercise 12.5.4. When trying to establish an integral comparison isomorphism, such as in Lemma 12.2.5, it sometimes suffices to establish a compatibility with perfect dualities at the integral level. Here is how it goes.

Let M, M' and N, N' be pairs of finite free modules over a discrete valuation ring R, all of the same positive rank, and there is given a pair of perfect R-bilinear duality pairings

$$M \times M' \to R, \quad N \times N' \to R.$$

Assume there are given R-linear maps $L : M \to N$ and $L' : M' \to N'$ such that the map of sets $L \times L' : M \times M' \to N \times N'$ is compatible with the perfect duality pairings. Prove that L and L' are isomorphisms.

Part IV. (φ, Γ)-modules and applications

13. Foundations

The theory of (φ, Γ)-modules is an improvement on the theory of étale φ-modules from §3. Recall that étale φ-modules classify all p-adic representations of G_E for any field E of characteristic p. This may not sound interesting if our main goal is to understand representations of G_K, but it is incredibly useful. The link is that if we choose an auxiliary infinitely ramified extension K_∞/K of a special type (such as $K_\infty = K(\mu_{p^\infty})$) then the theory of norm fields (which will be developed below from scratch in some basic cases of interest) provides a canonically associated equicharacteristic p local field $\mathbf{E}_{K_\infty}$ (called the *norm field* of K_∞/K) for which the Galois theories of K_∞ and $\mathbf{E}_{K_\infty}$ are *the same.* In particular, $G_E = G_{K_\infty}$, so we get a classification of p-adic representations of the closed subgroup G_{K_∞} in G_K. The utility of such a classification was seen in Part III.

When K_∞/K is infinitely ramified and Galois with Galois group Γ isomorphic to $\mathbf{Z}_p$ near the identity, there is a theory of étale (φ, Γ)-modules which goes a step further and provides a classification of *all* p-adic representations of the entire Galois group G_K, as well as all G_K-representations on finitely generated $\mathbf{Z}_p$-modules, in terms of semilinear data.

The theory of étale (φ, Γ)-modules was developed by Fontaine ([20], [13]) using the theory of norm fields. It has many applications in p-adic Hodge theory and in the study of families of p-adic representations, and it is a central tool in recent developments in the p-adic Langlands correspondance for GL_2. The aim of Part IV is to develop the ingredidents which make the theory work. It rests on earlier results and methods of Tate, and so we will begin with a discussion of Tate's ideas, then turn to the theory of norm fields (which is largely independent from our discussion of Tate's work), and finally bring the two topics together to set up the classification theory of G_K-representations by means of étale (φ, Γ)-modules.

13.1. **Ramification estimates.**

In Tate's work on p-adic representations arising from p-divisible groups over $\mathscr{O}_K$ (with an eye on the case of abelian varietes over K with good reduction), a fundamental insight he had was that ramification in an infinitely ramified $\mathbf{Z}_p$-extension K_∞/K can be rather precisely understood.

Example 13.1.1. Consider a 1-dimensional p-adic representation $\psi : G_K \to \mathbf{Q}_p^\times$, and assume that $\psi(I_K)$ is infinite; in other words, assume that the field K_∞/K corresponding to $\ker\psi$ is infinitely ramified. By continuity, $\psi(I_K)$ is a closed subgroup of $\mathbf{Z}_p^\times$. But it is infinite, so it must be open in $\mathbf{Z}_p^\times$ (as $\mathbf{Z}_p^\times$ near 1 is isomorphic to $\mathbf{Z}_p$ near 0 as topological groups, due to the p-adic exponential and logarithm maps, and all nontrivial closed subgroups of $\mathbf{Z}_p$ are open). Thus, $\mathrm{Gal}(K_\infty/K) = \psi(G_K)$ is an open subgroup of $\mathbf{Z}_p^\times$ containing $\psi(I_K)$ with finite index.

There is a natural rising tower of finite subextensions $\{K_n\}$ over K corresponding to $\ker(\psi \bmod p^n)$, and $K_\infty = \cup K_n$. (We will not be using the maximal unramified subextension of K here, so there seems no risk of confusion with our notation $K_0 = \mathrm{W}(k)[1/p]$ used elsewhere.) For example, if ψ is the p-adic cyclotomic character then $K_n = K(\zeta_{p^n})$. Since $\mathbf{Z}_p^\times$ and $\mathbf{Z}_p$ are locally isomorphic near their identity elements and $\psi(I_K)$ contains $1 + p^N\mathbf{Z}_p$ for some large N, there is an n_0 with two properties: I_{K_0} maps onto $\mathrm{Gal}(K_\infty/K_{n_0})$, and

$\mathrm{Gal}(K_\infty/K_{n_0})$ is identified with $\mathbf{Z}_p$ carrying $\mathrm{Gal}(K_\infty/K_n)$ to $p^{n-n_0}\mathbf{Z}_p$ for all $n \geqslant n_0$. In other words, K_∞/K_{n_0} is a totally ramified $\mathbf{Z}_p$-extension with layers K_n/K_{n_0} for $n \geqslant n_0$.

By using good estimates on ramification in the extensions K_n/K, Tate proved the following intrinsic property of K_∞ that may look like just a curiosity but is actually quite powerful (as we shall see):

Theorem 13.1.2 (Tate)**.** *Let K_∞/K be an infinitely ramified Galois extension such that $\mathrm{Gal}(K_\infty/K)$ is isomorphic to $\mathbf{Z}_p$ near the identity. For any finite extension M/K_∞, the image of the trace map $\mathrm{Tr}_{M/K_\infty} : \mathscr{O}_M \to \mathscr{O}_{K_\infty}$ contains $\mathfrak{m}_{K_\infty}$.*

The key to the proof of this theorem is a result to the effect that the extension of valuation rings $\mathscr{O}_{K_\infty} \to \mathscr{O}_M$ is "almost étale". In Exercise 13.7.1 we will explain the appropriateness of this terminology.

Remark 13.1.3. Tate's proof (which we will present below) rests on subtle arithmetic input such as Serre's geometric local class field theory and a detailed study of the higher ramification filtration, Faltings found a much simpler proof that instead uses commutative algebra (and so has the merit that it adapts to much more general settings, where it is a basic tool in Faltings' method for proving p-adic comparison isomorphisms involving p-adic cohomology theories). In Exercise 13.7.4 below, we outline Faltings' proof. It must be stressed that Tate's method gives sharp ramification estimates that Faltings' method does not, and this is essential for some applications (such as developing the theory of (φ, Γ)-modules) and underlies the powerful Tate–Sen method that we will explain and apply later.

Now we set the stage for Tate's work. Let $\{K_n\}_{n \geqslant 0}$ be any increasing sequence of finite Galois extensions of K inside of $\overline{K}$, and define $K_\infty = \cup K_n$. (We will not be making use of the maximal unramified extension of K, so there seems to be no risk of confusion with our new meaning for K_0.) Define $\Gamma = \mathrm{Gal}(K_\infty/K)$. Assume there exists $n_0 \geqslant 0$ so that for $n \geqslant n_0$, K_n/K_{n_0} is totally ramified with $\mathrm{Gal}(K_n/K_{n_0}) \simeq \mathbf{Z}/p^{n-n_0}\mathbf{Z}$. This implies that K_∞/K_{n_0} is $\mathbf{Z}_p$-extension that is totally ramified in the sense that all finite subextensions are totally ramified over K_{n_0}. We allow anything to happen in the layer K_{n_0}/K.

Example 13.1.4. Given any such tower $\{K_n\}$ over K, we can naturally make one over any finite extension L/K inside of $\overline{K}$ by defining $L_n = K_n L$. The union L_∞ of this tower is $K_\infty L$. (In the motivating example where K_∞ corresponds to the kernel of an infinitely ramified character ψ of G_K, L_∞ corresponds to the kernel of $\psi|_{G_L}$.) To see that this works, we need to relate the layers over L with layers over K, and we will need to pass to large n to eliminate the effect of overlap of L with K_n for some small n.

Let $F = L \cap K_\infty$, so there is an n_L such that F is contained in K_n when $n \geqslant n_L$. Hence, L is linearly disjoint over F from K_∞ and K_n for $n \geqslant n_L$. In particular, $L_\infty = K_\infty \otimes_F L$ and $L_n = K_n \otimes_F L$ for $n \geqslant n_L$. Therefore, $\mathrm{Gal}(L_\infty/L) = \mathrm{Gal}(K_\infty/F)$ is open in Γ (so it is isomorphic to $\mathbf{Z}_p$ near the identity and I_L has open image in $\mathrm{Gal}(L_\infty/L)$) and $\mathrm{Gal}(L_n/L) = \mathrm{Gal}(K_n/F)$ for $n \geqslant n_L$. It follows that L_∞/L_n is a totally ramified $\mathbf{Z}_p$-extension with $\mathrm{Gal}(L_{n'}/L_n) = \mathrm{Gal}(K_{n'}/K_n) \simeq \mathbf{Z}/p^{n'-n}\mathbf{Z}$ for $n' \geqslant n \geqslant n_0(L) := \max(n_L, n_0)$.

In the preceding example, we inferred that $L_\infty/L_{n_0(L)}$ is a totally ramified $\mathbf{Z}_p$-extension for sufficiently large $n_0(L)$ via a soft topological argument; we got no information on the precise amount of ramification in the layers $L_n/L_{n_0(L)}$. Moreover, although

$$L_n = K_n \otimes_F L = K_n \otimes_{K_{n_0(L)}} (K_{n_0(L)} \otimes_F L) = K_n \otimes_{K_{n_0(L)}} L_{n_0(L)}$$

for $n \geqslant n_0(L)$, so each layer L_n/K_n is obtained from the single layer $L_{n_0(L)}/K_{n_0(L)}$ via scalar extension to K_n (for $n \geqslant n_0(L)$), it is not at all clear how ramification in L_n/K_n behaves as $n \to \infty$.

Since the obstruction to perfectness of the trace pairing on valuation rings is encoded in the discriminant ideal, and the discriminant in a finite extension of local fields is the norm of the different ideal, we shall now study the behavior of $v(\mathfrak{D}_{L_n/K_n})$ and $v(\mathfrak{D}_{K_n/K}) = v(\mathfrak{D}_{K_n/K_{n_0}}) + v(\mathfrak{D}_{K_{n_0}}/K)$ as $n \to \infty$, where $\mathfrak{D}_{M'/M}$ denotes the different ideal [44, Ch. III]. The results we are after (Lemma 13.1.7, Lemma 13.1.10, Proposition 13.1.9, and Proposition 13.1.10) are insensitive to replacing K with any fixed K_m, so by making such a replacement we may and do arrange that K_∞/K is a totally ramified $\mathbf{Z}_p$-extension. In particular, we have $\Gamma := \mathrm{Gal}(K_\infty/K) \simeq \mathbf{Z}_p$ and $\mathrm{Gal}(K_n/K) \simeq \Gamma/p^n\Gamma \simeq \mathbf{Z}/p^n\mathbf{Z}$, so K_n is the fixed field of $p^n\Gamma$ on K_∞ for all $n \geqslant 0$.

In Galois extensions, ramification is best understood via the ramification filtration. For example, there is a formula for the valuation of the different in terms of orders of higher ramification groups. Thus, let us briefly review some basic facts concerning higher ramification groups for a finite Galois extension M/L of finite extensions of K inside of $\overline{K}$. Let

$$\{\mathrm{Gal}(M/L)_x\}_{x \geqslant -1}, \quad \{\mathrm{Gal}(M/L)^y\}_{y \geqslant -1}$$

respectively denote the filtration of $\mathrm{Gal}(M/L)$ by its ramification subgroups with the lower and upper numberings [44, Ch. IV]. Essentially by definition, $\mathrm{Gal}(M/L)_x = \mathrm{Gal}(M/L)^{\phi_{L/M}(x)}$ where $\phi_{M/L}$ is the Herbrand function. This function is the continous piecewise-linear self-map of $[-1, +\infty)$ whose slope on the interval $(i-1, i)$ is $\#\,\mathrm{Gal}(M/L)_i/\#\,\mathrm{Gal}(M/L)_0$ for all $i \geqslant 0$ [44, Ch. IV, §3]. In particular, $\phi_{M/L}(x) = x$ for $x \in [-1, 0]$ (so $\mathrm{Gal}(M/L)^0 = \mathrm{Gal}(M/L)_0 = I(M/L)$ is the inertia subgroup). The lower numbering is compatible with subgroups, whereas the upper numbering is compatible with taking quotients [44, Ch. IV, Prop. 4].

Since the ramification filtration with the upper numbering is compatible with passage to quotients, for infinite Galois extensions M/L it makes sense to pass to the limit to define a filtration $\{\mathrm{Gal}(M/L)^y\}_{y \geqslant -1}$ of $\mathrm{Gal}(M/L)$ which induces the upper-numbering filtration $\{\mathrm{Gal}(M_i/L)^y\})_{y \geqslant -1}$ on $\mathrm{Gal}(M_i/L)$ for every finite Galois subextension M_i/L in M/L. By construction the $\mathrm{Gal}(M/L)^y$ are *closed* subgroups of Γ, and $\cap\,\mathrm{Gal}(M/L)^y = 1$ (as can be checked by passage to the case of finite extensions, where we can switch to the lower numbers and see the vanishing of sufficiently high ramification groups). It is not obvious when the $\mathrm{Gal}(M/L)^y$ should be open!

Example 13.1.5. Consider the upper numbering filtration on $\Gamma = \mathrm{Gal}(K_\infty/K) \simeq \mathbf{Z}_p$. The nontrivial closed subgroups of Γ are $p^m\Gamma$ for $m \geqslant 0$, but it isn't a priori evident just from the definitions whether or not we may have $\Gamma^y = 0$ for all large y. Since Γ^0 is the inertia subgroup, at least Γ^0 is open! In general there is a monotonically increasing sequence $\{y_m\}_{m \geqslant -1}$ in $[-1, \infty)$ such that $y \in (y_{m-1}, y_m]$ precisely when $\Gamma^y = p^m\Gamma$. By the Hasse–Arf theorem [44, Ch. 5, Thm. 1], the y_m's all are integers; they are called the "jumps" of the ramification filtration.

Beware that we are *not* claiming a priori that every $p^m\Gamma$ actually arises as a Γ^y. In fact, we have $y_{m+1} = y_m$ precisely when it does not arise. Since $\cap\Gamma^y = \{0\}$, we see that Γ^y stabilizes for large y (and so is $\{0\}$) precisely when the sequence of integers $\{y_m\}$ is eventually constant. It is by no means obvious from the definitions if such terminal constancy cannot occur (and it would occur if we worked with an *unramified* $\mathbf{Z}_p$-extension).

It turns out that in the above example the Γ^y's are open, which is to say $y_m \to \infty$. In fact, although the specific values y_m for small m are erratic, for large m we get some nice explicit behavior thanks to class field theory:

Lemma 13.1.6. *There exists $m_0 \geqslant 0$ such that $y_{m+1} = y_m + e$ for all $m \geqslant m_0$, where $e = e(K)$ is the absolute ramification degree of K.*

Proof. The ramification filtration within the inertia group Γ^0 is insensitive to replacing K with $\widehat{K^{\mathrm{un}}}$ (see Exercise 1.4.4), so we may replace K with $\widehat{K^{\mathrm{un}}}$ to reduce to the case when the residue field k is algebraically closed. In this situation, Serre's geometric local class field theory [41] may be applied: there is a surjective continuous reciprocity morphism $r : \mathscr{O}_K^\times \to \Gamma$ that carries maps the filtration $\{1 + \mathfrak{m}_K^i\}_{i\geqslant 1}$ onto that given by $\{\Gamma^i\}_{i\geqslant 1}$ (akin to the classical case of ordinary local class field theory [44, Ch. 15, Cor. 3]). Since r must be a topological quotient map, this shows that the Γ^i's are open in $\Gamma \simeq \mathbf{Z}_p$, and so are nontrivial for all i. In particular, $y_m \to \infty$. However, we still do not know that the y_m's are pairwise distinct for all large m (i.e., $p^m\Gamma$ occurs as a Γ^y for all large m).

For $i > 2e/(p-1)$, the exponential map induces an isomorphism $\exp\colon \mathfrak{m}_K^i \simeq 1 + \mathfrak{m}_K^i$. Choose such an i_0 and let $\rho = r \circ \exp\colon \mathfrak{m}_K^{i_0} \to \Gamma \simeq \mathbf{Z}_p$, so the image of ρ is open. We prefer to work with ρ because it is "additive". For m big enough (say $m \geqslant m_0$), the jumps y_m correspond to the jumps in the sequence $\{\rho(\mathfrak{m}_K^i)\}_{i\geqslant i_0}$ of open subgroups of Γ. Here we have used that the y_m's are actually integers, so the ramification groups Γ^i with $i \in \mathbf{Z}$ (which are the ones described by class field theory) account for all subgroups Γ^y in Γ.

Let $i \geqslant i_0$ be a jump; that is, there exists $m \geqslant m_0$ such that $\rho(\mathfrak{m}_K^i) \not\subset p^m\Gamma$ and $\rho(\mathfrak{m}_K^{i+1}) \subset p^m\Gamma$. By openness of the higher ramification groups in our case, we can increase m so that $\rho(\mathfrak{m}_K^{i+1}) = p^m\Gamma$. We want to show that the next jump occurs at exactly $i+e$; more specifically, we will prove that $\rho(\mathfrak{m}_K^j) = p^m\Gamma$ for $i+1 \leqslant j \leqslant i+e$ and $\rho(\mathfrak{m}_K^{i+e+1}) = p^{m+1}\Gamma$. Note that $\mathfrak{m}_K^{i+e+1} = p\mathfrak{m}_K^{i+1}$. At this point all we can see is that $\rho(\mathfrak{m}_K^j) \subset p^m\Gamma$ for $j > i$.

Pick a uniformizer π of K, so it is the root of an Eisenstein polynomial over $\mathrm{W}(k)$. This gives

$$\pi^e \in p\,\mathrm{W}(k)^\times + p\pi\,\mathrm{W}(k) + \cdots + p\pi^{e-1}\,\mathrm{W}(k),$$

so $\mathfrak{m}_K^{i+e} \subset p\mathfrak{m}_K^i + p\mathfrak{m}_K^{i+1} + \cdots + p\mathfrak{m}_K^{i+e-1}$. As $\rho(p\mathfrak{m}_K^j) = p\rho(\mathfrak{m}_K^j) \subset p^{m+1}\Gamma$ for $j > i$, and $\rho(p\mathfrak{m}_K^i) = p\rho(\mathfrak{m}_K) \not\subset p^{m+1}\Gamma$, we have $\rho(\mathfrak{m}_K^{i+e}) \not\subset p^{m+1}\Gamma$. Hence, $\rho(\mathfrak{m}_K^{i+e}) = p^m\Gamma$, forcing $\rho(\mathfrak{m}_K^j) = p^m\Gamma$ for $i \leqslant j \leqslant i+e$. On the other hand, since $\pi^e \in p\mathscr{O}_K$ we have $\rho(\mathfrak{m}_K^{i+e+1}) \subset p\rho(\mathfrak{m}_K^{i+1}) \subset p^{m+1}\Gamma$. It follows that the jump after i occurs at $i+e$, with $\rho(\mathfrak{m}_K^{i+e+1}) = \rho(p\mathfrak{m}_K^{i+1}) = p\rho(\mathfrak{m}_K^{i+1}) = p^{m+1}\Gamma$. ∎

The preceding lemma has the following somewhat technical consequence which will not be used in the proof of Theorem 13.1.2 but will be invoked at the end of our later study of norm fields (and so could have been postponed until the end of §13.3). We first recall our standard notation that v is the valuation on $\mathbf{C}_K$ normalized by $v(p) = 1$.

Lemma 13.1.7. *For m_0 as in Lemma* 13.1.6*, there exists $m_K \geqslant m_0$ such that*

$$v((g-\mathrm{id})(t)) \geqslant \frac{1}{2(p-1)}$$

for all $m \geqslant m_K$, all $g \in \mathrm{Gal}(K_{m+1}/K_m)$, and all $t \in \mathscr{O}_{K_{m+1}}$.

Proof. Let γ be a topological generator of Γ. For any $m \geqslant 0$, the group $\mathrm{Gal}(K_{m+1}/K_m)$ is cyclic of order p and is generated by γ^{p^m}. We have to show that there exists $m_K \geqslant m_0$ such that

$$v((\gamma^{p^m}-\mathrm{id})(t)) \geqslant \frac{1}{2(p-1)}$$

for all $m \geqslant m_K$ and all $t \in \mathscr{O}_{K_{m+1}}$.

The successive quotients $\mathrm{Gal}(K_{m+1}/K)_i/\mathrm{Gal}(K_{m+1}/K)_{i+1}$ are killed by p for $i \geqslant 1$ [44, Ch. IV, Cor 3]. As these are subquotients of $\Gamma \simeq \mathbf{Z}_p$, they are either trivial or isomorphic to $\mathbf{Z}/p\,\mathbf{Z}$. Since K_{m+1}/K is totally ramified, this conclusion also holds for $i=0$. Since K_{m+1}/K is a totally ramified cyclic extension of degree p^{m+1}, it follows that *every* subgroup of $\mathrm{Gal}(K_{m+1}/K)$ arises as a ramification subgroup. Hence, the slopes of the Herbrand function $\phi_{m+1} := \phi_{K_{m+1}/K}$ (which are the indices of the ramification groups in the inertia group) range through precisely the values $\{1, p^{-1}, \dots, p^{-m-1}\}$. From the definition of y_m and the compatibility of upper numbering with quotients, for the unique number $x_m \geqslant 0$ such that $\phi_{K_{m+1}/K}(x_m)$ is equal to the ramification jump value y_m we have

$$\mathrm{Gal}(K_{m+1}/K)_{x_m} = \mathrm{Gal}(K_{m+1}/K)^{y_m} = p^m\Gamma/p^{m+1}\Gamma = \langle \gamma^{p^m}\rangle.$$

The lower numbering is defined in terms of the *normalized* valuation for the top field, and since $v(p)=1$ the normalized valuation on K_{m+1} is $e_{K_{m+1}}\cdot v$, where $e_{K_{m+1}} = e[K_{m+1}:K] = ep^{m+1}$ is the absolute ramification index of K_{m+1}. Thus, the above formula for the x_mth lower-numbering ramification group says exactly that

$$ep^{m+1}v((\gamma^{p^m}-\mathrm{id})(t)) \geqslant x_m+1$$

for all $t \in \mathscr{O}_{K_{m+1}}$

For $n \leqslant m$, the graph of the piecewise-linear continuous Herbrand function ϕ_{m+1} has slope p^{-n-1} over the interval (x_n, x_{n+1}) of x for which $\mathrm{Gal}(K_{m+1}/K)_x = \mathrm{Gal}(K_{m+1}/K)^{\phi_{m+1}(x)} = \mathrm{Gal}(K_{m+1}/K)^{y_{n+1}}$. Hence, if $n \leqslant m$ then $y_{n+1}-y_n = p^{-n-1}(x_{n+1}-x_n)$, so

$$x_m = x_{m_0} + \sum_{n=m_0}^{m-1} p^{n+1}(y_{n+1}-y_n)$$

for $m \geqslant m_0$. But for $m \geqslant m_0$ we also have $y_{m+1} = y_m + e$, due to how m_0 was chosen (in accordance with Lemma 13.1.6), so

$$x_m = x_{m_0} + ep\cdot\frac{p^m - p^{m_0}}{p-1}.$$

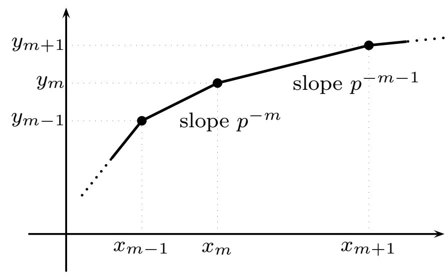
If $t \in \mathscr{O}_{K_{m+1}}$ we therefore have

$$v((\gamma^{p^m}-\mathrm{id})(t)) \geqslant \big(\frac{x_{m_0}+1}{e} - \frac{p^{m_0+1}}{p-1}\big)p^{-m-1} + \frac{1}{p-1}.$$

It is therefore enough to choose m_K such that

$$\left|\frac{x_{m_0}+1}{e}-\frac{p^{m_0+1}}{p-1}\right|p^{-m_K-1} \leqslant \frac{1}{2(p-1)},$$

where on the left side we use the usual *archimedean* notion of absolute on $\mathbf{Q}$. ■



Returning to the task of estimating growth of relative differents in towers, the following general integral formula for the ordinal of a different will be quite useful:

Lemma 13.1.8. *Let M and L be finite extensions of K with $L \subseteq M$ inside of $\overline{K}$. Then*

$$v(\mathfrak{D}_{M/L}) = \frac{1}{e_L}\int_{-1}^{\infty}\Big(1-\frac{1}{\#\operatorname{Gal}(M/L)^y}\Big)\mathrm{d}y$$

Keep in mind that v is normalized with $v(p) = 1$; it is generally not the normalized valuation for M or L. This is why there is a factor $1/e_L$ in the formula, for example.

Proof. By [44, Ch. IV, Prop. 4], we have $v(\mathfrak{D}_{M/L}) = (1/e_M)\sum_{i\geqslant 0}\big(\#\operatorname{Gal}(M/L)_i - 1\big)$, with the factor $1/e_M$ due to how we normalized v. This is trivially rewritten as a piecewise-linear integral:

$$v(\mathfrak{D}_{M/L}) = \frac{1}{e_M}\int_{-1}^{\infty}\big(\#\operatorname{Gal}(M/L)_x - 1\big)\mathrm{d}x$$

We now use change of variables to express this integral in terms of the upper numbering. By definition of the Herbrand function, $\operatorname{Gal}(M/L)_x = \operatorname{Gal}(M/L)^{\phi_{M/L}(x)}$. Since $\operatorname{Gal}(M/L)_0$ is the inertia group, we have $\#\operatorname{Gal}(M/L)_0 = e_M/e_L$.

For x not a corner point for $\phi_{M/L}$ we have

$$\phi'_{M/L}(x) = \frac{\#\operatorname{Gal}(M/L)_x}{\#\operatorname{Gal}(M/L)_0}.$$

Thus, the change of variables $y = \phi_{M/L}(x)$ yields:

$$\begin{aligned} v(\mathfrak{D}_{M/L}) &= \frac{1}{e_M}\int_{-1}^{\infty}\big(\#\operatorname{Gal}(M/L)^y - 1\big)\frac{\#\operatorname{Gal}(M/L)_0}{\#\operatorname{Gal}(M/L)^y}\,\mathrm{d}\,y \\ &= \frac{1}{e_L}\int_{-1}^{\infty}\Big(1-\frac{1}{\#\operatorname{Gal}(M/L)^y}\Big)\,\mathrm{d}\,y \end{aligned}$$

■

Our efforts finally pay off: we can prove that $v(\mathfrak{D}_{K_n/K})$ grow linearly in n to very good approximation:

Proposition 13.1.9. *There exist a constant c and a bounded sequence* $\{a_n\}_{n\geqslant 0}$ *(all depending on* K_∞/K*) such that* $v(\mathfrak{D}_{K_n/K}) = n + c + p^{-n}a_n$.

Proof. By Lemma 13.1.8 applied to the extension K_n/K, we have

$$v(\mathfrak{D}_{K_n/K}) = \frac{1}{e}\int_{-1}^{\infty}\Big(1 - \frac{1}{\#\,\mathrm{Gal}(K_n/K)^y}\Big)\,\mathrm{d}\,y$$

Compatibility of the upper numbering with quotients gives $\#\,\mathrm{Gal}(K_n/K)^y = \Gamma^y/(\Gamma^y \cap p^n\Gamma)$, so

$$\#\,\mathrm{Gal}(K_n/K)^y = \begin{cases} p^{n-m} & \text{if } y_{m-1} < y \leqslant y_m \text{ with } 0 \leqslant m \leqslant n, \\ 1 & \text{otherwise.}\end{cases} \tag{13.1.1}$$

The integral therefore becomes the following sum:

$$v(\mathfrak{D}_{K_n/K}) = \frac{1}{e}\sum_{m=0}^{n}(y_m - y_{m-1})(1 - p^{m-n}) \tag{13.1.2}$$

By Lemma 13.1.6, there is an integer m_0 such that $y_{m+1} = y_m + e$ for all $m \geqslant m_0$. Hence, for $n \geqslant m_0 + 1$ the formula (13.1.2) can be rewritten as:

$$\begin{aligned} v(\mathfrak{D}_{K_n/K}) &= \frac{1}{e}\sum_{m=0}^{m_0}(y_m - y_{m-1})(1 - p^{m-n}) + \frac{1}{e}\sum_{m=m_0+1}^{n} e(1 - p^{m-n}) \\ &= \frac{y_{m_0} - y_{-1}}{e} - \frac{p^{-n}}{e}\sum_{m=0}^{m_0} p^m(y_m - y_{m-1}) + (n - m_0) - \frac{p^{m_0+1-n} - p}{p-1} \\ &= n + c + p^{-n}a_n \end{aligned}$$

where $c = \frac{y_{m_0}-y_0}{e} - m_0 + 1 + \frac{p}{p-1}$ is independent of n and $a_n = -\frac{p^{m_0+1}}{p-1} - \frac{1}{e}\sum_{m=0}^{m_0-1} p^m(y_m - y_{m-1})$. For $n < m_0$ define $a_n = p^n\big(v(\mathfrak{D}_{K_n/K}) - n - c\big)$. (Note that $\{a_n\}$ is constant for large n.) ■

As another application of the integral formula in Lemma 13.1.8, we can show that the $v(\mathfrak{D}_{L_n/K_n})$'s are extremely small for any finite Galois extension L/K. This is the "almost étale" step.

Lemma 13.1.10. *Let L be a finite Galois extension of K, and define* $L_n = LK_n$ *for* $n \geqslant 0$*. The sequence* $\{p^n v(\mathfrak{D}_{L_n/K_n})\}_{n\geqslant 0}$ *is bounded.*

Proof. Arguing as in the discussion following Example 13.1.4, by replacing K with some K_n we can ensure that L is lineary disjoint from K_∞ over K. That is, $L_n = K_n \otimes_K L$ for all $n \geqslant 0$. Thus, $\mathrm{Gal}(L_n/K)$ decomposes as a direct product:

$$\mathrm{Gal}(L_n/K) \simeq \mathrm{Gal}(K_n/K) \times \mathrm{Gal}(L/K)$$

(where the projection to each factor is the natural quotient map, and so is compatible with upper numbering filtrations).

By transitivity of the different and applying Lemma 13.1.8 to K_n/K and L_n/K we have

$$v(\mathfrak{D}_{L_n/K_n}) = v(\mathfrak{D}_{L_n/K}) - v(\mathfrak{D}_{K_n/K}) = \frac{1}{e}\int_{-1}^{\infty}\Big(\frac{1}{\#\,\mathrm{Gal}(K_n/K)^y} - \frac{1}{\#\,\mathrm{Gal}(L_n/K)^y}\Big)\,\mathrm{d}\,y.$$

Choose $h \geqslant 0$ such that $\mathrm{Gal}(L/K)^y = \{1\}$ for all $y \geqslant h$; such an h exists since $\mathrm{Gal}(L/K)$ is finite. Hence, $\mathrm{Gal}(L_n/K)^y$ has trivial image in $\mathrm{Gal}(L/K)$, so the product decomposition for $\mathrm{Gal}(L_n/K)$ implies that $\mathrm{Gal}(L_n/K)^y$ injects into $\mathrm{Gal}(K_n/K)$. But its image in $\mathrm{Gal}(K_n/K)$ is $\mathrm{Gal}(K_n/K)^y$ due to quotient compatibility of the upper numbering, so we conclude that the natural map $\mathrm{Gal}(L_n/K)^y \twoheadrightarrow \mathrm{Gal}(K_n/K)^y$ is an isomorphism for all $y \geqslant h$. In other words, in the above integral formula for $v(\mathfrak{D}_{L_n/K_n})$ the integrand vanishes for $y \geqslant h$. Hence, we can end the integration at h to get:

$$v(\mathfrak{D}_{L_n/K_n}) = \frac{1}{e}\int_{-1}^{h}\Big(\frac{1}{\#\,\mathrm{Gal}(K_n/K)^y} - \frac{1}{\#\,\mathrm{Gal}(L_n/K)^y}\Big)\,\mathrm{d}\,y \leqslant \frac{1}{e}\int_{-1}^{h}\frac{\mathrm{d}\,y}{\#\,\mathrm{Gal}(K_n/K)^y}.$$

The integrand was computed in (13.1.1), with an especially nice formula when $y \leqslant y_n$, so by choosing n_0 large enough so that $y_{n_0} > h$ (as we may certainly do) we can replace the final integral over $[-1,h]$ with the analogous integral over $[-1, y_{n_0}]$ and use (13.1.1) to get

$$v(\mathfrak{D}_{L_n/K_n}) \leqslant \frac{1}{e}\sum_{m=0}^{n_0}(y_m - y_{m-1})p^{m-n} = \frac{p^{-n}}{e}\sum_{m=0}^{n_0}(y_m - y_{m-1})p^m.$$

Hence, $p^n v(\mathfrak{D}_{L_n/K_n})$ is bounded, as we claimed. ∎

Now we can finally prove Theorem 13.1.2:

Proof. Since the trace is transitive, we may enlarge M so that M/K_∞ is Galois. Replacing K by some K_n, we may arrange that K_∞/K is a totally ramified $\mathbf{Z}_p$-extension, and there is a large n and a finite Galois extension L/K_n inside of M such that $M = LK_\infty$. (To find L/K_n for some large n, we "descend" M/K_∞; see Exercise 13.7.2 below for a systematic treatment of "descending" structures over K_∞ to structures over K_n for some large n.) Replacing K by such a K_n, with n also large enough so that K_n contains the finite extension $L \cap K_\infty$ of K, we can ensure that L/K is not only finite Galois but also linearly disjoint over K from all K_m and K_∞. Hence,

$$M = L_\infty = K_\infty \otimes_K L = K_\infty \otimes_{K_m} (K_m \otimes_K L) = K_\infty \otimes_{K_m} L_m$$

for all $m \geqslant 0$. By Lemma 13.1.10 (!), we have $v(\mathfrak{D}_{L_n/K_n}) = p^{-n}c_n$ where $\{c_n\}_{n\geqslant 0}$ is a bounded sequence.

Now pick $\alpha \in \mathfrak{m}_{K_\infty}$, so $\alpha \in K_n$ for some n. Hence, for all $m \geqslant n$ we can write $\alpha\mathscr{O}_{K_m} = \mathfrak{m}_{K_m}^{i_m}$ where $i_m \geqslant 1$. Note that $i_m = p^{m-n}i_n$ for $m \geqslant n$ (because K_m/K_n is totally ramified), so $i_m \to \infty$. By [44, Ch. III, Prop. 7], we have

$$\begin{aligned}
\mathrm{Tr}_{L_m/K_m}(\mathfrak{m}_{L_m}^{j}) \subset \mathfrak{m}_{K_m}^{i} &\Leftrightarrow \mathfrak{m}_{L_m}^{j} \subset \mathfrak{m}_{K_m}^{i}\mathfrak{D}_{L_m/K_m}^{-1}\\
&\Leftrightarrow \frac{j}{e_{L_m/K_m}ep^m} \geqslant \frac{i}{ep^m} - \frac{c_m}{p^m}\\
&\Leftrightarrow j \geqslant e_{L_m/K_m}(i - ec_m).
\end{aligned}$$

In particular, $\mathrm{Tr}_{L_m/K_m}(\mathscr{O}_{L_m}) = \mathfrak{m}_{K_m}^{\lfloor ec_m \rfloor}$. Since $\{c_m\}_{m\geqslant 0}$ is bounded and $i_m \to \infty$, there exists $m \geqslant 0$ such that $i_m > ec_m$. It follows that $\alpha \in \mathrm{Tr}_{L_m/K_m}(\mathscr{O}_{L_m})$. But we arranged a linear disjointness property: $M = K_\infty \otimes_{K_m} L_m$. Hence, by compatibility of ring-theoretic trace with respect to extension of scalars, we see that $\mathrm{Tr}_{L_m/K_m}(x) = \mathrm{Tr}_{M/K_\infty}(x)$ for all $x \in \mathscr{O}_{L_m}$. This proves $\alpha \in \mathrm{Tr}_{M/K_\infty}(\mathscr{O}_M)$. ■

13.2. **Perfect norm fields.** The theory of norm fields is due to Fontaine and Wintenberger ([24], [51], [13, §4]). It sets up an equivalence of categories between the category of finite extensions of certain infinitely ramified extensions of a p-adic field and the category of finite separable extensions of an associated discretely-valued field of characteristic p (see Theorem 13.4.3).

There are two sides to the story: perfect norm fields and imperfect norm fields. We have already seen examples of each, without recognizing them as such (since the concept of a norm field has not yet been defined): the field $\mathrm{Frac}(R)$ from Theorem 4.3.5 is an example of a perfect norm field and the field $k(\!(u)\!)$ that arose in §11.1 in a somewhat explicit form is an example of an imperfect norm field. In neither of those cases did we see any "norms", nor did the constructions of those two fields look similar at all. Once we have explained how the norm field constructions work, we will recover both of these earlier classes of fields from a common point of view.

The case of perfect norm fields is somewhat easier to understand, as it amounts to just some simple generalizations of the work we did already in our study of R and $\mathrm{Frac}(R)$ in §4.3. Hence, we discuss this case first, and then (in §13.3) turn our attention to the imperfect norm fields. Both cases will be useful in the development of the theory of (φ, Γ)-modules.

Remark 13.2.1. We are now going to have to make extensive use of $\mathrm{W}(k)[1/p]$, as well as layers K_n of K_∞/K. Hence, to avoid confusion about the meaning of K_0, we now write F_0 to denote $\mathrm{W}(k)[1/p]$.

Let L/F_0 be an extension (not necessarily of finite degree) contained in $\overline{K}$. Fix a proper ideal $\mathfrak{a}$ in $\mathscr{O}_L$ that contains p and for which powers of $\mathfrak{a}$ cut out the p-adic topology (i.e., $\mathfrak{a}^N \subseteq p\mathscr{O}_L$ for some $N \geqslant 1$); this rules out taking $\mathfrak{a} = \mathfrak{m}_L$ when $L = \overline{K}$, for example.

Consider the inverse limit $\varprojlim_{n\geqslant 0} \mathscr{O}_L/\mathfrak{a}$ of $\mathbf{F}_p$-algebras using the transition maps $x \mapsto x^p$. This is the universal perfection $R_L := \underline{R}(\mathscr{O}_L/\mathfrak{a})$ from (4.2.1), so by Proposition 4.3.1 it is the same as $\underline{R}(\mathscr{O}_L/(p))$. For $L = \overline{K}$ it recovers the ring R that was studied in §4.3.

If $\widehat{\mathscr{O}}_L$ denotes the valuation ring of the completion $\widehat{L}$ of L (i.e., it is the p-adic completion of $\mathscr{O}_L$) then $\mathscr{O}_L/p\mathscr{O}_L = \widehat{\mathscr{O}}_L/p\widehat{\mathscr{O}}_L$, so Proposition 4.3.1 applied to $\widehat{\mathscr{O}}_L$ gives a natural multiplicative identification

$$R_L = \{(x^{(n)}) \in \prod_{n\geqslant 0} \widehat{\mathscr{O}}_L \mid (x^{(n+1)})^p = x^{(n)} \text{ for all } n \geqslant 0.\} \tag{13.2.1}$$

In what follows we will make frequent use of this identification. In particular, for $x \in R_L$ we write $x^{(n)}$ to denote the nth component of the corresponding p-power compatible sequence in $\widehat{\mathscr{O}}_L$.

The formula in (13.2.1) is functorial with respect to inclusions $L \subseteq L'$ among extensions of F_0 inside of $\overline{K}$, via the natural injective map

$$R_L = \underline{R}(\mathscr{O}_L/p\mathscr{O}_L) \hookrightarrow \underline{R}(\mathscr{O}_{L'}/p\mathscr{O}_{L'}) = R_{L'}.$$

In this way we may and do identify all R_L's with subrings of $R_{\overline{K}} = R$.

Recall that v_R on $\mathrm{Frac}(R)$ satisfies $v_R((x^{(n)})_{n\geqslant 0}) := v(x^{(0)})$, and this makes R the valuation ring for v_R on $\mathrm{Frac}(R)$. There is a similar result in general:

Lemma 13.2.2. *Let $L \subseteq \overline{K}$ be a subfield containing $F_0 = \mathrm{W}(k)[1/p]$, and let k' be its residue field inside of $\overline{k}$. Then R_L is the valuation ring in $\mathrm{Frac}(R_L)$ relative to the restriction of the valuation v_R on $\mathrm{Frac}(R)$, and its residue field is k'. In particular, the R_L's are normal domains.*

Beware that v_R may have trivial restriction to R_L, which is to say that R_L is a field (equivalently, $k' = R_L$). By Exercise 13.7.3(2), this happens whenever $[L : F_0]$ is finite.

Proof. Expressing elements of $\mathrm{Frac}(R_L)$ as ratios of elements of R_L, we have to show that if $x, y \in R_L - \{0\}$ and $x|y$ in $R_{\overline{K}}$ then $x|y$ in R_L. Passing to p-power compatible sequences in $\widehat{\mathscr{O}}_L$ respects multiplication, and so reduces the assertion to the evident claim that a divisibility condition $a|b$ in $\widehat{\mathscr{O}}_L$ may be checked in $\mathscr{O}_{\mathbf{C}_K}$.

To compute the residue field, we observe that the natural map $R_L \to \mathscr{O}_L/(p) \to k'$ defined by $x \mapsto x^{(0)} \bmod \mathfrak{m}_L$ is a surjective ring map; this follows from the definition of R_L and the perfectness of k'. ■

Since each R_L is perfect by construction, the corresponding fraction fields $\mathrm{Frac}(R_L)$ are also perfect. In Lemma 4.3.3 we proved that $R = R_{\overline{K}}$ is v_R-adically separated and complete, so it is also ϖ-adically separated and complete for any $\varpi \in \mathfrak{m}_R - \{0\}$.

The v_R-adic topology on $R = R_{\overline{K}}$ matches its subspace topology from the product topology on $\prod_{n\geqslant 0} \mathscr{O}_{\mathbf{C}_K}$ (using the description of R via p-power compatible sequences $(x^{(n)})_{n\geqslant 0}$ in $\mathscr{O}_{\mathbf{C}_K}$). Hence, each R_L is complete for the restriction of v_R since $\widehat{\mathscr{O}}_L$ is closed in $\mathscr{O}_{\mathbf{C}_K}$. In particular, R_L is ϖ-adically separated and complete for any $\varpi \in \mathfrak{m}_{R_L} - \{0\}$, and $\mathrm{Frac}(R_L)$ is complete for the v_R-adic topology.

Definition 13.2.3. The *perfect norm field* attached to L is $\mathrm{Frac}(R_L)$.

So far we have not mentioned any actual norms. When we construct the imperfect norm field attached to a finite extension M of K_∞ (for which we will see some norms), its perfect closure will turn out to be dense in the perfect norm field $\mathrm{Frac}(R_M)$ (thereby explaining the name "perfect norm field").

Consider the natural action by G_{F_0} on $\mathrm{Frac}(R)$ that preserves v_R. For any closed subgroup H and corresponding subfield $L = \overline{K}^H$, H acts trivially on the perfect norm field $\mathrm{Frac}(R_L)$ since H acts trivially on $\widehat{\mathscr{O}}_L$. We can do better:

Proposition 13.2.4. *For any closed subgroup $H \subseteq G_{F_0}$ with corresponding fixed field $L = \overline{K}^H$, $R^H = R_L$ and $\mathrm{Frac}(R)^H = \mathrm{Frac}(R_L)$.*

Proof. Using the identification

$$R = \{(x^{(n)}) \in \prod_{n\geqslant 0} \widehat{\mathscr{O}}_{\mathbf{C}_K} \mid (x^{(n+1)})^p = x^{(n)} \text{ for all } n \geqslant 0\}$$

we may pass to H-invariants to get that R^H is computed by the same formula as R except with $\mathscr{O}_{\mathbf{C}_K}$ replaced with $\mathscr{O}_{\mathbf{C}_K}^H = \mathscr{O}_{\mathbf{C}_K^H} = \mathscr{O}_{\widehat{L}}$ (see Proposition 2.1.2). But $\mathscr{O}_{\widehat{L}} = \widehat{\mathscr{O}}_L$, so we have proved $R^H = R_L$ inside of R. Since R and R_L are each valuation rings within their fraction fields for the same rank-1 valuation v_R on R, provided that v_R is nontrivial on R_L (i.e., R_L is not a field) we can invert a single non-unit in R_L to get the desired equality at the level of fraction fields.

Now consider the case when R_L is a field. In this case, if $x \in \mathrm{Frac}(R)^H$ we have to show that $x \in R$. If not then $1/x \in R$, so $1/x \in R^H = R_L$ with R_L a field. This forces $x \in R_L \subseteq R$ as well, contradicting the assumption that $x \notin R$. ■

Consider the valuation v_R on $\mathrm{Frac}(R)$ from Lemma 4.3.3 that makes $\mathrm{Frac}(R)$ complete with valuation ring R. The action of G_{F_0} on $\mathrm{Frac}(R)$ leaves this valuation invariant, so the G_{F_0}-action is continuous for the v_R-adic topology. This is akin to the fact that the G_{F_0}-action on $\mathscr{O}_{\mathbf{C}_K}$ is continuous for the p-adic topology.

Remark 13.2.5. Beware that the action by G_{F_0} on R and $\mathrm{Frac}(R)$ is not continuous for the discrete topology on these rings. There are two ways to see this. First, the natural surjective multiplicative map $R \to \mathscr{O}_{\mathbf{C}_K}$ defined by $x \mapsto x^{(0)}$ respects the absolute values and G_{F_0}-actions, so it suffices to show that the G_{F_0}-action on $\mathscr{O}_{\mathbf{C}_K}$ does not have open stabilizers. By Proposition 2.1.2, for any open subgroup $H \subseteq G_{F_0}$ the corresponding field of invariants $\mathbf{C}_K^H$ is the finite extension $\overline{K}^H$ of F_0, so any element of $\mathbf{C}_K$ not in $\overline{K}$ has non-open stabilizer; such elements exist since $\overline{K}$ is not complete [8, 3.4.3/1].

Another way to see the G_{F_0}-action on R is not continuous for the discrete topology is to exhibit an explicit element with non-open stabilizer. For example, the nontrivial element $\varepsilon \in 1 + \mathfrak{m}_R$ from Example 4.3.4 satisfies $g(\varepsilon) = \varepsilon^{\chi(g)}$ for $g \in G_{F_0}$. Since $1 + \mathfrak{m}_R$ with its natural $\mathbf{Z}_p$-module structure is torsion-free (as $\mathrm{char}(R) = p$), for $a, a' \in \mathbf{Z}_p$ we see that $\varepsilon^a = \varepsilon^{a'}$ if and only if $a = a'$. Hence, $g(\varepsilon) = \varepsilon$ if and only if $\chi(g) = 1$. Since χ does not have open kernel, G_{F_0} does not act on ε with an open kernel.

We can finally relate the Galois theory of an infinitely ramified extension of K to the Galois theory of a field of characteristic p, namely its associated perfect norm field. Our analysis will rest on an input from the theory of imperfect norm fields, so we give part of the proof now and then finish it later (see Corollary 13.3.12); the reader can check that no circular reasoning is involved. We fix an infinitely ramified Galois extension K_∞/K satisfying the properties given in the discussion immediately preceding Example 13.1.4.

Proposition 13.2.6. *Let M_1 and M_2 be finite extensions of K_∞ inside of $\overline{K}$, and assume that $M_2 \subseteq M_1$. The associated extension of perfect norm fields* $\mathrm{Frac}(R_{M_1})/\mathrm{Frac}(R_{M_2})$ *inside of* $\mathrm{Frac}(R)$ *is finite separable of degree* $[M_1 : M_2]$.

If M_1/M_2 is Galois then the natural action of G_{F_0} on $\mathrm{Frac}(R)$ *induces an isomorphism*

$$\mathrm{Gal}(M_1/M_2) \simeq \mathrm{Gal}(\mathrm{Frac}(R_{M_1})/\,\mathrm{Frac}(R_{M_2})).$$

The key to this proposition is that there is no "degree collapsing" when passing from M to $\mathrm{Frac}(R_M)$. This rests crucially on the fact that we work with fields M with infinite p-part in their ramification; see Exercise 13.7.3. Also, note that the corollary does not say that every finite (necessarily separable) extension of $\mathrm{Frac}(R_{K_\infty})$ inside of the algebraically closed field $\mathrm{Frac}(R)$ has the form $\mathrm{Frac}(R_M)$ for some finite extension M/K_∞ inside of $\overline{K}$. To get such an *equivalence* of Galois theories we will pass to imperfect norm fields in §13.4.

Proof. Since the $\mathrm{Frac}(R_M)$'s are perfect, all extensions among them are separable. To prove that $\mathrm{Frac}(R_{M_1})$ is finite over $\mathrm{Frac}(R_{M_2})$ with degree $[M_1 : M_2]$, choose a finite extension M_0/M_1 in $\overline{K}$ such that M_0/M_2 is Galois. Since $\mathrm{Frac}(R_{M_1})$ is an intermediate field in the extension $\mathrm{Frac}(R_{M_0})/\mathrm{Frac}(R_{M_2})$, it suffices to settle the Galois part of the claim (as we can then apply it to both M_0/M_1 and M_0/M_2). Hence, we now may and do assume that M_1/M_2 is Galois.

Let $H_i = \mathrm{Gal}(\overline{K}/M_i)$, so H_2/H_1 is naturally identified with $\mathrm{Gal}(M_1/M_2)$. By Proposition 13.2.4, we have $\mathrm{Frac}(R)^{H_i} = \mathrm{Frac}(R_{M_i})$, and G_{F_0} naturally acts on $\mathrm{Frac}(R_{M_1})$ via the G_{F_0}-action on the finite Galois extension M_1/F_0. Under this action H_1 acts trivially, so there is a natural action by H_2/H_1 on $\mathrm{Frac}(R_{M_1})$, and under the identification $H_2/H_1 = \mathrm{Gal}(M_1/M_2)$ this is the natural action induced by $\mathrm{Gal}(M_1/M_2)$. Hence,

$$\mathrm{Frac}(R_{M_1})^{\mathrm{Gal}(M_1/M_2)} = \mathrm{Frac}(R_{M_1})^{H_2/H_1} = (\mathrm{Frac}(R)^{H_1})^{H_2/H_1} = \mathrm{Frac}(R)^{H_2} = \mathrm{Frac}(R_{M_2}).$$

By Artin's lemma, if F is any abstract field equipped with an action by a finite group G, then F is finite Galois over the subfield F^G of G-invariants and the image of G in $\mathrm{Aut}(F)$ is $\mathrm{Gal}(F/F^G)$. Applying this with $F = \mathrm{Frac}(R_{M_1})$, we conclude that $\mathrm{Frac}(R_{M_1})/\mathrm{Frac}(R_{M_2})$ is indeed a finite Galois extension and that $\mathrm{Gal}(M_1/M_2)$ maps *onto* its Galois group. The only remaining problem is to prove that this surjective map of groups is an isomorphism, which is to say that if $g \in \mathrm{Gal}(M_1/M_2)$ acts trivially on R_{M_1} then $g = 1$. This will be proved in Corollary 13.3.12. ■

13.3. **Imperfect fields of norms: construction.** The theory of imperfect norm fields aims to describe the Galois theory of an infinitely ramified Galois extension K_∞/K inside of $\overline{K}$ of the type considered in the discussion immediately preceding Example 13.1.4. (Actually, the general theory as in [51] handles a much larger class of infinitely ramified extensions K_∞/K, namely those whose Galois closure has Galois group with the property that the higher ramification groups are all *open*; these are called *arithmetically profinite* extensions, and a theorem of Sen shows that this openness holds when the Galois group is a p-adic Lie group and has open inertia subgroup. This generality is important in work of Breuil and Kisin, which uses non-Galois extensions obtained by adjoining a compatible system of p-power roots of a uniformizer π, as in the integral p-adic Hodge theory in Part III.)

We now fix such an extension K_∞/K, and recall that it comes equipped with an exhaustive rising tower of finite Galois subextensions K_n/K ($n \geqslant 0$) such that for some $n_0 \geqslant 0$ each extension K_n/K_{n_0} is totally ramified and cyclic of degree p^{n-n_0} for all $n \geqslant n_0$. We let $\Gamma = \mathrm{Gal}(K_\infty/K)$, and note that the open normal subgroup $\mathrm{Gal}(K_\infty/K_{n_0})$ is topologically isomorphic to $\mathbf{Z}_p$. Since we are most interested in the case $K_\infty = K(\mu_{p^\infty})$ (as in the discussion at the beginning of §13.1), we definitely do not assume that $n_0 = 0$, nor do we assume that Γ has no nontrivial p-torsion (as we want to allow $K = \mathbf{Q}_p$ with $p = 2$). We also do not assume Γ is abelian, though this is not important for the applications we will give.

We fix a finite extension M/K_∞ inside of $\overline{K}$, and aim to associate to this an "imperfect norm field" inside of $\mathrm{Frac}(R)$. This will proceed by using a "finite approximation". That is, we choose a finite extension L/K such that $M = LK_\infty$, and we define $L_n = LK_n$ for all $n \geqslant 0$ and $L_\infty = LK_\infty = M$. By Example 13.1.4, the tower $\{L_n/L\}_{n\geqslant 0}$ satisfies the same axioms as $\{K_n/K\}$. In particular, there is an $n_0(L)$ such that $L_\infty/L_{n_0(L)}$ is a totally ramified $\mathbf{Z}_p$-extension with each $L_n/L_{n_0(L)}$ of degree $p^{n-n_0(L)}$ for $n \geqslant n_0(L)$. This $n_0(L)$ will be fixed throughout what follows (and we write n_0 to denote $n_0(K)$). Our construction of the imperfect norm field $\mathbf{E}_M \subseteq \operatorname{Frac}(R)$ attached to M will use the tower $\{L_n\}$, but in the end we will check that $\mathbf{E}_M$ is independent of the initial choice of L.

Much like perfect norm fields, the imperfect norm field associated to M will be the fraction field of a certain complete valuation ring inside of R. It will have perfect closure that is dense in $\operatorname{Frac}(R_M)$, so the construction of its valuation ring will be given inside of R_M.

Our first lemma begins to show why we use the terminology "norm fields" (and the complete justification will be given in Lemma 13.7.5).

Lemma 13.3.1. *There exist an integer $n_L \geqslant 0$ and a proper ideal $\mathfrak{a}_L$ in $\mathscr{O}_{L_{(n_L)}}$ containing p and cutting out the p-adic topology such that*

$$\mathrm{N}_{L_{n+1}/L_n}(x) \equiv x^p \mod \mathfrak{a}_L \mathscr{O}_{L_{n+1}}$$

for all $x \in \mathscr{O}_{L_{n+1}}$ and all $n \geqslant n_L$.

Proof. Applying Lemma 13.1.7 to the totally ramified $\mathbf{Z}_p$-extension $L_\infty/L_{n_0(L)}$, there is an integer $n_L \geqslant n_0(L)$ such that for all $n \geqslant n_L$ we have

$$v\big((g-\mathrm{id})(x)\big) \geqslant \frac{1}{2(p-1)}$$

for all $x \in \mathscr{O}_{L_{n+1}}$ and all $g \in \operatorname{Gal}(L_{n+1}/L_n)$. Since $L_\infty/L_{n_0(L)}$ is a totally ramified $\mathbf{Z}_p$-extension, by increasing n_L enough we can arrange that there is an element $y \in \mathscr{O}_{L_{n_L}}$ such that $0 < v(y) \leqslant \frac{1}{2(p-1)}$. Take $\mathfrak{a}_L = y\mathscr{O}_{L_{n_L}}$. Then for all $n \geqslant n_L$ and all $x \in \mathscr{O}_{L_{n+1}}$ we have

$$\mathrm{N}_{L_{n+1}/L_n}(x) = \prod_{g \in \operatorname{Gal}(L_{n+1}/L_n)} g(x) \equiv x^p \mod \mathfrak{a}_L \mathscr{O}_{L_{n+1}}.$$

■

Remark 13.3.2. In what follows, we always take n_L big enough as in the preceding proof with $n_L \geqslant n_0(L)$ as well, so L_∞/L_{n_L} is a totally ramified $\mathbf{Z}_p$-extension.

Lemma 13.3.1 shows that by working at the level of the finite extensions L_n/L we can relate p-power compatible sequences to norms at finite layer L_{n+1}/L_n far up in the tower of finite subextensions of L_∞/L. More specifically, by Proposition 4.3.1 we have

$$R_{L_\infty} = \underline{R}(\mathscr{O}_{L_\infty}/\mathfrak{a}_L\mathscr{O}_{L_\infty}) = \{(x_n) \in \prod_{n \geqslant 0} (\mathscr{O}_{L_\infty}/\mathfrak{a}_L\mathscr{O}_{L_\infty}) \,|\, x_{n+1}^p = x_n \text{ for all } n\},$$

and Lemma 13.3.1 says that if $n \geqslant n_L$ then the composite map

$$\mathscr{O}_{L_{n+1}} \xrightarrow{\mathrm{N}_{L_{n+1}/L_n}} \mathscr{O}_{L_n} \twoheadrightarrow \mathscr{O}_{L_n}/\mathfrak{a}_L\mathscr{O}_{L_n} \hookrightarrow \mathscr{O}_{L_{n+1}}/\mathfrak{a}_L\mathscr{O}_{L_{n+1}} \tag{13.3.1}$$

is a factorization of $x \mapsto x^p \bmod \mathfrak{a}_L\mathscr{O}_{L_{n+1}}$.

Definition 13.3.3. The ring $\mathbf{E}_L^+ \subseteq R_{L_\infty}$ is the subring of p-power compatible sequences $(x_n)_{n \geqslant 0}$ such that $x_n \in \mathscr{O}_{L_n} = \mathscr{O}_{LK_n}$ for sufficiently large n.

From the definition, we see that $\mathbf{E}_L^+$ is a local domain of characteristic p (its units are elements which are units in R_{L_∞}). In view of the factorization (13.3.1) of the p-power map in terms of a norm, the condition in this definition actually forces $x_n \in \mathscr{O}_{L_n}$ for *all* $n \geqslant n_L$ and $(x_n)_{n \geqslant n_L}$ is even "norm-compatible". Due to the closedness of $\mathscr{O}_{L_n}$ in $\widehat{\mathscr{O}}_{L_\infty}$ for all $n \geqslant n_L$ it follows that $\mathbf{E}_L^+$ is *closed* in R_{L_∞}, so it is complete for the v_R-adic topology. Note that we have not yet exhibited any nontrivial elements of R_{L_∞} beyond those coming from the residue field! Since we cannot expect to extract arbitrary p-power roots of elements of $\mathscr{O}_{L_\infty}$, in contrast with $\mathscr{O}_{\overline{K}}$, for general K_∞ it is not obvious at all how to make interesting elements of R_{L_∞} (aside from special cases such as $K_\infty = K(\mu_{p^\infty})$ for which we have $\varepsilon \in R_{K_\infty}$). In Lemma 13.3.6 we will make many interesting elements of $\mathbf{E}_L^+$.

Remark 13.3.4. The reason for the notation $\mathbf{E}_L^+$ is as follows: the "+" refers to the fact that this is a domain sitting inside of another ring of interest to be obtained by inverting a single distinguished element (much as we get B_{dR} from B_{dR}^+, B_{cris} from B_{cris}^+, and B_{st} from B_{st}^+). In fact it will turn out to be a *discrete* valuation ring, so inverting any nonzero non-unit will yields its fraction field (to be called $\mathbf{E}_L$). The "$\mathbf{E}$" refers to the fact that it is a ring of characteristic p (much like $\mathscr{E}$ and E from §3).

Now we check that the subring $\mathbf{E}_L^+$ in R_{L_∞} only depends on L_∞ rather than on L inside of $\overline{K}$. Indeed, if $L' \subseteq \overline{K}$ is another subextension of $\overline{K}/K$ finite over K such that $L'_\infty = L_\infty$ then there exist $m, m' \geqslant 0$ such that $L \subset L'_{m'} = L'K_{m'}$ and $L' \subseteq L_m = LK_m$, so for all $n \geqslant \max(m, m')$ we have

$$L_n = LK_n = (LK_m)K_n \supseteq L'K_n = L'_n, \;\; L'_n = L'K_n = (L'K_{m'})K_n \supseteq LK_n = L_n.$$

Since the definition of $\mathbf{E}_L^+$ only depends on the L_n's for large n, we get the asserted independence of L. Hence, the following definition (which is elaborated upon in Exercise 13.7.5) is well-posed:

Definition 13.3.5. Let M/K_∞ be a finite extension, and choose a finite extension L/K inside of M such that $M = LK_\infty = L_\infty$. Then define $\mathbf{E}_M^+ := \mathbf{E}_L^+$ inside of $R_{L_\infty} = R_M$. The fraction field $\mathbf{E}_M = \mathrm{Frac}(\mathbf{E}_M^+)$ is called the *field of norms* of M relative to K_∞/K. (We also denote it as $\mathbf{E}_L$.)

Whereas the valuation rings of the perfect norm fields are never noetherian in interesting cases, the characteristic p local domain $\mathbf{E}_L^+$ turns out to always be a complete discrete valuation ring. To analyze the structure of the rings $\mathbf{E}_L^+$, we need to find a uniformizer. The obvious strategy is to look for a compatible sequence of uniformizers in the L_n's for large n and to show that it "works". This is what we will do.

Choose $n \geqslant n_L$, so L_{n+1}/L_n is totally ramified and hence any uniformizer of L_{n+1} has norm in L_n that is a uniformizer of L_n. As Exercise 13.7.5 shows, we essentially need to go in the other direction: work our way up the tower to build a norm-compatible sequence (for large n). Actually, rather than working with norms, we can stick with the condition of p-power compatible sequences in the $\mathscr{O}_{L_n}/\mathfrak{a}\mathscr{O}_{L_n}$'s for large n, which is how we defined $\mathbf{E}_L^+$. It is not at all obvious how to find such sequences. Finding a p-power compatible sequence of uniformizers taken modulo $\mathfrak{a}_L$ comes down to some clever algebra:

Lemma 13.3.6. *For any $n \geqslant n_L$ and uniformizer π_{L_n} of L_n, there exists a uniformizer $\pi_{L_{n+1}}$ of L_{n+1} such that $\pi_{L_{n+1}}^p \equiv \pi_{L_n} \mod \mathfrak{a}_L \mathscr{O}_{L_{n+1}}$.*

Proof. First pick a uniformizer π of L_{n+1}, so $\mathrm{N}_{L_{n+1}/L_n}(\pi)$ is a uniformizer of L_n (as L_{n+1}/L_n is totally ramified, since $n \geqslant n_L$). Thus, the π_{L_n}-adic expansion of $\mathrm{N}_{L_{n+1}/L_n}(\pi)$ has vanishing constant term. Letting k' denote the common residue field k_{L_m} of all L_m's for $m \geqslant n_L$, we get Teichmüller coefficients $\{a_i\}_{i\geqslant 1}$ in k' such that

$$\mathrm{N}_{L_{n+1}/L_n}(\pi) = \sum_{i=1}^{\infty} [a_i]\pi_{L_n}^i,$$

and $a_1 \neq 0$ since $\mathrm{N}_{L_{n+1}/L_n}(\pi)$ is a uniformizer of L_n.

The uniformizer $\pi_{L_{n+1}}$ that we seek to construct in L_{n+1} must have a π-adic expansion $\sum_{j=1}^{\infty}[b_j]\pi^j$ for some sequence $\{b_j\}_{j\geqslant 1}$ in k' to be constructed (with $b_1 \neq 0$). Working modulo $\mathfrak{a}_L\mathscr{O}_{L_{n+1}}$, since $p \in \mathfrak{a}_L$ we use the property of $\mathfrak{a}_L$ from Lemma 13.3.1 to compute that modulo $\mathfrak{a}_L\mathscr{O}_{L_{n+1}}$ we have,

$$\begin{aligned}\pi_{L_{n+1}}^p \equiv \sum_{j=1}^{\infty}[b_j^p]\pi^{pj} \quad &\equiv \quad \sum_{j=1}^{\infty}[b_j^p]\Big(\sum_{i=1}^{\infty}[a_i]\pi_{L_n}^i\Big)^j \\ &\equiv \quad \sum_{m=1}^{\infty}\Big([b_m^p][a_1]^m + \sum_{j=1}^{m-1}[b_j^p]P_{m,j}([a_1],\dots,[a_m])\Big)\pi_{L_n}^m \mod \mathfrak{a}_L\mathscr{O}_{L_{n+1}}\end{aligned}$$

for some $P_{m,j} \in \mathbf{Z}[X_1,\dots,X_m]$. Since $a_1 \neq 0$ and k' is perfect, we can find $\{b_m\}_{m\geqslant 1}$ in k' solving the infinite system of equations

$$b_m^p a_1^m + \sum_{j=1}^{m-1} b_j^p P_{m,j}(a_1,\dots,a_m) = \delta_{m,1}$$

(where $\delta_{m,1}$ is the Kronecker symbol), noting that for $m=1$ we get $b_1 = a_1^{-1/p} \neq 0$. This defines a uniformizer $\pi_{L_{n+1}}$ such that $\pi_{L_{n+1}}^p \equiv \pi_{L_n} \mod \mathfrak{a}_L\mathscr{O}_{L_{n+1}}$. ■

Now fix a uniformizer $\pi_{L_{n_L}}$ of L_{n_L}. By Lemma 13.3.6 we can inductively construct a sequence $\{\pi_{L_n}\}_{n\geqslant n_L}$ of uniformizers in the L_n's for $n \geqslant n_L$ such that $\pi_{L_{n+1}}^p \equiv \pi_{L_n} \mod \mathfrak{a}_L\mathscr{O}_{L_{n+1}}$ for all $n \geqslant n_L$. To fill in the missing values for $n < n_L$, we proceed as in Exercise 13.7.5 by defining $\pi_{L_n} := \pi_{L_{n_L}}^{p^{n_L - n}}$ in $\mathfrak{m}_{L_{n_L}}$. (This may fail to lie in L_n, let alone to be a uniformizer of L_n, for $n < n_L$.) The sequence $\overline{\pi}_L := (\pi_{L_n})_{n\geqslant 0}$ is an element of $\mathbf{E}_L^+$. It depends on the specific choice of π_{L_n}'s for all $n \geqslant n_L$, but for our needs it is safe to use the suggestive notation $\overline{\pi}_L$.

What is $v_R(\overline{\pi}_L)$? Recall that by definition, for any ideal $\mathfrak{a} \subseteq \mathscr{O}_{\mathbf{C}_K}$ cutting out the p-adic topology and for any $x = (x_n) \in \underline{R}(\mathscr{O}_{\mathbf{C}_K}/\mathfrak{a}) = R$, $x^{(0)} = \lim \widehat{x}_n^{p^n}$ in $\mathscr{O}_{\mathbf{C}_K}$ for an *arbitrary* choice of lifts $\widehat{x}_n \in \mathscr{O}_{\mathbf{C}_K}$ of x_n for all $n \geqslant 0$ (no p-power compatibility requirement!). Hence, for $x = \overline{\pi}_L$ we can take $\widehat{x}_n = \pi_{L_n}$ for large n, so

$$v_R(\overline{\pi}_L) := v(\overline{\pi}_L^{(0)}) = \lim p^n v(\pi_{L_n}).$$

But for $n \geqslant n_L$ the extension L_n/L_{n_L} is totally ramified of degree p^{n-n_L}, so $v(\pi_{L_n}) = v(\pi_{L_{n_L}})/p^{n-n_L}$. Hence, for $n \geqslant n_L$ we see that the number $p^n v(\pi_{L_n}) > 0$ is *independent* of

n. In other words:

$$v_R(\overline{\pi}_L) = p^{n_L} v(\pi_{L_{n_L}}) = \frac{p^{n_L}}{e(L_{n_L})} > 0.$$

Our goal is to show that $\mathbf{E}_L^+$ is a complete discrete valuation ring with $\overline{\pi}_L$ as a uniformizer.

Example 13.3.7. Consider $L = K_0 := \mathrm{W}(k)[1/p]$, $K_\infty = K_0(\mu_{p^\infty})$, and $K_n = K_0(\zeta_{p^n})$. We then have $v(\pi_{K_n}) = 1/p^{n-1}(p-1)$ for any uniformizer π_{K_n} of K_n for any $n \geqslant 0$, so $v_R(\overline{\pi}_{K_0}) = p/(p-1) > 0$.

More specifically, in this case we can actually write down an explicit norm-compatible system of uniformizers at all levels: $\pi_{K_n} = \zeta_{p^n} - 1$ for $n \geqslant 1$. Viewed as a sequence in the ring $\mathscr{O}_{\overline{K}}/(p)$, we see that this corresponds to $\varepsilon - 1 \in \underline{R}(\mathscr{O}_{\overline{K}}) = R$, so the norm field $\mathbf{E}_{K_0(\mu_{p^\infty})}$ admits $\varepsilon - 1$ as a uniformizer. (Recall that in Example 4.3.4 we directly checked that $v_R(\varepsilon - 1) = p/(p-1)$ for all p, including $p = 2$.)

Since $v_R(\overline{\pi}_L) > 0$, the $\overline{\pi}_L$-adic topology on R is equal to v_R-adic topology with respect to which R is separated and complete. Since $\mathbf{E}_L^+$ is closed in R for the v_R-adic topology, it follows that $\mathbf{E}_L^+$ is separated and complete for the $\overline{\pi}_L$-adic topology. We have not yet proved that the $\overline{\pi}_L$-adic topology on $\mathbf{E}_L^+$ is as fine as the v_R-adic one; this will be easy to see once we show that $\mathbf{E}_L^+$ is an equicharacteristic complete discrete valuation ring with uniformizer $\overline{\pi}_L$.

To work out the structure of the characteristic p local domain $\mathbf{E}_L^+$, since we have a candidate for a uniformizer we should next determine its residue field. Since $L_\infty/L_{n_0(L)}$ is totally ramified (in the sense that all finite subextensions are totally ramified over $L_{n_0(L)}$), the residue field k_{L_∞} of $\mathscr{O}_{L_\infty}$ is the same as that of $\mathscr{O}_{L_{n_0(L)}}$; this is a finite extension k' of k, and it coincides with the residue field of $\mathscr{O}_{L_n}$ for all $n \geqslant n_0(L)$. We therefore have $\mathrm{W}(k') \subseteq \mathscr{O}_{L_n}$ for all $n \geqslant n_0(L)$, so $k' = \mathrm{W}(k')/(p)$ is naturally a subfield of $\mathscr{O}_{L_n}/\mathfrak{a}_L\mathscr{O}_{L_n}$ for all $n \geqslant n_0(L)$. Since k' is perfect, using p-power transition maps thereby identifies k' with a subring of $\mathbf{E}_L^+$. (In more concrete terms, we have just shown that $\mathbf{E}_L^+$ as a subring of the $\overline{k}$-algebra R contains k'; see 4.2.3.) In view of the $\overline{\pi}_L$-adic separatedness and completeness of $\mathbf{E}_L^+$, we now get a unique k_{L_∞}-algebra map

$$\theta_L \colon k_{L_\infty}[\![X]\!] \to \mathbf{E}_L^+$$

carrying X to $\overline{\pi}_L$, and it is continuous for the X-adic and $\overline{\pi}_L$-adic topologies. Since $\mathbf{E}_L^+$ is a domain and $\theta_L(\overline{\pi}_L) \neq 0$, it is clear that θ_L is injective (due to the structure of ideals in $k_{L_\infty}[\![X]\!]$). Even better:

Proposition 13.3.8. *The map θ_L is an isomorphism. In particular, $\mathbf{E}_L^+ \to R$ is a local map of valuation rings, so a pair of nonzero elements $x, y \in \mathbf{E}_L^+$ satisfy $x|y$ in $\mathbf{E}_L^+$ if and only if $v_R(x) \geqslant v_R(y)$.*

This proposition implies that the v_R-adic topology on $\mathbf{E}_L^+$ is the same as the $\overline{\pi}_L$-adic topology, and that the norm field $\mathbf{E}_L$ is exactly $\mathbf{E}_L^+[1/\overline{\pi}_L]$ inside of $\mathrm{Frac}(R_{L_\infty})$.

Proof. Let $k' = k_{L_\infty}$, and choose $n \geqslant n_L$. We have $k_{L_n} = k'$, so $\mathscr{O}_{L_n} = \mathrm{W}(k')[\pi_{L_n}]$. Let e_n denote the absolute ranification index of L_n and $\delta_L = v(\mathfrak{a}_L) \in (1/e_{n_L})\mathbf{Z}_{>0}$, so since $p \in \mathfrak{a}_L$ we have $\mathscr{O}_{L_n}/\mathfrak{a}_L\mathscr{O}_{L_n} = k'[\pi_{L_n}]/(\pi_{L_n}^{e_n\delta_L})$. Since L_n/L_{n_L} is totally ramified of degree p^{n-n_L}, we have $e_n = p^{n-n_L} e_{n_L}$. Thus, there is a unique isomorphism

$$\theta_{L,n} \colon k'[X_n]/(X_n^{p^{n-n_L} c_L}) \to \mathscr{O}_{L_n}/\mathfrak{a}_L \mathscr{O}_{L_n}$$

satisfying $X_n \mapsto \pi_{L_n}$ and $\theta_{L,n}(a) = a^{p^{-n}}$ for all $a \in k'$ (where $c_L := e_{n_L}\delta_L \in \mathbf{Z}_{>0}$ is an integer).

It is straightforward to verify (with the help of Lemma 13.3.1 and Lemma 13.3.6) the commutativity of the diagram

$$\begin{tikzcd}
k'[X_{n+1}]/(X_{n+1}^{p^{n+1-n_L} c_L}) \arrow[r, "\mathrm{pr}_n"] \arrow[d, "\theta_{L,n+1}"', "\simeq"] & k'[X_n]/(X_n^{p^{n-n_L} c_L}) \arrow[d, "\simeq"', "\theta_{L,n}"] \\
\mathscr{O}_{L_{n+1}}/\mathfrak{a}_L \mathscr{O}_{L_{n+1}} \arrow[r, "f_n"] & \mathscr{O}_{L_n}/\mathfrak{a}_L \mathscr{O}_{L_n}
\end{tikzcd}$$

where f_n is the p-power map and pr_n is the map of k'-algebras sending X_{n+1} to X_n. Passing to the inverse limit yields an isomorphism of rings

$$\varprojlim_{n \geqslant n_L} \theta_{L,n} \colon k'[\![X]\!] \simeq \varprojlim_{n \geqslant n_L} \mathscr{O}_{L_n}/\mathfrak{a}_L \mathscr{O}_{L_n}$$

mapping $X = \varprojlim_{n \geqslant n_L} X_n$ to $\overline{\pi}_L = \varprojlim_{n \geqslant n_L} \pi_{L_n}$. This map is θ_L precisely because of how the k'-algebra structure on $\mathbf{E}_L^+$ is *defined* (using compatible p-power root extractions in k', exactly as with the definition of the maps $\theta_{L,n}|_{k'}$). ■

In the setting of the cyclotomic extension as in Example 13.3.7 (with $K = \mathrm{W}(k)[1/p]$), we conclude that the uniformizers in $\mathbf{E}_{K_\infty}^+$ are precisely the elements with v_R-valuation equal to $p/(p-1)$.

Remark 13.3.9. Since v_R restricts to a multiple of the normalized valuation on the discretely-valued field $\mathbf{E}_L$, it follows that if L'/L is a finite extension inside of $\overline{K}$ (with L finite over $\mathrm{W}(k)[1/p]$) then $v_R(\overline{\pi}_{L'}) = v_R(\overline{\pi}_L)/e(\mathbf{E}_{L'} / \mathbf{E}_L)$ where $e(\mathbf{E}_{L'} / \mathbf{E}_L)$ denotes the ramification index of the "local" extension of discretely-valued fields $\mathbf{E}_{L'} / \mathbf{E}_L$ (which we have not yet shown to be a finite separable extension).

Our interest in the (imperfect) norm fields $\mathbf{E}_M$ for finite extensions M/K_∞ is because we will eventually prove that the correspondence $M \mapsto \mathbf{E}_M$ sets up a fully faithful bijection between the finite extensions of K_∞ inside of $\overline{K}$ and the finite separable extensions of $\mathbf{E}_{K_\infty} = \mathbf{E}_K$ inside of the algebraically closed field $\mathrm{Frac}(R)$. In particular, the Galois theory of K_∞ will coincide with that of the discretely-valued complete equicharacteristic field $\mathbf{E}_{K_\infty}$. To get a handle on the relations among these fields, it will be useful to relate them to the perfect norm fields. More precisely, we wish to see how the fraction field $\mathbf{E}_M$ of $\mathbf{E}_M^+$ is related to the perfect fraction field $\mathrm{Frac}(R_M)$ of R_M.

Choose L finite over $F_0 = \mathrm{W}(k)[1/p]$ inside of M so that $M = L_\infty$. We have a local inclusion $\mathbf{E}_M^+ := \mathrm{E}_L^+ \to R_M$ (in particular, R_M really is not a field!), and so by perfectness we see that R_M contains the ring $\varphi^{-\infty}(\mathbf{E}_M^+)$ of p-power roots of elements of $\mathbf{E}_M^+$ (and likewise at the level of fraction fields). We shall prove that $\varphi^{-\infty}(\mathbf{E}_M^+)$ is *dense* in R_M for the v_R-adic topology (and so likewise at the level of fraction fields). To do this, we need some more precise control on how "big" R_M is inside of $R = \underline{R}(\mathscr{O}_{\overline{K}}/(p)) = \underline{R}(\mathscr{O}_{\overline{K}}/\mathfrak{a}_L\mathscr{O}_{\overline{K}})$. This rests on a study of the following map: for $n \geqslant 0$, define

$$\rho_n \colon R \to \mathscr{O}_{\overline{K}}/\mathfrak{a}_L\mathscr{O}_{\overline{K}} \tag{13.3.2}$$

by the formula $(x_m)_{m\geqslant 0} \mapsto x_n$. The restriction $\rho_n|_{R_M}$ has image contained in $\mathscr{O}_M/\mathfrak{a}_L\mathscr{O}_M$, and we now describe the kernel and image of $\rho_n|_{R_M}$ more precisely:

Corollary 13.3.10. *Let M/K_∞ be a finite extension inside of $\overline{K}$, and choose L/F_0 finite inside of $\overline{K}$ such that $M = L_\infty$. The map ρ_n induces a surjection $R_M \to \mathscr{O}_M/\mathfrak{a}_L\mathscr{O}_M$ whose kernel is $\overline{\pi}_L^{p^{n-n_L}c_L}R_M$, where $c_L \geqslant 1$ is an integer depending only on L and $\mathfrak{a}_L$. If $n \geqslant n_L$ then this map carries $\mathbf{E}_M^+ := \mathbf{E}_L^+$ onto $\mathscr{O}_{L_n}/\mathfrak{a}_L\mathscr{O}_{L_n}$.*

Proof. Via the isomorphism θ_L of Proposition 13.3.8, if $n \geqslant n_L$ then the restriction of ρ_n to $\mathbf{E}_L^+$ is identified with the k_{L_∞}-algebra map map $k_{L_\infty}[\![X]\!] \to \mathscr{O}_{L_n}/\mathfrak{a}_L\mathscr{O}_{L_n}$ sending X to π_{L_n} for all $n \geqslant 0$. This is visibly surjective, and (with $\delta_L := v(\mathfrak{a}_L)$ and $e_n := e_{L_n}$) its kernel is generated by $X^{\delta_L e_n} = X^{p^{n-n_L}c_L}$, where $c_L = e_{n_L}\delta_L$, so the second statement is proved.

For arbitrary $n \geqslant 0$ we have $\rho_{n+1} \circ \varphi = \rho_n$ on R_M, where φ is the p-power map, so by invertibility of φ on the perfect R_M we see that all $\rho_n(R_M)$'s are the same. If we fix n and choose $m \geqslant \max(n, n_L)$ then

$$\mathscr{O}_{L_m}/\mathfrak{a}_L\mathscr{O}_{L_m} = \rho_m(\mathbf{E}_L^+) \subseteq \rho_m(R_M) = \rho_n(R_M).$$

Letting m grow, this proves that $\rho_n|_{R_M}$ has full image in $\mathscr{O}_M/\mathfrak{a}_L\mathscr{O}_M$.

It remains to determine the kernel of ρ_n on R_M, We have $v_R(\overline{\pi}_L) = p^{n_L}v(\pi_{L_{n_L}}) = p^{n_L}/e_{n_L}$ where e_{n_L} is the absolute ramification index of L_{n_L}. Hence, for $x = (x^{(m)})_{m\geqslant 0} \in R_M$ we have that $\rho_n(x) = 0$ if and only if $x^{(n)} \in \mathfrak{a}_L\widehat{\mathscr{O}}_M$, and since $v_R(x) := v(x^{(0)}) = p^n v(x^{(n)})$ we see that it is equivalent to have $v_R(x) \geqslant p^n\delta_L = p^{n-n_L}c_L v_R(\overline{\pi}_L)$. In other words, $\ker(\rho_n|_{R_M}) = \overline{\pi}_L^{p^{n-n_L}c_L}R_M$, as desired. ■

As an application of this corollary, we can establish the link between imperfect and perfect norm fields:

Proposition 13.3.11. *Let M/K_∞ be a finite extension inside of $\overline{K}$, and L/F_0 finite with $L_\infty = M$. The subring $\varphi^{-\infty}(\mathbf{E}_M^+)$ in R_M is dense for the v_R-adic (equivalently, $\overline{\pi}_L$-adic) topology.*

Proof. Pick an arbitrary element $x \in R_M$, so x is identified with a p-power compatible sequence $(x_m)_{m\geqslant 0}$ in $\mathscr{O}_M/\mathfrak{a}_L\mathscr{O}_M$. To approximate x by an element of $\varphi^{-\infty}(\mathbf{E}_M^+)$ we see to build some p-power compatible sequences (y_n) of elements $y_n \in \mathscr{O}_{L_n}/\mathfrak{a}_L\mathscr{O}_{L_n}$ for $n \geqslant n_L$. To this end, fix $n \geqslant n_L$ and consider $x_n \in \mathscr{O}_M/\mathfrak{a}_L\mathscr{O}_M$. Since $\mathscr{O}_M$ is the rising union of the valuation rings $\mathscr{O}_{L_m}$ for $m \geqslant 0$, we can pick $n' \geqslant n$ such that x_n lies in the subring $\mathscr{O}_{L_{n'}}/\mathfrak{a}_L\mathscr{O}_{L_{n'}}$.

By Corollary 13.3.10, there exists $y = (y_m)_{m\geqslant 0} \in \mathbf{E}_M^+ := \mathbf{E}_L^+$ such that $y_{n'} = x_n$. Hence, $\rho_n(x) = \rho_{n'}(y) = \rho_n(\varphi^{n-n'}(y))$, so

$$x - \varphi^{n-n'}(y) \in \ker \rho_n|_{R_M} = \overline{\pi}_L^{p^{n-n_L}c_L}R_M$$

(once again using Corollary 13.3.10). By taking n large, we get elements of $\varphi^{-\infty}(\mathbf{E}_M^+)$ arbitrarily close to x. ■

The loose end from the theory of perfect norm fields (at the end of the proof of Proposition 13.2.6) can now be filled in:

Corollary 13.3.12. *Let M'/M be a finite Galois extension of finite extensions of K_∞. If $g \in \mathrm{Gal}(M'/M)$ acts trivially on $R_{M'}$ then $g = 1$.*

Proof. Let L/K be a finite extension such that $L_\infty = M$ and $L = L_{n_L}$. Every element of $R_{M'}$ is represented by a unique p-power compatible sequence in $\widehat{\mathscr{O}}_{M'}$, so g acts trivially on all such sequences. The surjectivity in Corollary 13.3.10 gives that every element of $\mathscr{O}_{M'}/\mathfrak{a}_L\mathscr{O}_{M'}$. is the reduction of the initial term of an element of $R_{M'}$. In particular, the g-fixed part of $\widehat{\mathscr{O}}_{M'}$ maps surjectively to $\mathscr{O}_{M'}/\mathfrak{a}_L\mathscr{O}_{M'}$. But since $L = L_{n_L}$, we have that $\mathfrak{a}_L$ is a proper ideal in $\mathscr{O}_L$ (on which g acts trivially). Thus, by successive approximation we conclude that g acts trivially on $\widehat{\mathscr{O}}_{M'}$, so $g = 1$ as desired. ■

13.4. **Imperfect norm fields: Galois equivalence.** We can now finally study how the norm field $\mathbf{E}_M$ varies with M. By construction, if $M'/M/K_\infty$ is a finite tower inside of $\overline{K}$ then $\mathbf{E}_M$ is contained in $\mathbf{E}_{M'}$ inside of $\mathrm{Frac}(R)$ since we can write $M' = L'_\infty$ and $M = L_\infty$ with L'/L finite over $\mathrm{W}(k)[1/p]$ inside of $\overline{K}$. We eventually wish to show that all finite separable extensions of the norm field $\mathbf{E}_{K_\infty}$ have the form $\mathbf{E}_M$, and use this to identify the Galois theory of K_∞ and $\mathbf{E}_{K_\infty}$. It is not yet clear if such extensions of (imperfect) norm fields are separable, but let us first show that the degree is as desired:

Proposition 13.4.1. *For $M'/M/K_\infty$ finite inside of $\overline{K}$, the extension $\mathbf{E}_{M'}\,/\,\mathbf{E}_M$ has finite degree equal to $[M' : M]$.*

Proof. By choosing a finite Galois extension of M inside of $\overline{K}$ that contains M', we may use transitivity of field degree to reduce to the case when M'/L is Galois. Thus, we can choose a finite extension L'/L over $\mathrm{W}(k)[1/p]$ such that L'/L is Galois, $M' = L'_\infty$, $M = L_\infty$, and L' is linearly disjoint from L_∞ over L, so for $L_n = LK_n$ and $L'_n = L'K_n$ we have the equality of Galois groups

$$\mathrm{Gal}(L'/L) = \mathrm{Gal}(L'_n/L_n) = \mathrm{Gal}(M'/M)$$

for all n. We can also arrange that L has the same residue field as $L_\infty = M$ (i.e., all L_n/L are totally ramified). Since L'/L and M'/M have canonically identified Galois groups, there is a canonical bijection between their lattices of intermediate fields. In particular, by using transitivity of field degree, it suffices to separately treat the cases when L'/L is unramified or totally ramified. Moreover, in the totally ramified case we may assume that M' and M have the same residue field (as otherwise some L'_n/L_n has a nontrivial residue field degree, so by replacing L'/L with such an L'_n/L_n we could use another application of the unramified case and drop the degree of $[M' : M]$, and proceed by induction on this dgeree). Keep in mind that $\mathbf{E}_M = \mathbf{E}_L$ and $\mathbf{E}_{M'} = \mathbf{E}_{L'}$.

First assume that L'/L is unramified, so since L_∞/L is totally ramified it follows that L'_n/L_n is unramified for all n, so uniformizers of L_n are uniformizers of L'_n for all n. Hence, we can use $\overline{\pi}_L$ as $\overline{\pi}_{L'}$, so the extension $\mathbf{E}_{L'}\,/\,\mathbf{E}_L$ of completely discretely-valued fields has ramification index 1. Its residue field extension is $k_{L'_\infty}/k_{L_\infty}$, so $[\mathbf{E}_{L'} : \mathbf{E}_L] = [k_{L'_\infty} : k_{L_\infty}]$. Likewise, since L'/L is unramified, we have $[L' : L] = [k_{L'} : k_L]$. It has been arranged that $k_{L_\infty} = k_L$, and since $L'_n = L' \otimes_L L_n$ with L_n/L totally ramified and L'/L unramified we see that $k_{L'_n} = k_{L'}$ for all $n \geqslant 0$. Passing to the limit, $k_{L'_\infty} = k_{L'}$, so the desired equality of field degrees is proved in the unramified case.

Now we can assume L'/L is totally ramified with all L'_n, L_n, $L'_\infty = M'$, and $L_\infty = M$ having a common residue field k'. Thus, each L'_n/L_n is totally ramified with degree $e := [L'_\infty : L_\infty]$, so for any choice of uniformizers π_{L_n} or L_n and $\pi_{L'_n}$ of L'_n we have $\mathscr{O}_{L'_n} = \mathscr{O}_{L_n}[\pi_{L'_n}]$ and $\pi_{L_n} = \pi_{L'_n}^e u_n$ for some $u_n \in \mathscr{O}_{L'_n}^\times$. Choose the uniformizers so that for $n \geqslant \max(n_L, n_{L'})$ we have

$$\pi_{L_{n+1}}^p \equiv \pi_{L_n} \mod \mathfrak{a}_L \mathscr{O}_{L_{n+1}}, \ \pi_{L'_{n+1}}^p \equiv \pi_{L'_n} \mod \mathfrak{a}_L \mathscr{O}_{L'_{n+1}}.$$

Then $u_{n+1}^p \equiv u_n \mod \pi_{L'_n}^{-e} \mathfrak{a}_L \mathscr{O}_{L'_{n+1}}$ provided that n is big enough so $v(\pi_{L'_n}^e) \leqslant v(\mathfrak{a}_L)$ (and we can find such n since L'_∞/L'_m is a totally ramified $\mathbf{Z}_p$-extension for large enough m).

This compatibility property for the elements $u_n \in \mathscr{O}_{L'_n}^\times$ for large n defines an element $u \in (\mathbf{E}_{L'}^+)^\times$ such that $\overline{\pi}_L = \overline{\pi}_{L'}^e u$. Since $\mathbf{E}_{L'} = k'(\!(\overline{\pi}_{L'})\!)$ and $\mathbf{E}_L = k'(\!(\overline{\pi}_L)\!)$, we conclude that the local extension of complete discrete valuation rings $\mathbf{E}_L^+ \to \mathbf{E}_{L'}^+$ induces the extension $k' \to k'[t]/(t^e)$ modulo $\overline{\pi}_L$. Successive approximation therefore implies that it is a module-finite extension, necessarily then finite flat with degree e. Hence, $\mathbf{E}_{L'} / \mathbf{E}_L$ is a finite extension of fields with degree $e = [L'_\infty : L_\infty]$. ■

Recall from field theory that if F'/F is a finite extension of fields, then $\#\operatorname{Aut}(F'/F) \leqslant [F' : F]$ with equality if and only if F'/F is Galois. We use this to prove:

Proposition 13.4.2. *For every tower $M'/M/K_\infty$ of finite extensions, the extension of norm fields $\mathbf{E}_{M'} / \mathbf{E}_M$ is finite separable with degree $[M' : M]$, and if M'/M is Galois then so is $\mathbf{E}_{M'} / \mathbf{E}_M$, and there is a natural isomorphism* $\operatorname{Gal}(M'/M) \simeq \operatorname{Gal}(\mathbf{E}_{M'} / \mathbf{E}_M)$.

It will be proved shortly that there is a converse: if $\mathbf{E}_{M'} / \mathbf{E}_M$ is Galois then so is M'/M, and in general $\operatorname{Hom}_{K_\infty}(M_1, M_2) = \operatorname{Hom}_{\mathbf{E}_{K_\infty}}(\mathbf{E}_{M_1}, \mathbf{E}_{M_2})$ for all finite extensions M_1, M_2 of K_∞ inside of $\overline{K}$.

Proof. By transitivity of separability and field degree via a finite extension of M' that is Galois over M, it suffices to treat the case when M'/M is Galois. Proposition 13.4.1 shows that $[\mathbf{E}_{M'} : \mathbf{E}_M] = [M' : M]$, so we just have to construct a natural isomorphism $\operatorname{Gal}(M'/M) \to \operatorname{Aut}(\mathbf{E}_{M'} / \mathbf{E}_M)$.

We may and do find finite extensions $L'/L/K$ such that $M' = L'_\infty$, $M = L_\infty$, and L'/L is Galois with $L'_\infty = L' \otimes_L L_\infty$. In particular, $[M' : M] = [L' : L]$ and $\operatorname{Gal}(M'/M)$ is naturally identified with $\operatorname{Gal}(L'/L)$, with this group identification compatible with any further increase in L (and corresponding increase in L'). We also increase L so that we can take $L_{n_L} = L$, so $\mathfrak{a}_L$ is an ideal in $\mathscr{O}_L$ (proper and containing p). There is a natural map of groups

$$\operatorname{Gal}(L'/L) \to \operatorname{Aut}(\mathbf{E}_{L'} / \mathbf{E}_L) = \operatorname{Aut}(\mathbf{E}_{M'} / \mathbf{E}_M)$$

(the latter equality by definition of $\mathbf{E}_{M'}$ and $\mathbf{E}_M$), and it suffices to show that this is an isomorphism. The target has size $[\mathbf{E}_{M'} : \mathbf{E}_M] = [M' : M]$ and the source has size $[L' : L] = [M' : M]$, so it suffices to prove injectivity.

Consider $g \in \operatorname{Gal}(L'/L)$ that acts trivially on $\mathbf{E}_{L'}$, so its isometric action on $R_{L'_\infty} = R_{M'}$ is trivial on the subring $\varphi^{-\infty}(\mathbf{E}_{L'}^+)$ of p-powers of elements of $\mathbf{E}_{L'}^+ = \mathbf{E}_{M'}^+$. By Proposition

13.3.11 this subring is dense, so g acts trivially on $R_{M'}$. Hence, the element $g \in \mathrm{Gal}(L'/L) = \mathrm{Gal}(M'/M)$ acts trivially on $R_{M'}$, so by Corollary 13.3.12 we get $g = 1$ as desired. ∎

Theorem 13.4.3. *The functor*

$$\{\textit{finite extensions of } K_\infty \textit{ in } \overline{K}\} \to \{\textit{finite separable extensions of } \mathbf{E}_{K_\infty} \textit{ in } \mathrm{Frac}(R)\}$$
$$M \rightsquigarrow \mathbf{E}_M$$

is an equivalence of Galois-categories.

There are two things to be proved: the bijectivity on Hom-sets, and the essential surjectivity of the functor. First we consider bijectivity of the map

$$\mathrm{Hom}_{K_\infty}(M', M) \to \mathrm{Hom}_{\mathbf{E}_{K_\infty}}(\mathbf{E}_{M'}, \mathbf{E}_M).$$

Let M''/K_∞ be a finite Galois extension containing M and M'. Then $\mathrm{Hom}_{K_\infty}(M', M)$ is the set of $\mathrm{Gal}(M''/M)$-invariant elements in $\mathrm{Hom}_{K_\infty}(M', M'')$, and similarly on the norm-field side, so by applying Proposition 13.4.2 to M''/M and using functoriality we may replace M with M'' to reduce to the case when M/K_∞ is a Galois extension containing M' (inside of $\overline{K}$). In this case $\mathrm{Hom}_{K_\infty}(M', M)$ is naturally identified with $\mathrm{Gal}(M/K_\infty)/\mathrm{Gal}(M/M')$, and similarly on the norm-field side, so by functoriality we are done.

It remains to prove the essential surjectivity:

Proposition 13.4.4. *Let M/K_∞ be a finite extension inside of $\overline{K}$ and E a finite separable subextension of* $\mathrm{Frac}(R)/\mathbf{E}_M$. *There exists a finite extension M'/M inside of $\overline{K}$ such that $E = \mathbf{E}_{M'}$ inside of* $\mathrm{Frac}(R)$.

Proof. Since E is finite over $\mathbf{E}_M$, its ring of integers is a complete discrete valuation ring which is finite over that of $\mathbf{E}_M$. In particular, the extension of perfect residue fields k_E/k_M is finite separable, so there is a unique finite extension M_1/M inside of $\overline{K}$ for which $k_{M_1} = k_E$ and $[M_1 : M] = [k_E : k_M]$. (To make M_1, first choose a finite extension F of $\mathrm{W}(k)[1/p]$ inside of $\overline{K}$ such that $M = F_{|infty}$. Then pick n large enough so that F_∞ is a totally ramified $\mathbf{Z}_p$-extension of F_n. Let M_1 be the linearly disjoint compositum of M over F_n with the unramified extension of F_n inducing the residue field k_E.) We have $\mathbf{E}_M := \mathbf{E}_F = k_{F_\infty}(\!(\overline{\pi}_F)\!)$. The proof of Proposition 13.4.1 then gives that $\mathbf{E}_{M_1}^+ = k_{M_1}[\![\overline{\pi}_F]\!]$, so by replacing M with M_1 we may and do arrange that $k_E = k_M$.

Having arranged that $k_E = k_M$, now choose a finite extension L of $\mathrm{W}(k)[1/p]$ inside of $\overline{K}$ such that $M = L_\infty$. By Proposition 13.3.8, we have $\mathbf{E}_M := \mathbf{E}_L = k_{L_\infty}(\!(\overline{\pi}_L)\!)$. Since E is finite over $\mathbf{E}_L$, we have $E = k_E(\!(x)\!)$ with x the root of a separable Eisenstein polynomial $P \in k_E[\![\overline{\pi}_L]\!][X]$ [44, Ch. II, Thm. 2]. The meaning of the Eisenstein property is that

$$P = X^e + a_1 X^{e-1} + \cdots + a_e$$

with $a_1, \ldots a_e \in \overline{\pi}_L \, \mathbf{E}_L^+$ and $a_e \in \pi_L \cdot \left(\mathbf{E}_L^+\right)^\times$. To find the required L'/L we will make an Eisenstein polynomial over some $\mathscr{O}_{L_n}$ that "approximates" P and has a root in $\overline{K}$ generating an extension of $L_\infty = M$ whose associated (imperfect) norm field in $\mathrm{Frac}(R)$ will be E over $\mathbf{E}_L$.

To carry out the approximation, we need to use approximations slightly better than modulo $\mathfrak{a}_L$, so instead of using the ρ_n's as defined in (13.3.2) we use the variant ρ'_n with $\mathfrak{a}_L^2$ replacing

$\mathfrak{a}_L$; this is permissible since $R = \underline{R}(\mathscr{O}_{\overline{K}}/\mathfrak{a}\mathscr{O}_{\overline{K}})$ for any open proper ideal $\mathfrak{a}$ whose powers cut out the p-adic topology (e.g., we may use $\mathfrak{a}_L^2\mathscr{O}_{\overline{K}}$) and Corollary 13.3.10 remains valid for $\mathfrak{a}_L^2$ as well upon increasing some constants such as n_L. For each $n \geqslant n_L$ define

$$x_n = \rho_n'(x) \in \mathscr{O}_{L_n}/\mathfrak{a}_L\mathscr{O}_{L_n},\ \ \rho_n'(P) = X^e + \rho_n(a_1)X^{e-1} + \cdots + \rho_n(a_e) \in \big(\mathscr{O}_{L_n}/\mathfrak{a}_L\mathscr{O}_{L_n}\big)[X].$$

Let $P_n = X^e + \alpha_{1,n}X^{e-1} + \cdots + \alpha_{e,n} \in \mathscr{O}_{L_n}[X]$ be any lift of $\rho_{L,n}(P)$. We have $\alpha_{1,n}, \ldots, \alpha_{e,n} \in \pi_{L_n}\mathscr{O}_{L_n}$ and $\alpha_{e,n} \in \pi_{L_n}\mathscr{O}_{L_n}^\times$ (because this holds modulo $\mathfrak{a}_L$; we can increase n_L if necessary so that $v(\pi_{L_n}) < v(\mathfrak{a}_L)$ for all $n \geqslant n_L$). Thus P_n is an Eisenstein polynomial over L_n.

Let y be a root of P in R and define $y_n = \rho_n(y) \in \mathscr{O}_{\overline{K}}/\mathfrak{a}_L^2\mathscr{O}_{\overline{K}}$. We have $\rho_n(P(y)) = 0$, so $P_n(y_n) = 0$ in $\mathscr{O}_{\overline{K}}/\mathfrak{a}_L^2\mathscr{O}_{\overline{K}}$. By separability of P over $\mathrm{Frac}(R)$, $P'(y) \neq 0$ in R. Hence, taking n sufficiently large (depending on y), we can arrange that $P_n'(y_n) = \rho_n(P'(y))$ is "almost a unit" in $\mathscr{O}_{\overline{K}}/\mathfrak{a}_L^2\mathscr{O}_{\overline{K}}$. More specifically, we can ensure that $P_n'(y_n) \not\equiv 0 \bmod \mathfrak{a}_L\mathscr{O}_{\overline{K}}$. (Here we are using that if $r = (r^{(m)})_{m\geqslant 0} \in R$ is nonzero then $p^m v_R(r^{(m)}) = v_R(r^{(0)})$ is fixed and finite, so $v_R(r^{(m)}) \to 0$ as $m \to \infty$.)

There exists a finite subextension L_{n,y_n} of $\overline{K}/L_n$ such that $y_n \in \mathscr{O}_{L_{n,y_n}}/\mathfrak{a}_L^2\mathscr{O}_{L_{n,y_n}}$. Observe that $\mathscr{O}_{L_{n,y_n}}$ is a p-adic discrete valuation ring over which P_n is a monic polynomial with nonzero discriminant (which is generally not a unit in $\mathscr{O}_{L_{n,y_n}}$). We wish to prove (at least for big enough n) that there exists a $\widehat{y}_n \in \mathscr{O}_{L_{n,y_n}}$ that is a root of P_n and reduces to y_n modulo $\mathfrak{a}_L\mathscr{O}_{L_{n,y_n}}$. Note that in this lifting step the initial congruence modulo $\mathfrak{a}_L^2$ has been weakened to one modulo $\mathfrak{a}_L$ (which is why we had to impose the stronger congruence condition at the outset). To carry this out, the usual form of Hensel's Lemma is of no use since $P_n'(y_n)$ is generally not a unit. But we will show that it is "almost" a unit for sufficiently large n, so we will be able to succeed by using Lang's generalization [33, II, §2, Prop. 2] of Hensel's Lemma (incorporating a uniqueness aspect which is not stated by Lang and will not be helpful or relevant in what follows):

Lemma 13.4.5 (Lang)**.** *Let F be a field complete for a nontrivial $\mathbf{R}$-valued non-archimedean valuation v, let A be its valuation ring, and let $f \in A[X]$ be a monic polynomial. Suppose there exists $a_0 \in A$ for which $v(f(a_0)) > 2v(f'(a_0))$. (In particular, $f'(a_0) \neq 0$.) There is a root a of f in A satisfying $v(a - a_0) \geqslant v(f(a_0)) - 2v(f'(a_0))$. It also satisfies $v(a - a_0) \geqslant v(f'(a_0))$ and is unique as such.*

To apply this lemma, let $\delta_L = v(\mathfrak{a}_L)$. Consider any $z \in \mathscr{O}_{L_{n,y_n}}$ reducing to y_n modulo $\mathfrak{a}_L^2\mathscr{O}_{L_{n,y_n}}$, so $v(P_n(z)) \geqslant 2\delta_L$ since $P_n(z)$ reduces to $P_n(y_n) = 0$ modulo $\mathfrak{a}_L^2\mathscr{O}_{L_{n,y_n}}$. For sufficiently large n we will use Lang's criterion to find such a z for which $v(P_n'(z)) < \delta_L/2$, so $2v(P_n'(z)) < v(P_n(z)) - \delta_L \leqslant v(P_n(z))$. There will then exist a root $\widehat{y}_n$ of P_n in $\mathscr{O}_{L_{n,y_n}}$ such that

$$v(\widehat{y}_n - z) \geqslant v(P_n(z)) - 2v(P_n'(z)) \geqslant 2\delta_L - \delta_L = \delta_L,$$

so $\widehat{y}_n \equiv z \bmod \mathfrak{a}_L\mathscr{O}_{L_{n,y_n}}$ and hence $\widehat{y}_n \equiv y_n \bmod \mathfrak{a}_L\mathscr{O}_{L_{n,y_n}}$. That is, we will have found the desired $\widehat{y}_n$.

Since $P_n'(y_n) = \rho_n(P'(y))$, we have $P_n'(y_n)^4 = \rho_n(P'(y)^4)$. But $P'(y) \neq 0$ in the domain R (as y is a root of the polynomial $P \in R_L[X]$ with nonzero discriminant), so $P'(y)^4 \neq 0$, and hence we just need to check that for any nonzero $r = (r^{(m)})_{m\geqslant 0} \in R$, $\rho_m(r) = r^{(m)} \bmod \mathfrak{a}_L\mathscr{O}_{\mathbf{C}_K}$ is nonzero for m large enough. Since $p^m v_R(r^{(m)}) = v_R(r^{(0)})$ is fixed and finite (as $r^{(0)} \neq 0$), so $v_R(r^{(m)}) \to 0$ as $m \to \infty$, for big enough m we have $v_R(r^{(m)}) < v(\mathfrak{a}_L)$

(as $v(\mathfrak{a}_L) > 0$, due to $\mathfrak{a}_L$ cutting out the p-adic topology). This completes the construction of the desired root $\widehat{y}_n \in \mathscr{O}_{L_{n,y_n}}$ of P_n lifting y_n, with $\widehat{y}_n$ unique as such even in $\mathscr{O}_{\overline{K}}$.

The polynomial P is separable over $\mathbf{E}_L$ and monic over $\mathbf{E}_L^+$, so it has e distinct roots in R: for n big enough, say $n \geqslant n_E$, we have $v_R(y - y') < p^{n-n_L} c_L v_R(\overline{\pi}_L)$ for any distinct roots root y, y' of P, where c_L is as in Corollary 13.3.10. The description of the kernel in that corollary therefore gives that $y_n \neq y'_n$ in $\mathscr{O}_{\overline{K}}/\mathfrak{a}_L\mathscr{O}_{\overline{K}}$ for such n, so $v(\widehat{y}_n - \widehat{y}'_n) < \delta_L := v(\mathfrak{a}_L)$ where $\widehat{y}_n$ and $\widehat{y}'_n$ are the respective roots of P_n in $\mathscr{O}_{\overline{K}}$ found above that lift y_n and y'_n. In particular, as we vary y through the e roots of P in $\mathrm{Frac}(R)$, we get e pairwise distinct roots $\widehat{y}$ of P_n in $\mathscr{O}_{\overline{K}}$, so these must be all of the roots of P_n in $\mathscr{O}_{\overline{K}}$.

On the other hand, since $x_{n+1}^p = x_n$, we have $x_{n+1}^p - x_n = 0$ in $\mathscr{O}_{\overline{K}}/\mathfrak{a}_L\mathscr{O}_{\overline{K}}$, so $v(\widehat{x}_{n+1}^p - \widehat{x}_n) \geqslant \delta_L$. Thus, $v(\widehat{x}_{n+1}^p - \widehat{x}_n) > v(\widehat{y}_n - \widehat{x}_n)$ for all roots y of P distinct from our initial root x that we had at the start (the uniformizer of E). By Krasner's lemma [33, Ch. II, Prop. 3], this implies $\widehat{x}_n \in L_n\big(\widehat{x}_{n+1}^p\big)$.

The fields $L'_n := L_n(\widehat{x}_n)$ for $n \geqslant n_E$ will solve our problem as follows. Each L'_n is a totally ramified extension of L_n of degree e (as it is generated by the root of an Eisenstein polynomial P_n of degree e over L_n), so the residue field of each L'_n is k_{L_∞} and its valuation ring $\mathscr{O}_{L'_n}$ has the form $\mathscr{O}_{L_n}[\widehat{x}_n]$ where $\widehat{x}_n$ is a root of P_n. Moreover, $L'_n \subset L'_{n+1}$, so we have the following situation:

$$\begin{tikzcd}
 & L'_{n+1} \\
L_{n+1} \arrow[ur, dash, "e"] & \\
 & L'_n \arrow[uu, dash] \\
L_n \arrow[uu, dash, "p"] \arrow[ur, dash, "e"] &
\end{tikzcd}$$

This implies that $[L'_{n+1} : L'_n] = p$ and $L'_{n+1} = L_{n+1}L'_n$. In particular, $L'_n = L'K_n$ for all $n \geqslant n_E$, where $L' = L'_{n_E}$. By construction, the element $x = (x_n) \in R$ lies in $\mathbf{E}_{L'}^+$, so $E = k_{L_\infty}(\!(x)\!) \subset \mathbf{E}_{L'}$. But $[\mathbf{E}_{L'} : \mathbf{E}_L] = [L'_\infty : L_\infty] = e = [E : \mathbf{E}_L]$, so $E = \mathbf{E}_{L'}$ and we are done. ■

To summarize, we have proved that for each finite extension M/K_∞ inside of $\overline{K}$ there is a well-defined subfield $\mathbf{E}_M := \mathbf{E}_L$ inside of $\mathrm{Frac}(R)$ for any finite L/K (inside of M) satisfying $LK_\infty = M$, and that the subfield

$$\mathbf{E} = \bigcup_{\substack{K_\infty \subset M \subset \overline{K} \\ M/K \text{ finite}}} \mathbf{E}_M$$

is the separable closure of $\mathbf{E}_{K_\infty} := \mathbf{E}_K$ in $\mathrm{Frac}(R)$. Moreover, we have naturally

$$\mathrm{Gal}(\overline{K}/K_\infty) \simeq \mathrm{Gal}(\mathbf{E} \,/\, \mathbf{E}_{K_\infty})$$

carrying $\mathrm{Gal}(\overline{K}/M)$ to $\mathrm{Gal}(\mathbf{E} \,/\, \mathbf{E}_M)$ for all finite subextensions M of $\overline{K}/K_\infty$. (Equivalently, $\mathrm{Gal}(\overline{K}/L_\infty) \simeq \mathrm{Gal}(\mathbf{E} \,/\, \mathbf{E}_L)$ for all finite subextensions L of $\overline{K}/K$.)

By Lemma 13.1.10 we know that if L'/L are finite over K inside of $\overline{K}$ then the numbers $p^n v(\mathfrak{D}_{L'_n/L_n})$ are uniformly bounded for all $n \geqslant 0$. In fact, something much stronger is true:

this sequence eventually becomes constant, and the terminal value encodes ramification information for the corresponding extension $\mathbf{E}_{L'_\infty} / \mathbf{E}_{L_\infty}$ of (imperfect) norm fields:

Proposition 13.4.6. *Let M'/M be finite over K_∞ inside of $\overline{K}$, and let L'/L be finite over K inside of $\overline{K}$ such that $L'_\infty = M'$ and $L_\infty = M$. The extension $\mathbf{E}_{M'} / \mathbf{E}_M$ of complete discretely-valued fields is unramified if and only if M'/M is unramified, and in general $v_{\mathbf{E}}(\mathfrak{D}_{\mathbf{E}_{M'} / \mathbf{E}_M}) = p^n v(\mathfrak{D}_{L'_n/L_n})$ for all $n \geqslant n_{L'}$.*

Proof. Since $[\mathbf{E}_{M'} : \mathrm{E}_M] = [M' : M]$ and $\mathbf{E}_M$ (resp. $\mathbf{E}_{M'}$) has the same residue field as M (resp. M') inside of $\overline{k}$, the equivalence for unramifiedness is clear.

To relation the ramification information in characteristic 0 and in characteristic p, by transitivity of the different we can reduce to the case when M'/M is Galois (by comparing with a finite extension of M' that is Galois over M). We can also reduce to the case when the Galois extension M'/M is totally ramified (and hence $\mathbf{E}_{M'} / \mathbf{E}_M$ is totally ramified). By replacing L and L' with L_n and L'_n for sufficiently large n, we can also arrange that the residue fields of L'_∞ and L coincide (so all intermediate fields have the same residue field too), and we can arrange that L' is linearly disjoint from L_∞ over L and L'/L is Galois. Thus, L'_n/L_n are Galois with the same Galois group as L'/L. Let $G = \mathrm{Gal}(L'/L) = \mathrm{Gal}(M'/M) = \mathrm{Gal}(\mathbf{E}_{M'} / \mathbf{E}_M)$.

Let P (resp. P_n) be the minimal polynomial of $\overline{\pi}_{L'}$ over $\mathbf{E}_L$ (resp. of $\pi_{L'_n}$ over L_n), so

$$v_R(\mathfrak{D}_{\mathbf{E}_{L'} / \mathbf{E}_L}) = v_R(P'(\overline{\pi}_{L'})) = \lim_{n\to\infty} p^n v(P'_n(\pi_{L'_n})) = \lim_{n\to\infty} p^n v(\mathfrak{D}_{L'_n/L_n}).$$

On the other hand, we have arranged that $\mathrm{Gal}(L'_{n+1}/L_{n+1}) \simeq \mathrm{Gal}(L'_n/L_n) = G$. In particular, we have

$$v(\mathfrak{D}_{L'_{n+1}/L_{n+1}}) = \sum_{g\in G} v(g(\pi_{L'_{n+1}}) - \pi_{L'_{n+1}})$$

since L'_{n+1}/L_{n+1} is totally ramified with Galois group G. Since $\pi_{L'_{n+1}}^p \equiv \pi_{L'_n} \mod \mathfrak{a}_{L'}\mathscr{O}_{L'_{n+1}}$ and $v(\pi_{L'_{n+1}}) < v(\mathfrak{a}_{L'})$ for $n \geqslant n_{L'}$, we have $v(g(\pi_{L'_n}) - \pi_{L'_n}) = v(g(\pi_{L'_{n+1}}^p) - \pi_{L'_{n+1}}^p)$, so

$$v(\mathfrak{D}_{L'_n/L_n}) = \sum_{g\in G} pv(g(\pi_{L'_{n+1}}) - \pi_{L'_{n+1}}) = p \cdot v(\mathfrak{D}_{L'_{n+1}/L_{n+1}}).$$

■

13.5. **Some rings in characteristic zero.** In the period ring constructions of p-adic Hodge theory, the ring $\mathrm{W}(R)$ played a prominent role. Now that we see R is but a special case of the theory of norm fields, it is natural to look into Witt rings of valuation rings of other kinds of norm fields. Witt rings of imperfect rings are quite awful, so we restrict attention to perfect norm fields and their valuation rings.

Definition 13.5.1. Let L be a subextension of $\overline{K}/F_0$ (possibly of infinite degree over F_0). Define $\widetilde{\mathbf{A}}_L^+ = \mathrm{W}(R_L)$ and $\widetilde{\mathbf{A}}_L = \mathrm{W}(\mathrm{Frac}(R_L))$. In the special case $L = \overline{K}$ denote these as $\widetilde{\mathbf{A}}^+ = \mathrm{W}(R)$ and $\widetilde{\mathbf{A}} = \mathrm{W}(\mathrm{Frac}(R))$. Endow these rings with the Witt vector Frobenius φ.

We made extensive use of the ring $\widetilde{\mathbf{A}}^+ = \mathrm{W}(R)$ when constructing period rings, and used crucially that its natural action by $G_{F_0} = \mathrm{Gal}(\overline{K}/F_0)$ commutes with the action by φ. The ring $\widetilde{\mathbf{A}} = \mathrm{W}(\mathrm{Frac}(R))$ was useful in our initial development of the theory of $\mathfrak{S}$-modules in integral p-adic Hodge theory.

Remark 13.5.2. It can be quite difficult to remember all of the notation. Here are some rules to remember the above notation. Ring denoted "A" are always $\mathbf{Z}_p$-flat, and p is not a unit in them. Roughly speaking, rings with a "+" are "more integral" than the ones without. We view $\widetilde{\mathbf{A}}^+ := \mathrm{W}(R)$ are "more integral" than $\widetilde{\mathbf{A}} := \mathrm{W}(\mathrm{Frac}(R))$, for example.

Proposition 13.5.3. *If H is a closed subgroup of G_{F_0} and $L = \overline{K}^H$ then*

$$(\widetilde{\mathbf{A}}^+)^H = \widetilde{\mathbf{A}}_L^+, \ \ \widetilde{\mathbf{A}}^H = \widetilde{\mathbf{A}}_L.$$

In other words, $\mathrm{W}(R)^H = \mathrm{W}(R_L)$ *and* $\mathrm{W}(\mathrm{Frac}(R))^H = \mathrm{W}(\mathrm{Frac}(R_L))$.

Proof. Apply Proposition 13.2.4 to Witt coordinates. ■

Since R and its fraction field $\mathrm{Frac}(R)$ admit v_R-isometric actions by G_{F_0}, their Witt rings admit natural product topologies. In the case of $\mathrm{W}(R)$ this was rather crucial in defining the interesting topology on B_{dR}^+ in Exercise 4.5.3 which went beyond its discrete-valuation topology (and was necessary for B_{dR}^+ to "know" the valuation topology on its residue field $\mathbf{C}_K$, the topology one must use when applying the Tate–Sen theorems on H^1). We now give this topology a name:

Definition 13.5.4. The *weak topology* on $\widetilde{\mathbf{A}} = \mathrm{W}(\mathrm{Frac}(R))$ is defined as the product topology of the v_R-adic topology under the identification of sets $\mathrm{W}(\mathrm{Frac}(R)) = \prod_{n\geqslant 0}\mathrm{Frac}(R)$. Equivalently, it is the inverse limit of the product topologies on each $\mathrm{W}_n(\mathrm{Frac}(R)) = \mathrm{Frac}(R)^n$.

We define the weak topology on $\widetilde{\mathbf{A}}^+ = \mathrm{W}(R)$ similarly. (This is the subspace topology from the weak topology on $\widetilde{\mathbf{A}}$.) We make similar definitions for $\widetilde{\mathbf{A}}_L^+ = \mathrm{W}(R_L)$ and $\widetilde{\mathbf{A}}_L = \mathrm{W}(\mathrm{Frac}(R_L))$.

The importance of the topology on B_{dR}^+ from Exercise 4.5.3 makes it clear that the weak topology on $\mathrm{W}(\mathrm{Frac}(R))$ is a good thing to work with. It is has many more open sets than the p-adic topology (which corresponds to making $\mathrm{Frac}(R)$ discrete).

An important application of the rings R and $\mathrm{Frac}(R)$ in p-adic Hodge theory is to provide explicit Cohen rings for fields like $k(\!(u)\!)$, or more intrinsically (as we can now see) Cohen rings for imperfect norm fields. In Example 3.2.1 and the subsequent considerations in §3, as well as in Remark 10.4.6 and the subsequent consideration in §10.4, we saw how such Cohen rings are found inside of $\mathrm{W}(\mathrm{Frac}(R))$ and are related to the theory of étale φ-modules. (Strictly speaking, those earlier situations used non-Galois extensions K_∞/K and so require a more general theory of norm fields, as in [51].) Since we now better understand (via imperfect norm fields and Theorem 13.4.3) how étale φ-modules provide a semilinear algebra classification for p-adic representations of G_{K_∞}, before we move on to explain how to improve this to classify representations of G_K it is instructive to revisit the earlier constructions from Example 3.2.1 and Remark 10.4.6. to concretely construct (and then use!) Cohen rings for any imperfect norm field (in the setting considered above: K_∞/K essentially an infinitely ramified $\mathbf{Z}_p$-extension).

To make a concrete Cohen ring for the imperfect norm field $\mathbf{E}_{K_\infty} := \mathbf{E}_K$, we first give a concrete description of this field: by Proposition 13.3.8 there is a isomorphism of k_{K_∞}-algebras

$$\theta_{K_\infty} : k_{K_\infty}[\![X]\!] \to \mathbf{E}_K^+$$

defined by $X \mapsto \overline{\pi}_K$ where $\overline{\pi}_K$ can built from a system of norm-compatible uniformizers in the K_n's for large n. (For $K = \mathrm{W}(k)[1/p]$ we identified one explicit choice of $\overline{\pi}_K$ in Example 13.3.7: $\varepsilon - 1$.) The uniformizers of $\mathbf{E}_K^+$ are elements with a certain explicit v_R-valuation (given in the discussion preceding Example 13.3.7), but it is hard to make $\overline{\pi}_K$ very explicit. Let us assume such a choice has been made and do everything else concretely based on this choice.

Pick a lift $\widehat{\pi}_K \in \mathrm{W}(\mathrm{Frac}(R_{K_\infty})) = \widetilde{\mathbf{A}}_{K_\infty}$ of $\overline{\pi}_K$, such as the Teichmüller lift $[\overline{\pi}_K]$, so θ_{K_∞} lifts to an injective homomorphism of $\mathrm{W}(k_{K_\infty})$-algebras:

$$\begin{aligned}\Theta_{K_\infty} : \mathrm{W}(k_{K_\infty})[\![X]\!] &\to \widetilde{\mathbf{A}}_{K_\infty}\\ X &\mapsto \widehat{\pi}_K\end{aligned}$$

The element $\widehat{\pi}_K$ is a unit in $\widetilde{\mathbf{A}}_{K_\infty}$ because its image in the residue field $\mathrm{Frac}(R_{K_\infty})$ is the nonzero element $\overline{\pi}_K$. Thus, the lifted map Θ_{K_∞} uniquely extends to an injective homomorphism of $\mathrm{W}(k_{K_\infty})$-algebras

$$\mathrm{W}(k_{K_\infty})[\![X]\!][X^{-1}] \to \widetilde{\mathbf{A}}_{K_\infty} = \mathrm{W}(\mathrm{Frac}(R_{K_\infty})).$$

The target is a p-adic discrete valuation ring and the source is a Dedekind domain in which (p) is prime (the quotient by (p) being the field $k_{K_\infty}(\!(X)\!)$). Hence, there is an induced local injection $\mathrm{W}(k_{K_\infty})[\![X]\!][X^{-1}]_{(p)} \to \mathrm{W}(\mathrm{Frac}(R_{K_\infty}))$, so it extends uniquely to a local injective on the p-adic completion of the source ring:

$$j_{K_\infty} : \mathrm{W}(k_{K_\infty})[\![X]\!]\{X^{-1}\} \to \mathrm{W}(\mathrm{Frac}(R_{K_\infty})) = \widetilde{\mathbf{A}}_{K_\infty},$$

where $\mathrm{W}(k_{K_\infty})[\![X]\!]\{X^{-1}\}$ is the ring of integral formal Laurent series whose negative-degree coefficients tend p-adically to 0. This is analogous to the construction $\mathfrak{S}_{(p)}^{\wedge} \to \mathrm{W}(\mathrm{Frac}(R))$ that was used in integral p-adic Hodge theory.

Definition 13.5.5. Define $\mathbf{A}_K \subset \widetilde{\mathbf{A}}_{K_\infty} = \mathrm{W}(\mathrm{Frac}(R_{K_\infty}))$ to be the image of j_{K_∞}. This is a Cohen ring for the imperfect norm field $\mathbf{E}_K^+[1/\overline{\pi}_K] = \mathbf{E}_K$.

Remark 13.5.6. Beware that the subring $\mathbf{A}_K$ just defined inside of $\widetilde{\mathbf{A}}_{K_\infty}$ depends on the choice of $\widehat{\pi}_K$. Indeed, we can change $\widehat{\pi}_K$ by adding to it pw for an arbitrary $w \in \widehat{\mathbf{A}}_{K_\infty}$! We will regard this choice as fixed for all time.

For any finite extension L/K, we can make an analogous construction which avoids needing to make yet another non-canonical choice of a $\widehat{\pi}_L$. First, since $\mathbf{E}_L$ is a finite separable extension of the residue field $\mathbf{E}_K$ of $\mathbf{A}_K$, by Hensel's Lemma there is up to unique isomorphism a finite unramified extension $\mathbf{A}_L/\mathbf{A}_K$ inducing the extension $\mathbf{E}_L\,/\,\mathbf{E}_K$ on residue fields. In particular, $\mathbf{A}_L$ is a Cohen ring for the norm field $\mathbf{E}_L$. Moreover, since $\widetilde{\mathbf{A}}_{L_\infty} = \mathrm{W}(\mathrm{Frac}(R_{L_\infty}))$ is a p-adic discrete valuation ring containing $\widetilde{\mathbf{A}}_{K_\infty}$ and its residue field $\mathrm{Frac}(R_{L_\infty})$ contains the finite separable extension $\mathbf{E}_L$ of the norm field $\mathbf{E}_K$, by Hensel's Lemma there is a *unique* local $\mathbf{A}_K$-algebra map

$$\mathbf{A}_L \hookrightarrow \widetilde{\mathbf{A}}_{L_\infty} = \mathrm{W}(\mathrm{Frac}(R_{L_\infty}))$$

lifting the inclusion $\mathbf{E}_L \hookrightarrow \mathrm{Frac}(R_{L_\infty})$ on residue fields.

Consider the directed system of Cohen rings $\{\mathbf{A}_L\}$ inside of $\mathrm{W}(\mathrm{Frac}(R))$ as L runs through the finite extensions of K inside of $\overline{K}$. (It would be more elegant to index these subrings by the finite extensions M/K_∞.) The directed union of these rings is a discrete valuation ring with uniformizer p inside of $\mathrm{W}(\mathrm{Frac}(R))$ having residue field that is the union $\mathbf{E}$ of the $\mathbf{E}_L$'s (so $\mathbf{E}$ is the separable closure of $\mathbf{E}_K$ inside of $\mathrm{Frac}(R)$, by Theorem 13.4.3). Hence, the completion $\mathbf{A}$ of $\cup \mathbf{A}_L$ is a Cohen ring for $\mathbf{E} = (\mathbf{E}_K)_s$, so it is also the p-adic completion of the valuation ring of the maximal unramified extension of $\mathbf{A}_K[1/p] = \mathrm{Frac}(\mathbf{A}_K)$.

The theory of (φ, Γ)-modules involves how the G_K-action on $\mathrm{W}(\mathrm{Frac}(R))$ interacts with the subrings $\mathbf{A}_L$. This rests on:

Lemma 13.5.7. *For each finite extension L/K inside of $\overline{K}$, the G_{L_∞}-action on* $\mathrm{W}(\mathrm{Frac}(R))$ *is trivial on $\mathbf{A}_L$ and $\mathbf{A}^{G_{L_\infty}} = \mathbf{A}_L$.*

Proof. We first focus on the basic case $L = K$, and then build up everything from that. Since G_{K_∞} acting on R has trivial action on R_{K_∞}, and hence on $\mathbf{E}_K^+$, it follows that the G_{K_∞}-action on $\mathrm{W}(\mathrm{Frac}(R))$ is trivial on $\mathrm{W}(\mathrm{Frac}(R_{K_\infty}))$. But $\mathbf{A}_K$ lies inside of $\mathrm{W}(\mathrm{Frac}(R_{K_\infty}))$ by construction, so it has a trivial action under G_{K_∞}. In view of the uniqueness of the $\mathbf{A}_K$-algebra embeddings $\mathbf{A}_L \hookrightarrow \mathrm{W}(\mathrm{Frac}(R))$ lifting the canonical inclusions $\mathbf{E}_L \hookrightarrow \mathrm{Frac}(R)$ on residue fields, it follows that the G_{K_∞}-action shuffles the $\mathbf{A}_L$'s (with their embeddings) as it does the $\mathbf{E}_L$'s inside of $\mathrm{Frac}(R)$ via the canonical *isomorphism* $G_{K_\infty} \simeq \mathrm{Gal}(\mathbf{E}\,/\,\mathbf{E}_K)$ from Theorem 13.4.3. Thus, the G_{K_∞}-action must preserve the maximal unramified extension of $\mathbf{A}_K[1/p]$ and hence its p-adic completion $\mathbf{A}[1/p]$, and at the residue field level this induces the isomorphism of G_{K_∞} onto $\mathrm{Gal}(\mathbf{E}\,/\,\mathbf{E}_K)$. By the completed unramified descent from Lemma 3.2.6 (applied to the discrete valuation ring $\mathbf{A}_K$), we conclude that $\mathbf{A}^{G_{K_\infty}} = \mathbf{A}_K$. This settles the case $L = K$.

In general, G_{L_∞} inside of G_{K_∞} is identified with $\mathrm{Gal}(\mathbf{E}\,/\,\mathbf{E}_L)$ inside of $\mathrm{Gal}(\mathbf{E}\,/\,\mathbf{E}_K)$ via the norm field equivalence of Galois theories. Thus, the same argument via tracking residue field maps shows that G_{L_∞} must act trivially on $\mathbf{A}_L$, and since $\mathbf{A}$ is likewise identified with the completion of the maximal unramified extension of $\mathbf{A}_L$ we may again use completed unramified descent (now applied to $\mathbf{A}_L$) to see that $\mathbf{A}^{G_{L_\infty}} = \mathbf{A}_L$. ■

13.6. **(φ, Γ)-modules.** We can now use Theorem 13.4.3 and §3 to obtain the classification of G_{K_∞}-representations via étale φ-modules, as follows. Assume $\widehat{\pi}_K$ has been chosen so that $\mathbf{A}_K$ is stable under the φ-action; this holds if we choose the Teichmüller lifting $\widehat{\pi}_K := [\overline{\pi}_K]$, for example. It is automatic that this φ on $\mathbf{A}_K$ lifts the Frobenius on the residue field $\mathbf{E}_K$ (as may be checked more generally for the φ-action on $\mathrm{W}(\mathrm{Frac}(R))$), and by functoriality the completed maximal unramified extension $\mathbf{A}$ is also φ-stable. Together with the φ-equivariant G_{K_∞}-action on $\mathbf{A}$ that is trivial on $\mathbf{A}_K$ (Lemma 13.5.7), we are in *exactly* the setup axiomatized in §3 to classify p-adic representations of $G_{K_\infty} \simeq \mathrm{Gal}(\mathbf{E}\,/\,\mathbf{E}_K)$ via étale φ-modules over $\mathbf{A}_K$ (with $\mathbf{A}_K$ here playing the role of $\mathscr{O}_{\mathscr{E}}$ in §3, and likewise $\mathbf{E}_K$ playing the role of E there). In other words, we have just proved:

Theorem 13.6.1. *Assume $\widehat{\pi}_K$ is chosen so that $\mathbf{A}_K$ is φ-stable. There is an equivalence between $\mathrm{Rep}_{\mathbf{Z}_p}(G_{K_\infty})$ and the category of étale φ-modules over $\mathbf{A}_K$ via*

$$T \rightsquigarrow (\mathbf{A} \otimes_{\mathbf{Z}_p} T)^{G_{K_\infty}}, \;\; D \rightsquigarrow (\mathbf{A} \otimes_{\mathbf{A}_K} D)^{\varphi\otimes\varphi_D}.$$

There is an analogous equivalence between $\mathrm{Rep}_{\mathbf{Q}_p}(G_{K_\infty})$ and the category of étale φ-modules over the fraction field $\mathbf{B}_K = \mathbf{A}_K[1/p]$.

We want to extend this to a classification of $\mathrm{Rep}_{\mathbf{Z}_p}(G_K)$ and $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ in terms of semilinear algebra data. This requires one further hypothesis on $\mathbf{A}_K$: it is stable under the action of the full group G_K on $\mathrm{W}(\mathrm{Frac}(R))$. Assuming this, by tracking unramified extensions via residue fields we see as in the preceding arguments that $\mathbf{A}$ would then also be G_K-stable. Unfortunately, the subring $\mathbf{A}_K$ in $\mathrm{W}(\mathrm{Frac}(R_{K_\infty}))$ rested on a choice of $\widehat{\pi}_K$ about which we know very little, so it is hopeless in general how to find $\widehat{\pi}_K$ so that G_K preserves $\mathbf{A}_K$. A case which might be tractable is to use the Teichmüller lifting $\widehat{\pi}_K = [\overline{\pi}_K]$ (for which at least φ-stability of $\mathbf{A}_K$ is automatic) and to hope that the G_K-action on $\mathrm{Frac}(R)$ has a very simple description on $\overline{\pi}_K$. There is a fundamental example in which a slight variant on this works:

Example 13.6.2. Consider the *cyclotomic case* $F_\infty = F_0(\mu_{p^\infty})$ with $F_0 = \mathrm{W}(k)[1/p]$. In this case, let $F_n = F_0(\zeta_{p^n} - 1)$ with $\{\zeta_{p^n}\}$ a compatible system of primitive p^nth roots of unity. By Example 13.3.7, we may take $\overline{\pi}_{F_0} = \varepsilon - 1 \in R$, so we can take $\pi_\varepsilon := [\varepsilon] - 1$ as $\widehat{\pi}_{F_0}$. (This is *not* $[\varepsilon - 1]$, on which the G_{F_0}-action is a mess.) We can see the G_{F_0}- and φ-stability of the Cohen ring $\mathbf{A}_{F_0}$ thanks to the formulas

$$\varphi(\pi_\varepsilon) = [\varepsilon]^p - 1 = (1 + \pi_\varepsilon)^p - 1, \;\; g(\pi_\varepsilon) = [\varepsilon]^{\chi(g)} - 1 = (1 + \pi_\varepsilon)^{\chi(g)} - 1.$$

In general, for any K we can apply the above procedure to the maximal unramified subfield $F_0 = \mathrm{W}(k)[1/p]$ in K and then take $\mathbf{A}_K$ to be the unique unramified extension of $\mathbf{A}_{F_0}$ inside of $\mathrm{W}(\mathrm{Frac}(R))$ corresponding to the finite separable residue field extension $\mathbf{E}_{K_\infty} / \mathbf{E}_{F_\infty}$. The G_{F_0}-stability of $\mathbf{A}_{F_0}$ then implies the G_K-stability of $\mathbf{A}_K$, and likewise for the φ-stability. (We could then a-posteriori pick an element $\widehat{\pi}_K \in \mathbf{A}_K$ lifting $\overline{\pi}_K$ in the residue field $\mathbf{E}_K$ and thereby "reconstruct" $\mathbf{A}_K$ inside of $\mathrm{W}(\mathrm{Frac}(R))$ using this choice of $\widehat{\pi}_K$. However, such reverse-engineering is not necessary, since $\widehat{\pi}_K$ above was solely an intermediate device in the attempt to find a φ-stable and G_K-stable Cohen ring of $\mathbf{E}_K$ inside of $\mathrm{W}(\mathrm{Frac}(R))$.)

Now we assume we are in a case for which $\mathbf{A}_K$ is stable under G_K and φ (as can always be arranged by taking $K_\infty = K(\mu_{p^\infty})$), Necessarily $\mathbf{A}$ is stable under both of these as well, as is $\mathbf{A}_L$ for L/K finite Galois (and $\mathbf{A}_L$ is G_L-stable for every finite extension L/K inside of $\overline{K}$, as may be checked using residue field considerations). Keep in mind that the G_K-action on $\mathbf{A}$ commutes with the φ-action on $\mathbf{A}$, as these action both arise from $\mathrm{W}(\mathrm{Frac}(R))$ on which the two visibly commute. We identify G_{K_∞} with $\mathrm{Gal}(\mathbf{E} / \mathbf{E}_K)$.

Recall that $\Gamma = \mathrm{Gal}(K_\infty/K)$ (containing $\mathbf{Z}_p$ as an open subgroup). Since G_K preserves $\mathbf{A}_K$ by hypothesis and G_{K_∞} acts trivially on $\mathbf{A}_K$ (Lemma 13.5.7), we get a natural action of Γ on $\mathbf{A}_K$. Since étale φ-modules over $\mathbf{A}_K$ are finitely generated $\mathbf{A}_K$-modules, we are going to want to consider situations in which such modules have a Γ-action compatible with the Γ-action on $\mathbf{A}_K$. Here is the basic construction:

Example 13.6.3. For $T \in \mathrm{Rep}_{\mathbf{Z}_p}(G_K)$, consider the associated étale φ-module

$$\mathrm{D}(T) = (\mathbf{A} \otimes_{\mathbf{Z}_p} T)^{G_{K_\infty}}$$

over $\mathbf{A}_K$, endowed with its φ-action, and the induced action of $\Gamma = \mathrm{Gal}(K_\infty/K) = G_K/G_{K_\infty}$. which commutes with φ (as the φ-action comes entirely from $\mathbf{A}$, on which it commutes with the G_K-action).

Going in the reverse direction, consider an étale φ-module (D, φ_D) over $\mathbf{A}_K$, and assume it is endowed with an action by Γ that commutes with the φ-action and is compatible with the Γ-action on $\mathbf{A}_K$. We then form the $\mathbf{A}$-module $\mathbf{A} \otimes_{\mathbf{A}_K} D$ as usual, endowed with the diagonal action of G_K (using Γ on D!) which commutes with the diagonal Frobenius operator $\varphi \otimes \varphi_D$. Hence, the associated finitely generated $\mathbf{Z}_p$-module $\mathrm{T}(D) = (\mathbf{A} \otimes_{\mathbf{A}_K} D)^{\varphi \otimes \varphi_D}$ is endowed with an action of G_K.

To make effective use of Γ-action on finitely generated modules over $\mathbf{A}_K$ or its fraction field $\mathbf{B}_K = \mathbf{A}_K[1/p]$, it is crucial to also impose a continuity condition relative to a natural topology on this rings. (For example, this will be needed to ensure that any G_K-modules we construct actually have continuous G_K-action.) We do *not* use the p-adic topology of A_K, since the action of Γ is not continuous relative to this (and likewise for the G_K-action on $\mathbf{A}$). Indeed, using the p-adic topology would be tantamount to viewing the residue field $\mathbf{E}_K$ discretely, and it is not true in interesting cases that the Γ-action on $\mathbf{E}_K$ (arising from the *axiom* that G_K preserves $\mathbf{A}_K$ inside of $\mathrm{W}(\mathrm{Frac}(R))$) is continuous for the discrete topology. For example, in the cyclotomic case in Example 13.6.2 the action of $\mathrm{Gal}(K(\mu_{p^\infty})/K)$ on $\mathbf{E}_K = k(\!(X)\!)$ goes via $\gamma(X) = (1+X)^{\chi(\gamma)} - 1$, so $\gamma(X) \neq X$ whenever $\gamma \neq 1$. Exercise 13.7.9 takes care of the important topological aspects of $\mathbf{A}_K$, $\mathbf{A}$, and their fraction fields which are needed to make later proofs work (so that exercise should be looked at).

By keeping track of the Γ-action that remains when a G_{K_∞}-action is eliminated, we arrive at:

Definition 13.6.4. Assume $\mathbf{A}_K$ is G_K-stable inside of $\mathrm{W}(\mathrm{Frac}(R))$. A (φ, Γ)-*module* over $\mathbf{A}_K$ is a finitely generated $\mathbf{A}_K$-module D equipped with (i) a Frobenius operator φ_D that is semilinear over the φ on $\mathbf{A}_K$ (i.e., (D, φ_D) is a φ-module over $\mathbf{A}_K$) and (ii) a φ_D-equivariant action of Γ that is semilinear over the Γ-action on $\mathbf{A}_K$ and is continuous for the natural topology of finitely generated $\mathbf{A}_K$-modules (as in Exercise 13.7.9).

Such a (φ, Γ)-module is *étale* if the underlying φ-module over $\mathbf{A}_K$ is étale (i.e., the linearization of φ_D is an isomorphism). We denote by $\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_K}(\varphi, \Gamma)$ the category of étale (φ, Γ)-modules over $\mathbf{A}_K$.

The analogous definitions over $\mathbf{A}$ go the same way.

By Example 13.6.3, Exercise 13.7.9, and the equivalences of Fontaine in Theorem 3.2.5, we immediately obtain:

Theorem 13.6.5. *Assume $\mathbf{A}_K$ is G_K-stable and φ-stable in $\mathrm{W}(\mathrm{Frac}(R))$. Then the functor*

$$\begin{aligned} \mathrm{D}\colon \mathrm{Rep}_{\mathbf{Z}_p}(G_K) &\to \mathrm{Mod}^{\text{ét}}_{\mathbf{A}_K}(\varphi, \Gamma) \\ T &\rightsquigarrow (\mathbf{A} \otimes_{\mathbf{Z}_p} T)^{G_{K_\infty}} \end{aligned}$$

is an exact equivalence of categories, with quasi-inverse $(D, \varphi_D) \mapsto \big(\mathbf{A} \otimes_{\mathbf{A}_K} D\big)^{\varphi \otimes \varphi_D = \mathrm{id}}$.

In particular, the category $\mathrm{Rep}_{\mathbf{F}_p}(G_K)$ *is equivalent to the category of étale* (ϕ, Γ)*-modules over the impefect norm field* $\mathbf{A}_K/(p) = \mathbf{E}_K \simeq k'((u))$ *(with* k' *the residue field of* K_∞*).*

Example 13.6.6. Let's work out $\mathrm{D}(T)$ for $T = \mathbf{Z}_p(r)$ with $r \in \mathbf{Z}$. From the definition we compute $\mathrm{D}(T) = (\mathbf{A}(r))^{G_{K_\infty}}$. But G_{K_∞} acts trivially under χ^r, so the Tate twist pops out and we have $\mathrm{D}(T) = \mathbf{A}^{G_{K_\infty}}(r) = \mathbf{A}_K(r)$ as an $\mathbf{A}_K$-module, and its Γ-action is exactly the Tate twist of the usual one. Its Frobenius structure is the usual one on $\mathbf{A}_K$. In other words, $\mathrm{D}(\mathbf{Z}_p(r))$ has underlying φ-module $\mathbf{A}_K$, but the Γ-action is twist by χ^t.

We can adapt the $\mathbf{Z}_p$-theory to describe $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, out of analogy with Theorem 3.3.4, as follows. Let $\mathbf{B} = \mathbf{A}[p^{-1}]$ and $\mathbf{B}_K = \mathbf{A}_K[p^{-1}]$ be the fraction fields of the Cohen rings $\mathbf{A}$ and $\mathbf{A}_K$. These are complete discretely-valued fields endowed with a Frobenius endomorphism (lifting the Frobenius on their residue fields) and a compatible action of G_K.

The notion of a (φ, Γ)-module (without the étale condition) over $\mathbf{B}_K$ and $\mathbf{B}$ is defined exactly as for $\mathbf{A}_K$ and $\mathbf{A}$, including the continuity requirement on the Γ-action. To make sense of an analogue of Theorem 13.6.5 we first need to define the notion of an *'etale* (φ, Γ)-module over $\mathbf{B}_K$. The definition goes exactly as in Definition 3.3.2; i.e., we have to assume that there is an $\mathbf{A}_K$-lattice stable under φ and Γ and on which the φ-action is étale (i.e., linearizes to an isomorphism over $\mathbf{A}_K$). Note that the Γ-action on any Γ-stable $\mathbf{A}_K$-lattice is automatically continuous (Exercise 13.7.9).

Exactly as in Theorem 3.3.4, we deduce (from Theorem 13.6.5) that the functor

$$\begin{aligned} \mathrm{D}\colon \mathrm{Rep}_{\mathbf{Q}_p}(G_K) &\to \mathrm{Mod}^{\text{ét}}_{\mathbf{B}_K}(\varphi, \Gamma) \\ V &\rightsquigarrow (\mathbf{B} \otimes_{\mathbf{Q}_p} V)^H \end{aligned}$$

is an exact equivalence of categories, a quasi-inverse being $(D, \varphi_D) \mapsto \big(\mathbf{B} \otimes_{\mathbf{B}_K} D\big)^{\varphi \otimes \varphi_D = \mathrm{id}}$.

The importance of these equivalences is mainly theoretical (as it is quite difficult to do explicit calculations with general étale φ-modules over $\mathbf{A}_K$ or $\mathbf{B}_K$): they can be used to convert questions about G_K-representations into questions in semilinear algebra. Whereas semilinear algebra is not necessarily a simple thing, it at least opens the door to a large body of techniques (such as variation of coefficients, integral structures, etc.) that are much harder (or impossible) to work with in terms of the language of G_K-representations. For example, the concept of a (φ, Γ)-module makes sense without the étale condition, and it can be useful. In fact, although only the étale ones correspond to actual G_K-representations, it happens in some interesting situations that an irreducible p-adic representation of G_K has associated étale (φ, Γ)-module over $\mathbf{B}_K = \mathbf{A}_K[1/p]$ that becomes reducible as a (φ, Γ)-module over certain extension rings of $\mathbf{B}_K$ (allowing non-étale subobjects!). This comes up in Colmez' theory of trianguline representations, for example.

13.7. **Exercises.**

Exercise 13.7.1. By [44, Ch. III, §7, Prop. 14], if $A \to B$ is a finite extension of Dedekind domains that is separable at the level of fraction fields, the finite B-module $\Omega^1_{B/A}$ has a nonzero annihilator ideal equal to $\mathfrak{D}_{B/A}$. Since $A \to B$ is étale precisely when it is unramified, which is to say $\mathfrak{D}_{B/A} = B$, we could view B as being "approximately étale" if the nonzero

$\mathfrak{D}_{B/A}$ is not "too divisible". However, this is a silly concept for Dedekind domains since the valuations involved are discrete.

Things become rather more interesting over valuation rings like $\mathscr{O}_{K_\infty}$, since $\mathfrak{m}_{K_\infty}$ contains elements with arbitrarily small valuation > 0. In particular, if M/K_∞ is a finite extension then the non-discreteness of the valuation on $\mathscr{O}_M$ makes it more interesting to ask how "close" the ideal $\mathrm{ann}_{\mathscr{O}_M}(\Omega^1_{\mathscr{O}_M/\mathscr{O}_{K_\infty}})$ is to $\mathscr{O}_M$. That is, what is the infimum of the valuations of its elements?

(1) Pick a finite extension L/K inside of M such that $M = L_\infty$. Prove that the natural map $\varinjlim \Omega^1_{\mathscr{O}_{L_n}/\mathscr{O}_{K_n}} \to \Omega^1_{\mathscr{O}_M/\mathscr{O}_{K_\infty}}$ is an isomorphism. (We do not claim that the transition maps in the direct limit are injective.) Deduce that if $a \in \mathfrak{m}_M = \cup \mathfrak{m}_{L_n}$ then a kills $\Omega^1_{\mathscr{O}_M/\mathscr{O}_{K_\infty}}$ provided that $v(a) \geqslant v(\mathfrak{D}_{L_n/K_n})$ for all large n.

(2) Using Lemma 13.1.10, prove that $\Omega^1_{\mathscr{O}_M/\mathscr{O}_{K_\infty}}$ is annihilated by $\mathfrak{m}_M$. In other words, for any $\epsilon > 0$ there exists $a \in \mathscr{O}_M$ with $v(a) < \epsilon$ such that a kills $\Omega^1_{\mathscr{O}_M/\mathscr{O}_{K_\infty}}$, so this module is killed by elements that are "almost units". So we are justified to say that $\mathscr{O}_{K_\infty} \to \mathscr{O}_M$ is "almost étale". Another way to think about it is that essentially all of the ramification in the L_n's is eaten up by the K_n's, leaving very little for the relative ramification in L_n/K_n as $n \to \infty$.

Exercise 13.7.2. In the study of finite extensions of the field K_∞ that is rising unions of finite extensions of K, we want to "approximate" such extensions of K_∞ by finite extensions of K_n for large n. This descent to a K_n should also be compatible with properties of field extensions (such as being Galois), provided n is taken large enough. There are many ways in which this idea is implemented, and this exercise works out some ubiquitous operations along such lines.

The setup we consider at first is an *abstract* field K (of arbitrary characteristic), and an infinite-degree algebraic extension K_∞/K that is a directed union of a specified (not necessarily linearly ordered or even countable) collection of subfields $\{K_i/K\}$ of finite degree over K. All composite fields below are formed inside of a fixed algebraic closure $\overline{K}$ containing K_∞.

(1) Prove that every finite extension M/K_∞ has the form $M = LK_\infty$ for a finite extension L/K (so M satisfies the same axioms as K_∞/K, by using the fields $L_i = LK_i$ in place of the K_i's), and that by replacing L with LK_i for sufficiently large i it can be arranged that L contains some K_i with $L \otimes_{K_i} K_\infty \simeq M$ (i.e., L and K_∞ are linearly disjoint over K_i). Hint: first treat the case when M/K_∞ is primitive, and then use a suitable degree-induction.

(2) Let L and L' finite over some K_{i_0} be linearly disjoint from K_∞ over K_{i_0}, and define $L_\infty = LK_\infty$, $L'_\infty = L'K_\infty$, $L_i = LK_i$, and $L'_i = L'K_i$ (for $i \geqslant i_0$). Prove that $L_\infty = L'_\infty$ inside of $\overline{K}$ if and only if $L'_i = L_i$ for all large i, and that L_∞/K_∞ is separable (resp. primitive, resp. Galois) if and only if the same holds for L_i/K_i for all large i. Taking such a large i, in the Galois case use linear disjointness to show that $\mathrm{Gal}(L_i/K_i)$ is naturally identified with $\mathrm{Gal}(L_\infty/K_\infty)$.

(3) Now assume that K is a complete discretely-valued field, and endow all algebraic extensions with the corresponding valuation. Fix a finite extension M/K_∞ and choose

a finite extension L/K_{i_0} approximating it in (1) (including the linear disjointness condition). Prove that the residue field degree $[k_M : k_{K_\infty}]$ is equal to $[k_{L_i} : k_{K_i}]$ for all large i, and that k_M/k_{K_∞} is separable if and only if k_{L_i}/k_{K_i} is separable for all large i. Deduce that $e(L_i/K_i)$ becomes constant for large i (called the *ramification degree* $e(M/K_\infty)$), and that $e(M/K_\infty) = 1$ with k_M/k_{K_∞} and M/K_∞ separable if and only if L_i/K_i is unramified for all large i. In this latter case we say that M/K_∞ is *unramified*.

(4) Using finite-level approximations, prove the intrinsic statement that the unramified finite extensions of K_∞ are in functorial correspondence with finite separable extensions of k_{K_∞}, just as for K. (This can also be proved more directly using the theory of henselian local rings.)

(5) Now assume k_K is perfect. Prove that $[M : K_\infty] = e(M/K_\infty)[k_M : k_{K_\infty}]$, and call M/K_∞ *totally ramified* if $k_M = k_{K_\infty}$. Prove that M/K_∞ is totally ramified if and only if L_i/K_i is totally ramified for all large i, and that every finite extension M/K_∞ is uniquely a totally ramified extension of an unramified extension (just like for K).

Exercise 13.7.3. This exercise explains why the construction of R_L in (13.2.1) is only interesting when L/F_0 is infinitely wildly ramified (i.e., the p-part of the ramification degrees of the finite subextensions is unbounded).

(1) If k' is the residue field of L, explain why $\mathscr{O}_L/p\mathscr{O}_L$ is naturally a k'-algebra, and deduce that $R_L = \underline{R}(\mathscr{O}_L/p\mathscr{O}_L)$ is naturally a k'-algebra. (Hint: $\underline{R}(k') = k'$!) Relate this to the $\overline{k}$-algebra structure on R from (4.2.2).

(2) Let $L \subseteq \overline{K}$ be a subextension over F_0 for which there is bounded p-part in the ramification of finite subextensions (e.g., $[L : F_0]$ finite). Using (13.2.1), show that $R_L = k'$. In particular, show by example that there are nontrivial extensions $L \to L'$ inside of $\overline{K}$ over F_0 for which $R_L = R_{L'}$ (so R_L inside of R does not determine L inside of $\overline{K}$ in general).

(3) Can you construct other examples for which $R_L = k'$? Even better, can you make examples of $\mathbf{Z}_p$-extensions L/K such that $R_L \neq k'$?

Exercise 13.7.4. Proposition 13.1.9 gives a precise growth estimate on $v(\mathfrak{D}_{K_n/K})$. But the proof used some serious input, especially Serre's geometric local class field theory. If one is content with the weaker claim that $v(\mathfrak{D}_{K_n/K}) \to \infty$ then it is possible to proceed in a more "elementary" manner, using just commutative algebra. The following exercise outlines Faltings' argument along such lines.

Let A be a complete discrete valuation ring of mixed characteristic $(0, p)$, and let $\overline{A}$ be the valuation ring of a fixed algebraic closure of $\mathrm{Frac}(A)$. In what follows, the ring of integers of a finite extension of $\mathrm{Frac}(A)$ in $\mathrm{Frac}(\overline{A})$ will be called a *finite extension of* A. If B is a finite extension of A, we write k_B and $\mathfrak{m}_B$ to denote its residue field and maximal ideal respectively. We assume that $[k_A : k_A^p] = p^d$ (i.e., k_A has a finite p-basis); the case $d = 0$ corresponds to k_A being a perfect field.

(1) Show that $\dim_{k_B}(\Omega^1_{k_B/\mathbf{Z}}) = d$, and that $\Omega^1_{B/A}$ can be generated by $d + 1$ elements. (Hint: use the second fundamental exact sequence and Nakayama's lemma).

(2) Let $v\colon \overline{A} \to \mathbf{Q} \cup \{\infty\}$ be the valuation normalized by $v(p) = 1$. For $\delta \in v(B - \{0\})$, let p^δ denote an element in $B - \{0\}$ whose valuation is δ. In particular, the different $\mathfrak{D}_{B/A}$ has the form $p^{\delta_{B/A}} B$ for some $\delta_{B/A} \in \mathbf{Q}_{\geqslant 0}$ that we wish to estimate in certain cases. As a first step in that direction, explain why $\mathrm{length}_B(\Omega^1_{B/A}) = \mathrm{length}_B(B/p^{\delta_{B/A}} B)$.

(3) Assume now that we are given a sequence of finite extensions

$$A = A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots \subset \overline{A}$$

such that $\Omega^1_{A_n/A_{n-1}}$ admits a quotient isomorphic to $(A_n/pA_n)^{d+1}$ for all $n \geqslant 1$. If B is a finite extension of A, let B_n denote the normalization of the $B \otimes_A A_n$, which is to say the integral closure of A in $\mathrm{Frac}(B) \otimes_{\mathrm{Frac}(A)} \mathrm{Frac}(A_n)$. This is a product of finitely many discrete valuation rings $B_{n,i}$ finite over A (since A is complete). Let $\delta_n \in \mathbf{Q}_{\geqslant 0}$ be the maximum of the v-valuations of the ideals $\mathfrak{D}_{B_{n,i}/A_n}$. The rest of this exercise proves that $\delta_n \to \infty$.

First reduce to the case where B_n is a domain for all $n \in \mathbf{Z}_{\geqslant 0}$.

(4) Fix $n \in \mathbf{Z}_{\geqslant 0}$, and consider the composite

$$B_{n+1} \otimes_{B_n} \Omega^1_{B_n/A_n} \xrightarrow{a} \Omega^1_{B_{n+1}/A_n} \xrightarrow{b} \Omega^1_{B_{n+1}/A_{n+1}}$$

Using (2), show that the sequence

$$0 \to B_{n+1} \otimes_{A_{n+1}} \Omega^1_{A_{n+1}/A_n} \to \Omega^1_{B_{n+1}/A_n} \xrightarrow{b} \Omega^1_{B_{n+1}/A_{n+1}} \to 0$$

is exact.

Then using (1) and the elementary divisors theorem, deduce that $\ker(b)$ contains the kernel of multiplication by p, so that

$$\ker\left(B_{n+1} \otimes_{B_n} \Omega^1_{B_n/A_n} \xrightarrow{\times p} B_{n+1} \otimes_{B_n} \Omega^1_{B_n/A_n}\right) \subset \ker(b \circ a)$$

and $\mathrm{length}(\ker(b \circ a)) \geqslant \mathrm{length}(B_{n+1}/p^{\beta_n} B_{n+1})$ where $\beta_n = \min(1, \delta_n/(d+1))$.

(5) Using the definition of the discriminant, show that $p^{\delta_n - \delta_{n+1}} B_{n+1} \subset B_n \otimes_{A_n} A_{n+1} \subset B_{n+1}$. Deduce that $\mathrm{coker}(ba)$ is killed by $p^{\delta_n - \delta_{n+1}}$, and that

$$\mathrm{length}(\mathrm{coker}(b \circ a)) \leqslant \mathrm{length}\left(B_{n+1}/p^{(d+1)(\delta_n - \delta_{n+1})}\right)$$

(6) Use (4) and (5) to show that $\delta_n - \delta_{n+1} \geqslant \beta_n - (d+1)(\delta_n - \delta_{n+1})$. Deduce that $\delta_n \to \infty$.

Exercise 13.7.5. This exercise explains where where the terminology "field of norms" comes from; we work in the setup of §13.3, with L/K a finite extension for which L_∞ is a chosen finite extension M/K_∞.

Consider the inverse system $\{\mathscr{O}_{L_n}\}_{n \geqslant 0}$ using $\mathrm{N}_{L_{n+1}/L_n} : \mathscr{O}_{L_{n+1}} \to \mathscr{O}_{L_n}$ as the transition maps.

(1) For any norm-compatible sequence $(x^{(n)})_{n \geqslant n_L}$ in the $\mathscr{O}_{L_n}$'s, note that the reductions $x_n = x^{(n)} \bmod \mathfrak{a}\mathscr{O}_{L_n}$ for $n \geqslant n_L$ form a p-power compatible sequence in $\mathscr{O}_{L_\infty}/\mathfrak{a}\mathscr{O}_{L_\infty}$. This is not quite an element of $\underline{R}(\mathscr{O}_{L_\infty}/\mathfrak{a}_L\mathscr{O}_{L_\infty}) = R_{L_\infty}$ since we haven't fill in the terms x_n for $n < n_L$. Show that there is a unique way to fill in these missing terms to get an element of R_{L_∞}, and obtain a multiplicative map $\varprojlim \mathscr{O}_{L_n} \to \mathbf{E}_L^+ \to R_{L_\infty}$.

(2) Explain why the x_n as just artificially defined for small n usually do not lie in $\mathscr{O}_{L_n}$.

(3) Prove that the multiplicative map $\varprojlim \mathscr{O}_{L_n} \to \mathbf{E}_L^+$ is bijective. In the theory of norm fields a ring structure is directly defined on $\varprojlim \mathscr{O}_{L_n}$ in the spirit of the formulas in Remark 4.3.2 (except using norms instead of p-power maps), and it makes this map into a ring isomorphism. (Beware that this procedure for lifting elements of $\mathbf{E}_L^+$ to norm-compatible sequences in the $\mathscr{O}_{L_n}$'s gives a completely different output than the procedure that lifts elements of R_{L_∞} to p-power compatible sequences in $\widehat{\mathscr{O}}_{L_\infty}$!)

Exercise 13.7.6. This exercise proves that $\mathrm{Frac}(R)$ is the completion of the separable closure $\mathbf{E} = \cup\, \mathbf{E}_M$ of $\mathbf{E}_{K_\infty}$ (with M ranging through the finite extensions of K_∞ inside of $\overline{K}$, so $\mathbf{E}_M$ ranges through the finite separable extensions of $\mathbf{E}_{K_\infty}$ inside of $\mathrm{Frac}(R)$).

(1) Let E is a complete discrete valuation field of characteristic $p > 0$, $\overline{E}$ an algebraic closure and E_s the separable closure of E in $\overline{E}$. Show that $\widehat{E_s} = \widehat{\overline{E}}$. (Hint: Prove $\widehat{E_s}$ is perfect by approximating $X^p - a$ with $X^p - \pi^n X - a$ for a uniformizer π of F and large n.)

(2) Let $\mathbf{E}^+$ denote the valuation ring of $\mathbf{E}$. Using Proposition 13.3.11, show that the map $\rho_n : R \to \mathscr{O}_{\mathbf{C}_K}$ carries $\widehat{\mathbf{E}^+}$ onto $\mathscr{O}_{\mathbf{C}_K}$ for all $n \geqslant 0$, and deduce that $\mathbf{E}_K$ is dense in $\mathrm{Frac}(R)$.

Exercise 13.7.7. The weak topologies defined in Definition 13.5.4 satisfy a number of basic compatibilities that are used all the time without comment. This exercise develops these properties.

(1) Prove that the weak topologies on $\widetilde{\mathbf{A}}_L = \mathrm{W}(\mathrm{Frac}(R_L))$ and $\widetilde{\mathbf{A}}_L^+ = \mathrm{W}(R_L)$ are the subspace topologies from the weak topologies on $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{A}}^+$ respectively.

(2) Using perfectness of R_L, prove that $p : \widetilde{\mathbf{A}}_L \to \widetilde{\mathbf{A}}_L$ is a closed embedding for the weak topology on $\widetilde{\mathbf{A}}_L = \mathrm{W}(\mathrm{Frac}(R_L))$, and prove that the quotient topology on $\mathrm{Frac}(R_L) = \widetilde{\mathbf{A}}_L/(p)$ is the v_R-adic topology. Prove an analogue for $\widetilde{\mathbf{A}}_L^+ = \mathrm{W}(R_L)$, and show that $\widetilde{\mathbf{A}}_L^+$ is a closed (but not open!) subring of $\widetilde{\mathbf{A}}_L$.

(3) Prove that the action of G_{F_0} on $\widetilde{\mathbf{A}} = \mathrm{W}(\mathrm{Frac}(R))$ is continuous for the weak topology, but *not* for the p-adic topology. (Hint: In the discrete case, what is the induced topology on the quotient $\mathbf{C}_K$ by means of θ?)

(4) Prove that $\widetilde{\mathbf{A}}_L = \mathrm{W}(\mathrm{Frac}(R_L))$ is a Hausdorff topological ring with a countable base of opens around every point, so we can test openness and continuity using *sequences*. Prove that in $\widetilde{\mathbf{A}}_L$ there is a base of opens around 0 that are $\widetilde{\mathbf{A}}_L^+$-submodules.

(5) Prove that the topology on $\widetilde{\mathbf{A}}_L$ is *complete* in the sense that if $\{a_n\}$ is a sequence in $\widetilde{\mathbf{A}}_L$ converging to 0 then $\sum a_n$ converges.

Exercise 13.7.8. Prove that for the p-adic cyclotomic extension K_∞/K of a p-adic field K, the G_K-action on $\mathbf{A}_K$ factors through the quotient $\mathrm{Gal}(K_\infty/K)$. (Hint: reduce to the case $K = F_0 := \mathrm{W}(k)[1/p]$ and look at the induced action on the residue field $\mathbf{E}_K$, or at least its valuation ring $\mathbf{E}_K^+ \simeq k[\![X]\!]$.)

Exercise 13.7.9. Give $\mathbf{A}_K$ and $\mathbf{A}$ their natural subspace topologies from the weak topology on the Hausdorff topological ring $\mathrm{W}(\mathrm{Frac}(R))$ (so $\mathbf{A}_K$ gets the subspace topology from $\mathbf{A}$, and each has a countable base of opens around any point).

(1) Prove that the Γ-action on $\mathbf{A}_K$ is continuous, $p : \mathbf{A}_K \to \mathbf{A}_K$ is a closed embedding, and the quotient topology on the residue field $\mathbf{E}_K$ is its valuation topology. In particular, deduce that $p\mathbf{A}_K$ is *not* open in $\mathbf{A}_K$, but show that $\mathbf{A}_K$ has a base of opens around the identity which are ideals, and each of which contains some $p^n\mathbf{A}_K$. Also prove that $\mathbf{A}_K$ has the inverse limit topology from the $\mathbf{A}_K/(p^n)$'s. Do likewise for $\mathbf{A}$ with its G_K-action. (Hint: pull everything down from $\mathrm{W}(\mathrm{Frac}(R))$).

(2) Prove that $\mathbf{A}_K$ is closed in $\mathbf{A}$. (Hint: reduce to checking modulo p^n and carefully induct on n.)

(3) Prove that the set of units in $\mathbf{A}_K$ is open for the subspace topology. Is inversion continuous relative to the subspace topology?

(4) Let A be a complete discrete valuation ring endowed with a Hausdorff topological ring structure relative to which $\pi : A \to A$ is a closed embedding, for $\pi \in A$ a uniformizer. (B^+_{dR} with the topology from Exercise 4.5.3 is such an example!) Give every finite free A-module its canonical topological module structure using any A-basis. Using the structure theorem for modules over a discrete valuation ring, prove that if $M' \subseteq M$ is an inclusion of finite free A-modules then the subspace topology on M' is its canonical topology, and that M' is *closed* in M. Deduce that for any finitely generated A-module N, if we chose an A-linear presentation

$$0 \to M' \to M \to N \to 0$$

then the Hausdorff quotient topology on M/M' transferred to N is independent of the presentation and is functorial in N.

(5) Prove that for any short exact sequence $0 \to N' \to N \to N'' \to 0$ of finitely generated A-modules, if all three terms are endowed with their natural topology as just constructed, N' has the subspace topology from N and N'' has the quotient topology from N. Also if A is complete and the natural map of topological rings $A \to \varprojlim A/\mathfrak{m}_A^n$ is a homeomorphism, then prove the same for any finitely generated A-module N: the natural topologies on the $N/\mathfrak{m}_A^n N$'s have inverse limit that is identified with the natural topology on N.

(6) Part (2) gives a canonical topology to all finitely generated modules over $\mathbf{A}_K$ and $\mathbf{A}$. Prove that in Example 13.6.3, the Γ-action on $\mathrm{D}(T)$ is continuous.

(7) In the setup of Example 13.6.3, if the Γ-action on D is continuous then prove that the G_K-action on $\mathrm{T}(D)$ is *continuous* for the p-adic topology. (Using inverse limits, the key case is when D is a torsion $\mathbf{A}_K$-module, for which the claim is that $\mathrm{T}(D)$ is a discrete G_K-module.)

(8) Show that Exercise 4.5.3 carries over to define a topology on $\mathrm{W}(\mathrm{Frac}(R))[1/p]$ (inducing on $\mathrm{W}(\mathrm{Frac}(R))$ its weak topology as a closed but *not open* subring), relative to which the G_K-action is continuous. Prove that this makes $\mathrm{W}(\mathrm{Frac}(R))[1/p]$ a topological field. Giving the fraction fields $\mathbf{B}_K = \mathbf{A}_K[1/p]$ and $\mathbf{B} = \mathbf{A}[1/p]$ their subspace topologies, prove that both become topological fields with $\mathbf{B}_K$ and $\mathbf{A}$ closed subrings of $\mathbf{B}$, and prove analogues of (5) and (6) over $\mathbf{B}_K$ and $\mathbf{B}$. Prove that every $\mathbf{A}_K$-lattice in a finite-dimensional $\mathbf{B}_K$-vector space gets as its subspace topology exactly its own natural $\mathbf{A}_K$-module topology.

Exercise 13.7.10. Suppose k is algebraically closed, and consider a fundamental character $\psi : G_K \to \mathbf{F}_{p^f}^\times$ of level 1. The associated (φ, Γ)-module is 1-dimensional over $\mathbf{E}_K$. Can you describe it?

Exercise 13.7.11. This exercise gives properties of the equivalence D between $\mathrm{Rep}_{\mathbf{Z}_p}(G_K)$ and $\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_K}(\Gamma)$ to make Theorem 13.6.5 be useful.

(1) Prove that the equivalence D is compatible with tensor products, preserves rank and invariant factors (over $\mathbf{Z}_p$ and $\mathbf{A}_K$), and is compatible with duality on torsion objects and duality on finite free module objects.
(2) How does the restriction functor $\mathrm{Rep}_{\mathbf{Z}_p}(G_K) \to \mathrm{Rep}_{\mathbf{Z}_p}(G_{K'})$ translate through the equivalence D? As an application, characterize in terms of $\mathrm{D}(V)$ when the G_K-action on V is discrete (i.e., an open normal subgroup $G_{K'}$ acts trivially).
(3) How does the induction functor translate through the equivalence? (cf. Exercise 3.4.3)

Exercise 13.7.12. Let K be a p-adic field and let D be a (φ, Γ)-module over the local field $\mathbf{E}_K \simeq k'(\!(u)\!)$. This concept only uses the field structure of $\mathbf{E}_K$, not its valuation structure. By Exercise 3.4.1(6), D always contains a $k'[\![u]\!]$-lattice Δ which is φ-stable.

(1) Give a determinantal obstruction under which the linearization $\varphi^*(\Delta) \to \Delta$ of φ_D over $k'[\![u]\!]$ (not over $\mathbf{E}_K$!) can fail to be an isomorphism for all such Δ, and construct such an example in the 1-dimensional case (cf. Example 3.3.3).
(2) Adapting the proof of Lemma 1.2.6, show that Δ can be found so that it is also Γ-stable. In other words, in a suitable $\mathbf{E}_K$-basis of D we have that the matrix of φ_M is in $\mathrm{Mat}_d(k'[\![u]\!])$ and the (honest!) matrix of every $\gamma \in \Gamma$ lies in $\mathrm{GL}_d(k'[\![u]\!])$, where $d = \dim_{\mathbf{E}_K} D$.

Exercise 13.7.13. In the definition of an étale (φ, Γ)-module over $\mathbf{B}_K$, does it suffice that there is a φ-stable $\mathbf{A}_K$-lattice (with étale Frobenius structure)? That is, given such a lattice, does continuity of the Γ-action enable us to find one which is Γ-stable? (Keep in mind that $\mathbf{A}_K$ is *not* open in $\mathbf{B}_K$, so the solution to Exercise 13.7.12(2) does not apply.)

14. The Tate-Sen formalism and initial applications

In the early days of p-adic Hodge theory (before Fontaine came on the scene), the basic object of study was a finite-dimensional $\mathbf{C}_K$-vector space V equipped with a continuous semilinear action by G_K. The Hodge–Tate objects were quite well understood by Tate, and Sen studied the Galois cohomology set $\mathrm{H}^1(G_K, \mathrm{GL}_d(\mathbf{C}_K))$ (using continuous 1-cochains) which classifies isomorphism classes of *all* d-dimensional objects in the category $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ of finite-dimensional continuous semilinear representations of G_K over $\mathbf{C}_K$ (Exercise 14.4.1). Tate had studied the cohomology $\mathrm{H}^1(G_K, \mathbf{C}_K(r))$ which classifies 2-dimensional extension classes of $\mathbf{C}_K(r)$ by $\mathbf{C}_K$, a rather more special kind of problem.

The methods Sen developed (building on ideas of Tate) were adapted to other contexts (to prove the overconvergence of p-adic representations, to associate a differential module to a p-adic representation, etc.) Roughly speaking, the method is a descent followed by a "decompletion" (i.e., undoing a completion). To better understand the arguments, Colmez

([12, §3.3], [3, §3]) developed a general Tate–Sen formalism. In this section we explain the basic formalism and give several applications. Our exposition of the Tate–Sen formalism is modeled on the presentation by Berger and Colmez in [3, §3].

14.1. **The Tate-Sen conditions.** In Tate's initial work on p-divisible groups, he showed that in certain $\mathbf{C}_K$-semilinear situations one could kill a lot of cohomology by restricting to G_{K_∞}, where K_∞/K is an infinitely ramified Galois extension for which $\Gamma := \mathrm{Gal}(K_\infty/K)$ is isomorphic to $\mathbf{Z}_p$ near the identity. (The most important example is $K_\infty = K(\mu_{p^\infty})$, in which case we define $K_n = K(\zeta_{p^n})$.) Letting $H := G_{K_\infty} = \ker(G_K \twoheadrightarrow \Gamma)$, since $\mathbf{C}_K^H = \widehat{K}_\infty$ (Proposition 2.1.2) we have a left exact "inflation-restriction" sequence of pointed sets

$$1 \to \mathrm{H}^1(\Gamma, \mathrm{GL}_d(\widehat{K}_\infty)) \to \mathrm{H}^1(G_K, \mathrm{GL}_d(\mathbf{C}_K)) \to \mathrm{H}^1(H, \mathrm{GL}_d(\mathbf{C}_K)).$$

In other words, if $V \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ of some dimension $d > 0$ admits a $\mathbf{C}_K$-basis of H-invariant vectors (this is what is means to say the isomorphism class of V in $\mathrm{H}^1(G_K, \mathrm{GL}_d(\mathbf{C}_K))$ is killed under restriction to H; see Exercise 14.4.1) then the action of $\Gamma = G_K/H$ on such basis vectors must be described by matrices with coefficients in $\mathbf{C}_K^H = \widehat{K}_\infty$. That is, in such cases if we define $W = V^H$ then we would have $\mathbf{C}_K \otimes_{\widehat{K}_\infty} W \simeq V$ (so in particular, W is finite-dimensional over $\widehat{K}_\infty$ of dimension $\dim_{\mathbf{C}_K} V$). Note that it is not obvious at this point whether $W \neq 0$ when $V \neq 0$ for general V.

Following Serre's conventions in [45, Ch. I, §5] and [44, Ch. VII, App.], if an abstract group G acts on another group A on the left then a 1-*cochain* $a : G \to A$ is a function such that $a(gg') = a(g) \cdot g(a(g'))$. For example, taking $g = g' = 1$ gives $a(1) = 1$, and taking $g' = g^{-1}$ gives that $a(g)^{-1} = g(a(g^{-1}))$. We call two 1-cochains a and a' *cohomologous* if there is an $\alpha \in A$ such that $a'(g) = \alpha^{-1}a(g) \cdot g(\alpha)$ for all $g \in G$. This is an equivalence relation and the quotient set is denoted $\mathrm{H}^1(G, A)$. (For example, a is cohomologous to 1 precisely when $a(g) = \alpha^{-1}g(\alpha)$ for some $\alpha \in A$; these are the 1-*coboundaries*.) When G and A are topological groups and the action map $G \times A \to A$ is continuous we make similar definitions using continuous cochains, and usually write $\mathrm{H}^1(G, A)$ with this modified meaning (called "continuous cohomology").

Sen showed that for all V this procedure works: $V = \mathbf{C}_K \otimes_{\widehat{K}_\infty} (V^H)$, and the two functors

$$V \rightsquigarrow V^H, \;\; W \rightsquigarrow \mathbf{C}_K \otimes_{\widehat{K}_\infty} W$$

define quasi-inverse equivalences between $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ and $\mathrm{Rep}_{\widehat{K}_\infty}(\Gamma)$ (i.e., equivalence between categories of finite-dimensional *continuous* semilinear representation spaces, giving $\widehat{K}_\infty$ its valuation topology). In particular, he showed that the inflation map

$$\mathrm{H}^1(\Gamma, \mathrm{GL}_d(\widehat{K}_\infty)) \to \mathrm{H}^1(G_K, \mathrm{GL}_d(\mathbf{C}_K))$$

is bijective. Note that such bijectivity (as we vary d) is a weaker assertion than the equivalence of categories: it merely says that each object of $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ descents to an object in $\mathrm{Rep}_{\widehat{K}_\infty}(\Gamma)$ that is unique up to non-canonical isomorphism, but it says nothing about the important issue of descent for morphisms (i.e., functorial aspects of the descent).

Sen actually went a step further and developed a "decompletion" process to show that $\mathrm{Rep}_{\widehat{K}_\infty}(\Gamma)$ and $\mathrm{Rep}_{K_\infty}(\Gamma)$ are equivalent, where in this latter category of semilinear representations we require continuity relative to the valuation topology on the coefficient field K_∞

(i.e., we do not consider only discrete semilinear Γ-modules over K_∞, which are all trivial objects by Galois descent!).

To carry out the descent from $\mathbf{C}_K$ to $\widehat{K}_\infty$ and the "decompletion" from $\widehat{K}_\infty$ to K_∞, Sen adapted Tate's method of "normalized traces" which Tate had used to great effect in his study of the p-adic representations arising from abelian varieties over K with good reduction. The idea is that close work with the H-action should relate structures over $\mathbf{C}_K$ to structures over $\mathbf{C}_K^H = \widehat{K}_\infty$, but to get down to $K_\infty = \cup K_n$ one needs a family of "trace maps" $\widehat{K}_\infty \twoheadrightarrow K_n$ linear over K_n for all n, where $\{K_n\}$ is an exhaustion of K_∞ by an increasing sequence of finite extensions which become the layers of a $\mathbf{Z}_p$-extension for large n.

We now introduce the ingredients of the Tate–Sen formalism and simultaneously illustrate them in the setting of Sen's work (to be called *Sen's situation*).

Input 1: profinite groups. We fix a $\mathbf{Z}_p$-algebra A that is p-adically separated and complete (so $A^\times$ is open in A with subspace topology that makes it a topological group), and we fix a profinite group G_0 endowed with a continuous character $\psi \colon G_0 \to A^\times$ having image $\psi(G_0)$ that contains $\mathbf{Z}_p$ as an open subgroup. In other words, topologically $\psi(G_0) = \Gamma \times \mu$ with Γ a finitely generated $\mathbf{Z}_p$-module of rank 1 and μ a finite commutative group of order prime to p. (A typical example is $A = \mathbf{Z}_p$ and any ψ with infinite image in $\mathbf{Z}_p^\times$. But we also wish to allow maps valued in $\mathscr{O}_E^\times$ for finite extensions $E/\mathbf{Q}_p$ such that $\psi(G_0)$ is a 1-dimensional p-adic Lie group.) We write Γ_n to denote $p^n\Gamma$, so for sufficiently large N we have that $\Gamma_N \simeq \mathbf{Z}_p$ with Γ_{n+N} its unique subgroup of index p^n for each $n \geqslant 0$.

Let $H_0 = \ker(\psi)$ and for $g \in G_0$ define $n(g) \in \mathbf{Z}_{\geqslant 0} \cup \{\infty\}$ to be the least $n \geqslant 0$ such that $\psi(g) \notin \Gamma_{n+1}$. For example, $n(g) = +\infty$ precisely when $g \in H_0$ and $n(g) = 0$ if $\psi(g) \notin \Gamma$. [Sen's situation is $G_0 = G_K$, A the ring of integers of a finite extension of $\mathbf{Q}_p$ contained in K, and ψ any $A^\times$-valued character whose splitting field denoted K_∞/K has Galois group $\mathrm{Gal}(K_\infty/K) = \psi(G_K)$ that is commutative with open pro-p part Γ which contains $\mathbf{Z}_p$ as an open subgroup; in particular, K_∞ is a $\mathbf{Z}_p$-extension of a finite extension of K. In this case $H_0 = G_{K_\infty}$ with $K_\infty = \cup K_n$, where $K_n = \ker(\psi \bmod \Gamma_n)$ for $n \geqslant 0$. In this situation $n(g)$ is the biggest n for which g is trivial on K_n; we say $n(g) = +\infty$ if $g \in G_{K_\infty}$.]

For any open subgroup H in H_0 which is normal in G_0, we define $\widetilde{\Gamma}_H = G_0/H$, so there is an exact sequence

$$1 \to H_0/H \to \widetilde{\Gamma}_H = G_0/H \to G_0/H_0 = \psi(G_0) \to 1,$$

with H_0/H a finite group since H is open in H_0. [In Sen's situation, $H = G_{L_\infty}$ for a finite Galois extension L/K, in which case $\widetilde{\Gamma} = \mathrm{Gal}(L_\infty/K)$ and the above short exact sequence is the natural one from Galois theory, with $H_0/H = \mathrm{Gal}(L_\infty/K_\infty)$ and $G_0/H_0 = \mathrm{Gal}(K_\infty/K)$.]

The continuous map $\psi : G \to A^\times$ must be a quotient map onto its open image, so it is an open map onto its image. Thus, for any open subgroup G in G_0, the image $\psi(G)$ is open in $\psi(G) = \Gamma \times \mu$ and hence contains Γ_n for a minimal $n \geqslant 1$; when G is normal in G_0 we define $n_1(G)$ be this least such n. [In Sen's situation, $G = G_L$ for a finite Galois extension L/K, and $n_1(G)$ is the least $n \geqslant 1$ such that $L_n = LK_n$ is linearly disjoint from K_∞ over K_n.] This definition of $n_1(G)$ makes sense even if G is not normal in G_0, but to make later considerations work when G is not normal in G_0 one needs to actually generalize the definition to the non-normal case in a different way. (See Remark 14.1.8.)

Observe that the group $H := G \cap H_0 = \ker(\psi|_G)$ is normal in G_0 and open in H_0, and G_0 acts via conjugation on the subgroup $G/H = \psi(G)$ of $G_0/H = \widetilde{\Gamma}_H$ through conjugation by evaluation of the character $\psi : G_0 \to \psi(G_0)$. But $\psi(G_0)$ is commutative, so therefore G_0 acts trivially on G/H. Hence, G/H is *central* in $\widetilde{\Gamma}_H$. [In Sen's situation, this says that for L/K finite Galois, $\mathrm{Gal}(L_\infty/L)$ is central in $\mathrm{Gal}(L_\infty/K)$; this is obvious since $\mathrm{Gal}(L_\infty/K)$ acts on L_∞ through evaluation of a commutative character.]

Input 2: valued rings. We let $\widetilde{\Lambda}$ be an A-algebra (not necessarily a domain) equipped with a map $v_\Lambda \colon \widetilde{\Lambda} \to \mathbf{R} \cup \{+\infty\}$ such that for all $x, y \in \widetilde{\Lambda}$ the following weakening of the valuation axioms (in the spirit of a semi-norm) are satisfied:

(1) $v_\Lambda(x) = +\infty$ if and only if $x = 0$, and $v_\Lambda(\pm 1) = 0$;
(2) $v_\Lambda(x+y) \geqslant \min\big(v_\Lambda(x), v_\Lambda(y)\big)$;
(3) $v_\Lambda(xy) \geqslant v_\Lambda(x) + v_\Lambda(y)$;
(4) $v_\Lambda(p) > 0$ and $v_\Lambda(px) = v_\Lambda(p) + v_\Lambda(x)$.

We allow the possibility that $p = 0$ in $\widetilde{\Lambda}$ (in which case $v_\Lambda(p) = +\infty$ and axiom (4) is redundant). Axiom (3) implies that $v_\Lambda(-x) \geqslant v_\Lambda(x)$ for all x, so swapping x and $-x$ gives $v_\Lambda(-x) = v_\Lambda(x)$ for all x.

The ring $\widetilde{\Lambda}$ is endowed with a topological ring structure by using the additive subgroups $\widetilde{\Lambda}^{\geqslant a} := v_\Lambda^{-1}([a, +\infty])$ as a base of opens around 0; these are ideals under the open subring $\widetilde{\Lambda}^{\geqslant 0}$. This topology is Hausdorff by axiom (1) and every point has a countable base of open neighborhoods, so we can probe the topology using sequences. [In Sen's situation, $\widetilde{\Lambda} = \mathbf{C}_K$ with v_Λ given by the usual valuation, say with the normalization $v(p) = 1$. Note this does impose the right topology on $\mathbf{C}_K$.]

We assume that $\widetilde{\Lambda}$ is complete for this topology; this means that if $\{x_n\}_{n \geqslant 0}$ is a Cauchy sequence in $\widetilde{\Lambda}$ (in the sense that for all $C > 0$ there exists N such that $v_\Lambda(x_n - x_m) > C$ for all $n, m \geqslant N$) then it converges in $\widetilde{\Lambda}$. For example, if $\{a_n\}_{n \geqslant 0}$ is a sequence in $\widetilde{\Lambda}$ then $\sum a_n$ converges if and only if $a_n \to 0$. [Such completeness clearly holds in Sen's situation.]

We also assume that $\widetilde{\Lambda}$ is endowed with an A-algebra action by G_0 that leaves v_Λ invariant ("isometry") and is moreover a continuous action. Continuity is stronger than "isometry", since it requires that for each $x \in \widetilde{\Lambda}$ we have $v_\Lambda(g(x) - x) \to \infty$ as $g \to 1$, which does not seem to formally follow from the other running hypotheses. [In Sen's situation we use the usual G_K-action on $\mathbf{C}_K$, for which the isometry and continuity hypotheses are satisfied.]

We define a measure of "size" on matrices by applying v_Λ to the coefficients (and its elementary properties are worked out in Exercise 14.4.3):

Definition 14.1.1. For any $d \geqslant 1$ and $M = (m_{i,j}) \in \mathrm{Mat}_d(\widetilde{\Lambda})$ let

$$v_\Lambda(M) := \min_{(i,j)} v_\Lambda(m_{i,j}) \in \mathbf{R} \cup \{+\infty\}.$$

Remark 14.1.2. Since G_0 acts continuously on $\mathrm{GL}_d(\widetilde{\Lambda})$, it makes sense to form the pointed set of continuous cohomology $\mathrm{H}^1(G, \mathrm{GL}_d(\widetilde{\Lambda}))$ when G is any subgroup of G_0 (with the subspace topology). This classifies isomorphism classes of finite free $\widetilde{\Lambda}$-modules equipped with a semilinear action of G that is continuous for the natural topology of finite free $\widetilde{\Lambda}$-modules

(Exercise 14.4.1). For example, in Sen's situation with $G = G_0 = G_K$ this is exactly his problem of studying the pointed set $\mathrm{H}^1(G_K, \mathrm{GL}_d(\mathbf{C}_K))$ classifying d-dimensional continuous semilinear representations of G_K over $\mathbf{C}_K$.

Now we formulate the *Tate–Sen axioms*. These come in three parts and are rather complicated-looking at first sight, so we state each axiom in turn and verify each for Sen's situation before moving on to the next axiom .

As a review, Sen's situation ([49, §3], [40], [23, §2], [3, §4.1]) involves the following setup. We take $G_0 = G_K$ for a p-adic field K, ψ any infinitely ramified character valued in the units $A^\times$ of a finite extension of $\mathbf{Q}_p$ contained in K (especially $A = \mathbf{Z}_p$ with ψ the p-adic cyclotomic character) such that the splitting field K_∞/K is abelian and is a $\mathbf{Z}_p$-extension of a finite extension of K, and $H_0 = G_{K_\infty}$ with $K_\infty = \cup K_n$ for $K_n = K_\infty^{p^n\Gamma}$ where Γ is the pro-p part of $\psi(G_K)$. We also take $\widetilde{\Lambda} = \mathbf{C}_K$ with v_Λ the usual valuation v (i.e., the one for which $v(p) = 1$). In particular, for any finite Galois extension L/K and its corresponding open normal subgroup $G = G_L$ in G_K, we have $H := G \cap H_0 = G_{L_\infty}$ and $\widetilde{\Gamma}_H = \mathrm{Gal}(L_\infty/K)$. We define $L_n := LK_n$ for all $n \geqslant 0$. By Proposition 2.1.2, we have $\widetilde{\Lambda}^H = \widehat{L}_\infty$.

Axiom (TS1). *We assume there is a constant $c_1 \in \mathbf{R}_{>0}$ such that for all open subgroups $H_1 \subset H_2$ in H_0 that are normal in G, there exists an $\alpha \in \widetilde{\Lambda}^{H_1}$ satisfying $v_\Lambda(\alpha) > -c_1$ and $\sum_{\tau \in H_2/H_1} \tau(\alpha) = 1$.*

The sum in this axiom is a kind of trace, and this axiom is related to constructing "normalized traces" in later arguments. In Sen's situation it is a direct outgrowth of Tate's "almost étale" result in Theorem 13.1.2. To better understand what the axiom means, we now prove:

Lemma 14.1.3. *Axiom* (TS1) *is satisfied in Sen's situation with any $c_1 > 0$.*

Proof. Let $H_1 \subset H_2$ be open subgroups of $H_0 = G_{K_\infty}$. (We will not need normality.) These correspond to finite extensions $M_1/M_2/K_\infty$. We pick a finite extension L of K such that $M_2 = L_\infty$, and by Theorem 13.1.2 applied to the extension M_1 of $L_\infty = M_2$ we have $\mathfrak{m}_{M_2} \subset \mathrm{Tr}_{M_1/M_2}(\mathscr{O}_{M_1})$. In particular, since M_2/L is infinitely ramified there are elements of $\mathfrak{m}_{M_2}$ with arbitrarily small valuation. Hence, we can pick $a \in \mathfrak{m}_{M_2}$ with $v(a) < c_1$, and there exists $\alpha_0 \in \mathscr{O}_{M_1}$ such that $\mathrm{Tr}_{M_1/M_2}(\alpha_0) = a$. Hence, $\alpha = a^{-1}\alpha_0 \in M_1$ satisfies $\mathrm{Tr}_{M_1/M_2}(\alpha) = 1$ and $v(\alpha) = v(\alpha_0) - v(a) > -c_1$. ■

The next axiom is somewhat of a mouthful (it has five conditions, involving an infinite family of maps and closed subrings), so we will first work out the version in Sen's situation, and then state the axiom in general.

Pick a finite extension M/K_∞ Galois over K corresponding to an open subgroup $H \subseteq G_{K_\infty}$ normal in G_K, and choose a finite extension L/K such that $L_\infty = M$. Since M is Galois over K, it contains the Galois closure of L over K in $\overline{K}$, so by replacing L with this Galois closure we may and do arrange L/K to be Galois (so all $L_n = LK_n$ are also Galois over K). Replace L with some L_n if necessary so that $L/L_{n_0(L)}$ is a totally ramified $\mathbf{Z}_p$-extension for some $n_0(L) \geqslant 0$ and L is linearly disjoint from K_∞ over $K_{n_0(L)}$. Hence, for all $n \geqslant n_0(L)$ we have $L_n = K_n \otimes_{K_{n_0(L)}} L$, so $L_m = K_m \otimes_{K_n} L_n$ for $m \geqslant n \geqslant n_0(L)$. This ensures that Tr_{L_m/L_n} restricts to Tr_{K_m/K_n} on K_m whenever $m \geqslant n \geqslant n_0(L)$. We will use this compatibility shortly.

Since $[L_m : L_n] = p^{m-n}$ for all $m \geqslant n$, the following L_n-linear "normalized trace" map is well-defined:

$$R_{M,n} \colon M = L_\infty \to L_n$$
$$x \mapsto \frac{1}{p^{m-n}} \operatorname{Tr}_{L_m/L_n}(x) \quad \text{ if } x \in L_m$$

(The point is that $R_{M,n}(x)$ does not depend on the choice of $m \geqslant n$ for which $x \in L_m$.) Obvious $R_{M,n}|_{L_n}$ is the identity map. Also, since we arranged that $\operatorname{Tr}_{L_m/L_n}$ is a scalar extension of $\operatorname{Tr}_{K_m/K_n}$ for $m \geqslant n \geqslant n_0(L)$, if M'/M is a finite extension also Galois over K corresponding to an open subgroup $H' \subseteq H$ normal in G_K and if L'/L is a finite extension Galois over K such that $L'_\infty = M'$ (as we may always find for any M') then $R_{M',n}|_M = R_{M,n}$ for $n \geqslant \max(n_0(L), n_0(L'))$. This is a useful compatibility property of the $R_{M,n}$'s as we vary M and work with large n.

The utility of the normalized traces is that for large enough n they are bounded linear operators over L_m, and so define a *topological* splitting $M = L_\infty = L_n \oplus \ker(R_{M,n})$. Such boundedness, with an additional uniform control on the bounding constant, was already present in Tate's work and is the content of:

Lemma 14.1.4. *For any $c_2 \in \mathbf{R}_{>0}$ there exists $n(H) \geqslant n_0(L)$ such that $v(R_{M,n}(x)) \geqslant v(x) - c_2$ for all $n \geqslant n(H)$ and $x \in L_\infty$. Equivalently, if n is sufficiently large (depending on c_2 and H) then $R_{M,n}$ is a bounded L_n-linear operator with operator norm at most p^{c_2}.*

Proof. We apply Proposition 13.1.9 to the totally ramified $\mathbf{Z}_p$-extension $L_\infty/L_{n_0(L)}$ to get a constant c and a bounded sequence $\{a_n\}_{n \geqslant n_0(L)}$ such that $v(\mathfrak{D}_{L_n/L_{n_0(L)}}) = n + c + p^{-n}a_n$. Fix $n \geqslant n_0(L)$, so by transitivity of the different we get $v(\mathfrak{D}_{L_m/L_n}) = m - n - p^{-n}a_n + p^m a_m$ for all $m \geqslant n$. But we have the the following general equivalence ([44, Ch. III, Prop. 7] applied to L_m/L_n):

$$\operatorname{Tr}_{L_m/L_n}(\mathfrak{m}_{L_m}^j) \subset \mathfrak{m}_{L_n}^i \Leftrightarrow \mathfrak{m}_{L_m}^j \subset \mathfrak{m}_{L_n}^i \mathfrak{D}_{L_m/L_n}^{-1}.$$

We conclude that for for $x \in L_m$, $\operatorname{Tr}_{L_m/L_n}(x) \in \mathfrak{m}_{L_n}^i$ if and only if $x\mathfrak{D}_{L_m/L_n} \subset \mathfrak{m}_{L_n}^i \mathscr{O}_{L_m}$. Hence, $v(\operatorname{Tr}_{L_m/L_n}(x)) \geqslant v(x) + v(\mathfrak{D}_{L_m/L_n})$, which is to say $v(R_{M,n}(x)) \geqslant v(x) - p^{-n}a_n + p^m a_m$. We can therefore choose $n(H) \geqslant n_0(H)$ to simply be big enough such that $2p^{-n}|a_n| < c_2$ for all $n \geqslant n(H)$ (which can be done since $\{a_n\}$ is bounded). ■

Due to the boundedness of the L_n-linear operator $R_{M,n}$ provided by Lemma 14.1.4 (for $n \geqslant n(H)$), the map $R_{M,n}$ extends by continuity to an L_n-linear section $\widehat{R}_{M,n}$ of the inclusion $L_n \hookrightarrow \widehat{L}_\infty = \mathbf{C}_K^H$ for which the conclusion of Lemma 14.1.4 applies. In particular, $\widehat{R}_{M,n}$ splits off L_n as a *closed* subspace of $\widehat{L}_\infty$. That is, $\widehat{M} = L_n \oplus \ker(\widehat{R}_{M,n})$ as L_n-Banach spaces.

The following axiom (TS2) encodes information concerning the completed normalized traces $R_{H,n} := \widehat{R}_{M,n}$ as well as the conclusion of Lemma 14.1.4 applied to these completed maps (especially the "uniformity" of c_2 as H varies).

Axiom (TS2). *We assume there is a constant $c_2 \in \mathbf{R}_{>0}$ such that for all open subgroups $H \subset H_0$ that are normal in G there exists data $\{\Lambda_{H,n}, R_{H,n}\}_{n \geqslant n(H)}$ consisting of an increasing sequence of closed A-subalgebras $\Lambda_{H,n} \subseteq \widetilde{\Lambda}^H$ and A-linear maps $R_{H,n} \colon \widetilde{\Lambda}^H \to \Lambda_{H,n}$ for which*

(1) *if* $H_1 \subset H_2$ *and* $n \geqslant \max\{n(H_1), n(H_2)\}$, *then* $\Lambda_{H_2,n} \subset \Lambda_{H_1,n}$ *and the restriction of* $R_{H_1,n}$ *to* $\widetilde{\Lambda}^{H_2}$ *coincides with* $R_{H_2,n}$*;*
(2) $R_{H,n}$ *a* $\Lambda_{H,n}$*-linear section to the inclusion* $\Lambda_{H,n} \hookrightarrow \widetilde{\Lambda}^H$*;*
(3) $g(\Lambda_{H,n}) = \Lambda_{H,n}$ *and* $g(R_{H,n}(x)) = R_{H,n}(gx)$ *for all* $x \in \widetilde{\Lambda}^H$ *and* $g \in G_0$*;*
(4) $v_\Lambda(R_{H,n}(x)) \geqslant v_\Lambda(x) - c_2$ *for all* $x \in \widetilde{\Lambda}^H$*;*
(5) $\lim\limits_{n\to\infty} R_{H,n}(x) = x$ *for all* $x \in \widetilde{\Lambda}^H$.

We emphasize that by part (2), the map $R_{H,n}$ is a $\Lambda_{H,n}$-linear projector, so $X_{H,n} = \ker(R_{H,n})$ is a closed $\Lambda_{H,n}$-submodule of $\widetilde{\Lambda}^H$ and there is a *topological* decomposition $\widetilde{\Lambda}^H = \Lambda_{H,n} \oplus X_{H,n}$. Also, part (3) just says that an action by $\widetilde{\Gamma}_H = G_0/H$ makes sense on $R_{H,n}$ and is trivial.

Remark 14.1.5. It is natural to wonder why we begin the indexing with $n(H)$ instead of relabeling all indexing to begin at $n = 0$ for each H. The point is that in practice it happens that axiom (TS2) may continue to hold if we shrink c_2 provided that in axiom (4) (the only place where c_2 appears!) we drop the data $(\Lambda_{H,n}, R_{H,n})$ for some small n *depending on* H. So for simplicity of notation in such cases, it is best to give ourselves the flexibility of the parameter $n(H)$ that depends on H (and c_2).

In Sen's situation $\Lambda_{H,n} = L_n$ and $R_{H,n} = \widehat{R}_{M,n}$ for M/K_∞ corresponding to H. Axiom (TS2)(1) is exactly the compatibility condition $\widehat{R}_{M',n}|_M = \widehat{R}_{M,n}$ noted already for these (completed) normalized traces as we vary the finite extension M/K_∞, and axiom (TS2)(2) encodes the fact that the normalized traces (after completion) are sections to the inclusion map. This property has already been noted before completion and is certainly preserved by passage to the completion. Axiom (TS2)(4) in Sen's situation is exactly the conclusion of Lemma 14.1.4, combined with passage to the completion. For the remaining parts of (TS2), we now prove:

Proposition 14.1.6. *Axiom* (TS2) *is satisfied in Sen's situation with any* $c_2 > 0$.

Proof. We have already discussed parts (1), (2), and (4). Conditions (3) and (5) require some additional argument, as follows. Since L/K is Galois, so is $L_n = \Lambda_{H,n}$ for all n. Hence, for $m \geqslant n$ we have a conjugation action by $G_0 = G_K$ on the finite group $\mathrm{Gal}(L_m/L_n)$, so using the Galois-theoretic formula for trace maps we get $g \circ \mathrm{Tr}_{L_m/L_n} = \mathrm{Tr}_{L_m/L_n} \circ g$. Plugging this into the definition of the normalized traces $L_\infty \to L_n$ for $n \geqslant n(L)$ gives that they are also G_0-equivariant, and hence they remain so after passing to the completion.

It remains to treat part (5). Pick any $x \in \widehat{L}_\infty = \widetilde{\Lambda}^H$ and any $C \in \mathbf{R}_{\geqslant 0}$. We need to prove $v(x - \widehat{R}_{M,n}(x)) \geqslant C$ for all large enough n. Choose $y \in L_\infty$ such that $v(x - y) \geqslant C + c_2$, and pick $m \geqslant 0$ such that $y \in L_m$. Using part (4) to estimate $\widehat{R}_{M,n}(y - x)$, we get

$$v(x - \widehat{R}_{M,n}(x)) \geqslant \min\{v(x-y), v(y - R_{M,n}(y)), v(\widehat{R}_{M,n}(y-x))\} \geqslant \min\{C, v(y - R_{M,n}(y))\}.$$

But by taking $n \geqslant m$ as well, we have $R_{M,n}(y) = y$ by parts (1) and (2), so for such large n we have $v(x - \widehat{R}_{M,n}(x)) \geqslant C$ as required. ■

The final axiom of the Tate–Sen formalism describes how G_0 acts on the complement to $\Lambda_{H,n}$ in $\widetilde{\Lambda}^H$ provided by the splitting defined by the section $R_{H,n}$:

Axiom (TS3). *We assume there is a constant $c_3 \in \mathbf{R}_{>0}$ such that for every open normal subgroup $G \subset G_0$ (and associated open subgroup $H := G \cap H_0$ in H_0 that is normal in G) there is an integer $n(G) \geqslant \max(n_1(G), n(H))$ for which*

(i) *$\gamma - 1$ is invertible on $X_{H,n} := \ker(R_{H,n}) = (1 - R_{H,n})(\widetilde{\Lambda}^H)$,*
(ii) *$v_\Lambda(x) \geqslant v_\Lambda\big((\gamma - 1)(x)\big) - c_3$ for all $x \in X_{H,n}$*

for all $n \geqslant n(G)$ and all $\gamma \in \widetilde{\Gamma}_H = G_0/H$ satisfying $n(\gamma) := v(\psi(\gamma) - 1) \leqslant n$.

This axiom says that $\gamma - 1$ has a bounded linear inverse on $\ker(R_{H,n})$ (controlled by the constant c_3) provided that n is large enough (depending on G) and γ is not too close to 1 (depending on n). We have to restrict to $\ker(R_{H,n})$ to say anything about an inverse to $\gamma - 1$ since on the complement $\Lambda_{H,n}$ of $\ker(R_{H,n})$ in $\widetilde{\Lambda}^H$ the action of some open subgroup of $\widetilde{\Gamma}_H$ may be trivial (e.g., in Sen's situation $\Lambda_{H,n} = L_n$ has trivial action by $\mathrm{Gal}(L_\infty/L_n)$).

Note also that since we require $n \geqslant n(G) \geqslant n_1(G)$ in (TS3), necessarily $\Gamma_n \subseteq \psi(G)$. Thus, there are lots of elements γ in the open subgroup $G/H \subseteq \widetilde{\Gamma}_H$ for which $n(\gamma) = n$. The only purpose of requiring $n(G) \geqslant n_1(G)$ is to ensure that there are many $\gamma \in G/H$ with $n(\gamma) \leqslant n$. We will not actually use that $n(G) \geqslant n_1(G)$ in the proof.

Proposition 14.1.7. *Axiom* (TS3) *is satisfied in Sen's situation for any $c_3 \in \mathbf{R}_{>1}$.*

Proof. Fix $c_3 > 1$ and pick a finite Galois extension L/K. Let $G = G_L$ and define

$$H = \mathrm{Gal}(\overline{K}/L_\infty) = G \cap H_0$$

(open in H_0 and normal in G_0), and choose $n \geqslant n(H)$. Pick $\gamma \in \widetilde{\Gamma}_H = G_0/H$. If $x \in \widetilde{\Lambda}^H = \widehat{L}_\infty$ then $(\gamma - 1)(R_{H,n}(x)) = R_{H,n}((\gamma - 1)(x))$ by (TS2)(3). Thus, for $X_{H,n} = \ker(R_{H,n})$ we have $(\gamma - 1)(X_{H,n}) \subset X_{H,n}$. Moreover, $X_{H,n} \cap \ker(\gamma - 1) \subset X_{H,n} \cap L_{n(\gamma)} = \{0\}$ if γ satisfies $n(\gamma) \leqslant n$ (as $L_{n(\gamma)} \subseteq L_n$ for such γ). The map $\gamma - 1$ therefore induces an K-linear injection from $X_{H,n} \cap L_m$ to itself when $n(\gamma) \leqslant n \leqslant m$. But $X_{H,n} \cap L_m$ is finite-dimensional over K, so this injection is an isomorphism. Hence, $(\gamma - 1)(X_{H,n} \cap L_m) = X_{H,n} \cap L_m$ whenever $n(\gamma) \leqslant n \leqslant m$. Taking $m \to \infty$, if $n(\gamma) \leqslant n$ then $\gamma - 1$ induces a bijection from $X_{H,n} \cap L_\infty$ to itself. But $R_{H,n} = \widehat{R}_{L_\infty,n}$ is defined by passage to the completion of a projector on L_∞, so the complement $X_{H,n}$ to L_n defined by the projector $R_{H,n}$ is the completion of its intersection with L_∞. Hence, $\gamma - 1$ restricts to a bijection of $X_{H,n}$ to itself whenever $n(\gamma) \leqslant n$ *provided* that $\gamma - 1$ has a bounded inverse on $X_{H,n} \cap L_\infty$ (and hence the inverse in the completion $X_{H,n}$ is bounded using the same bounding constant).

It remains to prove that the inverse to $\gamma - 1$ on $X_{H,n} \cap L_\infty$ is bounded, with bounding constant governed by c_3 as in the statement of (ii) (for $x \in X_{H,n} \cap L_\infty$). To prove this, we shall use induction on $m \geqslant n$ to construct a sequence $\{d_m\}_{m \geqslant n}$ in $\mathbf{R}_{>0}$ such that $v(x - R_{H,n}(x)) \geqslant v((\gamma - 1)(x)) - d_m$ for all $x \in L_m$. These d_m's will be constructed so that $d_m \leqslant 1 + bp^{1-m}/(p - 1)$ for some $b \geqslant 0$, so it will then suffice to take $n(G)$ large enough such that $1 + bp^{1-n(G)}/(p - 1) < c_3$.

To find such d_m's, we will induct on m. First consider the case $m = n$. In this case the required estimate is $+\infty \geqslant v((\gamma - 1)(x)) - d_n$ for all $x \in L_n$, so we can take any finite d_n at all. For later purposes we take $d_n = 0$.

Now take $m \geqslant n$ (so $m \geqslant n(\gamma)$) and assume we have constructed the sequence up to stage m (so $d_m = 0$ if $m = n$). We need to find d_{m+1}. Pick $x \in L_{m+1}$ and let $y = \frac{1}{p}\operatorname{Tr}_{L_{m+1}/L_m}(x) \in L_m$. By definition of the normalized traces we have $R_{H,n}(x) = R_{H,n}(y)$, so

$$v(x - R_{H,n}(x)) \geqslant \min\{v(x-y), v(y - R_{H,n}(y))\}.$$

Thus, we seek suitable lower bounds on both $v(x-y)$ and $v(x - R_{H,n}(y))$.

To handle $v(x-y)$, recall that L_{m+1}/L_m is cyclic of degree p. Its Galois group is generated by $\gamma^{m-n(\gamma)}$, so

$$px - py = \sum_{i=1}^{p-1}(1 - \gamma^{ip^{m-n(\gamma)}})(x) = (1-\gamma)(P(\gamma)(x))$$

for some $P \in \mathbf{Z}[X]$. Thus, $v(p(x-y)) \geqslant v((\gamma-1)(x))$. This says $v(x-y) \geqslant v((\gamma-1)(x)) - 1$.

Turning to $v(x - R_{H,n}(y))$, we apply Proposition 13.1.9 to the totally ramified $\mathbf{Z}_p$-extension L_∞/L_n to get a constant c and bounded sequence $\{a_j\}_{j\geqslant 0}$ such that (using transitivity of the different) $v(\mathfrak{D}_{L_{j'}/L_j}) = j' - j + c + p^{j-j'}a_{j'}$ whenever $j' \geqslant j \geqslant 0$. In particular, we have

$$v(\mathfrak{D}_{L_{m+1}/L_m}) = v(\mathfrak{D}_{L_{m+1}/L_n}) - v(\mathfrak{D}_{L_m/L_n}) = 1 + p^{-m}b_m$$

where $\{b_m\}_{m\geqslant n}$ is a bounded sequence. Pick $b \in \mathbf{R}_{\geqslant 0}$ such that $b \geqslant |b_m|$ for all $m \geqslant n$. As in the proof of Lemma 14.1.4, we have

$$v(\operatorname{Tr}_{L_{m+1}/L_m}(z)) \geqslant v(z) + v(\mathfrak{D}_{L_{m+1}/L_m}) \geqslant v(z) + 1 - bp^{-m}$$

for all $z \in L_{m+1}$. Taking $z = (\gamma - 1)(x)$, this yields

$$v((\gamma-1)(py)) \geqslant v((\gamma-1)(x)) + 1 - bp^{-m},$$

which is to say $v((\gamma-1)(y)) \geqslant v((\gamma-1)(x)) - bp^{-m}$. But since $y \in L_m$ we have $v(y - R_{H,n}(y)) \geqslant v((\gamma-1)(y)) - d_m$ by the assumed existence of d_m. Thus, $v(y - R_{H,n}(y)) \geqslant v((\gamma-1)(x)) - d_m - bp^{-m}$.

Combining our two lower bounds, we get $v(x - R_{H,n}(x)) \geqslant v((\gamma-1)(x)) - d_{m+1}$ where $d_{m+1} := \max\{1, d_m + bp^{-m}\}$. This completes the inductive construction of the d_m's, and from the actual recursive definition (and the initial value $d_n = 0$) we see that

$$d_m \leqslant 1 + bp^{-n} + bp^{-n-2} + \cdots + bp^{-m+1} \leqslant 1 + b \cdot \frac{p^{1-n}}{p-1},$$

as required. This completes the proof. ■

Remark 14.1.8. The above axioms will suffice for our purposes, but we note that one can formulate modifications of Axioms (TS2) and (TS3) to allow the open subgroups $H \subseteq H_0$ and $G \subseteq G_0$ to be non-normal (and then drop the normality requirement in (TS1)). To make sense of this, some definitions need to be generalized. First, for general such G we define $\widetilde{\Gamma}_H$ to be the quotient $N_{G_0}(H)/H$, where $N_{G_0}(H)$ is the normalizer of H in G_0. For this to be useful we must check that $N_{G_0}(H)$ is actually open in G_0. Here is a proof of openness. Since a closed subgroup of a profinite group is profinite for the subspace topology, open subgroups of H_0 that are normal in G_0 are a base of open subgroups in H_0. Thus, since H is an open subgroup in H_0, there exists a subgroup $N \subseteq H$ that is open in H_0 and normal in G_0. Now consider the resulting containment of *finite* subgroups $H/N \subseteq H_0/N$ inside of G_0/N, with H_0/N normal in G_0/N. Since $(G_0/N)/(H_0/N) = \psi(G_0) \simeq \Gamma \times \mu$ acts continuously via conjugation on the finite H_0/N, some open subgroup of G_0/N must centralize H_0/N (as the permutation group of the finite set H_0/N is finite). The preimage of this in G_0 is an open subgroup that normalizes H, as desired. In particular, $\widetilde{\Gamma}_H$ is open in G_0/H in general, so $\psi : \widetilde{\Gamma}_H \to \psi(G_0)$ is an open mapping.

Having defined $\widetilde{\Gamma}_H$ for any open subgroup H in H_0, we next need to properly define $n_1(G)$ for any open subgroup G in G_0. This is defined to be the smallest integer $n \geqslant 1$ such that $\Gamma_n \subset \psi((G/H) \cap C_H)$, where C_H is the center of $\widetilde{\Gamma}_H$. (Recall that when G is normal in G_0 we proved that G/H is central in $\widetilde{\Gamma}_H$, so in the normal case we recover the earlier definition of $n_1(G)$.) To make sense of this definition of $n_1(G)$ as a finite integer it suffices to prove that $(G/H) \cap C_H$ is open in $\widetilde{\Gamma}_H$ (as we have already seen that $\psi : \widetilde{\Gamma}_H \to \psi(G_0)$ is an open mapping). Since G/H is certainly open in $\widetilde{\Gamma}_H$, the problem is to check that C_H is open in $\widetilde{\Gamma}_H$. To prove such openness for C_H, let N be the kernel of the restriction of $\psi^{\mu\cdot\#\Gamma_{\rm tor}}$ to $\widetilde{\Gamma}_H$, so N is finite and $\psi^{\mu\cdot\#\Gamma_{\rm tor}}$ induces a continuous injective map $\widetilde{\Gamma}_H/N \to \Gamma$ whose image is open and even isomorphic to $\mathbf{Z}_p$. Thus, $\widetilde{\Gamma}_H$ is topologically a semi-direct product of $\mathbf{Z}_p$ against the finite (discrete) group N. Its center C_H is therefore an open subgroup.

These two generalized definitions allow us to make sense of (TS2) and (TS3) in the non-normal case, provided we make one further modification in (TS2): for part (3) we should instead require $g(\Lambda_{H,n}) = \Lambda_{gHg^{-1},n}$ and $g(R_{H,n}(x)) = R_{gHg^{-1},n}(gx)$ for all $x \in \widetilde{\Lambda}^H$ and $g \in G_0$.

We conclude our introduction to the Tate–Sen axioms by proving a very useful lemma of Tate in the context of Sen's situation.

Lemma 14.1.9. *Let K_∞/K be an infinitely ramified $\mathbf{Z}_p$-extension, and γ_0 a topological generator of $\Gamma = \mathrm{Gal}(K_\infty/K)$. For any $\lambda \in 1 + \mathfrak{m}_K$ that is not a p-power root of unity, the bounded K-linear operator $\gamma - \lambda$ on $\widehat{K}_\infty$ is bijective.*

The proof even shows that the inverse K-linear operator is bounded.

Proof. By (TS3) there is a Γ-equivariant splitting $\widehat{K}_\infty = K \oplus X$ of K-Banach spaces such that $\gamma_0 - 1$ acts bijectively on X with $(\gamma_0 - 1)^{-1}$ bounded on X and having operator norm at most p^{c_3} (with c_3 as in the axioms (TS3)). Due to the uniformity when varying G in (TS3)(ii), if we pass to any K_n in place of K then $\gamma_0^{p^n} - 1$ acting on the corresponding a closed subspace $X_n \subseteq X$ complementary to K_n in $\widehat{K}_\infty$ has inverse whose operator norm is bounded above by the *same* constant p^{c_3}. This will be crucial.

Since $\lambda \neq 1$, we see that $\gamma_0 - \lambda$ acts invertibly on K. Thus, we just have to check that it acts bijectively on X. Consider the factorization

$$\gamma_0 - \lambda = (\gamma_0 - 1) - (\lambda - 1) = (\gamma_0 - 1)(1 - (\lambda - 1)(\gamma_0 - 1)^{-1}).$$

The factor $\gamma_0 - 1$ acts invertibly on X, so it would suffice that the other factor on the right acts invertibly on X. If $v(\lambda - 1)$ is large enough (depending on γ_0) then the bounded operator $(\lambda - 1)(\gamma_0 - 1)^{-1}$ on X has sup-norm strictly less than 1, so by completeness of X we can use a geometric series expansion in bounded K-linear operators to construct the required inverse on X.

It remains to deal with the possibility that $v(\lambda-1)$ is not big enough. But for large enough r the 1-unit λ^{p^r} is distinct from 1 and arbitrarily close to 1, so by our above observation concerning the uniformity aspect of (TS3)(ii) we see that the K-linear $\gamma_0^{p^r}-\lambda^{p^r}$ acts bijectively on $\widehat{K}_\infty$ if r is large enough. Since $\gamma_0-\lambda$ divides this operator in a commuting manner, it also acts bijectively on $\widehat{K}_\infty$. ■

14.2. **Consequences of the Tate–Sen axioms.** We now work in the general setting of the Tate–Sen axioms. For any open subgroup H in H_0 that is normal in G, define the "decompletion"

$$\Lambda_{H,\infty} = \varinjlim_{n\geqslant n(H)} \Lambda_{H,n} \subseteq \widetilde{\Lambda}^H.$$

[In Sen's situation, if $H = G_M$ for a finite extension M/K_∞ that is Galois over K, and a finite Galois L/K such that $L_\infty = M$, we have $\Lambda_{H,\infty} = \varinjlim L_n = L_\infty = M$ inside of $\widetilde{\Lambda}^H = \mathbf{C}_K^H = \widehat{M}$.] Consider the direct limit of inflation mappings given by

$$i_G : \varinjlim_{H} \varinjlim_{n} \mathrm{H}^1(\widetilde{\Gamma}_H, \mathrm{GL}_d(\Lambda_{H,n})) \to \mathrm{H}^1(G_0, \mathrm{GL}_d(\widetilde{\Lambda})), \tag{14.2.1}$$

where the limit is taken over $n\to\infty$ and then $H\to H_0$. Intuitively, we want to imagine that the left side is $\mathrm{H}^1(G_0/H_0, \mathrm{GL}_d(\Lambda_{H,\infty}))$, but in general we have no topological information about $\Lambda_{H,\infty}$ to justify such a passage of the direct limit through the cohomology. (In fact, this does not really matter, since the limit of the cohomologies will be entirely sufficient for our needs. It is just a psychological bonus if the limit can be moved inside.)

In the Sen situation the commuting of limit and cohomology is valid. To see this, first note that in this situation the map i_G is the inflation map

$$\varinjlim_{L/K} \varinjlim_{n} \mathrm{H}^1(\mathrm{Gal}(L_\infty/K), \mathrm{GL}_d(L_n)) \to \mathrm{H}^1(G_K, \mathrm{GL}_d(\mathbf{C}_K)).$$

By Exercise 14.4.4, the inner limit is $\mathrm{H}^1(\mathrm{Gal}(L_\infty/K), \mathrm{GL}_d(L_\infty))$ and the inflation map to it from $\mathrm{H}^1(\mathrm{Gal}(K_\infty/K), \mathrm{GL}_d(K_\infty))$ is an isomorphism. In other words, the entire left side is $\mathrm{H}^1(\mathrm{Gal}(K_\infty/K), \mathrm{GL}_d(K_\infty))$ (which is to say precisely that we have moved the entire limit inside of the cohomology)!

By consideration of Sen's situation, we see that a bijectivity result for the map i_G in general is to be viewed as a combination of descent and decompletion. It is precisely such a bijectivity result in the general setting of the Tate–Sen formalism that will be the main result of this section. To get there, we have to prove a number of technical lemmas. Since we are trying to carry out a descent and decompletion, we basically have to figure out ways to approximated cocycles over a "big" ring by those over a "small" ring. In Sen's situation, all approximations are p-adic. In the general axiomatic setting, p-adic approximation and v_Λ-adic approximation are two different things. Hence, the argument sort of becomes twice as long in the general setting.

Throughout the lemmas below, the constants c_1, c_2, c_3 are as in the Tate–Sen axioms. The first lemma says that if a 1-cocycle is close enough to the trivial cocycle then it is approximately a coboundary (and with bootstrapping will later be proved to really be a coboundary, with good control on a choice of 0-cochain yielding this coboundary). The strategy of the proof goes back to the classical proof of Hilbert's Theorem 90, which rests on a cocycle construction involving an element with trace 1.

Lemma 14.2.1. *Let H be an open normal subgroup of H_0 and choose a continuous 1-cocycle $h \mapsto U_h$ on H valued in $\mathrm{GL}_d(\widetilde{\Lambda})$. Pick $x \in (c_1, +\infty]$, and assume that $v_\Lambda(U_h - 1) \geqslant x$ and $U_h \in 1 + p^m \operatorname{Mat}_d(\widetilde{\Lambda})$ for all $h \in H$. Then there exists $B \in \mathrm{GL}_d(\widetilde{\Lambda})$ such that*

(1) *$v_\Lambda(B - 1) \geqslant x - c_1$ and $B \in 1 + p^m \operatorname{Mat}_d(\widetilde{\Lambda})$;*
(2) *$v_\Lambda(B^{-1}U_h h(B) - 1) \geqslant x + 1$ for all $h \in H$.*

Proof. First note that for $h \in H$ we have $v_\Lambda(U_h) \geqslant \min\{v_\Lambda(U_h - 1), v_\Lambda(1)\}$, so $v_\Lambda(U_h) \geqslant 0$. By continuity, there exists an open subgroup $H_1 \subset H$ (which we may then shrink to be normal in G_0) such that $v_\Lambda(U_h - 1) \geqslant x + c_1 + 1$ for all $h \in H_1$. By (TS1), there exists $\alpha \in \widetilde{\Lambda}^{H_1}$ such that $v_\Lambda(\alpha) > -c_1$ and $\sum_{\tau \in H/H_1} \tau(\alpha) = 1$.

Let $T \subset H$ be a set of representatives for H/H_1, and define $B = \sum_{\tau \in T} \tau(\alpha) U_\tau \in \operatorname{Mat}_d(\widetilde{\Lambda})$. We have $B - 1 = \sum_{\tau \in T} \tau(\alpha)(U_\tau - 1)$, so $v_\Lambda(B - 1) \geqslant x - c_1$ and $B \in 1 + p^m \operatorname{Mat}_d(\widetilde{\Lambda})$. But $x - c_1 > 0$, so by Exercise 14.4.3 we see that B is invertible and $v_\Lambda(B^{-1} - 1) > 0$. In particular, $v_\Lambda(B^{-1}) \geqslant 0$. Using the definition of B and the 1-cocycle condition, for each $h \in H$ we have

$$U_h h(B) = \sum_{\tau \in T} h\tau(\alpha) U_h h(U_\tau) = \sum_{\tau \in T} h\tau(\alpha) U_{h\tau}.$$

Fix $h \in H$. For $\tau \in T$ (as for any element of H), there exists unique $\tau' \in T$ and $h_1 \in H_1$ such that $h\tau = \tau' h_1$. Since

$$\begin{aligned} v_\Lambda\big(U_{\tau' h_1} - U_{\tau'}\big) = v_\Lambda\big(U_{\tau'} \tau'(U_{h_1} - 1)\big) &\geqslant v_\Lambda(U_{\tau'}) + v_\Lambda(U_{h_1} - 1) \\ &\geqslant x + c_1 + 1 \end{aligned}$$

and $h\tau(\alpha) U_{h\tau} = \tau'(\alpha) U_{\tau' h_1}$, we have $v_\Lambda\big(h\tau(\alpha) U_{h\tau} - \tau'(\alpha) U_{\tau'}\big) > x + 1$. Thus,

$$v_\Lambda(U_h h(B) - B) > x + 1,$$

so $v_\Lambda(B^{-1} U_h h(B) - 1) \geqslant x + 1$ since $v_\Lambda(B^{-1}) \geqslant 0$. ■

Now we bootstrap the preceding lemma to find B yielding a much better conclusion:

Proposition 14.2.2. *Under the hypotheses of Lemma 14.2.1, we can find B there so that in addition $B^{-1} U_h h(B) = 1$ for all $h \in H$. In other words, the 1-cocycle $h \mapsto U_h$ is a 1-coboundary of the form $U_h = B \cdot h(B)^{-1}$ ($h \in H$), where $B \equiv 1 \bmod p^m$ and $v_\Lambda(B - 1) \geqslant x - c_1$.*

Proof. We apply Lemma 14.2.1 repeatedly to construct a sequence $\{B_n\}_{n \geqslant 0}$ in $\mathrm{GL}_d(\widetilde{\Lambda})$ such that for all $n \geqslant 0$ we have

(i) $v_\Lambda(B_n - 1) \geqslant x + n - c_1$ and $B_n \in 1 + p^m \operatorname{Mat}_d(\widetilde{\Lambda})$;
(ii) $v_\Lambda((B_0 B_1 \cdots B_n)^{-1} U_h h(B_0 B_1 \cdots B_n)) \geqslant x + n$ for all $h \in H$

Since $\widetilde{\Lambda}$ is complete for the topology defined by v_Λ, property (i) implies that the product $B = \prod_{n=0}^{\infty} B_n$ converges in $\mathrm{GL}_d(\widetilde{\Lambda})$. By passage to the limit, we also have $B \in 1+p^m \mathrm{Mat}_d(\widetilde{\Lambda})$ and $v_\Lambda(B-1) \geqslant x - c_1$. Property (ii) then implies that $B^{-1}U_h h(B) = 1$ for all $h \in H$. ■

The next lemma says that if a matrix M over $\widetilde{\Lambda}^H$ is close to $\gamma(M)$ for $\gamma \in \widetilde{\Gamma}_H$ that is not too close to 1 (so γ is an approximate "topological generator" for the group $\widetilde{\Gamma}_H$ which is isomorphic to $\mathbf{Z}_p$ near the identity) then M has all entries in $\Lambda_{H,n}$ for a suitable n, and that if M were also invertible over $\widetilde{\Lambda}^H$ then it is also invertible over $\Lambda_{H,n}$. (This last point is a triviality in the Sen situation for which the coefficient rings are valued fields, but in other settings this descent of invertibility is not a tautology. For applications to descent of 1-cocycles valued in $\mathrm{GL}_d(\widetilde{\Lambda}^H)$, it is clearly essential to keep track of the invertibility property under descent of matrices.)

Lemma 14.2.3. *Let G be an open normal subgroup of G_0, and pick $n \geqslant n(G)$. Define $H = G \cap H_0$, and choose $\gamma \in \widetilde{\Gamma}_H$ such that $n(\gamma) \leqslant n$. For an $M \in \mathrm{Mat}_{d_2\times d_1}(\widetilde{\Lambda}^H)$ with $d_1, d_2 \geqslant 1$, assume there exist $U_i \in \mathrm{GL}_{d_i}(\Lambda_{H,n})$ such that:*

(1) *$v_\Lambda(U_1 - 1) > c_3$, $v_\Lambda(U_2 - 1) > c_3$;*
(2) *$\gamma(M) = U_1 M U_2$.*

Then $M \in \mathrm{Mat}_{d_2\times d_1}(\Lambda_{H,n})$. Moreover, if $d_1 = d_2 = d$ and $M \in \mathrm{GL}_d(\widetilde{\Lambda}^H)$ then $M \in \mathrm{GL}_d(\Lambda_{H,n})$.

Proof. Since M^{-1} satisfies the same hypotheses as M (upon swapping d_1 and d_2), it suffices to carry out the descent for M as a matrix (as then doing the same for M^{-1} when it exists will show that invertibility descends too). Define $C = M - R_{H,n}(M)$, a $d_2 \times d_1$ matrix with coefficients in $X_{H,n}$. Our task is exactly to prove that $C = 0$, so we assume not and seek a contradiction.

The operator $R_{H,n}$ is $\Lambda_{H,n}$-linear and commutes with $\gamma \in \widetilde{\Gamma}_H$, so $\gamma(C) = U_1 C U_2$. Hence,

$$(\gamma - 1)(C) = U_1 C U_2 - C = (U_1 - 1)CU_2 + U_1 C(U_2 - 1) - (U_1 - 1)C(U_2 - 1),$$

This implies $v_\Lambda((\gamma-1)(C)) \geqslant v_\Lambda(C) + \min\{v_\Lambda(U_1 - 1), v_\Lambda(U_2 - 1)\} > v_\Lambda(C) + c_3$, where the final strict inequality uses that $v_\Lambda(C)$ is finite (as we assumed $C \neq 0$). This exactly contradicts (TS3) (applied to a nonzero entry of C with minimal v_Λ-value), so $C = 0$ as desired. ■

In addition to considering approximate descent using approximations relative to v_Λ, as in Lemma 14.2.3, we also need to keep track of p-adic approximations when doing descent. A preliminary lemma in that direction (to then immediately be improved by bootstrapping in the subsequent proposition) is:

Lemma 14.2.4. *Let G be an open normal subgroup of G_0, and pick $n \geqslant n(G)$. Define $H = G \cap H_0$. Let $U = 1 + p^m(U_1 + U_2)$ for $U_1 \in \mathrm{Mat}_d(\Lambda_{H,n})$ and $U_2 \in \mathrm{Mat}_d(\widetilde{\Lambda}^H)$ such that*

$$\begin{cases} v_\Lambda(U_1) \geqslant x - v_\Lambda(p^m), \\ v_\Lambda(U_2) \geqslant y - v_\Lambda(p^m), \end{cases}$$

where $x \in [c_2 + c_3 + \delta, +\infty]$ and $y \in [\max(x + c_2, 2c_2 + 2c_3 + \delta), +\infty]$ for some $\delta \in (0, +\infty]$.

For any $\gamma \in \widetilde{\Gamma}_H$ satisfying $n(\gamma) \leqslant n$, there exists $B \in 1 + p^m \operatorname{Mat}_d\left(\widetilde{\Lambda}^H\right)$ such that $v_\Lambda(B-1) \geqslant y - c_2 - c_3$ and $B^{-1}U\gamma(B) = 1 + p^m(V_1+V_2)$, with $V_1 \in \operatorname{Mat}_d(\Lambda_{H,n})$ and $V_2 \in \operatorname{Mat}_d\left(\widetilde{\Lambda}^H\right)$ satisfying

$$\begin{cases} v_\Lambda(V_1) \geqslant x - v_\Lambda(p^m) \\ v_\Lambda(V_2) \geqslant y - v_\Lambda(p^m) + \delta \end{cases}$$

In this and subsequent lemmas the reason that we include the (generally trivial) case where various estimation parameters (x, y, δ) are infinite is that it later allows us to not need to make separate remarks when working with a 1-cocycle $g \mapsto U_g$ at a value U_γ that might equal 1 (i.e., $U_\gamma - 1 = 0$).

Proof. If $p = 0$ in $\widetilde{\Lambda}$ then the assertion is obvious, so we may and do assume $p \neq 0$ in $\widetilde{\Lambda}$. The given estimates on the $v_\Lambda(U_i)$'s force $v_\Lambda(U) = 0$. By (TS2)(4), we have

$$v_\Lambda(R_{H,n}(U_2)) \geqslant y - v_\Lambda(p^m) - c_2 \geqslant x - v_\Lambda(p^m).$$

By (TS3), there exists $V \in \operatorname{Mat}_d\left(\widetilde{\Lambda}^H\right)$ such that $(\gamma - 1)(V) = (R_{H,n} - 1)(U_2)$ and

$$v_\Lambda(V) \geqslant v_\Lambda(R_{H,n}(U_2) - U_2) - c_3 \geqslant \min(v_\Lambda(R_{H,n}(U_2)), v_\Lambda(U_2)) - c_3,$$

with $v_\Lambda(U_2) \geqslant y - v_\Lambda(p^m) > y - v_\Lambda(p^m) - c_2$. Hence, $v_\Lambda(V) \geqslant y - v_\Lambda(p^m) - c_2 - c_3$.

Define

$$V_1 = U_1 + R_{H,n}(U_2) \in \operatorname{Mat}_d(\Lambda_{H,n}) \quad \text{and} \quad B = 1 + p^m V \in \operatorname{Mat}_d(\widetilde{\Lambda}^H).$$

We then have $v_\Lambda(B-1) = v_\Lambda(p^m V) \geqslant y - c_2 - c_3 > 0$ (so B is invertible and $v_\Lambda(B) = 0$) and the matrix $V_1 \in \operatorname{Mat}_d(\Lambda_{H,n})$ satisfies

$$v_\Lambda(V_1) \geqslant \min\{v_\Lambda(U_1), v_\Lambda(R_{H,n}(U_2))\} \geqslant x - v_\Lambda(p^m).$$

Since $B^{-1} = 1 - p^m V + p^{2m} V^2 - \cdots$, one can write $B^{-1} = 1 - p^m V + p^{2m} V^2 C$ with $C \in \operatorname{Mat}_d(\widetilde{\Lambda}^H)$ satisfying $v_\Lambda(C) = 0$. We then compute:

$$\begin{aligned} B^{-1}U\gamma(B) - 1 - p^m V_1 &= (1 - p^m V + p^{2m} V^2 C) U (1 + p^m \gamma(V)) - (1 + p^m V_1) \\ &= U + p^m U\gamma(V) - p^m VU - p^{2m}(VU\gamma(V) - V^2 CU\gamma(B)) \\ &\qquad -1 - p^m U_1 - p^m R_{H,n}(U_2) \\ &= p^m\big(U_2 - R_{H,n}(U_2) + U\gamma(V) - VU \\ &\qquad -p^m(VU\gamma(V) - V^2CU\gamma(B))\big), \end{aligned}$$

where the final equality uses that $U = 1 + p^m(U_1 + U_2)$. Since $U_2 - R_{H,n}(U_2) = (1-\gamma)(V)$, we have $B^{-1}U\gamma(B) = 1 + p^m V_1 + p^m V_2$ with

$$V_2 := (U-1)\gamma(V) - V(U-1) - p^m(VU\gamma(V) - V^2CU\gamma(B)) \in \operatorname{Mat}_d(\widetilde{\Lambda}^H).$$

But $v_\Lambda(U-1) \geqslant v_\Lambda(p^m) + \min\{v_\Lambda(U_1), v_\Lambda(U_2)\}$ by the definition of U, and this is at least x since $y \geqslant x$. We obtained two lower bounds from this:

$$v_\Lambda((U-1)\gamma(V)) \geqslant y - v_\Lambda(p^m) - c_2 - c_3 + x \geqslant y - v_\Lambda(p^m) + \delta$$

and $v_\Lambda(V(U-1)) \geqslant y - v_\Lambda(p^m) + \delta$. Moreover, $v_\Lambda(VU\gamma(V) - V^2CU\gamma(B)) \geqslant 2v_\Lambda(V)$ because the "valuations" $v_\Lambda(\gamma(B)) = v_\Lambda(B)$, $v_\Lambda(C)$, and $v_\Lambda(U)$ all vanish. Thus,

$$\begin{aligned} v_\Lambda(p^m(VU\gamma(V) - V^2CU\gamma(B))) &\geqslant v_\Lambda(p^m) + 2(y - v_\Lambda(p^m) - c_2 - c_3) \\ &= y - v_\Lambda(p^m) + y - 2c_2 - 2c_3 \\ &\geqslant y - v_\Lambda(p^m) + \delta. \end{aligned}$$

This implies $v_\Lambda(V_2) \geqslant y - v_\Lambda(p^m) + \delta$, so we are done. ∎

Bootstrapping the lemma will now yield an improvement not involving any U_1, U_2, or x:

Proposition 14.2.5. *Let G be an open normal subgroup of G_0, and define $H = G \cap H_0$ and pick $n \geqslant n(G)$. Choose $U \in 1 + p^m \operatorname{Mat}_d(\widetilde{\Lambda}^H)$ for some $m \geqslant 0$ such that $v_\Lambda(U-1) \geqslant y$ with $y \in [2c_2 + 2c_3 + \delta, +\infty]$ some $\delta \in (0, +\infty]$. For any $\gamma \in \widetilde{\Gamma}_H$ satisfying $n(\gamma) \leqslant n$, there exists $B \in 1 + p^m \operatorname{Mat}_d\left(\widetilde{\Lambda}^H\right)$ such that $v_\Lambda(B-1) \geqslant y - c_2 - c_3$ and $B^{-1}U\gamma(B) \in 1 + p^m \operatorname{Mat}_d(\Lambda_{H,n})$.*

Note that since $v_\Lambda(B-1) \geqslant y - c_2 - c_3 > 0$, we must have $B \in \operatorname{GL}_d(\widetilde{\Lambda}^H)$. Hence, the appearance of B^{-1} in the conclusion makes sense.

Proof. The case $p = 0$ in $\widetilde{\Lambda}$ is trivial, so we may and do assume $p \neq 0$ in $\widetilde{\Lambda}$. Define $x := y - c_2 \geqslant c_2 + c_3 + \delta$, and also define $U_{1,1} = 0 \in \operatorname{Mat}_d(\Lambda_{H,n})$ and $U_{2,1} = U \in \operatorname{Mat}_d(\widetilde{\Lambda}^H)$. By repeatedly applying Lemma 14.2.4, we obtain a sequence $\{B_n\}$ on $B_n \in 1 + p^m \operatorname{Mat}_d(\widetilde{\Lambda}^H)$ such that

(i) $v_\Lambda(B_n - 1) \geqslant y - c_2 - c_3 + n\delta > 0$ (so $B_n \in \operatorname{GL}_d(\widetilde{\Lambda}^H)$);
(ii) $(B_0B_1 \cdots B_n)^{-1}U\gamma(B_0B_1\cdots B_n) = 1 + p^m(U_{1,n} + U_{2,n})$ with $U_{1,n} \in \operatorname{Mat}_d(\Lambda_{H,n})$ and $U_{2,n} \in \operatorname{Mat}_d(\widetilde{\Lambda}^H)$ satsifying $v_\Lambda(U_{1,n}) \geqslant x - v_\Lambda(p^m)$ and $v_\Lambda(U_{2,n}) \geqslant y - v_\Lambda(p^m) + n\delta$

for all $n \geqslant 0$.

Since $\widetilde{\Lambda}^H$ is complete for the topology defined by v_Λ, property (i) implies that the product $B = \prod_{n=0}^\infty B_n$ converges in $\operatorname{GL}_d(\widetilde{\Lambda}^H)$. We have $B \in 1 + p^m \operatorname{Mat}_d(\widetilde{\Lambda}^H)$ and $v_\Lambda(B-1) \geqslant y - c_2 - c_3 > 0$. In particular, $B \in \operatorname{GL}_d(\widetilde{\Lambda}^H)$. Property (ii) then implies that $B^{-1}U\gamma(B) \in 1 + p^m \operatorname{Mat}_d(\Lambda_{H,n})$ because $\Lambda_{H,n}$ is closed in $\widetilde{\Lambda}^H$ for the topology defined by v_Λ. ∎

We can now establish another result about the cohomological triviality of certain 1-cocycles, this time incorporating a p-adic estimate on the trivializing 0-cochain as well.

Proposition 14.2.6. *Let $U \colon G_0 \to \operatorname{GL}_d(\widetilde{\Lambda})$ be a continous cocycle, and assume that for some open normal subgroup G of G_0 we have $v_\Lambda(U_g - 1) > c_1 + 2c_2 + 2c_3$ and $U_g \in 1 + p^m \operatorname{Mat}_d(\widetilde{\Lambda})$ for all $g \in G$, with some $m \in \mathbf{Z}_{\geqslant 0}$.*

There exists $B \in 1 + p^m \operatorname{Mat}_d(\widetilde{\Lambda})$ such that $v_\Lambda(B-1) > c_2 + c_3$ (so $B \in \operatorname{GL}_d(\widetilde{\Lambda}^H)$) and the 1-cocycle $h \mapsto B^{-1}U_h h(B)$ is trivial on $H := G \cap H_0$ and has values in $\operatorname{GL}_d(\Lambda_{H,n(G)})$.

Proof. Since $v_\Lambda(U_g - 1) > 0$, we have $v_\Lambda(U_g) = 0$. By Proposition 14.2.2, there exists $B_1 \in 1 + p^m \operatorname{Mat}_d(\widetilde{\Lambda})$ such that $v_\Lambda(B_1 - 1) > 2c_2 + 2c_3 > 0$ (so $v_\Lambda(B_1) = 0$ and $B_1 \in \operatorname{GL}_d(\widetilde{\Lambda}^H)$) and the 1-cocycle $U' : g \mapsto U'_g := B_1^{-1}U_g g(B_1)$ is trivial on H. It is also clear that $v_\Lambda(U'_g) = 0$ for all $g \in G_0$, since $v_\Lambda(B_1 - 1) > 0$ and $v_\Lambda(U_g) = 0$. By triviality of restriction to H, the

1-cocycle U' is the inflation of a 1-cocycle $\widetilde{\Gamma}_H = G_0/H \to \mathrm{GL}_d\left(\widetilde{\Lambda}^H\right)$ that we still denote by U'.

Choose $\gamma \in G/H \subset \widetilde{\Gamma}_H$ such that $n(\gamma) = n(G)$. We have $U'_\gamma \in 1 + p^m \operatorname{Mat}_d(\widetilde{\Lambda}^H)$ and

$$\begin{aligned} v_\Lambda(U'_\gamma - 1) &= v_\Lambda\big(B_1^{-1}(U_\gamma - 1)\gamma(B_1) + B_1^{-1}\gamma(B_1 - 1) + B_1^{-1} - 1\big) \\ &\geqslant \inf\big\{v_\Lambda(U_\gamma - 1), v_\Lambda(B_1 - 1)\big\} \\ &> 2c_2 + 2c_3. \end{aligned}$$

Hence, we may apply Proposition 14.2.5 with $n = n(\gamma) = n(G)$, $U = U'_\gamma$, $y = v_\Lambda(U'_\gamma - 1)$ (which is infinite if $U'_\gamma = 1$!) and $\delta = y - 2c_2 - 2c_3 > 0$ to find $B_2 \in 1 + p^m \operatorname{Mat}_d(\widetilde{\Lambda}^H)$ such that $v_\Lambda(B_2 - 1) > c_2 + c_3 > 0$ (so $v_\Lambda(B_2) = 0$ and $B_2 \in \mathrm{GL}_d(\widetilde{\Lambda}^H)$) and $B_2^{-1}U'_\gamma\gamma(B_2) \in \mathrm{GL}_d(\Lambda_{H,n(G)})$.

Define $B := B_1B_2 \in 1 + p^m \operatorname{Mat}_d(\widetilde{\Lambda})$, so we have:

- $v_\Lambda(B - 1) \geqslant \min\{v_\Lambda(B_1 - 1), v_\Lambda(B_2 - 1)\} > c_2 + c_3$,
- the 1-cocycle $g \mapsto U''_g = B^{-1}U_g g(B)$ is trivial on H (so it is the inflation of a 1-cocycle $\widetilde{\Gamma}_H \to \mathrm{GL}_g(\widetilde{\Lambda}^H)$), $U''_\gamma \in \mathrm{GL}_d(\Lambda_{H,n(G)})$, and
- $$\begin{aligned} v_\Lambda(U''_\gamma - 1) &= v_\Lambda((B_2^{-1} - 1)U'_\gamma\gamma(B_2) + (U'_\gamma - 1)\gamma(B_2) + \gamma(B_2 - 1)) \\ &\geqslant \min\{v_\Lambda(B_2 - 1), v_\Lambda(U'_\gamma - 1)\} \\ &> c_2 + c_3 > c_3 > 0. \end{aligned}$$

  (In particular, $U''_\gamma \in \mathrm{GL}_d(\widetilde{\Lambda}^H)$.)

For any $g \in \widetilde{\Gamma}_H$ we have $g\gamma = \gamma g$ (because G/H lies in the center of $\widetilde{\Gamma}_H$), so $U''_g g(U''_\gamma) = U''_\gamma\gamma(U''_g)$; that is, $\gamma(U''_g) = U''^{-1}_\gamma U''_g g(U''_\gamma)$. Lemma 14.2.3 applied with $n = n(\gamma) = n(G)$, $M = U''_g$, $U_1 = U''^{-1}_\gamma$, and $U_2 = g(U''_\gamma)$ implies that $U''_g \in \mathrm{GL}_d(\Lambda_{H,n(G)})$ (note that $U_2 \in \mathrm{GL}_d(\Lambda_{H,n(G)})$ thanks to (TS2)(3)). Since g was arbitrary in $\widetilde{\Gamma}_H$, we are done. ∎

Finally we can give some interesting cohomological applications of the preceding technical results. In what follows, we consider continuous cohomology using the topology on $\widetilde{\Lambda}$ and its subrings defined by v_Λ.

Let G be an open normal subgroup of G_0, and as usual define $H = H_0 \cap G$. For $n \geqslant n(G)$, the inclusion $\Lambda_{H,n} \subset \widetilde{\Lambda}^H$ induces a map

$$\mathrm{H}^1\big(\widetilde{\Gamma}_H, \mathrm{GL}_d(\Lambda_{H,n})\big) \to \mathrm{H}^1\big(\widetilde{\Gamma}_H, \mathrm{GL}_d(\widetilde{\Lambda}^H)\big).$$

Composing with inflation $\mathrm{H}^1\big(\widetilde{\Gamma}_H, \mathrm{GL}_d(\widetilde{\Lambda}^H)\big) \to \mathrm{H}^1\big(G_0, \mathrm{GL}_d(\widetilde{\Lambda})\big)$ thereby defines a map

$$i_H : \varinjlim_n \mathrm{H}^1\big(\widetilde{\Gamma}_H, \mathrm{GL}_d(\Lambda_{H,n})\big) \to \mathrm{H}^1\big(G_0, \mathrm{GL}_d(\widetilde{\Lambda})\big).$$

Lemma 14.2.7. *The map i_H is injective.*

Proof. The inflation map $\mathrm{H}^1\big(\widetilde{\Gamma}_H, \mathrm{GL}_d(\widetilde{\Lambda}^H)\big) \to \mathrm{H}^1\big(G_0, \mathrm{GL}_d(\widetilde{\Lambda})\big)$ is injective, so our problem is to prove the injectivity of $\varinjlim_n \mathrm{H}^1\big(\widetilde{\Gamma}_H, \mathrm{GL}_d(\Lambda_{H,n})\big) \to \mathrm{H}^1\big(\widetilde{\Gamma}_H, \mathrm{GL}_d(\widetilde{\Lambda}^H)\big)$. Let

$$U, U' : \widetilde{\Gamma}_H \rightrightarrows \mathrm{GL}_d(\Lambda_{H,\infty})$$

be a pair of continuous 1-cocycles which become cohomologous in $\mathrm{GL}_d(\widetilde{\Lambda}^H)$. That is, we assume there exists $B \in \mathrm{GL}_d(\widetilde{\Lambda}^H)$ such that $U'_g = B^{-1}U_g g(B)$.

Choose $n_0 \geqslant n(G)$ such that U and U' are both valued in $\mathrm{GL}_d(\Lambda_{H,n_0})$. If $\gamma \in G/H$ satisfies $n(\gamma) \geqslant n_0$ then $\gamma(B) = U_\gamma^{-1} B U'_\gamma$. By continuity, we make choose γ with $n(\gamma)$ large and finite such that $v_\Lambda(U_\gamma - 1), v_\Lambda(U'_\gamma - 1) > c_3$. Hence, Lemma 14.2.3 may be applied with $n \geqslant n(\gamma)$, $M = B$, $U_1 = U_\gamma^{-1}$, and $U_2 = U'_\gamma$ to deduce that $B \in \mathrm{GL}_d(\Lambda_{H,n})$. Hence, U and U' are cohomologous as desired. ∎

To reduce clutter, we now *define* the notation

$$\mathrm{H}^1(\widetilde{\Gamma}_H, \mathrm{GL}_d(\Lambda_{H,\infty})) := \varinjlim_n \mathrm{H}^1(\widetilde{\Gamma}_H, \mathrm{GL}_d(\Lambda_{H,n})).$$

(This creates no risk of confusion, as we will never actually use the usual meaning of the H^1 on the left side, aside from Sen's situation where we have already proved in Exercise 14.4.4 that the limit pointed set can be passed inside of the cohomology.) Observe that elements of this set are represented by continuous 1-cocycles $\widetilde{\Gamma}_H \to \mathrm{GL}_d(\Lambda_{H,\infty})$ which happen to land in some $\mathrm{GL}_d(\Lambda_{H,n})$.

We wish to pass to the limit on the maps i_H as G shrinks. (Recall $H = G \cap H_0$.) To this end, consider an inclusion $G \subset G'$ of open normal subgroups of G_0, and define $H = H_0 \cap G$ and $H' = H_0 \cap G'$. There is a natural surjection $\widetilde{\Gamma}_H \twoheadrightarrow \widetilde{\Gamma}_{H'}$, and if $n \geqslant n' \geqslant \max(n(G), n(G'))$ then $\Lambda_{H',n'} \subset \Lambda_{H,n}$. In particular, we have an inflation map

$$i_{H,H'} \colon \mathrm{H}^1\big(\widetilde{\Gamma}_{H'}, \mathrm{GL}_d(\Lambda_{H',\infty})\big) \to \mathrm{H}^1\big(\widetilde{\Gamma}_H, \mathrm{GL}_d(\Lambda_{H,\infty})\big)$$

The sets $\mathrm{H}^1\big(\widetilde{\Gamma}_H, \mathrm{GL}_d(\Lambda_{H,\infty})\big)$ with $H = G \cap H_0$ and G varying through open normal subgroups of G_0 thus form an inductive system. Clearly $i_{H'} = i_H \circ i_{H,H'}$ for any such pair (G, G') as above, so we get a map

$$\varinjlim_G \mathrm{H}^1\big(\widetilde{\Gamma}_H, \mathrm{GL}_d(\Lambda_{H,\infty})\big) \to \mathrm{H}^1\big(G_0, \mathrm{GL}_d(\widetilde{\Lambda})\big)$$

in which the limit is taken over all open normal subgroups G in G_0.

Theorem 14.2.8. *This map is a bijection.*

Proof. Injectivity results from Lemma 14.2.7 and surjectivity from Proposition 14.2.6 (using $m = 0$). ∎

In terms of $\widetilde{\Lambda}$-representations, Theorem 14.2.8 has the following consequence for existence and uniqueness of descent, with a strong form of uniqueness.

Theorem 14.2.9. *Let W be a free $\widetilde{\Lambda}$-module of rank d equipped with a continuous semilinear action of G_0. Let $G \subset G_0$ be an open normal subgroup, and choose $c \in \mathbf{R}_{>c_1+2c_2+2c_3}$. Assume* *that W viewed as a G-representation admits a basis with respect to which the resulting* 1-*cocycle $g \mapsto U_g$ describing the action of G on W satisfies $v_\Lambda(U_g - 1) \geqslant c$ for all $g \in G$.*

Then there exists a unique finite free $\Lambda_{H,n(G)}$-submodule W' of W with rank d such that for $H := G \cap H_0$ we have

(1) *W' is stable under the action of G_0 and this action factors through $\widetilde{\Gamma}_H$;*
(2) *there exists some $c' > c_3$ such that W' admits a basis in which the* 1-*cocycle U' describing the action of $\widetilde{\Gamma}_H$ satisfies $v_\Lambda(U'_\gamma - 1) \geqslant c'$ for all $\gamma \in \widetilde{\Gamma}_H$;*
(3) *$W = \widetilde{\Lambda} \otimes_{\Lambda_{H,n(G)}} W'$ as $\widetilde{\Lambda}[G_0]$-modules.*

Note that the uniqueness for W' is as an actual subset of W, not merely up to abstract $\Lambda_{H,n(G)}$-linear $\widetilde{\Gamma}_H$-equivariant isomorphism.

Proof. The existence of W' follows from Proposition 14.2.6. For the uniqueness, assume W'' is a finite free $\Lambda_{H,n(G)}$-submodule of W with rank d having the same properties as W'. Choose $\Lambda_{H,n(G)}$-bases $\mathbf{e}'$ and $\mathbf{e}''$ of W' and W'' respectively. By property (3), these are also $\widetilde{\Lambda}$-bases of W, so we can define $B \in \mathrm{GL}_d(\widetilde{\Lambda})$ to be the matrix that converts $\mathbf{e}''$-coordinates into $\mathbf{e}'$-coordinates. The aim is to prove that actually $B \in \mathrm{GL}_d(\Lambda_{H,n(G)})$, as this says exactly that the respective $\Lambda_{H,n(G)}$-spans W' and W'' of $\mathbf{e}'$ and $\mathbf{e}''$ inside of W coincide.

The action of $\widetilde{\Gamma}_H$ on W is described by cocycles U' and U'' relative to the bases $\mathbf{e}'$ and $\mathbf{e}''$, and $U''_g = B^{-1}U'_g g(B)$ for all $g \in G_0$. By property (1), $U'_g = U''_g = 1$ if $g \in H$, so $B \in \mathrm{GL}_d(\widetilde{\Lambda}^H)$. Moreover, we have $v(U'_g - 1) > c_3$ and $v(U''_g - 1) > c_3$ for $g \in \widetilde{\Gamma}_H$ by property (2). Finally, since $g(B) = U'^{-1}_g B U''_g$ we have $B \in \mathrm{GL}_d(\Lambda_{H,n(G)})$ by Lemma 14.2.3, so we are done. ∎

14.3. **Descent of cohomology from G_K to Γ.** Tate [49, §3.2] proved $\mathbf{C}_K$-valued cohomological vanishing theorems for G_{K_∞} (e.g., $\mathrm{H}^1(G_{K_\infty}, \mathbf{C}_K) = 0$), and from this was able to get isomorphisms between G_K-cohomology and $\mathrm{Gal}(K_\infty/K)$-cohomology. This is quite remarkable, since $\mathrm{Gal}(K_\infty/K)$ is essentially $\mathbf{Z}_p$. In this section we rederive Tate's results, working within the broader context of the Tate–Sen formalism (following [1, App. I]).

First we describe the basic setup. Using notation as in §14.1 (the profinite group G_0, the ring $\widetilde{\Lambda}$, the "valuation" v_Λ, etc.), let W be a finite free $\widetilde{\Lambda}$-module of rank d. The choice of a basis identifies W with $\widetilde{\Lambda}^d$. Endowing the latter with the product topology (where the topology on each factor is defined by v_Λ) equips W with a topology, which is independent of the basis. For $m \geqslant 0$ define

$$\widetilde{\Lambda}_{\geqslant m} = \{x \in \widetilde{\Lambda} \,|\, v_\Lambda(x) \geqslant m\}$$

and $W_{\geqslant m} = \widetilde{\Lambda}_{\geqslant m} \cdot W$, so $\{W_{\geqslant m}\}_{m \geqslant 0}$ is a base of open neighborhoods of 0 in W.

Tate's case of interest was Sen's situation: $G = G_K$ and $\widetilde{\Lambda} = \mathbf{C}_K$ with its usual valuation. In what follows, as with Tate, we consider *continuous* cohomology of with values in W. To be precise, if $G \subset G_0$ is a closed subgroup and $n \geqslant 1$, we denote by $\mathscr{C}^n(G, W)$ the group of *continuous* r-cochains of G with values in W (i.e.,, continuous maps of topological spaces $G^r \to W$) and $\partial\colon \mathscr{C}^n(G, W) \to \mathscr{C}^{n+1}(G, W)$ is the usual the boundary map (which respects continuity, due to its explicit formula). We will be approximating cochains by other cochains, so we need a measure of uniform closeness. The natural definition is to assign to each $f \in \mathscr{C}^n(G, W)$ the supremum

$$v_\Lambda(f) = \inf\big\{v_\Lambda(f(g_1, \dots, g_r)) \,|\, g_1, \dots, g_r \in G\big\} \in \mathscr{R} \cup \{+\infty\}.$$

This depends on the choice of $\widetilde{\Lambda}$-basis of W, but that will not matter for our purposes.

The results below are due to Tate (*cf* [49, §3.2]) in the case of Sen's situation, and we follow the presentation of [1, Appendix I]. The key to everything is the following lemma, which handles the fact that continuous cohomology does not have good δ-functorial properties (due to the continuity conditions on cochains). Since Tate was aiming to relate G_K cohomology and $\mathrm{Gal}(K_\infty/K)$-cohomology, he wanted a version of the Hochschild-Serre spectral sequence. He also wanted to uniformly approximate continuous cochains by ones arising from finite groups (as in profinite group cohomology for discrete modules). Both of these aims are achieved in this lemma:

Lemma 14.3.1. *Assume* (TS1) *holds. Let H be an open subgroup of H_0 and $f \in \mathscr{C}^n(H, W)$.*

(1) *If there exists an open subgroup $H' \subset H$ such that f factors through H/H' then there exists $h \in \mathscr{C}^{n-1}(H/H', W)$ such that $v_\Lambda(f - \partial h) \geqslant v_\Lambda(\partial f) - c_1$ and $v_\Lambda(h) \geqslant v_\Lambda(f) - c_1$.*
(2) *There exists a sequence of open normal subgroups $(H_m)_{m>0}$ and $f_m \in \mathscr{C}^n(H/H_m, W)$ such that $v_\Lambda(f - f_m) \geqslant m$ for all $m \geqslant 1$.*

The second part of this lemma does not rest on (TS1) at all. Its proof is an elementary calculation.

Proof. First we prove (1), for which we may and do assume $n \geqslant 1$ By (TS1) we can choose $\alpha \in \widetilde{\Lambda}^{H'}$ such that $v_\Lambda(\alpha) > -c_1$ and $\sum_{\tau \in H/H'} \tau(\alpha) = 1$. Define $h \in \mathscr{C}^{n-1}(H/H', W)$ by:

$$h(g_1, \dots, g_{n-1}) = (-1)^n \sum_{\tau \in H/H'} g_1 \cdots g_{n-1} \tau(\alpha) f(g_1, \dots, g_{n-1}, \tau)$$

Since $v_\Lambda(\alpha) > -c_1$, we have

$$v_\Lambda(g_1 \cdots g_{n-1} \tau(\alpha) f(g_1, \dots, g_{n-1}, \tau)) \geqslant v_\Lambda(\alpha) + v_\Lambda(f(g_1, \dots, g_{n-1}, \tau)),$$

so $v_\Lambda(h) \geqslant v_\Lambda(f) - c_1$. (We include the case of equality to allow $f = 0$.) On the other hand, we have

$$\partial h(g_1, \dots, g_n) = \underbrace{g_1(h(g_2, \dots, g_n)) + \sum_{j=1}^{n-1} (-1)^j h(g_1, \dots, g_j g_{j+1}, \dots, g_n)}_{(-1)^n \Sigma} + (-1)^n h(g_1, \dots, g_{n-1})$$

We need to put the expression labeled as Σ into a more convenient form.

As $g_1(h(g_2, \dots, g_n)) = (-1)^n \sum_{\tau \in H/H'} g_1 \cdots g_n \tau(\alpha) g_1(f(g_2, \dots, g_n, \tau))$ and

$$h(g_1, \dots, g_j g_{j+1}, \dots, g_n) = (-1)^n \sum_{\tau \in H/H'} g_1 \cdots g_n \tau(\alpha) f(g_1, \dots, g_j g_{j+1}, \dots, g_n, \tau),$$

we get that Σ is equal to

$$\sum_{\tau\in H/H'} g_1\cdots g_n\tau(\alpha)\Big(g_1(f(g_2,\dots,g_n,\tau))+\sum_{j=1}^{n-1}(-1)^j f(g_1,\dots,g_jg_{j+1},\dots,g_n,\tau)\Big),$$

which expands out to become

$$\sum_{\tau\in H/H'} g_1\cdots g_n\tau(\alpha)\Big(\partial f(g_1,\dots,g_n,\tau)-(-1)^n f(g_1,\dots,g_{n-1},g_n\tau)-(-1)^{n+1}f(g_1,\dots,g_n)\Big).$$

But we also have (by change of variables)

$$\sum_{\tau\in H/H'} g_1\cdots g_n\tau(\alpha)f(g_1,\dots,g_{n-1},g_n\tau)=\sum_{\tau\in H/H'} g_1\cdots g_{n-1}\tau(\alpha)f(g_1,\dots,g_{n-1},\tau)$$

and $\sum_{\tau\in H/H'} g_1\cdots g_n\tau(\alpha)=1$ since $\sum_{\tau\in H/H'}\tau(\alpha)=1$, so we obtain the formula

$$\Sigma=\Big(\sum_{\tau\in H/H'} g_1\cdots g_n\tau(\alpha)\partial f(g_1,\dots,g_n,\tau)\Big)-h(g_1,\dots,g_{n-1})+(-1)^n f(g_1,\dots,g_n).$$

Thus, we compute

$$\partial h(g_1,\dots,g_n)-f(g_1,\dots,g_n)=(-1)^n\sum_{\tau\in H/H'} g_1\cdots g_n\tau(\alpha)\partial f(g_1,\dots,g_n,\tau),$$

so $v_\Lambda(\partial h-f)\geqslant v_\Lambda(\partial f)-c_1$ (equality allowing for the possibility $\partial f=0$). This proves (1).

Turning to the proof (2), since f is continuous we see that for each $m\geqslant 1$ there exists an open subgroup $H_m\subset H$ such that $\overline{f}_m\colon H^n\to W\to W/W_{\geqslant m}$ factors through a (discrete!) cochain $\overline{f})_m$ on the finite set $(H/H_m)^n$. If $s_m\colon W/W_{\geqslant m}\to W$ is any (set-theoretic) splitting of the projection, the cochain $f_m=s_m\circ\overline{f}_m$ is continuous and $v_\Lambda(f-f_m)\geqslant m$. ∎

Let $G\subset G_0$ be an open subgroup and $H=G\cap H_0=\ker(\psi|_G)$ as in Tate–Sen formalism.

Proposition 14.3.2. *Assume* (TS1) *holds. Then* $\mathrm{H}^n(H,W)=0$ *for all* $n\geqslant 1$*, and the inflation map* $\mathrm{H}^n(G/H,W^H)\to\mathrm{H}^n(G,W)$ *is an isomorphism for all* $n\geqslant 0$*.*

Proof. Choose $n\geqslant 1$ and $f\in\mathscr{C}^n(H,W)$. Pick sequence $(H_m)_{m\geqslant 1}$ and $\{f_m\}_{m\geqslant 1}$ as in Lemma 14.3.1(2). For each $m\geqslant 1$ let $h_m\in\mathscr{C}^{n-1}(H/H_m,W)$ as in Lemma 14.3.1 (1) be such that $v_\Lambda(f_m-\partial h_m)\geqslant v_\Lambda(\partial f_m)-c_1$ and $v_\Lambda(h_m)\geqslant v_\Lambda(f_m)-c_1$. Since the sequence $(h_m)_{m\in\mathbf{N}_{>0}}$ is Cauchy, it converges to some $h\in\mathscr{C}^{n-1}(H,W)$ (by the completeness of W).

Now assume f is a cocycle; i.e., $\partial f=0$. Since $v_\Lambda(f-f_m)\geqslant m$, we have $v_\Lambda(\partial f_m)\geqslant m$, so $v_\Lambda(f_m-\partial h_m)\geqslant v_\Lambda(\partial f_m)-c_1\geqslant m-c_1$. Passing to the limit as m goes to infinity gives $f=\partial h$, which proves $\mathrm{H}^n(H,W)=0$.

The isomorphism claim for the inflation map is clear for $n=0$, so to prove this claim in general we may assume $n\geqslant 1$. The case $n=1$ is proved by the classical method of proof with 1-cocycles. Thus, we now assume $n\geqslant 2$. There is no Hochschild-Serre spectral sequence, due to the continuity conditions, but we can adapt the idea by working "by hand" as follows. To handle surjectivity for the H^n's, by thinking in terms of continuous cocycles we just need the restriction map $\mathscr{C}^{n-1}(G,W)\to\mathscr{C}^{n-1}(H,W)$ to be surjective. Such surjectivity follows from Lemma 14.3.1(2).

For injectivity of the inflation mapping in degree $n \geqslant 2$, consider $f \in \mathscr{C}^n(G/H, W^H)$ such that $\partial f = 0$ and the composite mapping

$$G^n \twoheadrightarrow (G/H)^n \xrightarrow{f} W^H \hookrightarrow W$$

has the form ∂F for some $F \in \mathscr{C}(G^{n-1}, W)$ (which says that the class of f in $\mathrm{H}^n(G/H, W^H)$ inflates to 0 on $\mathrm{H}^n(G, W)$. We then have that $F|_{H^{n-1}}$ has n-cocycle boundary $\partial f|_{H^n} = 0$, so $F|_{H^{n-1}} \in \mathscr{Z}^{n-1}(H, W)$. But $n - 1 \geqslant 1$, so the proved vanishing of higher H-cohomology of W gives that $F|_{H^{n-1}} = \partial F'$ for some $F' \in \mathscr{C}^{n-2}(H, W)$. By Lemma 14.3.1(2) this h' lifts to some $f' \in \mathscr{C}^{n-2}(G, W)$, so if we replace (as we may) F with $F - \partial f'$ then we have arranged that $F|_{H^{n-1}} = 0$. [8] ■

Proposition 14.3.3. *If $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$, then $\mathrm{H}^n(H, W) = 0$ for all $n \geqslant 1$ and the inflation map $\mathrm{H}^n(G/H, W^H) \to \mathrm{H}^n(G, W)$ is an isomorphism for all $n \geqslant 0$.*

Proof. We apply Proposition 14.3.2 to the field $\widetilde{\Lambda} = \mathbf{C}_K$, the group $H = G_{K_\infty}$, and $v_\Lambda = v$. Condition (TS1) is fulfilled by Proposition 14.1.3. ■

Now we can finally give a proof of Theorem 2.2.7. Let us also recall the statement.

Theorem 14.3.4. *Let K be a p-adic field and $\eta : G_K \to \mathscr{O}_K^\times$ a continuous character such that $\eta(G_K)$ is a p-adic Lie group of dimension at most 1. Let $\mathbf{C}_K(\eta)$ denote $\mathbf{C}_K$ with the twisted G_K-action $g.c = \eta(g)g(c)$.*

If $\eta(I_K)$ is infinite then $\mathrm{H}^i_{\mathrm{cont}}(G_K, \mathbf{C}_K(\eta)) = 0$ for $i = 0, 1$ and these cohomologies are 1-dimensional over K when $\eta(I_K)$ is finite (i.e., when the splitting field of η over K is finitely ramified).

Proof. First assume that $\eta(I_K)$ is infinite. In this case the Tate–Sen formalism applies, so Proposition 14.3.3 gives

$$\mathrm{H}^i(G_K, \mathbf{C}_K(\eta)) \simeq \mathrm{H}^i(\mathrm{Gal}(K_\infty/K), \mathbf{C}_K(\eta)^{G_{K_\infty}}) = \mathrm{H}^i(\mathrm{Gal}(K_\infty/K), \widehat{K}_\infty(\eta)).$$

Thus, we wish to prove $\mathrm{H}^i(\mathrm{Gal}(K_\infty/K), \widehat{K}_\infty(\eta)) = 0$ for $i = 0, 1$. For any finite Galois extension K'/K inside of K_∞, the usual proof of inflation-restriction in degree 1 works with continuous 1-cocycles to give a left exact sequence

$$0 \to \mathrm{H}^1(\mathrm{Gal}(K'/K), K'(\eta)) \to \mathrm{H}^1(\mathrm{Gal}(K_\infty/K), \widehat{K}_\infty(\eta)) \to \mathrm{H}^1(\mathrm{Gal}(K_\infty/K'), \widehat{K}_\infty(\eta)).$$

Finite group cohomology on a $\mathbf{Q}$-vector space vanishes, so to prove the desired vanishing in degree 1 it is harmless to replace K with any such K'. Likewise, if $\widehat{K}_\infty(\eta)$ has vanishing space of invariants for $\mathrm{Gal}(K_\infty/K')$ then it certainly has vanishing space of invariants for $\mathrm{Gal}(K_\infty/K)$. Hence, for our treatment of the infinitely ramified case it is harmless to replace K with such a K'. We may therefore assume that $\mathrm{Gal}(K_\infty/K) \simeq \mathbf{Z}_p$ with topological generator γ. The infinitely ramified hypothesis implies that the unit $\lambda = \eta(\gamma) \in \mathscr{O}_K^\times$ is not a root of unity, and by replacing K with a finite extension inside of K_∞ we can arrange that $v(\lambda - 1) > 0$. That is, $\lambda \in 1 + \mathfrak{m}_K$.

We identify $\widehat{K}_\infty(\eta)$ with the K-Banach space $\widehat{K}_\infty$ on which each $g \in \mathrm{Gal}(K_\infty/K)$ acts via $g.c = \eta(g) \cdot g(c)$ (using the natural action of $\mathrm{Gal}(K_\infty/K)$ on $\widehat{K}_\infty$ via isometries). Since

[8] need to finish writing the argument

λ^{-1} is a 1-unit that is not a root of unity, by Lemma 14.1.9 the operator $x \mapsto \gamma(x) - \lambda^{-1}x$ on $\widehat{K}_\infty$ is bijective. Multiplying on the left by λ, this says that $x \mapsto \gamma.x - x$ is a bijective self-map of $\widehat{K}_\infty$. The denseness of $\gamma^{\mathbf{Z}}$ in $\mathrm{Gal}(K_\infty/K)$ implies that $\mathrm{H}^0(\mathrm{Gal}(K_\infty/K), \widehat{K}_\infty(\eta))$ is exactly the space of $c \in \widehat{K}_\infty$ such that $\gamma.c = c$, which is to say that this is the kernel of the injective map $x \mapsto \gamma.x - x$ on $\widehat{K}_\infty$. This proves the desired vanishing result in degree 0.

For degree 1, first note that if $c : \mathrm{Gal}(K_\infty/K) \to \widehat{K}_\infty(\eta)$ is a 1-cocycle then $(\gamma-1).c(\gamma^n) = (\gamma^n-1).c(\gamma)$ by forwards and backwards induction on $n \in \mathbf{Z}$ (using that $c(1) = 0$). Hence, if c is also continuous then $(\gamma-1).c(g) = (g-1).c(\gamma)$ for all $g \in \mathrm{Gal}(K_\infty/K)$. By surjectivity of $x \mapsto \gamma.x - x$ we can write $c(\gamma) = \gamma.x_0 - x_0$ for some $x_0 \in \widehat{K}_\infty$, and so if we subtract the continuous 1-cocycle $g \mapsto g.x_0 - x_0$ from c (as we may do without changing its degree-1 cohomology class) then we are reduced to the case when $c(\gamma) = 0$. But in such cases, for all $g \in \mathrm{Gal}(K_\infty/K)$ we have $(\gamma-1).c(g) = 0$. By injectivity of $x \mapsto \gamma.x - x$ on $\widehat{K}_\infty$, we conclude that c is identically zero as a function. This completes the infinitely ramified case.

Now suppose that $\eta(I_K)$ is finite. In this case the Tate–Sen formalism does not apply, so instead we will use completed unramified descent as in Lemma 3.2.6. First assume that $\eta(I_K)$ is trivial, which is to say that η is unramified. We may and do view η as a continuous character $G_k \to \mathscr{O}_K^\times$. In this case $\mathbf{C}_K(\eta)^{I_K} = (\mathbf{C}_K)^{I_K}(\eta) = \widehat{K^{\mathrm{un}}}(\eta)$. Thus,

$$\mathbf{C}_K(\eta)^{G_K} = (\mathscr{O}_{\widehat{K^{\mathrm{un}}}}(\eta))^{G_k}[1/p].$$

By Lemma 3.2.6, the space of G_k-invariants on the finite free rank-1 $\mathscr{O}_{\widehat{K^{\mathrm{un}}}}$-module $\mathscr{O}_{\widehat{K^{\mathrm{un}}}}(\eta)$ is free of rank 1 over $\mathscr{O}_K$, so inverting p gives that $\mathbf{C}_K(\eta)^{G_K}$ is 1-dimensional over K. To handle degree-0 in general, let K'/K be a finite Galois extension in the splitting field of η such that the open normal subgroup $I_{K'}$ in I_K is contained in $\ker(\eta|_{I_K})$. That is, $\eta|_{G_{K'}}$ is unramified. It follows that $\mathbf{C}_K(\eta)^{G_{K'}}$ is a 1-dimensional K'-vector space, and its natural $\mathrm{Gal}(K'/K)$-action is visibly K'-semilinear. Thus, by ordinary Galois descent, $\mathbf{C}_K(\eta)^{G_K} = ((\mathbf{C}_K(\eta)^{G_{K'}})^{\mathrm{Gal}(K'/K)}$ is 1-dimensional over K.

Moving on to degree 1, we first handle the case $\eta = 1$, and then we shall deduce the general case. That is, we first prove $\mathrm{H}^1(G_K, \mathbf{C}_K) = 0$. For a finite Galois extension K'/K contained in $\overline{K}$, we have the left exact sequence

$$0 \to \mathrm{H}^1(\mathrm{Gal}(K'/K), K') \to \mathrm{H}^1(G_K, \mathbf{C}_K) \to \mathrm{H}^1(G_{K'}, \mathbf{C}_K),$$

so since $\mathrm{H}^1(\mathrm{Gal}(K'/K), K') = 0$ it suffices to treat the case of K' in place of K. Let K_∞/K be the infinitely ramified p-adic cyclotomic extension. By axiom (TS3), we may replace K with some K_n so that $\mathrm{Gal}(K_\infty/K) \simeq \mathbf{Z}_p$ with topological generator γ such that $\gamma - 1$ acts bijectively on the kernel $X \subseteq \widehat{K}_\infty$ of the normalized trace to K. By Proposition 14.3.3 applied to K_∞/K, we have

$$\mathrm{H}^1(G_K, \mathbf{C}_K) \simeq \mathrm{H}^1(\mathrm{Gal}(K_\infty/K), \widehat{K}_\infty) \simeq \mathrm{H}^1(\mathrm{Gal}(K_\infty/K), K) \oplus \mathrm{H}^1(\mathrm{Gal}(K_\infty/K), X),$$

so $\mathrm{H}^1(G_K, \mathbf{C}_K) \simeq \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p, K)$ is 1-dimensional over K.

Now turning to the general case, let K'/K be a finite subextension of the splitting field of η which absorbs the ramification of η, and let $L = K'^{\mathrm{un}}$. If we ignore continuity conditions on cocycles, the Hochschild–Serre spectral sequence provides isomorphisms $\mathrm{H}^n_{\mathrm{alg}}(G_K, \mathbf{C}_K(\eta)) \simeq \mathrm{H}^n_{\mathrm{alg}}(G_{K'}, \mathbf{C}_K(\eta))^{\mathrm{Gal}(K'/K)}$ via restriction for all $n \geqslant 1$ since the $\mathrm{Gal}(K'/K)$-cohomology of a $\mathbf{Q}$-vector space vanishes. Since $G_{K'}$ is open in G_K, it follows that for $n = 1$ this restriction isomorphism induces an isomorphism $\mathrm{H}^1(G_K, \mathbf{C}_K(\eta)) \simeq \mathrm{H}^1(G_{K'}, \mathbf{C}_K(\eta))^{\mathrm{Gal}(K'/K)}$ in continuous cohomology as well. Hence, provided that $\mathrm{H}^1(G_{K'}, \mathbf{C}_K(\eta))$ is a 1-dimensional K'-vector space, the invariants under a K'-semilinear action by $\mathrm{Gal}(K'/K)$ constitute a 1-dimensional K-vector space via usual Galois descent. It is therefore enough to solve the problem for K' in place of K, so we may assume that η is unramified. By completed unramified descent in Lemma 3.2.6, $\widehat{K^{\mathrm{un}}}(\eta) \simeq \widehat{K^{\mathrm{un}}}$ as $\widehat{K^{\mathrm{un}}}$-vector spaces equipped with a semilinear action by $G_k = G_K/I_K$. Thus, $\mathbf{C}_K(\eta) \simeq \mathbf{C}_K$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$. We have already proved that $\mathrm{H}^1(G_K, \mathbf{C}_K)$ is 1-dimensional over K, so we are done. ■

14.4. **Exercises.**

Exercise 14.4.1. Let R be a topological ring, G a topological group equipped with a continuous action on R (i.e., $G \times R \to R$ is continuous). Endow any finite free R-module M with its natural topological module structure using a basis of R (this topology is independent of the basis; why?). Assume M is endowed with a semilinear action by G; we say it is *continuous* if the action map $G \times M \to M$ is continuous. We wish to classify such M (especially with continuous action) in terms of a suitable cohomology set.

(1) Let d be the rank of M and choose an R-basis $\mathbf{e}$ of M. Make no continuity assumption on the action of G. Define the associated function $U_{\mathbf{e}} : G \to \mathrm{GL}_d(R)$ by setting $U_{\mathbf{e}}(g) = (r_{ij}(g))$ where $g(e_j) = \sum r_{ij}(g)e_i$. Prove that $U_{\mathbf{e}}$ is a 1-cocycle and that as we vary through all choices of $\mathbf{e}$ the $U_{\mathbf{e}}$ sweep out exactly a single cohomology class of 1-cocycles. Prove that $U_{\mathbf{e}} : G \to \mathrm{Mat}_d(R)$ is continuous precisely when the action of G on M is continuous, and that in such cases $g \mapsto U_{\mathbf{e}}(g)^{-1}$ is also continuous as a map $G \to \mathrm{Mat}_d(R)$. (Beware that $\mathrm{GL}_d(R)$ may not be a group under the subspace topology from $\mathrm{Mat}_d(R)$; consider $d = 1$ and R an adele ring of a global field!)

(2) There is a natural way to topologize $\mathrm{GL}_d(R)$ so that it is a topological group (recovering the subspace topology when $R^\times$ has continuous inversion with respect to the subspace topology from R). The "bare hands" approach, which is rather artificial-looking, is to observe that although GL_d is not Zariski-closed in Mat_d (viewed as affine R-schemes of finite type), GL_d is Zariski-closed in $\mathrm{Mat}_d \times \mathrm{Mat}_d$ via the anti-diagonal mapping $g \mapsto (g, g^{-1})$. So topologize $\mathrm{GL}_d(R)$ using the subspace topology from $\mathrm{Mat}_d(R)^2$ via the anti-diagonal map, and show it makes $\mathrm{GL}_d(R)$ a topological group. (In case you think that this construction is too ad hoc, see [16, §2] for a more functorial approach.)

   Prove that the G-action on M is continuous if and only if $g \mapsto U_{\mathbf{e}}(g)$ is continuous as a map $G \to \mathrm{GL}_d(R)$. (Consequence for peace of mind: provided we have a 1-cocycle *algebraically*, it is completely safe to use the topology of $\mathrm{Mat}_d(R)$ to track continuity for $U_{\mathbf{e}}$!) Conclude that the pointed set $\mathrm{H}^1(G, \mathrm{GL}_d(R))$ of continuous cohomology is in a natural bijection, *functorially in G and R!*, with the set of isomorphism classes of continuous semilinear representations of G on finite free R-modules of rank d. To what isomorphism class does the trivial cohomology class correspond?

(3) If H is a normal subgroup of G (with the subspace topology), construct and interpret a left-exact sequence of pointed sets
$$1 \to \mathrm{H}^1(G/H, \mathrm{GL}_d(R^H)) \to \mathrm{H}^1(G, \mathrm{GL}_d(R)) \to \mathrm{H}^1(H, \mathrm{GL}_d(R)).$$

(4) Take $G = \mathrm{Gal}(F'/F)$ for a Galois extension of fields and $R = F'$ with the discrete topology and the usual G-action (which is continuous!). Using Galois descent for vector spaces (as in (2.4.3)), infer that $\mathrm{H}^1(\mathrm{Gal}(F'/F), \mathrm{GL}_d(F')) = \{1\}$. Can you adapt the above arguments to work with GL_d replaced by Sp_{2d}? Or SL_d? Or GSp_{2d}? (Don't forget that $R^\times$ may not be a group for the subspace topology of R.)

Exercise 14.4.2. In the ring-theoretic input to the Tate–Sen formalism, prove that if $p \neq 0$ in $\widetilde{\Lambda}$ then $\widetilde{\Lambda}$ is $\mathbf{Z}_p$-flat and the multiplication map $p : \widetilde{\Lambda} \to \widetilde{\Lambda}$ is a topological embedding. Using completeness, show it is even a closed embedding.

This ensures that being a multiple of a specified power of p is a closed condition on $\widetilde{\Lambda}$ (a triviality when $p = 0$ in $\widetilde{\Lambda}$). It is a property that is used all the time without comment when working with various kinds of p-adic rings and dividing by p in the formation of limits.

Exercise 14.4.3. Verify the following properties of v_Λ on $\mathrm{Mat}_d(\widetilde{\Lambda})$ (from Definition 14.1.1) for any $d \geqslant 1$.

(1) Prove that v_Λ on $\mathrm{Mat}_d(\widetilde{\Lambda})$ satisfies analogues of the axioms (1)–(4) for v_Λ on $\widetilde{\Lambda}$ (the case $d = 1$), as well as the completeness axiom and the continuity and "isometry" axioms for the G_0-action through matrix entries. (Beware that we do not claim that $\mathrm{GL}_d(\widetilde{\Lambda})$ with its subspace topology is a topological group, as in the case $d = 1$ this might fail. Compare with the case of adele rings of global fields.)

(2) Consider $M \in \mathrm{GL}_d(\widetilde{\Lambda})$ for which $v_\Lambda(1-M) > 0$. Prove that $v_\Lambda(M) = 0$, $M \in \mathrm{GL}_d(\widetilde{\Lambda})$, and $v_\Lambda(1 - M^{-1}) = v_\Lambda(1 - M)$. (Hint: consider the infinite series $\sum_{n \geqslant 0} (1 - M)^n$.)

Exercise 14.4.4. Let F be a field complete with respect to a nontrivial discrete valuation, and let F'/F be a Galois extension (with possibly infinite degree). Let $\{F'_i\}$ be a directed system of subfields of F' Galois over F with $F' = \cup F'_i$.

(1) Prove that the limit $\varinjlim \mathrm{H}^1(\mathrm{Gal}(F'/F), \mathrm{GL}_d(F'_i))$ (using continuous cochains, as always) is identified with the set of isomorphism classes of continuous semilinear d-dimensional representation spaces W' of $\mathrm{Gal}(F'/F)$ over F' with the property that the $\mathrm{Gal}(F'/F)$-action on W is defined over some F'_i (i.e., $W' = F' \otimes_{F'_i} W'_i$ for some d-dimensional continuous semilinear representation space W'_i over F'_i for $\mathrm{Gal}(F'/F)$).

(2) Deduce that the map $\varinjlim \mathrm{H}^1(\mathrm{Gal}(F'/F), \mathrm{GL}_d(F'_i)) \to \mathrm{H}^1(\mathrm{Gal}(F'/F), \mathrm{GL}_d(F'))$ is injective.

(3) Assume that there are only countably many F'_i. Use the Baire category theorem (!) to prove that for any profinite group G, a continuous 1-cocycle $G \to \mathrm{GL}_d(F')$ has image contained in some $\mathrm{GL}_d(F'_i)$. (Hint: by continuity the image of a continuous 1-cocycle is compact.) Deduce that the map in (2) is also surjective.

(4) As an example of the preceding part, prove that a continuous homomorphism $\Gamma \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ from a profinite group Γ lands in $\mathrm{GL}_n(K)$ for some finite extension $K/\mathbf{Q}_p$. Using $\Gamma = \mathbf{Z}_p$ and $s \mapsto x^s$ for suitable $x \in 1 + p\mathscr{O}_{\mathbf{C}_p}$, show that this conclusion fails if we replace $\overline{\mathbf{Q}}_p$ with $\mathbf{C}_p$.

(5) Let $L \subseteq F'$ be a subfield containing F such that F'/L is finite (necessarily Galois) and L/F is Galois. Using the vanishing of the pointed set $\mathrm{H}^1(\mathrm{Gal}(F'/L), \mathrm{GL}_d(F'))$ (Galois descent!), deduce that the inflation map

$$\mathrm{H}^1(\mathrm{Gal}(L/F), \mathrm{GL}_d(L)) \to \mathrm{H}^1(\mathrm{Gal}(F'/F), \mathrm{GL}_d(F'))$$

is bijective. In other words, we can replace F' by any such L that we please (but we cannot go all the way down to $\cap L = F$; this H^1 is generally very nontrivial, as we shall see in Sen's situation).

15. p-adic representations and formal linear differential equations

The "classical" Sen theory is the following setup. Let $\psi : G_K \to \mathbf{Z}_p^\times$ be an infinitely ramified character, K_∞/K its splitting field, $\Gamma = \mathrm{Gal}(K_\infty/K)$, and $H = G_{K_\infty}$. Finally, as usual, let $K_n = \ker(\psi \bmod p^n)$. We have checked in §14.1 that this situation satisfies the Tate–Sen axioms, and now we shall deduce two kinds of consequences: a descent and decompletion result that sets up an equivalence between $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ and $\mathrm{Rep}_{K_\infty}(\Gamma_K)$, and the theory of the Sen operator (which generalizes the Hodge–Tate decomposition to arbitrary continuous finite-dimensional $\mathbf{C}_K$-semilinear representations of G_K, and provides a link between p-adic representations and p-adic differential equations).

15.1. **Classical Sen theory.** The first step toward a functorial theory of descent and decompletion for $\mathbf{C}_K$-semilinear representations of G_K is to work at the level of isomorphism classes of objects:

Theorem 15.1.1. *The natural inflation map* $\mathrm{H}^1\big(\Gamma, \mathrm{GL}_d(K_\infty)\big) \to \mathrm{H}^1\big(G_K, \mathrm{GL}_d(\mathbf{C}_K)\big)$ *is bijective for all* $d \geqslant 1$.

Proof. By Theorem 14.2.8, the natural map

$$\varinjlim_L \varinjlim_n \big(\mathrm{Gal}(L_\infty/K), \mathrm{GL}_d(L_n)\big) \to \mathrm{H}^1\big(G_K, \mathrm{GL}_d(\mathbf{C}_K)\big)$$

is bijective (with L running over the finite Galois extensions of K inside of $\overline{K}$). Exercise 14.4.4 identifies the left side with $\mathrm{H}^1(\Gamma, \mathrm{GL}_d(K_\infty))$ compatibly with inflation maps. ■

A more precise version of Theorem 15.1.1 is:

Theorem 15.1.2. *Let W be a continuous semilinear $\mathbf{C}_K$-representation of G_K of dimension $d \geqslant 1$. There exists a unique G_K-stable d-dimensional K_∞-subspace $\mathrm{D}_{\mathrm{Sen}}(W)$ in W on which H acts trivially and for which $\mathbf{C}_K \otimes_{K_\infty} \mathrm{D}_{\mathrm{Sen}}(W) = W$ (i.e., a K_∞-basis of $\mathrm{D}_{\mathrm{Sen}}(W)$ is a $\mathbf{C}_K$-basis of W).*

Moreover, $W^H = \widehat{K}_\infty \otimes_{K_\infty} \mathrm{D}_{\mathrm{Sen}}(W)$ and $\mathrm{D}_{\mathrm{Sen}}(W)$ descends to a continuous semilinear representation of Γ over some K_n. In particular, $\mathrm{D}_{\mathrm{Sen}}(W) \in \mathrm{Rep}_{K_\infty}(\Gamma)$.

Observe that $\mathrm{D}_{\mathrm{Sen}}(W)$ depends on the subfield K_∞/K inside of $\overline{K}$, but not on the specific infinitely ramified character $\psi : G_K \to \mathbf{Z}_p^\times$ for which it is the splitting field.

Proof. By Theorem 15.1.1 there exists a d-dimensional $D \in \mathrm{Rep}_{K_\infty}(\Gamma)$ for which there is an isomorphism $\mathbf{C}_K \otimes_{K_\infty} D \simeq W$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ (where G_K acts on D through its quotient

Γ). In particular, since H acts trivially on K_∞ and on D, we may use a K_∞-basis of D to compute

$$W^H = \mathbf{C}_K^H \otimes_{K_\infty} D = \widehat{K}_\infty \otimes_{K_\infty} D$$

(with $\mathbf{C}_K^H = \widehat{K}_\infty$ by Proposition 2.1.2).

Observe also that if D' is a d-dimensional K_∞-subspace of W on which H acts trivially and for which the natural map $\mathbf{C}_K \otimes_{K_\infty} D' \to W$ is an isomorphism then a K_∞-basis of D' is a $\mathbf{C}_K$-basis of W. Hence, in such cases the subspace topology on D' from W is its natural topology as a finite-dimensional K_∞-vector space (as K_∞ gets its valuation topology as the subspace topology from $\mathbf{C}_K$). It therefore follows from the continuity of the $\mathbf{C}_K$-semilinear action of G_K on W that the K_∞-semilinear action by $\Gamma = G_K/H$ on D' is continuous for the K_∞-linear topology on D'. In other words, necessarily $D' \in \mathrm{Rep}_{K_\infty}(\Gamma)$.

It remains to show that if $D_1, D_2 \in \mathrm{Rep}_{K_\infty}(\Gamma)$ satisfy $\mathbf{C}_K \otimes_{K_\infty} D_i = W$ then $D_1 = D_2$ inside of W. To descend the equality, we first note that it suffices to descend it to an equality $M \otimes_{K_\infty} D_1 = M \otimes_{K_\infty} D_2$ inside of W for some finite Galois extension M/K_∞ inside of $\mathbf{C}_K$. Indeed, if we can prove such a descent then equivariance with respect to the action of the finite quotient $\mathrm{Gal}(M/K_\infty)$ of $G_{K_\infty} = H$ implies that an equality of K_∞-subspaces of $\mathrm{Gal}(M/K_\infty)$-invariant elements. But such invariant points in $L_\infty \otimes_{K_\infty} D_i$ are exactly the elements in D_i. Since any such M/K_∞ has the form L_∞/K_∞ where L/K is finite Galois, we may replace G_K with any open normal subgroup (perhaps depending on D_1 and D_2).

The 1-cocycles describing the G_K-action on W relative to the $\mathbf{C}_K$-bases arising from K_∞-bases of the D_i's are continuous, and so on a sufficiently small open normal subgroup of G_K these 1-cocycles can be made as uniformly close to 1 as we please. Hence, by replacing G_K with a sufficiently small open normal subgroup we can arrange that both 1-cocycles (arising from D_1 and D_2) satisfying the hypothesis "$v(U_g - 1) \geqslant c$ for all $g \in G_K$" in Theorem 14.2.9 (applied in Sen's situation). The strong form of the uniqueness in that result then says that $D_1 = D_2$ inside of W, as desired. ■

Although we have not yet shown that $\mathrm{D}_{\mathrm{Sen}}$ is functorial, we can establish some elementary properties as if it were a good functor. The third of these properties will useful in the proof of functoriality.

Lemma 15.1.3. *Choose $W_1, W_2 \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ and let $D_i = \mathrm{D}_{\mathrm{Sen}}(W_i) \in \mathrm{Rep}_{K_\infty}(\Gamma)$ as a G_K-stable K_∞-subspace of W_i. For any $D \in \mathrm{Rep}_{K_\infty}(\Gamma)$ let $D_{\mathbf{C}_K}$ denote the corresponding scalar extension $\mathbf{C}_K \otimes_{K_\infty} D$ in $\mathrm{Rep}_{G_K}(\mathbf{C}_K)$. We have the following equalities:*

- $\mathrm{D}_{\mathrm{Sen}}(W_1 \oplus W_2) = D_1 \oplus D_2$ *inside of* $W_1 \oplus W_2 = (D_1 \oplus D_2)_{\mathbf{C}_K}$,
- $\mathrm{D}_{\mathrm{Sen}}(W_1 \otimes_{\mathbf{C}_K} W_2) = D_1 \otimes_{K_\infty} D_2$ *inside of* $W_1 \otimes_{\mathbf{C}_K} W_2 = (D_1 \otimes_{K_\infty} D_2)_{\mathbf{C}_K}$.
- $\mathrm{D}_{\mathrm{Sen}}(\mathrm{Hom}_{\mathbf{C}_K}(W_1, W_2)) = \mathrm{Hom}_{K_\infty}(D_1, D_2)$ *inside of the common* $\mathbf{C}_K$*-vector space* $\mathrm{Hom}_{\mathbf{C}_K}(W_1, W_2) = \mathrm{Hom}_{K_\infty}(D_1, D_2)_{\mathbf{C}_K}$,

Proof. In each case, if we let D denote the right side of the desired equality then there is a natural $\mathbf{C}_K$-linear G_K-equivariant identification of $D_{\mathbf{C}_K}$ (using Γ-action on D) with the object $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ for which the left side is written as $\mathrm{D}_{\mathrm{Sen}}(W)$. The uniqueness in Theorem 15.1.2 therefore implies that under this identification $D = \mathrm{D}_{\mathrm{Sen}}(W)$. ■

In the case of Hom-spaces, the identification in the lemma says exactly that the natural $\widehat{K}_\infty$-linear Γ-equivariant map
(15.1.1)
$$\widehat{K}_\infty \otimes_{K_\infty} \mathrm{Hom}_{K_\infty}(\mathrm{D}_{\mathrm{Sen}}(W_1), \mathrm{D}_{\mathrm{Sen}}(W_2)) \to \mathrm{Hom}_{\mathbf{C}_K}(W_1, W_2)^H = \mathrm{Hom}_{\mathbf{C}_K[H]}(W_1, W_2)$$

defined by $a \otimes T \mapsto (c \otimes w_1 \mapsto ac \otimes T(w_1))$ is an isomorphism. Now pass to the Γ-invariants on both sides. On the right side we get the K-vector space $\mathrm{Hom}_{\mathbf{C}_K[G_K]}(W_1, W_2)$. On the left side get the K-vector space of Γ-invariants in $\widehat{K}_\infty \otimes_{K_\infty} \mathrm{Hom}_{K_\infty}(D_1, D_2)$, where $D_i = \mathrm{D}_{\mathrm{Sen}}(W_i)$. This space of Γ-invariants certainly contains $\mathrm{Hom}_{K_\infty[\Gamma]}(D_1, D_2)$, but is it any bigger? If not, then we will have established that $\mathrm{D}_{\mathrm{Sen}}$ is actually functorial, and even fully faithful as such. More generally, we have:

Proposition 15.1.4. *For any $D \in \mathrm{Rep}_{K_\infty}(\Gamma)$, $(\widehat{K}_\infty \otimes_{K_\infty} D)^\Gamma = D^\Gamma$. In particular, $\mathrm{D}_{\mathrm{Sen}}$ is a fully faithful functor from $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ to $\mathrm{Rep}_{K_\infty}(\Gamma)$.*

Proof. Let $\mathbf{e} = \{e_1, \dots, e_d\}$ be a K_∞-basis of D, and consider $x \in (\widehat{K}_\infty \otimes_{K_\infty} D)^\Gamma$. Thus, $x = \sum x_i e_i$ with $x_i \in \widehat{K}_\infty$, so the column vector $\vec{x}$ of x_i's in $\widehat{K}_\infty^d$ satisfies $U_\gamma \cdot \gamma(\vec{x}) = \vec{x}$ for all $\gamma \in \Gamma$. By replacing K with K_N for a sufficiently large N, we can arrange that $v(U_\gamma^{-1} - 1) > c_3$ for all $\gamma \in \Gamma$. Thus, we can apply Lemma 14.2.3 with $U_1 = U_\gamma^{-1}$ and $U_2 = 1$ to get that $\vec{x} \in K_n^d$ for some $n \geqslant 1$. Hence, $x_i \in K_\infty$ for all i, so $x \in D$ as desired. ■

The following theorem is an important generalization of Proposition 15.1.4, and it gives a Galois-theoretic characterization of the "decompletion" D inside of $\widehat{K}_\infty \otimes_{K_\infty} D$.

Theorem 15.1.5. *For any $D \in \mathrm{Rep}_{K_\infty}(\Gamma)$, D is the K_∞-subspace of points $x \in \widehat{D}$ whose Γ-orbit has K-span with finite K-dimension.*

Proof. We first check that all Γ-orbits in D have K-span with finite K-dimension. We may assume $D \neq 0$ and consider the Γ-orbit of a nonzero $x \in D$. We choose a K_∞-basis $\{x = e_1, \dots, e_d\}$. The Γ-action on D is described relative to this basis by a continuous 1-cocycle $U : \Gamma \to' \mathrm{GL}_d(K_\infty)$. By Exercise 14.4.1, this lands in $\mathrm{GL}_d(K_n)$ for some n. Hence, the K_n-span of the e_i's is Γ-stable, and so this is a finite-dimensional K-subspace of D containing the Γ-orbit of $x = e_1$.

It remains to show that any point in $\widehat{D}$ whose Γ-orbit has finite-dimensional K-span must lie in D. To prove this it is harmless to replace K with any K_n and Γ with the corresponding open subgroup. In particular, we can arrange that $\Gamma \simeq \mathbf{Z}_p$. Pick a K_∞-basis $\mathbf{e} = \{e_i\}$ of D, so it is also a $\widehat{K}_\infty$-basis of $\widehat{D}$. As we have seen above, the Γ-action on the e_i's is described by matrices in $\mathrm{GL}_d(K_n)$ for some n, and so by replacing K with such a K_n we may assume that the 1-cocycle is valued in $\mathrm{GL}_d(K)$.

For any $x \in \widehat{D}$, consider the unique expansion $x = \sum c_i e_i$ with $c_i \in \widehat{K}_\infty$. For any $\gamma \in \Gamma$ we have $\gamma(e_i) = \sum u_{ji}(\gamma) e_i$ with $(u_{ij}(\gamma)) \in \mathrm{GL}_d(K)$, so

$$\gamma(x) = \sum \gamma(c_i)\gamma(e_i) = \sum_j \big(\sum_i u_{ji}(\gamma) \cdot \gamma(c_i) \big) e_j.$$

Hence, the $\gamma(c_i)$'s are a K-linear combination of the $\mathbf{e}$-coefficients of $\gamma(x)$. In particular, the Γ-orbit of the $\mathbf{e}$-coefficients of x is contained in the K-span of the $\mathbf{e}$-coefficients of the points in the Γ-orbit of x.

Now assume that the Γ-orbit of x has finite-dimensional K-span D_0 inside of $\widehat{D}$. We aim to prove that in such cases, all c_i lie in K_∞ (as then $x \in \sum K_\infty e_i = D$, as desired). The first step is to show that these coefficients satisfy the finiteness hypothesis relative to $\widehat{K}_\infty$ that x does relative to $\widehat{D}$; that is, we will reduce our problem to the special case $D = K_\infty$.

The $\mathbf{e}$-coefficients of *all* points in D_0 are collectively contained in a finite-dimensional K-subspace of $\widehat{K}_\infty$ (namely, contained in the K-span of the $\mathbf{e}$-coefficients of a K-basis of D_0). In particular, the $\mathbf{e}$-coefficients of all of the points $\gamma(x)$ (as we vary $\gamma \in \Gamma$) are contained in a finite-dimensional K-subspace of $\widehat{K}_\infty$. But we have already seen that the Γ-orbit of each c_i is in the K-span of the $\mathbf{e}$-coefficients of all points $\gamma(x)$. Hence, the Γ-orbit of each c_i has finite-dimensional K-span inside of $\widehat{K}_\infty$.

We have now reduced ourselves to the special case $D = K_\infty$. Since the K-span of the Γ-orbit of a point in $\widehat{K}_\infty$ is visibly Γ-stable, it is equivalent to prove that any Γ-stable K-subspace $W \subseteq \widehat{K}_\infty$ with finite K-dimension is contained in K_∞. (This property makes sense without requiring as we do that Γ is 1-dimensional as a p-adic Lie group, but Tate gave counterexamples in such wider generality with K_∞/K replaced with Galois extensions M/K for which $\mathrm{Gal}(M/K)$ is a higher-dimensional p-adic Lie group.) Since $\widehat{K}_\infty \cap \overline{K} = K_\infty$ (Exercise 2.5.1), it suffices to prove that all elements of W are *algebraic* over K.

Recall that we arranged $\Gamma \simeq \mathbf{Z}_p$, so it has a topological generator γ_0. This acts on W over K with some characteristic polynomial that splits over a finite Galois extension K'/K inside of $\overline{K}$. Observe that γ_0 preserves $K'W = \mathrm{image}(K' \otimes_K W \to K'\widehat{K}_\infty = \widehat{K}'_\infty)$ inside of $\widehat{K}'_\infty$, though it generally does not act K'-linearly. But consider the situation after replacing K with K', K_∞ with K'_∞, W with $K'W$, and Γ with the corresponding open subgroup (identified with $\mathrm{Gal}(K'_\infty/K')$) which leaves all elements of K' invariant. In this case γ_0 is replaced with some $\gamma_0^{p^r}$, and since $K'W$ is spanned over K' by W we see that the eigenvalues of $\gamma_0^{p^r}$ viewed as a K'-linear endomorphism of $K'W$ are G_K-conjugates of the p^r-powers of the eigenvalues of γ_0 on W. In particular, by replacing K with K' in this way (which is harmless for the purposes of the algebraicity property that we need to establish) we are reduced to the case when γ_0 acts with a full set of eigenvalues in $K^\times$.

Let $w \in W$ be an eigenvector for the γ_0-action, say with corresponding eigenvalue λ. Since $\gamma_0^{p^a} \to 1$ in Γ as $a \to +\infty$, by continuity of the Γ-action on $\widehat{K}_\infty$ we get $\gamma_0^{p^a} w \to w$. That is, $\lambda^{p^a} w \to w$. Since $w \neq 0$, multiplying by $1/w$ on $\widehat{K}_\infty$ gives $\lambda^{p^a} \to 1$ in $\widehat{K}_\infty$ and hence in K. We conclude that $\lambda \in 1 + \mathfrak{m}_K$. By Lemma 14.1.9, all eigenvalues λ of γ_0 on W are p-power roots of unity (as otherwise $\gamma_0 - \lambda$ acts injectively on $\widehat{K}_\infty$). Thus, there is an r so large that $\lambda^{p^r} = 1$ for all eigenvalues λ of γ_0. By replacing K with K_r and γ_0 with $\gamma_0^{p^r}$, we are brought to the case when γ_0 has all eigenvalues on W equal to 1. That is, $\gamma_0 - 1$ is a nilpotent operator on W. But by (TS3), $\gamma_0 - 1$ acting on $\widehat{K}_\infty = K \oplus X$ has $\gamma_0 - 1$ acting invertibly on X and of course as 0 on K. We conclude that the only vectors on which $\gamma_0 - 1$ acts in a nilpotent manner are those of K, so $W \subseteq K$. (Note that the current K is a finite extension of the original one, so we have not proved anything stronger than initially expected.) This completes the proof of Theorem 15.1.5. ■

Corollary 15.1.6. *The functor* $\mathrm{D}_{\mathrm{Sen}} : \mathrm{Rep}_{\mathbf{C}_K}(G_K) \to \mathrm{Rep}_{K_\infty}(\Gamma)$ *is an equivalence of categories, quasi-inverse to* $\mathbf{C}_K \otimes_{K_\infty} (\cdot)$*. Likewise, the functor of H-invariants is an equivalence* $\mathrm{Rep}_{\mathbf{C}_K}(G_K) \to \mathrm{Rep}_{\widehat{K}_\infty}(\Gamma)$*, quasi-inverse to* $\mathbf{C}_K \otimes_{\widehat{K}_\infty} (\cdot)$*.*

Proof. Combining Theorem 15.1.5 and the discussion following Lemma 15.1.3, we have seen that $\mathrm{D}_{\mathrm{Sen}}$ makes sense as a functor and is fully faithful. By construction it inverts the scalar extension functor, and by Theorem 15.1.2 it is essentially surjective. Theorem 15.1.2 likewise gives that $W = \mathbf{C}_K \otimes_{\widehat{K}_\infty} W^H$ for any $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$, and Proposition 2.1.2 gives that for any $\widehat{D} \in \mathrm{Rep}_{\widehat{K}_\infty}(\Gamma)$ we have

$$(\mathbf{C}_K \otimes_{\widehat{K}_\infty} \widehat{D})^H = \mathbf{C}_K^H \otimes_{\widehat{K}_\infty} \widehat{D} = \widehat{D}.$$

This establishes the other equivalence. ■

To get a deeper understanding of the Γ-action on general objects D in $\mathrm{Rep}_{K_\infty}(\Gamma)$, the main tool is the *Sen operator*. It is the focus of the following result, and will depend in a mild manner on the specific $\psi : G_K \to \mathbf{Z}_p^\times$ that we used at the start.

Theorem 15.1.7 (Sen)**.** *For each $D \in \mathrm{Rep}_{K_\infty}(\Gamma)$ there is a unique K_∞-linear endomorphism $\Theta_D = \Theta_{D,\mathrm{Sen}}$ on D such that for all $x \in D$ we have*

$$\gamma(x) = \exp\big(\log(\psi(\gamma)) \cdot \Theta_{D,\mathrm{Sen}}\big)(x) \tag{15.1.2}$$

for all γ in an open subgroup Γ_x of Γ.

Remark 15.1.8. Before we prove the theorem, it will be helpful to make several remarks on what the theorem means.

(1) It may look like a semilinear K_∞-analogue of Grothendieck's Theorem 8.2.4 in the ℓ-adic case, but it really is not since (i) $\Theta_{D,\mathrm{Sen}}$ is not a nilpotent operator, (ii) (15.1.2) only holds for γ in a neighborhood of 1 depending on x.. If D were a linear representation space rather a semilinear one over K_∞ then the formula (15.1.2) with the factor $\log \psi(\gamma)$ removed would reflect nothing more or less than the fact that Γ is a 1-dimensional p-adic Lie group (so the mapping $\Gamma \to \mathrm{GL}(D)$ would be its own "1-parameter subgroup", up to the fact that the coefficient field K_∞ is not complete). The interesting thing is that we can get such a formula even though the action is just semilinear.
(2) It may initially seem that the formulas for $\gamma(cx)$ and $\gamma(x)$ are inconsistent for $x \in D$ and $c \in K_\infty$, as the right side depends K_∞-linearly on x whereas the left side is merely Γ-semilinear. But recall that (15.1.2) is only valid for γ in a neighborhood of 1 depending on x. For any particular $c \in K_\infty$ we have $c \in K_n$ for some large n, or in other words $\gamma(c) = c$ if we take γ near enough to 1 in Γ.
(3) Next, we address the meaning of $\exp(cT)$ as in the statement of the result (with $c \in K_\infty$ and $T : D \to D$ a linear self-map). First of all, if we pick a K_∞-basis of D then the matrix for T involves only finitely many elements of K_∞, so they all occur in some K_n and hence the convergence issues really take place over the field K_n which

has the virtue (unlike K_∞!) of being complete. Hence, the convergence aspect is insensitive to the fact that K_∞ is not complete. But does convergence make sense even at the level of some K_n-model for the situation?

Over a p-adic field (or really any complete non-archimedean extension field over $\mathbf{Q}_p$) the p-adic exponential on endomorphisms of a finite-dimensional vector space D has finite (positive) radius of convergence near 0, so for any linear endomorphism T of D if we take γ sufficiently near to 1 (depending on T) then $\log(\psi(\gamma))T$ can be made as close as we wish to 0. Thus, $\exp(\log(\psi(\gamma))T)$ always makes sense for γ close to 1, depending on T. (This is not the only reason why Γ_x depending on x intervenes in the theorem, as the proof will show.)

Now that we have parsed the meaningfulness of the theorem, we are ready to prove it.

Proof. First we check uniqueness. In general, if $T, T' : D \rightrightarrows D$ are linear endomorphisms and $\exp(\log\psi(\gamma)\cdot T) = \exp(\log\psi(\gamma)\cdot T')$ for all $\gamma \in \Gamma$ sufficiently near to 1, then we claim that $T = T'$. To see this, we observe that for γ near enough to 1 (depending on T and T') this common exponential is as close to 1 as we please. Hence, for such γ we can apply the p-adic logarithm; as with the exponential, there are no convergence problems despite working over the non-complete field K_∞ since the relevant calculations all occur over some K_n with n sufficiently large. This gives $\log\psi(\gamma)\cdot T = \log\psi(\gamma)\cdot T'$ for all γ sufficiently near to 1 in Γ, and choosing such γ for which $\psi(\gamma) \neq 1$ but $\psi(\gamma)$ is near to 1 (e.g., $\psi(\gamma) \equiv 1 \bmod p^2$) ensures that $\log\psi(\gamma) \neq 0$. Thus, $T = T'$. The proves uniqueness.

Now we prove existence of the Sen operator. Pick a K_∞-basis $\mathbf{e} = (e_1, \dots, e_d)$ of D, and let $U\colon \Gamma \to \mathrm{GL}_d(K_\infty)$ be the continuous 1-cocycle describing the action of Γ relative to $\mathbf{e}$. By Exercise 14.4.4(3), U has image contained in $\mathrm{GL}_d(K_n)$ for some n. Also, by continuity of the 1-cocycle, for $\gamma \in \Gamma$ sufficiently near 1 we have $v(U_\gamma - 1) \geqslant c > 0$ with a fixed constant c (e.g., $c = 1$). Consider such γ with $n(\gamma) \geqslant n$; i.e., γ acts trivially on K_n. This cuts out an open subgroup Γ' in Γ acting trivially on K_n, and by shrinking Γ' some more we may arrange that also $\Gamma' \simeq \mathbf{Z}_p$. On this open subgroup subgroup the 1-cocycle condition becomes $U_{\gamma_1\gamma_2} = U_{\gamma_1}U_{\gamma_2}$ in $\mathrm{GL}_d(K_n)$ for all $\gamma_1, \gamma_2 \in \Gamma'$, so $\gamma \mapsto U_\gamma$ is a continuous linear representation of Γ' over K_n . In particular, if $x = \sum c_i e_i$ with all $c_i \in K_n$ then for all $\gamma \in \Gamma'$ the point $\gamma(x) = \sum c_i\gamma(e_i)$ has $\mathbf{e}$-coordinates $U_\gamma(\mathbf{c})$ where $\mathbf{c} = (c_1, \dots, c_d) \in K_n^d$.

For $\gamma \in \Gamma'$, the sum

$$\log(U_\gamma) := \sum_{m \geqslant 1} \frac{(-1)^{m-1}}{m}(U_\gamma - 1)^m \in \mathrm{GL}_d(K_n)$$

converges and depends continuously on γ since U is a continuous map and $v(U_\gamma - 1)$ is uniformly bounded away from 0 on Γ'. It is easy to check that $\log(U_{\gamma\gamma'}) = \log(U_\gamma) + \log(U_{\gamma'})$ for all $\gamma, \gamma' \in \Gamma'$, so in particular $\log(U_{\gamma^n}) = n\log(U_\gamma)$ for any $\gamma \in \Gamma'$ and $n \in \mathbf{Z}$. But we also have $\log(\psi(\gamma^n)) = n\log(\psi(\gamma))$ for any $\gamma \in \Gamma'$ and $n \in \mathbf{Z}$. Hence, if we pick a topological generator γ_0 of $\Gamma' \simeq \mathbf{Z}_p$ then for any nonzero $n \in \mathbf{Z}$ we have $\log(\psi(\gamma_0^n)) \neq 0$ and the ratio $\log(U_{\gamma_0^n})/\log(\psi(\gamma_0^n))$ is independent of n. That is, the continuous map $\Gamma' - \{1\} \to \mathrm{GL}_d(K_n)$ defined by $\gamma \mapsto \log(U_\gamma)/\log(\psi(\gamma))$ is *constant* on the dense subset of γ_0^n's with $n \in \mathbf{Z} - \{0\}$, whence it is constant. (Note that $\log(\psi(\gamma)) \neq 0$ for any $\gamma \in \Gamma' - \{1\}$.)

We have proved that it is well-posed to define

$$\Theta_{\mathrm{Sen},D} = \frac{\log(U_\gamma)}{\log(\psi(\gamma))} \in \mathrm{Mat}_d(K_n) \subseteq \mathrm{Mat}_d(K_\infty)$$

for *any* $\gamma \in \Gamma' - \{1\}$. We then have

$$\log(U_\gamma) = \log(\psi(\gamma)) \cdot \Theta_{\mathrm{Sen},D}$$

for all $\gamma \in \Gamma'$, even $\gamma = 1$ (since $U_1 = 1$ by the cocycle relation). By shrinking Γ' a bit more in order that exponentiation of both sides makes sense and $\exp(\log(U_\gamma)) = U_\gamma$, we get $U_\gamma = \exp(\log(\psi(\gamma)) \cdot \Theta_{\mathrm{Sen},D})$ in $\mathrm{Mat}_d(K_n)$. For any $x \in \sum_i K_n e_i$, applying U_γ to the vector of $\mathbf{e}$-coordinates of x in K_n yields the $\mathbf{e}$-coordinates of $\gamma(x)$. In more intrinsic terms, we have prove (15.1.2) for all $x \in \sum K_n e_i$. To handle $x \in \sum K_m e_i$ for $m \geqslant n$ we simply shrink Γ' some more so that it acts trivially on the K_m coefficients. This gives Γ_x depending on x, as each specific $x \in D = \sum K_\infty e_i$ lies in $\sum K_m e_i$ for some large $m \geqslant n$ (depending on x and the choice of $\mathbf{e}$). This proves the existence. ■

Fix $x \in D$, and view the formula (15.1.2) as an identity of continuous maps $\Gamma_x \to D$. By differentiating it at $\gamma = 1$ we arrive at the formula

$$\Theta_{\mathrm{Sen},D}(x) = \lim_{\gamma \to 1} \frac{\gamma(x) - x}{\log(\psi(\gamma))} \tag{15.1.3}$$

in D. (It is not obvious a-priori that this limit exists.) This formula also makes explicit how the Sen operator changes if we change the initial choice of ψ (the effect is a constant scaling); the notation $\Theta_{\mathrm{Sen},D,\psi}$ would be more accurate (though ψ is fixed throughout Sen's theory). The traditional case is to take $K_\infty = K(\mu_{p^\infty})$ and ψ to be the p-adic cyclotomic character.

Remark 15.1.9. In view of the property in Theorem 15.1.7 that uniquely characterizes the Sen operator, or by using (15.1.3), this operator is unaffected by replacing K with a finite extension L inside of $\overline{K}$ (and replacing D with $L \otimes_K D$, due to Exercise 15.5.2).

Corollary 15.1.10. *For $D \in \mathrm{Rep}_{K_\infty}(\Gamma)$, the K_∞-linear operator $\Theta_D = \Theta_{\mathrm{Sen},D}$ satisfies the following properties.*

1. *The operator Θ_D is Γ-equivariant, and it is functorial in D. In particular, its characteristic polynomial has coefficients in K.*
2. *The kernel $\ker \Theta_D$ is equal to $K_\infty \otimes_K D^\Gamma$ and consists of precisely those $x \in D$ whose Γ-orbit is finite (i.e., it is the maximal K_∞-subspace of D on which the Γ-action is discrete). In particular, Θ_D is an isomorphism if and only if $D^\Gamma = 0$, $\Theta_D = 0$ if and only if D has discrete Γ-action, and $\dim_K D^\Gamma \leqslant \dim_{K_\infty} D$ with equality if and only if Θ_D is an isomorphism.*

Proof. Since Γ is commutative, the Γ-equivariance of Θ_D follows from (15.1.3). The functoriality follows from (15.1.3) as well.

Now we consider (2). First observe that since Γ acts on K_∞ with finite orbits, we can see just from definitions that the set of $x \in D$ with a finite Γ-orbit is a K_∞-subspace. By continuity of the Γ-action on D, a point $x \in D$ has a finite Γ-orbit if and only if an open subgroup of Γ fixes x. Equivalently, x has a finite Γ-orbit if and only if for all $\gamma \in \Gamma$ sufficiently near 1 (and at least requiring $\gamma \in \Gamma_x$), $\exp(\log(\psi(\gamma))\Theta_D)(x) = x$. If $\Theta_D(x) = 0$

then this latter identity certainly holds for all $\gamma \in \Gamma_x$. Conversely, consider $x \in D$ with a finite Γ-orbit. We have $\gamma(x) = x$ for all $\gamma \in \Gamma_x$ that are sufficiently near to 1, so by the limit formula (15.1.3) we see $\Theta_D(x) = 0$.

We have shown that $D' := \ker \Theta_D$ is the K_∞-subspace of vectors with discrete Γ-action. In particular, it contains the K-subspace D^Γ, so $D'^\Gamma = D^\Gamma$. But in view of the discreteness of the semilinear action of $\Gamma = \mathrm{Gal}(K_\infty/K)$ on D', we can apply classical Galois descent to conclude that $D' = K_\infty \otimes_K D'^\Gamma$. ■

Here is an analogue for Θ_{Sen} of the compatibility properties in Lemma 15.1.3 for $\mathrm{D}_{\mathrm{Sen}}$. The formulas are exactly as in the theory of Lie algebra representations, which makes sense since (15.1.2) shows that the Sen operator is like a derivative at $\gamma = 1$ for a representation.

Lemma 15.1.11. *For $D_1, D_2 \in \mathrm{Rep}_{K_\infty}(\Gamma)$, the Sen operators on $D_1 \oplus D_2$, $D_1 \otimes_{K_\infty} D_2$, and $\mathrm{Hom}_{K_\infty}(D_1, D_2)$ are respectively given by:*

- $\Theta_{D_1 \oplus D_2} = \Theta_{D_1} \oplus \Theta_{D_2}$,
- $\Theta_{D_1 \otimes D_2} = \Theta_{D_1} \otimes 1 + 1 \otimes \Theta_{D_2}$,
- $\Theta_{\mathrm{Hom}(D_1,D_2)}(T) = \Theta_{D_2} \circ T - T \circ \Theta_{D_1}$. *In particular,* $\Theta_{D^\vee}(\ell) = -\ell \circ \Theta_D = -\Theta_D^\vee(\ell)$.

Proof. In view of the unique characterization, in all cases it suffices to check that the right side "works". The cases of direct sums and tensor products therefore go exactly as in the calculation of ℓ-adic monodromy in the discussion following Definition 8.2.5. To handle $\mathrm{Hom}(D_1, D_2)$, the isomorphism $\mathrm{Hom}(D_1, D_2) \simeq D_2 \otimes D_1^\vee$ reduces this to the case of duals. Finally, to prove $\Theta_D^\vee$ is the Sen operator on $D^\vee$ we again argue as in the case of ℓ-adic monodromy. ■

It is natural to ask how much information is lost by passing from an object $D \in \mathrm{Rep}_{K_\infty}(\Gamma)$ to the associated pair (D, Θ_D) in the category $\mathscr{S}_{K_\infty}$ consisting of a finite-dimensional K_∞-vector spaces equipped with a linear endomorphism (i.e., the category of $K_\infty[X]$-modules with finite K_∞-dimension). There are nontrivial constraints on the possibilities for Θ_D (e.g., its characteristic polynomial must have coefficients in K), so this functor is not essentially surjective in general (but see [40, §2.5] for a description of the essential image when k is algebraically closed). This functor $\mathrm{Rep}_{K_\infty}(\Gamma) \to \mathscr{S}_{K_\infty}$ also cannot be fully faithful since Hom-modules in $\mathrm{Rep}_{K_\infty}(\Gamma)$ are merely K-vector spaces (due to the action of Γ being K_∞-semilinear rather than K-linear) whereas K_∞ acts on everything in $\mathscr{S}_{K_\infty}$. But this discrepancy is easy to explain:

Proposition 15.1.12. *For $D_1, D_2 \in \mathrm{Rep}_{K_\infty}(\Gamma)$, the natural map*

$$K_\infty \otimes_K \mathrm{Hom}_{\mathrm{Rep}_{K_\infty}(\Gamma)}(D_1, D_2) \to \mathrm{Hom}_{\mathscr{S}_{K_\infty}}\left((D_1, \Theta_{\mathrm{Sen},1}), (D_2, \Theta_{\mathrm{Sen},2})\right)$$

is an isomorphism.

Proof. Consider $D = \mathrm{Hom}_{K_\infty}(D_1, D_2)$ as an object in $\mathrm{Rep}_{K_\infty}(\Gamma)$. By Lemma 15.1.11(3), this map is $K_\infty \otimes_K D^\Gamma \to \ker(\Theta_D)$. Hence, it is an isomorphism by Corollary 15.1.10(2). ■

Despite the functor $D \rightsquigarrow (D, \Theta_D)$ not being fully faithful, it at least retains information about isomorphism classes, as follows.

Corollary 15.1.13. *Objects $D_1, D_2 \in \mathrm{Rep}_{K_\infty}(\Gamma)$ are isomorphic if and only if (D_1, Θ_{D_1}) and (D_2, Θ_{D_2}) are isomorphic in $\mathscr{S}_{K_\infty}$.*

Proof. Functoriality of the Sen operator proves the "only if" direction. For the converse, consider $D = \mathrm{Hom}_{K_\infty}(D_1, D_2)$ as an object in $\mathrm{Rep}_{K_\infty}(\Gamma)$ and assume that the (D_i, Θ_{D_i})'s are isomorphic (so $\dim D_1 = \dim D_2$). This says precisely that $\ker \Theta_D$ contains an element of D that is a linear isomorphism. The condition that D_1 and D_2 be isomorphic is precisely the condition that D^Γ contains an element that is a linear isomorphism. By Corollary 15.1.10(2) we have $\ker \Theta_D = K_\infty \otimes_K D^\Gamma$.

Our situation is now an instance of the following. Let F'/F be an algebraic extension of fields with F infinite (such as K_∞/K), and let V_1' and V_2' be finite-dimensional vector spaces over F' with the same dimension (such as D_1 and D_2 over K_∞). Consider a finite-dimensional F-subspace V inside of $\mathrm{Hom}_{F'}(V_1', V_2')$ such that $V' := F' \otimes_F V \to \mathrm{Hom}_{F'}(V_1', V_2')$ is injective (e.g., D^Γ inside of D). We wish to prove that if V' contains an F'-linear isomorphism $V_1' \simeq V_2'$ then V also contains a (possibly different) F'-linear isomorphism. The basic idea is that the isomorphism condition is a Zariski-open condition, and a non-empty Zariski-open locus in an affine space over an infinite field always has a rational point.

To be more precise, by expressing F' as a direct limit of its finite subextensions over F, we reduce to the case when $[F' : F] < \infty$. Now let $\underline{V}$ be the affine space over F corresponding to V, so the base change $\underline{V}_{F'}$ is related in the same way to V'. Suppose there is given a *non-empty* Zariski-open set U' in $\underline{V}'$ (e.g., the overlap of $\underline{V}'$ and the locus of isomorphisms, in the motivating example). We want $U'(F')$ to contain a point which comes from $\underline{V}(F)$. Every non-empty Zariski-open locus in an affine space over an infinite field contains a rational point, so it suffices to show that U' contains the preimage of a non-empty Zariski-open set U in $\underline{V}$. Equivalently, we want the proper Zariski-closed complement $Z' = \underline{V}_{F'} - U'$ to have non-dense image in $\underline{V}$. This follows from dimension and irreducibility considerations, applied to the natural map $\underline{V}_{F'} \to \underline{V}$.

■

We promised at the outset that the Sen operator would allow us to generalize Hodge–Tate decompositions to arbitrary objects in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$. To see how this goes, look at Exercise 15.5.4.

15.2. **Sen theory over B_{dR}^+: the descent step.** Now we prepare to recast the classical Sen theory in a manner that will relate p-adic representations to p-adic linear differential equations, and thereby give an entirely different way of characterizing the de Rham condition; this is due to Fontaine [23, §3]. Briefly, we will replace $\mathbf{C}_K$ in Sen's work with B_{dR}^+, essentially recovering Sen's theory as the t-torsion case (though it must be noted that Sen's work over $\mathbf{C}_K = \mathrm{B}_{\mathrm{dR}}^+/(t)$ will be essential for getting the B_{dR}^+-version off the ground). The differential equations we eventually get will be formal; we will work over formal power series and formal Laurent series rings, rather than over an open disk or open punctured disk. In §16 we will prove the "overconvergence" of p-adic representations, which says that these formal differential equations really converge with positive radius.

The starting point is the observation from Remark 5.2.3 that a p-adic representation V of G_K is de Rham if and only if $B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$ viewed as a semilinear representation space for G_K over B_{dR} is identified with a 'trivial" such object, namely B_{dR}^n (with $n = \dim V$). If V is not de Rham then $B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$ is a rank-n finite free B_{dR}-module equipped with a semilinear G_K-action, but it is not G_K-equivariantly isomorphic to a direct sum of copies of B_{dR}. Thus, it is natural to ask what can be said about the general structure of semilinear G_K-representations on finite-dimensional B_{dR}-vector spaces. The same question with $\mathbf{C}_K$ replacing B_{dR} was the focus of Sen's work in §15.1.

In the work of Tate and Sen, continuity conditions were essential. Unfortunately, B_{dR} has no evident topology compatible with the very useful one on B_{dR}^+. Thus, we will study continuous B_{dR}^+-semilinear representations of G_K. Another way to think about working with B_{dR}^+ rather than B_{dR} is that B_{dR}^+ is a complete discrete valuation ring with residue field $\mathbf{C}_K$, so working over B_{dR}^+ is akin to "lifting" Sen's theory over $\mathbf{C}_K$ up into Fontaine's theory. The proofs will show that this is exactly what happens: many proofs will rest on inductive arguments for which the base of the induction is Sen's work over $\mathbf{C}_K$.

For technical reasons, it will be convenient to consider B_{dR}^+-semilinear representations of G_K on finitely generated B_{dR}^+-modules that may not be free. Just as Sen's work passed between $\mathbf{C}_K$ and $\widehat{K}_\infty$ via the formation of H_K-invariants, we will now try to do the same with B_{dR}^+-coefficients and the subring $\mathrm{L}_{\mathrm{dR}}^+ = (B_{\mathrm{dR}}^+)^{H_K}$. Since we now wish to be in the setting of interest for p-adic Hodge theory, Sen's theory is of interest with the basic infinitely ramified character $\psi : G_K \to \mathbf{Z}_p^\times$ in that theory taken to be the p-adic cyclotomic character χ. Hence, in this section we work with the subfields

$$K_\infty = K(\mu_{p^\infty}), \;\; K_n = K(\zeta_{p^n})$$

in $\overline{K}$ and the groups

$$\Gamma_K = \mathrm{Gal}(K_\infty/K), \;\; H_K = G_{K_\infty} = \ker(G_K \twoheadrightarrow \Gamma_K).$$

Recall from Exercise 4.5.3 that B_{dR}^+ has an interesting topology defined via the identification $B_{\mathrm{dR}}^+ = \varprojlim B_m$, where $B_m = \mathrm{W}(R)[1/p]/\ker(\theta_{\mathbf{Q}})^m$. Explicitly, in Exercise 4.5.3 we topologized B_{dR}^+ using the inverse limit topology from the topology on the B_m's defined by a "decay of negative-degree Witt coordinates" topology on $\mathrm{W}(R)[1/p]$ (which made $\mathrm{W}(R)$ a closed subring having its weak topology from Definition 13.5.4). It was also seen in Exercise 4.5.3 that the action of G_K on B_{dR}^+ is continuous for this topology, the topology is complete (with a countable base of open $\mathrm{W}(R)$-submodules around 0), the residue field $\mathbf{C}_K$ of B_{dR}^+ gets its valuation topology as the quotient topology, and the multiplication map on B_{dR}^+ by any uniformizer (e.g., t) is a closed embedding. Finally, we recall also from Lemma 4.4.10 that B_{dR}^+ has a canonical G_K-equivariant structure of $\overline{K}$-algebra (but recall from Remark 4.4.11 that the structure map from $\overline{K}$ is *not* continuous, wherereas the structure map is continuous on any finite extension of K_0 or $\widehat{K_0^{\mathrm{un}}}$ by Lemma 4.4.10).

Lemma 15.2.1. *The ring of invariants* $\mathrm{L}_{\mathrm{dR}}^+ := (B_{\mathrm{dR}}^+)^{H_K}$ *is a closed* K_∞*-subalgebra of* B_{dR}^+ *that is a complete discrete valuation ring with uniformizer* t *and residue field* $\widehat{K}_\infty$ (*equipped with its valuation topology as the quotient topology*)*. Moreover, the multiplication map* $t : \mathrm{L}_{\mathrm{dR}}^+ \to \mathrm{L}_{\mathrm{dR}}^+$ *is a closed embedding.*

The topological ring $\mathrm{L}_{\mathrm{dR}}^+$ *is separated and complete for its subspace topology from* B_{dR}^+.

The proof of this lemma may seem longer than expected, but it is crucial to keep track of the topological structures as we do in this lemma. Otherwise later considerations would run into a brick wall.

Proof. By continuity of the action of G_K on B_{dR}^+ we see that $\mathrm{L}_{\mathrm{dR}}^+$ is closed. It then follows that $\mathrm{L}_{\mathrm{dR}}^+$ is separated and complete for it subspace topology, as this holds for B_{dR}^+. By definition this subring contains $\overline{K}^{H_K} = K_\infty$, and since G_K acts on t through the character χ whose kernel is H_K we see that $t \in \mathrm{L}_{\mathrm{dR}}^+$. Since t is a uniformizer of B_{dR}^+, it therefore follows from the definition of $\mathrm{L}_{\mathrm{dR}}^+$ that an element of $\mathrm{L}_{\mathrm{dR}}^+$ is a non-unit if and only if it is in $t\mathrm{L}_{\mathrm{dR}}^+$. Thus, $\mathrm{L}_{\mathrm{dR}}^+$ is a discrete valuation ring with uniformizer t.

By the completeness of B_{dR}^+ as a discrete valuation ring, there is a unique local map $K_\infty[\![T]\!] \to B_{\mathrm{dR}}^+$ as K_∞-algebras satisfying $T \mapsto t$, and (in view of the ideal theory of $K_\infty[\![T]\!]$) it is injective. We may therefore identify $K_\infty[\![T]\!]$ with a subring $K_\infty[\![t]\!]$. (This subring is canonical, though t is only canonical up to $\mathbf{Z}_p^\times$-multiple.) By t-adic separatedness of B_{dR}^+, the subring $K_\infty[\![t]\!]$ is contained in $\mathrm{L}_{\mathrm{dR}}^+$.

Since B_{dR}^+ is t-adically separated and complete, with its topology finer than the t-adic one, the subring $\mathrm{L}_{\mathrm{dR}}^+$ that is closed for this finer topology is also closed for the t-adic topology. That is, $\mathrm{L}_{\mathrm{dR}}^+$ is a complete discrete valuation ring. Since $\mathrm{L}_{\mathrm{dR}}^+$ is closed in B_{dR}^+ and the t-multiplication map on B_{dR}^+ is a closed embedding, it follows that t-multiplication on $\mathrm{L}_{\mathrm{dR}}^+$ is also a closed embedding.

It remains to show that $\mathrm{L}_{\mathrm{dR}}^+$ has residue field identified (topologically, using quotient topology) with the subfield $\widehat{K}_\infty$ inside of the residue field $\mathbf{C}_K$ of B_{dR}^+. Note that it is harmless to replace K with K_n for any single n, so we can arrange that K_∞/K is a totally ramified $\mathbf{Z}_p$-extension. The G_K-equivariant projection map $B_{\mathrm{dR}}^+ \twoheadrightarrow \mathbf{C}_K$ of $\mathrm{W}(k)[1/p]$-algebras is a $\overline{K}$-algebra map due to how the $\overline{K}$-algebra structure is defined, so it carries $\mathrm{L}_{\mathrm{dR}}^+$ into $\mathbf{C}_K^{H_K} = \widehat{K}_\infty$ (Proposition 2.1.2). Hence, the residue field $\mathrm{L}_{\mathrm{dR}}^+/(t)$ is a subfield of $\widehat{K}_\infty$ containing K_∞. To show that $\mathrm{L}_{\mathrm{dR}}^+/(t) = \widehat{K}_\infty$ as fields (setting aside the quotient topology aspect for a moment), it suffices to show that $\mathrm{L}_{\mathrm{dR}}^+$ hits $\mathscr{O}_{\widehat{K}_\infty} = \widehat{\mathscr{O}}_{K_\infty}$ under $\theta_{\mathbf{Q}} : B_{\mathrm{dR}}^+ \to \mathbf{C}_K$. This requires constructing "enough" elements in $\mathrm{L}_{\mathrm{dR}}^+$. The theory of perfect norm fields will provide what we need.

The subring $\mathrm{W}(R_{K_\infty})$ inside of $\mathrm{W}(R) \subseteq B_{\mathrm{dR}}^+$ is H_K-invariant and so is contained in $\mathrm{L}_{\mathrm{dR}}^+$. By H_K-equivariance, the map $\theta : \mathrm{W}(R) \to \mathscr{O}_{\mathbf{C}_K}$ carries $\mathrm{W}(R_{K_\infty})$ into $\mathscr{O}_{\mathbf{C}_K}^{H_K} = \mathscr{O}_{\widehat{K}_\infty}$. Moreover, by Corollary 13.3.10 this map

$$\mathrm{W}(R_{K_\infty}) \to \mathscr{O}_{\widehat{K}_\infty} \tag{15.2.1}$$

is surjective modulo $\mathfrak{a}\mathscr{O}_{\widehat{K}_\infty}$ for some nonzero proper ideal $\mathfrak{a}$ of some $\mathscr{O}_{K_n}$. Since $\mathfrak{a}^N \subseteq p\mathscr{O}_{K_n}$ for some $N \geqslant 1$ and H_K acts trivially on the copy of K_∞ inside of B_{dR}^+, by iterating the surjectivity of (15.2.1) finitely many times we see that for some large n, the map

$$\mathscr{O}_{K_n} \cdot \mathrm{W}(R_{K_\infty}) \to \mathscr{O}_{\widehat{K}_\infty}/(p) \tag{15.2.2}$$

is surjective. But $\mathscr{O}_{K_n} \cdot \mathrm{W}(R_{K_\infty})$ is identified with $\mathscr{O}_{K_n} \otimes_{\mathrm{W}(k)} \mathrm{W}(R_{K_\infty})$ due to K_n/K being totally ramified and $W(\mathrm{Frac}(R_{K_\infty}))$ having absolute ramification degree 1 over $\mathrm{W}(k)$. Thus, $\mathscr{O}_{K_n} \mathrm{W}(R_{K_\infty})$ is p-adically separated and complete (here using that R_{K_∞} is perfect). We may therefore use p-adic successive approximation with (15.2.1) to infer that $\mathscr{O}_{K_n}\mathrm{W}(R_{K_\infty})$ is carried onto $\mathscr{O}_{\widehat{K}_\infty}$ by θ. Since K_∞ and $\mathrm{W}(R_{K_\infty})$ are contained in $(B_{\mathrm{dR}}^+)^{H_K} = \mathrm{L}_{\mathrm{dR}}^+$, we finally get the required image

The preceding argument also shows that $\mathrm{L}_{\mathrm{dR}}^+ \to \widehat{K}_\infty$ is an open mapping (when using the valuation topology on $\widehat{K}_\infty$), so the quotient topology on the residue field $\widehat{K}_\infty$ of $\mathrm{L}_{\mathrm{dR}}^+$ is its valuation topology. ■

Exercise 13.7.9(4) applies to both B_{dR}^+ and $\mathrm{L}_{\mathrm{dR}}^+$ as topologized discrete valuation rings (with the finer topologies as discussed above that make their residue fields $\mathbf{C}_K$ and $\widehat{K}_\infty$ acquire the valuation topology as quotient topology) . Hence, finitely generated modules over these rings admit a functorial Hausdorff topological module structure, and Exercise 13.7.9(5) shows that these topologies are compatible with short exact sequences. It is important to check the topological compatibility of inverse limits as well:

Lemma 15.2.2. *Let N be a finitely generated module over the topologized complete discrete valuation ring $A \in \{B_{\mathrm{dR}}^+, \mathrm{L}_{\mathrm{dR}}^+\}$, and endow N with its natural topology. Letting $\mathfrak{m}$ denote the maximal ideal of A, the natural continuous linear bijection*

$$N \to \varprojlim N/\mathfrak{m}^i N$$

is a homeomorphism.

Moreover, for each $n \geqslant 1$ the quotient topology on $\mathrm{L}_{\mathrm{dR}}^+/t^n\mathrm{L}_{\mathrm{dR}}^+$ coincides with its subspace topology from $B_{\mathrm{dR}}^+/t^nB_{\mathrm{dR}}^+$.

Proof. By Exercise 13.7.9(5), for the claim concerning inverse limits it suffices to treat the case $N = A$. For $A = B_{\mathrm{dR}}^+$ the desired compatibility expresses the definition of the topology on B_{dR}^+ as an inverse limit. For $A = \mathrm{L}_{\mathrm{dR}}^+$, the topological identification $B_{\mathrm{dR}}^+ = \varprojlim B_{\mathrm{dR}}^+/t^nB_{\mathrm{dR}}^+$ carries $\mathrm{L}_{\mathrm{dR}}^+$ over to $\varprojlim \mathrm{L}_{\mathrm{dR}}^+/t^n\mathrm{L}_{\mathrm{dR}}^+$. Hence, the problem comes down to the second part of the lemma: checking that for each $n \geqslant 1$, the quotient topology on $\mathrm{L}_{\mathrm{dR}}^+/t^n\mathrm{L}_{\mathrm{dR}}^+$ coincides with its subspace topology from $B_{\mathrm{dR}}^+/t^nB_{\mathrm{dR}}^+$. The case $n = 1$ is part of Lemma 15.2.1.

In general we proceed by induction, so suppose the result is true for some $n \geqslant 1$. We can characterize the two candidate topologies on $\mathrm{L}_{\mathrm{dR}}^+/(t^{n+1})$ (not yet shown to be the same) in terms of convergence of sequences, so suppose $\{x_i\}$ is a sequence in $\mathrm{L}_{\mathrm{dR}}^+/(t^{n+1})$ that converges to 0 in $\mathrm{B}_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+$. We need to prove it conveges to 0 in $\mathrm{L}_{\mathrm{dR}}^+/t^{n+1}\mathrm{L}_{\mathrm{dR}}^+$ with respect to the natural (quotient) topology.

Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{L}_{\mathrm{dR}}^+/t\mathrm{L}_{\mathrm{dR}}^+ \arrow[r, "t^n"] \arrow[d] & \mathrm{L}_{\mathrm{dR}}^+/t^{n+1}\mathrm{L}_{\mathrm{dR}}^+ \arrow[r] \arrow[d] & \mathrm{L}_{\mathrm{dR}}^+/t^n\mathrm{L}_{\mathrm{dR}}^+ \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_{\mathrm{dR}}^+/tB_{\mathrm{dR}}^+) \arrow[r, "t^n"] & B_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+ \arrow[r] & B_{\mathrm{dR}}^+/t^nB_{\mathrm{dR}}^+ \arrow[r] & 0
\end{tikzcd}$$

in which the outer vertical maps are topological embeddings (by induction) and the rows are topologically exact. Since $\{x_i\}$ converges to 0 in the middle term along the bottom, it does so in the lower-right term as well. But the right-most vertical map is a topological embedding, so $\{x_i \bmod t^n\}$ converges to 0 in $\mathrm{L}_{\mathrm{dR}}^+/t^n\mathrm{L}_{\mathrm{dR}}^+$ with its quotient topology. Hence, this lifts to a sequence $\{x_i'\}$ in $\mathrm{L}_{\mathrm{dR}}^+/t^{n+1}\mathrm{L}_{\mathrm{dR}}^+$ that converges to 0.

Since we aim to prove that $x_i \to 0$ in $\mathrm{L}_{\mathrm{dR}}^+/t^{n+1}\mathrm{L}_{\mathrm{dR}}^+$, it is harmless to replace x_i with $x_i - x_i'$ to reduce to the case when $\{x_i\}$ comes from the upper left term in the diagram. But then to check convergence to 0 within that term (which suffices for our needs), we can push the problem into the lower left term of the diagram. This problem can be settled by checking in the middle term along the bottom, where the convergence to 0 was our initial hypothesis. ■

Definition 15.2.3. The category $\mathrm{Rep}_{B_{\mathrm{dR}}^+}(G_K)$ is the category of continuous semilinear representations of G_K on finitely generated B_{dR}^+-modules. The category $\mathrm{Rep}_{\mathrm{L}_{\mathrm{dR}}^+}(\Gamma_K)$ is defined similarly.

We willcompare these categories; Sen's equivalence $\mathrm{Rep}_{\mathbf{C}_K}(G_K) \simeq \mathrm{Rep}_{\widehat{K}_\infty}(\Gamma_K)$ (without the decompletion step) will be the comparison on t-torsion objects.

Proposition 15.2.4. *For any $W \in \mathrm{Rep}_{B_{\mathrm{dR}}^+}(G_K)$, the $\mathrm{L}_{\mathrm{dR}}^+$-module W^{H_K} is finitely generated with a continuous Γ_K-action for its natural topology as a finitely generated $\mathrm{L}_{\mathrm{dR}}^+$-module, and the natural $\mathrm{B}_{\mathrm{dR}}^+$-linear map*

$$\alpha_W : B_{\mathrm{dR}}^+ \otimes_{\mathrm{L}_{\mathrm{dR}}^+} W^{H_K} \to W \tag{15.2.3}$$

is an isomorphism. In particular, the rank and invariant factors of W^{H_K} over $\mathrm{L}_{\mathrm{dR}}^+$ coincide with those of W over B_{dR}^+.

Proof. We initially treat the case when W is a torsion B_{dR}^+-module, and more general cases will be inferred from the torsion case by passage to inverse limits. We will argue by induction on the power of t killing W that (in the torsion case) the natural map α_W is an isomorphism and the continuous cohomology group $\mathrm{H}^1(H_K, W)$ vanishes. The overall method is similar to the proof of completed unramified descent in Lemma 3.2.6, except that Hilbert 90 there has to be replaced with results of Sen and Tate.

First suppose that $tW = 0$, which is to say that $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$. (Here we have used crucially that $\mathbf{C}_K$ gets its natural topology as the quotient topology of B_{dR}^+; that is, the topology put on W through its structure as a finitely-generated B_{dR}^+-module matches its natural topology as a finite-dimensional $\mathbf{C}_K$-vector space! This latter topology is what is used to define the continuity condition for the G_K-action on objects in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$.) In this case the comparison map α_W is the natural map $\mathbf{C}_K \otimes_{\widehat{K}_\infty} W^{H_K} \to W$, and $W^{H_K} = \widehat{K}_\infty \otimes_{K_\infty} \mathrm{D}_{\mathrm{Sen}}(W)$ by Theorem 15.1.2. Thus, the comparison map is $\mathbf{C}_K \otimes_{K_\infty} \mathrm{D}_{\mathrm{Sen}}(W) \to W$, and this is an isomorphism by Theorem 15.1.2. The vanishing of $\mathrm{H}^1(H_K, W)$ is part of Proposition 14.3.3.

For the general torsion case, we assume W is killed by t^{n+1} for some $n \geqslant 1$, with the result known for t^n-torsion objects. Consider the exact sequence

$$0 \to tW \to W \to W/tW \to 0 \tag{15.2.4}$$

in $\mathrm{Rep}_{B_{\mathrm{dR}}^+}(G_K)$. Since the outer terms are killed by t^n (as $n \geqslant 1$), by induction we have $\mathrm{H}^1(H_K, tW) = 0$. Thus, by Exercise 2.5.3, the Γ_K-equivariant sequence of $\mathrm{L}_{\mathrm{dR}}^+$-modules

$$0 \to (tW)^{H_K} \to W^{H_K} \to (W/tW)^{H_K} \to 0$$

is exact. In particular, this shows that W^{H_K} is finitely generated over $\mathrm{L}_{\mathrm{dR}}^+$. We now have the following B_{dR}^+-linear diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{B}_{\mathrm{dR}}^+ \otimes_{\mathrm{L}_{\mathrm{dR}}^+} (tW)^{H_K} \arrow[r] \arrow[d, "\alpha_{tW}"'] & \mathrm{B}_{\mathrm{dR}}^+ \otimes_{\mathrm{L}_{\mathrm{dR}}^+} W^{H_K} \arrow[r] \arrow[d, "\alpha_W"'] & \mathrm{B}_{\mathrm{dR}}^+ \otimes_{\mathrm{L}_{\mathrm{dR}}^+} (W/tW)^{H_K} \arrow[r] \arrow[d, "\alpha_{W/tW}"'] & 0 \\
0 \arrow[r] & tW \arrow[r] & W \arrow[r] & W/tW \arrow[r] & 0
\end{tikzcd}$$

in which the top row is exact because the scalar extension $\mathrm{L}_{\mathrm{dR}}^+ \to B_{\mathrm{dR}}^+$ is flat (as it is an injective map of discrete valuation rings). The maps α_{tW} and $\alpha_{W/tW}$ are isomorphisms by induction, so α_W is also an isomorphism. Moreover, since (15.2.4) is *topologically* exact (especially the left term has the subspace topology from the middle term), we have the exact sequence

$$\mathrm{H}^1(H_K, tW) \to \mathrm{H}^1(H_K, W) \to \mathrm{H}^1(H_K, W/tW)$$

by Exercise 2.5.3. Hence, $\mathrm{H}^1(H_K, W) = 0$ because $\mathrm{H}^1(H_K, tW) = \mathrm{H}^1(H_K, W/tW) = 0$.

We have now settled the general torsion case, and in particular we note that the functor of H_K-invariants is exact on the category of torsion objects, due to either the H^1-vanishing established in that case or because we use the comparison isomorphismm α_W for torsion W and the faithful flatness of B_{dR}^+ over $\mathrm{L}_{\mathrm{dR}}^+$ (as it is a local extension of discrete valuation rings). To establish the isomorphism result and finite generation of W^{H_K} over $\mathrm{L}_{\mathrm{dR}}^+$ in the general case, since W is finitely generated over the complete discrete valuation ring B_{dR}^+ and the map $\mathrm{L}_{\mathrm{dR}}^+ \to B_{\mathrm{dR}}^+$ is an injection between discrete valuation rings with a common uniformizer (such as t), we can carry over the same argument building up from the torsion case as at the end of the proof of Lemma 3.2.6 (beginning at (3.2.1)). ■

Corollary 15.2.5. *If $W \in \mathrm{Rep}_{\mathrm{B}_{\mathrm{dR}}^+}(G_K)$ then the finite $\mathrm{L}_{\mathrm{dR}}^+$-module W^{H_K} equipped with its natural Γ_K-action and natural $\mathrm{L}_{\mathrm{dR}}^+$-module topology has continuous Γ_K-action. The resulting functor*

$$\mathrm{Rep}_{\mathrm{B}_{\mathrm{dR}}^+}(G_K) \to \mathrm{Rep}_{\mathrm{L}_{\mathrm{dR}}^+}(\Gamma_K)$$
$$W \rightsquigarrow W^{\mathrm{H}_K}$$

is an equivalence of categories. A quasi-inverse is given by $X \rightsquigarrow \mathrm{B}_{\mathrm{dR}}^+ \otimes_{\mathrm{L}_{\mathrm{dR}}^+} X$.

Proof. We saw in Proposition 15.2.4 that W^{H_K} is a finitely generated $\mathrm{L}_{\mathrm{dR}}^+$-module, and that the natural comparison morphism

$$\alpha_W : B_{\mathrm{dR}}^+ \otimes_{\mathrm{L}_{\mathrm{dR}}^+} W^{H_K} \to W$$

is an isomorphism (recovering the inclusion of W^{H_K} into W). But rather generally, from Lemma 15.2.1 and Lemma 15.2.2 it follows (by passage to the case of cyclic modules) that for any finitely generated $\mathrm{L}_{\mathrm{dR}}^+$-module M, the natural continuous injective $M \to B_{\mathrm{dR}}^+ \otimes_{\mathrm{L}_{\mathrm{dR}}^+} M$ is a homeomorphism onto its image (using the natural topologies on finitely generated modules over $\mathrm{L}_{\mathrm{dR}}^+$ and B_{dR}^+). Hence, due to the B_{dR}^+-linear isomorphism α_W we see that the subspace topology on W^{H_K} from W *is* its natural topology as a finitely generated $\mathrm{L}_{\mathrm{dR}}^+$-module. The continuity of the G_K-action on W therefore implies the continuity of the Γ_K-action on W^{H_K} relative to the $\mathrm{L}_{\mathrm{dR}}^+$-module topology on W^{H_K}, so the proposed functor indeed makes sense.

Consider any $X \in \mathrm{Rep}_{\mathrm{L}_{\mathrm{dR}}^+}(\Gamma_K)$. Then $V := B_{\mathrm{dR}}^+ \otimes_{\mathrm{L}_{\mathrm{dR}}^+} X$ with its natural topology as a finitely generated B_{dR}^+-module has continuous G_K-action (due to continuity of the G_K-actions on B_{dR}^+ and X, coupled with the description of the topology on X in terms of the structure theorem for finitely generated $\mathrm{L}_{\mathrm{dR}}^+$-modules). Hence, $V \in \mathrm{Rep}_{B_{\mathrm{dR}}^+}(G_K)$. Since the H_K-action on V leaves elements of X invariant, we get $V^{H_K} = X$ by the definition of $\mathrm{L}_{\mathrm{dR}}^+$ (and calculation relative to a cyclic module decomposition of X). Together with the comparison isomorphism from Proposition 15.2.4, we have now shown that the functors in both directions are quasi-inverse to each other. ■

15.3. **Sen theory over B_{dR}^+: decompletion.** Now we explain Fontaine's B_{dR}^+-version of Sen's decompletion process. Inspired by Theorem 15.1.5 we are led to the following definition.

Definition 15.3.1. For $X \in \mathrm{Rep}_{\mathrm{L}_{\mathrm{dR}}^+}(\Gamma_K)$, if X is torsion (equivalently, killed by a power of t) then define X_{f} to be the directed union of Γ_K-stable finite-dimensional K-subspaces of X, and give it the subspace topology from X. In general, define $X_{\mathrm{f}} = \varprojlim (X/t^m X)_{\mathrm{f}}$ and give it the subspace topology from $X = \varprojlim X/t^m X$. topologies.

Remark 15.3.2. Since $X = \varprojlim X/t^m X$ topologically, by Lemma 15.2.2, the topology on X_{f} in general is also the inverse limit topology from the $(X/t^m X)_{\mathrm{f}}$'s.

Note that if X is killed by t^m then X_{f} is a $K_\infty[t]/(t^m)$-module. Hence, in general X_{f} is a $K_\infty[\![t]\!]$-module, and so we may view $X \rightsquigarrow X_{\mathrm{f}}$ as a functor from $\mathrm{Rep}_{\mathrm{L}_{\mathrm{dR}}^+}(\Gamma_K)$ to the category of $K_\infty[\![t]\!]$-modules. The subring $K_\infty[\![t]\!] \subseteq \mathrm{L}_{\mathrm{dR}}^+$ is regarded as a decompletion, and we give it the subspace topology from $\mathrm{L}_{\mathrm{dR}}^+$. Since t is ambiguous only up to $\mathbf{Z}_p^\times$-multiple, the topology viewed on the abstract ring $K_\infty[\![T]\!]$ is independent of the choice of t; we will not comment on this again.

Exactly as in our analysis of the subspace topology on $\mathrm{L}_{\mathrm{dR}}^+$ from B_{dR}^+ in Lemma 15.2.1, we get the following analogous result relating the topologies on $\mathrm{L}_{\mathrm{dR}}^+$ and $K_\infty[\![t]\!]$.

Lemma 15.3.3. *The topological ring structure on $K_\infty[\![t]\!]$ makes the Γ_K-action continuous, t-multiplication a closed embedding, and the residue field K_∞ have its valuation topology the quotient topology. Moreover, on each $K_\infty[\![t]\!]/t^m K_\infty[\![t]\!]$ the quotient topology is the subspace topology from $\mathrm{L}_{\mathrm{dR}}^+/t^m\mathrm{L}_{\mathrm{dR}}^+$ (equivalently, from $B_{\mathrm{dR}}^+/t^m B_{\mathrm{dR}}^+$), and when finitely generated $K_\infty[\![t]\!]$-modules are topologized in accordance with Exercise* 13.7.9(4), (5) *the conclusion of Lemma* 15.2.2 *applies.*

In particular, the identification $K_\infty[\![t]\!] \simeq \varprojlim K_\infty[\![t]\!]/t^m K_\infty[\![t]\!]$ of K_∞-algebras is a homeomorphism.

Example 15.3.4. Consider the unit object $X = \mathrm{L}_{\mathrm{dR}}^+$ in $\mathrm{Rep}_{\mathrm{L}_{\mathrm{dR}}^+}(\Gamma_K)$. Then

$$(X/t^m X)_{\mathrm{f}} = (\mathrm{L}_{\mathrm{dR}}^+/t^m \mathrm{L}_{\mathrm{dR}}^+)_{\mathrm{f}} \supseteq K_\infty[\![t]\!]/t^m K_\infty[\![t]\!], \tag{15.3.1}$$

and we claim that this containment is an equality. In particular, passing to the limit, we would get $X_{\mathrm{f}} = K_\infty[\![t]\!]$ with the subspace topology from $\mathrm{L}_{\mathrm{dR}}^+$ (equivalently, from B_{dR}^+).

For $m = 1$, the assertion that (15.3.1) is an equality says exactly that $(\widehat{K}_\infty)_{\mathrm{f}} \stackrel{?}{=} K_\infty$, a property that was established in the proof of Theorem 15.1.5. In general we proceed by induction on m as follows. Granting the result for $m \geqslant 1$, we consider $x \in (\mathrm{L}_{\mathrm{dR}}^+/t^{m+1}\mathrm{L}_{\mathrm{dR}}^+)_{\mathrm{f}}$. By induction $x \bmod t^m \in K_\infty[\![t]\!]/t^m K_\infty[\![t]\!]$ inside of $(\mathrm{L}_{\mathrm{dR}}^+/t^m\mathrm{L}_{\mathrm{dR}}^+)_{\mathrm{f}}$. Thus, if $\widehat{x} \in K_\infty[t]/t^{m+1}K_\infty[\![t]\!]$ is a lift of $x \bmod t_m$ then by replacing x with $x - \widehat{x}$ we are brought to the case $x \in t^m\mathrm{L}_{\mathrm{dR}}^+/t^{m+1}\mathrm{L}_{\mathrm{dR}}^+)_{\mathrm{f}}$. Writing $x = t^m x_0$ with $x_0 \in \mathrm{L}_{\mathrm{dR}}^+/t\mathrm{L}_{\mathrm{dR}}^+ = \widehat{K}_\infty$, since Γ_K acts on t through a $\mathbf{Z}_p^\times$-valued character we see that t^m-multiplication carries the Γ_K-orbit of x_0 bijectively onto the Γ_K-orbit of x. Thus, $x_0 \in (\widehat{K}_\infty)_{\mathrm{f}} = K_\infty$, so the induction is complete.

Example 15.3.5. Consider a t-torsion $X \in \mathrm{Rep}_{\mathrm{L}_{\mathrm{dR}}^+}(\Gamma_K)$, so equivalently $X \in \mathrm{Rep}_{\widehat{K}_\infty}(\Gamma_K)$. (Note once again that there is no topological problem here, precisely because the quotient topology on $\widehat{K}_\infty$ as the residue field of $\mathrm{L}_{\mathrm{dR}}^+$ is the valuation topology.) Let $W = \mathbf{C}_K \otimes_{\widehat{K}_\infty} X$ be the corresponding object in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ under the equivalence in Corollary 15.1.6, so $X = W^{H_K}$.

In such cases, by Theorem 15.1.2 and Theorem 15.1.5 we have $X_{\mathrm{f}} = \mathrm{D}_{\mathrm{Sen}}(W)$, which is finite-dimensional over K_∞ with subspace topology equal to its natural topology as a finite-dimensional K_∞-vector space. Theorem 15.1.2 also gives that the natural map $\widehat{K}_\infty \otimes_{K_\infty} X_{\mathrm{f}} \to X$ is an isomorphism.

In view of Lemma 15.3.3, we are motivated to make the following definition:

Definition 15.3.6. The category $\mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)$ consists of finitely generated $K_\infty[\![t]\!]$-modules endowed with a semilinear action of Γ_K that is continuous relative to the natural topology on finitely generated $K_\infty[\![t]\!]$-modules (as in Exercise 13.7.9(4),(5)) when $K_\infty[\![t]\!]$ is endowed with the subspace topology from B_{dR}^+.

This definition is not so nice, since we do not have a good way of describing the "de Rham" topology on $K_\infty[\![t]\!]$ (acquired from how it sits in B_{dR}^+) in more direct terms. For example, we do not even understand a convenient way to describe the topology on the subfield of constants K_∞ acquired from B_{dR}^+! (It is almost surely not the valuation topology.) The reason we use this topology in the definition of $\mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)$ is that it is what we must use to establish Fontaine's lifting of Sen's theory. But the good news is that we get the same category of continuous representations if we use a more accessible "linear" topology on $K_\infty[\![t]\!]$ that happens to also be what we must use when we make the link with formal p-adic linear differential equations. The insensitivity of $\mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)$ to such a switch in topologies on $K_\infty[\![t]\!]$ is the content of the following lemma.

Lemma 15.3.7. *Let M be a finitely generated $K_\infty[\![t]\!]$-module. Let τ_{dR} denote the topology on M acquired from topologizing $K_\infty[\![t]\!]$ by its subspace topology from B_{dR}^+, and let τ_{can} denote the topology on M acquired from topologizing $K_\infty[\![t]\!]$ with the product topology of the valuation topology on K_∞.*

A $K_\infty[\![t]\!]$-semilinear Γ_K-action on M is continuous with respect to τ_{can} if and only if it is continuous with respect to τ_{dR}.

For t-torsion objects M thw two topologies coincide, but otherwise they seem to be incompatible in both directions (i.e., neither is finer than the other).

Proof. In both cases, the topologies are inverse limits of the topologies on M/t^rM (see Lemma 15.3.3 for τ_{dR}), so we can assume that M is a $K_\infty[\![t]\!]/(t^m)$-module for some $m \geqslant 1$.

The topology $\tau_{\rm can}$ is then the linear topology as a finite-dimensional vector space over the valued field K_∞ (so we shall call it the "linear topology"), whereas $\tau_{\rm dR}$ is rather mysterious but we have gained some understanding of its properties in our preceding work (so we shall call it the "de Rham topology").

Consider a $K_\infty[\![t]\!]/(t^m)$-semilinear action of Γ_K on M. We must proved that it is continuous for the de Rham topology if and only if it is continuous for the linear topology. We can assume $M \neq 0$. The key point is that any such action descends to some K_n, and both topologies agree on every K_n. To make this precise, first choose a minimal generating set $\{x_1, \dots, x_d\}$ for M over $K_\infty[\![t]\!]/(t^m)$; thus, M is a direct sum of cyclic modules $(K_\infty[\![t]\!]/(t^{m_i}))\cdot x_i$ where (t^{m_i}) is the annihilator ideal of x_i (with $1 \leqslant m_i \leqslant m$). For each n, let $M_n := \oplus(K_n[\![t]\!]/(t^{m_i}))\cdot x_i$, so this is a $K_n[\![t]\!]/(t^m)$-descent of M in the sense that

$$(K_\infty[\![t]\!]/(t^m)) \otimes_{K_n[\![t]\!]/(t^m)} M_n = M$$

for all n. We also have formulas $\gamma(x_j) = \sum a_{ij}x_i$ with $a_{ij} \in K_\infty[\![t]\!]/(t^{m_i})$ that are *unique* (due to the cyclic structure), so there is a large n_0 such that $a_{ij} \in K_{n_0}[\![t]\!]/(t^{m_i})$ for the finitely many (i,j)'s. Hence, the Γ_K-action preserves M_n for all $n \geqslant n_0$.

For $n \geqslant n_0$, the continuity for the Γ_K-action on M relative to the linear (resp. de Rham) topology on M is equivalent to the same relative to the analogous topology on M_n (defined by replacing K_∞ with K_n everywhere), by adapting the proof of Lemma 15.2.2. Thus, it suffices to prove that on M_n these topologies *coincide*. In view of the cyclic decomposition of such an M_n, this problem reduces to the cyclic parts, so finally we are reduced to checking that when $K_n[\![t]\!]/(t^m)$ is viewed inside of $B^+_{\rm dR}/t^mB^+_{\rm dR}$ its subspace topology is its K_n-linear topology. Since $[K_n : K]$ is finite, so the K_n-linear topology is the K-linear topology, it remains to recall the general fact that any finite-dimensional K-subspace of the topological ring $B^+_{\rm dR}/t^mB^+_{\rm dR}$ has its K-linear topology as the subspace topology (as the linear topology is the unique Hausdroff topological vector space structure in the finite-dimensional case [7, I, §3, Thm. 2]. ■

Here is Fontaine's lifting of Sen's decompletion theory:

Theorem 15.3.8. *For any $X \in \mathrm{Rep}_{\mathrm{L}^+_{\rm dR}}(\Gamma_K)$, the $K_\infty[\![t]\!]$-submodule $X_{\rm f}$ is finitely generated with continuous Γ_K-action for its natural topology as a finitely generated $K_\infty[\![t]\!]$-module (so $X_{\rm f} \in \mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)$). Moreover, the natural map*

$$\beta_X : \mathrm{L}^+_{\rm dR} \otimes_{K_\infty[\![t]\!]} X_{\rm f} \to X$$

is an isomorphism.

In particular, $X_{\rm f}$ is dense in X and its natural $K_\infty[\![t]\!]$-module topology coincides with its subspace topology from X.

Since $\mathrm{L}^+_{\rm dR}$ is complete for its own topology, we view this proposition as saying that X is "the completion" of $X_{\rm f}$. We do not make a general intrinsic definition of completion in this setting (nor will it be necessary to do so later).

Proof. Example 15.3.5 settles the t-torsion case (as β_X is then exactly Sen's isomorphism $\widehat{K}_\infty \otimes_{K_\infty} \mathrm{D}_{\rm Sen}(W) \simeq W^{H_K}$ for $W = \mathbf{C}_K \otimes_{\widehat{K}_\infty} X$).

Step 1: injectivity of β_X (torsion case). We now prove that for a general torsion X, the natural map β_X is injective. The argument is by induction on the power of t that kills X. F filtering with $X' = tX$ and $X'' = X/tX$, by left-exactness of inverse limits we have a left-exact sequence of $K_\infty[\![t]\!]$-modules $0 \to X'_{\mathrm{f}} \to X_{\mathrm{f}} \to X''_{\mathrm{f}}$. By flatness of the scalar extension $K_\infty[\![t]\!] \to \mathrm{L}_{\mathrm{dR}}^+$ we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{L}_{\mathrm{dR}}^+ \otimes_{K_\infty[\![t]\!]} X'_{\mathrm{f}} \arrow[r] \arrow[d, "\beta_{X'}"'] & \mathrm{L}_{\mathrm{dR}}^+ \otimes_{K_\infty[\![t]\!]} X_{\mathrm{f}} \arrow[r] \arrow[d, "\beta_X"'] & \mathrm{L}_{\mathrm{dR}}^+ \otimes_{K_\infty[\![t]\!]} X''_{\mathrm{f}} \arrow[d, "\beta_{X''}"'] & \\
0 \arrow[r] & X' \arrow[r] & X \arrow[r] & X'' \arrow[r] & 0
\end{tikzcd}$$

Injectivity for $\beta_{X'}$ and $\beta_{X''}$ then implies the same for β_X, as required. Hence, for torsion X we see that $\mathrm{L}_{\mathrm{dR}}^+ \otimes_{K_\infty[\![t]\!]} X_{\mathrm{f}}$ is a finitely generated $\mathrm{L}_{\mathrm{dR}}^+$-module. But the scalar extension $K_\infty[\![t]\!] \to \mathrm{L}_{\mathrm{dR}}^+$ is a local injective map of discrete valuation rings, so it is *faithfully* flat. Hence, the finite generatedness descends (Exercise 15.5.7), so X_{f} is finitely generated over $K_\infty[\![t]\!]$ for any torsion object $X \in \mathrm{Rep}_{\mathrm{L}_{\mathrm{dR}}^+}(\Gamma_K)$.

In view of the fact that we can describe the topologies on finitely generated modules over the topologized discrete valuation rings $K_\infty[\![t]\!]$ and $\mathrm{L}_{\mathrm{dR}}^+$ in terms of any finite presentation, it follows from the topological aspects of Example 15.3.4 that for torsion X the subspace topology on X_{f} coincides with its natural topology as a finitely generated $K_\infty[\![t]\!]$-module. In particular, the natural Γ_K-action on X_{f} is continuous since continuity holds for the Γ_K-action on X. That is, $X_{\mathrm{f}} \in \mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)$ when X is torsion.

Step 2: surjectivity of β_X (torsion case). To complete our treatment of the torsion case, it remains to show that the injective β_X is an isomorphism for torsion X. We know this when X is t-torsion, and once again we induct on the power of t that kills X. We will need a technical cocycle lemma that replaces the role of the H^1-vanishing ingredient which was used in the proof of Proposition 15.2.4.

Since the t-torsion case is settled, we may assume $X \neq 0$ and that X is killed by t^{m+1} with some $m \geqslant 1$ such that the comparison map is known to be an isomorphism in the t^m-torsion case. Consider the exact sequence

$$0 \to X' \to X \to X'' \to 0$$

in $\mathrm{Rep}_{\mathrm{L}_{\mathrm{dR}}^+}(\Gamma_K)$ with $X'' = X/t^m X$ killed by t^m and $X' = t^m X$ killed by t. In particular, $X' \in \mathrm{Rep}_{\widehat{K}_\infty}(\Gamma_K)$. The maps $\beta_{X'}$ and $\beta_{X''}$ are isomorphisms by induction, so a minimal generating set $\{x_1, \ldots, x_d\}$ for X''_{f} over $K_\infty[\![t]\!]$ (i.e., a subset that lifts a basis of $X''_{\mathrm{f}}/tX''_{\mathrm{f}}$ over K_∞) is also a minimal generating set for $X'' = X/t^m X$ over $\mathrm{L}_{\mathrm{dR}}^+$. Hence, if we lift this to a subset $\{\widehat{x}_1, \ldots, \widehat{x}_d\}$ of X we know that the $\widehat{x}_i$'s are a minimal generating set of X over $\mathrm{L}_{\mathrm{dR}}^+$ (as the quotient map $X \to X/t^m X$ induces an isomorphism modulo t). If We fix a choice of $\widehat{x}_i$'s, and will use them to find another choice $\{\widehat{x}'_i\}$ that is also a minimal generating set of X_{f} over $K_\infty[\![t]\!]$, thereby establishing the surjectivity of β_X. To find the $\widehat{x}'_i$'s we study the Γ_K-action (much as we used vanishing of higher Γ_K-cohomology in the proof of Proposition 15.2.4).

Pick $\gamma \in \Gamma_K$, so $\gamma(\widehat{x}_j) = \sum a_{ij}\widehat{x}_i$ for $a_{ij} \in \mathrm{L}_{\mathrm{dR}}^+/t^{m+1}\mathrm{L}_{\mathrm{dR}}^+$. Since

$$X = \oplus_i (\mathrm{L}_{\mathrm{dR}}^+/t^{m_i}\mathrm{L}_{\mathrm{dR}}^+)\widehat{x}_i$$

with (t^{m_i}) the proper $\mathrm{L}_{\mathrm{dR}}^+$-annihilator of $\widehat{x}_i$ in X, each a_{ij} is well-defined modulo t^{m_i} and only matters modulo t^{m_i}. Also, the matrix $A := (a_{ij}) \in \mathrm{Mat}_d(\mathrm{L}_{\mathrm{dR}}^+/(t^{m+1}))$ is invertible, as such invertibility may be checked modulo t. In view of the uniqueness of the coefficients modulo the appropriate annihilators, and the fact that the flat extension $K_\infty[\![t]\!] \to \mathrm{L}_{\mathrm{dR}}^+$ commutes with the formation of annihilator ideals, since the x_i's are a minimal generating set for the $K_\infty[\![t]\!]$-structure X''_{f} of $X'' = X/t^m X$, we can arrange that each $a_{ij} \bmod t^m$ lies in $K_\infty[\![t]\!]/t^m K_\infty[\![t]\!]$. In other words (since $m \geqslant 1$), we may assume $A = A_0 + t^m A_1$ where $A_0 \in \mathrm{GL}_d(K_\infty[\![t]\!]/t^{m+1}K_\infty[\![t]\!])$ and $A_1 \in \mathrm{Mat}_d(\widehat{K}_\infty)$, with this $\widehat{K}_\infty$ being exactly the residue field of $\mathrm{L}_{\mathrm{dR}}^+$.

Let $U = A \bmod t = A_0 \bmod t \in \mathrm{GL}_d(K_\infty)$. This computes the γ-action on $X/tX \in \mathrm{Rep}_{\widehat{K}_\infty}(\Gamma_K)$ relative to the basis $\{x_i \bmod tX\}$. In particular, it depends continuously on γ because the Γ_K-action on X/tX is continuous. Thus, by taking γ sufficiently close to 1 in Γ_K we may and do arrange that $v(U-1) > c_3$ and $n(\gamma) > c_3$, where c_3 is as in (TS3). We also may and do arrange that γ is a topological generator of an open subgroup $\mathbf{Z}_p$ inside of Γ_K; fix this γ.

Step 3: cocycle arguments (torsion case). For $B := 1 + t^m M \in \mathrm{GL}_d(\mathrm{L}_{\mathrm{dR}}^+/t^{m+1}\mathrm{L}_{\mathrm{dR}}^+)$ with $M \in \mathrm{Mat}_d(\widehat{K}_\infty)$, consider the effect of applying B to the $\widehat{x}_i$'s. The effect on the matrix A describing the γ-action is to replace it with

$$\begin{aligned} B^{-1}A\gamma(B) &= (1 - t^m M)A(1 + \chi(\gamma)^m t^m \gamma(M)) \\ &= A - t^m(MA - \chi(\gamma)^m A\gamma(M)) \\ &= A - t^m(MU - \chi(\gamma)^m U\gamma(M)) \end{aligned}$$

in $\mathrm{Mat}_d(\mathrm{L}_{\mathrm{dR}}^+/t^{m+1}\mathrm{L}_{\mathrm{dR}}^+)$. We want this to lie in $\mathrm{GL}_d(K_\infty[\![t]\!]/t^{m+1}K_\infty[\![t]\!])$ for a suitable choice of M, so we first find a more manageable expression for the multiplier against t^m (which is a matrix in $\mathrm{Mat}_d(\widehat{K}_\infty)$) for general M.

Consider $n \geqslant \max(n(\gamma), n(G_K))$ large enough so that $U \in \mathrm{GL}_d(K_n)$ (as we may do since $U \in \mathrm{GL}_d(K_\infty)$). Hence, the matrix $(1 - R_{H_K,n})(A_1) \cdot U^{-1} \in \mathrm{Mat}_d(\widehat{K}_\infty)$ is killed entrywise by the K_n-linear projector $R_{H_K,n} : \widehat{K}_\infty \to K_n$, so we can apply Lemma 15.3.9 with $V = U^{-1}$ to get that

$$(1 - R_{H_K,n})(A_1) \cdot U^{-1} = M' - \chi(\gamma)^m U\gamma(M')U^{-1}$$

for some matrix $M' \in \mathrm{Mat}_d(\widehat{K}_\infty)$ with entries in the kernel of the projector $R_{H_K,n}$. Multiplying on the right by U gives $M'U - \chi(\gamma)^m U\gamma(M') = (1 - R_{H_K,n})(A_1)$. Hence, if we take such an M' as the choice for M in the above definition of B we arrive at the formula

$$B^{-1}A\gamma(B) = A_0 + t^m R_{H_K,n}(A_1) \in \mathrm{GL}_d(K_\infty[t]/(t^{m+1})).$$

In other words, with such a choice for B we have found a minimal generating set $\{\widehat{x}'_i\}$ for X over $\mathrm{L}_{\mathrm{dR}}^+$ on which the action by our fixed γ is described by a matrix in $\mathrm{GL}_d(K_\infty[t]/(t^{m+1}))$ Only finitely many elements of K_∞ arises in this matrix, so for some big N this matrix lies in $\mathrm{GL}_d(K_N[t]/(t^{m+1}))$. Thus, the continuous 1-cocycle map $\Gamma_K \to \mathrm{GL}_d(L_{\mathrm{dR}}^+/t^{m+1}\mathrm{L}_{\mathrm{dR}}^+)$ has restriction to the $\gamma^{\mathbf{Z}_p}$ that lands in $\mathrm{GL}_d(K_N[t]/(t^{m+1}))$ on the dense subset $\gamma^{\mathbf{Z}}$. But $K_N[t]/(t^{m+1})$ is closed in $\mathrm{L}_{\mathrm{dR}}^+/t^{m+1}\mathrm{L}_{\mathrm{dR}}^+$ (with the complete K_N-linear linear topology as its subspace topology), as may be checked by working inside of $B_{\mathrm{dR}}^+/t^{m+1}B_{\mathrm{dR}}^+$ and applying

Lemma 15.2.2 and Lemma 4.4.12. Hence, the entire $\gamma^{\mathbf{Z}_p}$ is carried into $\mathrm{GL}_d(K_N[t]/(t^{m+1}))$, so the $\gamma^{\mathbf{Z}_p}$-orbit of the $\widehat{x}_i'$'s is contained in a finitely generated $K_N[t]/(t^{m+1})$-submodule of X.

Since $\gamma^{\mathbf{Z}_p}$ has finite index in Γ_K, we conclude that the Γ_K-orbit of each $\widehat{x}_i'$ has finitely generated $K_N[t]/(t^{m+1})$-span. In particular, each $\widehat{x}_i'$ has Γ_K-orbit contained in a finite-dimensional K-vector space, so each $\widehat{x}_i'$ lies in X_{f}. This is a (minimal) generating set of X over $\mathrm{L}_{\mathrm{dR}}^+$ that lies in X_{f}, so the injective map β_X is also surjective. This completes our treatment of the torsion case.

Observe also that since $\mathrm{L}_{\mathrm{dR}}^+$ is faithfully flat over $K_\infty[\![t]\!]$ and β_X is an isomorphism in the torsion case, the functor $X \rightsquigarrow X_{\mathrm{f}}$ from torsion objects in $\mathrm{Rep}_{\mathrm{L}_{\mathrm{dR}}^+}(\Gamma_K)$ to torsion objects in $\mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)$ is an exact (as exactness may be checked after faithfully flat extension of scalars).

Step 4: general case. The general case is inferred from the torsion case via passage to inverse limits, as follows. We *defined* $X_{\mathrm{f}} = \varprojlim (X/t^m X)_{\mathrm{f}}$, and the results in the torsion case tell us that each $K_\infty[t]/(t^m)$-module $Y_m := (X/t^m X)_{\mathrm{f}}$ is finitely generated such that $Y_{m+1}/t^m Y_{m+1} \simeq Y_m$ for all $m \geqslant 1$. Hence, by general principles in commutative algebra (Exercise 15.5.7), the inverse limit X_{f} is finitely generated over $K_\infty[\![t]\!]$ with $X_{\mathrm{f}}/t^m X_{\mathrm{f}} \simeq Y_m = (X/t^m X)_{\mathrm{f}}$ for all $m \geqslant 1$. In particular, the mod-t^m reduction of β_X is identified with $\beta_{X/t^m X}$ for all $m \geqslant 1$, so these reductions are all isomorphisms of $\mathrm{L}_{\mathrm{dR}}^+$-modules. Thus, β_X is an isomorphism in general.

The isomorphism property for β_X in general lets us settle the topological assertions. First, the $K_\infty[\![t]\!]$-module topology on X_{f} is the subspace topology from the $\mathrm{L}_{\mathrm{dR}}^+$-module topology on $\mathrm{L}_{\mathrm{dR}}^+ \otimes_{K_\infty[\![t]\!]} X_{\mathrm{f}}$ (as the structure theorem for finitely generated modules lets us check by reducing to cyclic modules, which we analyzed in Example 15.3.4). But β_X is a linear isomorphism over $\mathrm{L}_{\mathrm{dR}}^+$ so it is automatically a homeomorphism, and its restriction to X_{f} is the canonical inclusion. This proves that the natural $K_\infty[\![t]\!]$-module topology on X_{f} is the same as its subspace topology from X, so in particular the continuity of the Γ_K-action on X_{f} is inherited from the continuity of the action on X. ■

The following technical lemma on cocycles was used in the preceding proof. In the statement we use the notation as in axiom (TS3) (which we proved in Sen's situation; see Proposition 14.1.7).

Lemma 15.3.9. *Choose $U, V \in \mathrm{Mat}_d(K_\infty)$ such that $v(U-1) > c_3$ and $v(V-1) > c_3$, with c_3 as in* (TS3). *Choose $n \geqslant n(G_K)$ such that $n > c_3$ and $U, V \in \mathrm{Mat}_d(K_n)$. For any $m \geqslant 1$ and any $\gamma \in \Gamma_K$ satsifying $c_3 < n(\gamma) \leqslant n$, the map*

$$\begin{aligned} f\colon\ \mathrm{Mat}_d(\widehat{K}_\infty) &\to \mathrm{Mat}_d(\widehat{K}_\infty) \\ M &\mapsto M - \chi(\gamma)^m U \gamma(M) V \end{aligned}$$

restricts to a bijective self-map of the space of matrices whose entries lie in the kernel of the projector $R_{H_K,n} : \widehat{K}_\infty \to K_n$.

Note that the assumption on γ forces $n(\gamma) > 0$, so $\chi(\gamma) \in 1 + p\mathbf{Z}_p$.

Proof. Since $R_{H_K,n}$ is K_n-linear and Γ_K-equivariant, and U and V have entries in K_n, the image of the mapping f has matrix entries contained in the kernel $X_{H_K,n}$ of $R_{H_K,n}$. Thus, f

is a K_n-linear self-map of the K_n-Banach space of $d \times d$ matrices with entries in $X_{H_K,n}$. We can therefore try to use a contraction mapping argument (with the sup-norm on matrices), which is what we now do.

To streamline the notation, we observe that $f(M) = (1-\gamma)(M) + h(\gamma(M))$ where $h(N) = N - \chi(\gamma)^m UNV$ for any $N \in \mathrm{Mat}_d(\widehat{K}_\infty)$ with entries in $X_{H_K,n}$. For any N we have

$$h(N) = (N - \chi(\gamma)^m N) + \chi(\gamma)^m((N - UN) + UN(1 - V))$$

with $v(U) = 0$, so $v(h(N)) \geqslant \min\{v((1 - \chi(\gamma)^{r-1})N), v((U - 1)N), v(N(V - 1))\}$. Hence, $v(h(N)) \geqslant v(N) + \delta$ where $\delta = \min\{n(\gamma), v(U - 1), v(V - 1)\} > c_3$. But by (TS3) in Sen's situation (Proposition 14.1.7), the K_n-linear operator $1-\gamma$ on $X_{H_K,n}$ is bijective, so it suffices to prove bijectivity for that the self-map $\widetilde{f} = f \circ (1 - \gamma)^{-1}$ on the space of matrices with entries in $X_{H_K,n}$.

Since $\widetilde{f}(N) = N + h(\gamma(1 - \gamma)^{-1}(N))$ and $v((1 - \gamma)^{-1}(N)) \geqslant v(N) - c_3$, we see that $v(\widetilde{f}(N) - N) \geqslant v((1 - \gamma)^{-1}(N)) + \delta \geqslant v(N) + \delta - c_3$. As $\delta - c_3 > 0$, the K-linear operator $\widetilde{f} - \mathrm{id}$ is a contraction mapping with sup-norm strictly less than 1. Hence, $\widetilde{f} = \mathrm{id} - (\mathrm{id} - \widetilde{f})$ is bijective thanks to the usual geometric series expansion and the K-Banach property of $X_{H_K,n}$ relative to the sup-norm. ■

We can now deduce the main result we have been after, together with a nice alternative characterization of X_{f}.

Corollary 15.3.10. *The functor* $\mathrm{Rep}_{\mathrm{L}^+_{\mathrm{dR}}}(\Gamma_K) \to \mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)$ *defined by* $X \rightsquigarrow X_{\mathrm{f}}$ *is an equivalence of abelian categories, with quasi-inverse given by* $Y \rightsquigarrow \mathrm{L}^+_{\mathrm{dR}} \otimes_{K_\infty[\![t]\!]} Y$. *This equivalence preserves ranks and invariant factors. In particular,* X_{f} *is exact in* X.

Moreover, for any such X, *the subset* X_{f} *is the directed union of the finitely generated* $K_\infty[\![t]\!]$*-submodules of* X *that are stable under the action of* Γ_K. *In particular, all such submodules have a continuous* Γ_K*-action for their natural* $K_\infty[\![t]\!]$*-module topology, and there is one such submodule* (*namely,* X_{f}) *that contains all others.*

Proof. The equivalence aspect is immediate from Theorem 15.3.8, coupled with Example 15.3.4 (to verify the quasi-inverse property relative to the scalar extension functor in the opposite direction). In particular, X_{f} is exact in X.

It remains to show that any Γ_K-stable finitely generated $K_\infty[\![t]\!]$-submodule X'_{f} of X is contained in X_{f}. By the exactness of X_{f} in X, the image of X_{f} in $X/t^m X$ is $(X/t^m X)_{\mathrm{f}}$, so it suffices to check that the image of X'_{f} in $X/t^m X$ is contained in $(X/t^m X)_{\mathrm{f}}$ for all $m \geqslant 1$ (as we have compatibly $X = \varprojlim(X/t^m X)$ and $X_{\mathrm{f}} = \varprojlim X_{\mathrm{f}}/t^m X_{\mathrm{f}} = \varprojlim(X/t^m X)_{\mathrm{f}}$). This reduces us to considering the case when X is a torsion object.

Say X is killed by t^m for some $m \geqslant 1$, so X'_{f} is a Γ_K-stable finitely generated $K_\infty[t]/(t^m)$-submodule of the $\mathrm{L}^+_{\mathrm{dR}}/(t^m)$-module X. We need to show that each element of X'_{f} has Γ_K-orbit contained in a finite-dimensional K-subspace of X (as then $X'_{\mathrm{f}} \subseteq X_{\mathrm{f}}$, by the definition of X_{f}).

Pick $\gamma \in \Gamma_K$ that topologically generates an open subgroup of the form $\mathbf{Z}_p$, and consider the γ-action on X'_{f}. If we pick a finite generating set $x'_1, \dots, x'_N$ of X'_{f} over $K_\infty[t]/(t^m)$ then $\gamma(x'_j) = \sum a_{ij} x'_i$ for some $a_{ij} \in K_\infty[t]/(t^m)$. There are only finitely many a_{ij}, and each is represented by a polynomial over K_∞ with degree at most $m - 1$, so for some large n we have that $a_{ij} \in K_n[t]/(t^m)$ for all i, j. Hence, the $K_n[t]/(t^m)$-span Y of the x_i''s is γ-stable and hence $\gamma^{\mathbf{Z}}$-stable. This is a finite-dimensional K-subspace of X; suppose for a moment that it is *closed* in X, so it is $\gamma^{\mathbf{Z}_p}$-stable. We would have then produced a finite-dimensional K-subspace of X which contains a generating set of X_{f}' and is stable under an open subgroup Γ' of Γ_K. Applying a set of coset representatives of the finite set Γ_K/Γ' would then provide a Γ_K-stable finite-dimensional K-subspace of X containing generators of X_{f}'. That is, X_{f}' is generated over $K_\infty[\![t]\!]$ by a set of elements of X_{f}, so $X_{\mathrm{f}}' \subseteq X_{\mathrm{f}}$ as desired.

It now remains to check the closedness condition mentioned above, or more generally that *every* finite-dimensional K-subspace of X is closed. Even better, we claim such a subspace has its natural K-linear topology as the subspace topology, so closedness is forced by the completeness of K. To check this claim about the subspace topology, it suffices to pass to a larger finite-dimensional K-subspace of X. Since X is a direct sum of copies of $\mathrm{L}_{\mathrm{dR}}^+$ or various quotients $\mathrm{L}_{\mathrm{dR}}^+/(t^m)$, we are reduced to checking that all finite-dimensional K-subspaces of $\mathrm{L}_{\mathrm{dR}}^+$ and $\mathrm{L}_{\mathrm{dR}}^+/(t^m)$ (any $m \geqslant 1$) have the K-linear topology as their subspace topology. By Lemma 15.2.2, it suffices to do the same with the abstract $\mathrm{L}_{\mathrm{dR}}^+$ replaced by the more accessible B_{dR}^+. This is Lemma 4.4.12. ∎

Theorem 15.3.11. *The functor*

$$\begin{aligned} \mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K) &\to \mathrm{Rep}_{\mathrm{B}_{\mathrm{dR}}^+}(G_K) \\ Y &\rightsquigarrow \mathrm{B}_{\mathrm{dR}}^+ \otimes_{K_\infty[\![t]\!]} Y \end{aligned}$$

is an equivalence of categories. A quasi-inverse is given by $W \rightsquigarrow \left(W^{H_K}\right)_{\mathrm{f}}$.

Proof. This follows by combining the equivalences in Corollary 15.2.5 and Corollary 15.3.10. ∎

15.4. **Fontaine's functor** $\mathrm{D}_{\mathrm{dif}}$**.** The equivalence in Theorem 15.3.11 will create a link with formal p-adic linear differential equations. Before explaining this, we make an important topological observation. The mysterious "de Rham" topology on $K_\infty[\![t]\!]$ has done its work, and now we want to forget about it and interpret the continuity condition in the definition of $\mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)$ by using the product topology on $K_\infty[\![t]\!]$ via the valuation topology on K_∞. That is, for a finitely generated $K_\infty[\![t]\!]$-module M, we wish to work with the topology on M that is *the inverse limit of the usual* K_∞*-linear topologies on the finite-dimensional quotients* M/t^mM $(m \geqslant 1)$. Lemma 15.3.7 assures that this switch of topologies does not affect the continuity condition on semilinear representations of Γ_K! So from now on we can and will work with the more accessible topology that is a mixture of t-adic and K_∞-linear topologies.

Remark 15.4.1. When we work with K_∞-linear structures (such as certain connections below), the only topology that will generally matter is the t-adic one, since modulo any power of t our modules will become finite-dimensional over K_∞ and hence continuity conditions will be satisfied for K_∞-linear structures.

To discuss differential equations, we need a suitable module of Kähler differentials for $K_\infty[\![t]\!]$ over K_∞ that accounts for the topology on $K_\infty[\![t]\!]$:

Lemma 15.4.2. *Consider pairs (M, ∂) consisting of a finitely generated $K_\infty[\![t]\!]$-module M and a K_∞-linear derivation $\partial : K_\infty[\![t]\!] \to M$ that is continuous relative to the natural topology on finitely generated $K_\infty[\![t]\!]$-modules.*

Among such pairs there is an initial one $(\Omega^1_{K_\infty[\![t]\!]/K_\infty}, \mathrm{d})$, and it is free of rank 1 on the basis $\mathrm{d}t$, with $\mathrm{d}f = f'\mathrm{d}t$.

By Remark 15.4.1 (or the proof below), nothing would be affected if we worked just with the t-adic topology and ignored the valuation topology on K_∞. When thinking in such purely algebraic terms, for which K_∞ can be an arbitrary field F, we write $\widehat{\Omega}^1_{F[\![t]\!]/F}$ instead.

Proof. The content of the lemma is that $\partial(f) = f'\partial(t)$ and that the value of $\partial(t)$ may be assigned arbitrarily. Since the topology on M is the inverse limit of the linear topologies on the $M/(t^m)$'s, it suffices to treat the case when M is torsion. In this case even algebraically we have the existence and uniqueness, so the only issue is to check that for any $v \in M$ the K_∞-linear derivation $f \mapsto f'v$ is actually a continuous map. This map kills (t^{m+1}), so it factors through a linear map between finite-dimensional K_∞-vector spaces, and such maps are always continuous for the linear topology. ∎

The following variants on $\Omega^1_{K_\infty[\![t]\!]/K_\infty}$ will be more useful for our purposes.

Definition 15.4.3. The module of *meromorphic Kähler differentials* is

$$\Omega^1_{K_\infty(\!(t)\!)/K_\infty} := K_\infty(\!(t)\!) \otimes_{K_\infty[\![t]\!]} \Omega^1_{K_\infty[\![t]\!]/K_\infty} = (\Omega^1_{K_\infty[\![t]\!]/K_\infty})[1/t]$$

equipped with $\mathrm{d} : K_\infty(\!(t)\!) \to \Omega^1_{K_\infty(\!(t)\!)/K_\infty}$ defined by unique localization extension of the universal derivation.

The module of *logarithmic Kähler differentials* is $\Omega^+ = t^{-1} \cdot \Omega^1_{K_\infty[\![t]\!]/K_\infty}$ inside of $\Omega^1_{K_\infty(\!(t)\!)/K_\infty}$, equipped with the natural map $\mathrm{d} : K_\infty[\![t]\!] \to \Omega^+$.

Observe that Ω^+ is a finitely generated $K_\infty[\![t]\!]$-module (even free of rank 1 with basis $\mathrm{d}t/t$), so it has a natural topology that mixes the t-adic topology and the valuation topology on K_∞. On the other hand, $\Omega^1_{K_\infty(\!(t)\!)/K_\infty}$ does not have a topology of this sort, but it has a useful t-adic topology.

The above modules of differentials will allow us to define various notions of "module with connection". The motivation for bringing (the algebraic theory of) connections into the picture comes from the correspondence in differential geometry between monodromy representations of topological fundamental groups and vector bundles equipped with a flat connection.

Definition 15.4.4. Let F be a field of characteristic 0. A *logarithmic connection* on a finitely generated $F[\![t]\!]$-module M is an F-linear map

$$\nabla \colon M \to M \otimes_{F[\![t]\!]} (t^{-1}\widehat{\Omega}^1_{F[\![t]\!]/F}) = M\frac{\mathrm{d}t}{t}$$

that is continuous relative to the t-adic topology and satisfies the *Leibniz Rule*: $\nabla(\lambda m) = m \otimes \mathrm{d}\,\lambda + \lambda\nabla(m)$ for all $\lambda \in F[\![t]\!]$ and $m \in M$.

A *meromorphic connection* on a finite-dimensional $F((t))$-vector space V is an F-linear map $\nabla : V \to V \otimes_{F((t))} \Omega^1_{F((t))/F}$ that is continuous relative to the natural topology of finite-dimensional $F((t))$-vector spaces and satisfies the Leibniz Rule $\nabla(\lambda v) = v \otimes \mathrm{d}\lambda + \lambda \nabla(v)$ for all $\lambda \in F((t))$ and $v \in V$.

The "logarithmic" aspect refers to the appearance of $\mathrm{d}t/t$ ("$\mathrm{d}(\log t)$") in the definition. The analogous definition without allowing the simple pole at t will not be useful for our purposes.

Beware that although the pairs (M, ∇) or either logarithmic or meromorphic type form an F-linear abelian category, one *cannot* form general F-linear combinations in connections: such combinations generally ruin the Leibniz Rule (look at the $m \otimes \mathrm{d}\,\lambda$ term), unless the coefficients of the linear combination add up to 1.

Let us make the above notions of connection more explicit. First consider the logarithmic case. We may uniquely write $\nabla(m) = \nabla_0(m) \otimes (\mathrm{d}\,t/t)$ with $\nabla_0(m) \in M$ that depends F-linearly on m. The necessary and sufficient conditions on $\nabla_0 : M \to M$ are that it is continuous and F-linear and satisfies $\nabla_0(\lambda m) = t(\mathrm{d}\,\lambda/\mathrm{d}\,t)m + \lambda\nabla_0(m)$ for all $m \in M$ and $\lambda \in F[\![t]\!]$. In other words, $\nabla_0 : M \to M$ is a derivation over the derivation $t \cdot \mathrm{d}/\mathrm{d}t$ on the coefficient ring $F[\![t]\!]$. When M is finite and free, such connections can be described even more explicitly; see Exercise 15.5.8. In the meromorphic case we write $\nabla = \nabla_0 \otimes \mathrm{d}t$ instead, and the condition is that $\nabla_0 : V \to V$ is F-linear, continuous, and satisfies $\nabla_0(fv) = f'v + f\nabla(v)$ for all $f \in F((t))$ and $v \in V$.

One good feature of logarithmic connections is that $\nabla_0(t^r M) \subseteq t^r M$ for any $r \geqslant 1$. (In the non-logarithmic case over $F[\![t]\!]$ with $\nabla = \nabla_0 \otimes \mathrm{d}t$ we would only have $\nabla_0(t^r M) \subseteq t^{r-1}M$, which is sometimes not good enough.) Hence, logarithmic ∇'s are rather nicely-behaved with respect to t-adic considerations. For example, if ∇ is a logarithmic connection on M then for any $n \geqslant 1$ there is a well-defined logarithmic connection on M/t^nM satisfying $m \bmod t^nM \mapsto \nabla(m) \bmod t^nM$. This collection of connections for all $n \geqslant 1$ uniquely determines ∇, and so allows us to reduce to some problems to the case of torsion M.

Now the Sen operator works its magic:

Proposition 15.4.5. *For any $M \in \mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)$ there exists a unique connection $\nabla_M : M \to M \otimes_{K_\infty[\![t]\!]} \Omega^+$ such that for all $r \geqslant 1$ and all $v \in M$, we have*

$$\gamma(v) \equiv \exp(\log(\chi(\gamma)) \cdot \nabla_{M,0})(v) \mod t^r M$$

for all γ in an open subgroup $\Gamma_{K,r,v} \subseteq \Gamma_K$, where $\nabla_{M,0} : M \to M$ is the K_∞-linear map for which $\nabla = \nabla_0 \otimes \mathrm{d}t/t$.

The congruential criterion makes sense since logarithmic connections are compatible with reduction modulo t^r for any $r \geqslant 1$, and the exponentiation makes sense since $\nabla_{M,0} \bmod t^r M$ is a K_∞-linear endomorphism of the *finite-dimensional* K_∞-vector space M/t^rM.

Proof. Since logarithmic connections are compatible with reduction mod t^m for any $m \geqslant 1$, to prove the existence and uniqueness it suffices to treat the case when M is a torsion object. Thus, M is a finitely generated $K_\infty[t]/(t^m)$-module for some $m \geqslant 1$, so its topology when viewed in $\mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)$ is its linear topology. That is, the Γ_K-action is continuous relative to the linear topology on M as a finite-dimensional K_∞-vector space. Hence, we may also view M as an object in $\mathrm{Rep}_{K_\infty}(\Gamma_K)$, and so as such it admits (by Theorem 15.1.7) a unique K_∞-linear Sen operator $\Theta : M \to M$ such that for all $v \in M$ there is an identity $\gamma(v) = \exp(\log(\chi(\gamma))\Theta)(v)$ in M for $\gamma \in \Gamma_K$ sufficiently near 1 (depending on v).

The conditions being imposed on the connection modulo $t^r M$ for all $r \geqslant 1$ need only be checked for large r, and so by taking $r = m$ the condition on $\nabla_{M,0}$ is that for each $v \in M$ there is an open subgroup $\Gamma_{K,v}$ in Γ_K such that

$$\gamma(v) = \exp(\log(\chi(\gamma))\nabla_{M,0})(v)$$

in M for all γ sufficiently near 1 (depending on v). By differentiation, the only possibility for the K_∞-linear map $\nabla_{M,0} : M \to M$ is that it is Θ. Hence, the uniqueness is settled and for existence we have to prove that the Sen operator is a derivation over $t \cdot \mathrm{d}/\mathrm{d}t$. In effect, when there is semilinearity of the Γ_K-action relative to a $K_\infty[t]/(t^m)$-module structure (where $\gamma(t) = \chi(\gamma)\cdot t$) we need to understand how the Sen operator interacts with the module structure.

That is, for $\lambda \in K_\infty[t]/(t^m)$ and $v \in M$ we want $\Theta(\lambda v) = t\lambda' v + \lambda\Theta(v)$, where $\lambda' := \mathrm{d}\lambda/\mathrm{d}t$. By K_∞-linearity it suffices to check this for a monomial $\lambda = t^e$ with $0 \leqslant e < m$, and the case $e = 0$ is trivial. For $e \geqslant 1$ the desired formula is

$$\Theta(t^e v) \stackrel{?}{=} et^e v + t^e\Theta(v),$$

and if this holds for all $v \in M$ with $e = 1$ then by a straightforward induction on e we would get the desired formula in general.

It now remains to prove that $\Theta(tv) = t(v + \Theta(v))$ for all $v \in M$. We plug into the limit formula (15.1.3): for any $v \in M$,

$$\begin{aligned}\Theta(tv) &= \lim_{\gamma\to 1}\frac{\gamma(tv) - tv}{\log(\chi(\gamma))}\\ &= \lim_{\gamma\to 1}\frac{\chi(\gamma) - 1}{\log(\chi(\gamma))}\cdot t\gamma(v) + t\lim_{\gamma\to 1}\frac{\gamma(v) - v}{\log(\chi(\gamma))}\\ &= tv + t\Theta(v),\end{aligned}$$

where the final step uses the continuity of the Γ_K-action (to infer that $\gamma(v) \to v$ in M as $\gamma \to 1$ in Γ_K). ■

Thanks to Proposition 15.4.5, the additive and K-linear functor $M \rightsquigarrow (M, \nabla_M)$ from $\mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)$ to the category of finitely generated $K_\infty[\![t]\!]$-modules equipped with a logarithmic connection is visibly exact and faithful.

Inspired by the theory of coherent sheaves in algebraic geometry, we regard a finitely generated $K_\infty[\![t]\!]$-module M as analogous to a family of finite-dimensional K_∞-vector spaces parameterized by a small open disk centered at the origin, with M/tM being the fiber over the origin. We view a logarithmic connection ∇ on M as analogous to a system of first-order linear ordinary differential equations (see Exercise 15.5.8), and the kernel $M^{\nabla=0}$ of the connection (or equivalently, the kernel of the associated K_∞-linear endomorphism $\nabla_0 : M \to M$) as analogous to the global solutions to the differential equations. This "solution space" is a K_∞-subspace of M, and it is not obvious just from the definitions if it is finite-dimensional in general. Experience from the theory of ordinary differential equations suggests asking if this solution space has dimension at most that of a "generic fiber" (solution to an ODE is determined by its initial conditions).

For the connections arising above from Γ_K-representations, things work out very nicely:

Proposition 15.4.6. *For any $Y \in \mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)$ the K_∞-vector space $Y^{\nabla_Y=0}$ is finite-dimensional there is an equality*

$$K_\infty \otimes_K Y^{\Gamma_K} = Y^{\nabla_Y=0}$$

inside of Y. In particular, $\dim_K(Y^{\Gamma_K})$ is finite. Moreover:

1. *If Y is free as a $K_\infty[\![t]\!]$-module then $\dim_K(Y^{\Gamma_K}) \leqslant \dim_{K_\infty}(Y/tY)$.*
2. *For $Y_1, Y_2 \in \mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)$,*
$$K_\infty \otimes_K \mathrm{Hom}_{\mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)}(Y_1, Y_2) \simeq \mathrm{Hom}_{\mathscr{R}_{K_\infty}}\big((Y_1, \nabla_{Y_1}), (Y_2, \nabla_{Y_2})\big),$$
where $\mathscr{R}_{K_\infty}$ is the category of finitely generated $K_\infty[\![t]\!]$-modules equipped with a logarithmic connection. If Y_1 and Y_2 are free as $K_\infty[\![t]\!]$-modules then
$$\dim_K\big(\mathrm{Hom}_{\mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)}(Y_1, Y_2)\big) \leqslant \dim_{K_\infty}(Y_1/tY_1)\dim_{K_\infty}(Y_2/tY_2).$$
3. *For $Y_1, Y_2 \in \mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)$, they are isomorphic if and only if (Y_1, ∇_{Y_1}) and (Y_2, ∇_{Y_2}) are isomorphic as finitely generated $K_\infty[\![t]\!]$-modules equipped with a logarithmic connection.*

Proof. Let $Y_r = Y/t^rY$ with $r \geqslant 1$. Since $\nabla_{Y_r,0}$ is the Sen operator of $Y_r \in \mathrm{Rep}_{K_\infty}(\Gamma_K)$, it follows from Corollary 15.1.10(2) that the natural map $K_\infty \otimes_K Y_r^{\Gamma_K} \to \ker(\nabla_{Y_r,0})$ is bijective. For fixed $n \geqslant 1$, we have

$$\varprojlim_r (K_n \otimes_K Y_r^{\Gamma_K}) = K_n \otimes_K \varprojlim_r Y_r^{\Gamma_K} = K_n \otimes_K Y^{\Gamma_K}$$

(we have to work with K_n rather than K_∞ to justify passing the scalar extension through the inverse limit), and by left-exactness considerations we have $\varprojlim_r Y_r^{\nabla_{Y_r}=0} = Y^{\nabla_Y=0}$. Hence, we get an injection $K_n \otimes_K Y^{\Gamma_K} \subseteq Y^{\nabla_Y=0}$ inside of Y. Passing to the direct limit on n, we get

$$K_\infty \otimes_K Y^{\Gamma_K} \subseteq Y^{\nabla_Y=0}. \tag{15.4.1}$$

Before we prove this is always an equality (as have been established in the torsion case), we prove that $\dim_K Y^{\Gamma_K}$ is always finite. For this, by (15.4.1) it suffices to show that $Y^{\nabla_Y=0}$ has finite K_∞-dimension. Consider the canonical short exact sequence

$$0 \to Y' \to Y \to Y'' \to 0$$

with $Y' = Y_{\mathrm{tor}}$ the torsion submodule and Y'' its maximal $K_\infty[\![t]\!]$-free quotient. From the Leibniz rule we see that $\nabla_{Y,0}$ preserves the torsion submodule, so for the finiteness of the K_∞-dimension of $Y^{\nabla_Y=0}$ it suffices to separately treat the torsion and free cases. The torsion case is trivial since then even $\dim_{K_\infty} Y$ is finite. When Y is a finite free $K_\infty[\![t]\!]$-module, we claim that $\dim_{K_\infty} Y^{\nabla_Y=0}$ is bounded above by the rank of Y over $K_\infty[\![t]\!]$. More precisely, upon inverting t we can apply:

Lemma 15.4.7. *For any field F of characteristic 0 and any finite-dimensional $F(\!(t)\!)$-vector space V equipped with a meromorphic connection ∇, the natural map*

$$F(\!(t)\!) \otimes_F M^{\nabla=0} \to M[1/t]$$

is injective.

Proof. Consider a nonzero element of the kernel (if one exists) admitting a minimal-length expression $\sum f_i \otimes v_i$ in elementary tensors. In particular, the f_i's are nonzero and the v_i's are linearly independent over K_∞. We may and do scale so that $f_1 = 1$. Applying $\nabla_{V,0}$ then gives

$$0 = \sum (f_i' v_i + f_i \nabla_{V,0}(v_i)) = \sum f_i' v_i$$

since all $\nabla_{V,0}(v_i) = 0$. But $f_1' = 0$, so by minimality of the dependence relation we must have $f_i' = 0$ for all i. As we are in characteristic 0, this forces all $f_i \in K_\infty$. Hence, the relation $v_1 = -\sum_{i>1} f_i v_i$ is a nontrivial linear dependence relation over K_∞, a contradiction. ∎

We now know that Y^{Γ_K} is always finite-dimensional over K, and also we have $Y^{\Gamma_K} = \varprojlim Y_r^{\Gamma_K}$ with the $Y_r^{\Gamma_K}$'s all of finite K-dimension (at most $\dim_{K_\infty} Y_r$), so by a Mittag-Leffler argument we get some big N such that for all sufficiently large r, the natural map

$$Y^{\Gamma_K} \to \mathrm{image}(Y_{r+N}^{\Gamma_K} \to Y_r^{\Gamma_K})$$

is a K-linear isomorphism. Extending scalars to K_∞ then gives that

$$K_\infty \otimes_K Y^{\Gamma_K} \simeq \mathrm{image}(K_\infty \otimes_K Y_{r+N}^{\Gamma_K} \to K_\infty \otimes_K Y_r^{\Gamma_K}).$$

But we already know that $K_\infty \otimes_K Y_r^{\Gamma_K} = Y_r^{\nabla_Y=0}$ for all r since all Y_r are torsion objects, so

$$K_\infty \otimes_K Y^{\Gamma_K} \simeq \mathrm{image}(Y_{r+N}^{\nabla_Y=0} \to Y_r^{\nabla_Y=0})$$

for all large r.

The identification of these latter images (for large r) with the common space $K_\infty \otimes_K Y^{\Gamma_K}$ shows that for sufficiently large r, the transition map $Y_{r+1} \to Y_r$ induces an isomorphism

$$K_\infty \otimes_K Y_{r+1}^{\Gamma_K} \to K_\infty \otimes_K Y_r^{\Gamma_K}$$

Hence, these image spaces are all compatibly isomorphic to $\varprojlim Y_r^{\nabla_Y=0} = Y^{\nabla_Y=0}$, so the injective map $K_\infty \otimes_K Y^{\Gamma_K} \hookrightarrow Y^{\nabla_Y=0}$ must be an isomorphism. In the special case that Y is a free $K_\infty[\![t]\!]$-module, Lemma 15.4.7 ensures that $\dim_{K_\infty}(Y^{\nabla_Y=0})$ is at most the rank of Y, which is $\dim_{K_\infty}(Y/tY)$. This proves part (1).

It remains to prove (2) and (3). For (2) we just apply the above conclusions to $Y := \mathrm{Hom}_{K_\infty[\![t]\!]}(Y_1, Y_2) \in \mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)$, which is endowed with a logarithmic connection in the habitual manner akin to what we have already seen for monodromy operators: $\nabla_Y(f) = \nabla_{Y_2} \circ f - (f \otimes 1) \circ \nabla_{Y_1}$ for $f \in Y$. We compute that $Y^{\Gamma_K} = \mathrm{Hom}_{\mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)}(Y_1, Y_2)$ and $Y^{\nabla_Y=0} = \mathrm{Hom}_{\mathscr{R}_{K_\infty}}\big((Y_1, \nabla_{Y_1}), (Y_2, \nabla_{Y_2})\big)$, and the natural equality between $K_\infty \otimes_K Y^{\Gamma_K}$ and $Y^{\nabla_Y=0}$ inside of Y translates into the desired natural map relating Hom-spaces. Thus, (2) is proved.

To prove (3) we cannot apply the inverse limit method to reduce to the torsion case since we cannot ensure the isomorphisms to be constructed (by the method of proof of Proposition 15.1.13) are compatible with change in torsion level. In general necessity is obvious, so we just have to show that if $\mathrm{Hom}_{\mathscr{R}_{K_\infty}}\big((Y_1,\nabla_{Y_1}),(Y_2,\nabla_{Y_2})\big)$ contains an isomorphism, then Y_1 and Y_2 are isomorphic in $\mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)$, Let $\{x_1,\dots,x_d\}$ and $\{y_1,\dots,y_{d'}\}$ be minimal $K_\infty[\![t]\!]$-module generating sets of Y_1 and Y_2 respectively. Since Y_1 and Y_2 are isomorphic as modules with connection, so in particular as $K_\infty[\![t]\!]$-modules, necessarily $d'=d$ and a $K_\infty[\![t]\!]$-linear map $f\colon Y_1\to Y_2$ is an isomorphism if and only if its reduction $\overline{f}$ modulo t is an isomorphism.

Let $\{f_1,\dots,f_n\}$ be a K-basis of $\mathrm{Hom}_{\mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)}(Y_1,Y_2)$, and let

$$\overline{f}_i\in\mathrm{Hom}_{K_\infty}(Y_1/tY_1,Y_2/tY_2)$$

be the reduction of f_i modulo t. Using the bases $\{x_j \bmod tY_1\}$ and $\{y_i \bmod tY_2\}$, each $\overline{f}_i$ is described by a $d\times d$ matrix over K_∞. By (2), $\{f_1,\dots,f_n\}$ is a K_∞-basis of

$$\mathrm{Hom}_{\mathscr{R}_{K_\infty}}\big((Y_1,\nabla_{Y_1}),(Y_2,\nabla_{Y_2})\big),$$

but this latter Hom-space contains an isomorphism. Hence, there exist $\lambda_1,\dots,\lambda_n\in K_\infty$ such that $\det(\lambda_1\overline{f}_1+\cdots+\lambda_n\overline{f}_n)\neq 0$. This implies that the polynomial $\det(X_1\overline{f}_1+\cdots+X_n\overline{f}_n)\in K_\infty[X_1,\dots,X_n]$ is non zero.

As K is infinite, there exist $\mu_1,\dots,\mu_n\in K$ such that $\det(\overline{f})\neq 0$ where $f=\mu_1f_1+\cdots+\mu_nf_n$. But $f\in\mathrm{Hom}_{\mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)}(Y_1,Y_2)$, so f is an isomorphism in $\mathrm{Rep}_{K_\infty[\![t]\!]}(\Gamma_K)$. ■

We now pass to the situation with $B_{\mathrm{dR}}=B_{\mathrm{dR}}^+[1/t]$ rather than B_{dR}^+, and likewise work with $K_\infty(\!(t)\!)$ rather than $K_\infty[\![t]\!]$. This amounts to inverting t in the preceding theory, so we will work with the following "isogeny categories":

Definition 15.4.8. The category $\mathrm{Rep}_{B_{\mathrm{dR}}}(G_K)$ is the t-isogeny category of $\mathrm{Rep}_{B_{\mathrm{dR}}^+}(G_K)$, which is to say that it consists of finite-dimensional B_{dR}-semilinear representations of G_K for which there is a G_K-stable B_{dR}^+-lattice on which the G_K-action is continuous relative to the natural topology of the lattice as a finite free B_{dR}^+-module. (All G_K-stable B_{dR}^+-lattices have the continuity property if one does, as we see by t-power scaling.)

The category $\mathrm{Rep}_{K_\infty(\!(t)\!)}(\Gamma_K)$ is defined similarly, using $K_\infty[\![t]\!]$ in place of B_{dR}^+.

Example 15.4.9. For $V\in\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, we have $B_{\mathrm{dR}}\otimes_{\mathbf{Q}}V\in\mathrm{Rep}_{B_{\mathrm{dR}}}(G_K)$ since it is $(B_{\mathrm{dR}}^+\otimes_{\mathbf{Q}_p}V)[1/t]$.

We also will work with the subring $\mathrm{L}_{\mathrm{dR}}=B_{\mathrm{dR}}^{H_K}=(B_{\mathrm{dR}}^+)^{H_K}[1/t[=\mathrm{L}_{\mathrm{dR}}^+[1/t]$, and we define $\mathrm{Rep}_{\mathrm{L}_{\mathrm{dR}}}(\Gamma_K)$ exactly as we defined $\mathrm{Rep}_{B_{\mathrm{dR}}}(G_K)$ above. For $X\in\mathrm{Rep}_{\mathrm{L}_{\mathrm{dR}}}(G_K)$, we denote by X_{f} the union of its $K_\infty[\![t]\!]$-submodules that are finitely generated and stable under the action of Γ_K. (This is inspired by the second half of Corollary 15.3.10.)

By combining Corollary 15.3.10 and Theorem 15.3.11, upon inverting t we obtain:

Theorem 15.4.10. *The functor*

$$\begin{aligned}\mathrm{Rep}_{K_\infty(\!(t)\!)}(\Gamma_K)&\to\mathrm{Rep}_{\mathrm{B}_{\mathrm{dR}}}(G_K)\\ Y&\rightsquigarrow\mathrm{B}_{\mathrm{dR}}\otimes_{K_\infty(\!(t)\!)}Y\end{aligned}$$

is an equivalence of categories. A quasi-inverse is given by $W\mapsto\big(W^{H_K}\big)_{\mathrm{f}}$.

Definition 15.4.11. A meromorphic connection ∇ on a finite-dimensional $K_\infty(\!(t)\!)$-vector space V is *regular* if there exists a $K_\infty[\![t]\!]$-lattice $M\subseteq V$ (called a *regular lattice*) that is stable under ∇_0, which is to say that ∇ restricts to a logarithmic connection on M.

We say that a module with meromorphic connection (M, ∇) over $K_\infty(\!(t)\!)$ is *trivial* (or that ∇ is *flat*) when the natural injective map $K_\infty(\!(t)\!) \otimes_K M^{\nabla=0} \hookrightarrow M$ is an isomorphism. (Equivalently, $\dim_K M^{\nabla=0} = \dim_{K_\infty(\!(t)\!)} M$.)

Upon inverting t, Proposition 15.4.5 now implies:

Proposition 15.4.12. *Choose $Y \in \mathrm{Rep}_{K_\infty(\!(t)\!)}(\Gamma_K)$. There exists a unique regular connection $\nabla_Y = \nabla_{Y,0} \otimes \frac{\mathrm{d}t}{t} \colon Y \to Y \otimes_{K_\infty(\!(t)\!)} \Omega^1_{K_\infty(\!(t)\!)/K_\infty}$ such that for all regular lattices $\mathcal{Y}$ in Y, all $r \geqslant 1$, and all $v \in \mathcal{Y}$, there exists an open subgroup $\Gamma_{K,r,v} \subset \Gamma_K$ such that*

$$\gamma(v) \equiv \exp(\log(\chi(\gamma)) \cdot \nabla_{Y,0})(y) \mod t^r \mathcal{Y}$$

We thus have an additive and K-linear functor

$$\begin{aligned} \mathrm{Rep}_{K_\infty(\!(t)\!)}(\Gamma_K) &\to \mathscr{R}_{K_\infty,t} \\ Y &\mapsto (Y, \nabla_Y) \end{aligned}$$

Moreover, upon inverting t, Proposition 15.4.6 implies:

Proposition 15.4.13. (1) *If $Y \in \mathrm{Rep}_{K_\infty(\!(t)\!)}(\Gamma_K)$, then $K_\infty \otimes_K Y^{\Gamma_K} \simeq Y^{\nabla_Y=0}$.*

(2) *If $Y_1, Y_2 \in \mathrm{Rep}_{K_\infty(\!(t)\!)}(\Gamma_K)$, then*

$$K_\infty \otimes_K \mathrm{Hom}_{\mathrm{Rep}_{K_\infty(\!(t)\!)}(\Gamma_K)}(Y_1, Y_2) \simeq \mathrm{Hom}_{\mathscr{R}_{K_\infty,t}}\big((Y_1, \nabla_{Y_1}), (Y_2, \nabla_{Y_2})\big).$$

(3) *If $Y_1, Y_2 \in \mathrm{Rep}_{K_\infty(\!(t)\!)}(\Gamma_K)$, then Y_1 and Y_2 are isomorphic if and only if (Y_1, ∇_{Y_1}) and (Y_2, ∇_{Y_2}) are isomorphic in $\mathscr{R}_{K_\infty,t}$.*

Definition 15.4.14. For $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, we define $\mathrm{D}_{\mathrm{dif}}(V) \in \mathscr{R}_{K_\infty,t}$ to be the object associated to the canonical $K_\infty(\!(t)\!)$-descent of $(B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{H_K} \in \mathrm{Rep}_{\mathrm{L}_{\mathrm{dR}}}(\Gamma_K)$.

Fontaine's main result linking de Rham representations and differential equations is expressed in terms of the functor

$$\mathrm{D}_{\mathrm{dif}} \colon \mathrm{Rep}_{\mathbf{Q}_p}(G_K) \to \mathscr{R}_{K_\infty,t}$$

as follows:

Proposition 15.4.15. *For any $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, V is de Rham if and only if the meromorphic connection on $\mathrm{D}_{\mathrm{dif}}(V)$ is flat.*

Proof. The p-adic representation V of G_K is de Rham if and only if the B_{dR}-representation $\mathrm{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$ is isomorphic to $\mathrm{B}_{\mathrm{dR}}^d$ compatibly with the B_{dR}-module structure and G_K-actions. By Proposition 15.4.13(3), this is equivalent to the flatness of the connection on $\mathrm{D}_{\mathrm{dif}}(V)$. ∎

The equivalence in Proposition 15.4.15 opens up a whole new body of techniques (centered on p-adic differential equations), *provided* we can show that the differential equations arising from p-adic representations are not merely formal, but actually convergent on some open disk (centered at 0). This convergence property is one of the aims of §16.

15.5. **Exercises.**

Exercise 15.5.1. Let $\psi : G_K \to \mathbf{Z}_p^\times$ be a continuous infinitely ramified character, with K a p-adic field. We consider the associated functor $\mathrm{D}_{\mathrm{Sen}}$ on $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$.

(1) Let $W = \mathbf{C}_K(\psi^r)$ with $r \in \mathbf{Z}$. Prove that $\mathrm{D}_{\mathrm{Sen}}(W)$ is the canonical copy of $K_\infty(\psi^r)$ inside of W.
(2) What if $W = \mathbf{C}_K(\psi^{2(p-1)s})$ for $s \in \mathbf{Z}_p - \mathbf{Z}$? (Note that $\psi^{2(p-1)}$ is valued in $1 + p\mathbf{Z}_p$, so raising it to a p-adic exponent makes sense.)
(3) Suppose instead that $W = \mathbf{C}_K(\eta)$ for a character $\eta : G_K \to \mathbf{Z}_p^\times$ of finite order. Show that if η factors through Γ then $\mathrm{D}_{\mathrm{Sen}}(W)$ is the canonical $K_\infty(\eta)$, but that this is false otherwise. Can you describe a nice K_∞-spanning vector of $\mathrm{D}_{\mathrm{Sen}}(W)$ in these other cases?

Exercise 15.5.2. This exercise addresses how $\mathrm{D}_{\mathrm{Sen}}$ behaves under finite extension on K within $\overline{K}$. Thus, to avoid ambiguity, for $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ let us now write $\mathrm{D}_{\mathrm{Sen},K}(W)$ rather than $\mathrm{D}_{\mathrm{Sen}}(W)$.

For a finite extension K'/K inside of $\overline{K}$, we can view W in $\mathrm{Rep}_{\mathbf{C}_K}(G_{K'})$. Using $\psi|_{G_{K'}}$ and $\Gamma' = \mathrm{Gal}(K'_\infty/K')$ thereby gives an object $\mathrm{D}_{\mathrm{Sen},K'}(W) \in \mathrm{Rep}_{K'_\infty}(\Gamma')$ that is naturally a K'_∞-structure on the $\mathbf{C}_K$-vector space W (compatibly with $G_{K'}$-actions). Prove that $K' \otimes_K \mathrm{D}_{\mathrm{Sen},K}(W) = \mathrm{D}_{\mathrm{Sen},K'}(W)$ inside of W.

Exercise 15.5.3. Choose $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ and let $D = \mathrm{D}_{\mathrm{Sen}}(W) \in \mathrm{Rep}_{K_\infty}(\Gamma)$. Let $\Theta_D : D \to D$ be the corresponding Sen operator.

(1) Prove that a subspace of $W' \subseteq W$ is stable under some open subgroup of G_K if and only if W' is stable under Θ_D. (Why does the analogy with Lie group representations make this plausible?) Deduce that W is semisimple as a $\mathbf{C}_K$-semilinear representation of G_K if and only if Θ_D is a semisimple operator on $\mathrm{D}_{\mathrm{Sen}}(W)$. Keep in mind that the G_K-action here is $\mathbf{C}_K$-semilinear rather than $\mathbf{C}_K$-linear.

   (By Theorem 2.2.7 we have $\mathrm{H}^1(G_K, \mathbf{C}_K) \neq 0$, which is to say that there exists a 2-dimensional W which is a non-split extension of $\mathbf{C}_K$ by $\mathbf{C}_K$, and $\Theta_{\mathrm{D}_{\mathrm{Sen}}(W)}$ is a nonzero nilpotent operator for such W.)
(2) The operator Θ_D depends on the initial choice of infinitely ramified character $\psi : G_K \to \mathbf{Z}_p^\times$ that got the theory started. Using (15.1.3), show that twisting ψ by a finite-order character of Γ has no effect on the Sen operator, so by twisting away the Teichmüller factor we now suppose that ψ is valued in $1 + p\mathbf{Z}_p$ (so ψ^s makes sense for any $s \in \mathscr{O}_{\mathbf{C}_K}$, though it is valued in $\mathbf{Z}_p^\times$ only for $s \in \mathbf{Z}_p$).

   Prove that $s \in \mathscr{O}_{\mathbf{C}_K}$ is an eigenvalue of Θ_D if and only if $\mathbf{C}_K(\psi^s)$ occurs as a subobject of W. (Keep in mind that (15.1.3) only holds for γ near 1.) How about generalized eigenspaces?
(3) Writing $\Theta_{D,\psi}$ to record the dependence on ψ, how is Θ_{D,ψ^s} related to $\Theta_{D,\psi}$ for nonzero $s \in \mathbf{Z}_p$?

Exercise 15.5.4. Choose $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ and let $D = \mathrm{D}_{\mathrm{Sen}}(W) \in \mathrm{Rep}_{K_\infty}(\Gamma)$ (so $W = \mathbf{C}_K \otimes_{K_\infty} D$). Let $\Theta_D : D \to D$ be the Sen operator.

(1) Using scalar extension we get a $\mathbf{C}_K$-linear endomorphism $(\Theta_D)_{\mathbf{C}_K}$ of W. In general this is hard to describe (e.g., no analogue of (15.1.3), even over $\widehat{K}_\infty$). But show that its kernel is W^{G_K}.
(2) Prove that if W is a direct sum of copies of $\mathbf{C}_K(\psi^r)$ if and only if $\Theta_D = r \cdot \mathrm{id}_D$.
(3) Using functorial properties of $\mathrm{D}_{\mathrm{Sen}}$ and Θ_D, deduce that W is a direct sum of copies of $\mathbf{C}_K(\psi^{r_i})$'s with various $r_i \in \mathbf{Z}$ if and only if Θ_D is semisimple with all eigenvalues equal to integers. Taking ψ to be the p-adic cyclotomic character, this characterizes when W is Hodge–Tate.

Exercise 15.5.5. For $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, define $D^+_{\mathrm{dR}}(V) = (B^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K} \subseteq D_{\mathrm{dR}}(V)$. Prove that this inclusion is an equality if V has no Hodge–Tate weights > 0, and that in such cases V is de Rham if and only if the natural comparison map

$$B^+_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V) \to B^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$$

is an isomorphism. (Beware that B^+_{dR} is not $(\mathbf{Q}_p, G_K)$-regular: the line $\mathbf{Q}_p t$ is G_K-stable and t is not a unit in B^+_{dR}.)

Exercise 15.5.6. Here are some topological exercises, the first of which should give you more appreciation for the other parts.

(1) Choose a compatible system $\{\zeta_{p^n}\}_{n \geqslant 0}$ of primitive p^nth roots of unity, and let $\varepsilon = (\zeta_{p^n} \bmod p\mathscr{O}_{\mathbf{C}_K})_{n \geqslant 0} \in R$. Observe that $\varepsilon^{1/p^m} = (\zeta_{p^{n+m}} \bmod p\mathscr{O}_{\mathbf{C}_K})_{n \geqslant 0}$ for all $m \geqslant 0$. Let $t = \log[\varepsilon]$ as usual.

 Prove that in B^+_{dR},
$$[\varepsilon^{1/p^m}] = \zeta_{p^m} \exp(t/p^m)$$
for all $m \geqslant 0$, where exp has the usual meaning as on any complete discrete valuation ring of residue characteristic 0. Also show that $p^m[\varepsilon^{1/p^{2m}}] \to 0$ in B^+_{dR}, and by expanding the exponential to order t^2 deduce that $\zeta_{p^m} t \to 0$ in $K_\infty[\![t]\!]/t^2 K_\infty[\![t]\!]$ using the subspace topology from $\mathrm{L}^+_{\mathrm{dR}}/t^2\mathrm{L}^+_{\mathrm{dR}}$. Deduce that this subspace topology is *not* the natural topology as a 2-dimensional vector space over the valued field K_∞, and that the subspace topology on the subfield of constants K_∞ is not its valuation topology!
(2) Carry out the verification of Lemma 15.3.3.
(3) Prove that $K_\infty[\![t]\!]$ is a dense subring of $\mathrm{L}^+_{\mathrm{dR}}$ (thereby justifying that we view it as a decompletion of $\mathrm{L}^+_{\mathrm{dR}}$).

Exercise 15.5.7. The following two useful assertions in commutative algebra were used in the proof of Theorem 15.3.8.

(1) Let $A \to A'$ be a faithfully flat map of commutative rings, and M an A-module. If $M' := A' \otimes_A M$ is finitely generated as an A'-module, prove that M is finitely generated as an A-module. (Hint: express M as the direct limit of its finitely generated A-submodules.) Give a counterexample if faithful flatness is relaxed to flatness.
(2) Let A be a noetherian ring that is separated and complete for the topology defined by an ideal I. (An important example is a complete local noetherian ring, with I the maximal ideal.) Let $A_n = A/I^{n+1}$ for $n \geqslant 0$, and let $\{M_n\}$ be an inverse system of modules over the inverse system $\{A_n\}$ such that the natural map $M_{n+1}/I^{n+1}M_{n+1} \to M_n$ is an isomorphism for all $n \geqslant 0$. Prove that $M = \varprojlim M_n$ is a finitely generated

A-module and that the natural map $M/I^{n+1}M \to M_n$ is an isomorphism for all n. Feel free to restrict attention to the case when A is a discrete valuation ring with maximal ideal I, as this is the case relevant to the proof of Theorem 15.3.8.

Exercise 15.5.8. Let M be a finite free $K_\infty[\![t]\!]$-module, and let $\mathbf{e} = \{e_1, \dots, e_n\}$ be a basis. Consider a general logarithmic connection ∇ on M, so $\nabla(e_j) = \sum_k \Gamma_j^k e_k \otimes \mathrm{d}t$ for some $\Gamma_j^k \in K_\infty(\!(t)\!)$ with at worst simple poles. These Γ_j^k's are called the *Christoffel symbols* of the connection, and they depend on the basis $\mathbf{e}$. (If we were working over a higher-dimensional base with parameters $\{t_i\}$ then the formula would be $\nabla(e_j) = \sum_k \Gamma_{ij}^k e_k \otimes \mathrm{d}t_i$, but in our present circumstances we only have $i = 1$ and so drop it from the notation.)

(1) For $f_1, \dots, f_n \in K_\infty[\![t]\!]$, compute a formula for $\nabla(\sum f_j e_j)$ in terms of the Christoffel symbols, the f_j's, and the derivatives of the f_j's.
(2) Compute the formula for how the Christoffel symbols transform under a change of basis on M. (It could get quite messy, and is never needed below, but is worth a try.)

16. Overconvergence of p-adic representations (to be added in!)

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