DELIGNE'S NOTES ON NAGATA COMPACTIFICATIONS

BRIAN CONRAD

Abstract. We provide a proof of Nagata's compactification theorem: any separated map of finite type between quasi-compact and quasi-separated schemes (e.g., noetherian schemes) factors as an open immersion followed by a proper morphism. This is a detailed exposition of private notes of Deligne that translate Nagata's method into modern terms, and includes some applications of general interest in the theory of rational maps, such as refined versions of Chow's Lemma and the elimination of indeterminacies in a rational map, as well as a blow-up characterization of when a proper morphism (to a rather general base scheme) is birational.

Introduction

It is a fundamental theorem of Nagata ([N1], [N2]) that if S is a noetherian scheme and $f : X \to S$ is separated and of finite type then there exists a proper S-scheme $\overline{X}$ and an open immersion $j : X \hookrightarrow \overline{X}$ over S. For example, this theorem is used to define the higher direct image functors with proper support in étale cohomology. In [SGA4, XVII, §3.2ff] this theorem was avoided by developing a theory of such higher direct image functors for "compactifiable" morphisms f (i.e., those admitting a factorization as in Nagata's theorem). In the case of a noetherian scheme S, the compactifiable S-schemes are exactly the separated finite type S-schemes once one knows that Nagata's theorem is true. Another application of Nagata's theorem is in Grothendieck duality, where it is used in the approach of Deligne and Verdier to constructing the twisted inverse image functor $f^!$ for a separated map $f : X \to S$ of finite type with a general noetherian S (that may not admit a dualizing complex). Unfortunately, Nagata's proof is given in terms of pre-Grothendieck algebro-geometric terminology that is difficult for a modern reader to understand. At the beginning of [N2], Nagata writes "... the usual definition of a scheme is not nicely suited to our proof." His arguments use the Zariski–Riemann space attached to a function field, together with arguments proceeding by induction on the rank of a valuation.

In graduate school I was told by an expert in algebraic geometry that there was some uncertainty about the validity of Nagata's theorem over a general noetherian base because modern algebraic geometers could not understand Nagata's proof. I asked Deligne about this a few years later, and he said that by translating Nagata's methods he had worked out a scheme-theoretic version of Nagata's proof in which S is permitted to be an arbitrary quasi-compact and quasi-separated base scheme, with no noetherian or finite presentation hypotheses. (Recall that a scheme is *quasi-separated* if the overlap of any two quasi-compact opens is again quasi-compact, so this includes all locally noetherian schemes. Also, away from the locally noetherian case a finite type morphism may not be finitely presented; e.g., a closed immersion.) Deligne offered me a photocopy of his personal notes on these matters [D]. Since Deligne wrote these notes for himself, many proofs in his notes are merely sketched. Due to the importance of Nagata's theorem, I decided to write out complete proofs of the assertions in [D].

I later found out that Lütkebohmert [L] published a short proof of the theorem in the noetherian case, but Nagata's method is different and his study of quasi-dominations (which we give in §2) yields interesting refinements of and improvements upon classical blow-up techniques such as elimination of the indeterminacy locus of a rational map and Chow's Lemma (see Remark 2.5 and Corollary 2.6 respectively). These arguments also give a characterization of proper birational morphisms that applies under much weaker hypotheses than

Date: August 10, 2007.
1991 *Mathematics Subject Classification.* Primary 14A15; Secondary 14E99.
Key words and phrases. compactification, blow-up.
The author was supported by NSF grant DMS-0093542 during work on this paper.

the classical fact [H, II, 7.17] that birational projective maps between integral quasi-projective schemes over a field are blow-ups: if a proper map $f : X \to Y$ between noetherian schemes is an isomorphism over a dense open $U \subseteq Y$ with dense preimage in X then there are blow-ups $\widetilde{X} \to X$ and $\widetilde{Y} \to Y$ away from U and an isomorphism $\widetilde{f} : \widetilde{X} \simeq \widetilde{Y}$ covering f; see Theorem 2.11 (and Remark 2.12). In Corollary 4.4 we record some related results that describe separated but possibly non-proper birational maps of finite type after suitable blow-up. (I am grateful to S. Nayak for pointing out these consequences of Nagata's methods.) These general results concerning rational maps are not given in [L] and do not seem to be widely known even in the case of abstract varieties over a field. Roughly speaking, they say that separated birational maps of finite type always arise from blow-ups and dense open immersions, much as Zariski's Main Theorem says that separated quasi-finite maps always arise from finite maps and open immersions. We emphasize that these results exert good control over the blow-up locus.

In recent work of Temkin [T] on applying valuation-theoretic methods to problems in algebraic geometry, the original viewpoint of Nagata (Zariski–Riemann spaces) is resurrected in a modern form. In fact, Temkin gives a new proof of Nagata's theorem with a general quasi-compact and quasi-separated base scheme S as a consequence of a remarkable factorization theorem: every quasi-compact and separated map $X \to S$ to such a scheme (no finite-type assumption on the map!) factors as an affine morphism followed by a proper map. In Temkin's work some ideas related to quasi-dominations as in Deligne's notes play an important role and so it is convenient for him to refer to [D] (or rather, to this exposition of [D]). Since Nagata's proof yields results of general interest concerning rational maps and (in contrast with the powerful but abstract valuation-theoretic techniques of Temkin) it only uses techniques that are widely known to modern algebraic geometers, it seems appropriate to make Deligne's notes available in a more permanent form as we do here.

In addition to the enormous thanks I owe to Deligne, both for allowing me to see his notes and for his permission to disseminate an exposition of them, I want to thank M. Kisin and M. Raynaud for helpful discussions, S. Nayak for instructive suggestions and a careful reading of the entire manuscript, and M. Temkin for his encouragement.

Noetherian hypotheses. Since most readers will only be interested in the noetherian case, whenever a proof in the noetherian case can avoid a technical step I first give the relevant part of the argument in the noetherian case and then direct the "noetherian reader" to skip ahead to a later paragraph so as to bypass complications caused by working without noetherian assumptions. The phrase "finite type quasi-coherent sheaf" should be read as "coherent sheaf" by a noetherian reader, and such a person should ignore all mention of schemes or morphisms being "quasi-compact and quasi-separated" or open subschemes and open immersions being "quasi-compact".

The notes of Deligne are written almost entirely in the noetherian case, and at the end he observes that one can modify the arguments so that they work over any quasi-compact and quasi-separated base scheme. One simply has to be careful to only use open subschemes that are quasi-compact and use closed subschemes that are defined by finite type quasi-coherent ideal sheaves, and it is necessary to use direct limit arguments when working with scheme-theoretic closures. For the sake of maximal utility as a reference, I have made the necessary modifications to incorporate such generality right from the start in this exposition: all preliminary results are stated and proved without noetherian hypotheses.

In Deligne's translation of the proof of Nagata's theorem there is one key step that seems (to me) to be difficult to carry out in the absence of a noetherian condition, so I have written the proof of the theorem in such a way that this step is isolated at the end. The difficulty is circumvented by a trick using some remarkable results of Thomason and Trobaugh [TT, App. C]. More specifically, we use [TT] to deduce Nagata's theorem in general as a formal consequence of its validity in the noetherian case. A reader who is familiar with [L] (or [N2]!) and only wants to see how to eliminate noetherian hypotheses in Nagata's theorem can read just this self-contained reduction step (see Theorem 4.3 and the last two paragraphs of the proof of Theorem 4.1). However, as we have noted, Nagata's method gives intermediate general results for rational maps that are of independent interest, even for abstract varieties over a field.

Guide to the proof. To aid the reader in following the rather complicated series of steps leading to the proof of Nagata's theorem, here is a detailed sketch how the argument goes. Suppose for simplicity that S is noetherian and affine and that there is a covering of X by finitely many *dense* open subschemes U_i that are quasi-projective over S. (For example, if X is irreducible then we can take the U_i's to be affine. In the general noetherian case such U_i's can always be found by using finite disjoint unions of quasi-affine opens, as we explain near the end of the proof of Theorem 4.1.) This key use of quasi-projective opens seems to prevent Nagata's method from carrying over to algebraic spaces. If there is to be a compactification $\overline{X}$ of X then it also serves as a compactification of each dense open U_i, and we may hope that the boundary $\overline{X} - X$ admits an open covering $\{Y_i\}$ so that $U_i \cup Y_i$ is an open subset of $\overline{X}$ that can be rediscovered (with its open subscheme structure) as an open subscheme in a more elementary (e.g., projective) compactification $\overline{U}_i$ of the quasi-projective U_i. Proceeding in reverse, each U_i is quasi-projective over S and so admits a projective compactification $\overline{U}_i$ over S. We want to choose the $\overline{U}_i$'s carefully so that if we append a suitable open subset Y_i of the boundary $\overline{U}_i - U_i$ to U_i then these can be glued together to give a compactification of $X = \cup U_i$. More precisely, we seek a closed subset $Z_i \subseteq \overline{U}_i - U_i$ so that the gluing M_i of $\overline{U}_i - Z_i \supseteq U_i$ to X along U_i is a *separated* S-scheme and the gluing of the M_i's along X is the desired compactification.

Before we find a good choice for $\overline{U}_i$, we note that the possible failure of separatedness under gluing is a basic difficulty that has to be overcome. Recall that for any scheme S, if we glue two separated S-schemes T_1 and T_2 along an S-isomorphism $\iota : V_1 \simeq V_2$ between open subschemes, then the resulting S-scheme T is separated if and only if the subscheme graph $\Gamma_\iota \subseteq T_1 \times_S T_2$ of ι is closed. Such closedness is a stronger assertion than the closedness of the graph in $V_1 \times_S V_2$ (which is an obvious consequence of the separatedness of the V_j's over S). To prove that this stronger closedness property is necessary and sufficient for separatedness of T over S, observe that $\{T_i \times_S T_j\}$ is an open covering of $T \times_S T$ and the intersection of $\Delta_{T/S}$ with the elements of this covering are $\Delta_{T_1/S}$, $\Delta_{T_2/S}$, Γ_ι, and $\Gamma_{\iota^{-1}}$. Since closedness can be checked over an open covering of a topological space, we see the assertion.

Thus, to maintain separatedness in the gluing procedure we have to be attentive to closedness of graphs in larger product spaces. It is therefore natural to consider the following concept: for a noetherian scheme S and a pair of S-schemes X and X' of finite type with X' separated over S, a *quasi-domination* of X over X' is a dense open subscheme $U \subseteq X$ and an S-map $f : U \to X'$ such that the graph Γ_f is closed in $X \times_S X'$ (not only closed in $U \times_S X'$, as is automatic by S-separatedness of X'). Nagata's study of quasi-dominations in §2 yields several useful consequences, among which is a striking refinement of Chow's Lemma (Corollary 2.6) that gives (in our setup above for Nagata's theorem) a blow-up $q_i : X_i \to X$ away from the dense open quasi-projective U_i such that X_i is compactifiable to a projective $\overline{X}_i$ over S. We consider $\overline{X}_i$ to be a good first choice for $\overline{U}_i$ since it knows about all of X (in the sense that it contains the open subscheme X_i that has a proper birational map q_i onto X), and we view the closed set $Y_i = X_i - U_i$ as an exceptional locus for the blow-up q_i. It makes sense to form the gluing M_i of X and $\overline{X}_i - \overline{Y}_i$ along $X_i - Y_i = U_i$, where $\overline{Y}_i$ is the closure of Y_i in $\overline{X}_i$; we hope that these M_i's can be an open cover of the sought-after compactification of X (i.e., glue the M_i's along X). At least each gluing M_i is S-separated, which amounts to the diagonal image of U_i in $X \times_S (\overline{X}_i - \overline{Y}_i)$ being closed. To verify such closedness, one checks that the preimage of this diagonal under the proper surjection $q_i \times 1_{\overline{X}_i - \overline{Y}_i}$ is the overlap of $X_i \times_S (\overline{X}_i - \overline{Y}_i)$ with the closed subset $\Delta_{\overline{X}_i/S}$ in $\overline{X}_i \times_S \overline{X}_i$.

The gluing of the M_i's along X will usually not be separated, so we have to modify the M_i's away from X. The most difficult result in Nagata's treatment (Theorem 2.8 and its inductive refinement in Corollary 2.10) gives a general procedure to make a separated gluing after blow-up of the open pieces. This provides blow-ups M'_i of M_i away from X and a gluing M of the M'_i's along open subschemes containing X so that M is S-separated (and contains X as a dense open subscheme, so $U = \cap U_i$ is also a dense open in M). Roughly speaking, the role of blow-ups is to separate apart certain closed subschemes so as to force gluings to be separated over the base. Since M is separated, to try to prove it is proper (and hence is a compactification of X over S) we seek a (projective) compactification $\overline{U}$ of U dominating every $\overline{X}_i$ so that some blow-up U^* of $\overline{U}$ away from U resolves indeterminacies of the rational map from $\overline{U}$ to M arising from the inclusion $U \hookrightarrow M$, so the compactification U^* of U over S admits an S-map $U^* \to M$ extending the inclusion $U \to M$. Such an S-map to M is dominant (by denseness of U in M) and proper (since U^* is S-proper and M is S-separated), hence surjective, and so this would show that the separated M is S-proper. Unfortunately, it may not be possible to find such a $\overline{U}$ (and in fact M may not be proper over S) unless we make better choices of the $\overline{X}_i$'s at the start.

To explain the difficulty, consider the S-proper closure $\overline{U}$ of the image of $U = \cap U_i$ in $\prod \overline{X}_i$. This is a (projective) compactification of U and it dominates each $\overline{X}_i$ via the projection $\pi_i : \overline{U} \to \overline{X}_i$ (extending the dense open immersion of U into X_i). We want to construct a blow-up $q : U^* \to \overline{U}$ away from U so that there is an S-map $g : U^* \to M$ extending the inclusion of U into M. One can make a reasonable candidate for U^* so that U is schematically dense in U^* and on the open $U_i^* = q^{-1}(\overline{U} - \pi_i^{-1}(\overline{Y}_i)) \subseteq U^*$ that meets U in U_i there is an S-map $g_i : U_i^* \to M$ extending the inclusion $U_i \hookrightarrow M$. These agree on overlaps (by comparing on the schematically dense open $U = \cap U_i$ in U^*), so we can glue the g_i's to get an S-map $g : \cup U_i^* \to M$ extending $U \hookrightarrow M$. However, the U_i^*'s may fail to cover U^*, or equivalently $\cap \pi_i^{-1}(\overline{Y}_i)$ in $\overline{U}$ may be non-empty. Such non-emptiness (if it occurs) can be partially controlled in the following sense: by using separatedness of X one can compute that the preimages $\pi_i^{-1}(Y_i)$ satisfy $\cap \pi_i^{-1}(Y_i) = \emptyset$. The difficulties are therefore happening over the boundaries $\overline{X}_i - X_i$ (since $\overline{Y}_i \cap X_i = Y_i = X_i - U_i$). A general result on separation of closures after blow-up (Lemma 3.2, whose proof rests on Theorem 2.4, the most fundamental ingredient in the entire proof and the hardest step in Deligne's notes) enables us to use the emptiness of $\cap \pi_i^{-1}(Y_i)$ to construct blow-ups $\overline{q}_i : \overline{X}'_i \to \overline{X}_i$ away from X_i such that the closures $\overline{Y}'_i$ of $Y_i = X_i - U_i$ in $\overline{X}'_i$ satisfy $\cap(\pi'_i)^{-1}(\overline{Y}'_i) = \emptyset$, where $\pi'_i : \overline{U'} \to \overline{X}'_i$ is the projection from the closure $\overline{U'}$ of the image of $U' = U$ in $\prod \overline{X}'_i$. Thus, if we work with $\overline{X}'_i$ rather than with $\overline{X}_i$ at the start then the U_i^*'s will cover U^*, so the S-separated M can indeed be dominated by the S-proper U^*, and hence M is S-proper. This M is therefore the desired compactification of X over S.

Notation and terminology. For a scheme X and a quasi-coherent ideal sheaf $\mathscr{I}$ on X, we let $V(\mathscr{I}) = \mathrm{Spec}(\mathscr{O}_X/\mathscr{I})$ denote the associated closed subscheme. If $f : X \to Y$ is a map of schemes and $\mathscr{I}$ is a quasi-coherent ideal sheaf on Y, we let $\mathscr{I} \cdot \mathscr{O}_X$ denote the quasi-coherent pullback ideal sheaf $f^{-1}(\mathscr{I}) \cdot \mathscr{O}_X$ on X. A quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is of *finite type* if it is locally finitely generated (and so coherent in the locally noetherian case).

If Z is a subscheme of X, we sometimes denote this by writing $Z \subseteq X$. We let $X - Z$ denote the set-theoretic complement of Z in X, understood to have its canonical open subscheme structure if Z is closed in X. Also, if Z is a closed subscheme of X, we let $\mathscr{I}_Z$ denote the corresponding quasi-coherent ideal sheaf on X. If $\mathscr{I}$ and $\mathscr{K}$ are quasi-coherent ideal sheaves on X, we say that $\mathscr{K}$ is a *subideal sheaf in* $\mathscr{I}$ if $\mathscr{K} \subseteq \mathscr{I}$ as subsheaves of $\mathscr{O}_X$.

When we refer to the *image* of a morphism $f : X \to Y$, we mean the image $f(X)$ on the level of underlying topological spaces. When we wish to discuss the scheme-theoretic image (when it exists — e.g., if f is quasi-compact and quasi-separated), we will always use the adjective "scheme-theoretic". A similar comment applies when we discuss closures of subschemes.

We define the concepts of *quasi-projective* and *projective* for morphisms of schemes as in [EGA, II, 5.3, 5.5] (also see [EGA, IV$_1$, 1.7.19]). For example, if S is quasi-compact and quasi-separated then a map of schemes $f : X \to S$ is *quasi-projective* (resp. *projective*) if it factors as a quasi-compact immersion (resp. closed immersion) into an S-scheme of the form $\mathbf{P}(\mathscr{E}) := \mathrm{Proj}(\mathrm{Sym}(\mathscr{E}))$ for a quasi-coherent sheaf $\mathscr{E}$ of finite type on S. If S is affine then this coincides with the more familiar definition (as in [H, II, §7]) using $\mathbf{P}^n_S$. We refer to [EGA, II, 5.1] (and [EGA, IV$_1$, 1.7.16]) for the definition and properties of *quasi-affine* morphisms.

1. Review of Blow-ups

Let A be a ring and I an ideal. For $a \in I$, define $U(a) = \mathrm{Spec}(A[Ia^{-1}])$, with $A[Ia^{-1}] \subseteq A[a^{-1}] = A_a$ denoting the A-subalgebra of A_a generated by elements of the form b/a, with $b \in I$. Though a is usually not a unit in $A[Ia^{-1}]$, it is at least not a zero divisor. For $a_1, a_2 \in I$, we have a unique isomorphism of A-algebras

$$A[Ia_1^{-1}]_{a_2/a_1} \simeq A[Ia_2^{-1}]_{a_1/a_2} \tag{1.1}$$

(equality inside of $A_{a_1a_2}$). In terms of universal properties, $U(a)$ is the final object in the category of A-schemes X such that $I\cdot\mathscr{O}_X$ is an invertible ideal sheaf with global basis given by (the pullback of) $a\in I$. In this sense the isomorphism (1.1) corresponds to identifying open subschemes in $U(a_1)$ and $U(a_2)$ with the same universal property, namely being final among A-schemes X such that $I\cdot\mathscr{O}_X$ has both a_1 and a_2 as a global basis. Gluing along the isomorphisms (1.1) thereby yields an A-scheme $\mathrm{Bl}_I(A)$ called the *blow-up of A along I* (called a *dilitation* by Nagata in [N1, §2]) with the universal property that it is final in the category of A-schemes X such that $I\cdot\mathscr{O}_X$ is invertible. The open subscheme $U(a)$ in $\mathrm{Bl}_I(A)$ represents the open subfunctor as described above, so if a runs through a set of generators of the ideal I then the schemes $U(a)$ give an open affine covering of $\mathrm{Bl}_I(A)$.

The description of $\mathrm{Bl}_I(A)$ via $U(a)$'s glued along isomorphisms as in (1.1) will be useful in later calculations, but for other purposes it is convenient to have an alternative description via Proj, as follows. The A-algebra $\mathrm{Gr}_I(A)=\oplus_{n\geq 0}I^n$ is graded and is generated by the first graded piece. The natural map of A-algebras $\mathrm{Gr}_I(A)\to A$ defined by the inclusion on each graded piece carries $a_1:=a\in I=(\mathrm{Gr}_I(A))_1$ to $a\in A$, so we get a natural A-algebra map $(\mathrm{Gr}_I(A))_{a_1}\to A_a$. This latter map is injective on the 0th graded piece of the source, so it induces an A-algebra isomorphism

$$(\mathrm{Gr}_I(A))_{(a)}\simeq A[Ia^{-1}].$$

Thus, we see that $\mathrm{Bl}_I(A)\simeq\mathrm{Proj}(\mathrm{Gr}_I(A))$ as A-schemes, and this isomorphism is unique since $\mathrm{Bl}_I(A)$ has no non-trivial A-automorphisms. In particular, $\mathrm{Bl}_I(A)$ is projective over A when I is finitely generated, but usually not otherwise (as it may not even be quasi-compact).

For any ideal J in A, the *strict transform* $\widetilde{V(J)}$ of $V(J)=\mathrm{Spec}(A/J)$ in $\mathrm{Bl}_I(A)$ is a closed subscheme of the pullback of $V(J)$ under $\mathrm{Bl}_I(A)\to\mathrm{Spec}(A)$ and it is $V(J)$-isomorphic to the blow-up of $V(J)$ along the pullback ideal $(I+J)/J\subseteq A/J$. Explicitly, over an open $U(a)=\mathrm{Spec}(A[Ia^{-1}])$ in $\mathrm{Bl}_I(A)$, $\widetilde{V(J)}$ cuts out the closed subscheme $V(J_a\cap A[Ia^{-1}])\subseteq U(a)$.

We can globalize this to the setting of a scheme X and a quasi-coherent ideal sheaf $\mathscr{I}$ on X (or, equivalently, a closed subscheme Y of X, with $\mathscr{I}_Y=\mathscr{I}$) as follows. We define the X-scheme

$$\mathrm{Bl}_{\mathscr{I}}(X)=\mathrm{Bl}_Y(X):=\mathrm{Proj}(\mathrm{Gr}_{\mathscr{I}}(\mathscr{O}_X)),$$

often denoted $\widetilde{X}$ if $\mathscr{I}$ is understood. Note that if $\mathscr{I}$ is of finite type then the map $\mathrm{Bl}_{\mathscr{I}}(X)\to X$ is projective in the sense of [EGA, II, 5.5.2] (so in particular, this map is proper). In the non-noetherian case it may happen that $\mathscr{I}$ is not of finite type, so the blow-up morphism may fail to be projective (or even quasi-compact). Thus, we have to be careful about which ideals we blow up. We call Y the *center* of the blow-up, so the structure map $q:\widetilde{X}\to X$ is an isomorphism over the open complement $U=X-Y$ of the center of the blow-up. Strictly speaking, the blow-up morphism does not uniquely determine the center as we have defined it (e.g., we do not require the center to have support equal to the minimal closed set in the base away from which the blow-up morphism is an isomorphism). This will not matter for us since we use the notion of "center" purely as a linguistic device.

By construction, the quasi-coherent ideal $\mathscr{I}\cdot\mathscr{O}_{\widetilde{X}}$ is an invertible ideal sheaf on $\widetilde{X}=\mathrm{Bl}_{\mathscr{I}}(X)$, so the open subscheme $j:U\hookrightarrow\widetilde{X}$ is *schematically dense* in the sense that $\mathscr{O}_{\widetilde{X}}\to j_*\mathscr{O}_U$ is injective [EGA, IV_3, 11.10.2]; this amounts to the fact that for a ring R and $r\in R$ that is not a zero divisor, the map $R\to R_r$ is injective. By [EGA, IV_3, 11.10.4], the schematic density implies that U is also dense as a subspace of the topological space $\widetilde{X}$. If $\mathscr{I}$ is of finite type then $U=X-Y\to X$ is quasi-compact and hence the open immersion $U\to\widetilde{X}$ is *quasi-compact* [EGA, IV_1, 1.1.2(v)]. This sort of reasoning will be used frequently below, and it is another reason why we generally only blow up finite type quasi-coherent ideal sheaves when avoiding noetherian hypotheses.

As in the case of affine X that we considered at the outset, in general the X-scheme $\widetilde{X}$ is the final object in the category of X-schemes $f:X'\to X$ such that $\mathscr{I}\cdot\mathscr{O}_{X'}$ is invertible. This is easily reduced to checking locally on X, so we can assume X is affine, and then we can work locally on $\widetilde{X}$ by using the affine open covering by $U(a)$'s for $a\in I=\Gamma(X,\mathscr{I})$. We omit the straightfoward details. For a closed subscheme Z in X the *strict transform* $\widetilde{Z}$ of Z in X is defined to be $\mathrm{Bl}_{\mathscr{I}\cdot\mathscr{O}_Z}(Z)$, and the canonical map $\widetilde{Z}\to\widetilde{X}$ over $Z\hookrightarrow X$ is a closed immersion because we can check this over an open affine covering of X (where it has been checked above). For example, if $\mathscr{I} = \mathscr{I}_Z$ then $\widetilde{Z}$ is empty. We can give an alternative (and useful!) description of the strict transform of Z in $\widetilde{X}$, as follows:

Lemma 1.1. *Let X be a scheme, Y and Z closed subschemes of X, and $U = X - Y$ the open complement of Y. We view $Z \cap U$ as a closed subscheme of U, which in turn is an open subscheme of $\widetilde{X} = \mathrm{Bl}_Y(X)$. The immersion $j : Z \cap U \to \widetilde{X}$ is quasi-compact (even affine) and its scheme-theoretic closure in $\widetilde{X}$ is the closed subscheme $\widetilde{Z}$.*

Proof. This can be checked locally on X, so we may assume $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(A/I)$, $Z = \mathrm{Spec}(A/J)$, so $\widetilde{X} = \mathrm{Proj}(\mathrm{Gr}_I(A))$. The scheme $Z-(Z\cap Y) = Z\cap U$ is is covered by open affines of the form $\mathrm{Spec}((A/J)_a)$, with $a \in I$. For $a \in I$, we let $Z_a = \mathrm{Spec}((A/J)_a)$ and $X_a = \mathrm{Spec}(A[Ia^{-1}])$ (this was denoted $U(a)$ above). We have a canonical map

$$Z_a \to X_a \hookrightarrow \widetilde{X}.$$

If we can show that $j^{-1}(X_a) = Z_a$ then j is affine and the kernel of $\mathscr{O}_{\widetilde{X}} \to j_*(\mathscr{O}_{Z\cap U})$ over X_a is exactly the quasi-coherent sheaf associated to the kernel of $A[Ia^{-1}] \to (A/J)_a = A_a/J_a$. Since this kernel is just $J_a \cap A[Ia^{-1}]$, we will then have what we wanted (due to the earlier explicit description of strict transforms in the affine setting).

In order to see that $j^{-1}(X_a) = Z_a$, we compute that for any $a' \in I$, $j^{-1}(X_a) \cap Z_{a'} \simeq Z_{a'} \times_{X_{a'}} (X_{a'} \cap X_a)$ is exactly

$$\mathrm{Spec}((A/J)_{a'} \otimes_{A[Ia'^{-1}]} A[Ia'^{-1}]_{a/a'}) = \mathrm{Spec}(((A/J)_{a'})_{a/a'}) = \mathrm{Spec}((A/J)_{aa'}).$$

This is exactly the open subscheme $Z_{a'} \cap Z_a$ inside of $Z \cap U$. Since $Z \cap U$ is covered by the open subschemes $Z_{a'}$ for $a' \in I$, taking the union over all $a' \in I$ yields $j^{-1}(X_a) = Z_a$, as desired. ■

For a blow-up $q : \widetilde{X} = \mathrm{Bl}_{\mathscr{I}}(X) \to X$ and a section s of $\mathscr{I}$ over an open subscheme $W \subseteq X$, we define $U(s)$ to be the open subscheme of $q^{-1}(W)$ over which $q^*(s)$, viewed as a section of $\mathscr{I} \cdot \mathscr{O}_{\widetilde{X}}$, is a local generator. We will sometimes abuse notation and speak of s, rather than $q^*(s)$, as a section of $\mathscr{I} \cdot \mathscr{O}_{\widetilde{X}}$ when there is no possibility of confusion. When W is affine, this definition for $U(s)$ coincides with the one given earlier.

A couple of points concerning composites of blow-ups will be useful in our later considerations. Let $\mathscr{I}_1$ and $\mathscr{I}_2$ be two quasi-coherent ideal sheaves on X and let $\widetilde{X}_i = \mathrm{Bl}_{\mathscr{I}_i}(X)$. There is a natural commutative diagram

(1.2)
$$\begin{array}{ccc} \widetilde{X}_{12} & \longrightarrow & \widetilde{X}_1 \\ \downarrow & & \downarrow \\ \widetilde{X}_2 & \longrightarrow & X \end{array}$$

in which $\widetilde{X}_{12} = \mathrm{Bl}_{\mathscr{I}_1 \mathscr{I}_2}(X)$. To make this diagram, first observe that $\widetilde{X}_{12}$ is covered by opens $U(s_1 s_2)$ for s_1 and s_2 respective sections of $\mathscr{I}_1$ and $\mathscr{I}_2$ over a common open W in X. On $U(s_1 s_2)$, neither s_1 nor s_2 is a zero divisor. For any section t_1 of $\mathscr{I}_1$ over W, $t_1 s_2$ is a section of $\mathscr{I}_1 \mathscr{I}_2 \cdot \mathscr{O}_{\widetilde{X}_{12}}$ over $U(s_1 s_2)$ and so $t_1 s_2$ is a multiple of $s_1 s_2$ here. Thus, t_1 is a multiple of s_1, so $\mathscr{I}_1|_{U(s_1 s_2)}$ has s_1 as a basis. In this way, we see that $\mathscr{I}_1 \cdot \mathscr{O}_{\widetilde{X}_{12}}$ is invertible on $\widetilde{X}_{12}$, and likewise with $\mathscr{I}_2$ replacing $\mathscr{I}_1$. The existence of (1.2) then follows from the universal property of blow-ups, and the map $\widetilde{X}_{12} \to \widetilde{X}_j$ carries $U(s_1 s_2)$ into $U(s_j)$. Moreover, the resulting natural map from $\widetilde{X}_{12}$ to the blow-up of $\widetilde{X}_1$ along $\mathscr{I}_2 \cdot \mathscr{O}_{\widetilde{X}_1}$ carries $U(s_1 s_2)$ into $(U(s_1))(s_2)$, and for $W = \mathrm{Spec}(A) \subseteq X$ the $U(s_1)$-map $U(s_1 s_2) \to (U(s_1))(s_2)$ corresponds to the A-algebra map $(A[I_1^{-1} s_1])[I_2 s_2^{-1}] \to A[I_1 I_2 (s_1 s_2)^{-1}]$ that is an isomorphism by inspection (equality inside $A_{s_1 s_2}$). Since the $U(s_1 s_2)$'s (resp. the $(U(s_1))(s_2)$'s) cover $\widetilde{X}_{12}$ (resp. $\mathrm{Bl}_{\mathscr{I}_2 \cdot \mathscr{O}_{\widetilde{X}_1}}(\widetilde{X}_1)$) as we vary W and s_1, s_2, by using the known gluing data on such opens we thereby see that the natural map $\widetilde{X}_{12} \to \mathrm{Bl}_{\mathscr{I}_2 \cdot \mathscr{O}_{\widetilde{X}_1}}(\widetilde{X}_1)$ is an isomorphism. Likewise, $\widetilde{X}_{12}$ is identified with the blow-up of $\widetilde{X}_2$ along $\mathscr{I}_1 \cdot \mathscr{O}_{\widetilde{X}_2}$. In particular, by induction there are unique X-isomorphisms $\mathrm{Bl}_{\mathscr{I}^n}(X) \simeq \mathrm{Bl}_{\mathscr{I}}(X)$ for all $n \geq 1$; these isomorphisms are induced by applying relative Proj to the obvious explicit map on the level of quasi-coherent graded $\mathscr{O}_X$-algebras.

We begin our sequence of blow-up lemmas with an important result which essentially says that a composite of blow-ups is the same as a single blow-up. Though some later results could be stated in terms of a finite sequence of blow-ups if we wished to avoid Lemma 1.2 below, in other places (e.g., applications of Lemma 2.7) it seems essential that we only require a single blow-up. Lemma 1.2 is not explicitly mentioned in [D], but Deligne must have had it in mind. First, we introduce some terminology. If X is a scheme and $U \subseteq X$ is an open subscheme, we say that an X-scheme $f : X' \to X$ is a *U-admissible blow-up* of X if it is X-isomorphic to $\mathrm{Bl}_{\mathscr{I}}(X)$, where $\mathscr{I}$ is a quasi-coherent ideal sheaf on X of *finite type* (automatic if X is locally noetherian) and $V(\mathscr{I})$ is disjoint from U. In particular, f is proper (even projective).

Lemma 1.2. [GR, Lemma 5.1.4] *Let X be a quasi-compact and quasi-separated scheme, $U \hookrightarrow X$ a quasi-compact open immersion, $f : X' \to X$ a U-admissible blow-up, and $g : X'' \to X'$ an $f^{-1}(U)$-admissible blow-up. Then $f \circ g : X'' \to X$ is a U-admissible blow-up.*

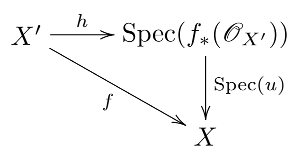
Any noetherian scheme X and open subscheme $U \subseteq X$ satisfy the initial topological hypotheses in this lemma.

Proof. We give a more detailed version of the proof in [GR] because we include an additional argument due to Raynaud which plays a critical role in the proof (and which is not mentioned in [GR]). We have $X' = \mathrm{Bl}_{\mathscr{I}}(X)$ as an X-scheme and $X'' = \mathrm{Bl}_{\mathscr{K}}(X')$ as an X'-scheme with $\mathscr{I}$ and $\mathscr{K}$ quasi-coherent and finite type ideal sheaves on X and X' respectively such that $V(\mathscr{I}) \subseteq X - U$ and $V(\mathscr{K}) \subseteq X' - f^{-1}(U)$ respectively (as sets). Since $\mathscr{I}$ is of finite type, $X' \to X$ is projective and $\mathscr{I}' = \mathscr{I} \cdot \mathscr{O}_{X'}$ is an X-ample invertible sheaf [EGA, II, 5.5.1, 8.1.7]. Thus, by [EGA, II, 4.6.8; IV_1, 1.7.15], the natural map

$$f^* f_*(\mathscr{K} \otimes {\mathscr{I}'}^{\otimes n}) \to \mathscr{K} \otimes {\mathscr{I}'}^{\otimes n}$$

is surjective for all large n. But $\mathscr{K} \otimes {\mathscr{I}'}^{\otimes n} \simeq \mathscr{K}\, {\mathscr{I}'}^{n}$ (since $\mathscr{I}'$ is invertible) and ${\mathscr{I}'}^{n} = \mathscr{I}^n \cdot \mathscr{O}_{X'}$, so replacing $\mathscr{I}$ by a suitable power allows us to assume that the natural map $f^* f_*(\mathscr{K}\, \mathscr{I}') \to \mathscr{K}\, \mathscr{I}'$ is surjective.

Since f is quasi-compact and separated (even proper), $f_* \mathscr{O}_{X'}$ is a quasi-coherent $\mathscr{O}_X$-algebra. Hence, the natural map $u : \mathscr{O}_X \to f_* \mathscr{O}_{X'}$ gives rise to a commutative diagram



If X is noetherian then $f_*(\mathscr{O}_{X'})$ is coherent, so $\mathrm{Spec}(u)$ is finite. Moreover, in this case $\mathrm{Spec}(u)$ is an isomorphism over U and (for similar reasons) the coherent $\mathscr{O}_X$-module $f_*(\mathscr{K}\, \mathscr{I}')$ coincides with $\mathscr{O}_X$ over U. The noetherian reader should define $\mathscr{N} = f_*(\mathscr{K}\, \mathscr{I}')$ and skip the rest of this paragraph and the next one. In the general case we claim that $\mathrm{Spec}(u)$ is integral. To check this, we may assume that $X = \mathrm{Spec}(A)$ is affine and we want to show that for a finitely generated ideal $I = (a_0, \ldots, a_m)$ in A, every global function on $P = \mathrm{Proj}(\mathrm{Gr}_I(A)) \subseteq \mathbf{P}^m_A$ is integral over A. If A is noetherian, then the A-module of global functions is finite and thus we have the integrality assertion. Let's reduce to the noetherian case, even though P may not be finitely presented over A. This is a standard limit argument, as we now explain. Choose a global function s on P. This is equivalent to specifying $s_i \in (\mathrm{Gr}_I(A))_{(a_i)}$ for $0 \leq i \leq m$, with $s_i = s_j$ under the canonical identification $((\mathrm{Gr}_I(A))_{(a_i)})_{a_j/a_i} = ((\mathrm{Gr}_I(A))_{(a_j)})_{a_i/a_j}$. Because of the direct sum in the definition of $\mathrm{Gr}_I(A)$, this data only involves a *finite* number of elements of A. Thus, if we consider a suitable finite-type $\mathbf{Z}$-subalgebra A_0 in A depending on s and containing $a_0, \ldots, a_m$, and let I_0 be the ideal generated by the a_i's in A_0, then we get a global function s_0 on $P_0 = \mathrm{Proj}(\mathrm{Gr}_{I_0}(A_0))$ inducing s after pullback along the natural map $r : P \to P_0$ over $\mathrm{Spec}(A) \to \mathrm{Spec}(A_0)$. Note that r is "defined everywhere" since $I_0 A = I$. The induced map of rings

$$\Gamma(P_0, \mathscr{O}_{P_0}) \to \Gamma(P_0, r_* \mathscr{O}_P) = \Gamma(P, \mathscr{O}_P)$$

over $A_0 \to A$ takes s_0 to s. Since s_0 is integral over A_0, it follows that s is integral over A.

Clearly $\mathrm{Spec}(u)$ is an isomorphism over U and the quasi-coherent $\mathscr{O}_X$-module $f_*(\mathscr{K}\mathscr{I}')$ coincides with $\mathscr{O}_X$ over U. We claim that there is a quasi-coherent finite type $\mathscr{O}_X$-submodule $\mathscr{N} \subseteq f_*(\mathscr{K}\mathscr{I}')$ such that $\mathscr{N}|_U = \mathscr{O}_X|_U$ and the natural map $f^*(\mathscr{N}) \to \mathscr{K}\mathscr{I}'$ is surjective. By [EGA, II, 9.4.7ff; IV_1, 1.7.7], we have $f_*(\mathscr{K}\mathscr{I}') = \varinjlim \mathscr{N}_i$, with $\mathscr{N}_i \subseteq f_*(\mathscr{K}\mathscr{I}')$ running through the finite type quasi-coherent $\mathscr{O}_X$-submodules equal to $\mathscr{O}_X|_U$ over U. Since

$$\varinjlim f^*(\mathscr{N}_i) \simeq f^*(\varinjlim \mathscr{N}_i) \simeq f^* f_*(\mathscr{K}\mathscr{I}') \to \mathscr{K}\mathscr{I}'$$

is surjective, $\mathscr{K}\mathscr{I}'$ is of finite type, and X is quasi-compact, for suitably large i_0 we have that $\mathscr{N} \stackrel{\mathrm{def}}{=} \mathscr{N}_{i_0}$ is a quasi-coherent $\mathscr{O}_X$-submodule of $f_*(\mathscr{K}\mathscr{I}')$ which is of finite type over $\mathscr{O}_X$, satisfies $\mathscr{N}|_U = \mathscr{O}_X|_U$, and has the property that the natural map $f^*(\mathscr{N}) \to \mathscr{K}\mathscr{I}'$ is surjective.

The integrality of $\mathrm{Spec}(u)$ ensures that the quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{O}_{X_0} \subseteq f_*(\mathscr{O}_{X'})$ generated by the finite type $\mathscr{O}_X$-module $\mathscr{N}$ (i.e., the image of $\oplus_{m \geq 0} \mathscr{N}^{\otimes m} \to f_*(\mathscr{O}_{X'})$) is a *finite* $\mathscr{O}_X$-algebra. In the noetherian case (with $\mathscr{N} = f_*(\mathscr{K}\mathscr{I}')$) we have $\mathscr{O}_{X_0} = \mathscr{O}_X + f_*(\mathscr{K}\mathscr{I}') \subseteq f_*(\mathscr{O}_{X'})$. In general, let $\mathscr{N}_0 \subseteq \mathscr{O}_{X_0}$ be the finite type ideal generated by $\mathscr{N}$ (i.e., the image of $\oplus_{m \geq 1} \mathscr{N}^{\otimes m} \to \mathscr{O}_{X_0}$). Note that $\mathscr{O}_X|_U \simeq \mathscr{O}_{X_0}|_U$. Now comes the most critical step in the proof. We claim that there exists a positive integer m such that

$$\mathscr{N}_0^m = u(\mathscr{M}) \cdot \mathscr{O}_{X_0} \tag{1.3}$$

for some quasi-coherent finite type ideal sheaf $\mathscr{M}$ in $\mathscr{O}_X$ with $\mathscr{M}|_U = \mathscr{O}_X|_U$ (here we view u as a map $\mathscr{O}_X \to \mathscr{O}_{X_0}$). The proof of the existence of m in [GR] is unclear. Assume this existence for now, and let's see how this enables us to finish the proof.

Using the commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & \mathrm{Spec}(f_*(\mathscr{O}_{X'})) \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \longleftarrow & \mathrm{Spec}(\mathscr{O}_{X_0}) \end{array}$$

we see that $\mathscr{M} \cdot \mathscr{O}_{X'}$ is the mth power of the ideal generated by the image of $f^*(\mathscr{N}) \to \mathscr{O}_{X'}$, which is nothing other than $(\mathscr{K}\mathscr{I}')^m$. Since $\mathscr{I}'$ is invertible on X', X'' is X'-isomorphic to the blow-up of X' along $(\mathscr{K}\mathscr{I}')^m = \mathscr{M} \cdot \mathscr{O}_{X'}$. Thus, $X'' \simeq \mathrm{Bl}_{\mathscr{M}\mathscr{I}}(X)$ as X-schemes. Since $\mathscr{M}\mathscr{I}$ is a quasi-coherent finite type ideal in $\mathscr{O}_X$ and restricts to $\mathscr{O}_X|_U$ on U, this blow-up is U-admissible and we are done.

It remains to find an integer m for which (1.3) holds. The following argument is due to Raynaud. We will find an $m > 0$ so that $\mathscr{N}_0^m \subseteq u(\mathscr{O}_X)$ inside of $\mathscr{O}_{X_0}$. This is somewhat stronger than the original assertion, as we'll soon see, and this condition has the important technical advantage that it can be checked locally on our quasi-compact base X (so we will be able to reduce to the case of an affine base). Let's show that any m such as we just described is also adequate for our original needs above. Define $\mathscr{M}' = u^{-1}(\mathscr{N}_0^m) \subseteq \mathscr{O}_X$, so $\mathscr{M}'$ is a quasi-coherent ideal sheaf on X satisfying $\mathscr{M}'|_U = \mathscr{O}_X|_U$. Thus, $\mathscr{M}'$ satisfies all of the requirements except that in the non-noetherian case it may not be of finite type. The noetherian reader should skip to the next paragraph. To handle the general case we again use a simple direct limit argument, as follows. We may write $\mathscr{M}' = \varinjlim \mathscr{M}_j$ with $\mathscr{M}_j \subseteq \mathscr{M}'$ running through the finite type quasi-coherent ideal sheaves in $\mathscr{M}'$ such that $\mathscr{M}_j|_U = \mathscr{O}_X|_U$. Since $\varinjlim u(\mathscr{M}_j) = \mathscr{N}_0^m$ and $\mathscr{N}_0^m$ is *finite* as an $\mathscr{O}_X$-module (since $\mathscr{O}_{X_0}$ and $\mathscr{N}$ are), we conclude from the quasi-compactness of X that $u(\mathscr{M}_j) = \mathscr{N}_0^m$ for large j. For large j, taking $\mathscr{M} = \mathscr{M}_j$ gives what we needed above.

To find m such that $\mathscr{N}_0^m \subseteq u(\mathscr{O}_X)$ inside of $\mathscr{O}_{X_0}$, we can work locally on the quasi-compact X and so may now assume $X = \mathrm{Spec}(A)$. Let $I = \Gamma(X, \mathscr{I})$, $K = \Gamma(X, \mathscr{N})$, $B = \Gamma(X, \mathscr{O}_{X_0})$. Since the ideal K in B is finitely generated as an A-module, it suffices to choose $b \in K \subseteq B$ and to show that $b^N B \subseteq u(A)$ for some large N (perhaps depending on b). We first claim that the open affine $W = \mathrm{Spec}(B_b) \subseteq \mathrm{Spec}(B)$ lies over an open subscheme V in $\mathrm{Spec}(A)$ over which $X' \to X$ is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & \mathrm{Spec}(B) \\ & \searrow^{f} & \downarrow \\ & & X \end{array}$$

Since the ideal generated by K in B contains b and pulls back (under g) to the ideal sheaf $\mathscr{K}\,\mathscr{I}'$ on X' (by the choice of $\mathscr{N}$), we see that $\mathscr{O}_{X'}|_{g^{-1}(W)} = \mathscr{K}\,\mathscr{I}'|_{g^{-1}(W)} \subseteq \mathscr{I}'|_{g^{-1}(W)}$. Thus, $\mathscr{I}'|_{g^{-1}(W)} = \mathscr{O}_{X'}|_{g^{-1}(W)}$, so $g^{-1}(W)$ lies over $X - V(\mathscr{I})$. But g is proper (since f is) and dominant (since $B \to \mathscr{O}_{X'}(X')$ is injective by construction of g, so [EGA, I, 9.5.4] applies). Hence, g is surjective, so W also lies over $X - V(\mathscr{I})$. That is, $\mathrm{Spec}(B_b)$ lies over an open $V \subseteq \mathrm{Spec}(A)$ for which $\mathscr{O}_X|_V \simeq f_*\mathscr{O}_{X'}|_V$. In particular, since $u(\mathscr{O}_X) \subseteq \mathscr{O}_{X_0} \subseteq f_*\mathscr{O}_{X'}$, we have $\mathscr{O}_X|_V \simeq \mathscr{O}_{X_0}|_V$. But f is an isomorphism over V, so $\mathrm{Spec}(B_b) \to \mathrm{Spec}(A)$ is an open immersion.

Choose a finite open affine covering $\{\mathrm{Spec}(A_{f_i})\}$ of the open subset of $\mathrm{Spec}(A)$ given by the image of $\mathrm{Spec}(B_b)$, so the f_i's generate the unit ideal in B_b, which is to say that there exists some large N for which

$$b^N B \subseteq \sum f_i B.$$

Since $X' \to X$ is an isomorphism over V and g is surjective, $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ restricts to an isomorphism over the open subscheme $\bigcup \mathrm{Spec}(A_{f_i}) \subseteq V$ and hence the map $A_{f_i} \to B_{f_i}$ is an isomorphism for all i. Let $b_1, \dots, b_t$ generate B as an A-module. Via $A_{f_i} \simeq B_{f_i}$, we have $b_j = a_j/f_i^M$ in B_{f_i}, for $a_j \in A$ and M large. Thus, there is a large M' such that

$$(f_i^M b_j - a_j) f_i^{M'} = 0$$

in B for all i and j, so for $N' = M + M'$ we have $f_i^{N'} b_j \in u(A)$ inside B for all i and j. This implies that $f_i^{N'} B \subseteq u(A)$ for all i. Since we also have that $b^N B \subseteq \sum f_i B$ for some large N, we conclude that there is a large N'' such that $b^{N''} B \subseteq u(A)$. This is what we needed to prove. ■

Since we are primarily interested only in blow-ups along finitely presented closed subschemes (i.e., along finite type quasi-coherent ideal sheaves), we now give a simple criterion for the complement of an open set to admit the structure of a finitely presented closed subscheme. The following lemma should be skipped by the noetherian reader.

Lemma 1.3. *Let X be a quasi-compact and quasi-separated scheme, U an open subscheme which is quasi-compact. There exists a finite type quasi-coherent ideal sheaf $\mathscr{I}$ such that $V(\mathscr{I}) = X - U$.*

By quasi-separatedness of X, the quasi-compactness condition on U is equivalent to saying that the open immersion $U \to X$ is quasi-compact.

Proof. Let $\mathscr{K}$ be the quasi-coherent ideal sheaf associated to the canonical reduced closed subscheme structure on the closed subset $X - U$ in X. By [EGA, I, 9.4.9; IV_1, 1.7.7], $\mathscr{K} = \varinjlim \mathscr{K}_i$, with $\{\mathscr{K}_i\}$ the set of finite type quasi-coherent subideals in $\mathscr{K}$. We can consider $\{V(\mathscr{K}_i)\}$ as a decreasing inverse system of closed subsets of X, with intersection $V(\mathscr{K})$ (as sets). We want to check that this inverse system terminates for large i. Since X is quasi-compact and quasi-separated, we can assume $X = \mathrm{Spec}(A)$ is affine. Let $J = \Gamma(X, \mathscr{K})$, $J_i = \Gamma(X, \mathscr{K}_i)$, so the J_i are a cofinal system of finitely generated ideals in J. The quasi-compactness of U allows us to cover U by finitely many open affines $\mathrm{Spec}(A_{f_j})$ with $f_j \in J$. Taking i so large that J_i contains all of the f_j's, we have $V(J_i) = V(J)$ as subsets of X. ■

Lemma 1.4. [D, Lemme 0.2] *For closed subschemes Y and Y' in a scheme X, the strict transform $\widetilde{Y}$ of Y in $\widetilde{X} = \mathrm{Bl}_{Y \cap Y'}(X)$ lies inside of the union of the $U(s)$'s as s ranges through all sections of $\mathscr{I}_{Y'}$ over all opens. Moreover, if $Y_1, \dots, Y_n$ are closed subschemes of X then the strict transforms $\widetilde{Y}_j$ in $\mathrm{Bl}_{\cap Y_i}(X)$ satisfy $\cap \widetilde{Y}_j = \emptyset$.*

The mutual disjointness of strict transforms in the blow-up corresponds to [N1, Prop. 2.4].

Proof. We may assume $X = \mathrm{Spec}(A)$ is affine. For the first claim, let $I' = \Gamma(X, \mathscr{I}_{Y'})$ and $I = \Gamma(X, \mathscr{I}_Y)$. Clearly $\widetilde{X}$ is covered by $U(s)$ for $s \in I \cup I'$, so it is enough to show that for $s \in I$, $U(s)$ is disjoint from the strict transform $\widetilde{V(I)}$. In terms of commutative algebra, the overlap of $U(s)$ and $\widetilde{V(I)}$ in $\widetilde{X}$ is the affine scheme attached to the ring

$$A[(I+J)s^{-1}]/(I_s \cap A[(I+J)s^{-1}]),$$

with $I_s \cap A[(I+J)s^{-1}]$ taken inside of A_s. Since $s \in I$ we have $1 \in I_s \cap A[(I+J)s^{-1}]$, so the overlap is empty. For the second claim, let $I_j = \Gamma(X, \mathscr{I}_{Y_j})$, so since $\mathscr{I}_{\cap Y_j} = \sum \mathscr{I}_{Y_j}$ we have that the $U(s)$'s for $s \in \cup I_j$ cover $\widetilde{X} = \mathrm{Bl}_{\cap Y_j}(X)$. It therefore suffices to show that for $s \in I_1$ the open set $U(s)$ is disjoint from $\widetilde{V(I_1)}$. This goes exactly as in the preceding calculation, setting $I = I_1$ and $J = I_2 + \cdots + I_n$ (so $J = 0$ if $n = 1$). ■

Now we come to the first of the separatedness lemmas which we will need to prove. The lemma has a disjointness hypothesis and asserts that after a certain admissible blow-up we can get a stronger disjointness property for closures. We will later have more subtle lemmas of this sort.

To set up the necessary notation, consider a commutative diagram of schemes

$$\begin{array}{ccccc} Y & & \xrightarrow{\quad f \quad} & & Z \\ \downarrow & & & & \downarrow{\scriptstyle p} \\ U & \xrightarrow[i]{} & V & \xrightarrow[j]{} & X \end{array}$$

We assume that i and j are open immersions and f is a quasi-compact immersion. Obviously Y and $p^{-1}(V-U)$ are disjoint in Z, but $\overline{Y}$ and $p^{-1}(\overline{V-U})$ may meet. We will eliminate this via base change by a blow-up of X with center in $X - V$. Let $\mathscr{I}_1$ be a quasi-coherent ideal sheaf on X with $\overline{V-U} \subseteq V(\mathscr{I}_1) \subseteq X - U$ as sets. (In Case 3 in the proof of Theorem 2.4 we will use $\mathscr{I}_1$ satisfying $V(\mathscr{I}_1) = \overline{V-U}$ in the noetherian case. In the proof of Lemma 3.1 we will use $\mathscr{I}_1$ satisfying $V(\mathscr{I}_1) = X - U$.) Define $T_1 = V(\mathscr{I}_1^2)$, roughly a square-zero thickening of $\overline{V-U}$.

Let $\mathscr{I}_2$ be a quasi-coherent ideal sheaf on X such that $T_2 = V(\mathscr{I}_2)$ lies inside of $T_1 \cap (X - V)$ as a set. Setting $\overline{Y}$ to be the scheme-theoretic image of Y in Z under the quasi-compact immersion f, we assume that the closed subscheme $p^{-1}(T_1) \cap \overline{Y}$ in Z (which is set-theoretically contained in $\overline{Y} - Y$ since $T_1 \cap U = \emptyset$) factors through the closed subscheme $p^{-1}(T_2)$. Loosely speaking, we have

$$p(p^{-1}(T_1) \cap \overline{Y}) \subseteq T_2 \subseteq T_1 \cap (X - V),$$

with the first inclusion as schemes and the second as sets. This strange-looking hypothesis is motivated by specific situations that arise in the proofs of Theorem 2.4 and Lemma 3.1.

Let $q : X' \to X$ be the blow-up of X along $\mathscr{I}_1 + \mathscr{I}_2$ (an ideal cutting out $T_1 \cap T_2$ as a set, but usually not as a scheme). Let Y', Z', p', etc. denote the base changes of Y, Z, p, etc. by q. Note that q is an isomorphism over V, since $V(\mathscr{I}_1 + \mathscr{I}_2) \subseteq V(\mathscr{I}_2) = T_2$ is disjoint from V. After base change by q, we have a disjointness assertion:

Lemma 1.5. [D, Lemme 0.3] *With the above notation and hypotheses, the closed subsets*

$$\overline{Y'},\ \ {p'}^{-1}(\overline{q^{-1}(V) - q^{-1}(U)}) = {p'}^{-1}(\overline{q^{-1}(V-U)}) \subseteq Z'$$

are disjoint. In other words, after the base change by q, $p(\overline{Y})$ and $\overline{V-U}$ are disjoint.

Proof. Since the formation of closures in topological spaces (resp. scheme-theoretic images of quasi-compact immersions) commutes with passage to an open subspace (resp. open subscheme), our problem is local on both X and Z, so we may assume $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$. Let $I_j = \Gamma(X, \mathscr{I}_j)$. In this setting, we will try to locally construct functions on Z' that vanish on ${p'}^{-1}(\overline{q^{-1}(V-U)})$ and equal 1 on $\overline{Y'}$.

First, we claim that the opens $U(s_2) \subseteq X'$ for $s_2 \in I_2$ cover $\overline{q^{-1}(V-U)}$. To show this, note that the strict transform of $V(\mathscr{I}_1)$ is a closed subscheme of X'. Since the underlying space of $V(\mathscr{I}_1)$ contains $\overline{V-U} \supseteq V - U$ and we have $q^{-1}(V-U) \simeq V - U$ as topological spaces, we see that $\overline{q^{-1}(V-U)}$ lies inside of the strict transform of $V(\mathscr{I}_1)$. By the proof of Lemma 1.4 in the affine case, we conclude that $\overline{q^{-1}(V-U)}$

is covered by the $U(s_2)$'s for $s_2 \in I_2$, as desired. Thus, the ${p'}^{-1}(U(s_2))$'s for $s_2 \in I_2$ give an open covering of ${p'}^{-1}(\overline{q^{-1}(V-U)})$. It therefore is enough to show that for all $s_2 \in I_2$,

$$ {p'}^{-1}(\overline{q^{-1}(V-U)}) \cap {p'}^{-1}(U(s_2)) \cap \overline{Y'} = \emptyset. $$

Since we have the set-theoretic inclusions

$$ {p'}^{-1}(\overline{q^{-1}(V-U)}) \subseteq {p'}^{-1}(q^{-1}(\overline{V-U})) \subseteq {p'}^{-1}(q^{-1}(T_1)), $$

it suffices to show that for all $s_2 \in I_2$, we have set-theoretically in Z' that

$$ {p'}^{-1}(q^{-1}(T_1)) \cap {p'}^{-1}(U(s_2)) \cap \overline{Y'} = \emptyset. \tag{1.4} $$

By hypothesis, $p^{-1}(T_1) \cap \overline{Y} \subseteq p^{-1}(T_2)$ as closed subschemes of $Z = \mathrm{Spec}(B)$, so $p^*(s_2) \in B$ vanishes on $p^{-1}(T_2) \cap \overline{Y}$. Thus,

$$ p^*(s_2) = t_1 + g $$

in B, with t_1 vanishing on the closed subscheme $p^{-1}(T_1)$ cut out by $I_1^2 B$ and g vanishing on the closed subscheme $\overline{Y}$. The quasi-compact immersion $Y \to Z$ makes Y an open subscheme of $\overline{Y}$, so if we write $t_1 = \sum b_i s_{1i} r_{1i}$ with $s_{1i}, r_{1i} \in I_1$ and $b_i \in B$, we have

$$ s_2|_Y = \sum b_i s_{1i} r_{1i} $$

on the subscheme $Y \hookrightarrow Z = \mathrm{Spec}(B)$.

Because Y lives over $U \subseteq V$ and V does not meet T_2, we see that

$$ Y' \cap {p'}^{-1}(U(s_2)) \subseteq {p'}^{-1}(U(s_2) \cap (X' - q^{-1}(T_2))). $$

However, s_2 doesn't vanish at any points on $U(s_2) \cap (X' - q^{-1}(T_2))$, so s_2 is a unit here. Thus, the pullback of s_2 to Z' is a unit on the subscheme $Y' \cap {p'}^{-1}(U(s_2))$, whence

$$ \sum b_i \cdot \left(\frac{s_{1i}}{s_2}\right) \cdot r_{1i} = 1 $$

on the subscheme $Y' \cap {p'}^{-1}(U(s_2))$. But $h = \sum b_i (s_{1i}/s_2) r_{1i}$ is a global function on the open subscheme ${p'}^{-1}(U(s_2))$ in Z', so since $h = 1$ on the subscheme $Y' \cap {p'}^{-1}(U(s_2))$, we see that $h = 1$ on its scheme-theoretic image $\overline{Y'} \cap {p'}^{-1}(U(s_2))$ in ${p'}^{-1}(U(s_2))$ (recall that $Y' \cap {p'}^{-1}(U(s_2)) \to {p'}^{-1}(U(s_2))$ is a quasi-compact immersion and use [EGA, I, 9.5.4, 9.5.8; IV_1, 1.7.8]).

On the other hand, h vanishes on the subscheme ${p'}^{-1}(q^{-1}(V(\mathscr{I}_1))) \cap {p'}^{-1}(U(s_2))$, since all $r_{1i} \in I_1$. Thus, the subschemes ${p'}^{-1}(q^{-1}(V(\mathscr{I}_1))) \cap {p'}^{-1}(U(s_2))$ and $\overline{Y'} \cap {p'}^{-1}(U(s_2))$ of ${p'}^{-1}(U(s_2))$ are disjoint. Passing to the underlying sets and noting that $V(\mathscr{I}_1) = V(\mathscr{I}_1^2) = T_1$ as sets, we obtain the desired disjointness (1.4), since the function h separates the sets under consideration (at least on ${p'}^{-1}(U(s_2))$). ■

There are other blow-up lemmas that will be needed for the proof of the Nagata compactification theorem. However, what we have already is adequate to prove some results that are necessary in the proof of Nagata's theorem and are interesting on their own. Thus, we will postpone the remaining blow-up lemmas until §3 and in the next section we give applications of the results so far.

2. Some Preliminary Theorems

We begin by introducing a notion that is critical in the constructions needed for the proof of Nagata's theorem.

Definition 2.1. Let S be a scheme and let X and Y be S-schemes with Y separated over S. A *quasi-domination* of X over Y is a pair (U, f) with $U \subseteq X$ a dense open subscheme and $f : U \to Y$ an S-morphism such that the graph subscheme $\Gamma_f \hookrightarrow U \times_S Y$ is closed when viewed as a subscheme of $X \times_S Y$.

Note that this graph is closed in $U \times_S Y$ since Y is S-separated, so the condition to be a quasi-domination of X over Y is a stronger closedness property. Also, since the subscheme Γ_f maps isomorphically to the open subscheme $U \subseteq X$, it encodes the data of the pair (U, f). If $f : U \to Y$ is proper, we say that (U, f) is a *proper* quasi-domination of X over Y. If $U_0 \subseteq X$ is an open subscheme and $f_0 : U_0 \to Y$ is an S-morphism, a *quasi-domination of X over Y extending* f_0 is a quasi-domination (U, f) of X over Y with $U_0 \subseteq U \subseteq X$ and $f|_{U_0} = f_0$. The reason that the notion of a quasi-domination is of interest to us is because we will need to glue separated S-schemes along S-isomorphic open subschemes and so we need to make sure that the relevant graphs are closed (for reasons given in the Introduction).

Example 2.2. The notion of quasi-domination is related to viewing rational maps on their (maximal) domain of definition, as follows. Suppose that Y is separated over S. We view an S-map $f : U \to Y$ on a schematically dense (hence topologically dense) open subscheme $U \subseteq X$ as a rational map from X to Y over S, and its *domain of definition* is the maximal open $U' \subseteq X$ containing U to which f (necessarily uniquely) extends as an S-map $f' : U' \to Y$. If f is a quasi-domination of X over Y then U coincides with this domain of definition. Indeed, $\Gamma_{f'} \subseteq U' \times_S Y$ is closed and contains $\Gamma_f \subseteq U \times_S Y$, but Γ_f is closed in $X \times_S Y$ so it is also closed in $U' \times_S Y$. Since the closed immersion $\Gamma_f \hookrightarrow \Gamma_{f'}$ is identified with the open immersion $U \hookrightarrow U'$ that is schematically dense, we deduce that $\Gamma_f = \Gamma_{f'}$, so $U = U'$.

If the inclusion of U into X is quasi-compact then we can generalize this: f is a quasi-domination of X over Y if and only if for every separated map $h : X' \to X$ that is an isomorphism over U and for which $h^{-1}(U) = U$ is a schematically dense open subscheme in X' (e.g., a U-admissible blow-up of X) the domain of definition of $f' = f \circ (h|_{h^{-1}(U)})$ (viewed as a rational S-map from X' to Y) coincides with U. The necessity follows from the preceding argument and the fact that $\Gamma_{f'} \subseteq X' \times_S Y$ is the pullback of $\Gamma_f \subseteq X \times_S Y$, and for sufficiency we take X' to be the schematic closure of Γ_f in $X \times_S Y$ (and h to be its natural projection to X); this schematic closure makes sense since $U \to X$ is assumed to be quasi-compact.

Example 2.3. Consider an S-map $\overline{f} : X \to \overline{Y}$ to a separated S-scheme and let $Y \subseteq \overline{Y}$ be an open subscheme. The restriction $f : U = \overline{f}^{-1}(Y) \to Y$ is a quasi-domination of X over Y if this U is dense in X, since $\Gamma_f = (X \times_S Y) \cap \Gamma_{\overline{f}}$.

Now replace the density condition on U by the condition (automatic in the noetherian case) that the open immersion $U \hookrightarrow X$ is quasi-compact. Let $\overline{U}$ be the scheme-theoretic image of U in X, so U is a dense open subscheme of $\overline{U}$ and is schematically dense in $\overline{U}$ as well. Clearly $U \hookrightarrow \overline{U}$ is quasi-compact. Let $f : U \to Y$ be an S-morphism. Consider a quasi-domination $(V, \overline{f})$ of $\overline{U}$ over Y extending f, with V an open subscheme of $\overline{U}$ containing U and thus automatically dense in $\overline{U}$. We claim that the graph $\Gamma_{\overline{f}}$, as a (closed) subscheme of $\overline{U} \times_S Y$, must be the schematic image $\overline{\Gamma}_f$ of the quasi-compact immersion

$$\Gamma_f \hookrightarrow U \times_S Y \hookrightarrow \overline{U} \times_S Y$$

(note that the first immersion is quasi-compact because it is a closed immersion). This implies in particular that the pair $(V, \overline{f})$ is unique if it exists.

In order to prove that $\overline{\Gamma}_f = \Gamma_{\overline{f}}$, consider the quasi-compact immersions

$$\varphi_1 : U \xrightarrow{\Gamma_f} U \times_S Y \hookrightarrow \overline{U} \times_S Y$$

and

$$\varphi_2 : V \xrightarrow{\Gamma_{\overline{f}}} V \times_S Y \hookrightarrow \overline{U} \times_S Y.$$

Although $V \hookrightarrow \overline{U}$ might not be quasi-compact, the immersion φ_2 is nevertheless quasi-compact because of the hypothesis that it is a closed immersion. The scheme-theoretic images of these are $\overline{\Gamma}_f$ and $\Gamma_{\overline{f}}$ respectively. Since $\varphi_1 = \varphi_2 \circ j$, with $j : U \to V$ the canonical open immersion, in order to establish equality of schematic images it is enough to prove that j is schematically dense. But j factors through $U \hookrightarrow \overline{U}$, which is schematically dense by construction of $\overline{U}$, so we are done.

Note that if Y is proper over S, then a quasi-domination (U, f) of X over Y must have $U = X$. Indeed, U is the image of the closed subscheme $\Gamma_f \hookrightarrow X \times_S Y$ under the proper projection to X, yet U is a dense open subscheme, so therefore $U = X$. Because of this, the following theorem seems to be a powerful tool in the elimination of indeterminacies of rational S-maps to an S-proper target. Indeed, Theorem 2.4 below asserts that (under mild restrictions) after making a suitable U-admissible blow-up on X, we can make any such map be defined everywhere. This theorem is the most important step in the proof of Nagata's theorem, but its proof is rather long and technical; the reader may prefer to skip it on a first reading.

Theorem 2.4. [N1, Theorem 3.2], [D, Théorème 1.2] *Let S be a quasi-compact and quasi-separated scheme, X and Y two quasi-compact and quasi-separated S-schemes. Let $U \subseteq V \subseteq X$ be dense open subschemes of X such that the open immersions of U and V into X are quasi-compact. Assume $Y \to S$ is separated and* of finite type. *Let (U, f) be a quasi-domination of V over Y in the sense of Definition* 2.1. *There exists a V-admissible blow-up $\widetilde{X}$ of X such that $\widetilde{X}$ quasi-dominates Y in a manner extending $f : U \to Y$.*

The quasi-compactness conditions on the immersions of U and V into X are equivalent to assuming that U and V are quasi-compact as schemes. Note also that U is dense in any V-admissible (even U-admissible) blow-up $\widetilde{X}$ of X (as we explain in Case 1 of the proof below), so Definition 2.1 makes sense for such an $\widetilde{X}$, Y, and $f : U \to Y$.

Remark 2.5. Consider the interesting special case $U = V$. This is the only case we will need in applications. In this case, any S-morphism $f : U \to Y$ is a quasi-domination of U over Y since $Y \to S$ is separated. Thus, if Y is S-proper (so for a scheme X' containing U as a dense open subscheme, any quasi-domination of X' over Y extending f must have domain of definition X') then the theorem says that f extends to an S-morphism $\widetilde{X} \to Y$ on some U-admissible blow-up $\widetilde{X}$ of X. This special case for noetherian S is [L, Lemma 2.2]. If we require $U = V$ in the statement of the theorem then the proof does not work. It is essential for the success of the argument that we have the flexibility in certain constructions to allow for the possibility that $U \neq V$. We will point out the step in the proof where this extra generality is needed (it occurs in the treatment of the "general case").

Proof. Note that f is automatically quasi-compact and quasi-separated, by [EGA, IV$_1$, 1.1.2(v), 1.2.2(v)]. Replacing Y by $Y \times_S X$, we can assume $X = S$. Thus, $Y \to S = X$ is separated and finite type, with $f : U \to Y$ a section over U whose set-theoretic image is closed in the part of Y that lies over $V \subseteq X$. We prove the theorem by handling a series of special cases, which we make more general one step at a time.

Case 1: $U = V$, $p : Y \to X$ quasi-affine, X affine.

Since p is *finite type* and quasi-affine, by [EGA, II, 5.1.9] we deduce the existence of a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\;i\;} & \mathbf{A}^n_X \\ & {}_{p}\searrow & \downarrow \\ & & X \end{array}$$

with i a quasi-compact immersion. Consider the immersion $F = i \circ f : U \to \mathbf{A}^n_X$. Since $U = V$ and $\mathbf{A}^n_X \to X$ is separated, F is obviously a quasi-domination of V over $\mathbf{A}^n_X$. Assume temporarily that in the case $Y = \mathbf{A}^n_X$ (i.e., i an isomorphism) there exists a V-admissible blow-up $\widetilde{X}$ which works. That is, suppose we have a finite type quasi-coherent ideal sheaf $\mathscr{I}$ on X with $V(\mathscr{I})$ disjoint from $V = U$, an open subscheme $\widetilde{W}$ in $\widetilde{X} = \mathrm{Bl}_{\mathscr{I}}(X)$ containing U, and an X-morphism $\widetilde{F} : \widetilde{W} \to \mathbf{A}^n_X$ extending f such that the graph subscheme

$$\Gamma_{\widetilde{F}} \subseteq \widetilde{W} \times_X \mathbf{A}^n_X \subseteq \widetilde{X} \times_X \mathbf{A}^n_X$$

is a closed subscheme. Note that $\widetilde{W}$ is automatically dense in $\widetilde{X}$. To see this, it is enough to check that U is dense in *any* U-admissible blow-up $\mathrm{Bl}_{\mathscr{I}}(X)$ of X. Since U is dense in X it is dense in $X - V(\mathscr{I}) \simeq \mathrm{Bl}_{\mathscr{I}}(X) - V(\mathscr{I}')$, with $\mathscr{I}' = \mathscr{I} \cdot \mathscr{O}_{\widetilde{X}}$ the invertible pullback ideal of $\mathscr{I}$. Since $\mathscr{I}'$ is of finite type, the open immersion $\mathrm{Bl}_{\mathscr{I}}(X) - V(\mathscr{I}') \hookrightarrow \mathrm{Bl}_{\mathscr{I}}(X)$ is quasi-compact. But this open immersion is also schematically dense since $\mathscr{I}'$ is invertible. Hence, a consideration of the scheme-theoretic image shows that the complement of $V(\mathscr{I}')$ is topologically dense in $\mathrm{Bl}_{\mathscr{I}}(X)$, so U is topologically dense in here also.

Since $U \to \widetilde{X}$ is a quasi-compact open immersion, there is a finitely presented closed subscheme in $\widetilde{X}$ with support equal to the complement of U (in the non-noetherian case, use Lemma 1.3). Let $q : \widetilde{X}' \to \widetilde{X}$ be the U-admissible blow-up along such a closed subscheme. The open subscheme $\widetilde{W}' = q^{-1}(\widetilde{W})$ in $\widetilde{X}'$ contains U, so it is dense. Also,

$$\widetilde{F}' : \widetilde{W}' \xrightarrow{q} \widetilde{W} \xrightarrow{\widetilde{F}} \mathbf{A}^n_X$$

has a graph $\Gamma_{\widetilde{F}'} \hookrightarrow \widetilde{X}' \times_X \mathbf{A}^n_X$ that is a closed subscheme since it is the base change of $\Gamma_{\widetilde{F}}$. Note that U is *schematically dense* in $\widetilde{X}'$, due to how the center of the blow-up $q : \widetilde{X}' \to \widetilde{X}$ was chosen. By Lemma 1.2, the map $\widetilde{X}' \to X$ is a U-admissible blow-up, so we can replace $\widetilde{X}$ with $\widetilde{X}'$ to get to the case when U is schematically dense in $\widetilde{X}$. This will be important in our attempt to recover the result for our original Y from the (temporarily assumed) result for $\mathbf{A}^n_X$.

The morphism $\widetilde{F}|_U : U \to \mathbf{A}^n_X$ is just F, which factors through the immersion $i : Y \hookrightarrow \mathbf{A}^n_X$. Since i is an immersion, there is an open subscheme $U_0 \subseteq \mathbf{A}^n_X$ such that i factors through a closed immersion $j : Y \hookrightarrow U_0$. Define $W = \widetilde{F}^{-1}(U_0) \subseteq \widetilde{X}$. Since $U \subseteq W$ is an open subscheme, we see that W is open and schematically dense in $\widetilde{X}$ (because $U \subseteq \widetilde{X}$ is so). Also, since the graph subscheme $\Gamma_{\widetilde{F}} \subseteq \widetilde{X} \times_X \mathbf{A}^n_X$ is closed, we see that the graph subscheme $\Gamma_{\widetilde{F}|_W} \subseteq \widetilde{X} \times_X U_0$ is closed.

We claim that it is enough to show that $\widetilde{F}|_W = j \circ \widetilde{f}$ for some (necessarily unique) $\widetilde{f} : W \to Y$. Indeed, if such an $\widetilde{f}$ exists, then $\widetilde{f}|_U = f$ (since $\widetilde{F}|_U = F$) and the graph subscheme $\Gamma_{\widetilde{f}} \subseteq \widetilde{X} \times_X Y$ is exactly $(1_X \times j)^{-1}(\Gamma_{\widetilde{F}|_W})$, which is closed. Thus, $\widetilde{f} : W \to Y$ is the desired quasi-domination of $\widetilde{X}$ over Y extending $f : U \to Y$.

As for the existence of $\widetilde{f}$, we know that $\widetilde{F}|_U = j \circ f$ as maps from U to U_0, so since U is an open and schematically dense subscheme of W it follows that the pullback of the closed immersion $j : Y \hookrightarrow U_0$ under $\widetilde{F}|_W : W \to U_0$ must be W (i.e., the only closed subscheme of W which contains U as an open subscheme is W). This shows that $\widetilde{F}|_W$ factors as $j \circ \widetilde{f}$ in the manner desired. Observe that this argument shows that in Case 1, we can even find $\widetilde{X}$ in which U is schematically dense. This will be useful later.

For Case 1, it remains to settle the case $Y = \mathbf{A}^n_X$. Since $U \hookrightarrow X = \mathrm{Spec}(A)$ is quasi-compact, we can cover U by $\mathrm{Spec}(A_{u_i})$'s for $i = 1, \ldots, m$, so the finitely generated ideal $(u_1, \ldots, u_m)$ in A cuts out the closed subset $X - U$ in X. The X-morphism $\mathrm{Spec}(A_{u_i}) \subseteq U \to \mathbf{A}^n_X$ corresponds to an A-algebra map

$$\begin{array}{ccc} A[t_0, \ldots, t_{n-1}] & \to & A_{u_i} \\ & & \\ t_j & \mapsto & g_{ij}/u_i^N \end{array}$$

for $g_{ij} \in A$ and some large N. The compatibility on overlaps $\mathrm{Spec}(A_{u_i}) \cap \mathrm{Spec}(A_{u_{i'}})$ says that for some large M,

$$(u_i u_{i'})^M (u_i^N g_{i'j} - u_{i'}^N g_{ij}) = 0$$

in A. Replace g_{ij} by $u_i^M g_{ij}$ and u_i^N by u_i^{N+M} (and N by $N+M$), so we can assume that $u_i^N g_{i'j} = u_{i'}^N g_{ij}$ for all i, i', j ($1 \le i, i' \le m$, $0 \le j \le n-1$). Call this common element $g_{i,i',j} = g_{i',i,j} \in A$.

Let $I = (u_i^N) + (g_{i,i',j})$ be the ideal of A generated by all u_i^N and $g_{i,i',j}$, and let $\mathscr{I}$ be the associated quasi-coherent ideal sheaf on $X = \mathrm{Spec}(A)$. Let $\widetilde{X} = \mathrm{Bl}_{\mathscr{I}}(X)$, so $\widetilde{X} \to X$ is a U-admissible blow-up. We will extend $f : U \to \mathbf{A}^n_X$ to an X-morphism $\widetilde{f} : \widetilde{X} \to \mathbf{P}^n_X$ (where $\mathbf{A}^n_X$ embeds $\mathbf{P}^n_X$ in the classical manner as the complement of the hyperplane $\{T_n = 0\}$). Assume we have such an $\widetilde{f}$. We can then argue as in Example 2.3 to get the required quasi-domination. More precisely, since $\mathbf{P}^n_X \to X$ is separated, the graph $\Gamma_{\widetilde{f}} \subseteq \widetilde{X} \times_X \mathbf{P}^n_X$ is closed. Also, $W = \widetilde{f}^{-1}(\mathbf{A}^n_X)$ is an open subscheme of $\widetilde{X}$ containing U, so W is automatically dense in the U-admissible blow-up $\widetilde{X}$. The X-morphism $W \to \mathbf{A}^n_X$ induced by $\widetilde{f}$ is a quasi-domination of $\widetilde{X}$ over $\mathbf{A}^n_X$ extending f, as desired.

All we need to do now is construct $\widetilde{f}$. Intuitively, on $\mathrm{Spec}(A_{u_i}) \subseteq U \subseteq \widetilde{X}$ we want to consider the A-map to $\mathbf{P}^n_A$ given by homogeneous coordinates $[g_{i1}, \ldots, g_{in}, u_i^N]$, and we want to prove that these agree on overlaps and (more importantly) extend to meaningful homogeneous coordinate formulas locally over all of $\widetilde{X} = \mathrm{Bl}_{\mathscr{I}}(X)$ by using that $g_{ij}/u_i^N = g_{i'j}/u_{i'}^N$ in $A_{u_i u_{i'}}$ for all i, i', j. To be precise, consider the open covering of $\widetilde{X}$ by the open affine subschemes $\mathrm{Spec}(A[I(u_i^N)^{-1}])$ (all i) and $\mathrm{Spec}(A[Ig_{i,i',j}^{-1}])$ (all i, i', j). Writing $\mathbf{P}_X^n = \mathrm{Proj}(A[T_0, \dots, T_n])$, define the X-map $\mathrm{Spec}(A[I(u_i^N)^{-1}]) \to \mathbf{A}_X^n \simeq D_+(T_n)$ by $T_j/T_n \mapsto g_{i,i,j}/(u_i^N)^2$. Meanwhile, we define an X-map $\mathrm{Spec}(A[Ig_{i,i',j}^{-1}]) \to D_+(T_j) = \mathrm{Spec}(A[T_0/T_j, \dots, T_n/T_j])$ by $T_r/T_j \mapsto g_{i,i',r}/g_{i,i',j}$ for $r \leq n-1$ and $T_n/T_j \mapsto u_{i'}^N u_i^N/g_{i,i',j}$. It is straightfoward (though somewhat tedious) to check compatibility on the overlaps; one simply applies the relation $u_i^N g_{i'j} = u_{i'}^N g_{ij}$ in A many times. Thus, these maps glue to give an X-morphism $\widetilde{f} : \widetilde{X} \to \mathbf{P}_X^n$, and $\widetilde{f}$ is easily seen to extend the given map $f : U \to \mathbf{A}_X^n$. This completes the proof of Case 1.

This case is the only place where the finite type condition on Y is explicitly needed (aside from the fact that everything else will ultimately reduce back to Case 1, as we shall see).

Case 2: $U = V$, p is quasi-affine.

Let $\{W_i\}_{0 \leq i \leq n}$ be a finite open affine covering of X, $p_i : p^{-1}(W_i) \to W_i$ the induced quasi-affine map over W_i, $U_i = U \cap W_i$ the induced quasi-compact open dense subscheme in W_i. The pullback $f_i : U_i \to Y_i = p^{-1}(W_i)$ of f over U_i is a quasi-domination of U_i over Y_i. By Case 1, there exists a finite type quasi-coherent ideal sheaf $\mathscr{I}_i$ on W_i such that $V(\mathscr{I}_i) \subseteq W_i - U_i \subseteq X - U = X - V$ as sets and $\mathrm{Bl}_{\mathscr{I}_i}(W_i)$ quasi-dominates Y_i in a manner extending f_i. In addition, as we noted in Case 1, we can (and do) also arrange that the open subscheme U_i in $\mathrm{Bl}_{\mathscr{I}_i}(W_i)$ is schematically dense.

We claim that $\mathscr{I}_i$ extends to a quasi-coherent finite type ideal sheaf $\overline{\mathscr{I}}_i$ on X with $V(\overline{\mathscr{I}}_i) \subseteq X - V$ as sets. In the noetherian case we take $\overline{\mathscr{I}}_i$ to define the scheme-theoretic closure of $V(\mathscr{I}_i)$ in X. The noetherian reader should skip ahead to the next paragraph. The construction of $\overline{\mathscr{I}}_i$ in general is given by a standard technique that we will need to use frequently when avoiding noetherian hypotheses, and so we explain how it goes in the present setting. Consider the scheme-theoretic closure $V(\mathscr{K}_i)$ of $V(\mathscr{I}_i)$ in X under the quasi-compact immersion $W_i \hookrightarrow X$ (recall that X is quasi-separated). Since $W_i - U_i = (X - V) \cap W_i$ (as sets), we see that $V(\mathscr{K}_i) \subseteq X - V$ as sets and $\mathscr{K}_i|_{W_i} = \mathscr{I}_i$. Write $\mathscr{K}_i$ as the direct limit of quasi-coherent finite type subideal sheaves $\mathscr{K}_{i,\alpha}$ which restrict to $\mathscr{I}_i$ on W_i. Since $\{V(\mathscr{K}_{i,\alpha}) \cap V\}$ is a decreasing inverse system of closed subsets in the quasi-compact space V and the intersection of all of these closed subsets is empty, we conclude that for some large α_0, $V(\mathscr{K}_{i,\alpha_0})$ is disjoint from V. Define $\overline{\mathscr{I}}_i = \mathscr{K}_{i,\alpha_0}$.

Define $\mathscr{I} = \prod \overline{\mathscr{I}}_i$, a quasi-coherent finite type ideal sheaf on X with $V(\mathscr{I})$ disjoint from V. Define $\widetilde{X} = \mathrm{Bl}_{\mathscr{I}}(X)$, $\widetilde{X}_i = \mathrm{Bl}_{\overline{\mathscr{I}}_i}(X)$. By definition of $\mathscr{I}_i$ (in terms of a quasi-domination property for $\widetilde{X}_i|_{W_i} = \mathrm{Bl}_{\mathscr{I}_i}(W_i)$), there is an open dense subscheme $T_i \subseteq \mathrm{Bl}_{\mathscr{I}_i}(W_i) = \widetilde{X}_i|_{W_i}$ containing U_i and an X-morphism $f_i : T_i \to Y_i \hookrightarrow Y$ which extends $f|_{U_i}$ such that the composite $\Gamma_{f_i} \subseteq T_i \times_{W_i} Y_i \subseteq (\widetilde{X}_i|_{W_i}) \times_{W_i} Y_i$ of a closed immersion and an open immersion is a closed subscheme. As we noted above, U_i is schematically dense in $\widetilde{X}_i|_{W_i}$, so U_i is schematically dense in T_i.

Consider the canonical morphisms $\pi_i : \widetilde{X} \to \widetilde{X}_i$ over X as in (1.2), so this can be viewed as a blow-up along a finite type quasi-coherent ideal sheaf whose zero scheme is disjoint from $V = U \supseteq U_i$. In particular, we can view U_i as an open subscheme of $\widetilde{X}$. Let $T_i' = {\pi_i}^{-1}(T_i) \subseteq \widetilde{X}$, an open subscheme of $\widetilde{X}$ which contains U_i as an open subscheme. We claim that U_i is schematically dense in T_i'. Since π_i is a blow-up map, the overlap of T_i' with the complement of the center of the blow-up π_i is schematically dense in T_i'. But this overlap is an open subscheme of T_i which contains U_i, so in view of the schematic density of U_i in T_i, we obtain the schematic density of U_i in T_i'.

The open subscheme $T = \bigcup T_i'$ in $\widetilde{X}$ contains the open subscheme $\bigcup U_i = U = V$, so T is dense in $\widetilde{X}$ since $\widetilde{X} \to X$ is a V-admissible blow-up. We will now check that the X-morphisms $f_i \circ \pi_i : T_i' \to Y_i \subseteq Y$ agree on overlaps in $\widetilde{X}$, so they glue to give an X-morphism $\overline{f} : T \to Y$ extending f. Since U_i is a schematically dense open in T_i', $U_i \cap U_j$ is a schematically dense open in $T_i' \cap T_j'$. Combining this with the separatedness of $Y \to X$ and the obvious fact that $f_i \circ \pi_i$ and $f_j \circ \pi_j$ coincide on $U_i \cap U_j$, it follows that these maps coincide on $T_i' \cap T_j'$.

Now we check that the graph $\Gamma_{\overline{f}} \subseteq T \times_X Y$ is closed in $\widetilde{X} \times_X Y$. This can be checked locally on X, as follows. Over the open $W_i \subseteq X$, we have

$$\Gamma_{\overline{f}}|_{W_i} = \pi_{i,Y}^{-1}(\Gamma_{f_i}),$$

where $\pi_{i,Y} : (\widetilde{X} \times_X Y)|_{W_i} \to (\widetilde{X}_i|_{W_i}) \times_{W_i} Y_i$ is the natural map induced by π_i on the first factor. Since we have already noted (by the quasi-domination property for f_i) that Γ_{f_i} is closed in $(\widetilde{X}|_{W_i}) \times_{W_i} Y_i$, we get closedness for $\Gamma_{\overline{f}}|_{W_i}$ in $(\widetilde{X} \times_X Y)|_{W_i}$. Thus, $\overline{f}$ is the quasi-domination we sought.

Case 3: p is quasi-affine.

By Lemma 1.2, at the start we may apply a base change by a V-admissible blow-up $\widetilde{X} \to X$ (note that such a blow-up does not cause us to lose the denseness hypotheses on U and V). Thus, before we settle Case 3, let's first show that after a base change on X by a suitable such V-admissible blow-up of X, we have $p(\overline{f(U)}) \cap \overline{V - U} = \emptyset$. We will then prove Case 3 with this additional condition.

Consider the commutative diagram

$$\begin{array}{ccc} f(U) & \xrightarrow{\qquad\qquad} & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ U \longrightarrow & V & \longrightarrow X \end{array} \tag{2.1}$$

In this diagram, $f(U) = \Gamma_f$ is the quasi-compact subscheme of Y cut out by the section f (so the left column is the canonical isomorphism). We wish to apply Lemma 1.5 to (2.1). To get started, let $\mathscr{I}_1$ be any quasi-coherent ideal sheaf on X with $V(\mathscr{I}_1) = \overline{V - U} \subseteq X - U$ as a set, and define $T_1 = V(\mathscr{I}_1^2)$ (e.g., we can take $\mathscr{I}_1$ to correspond to the canonical reduced structure on the closed subset $\overline{V - U}$ in X). The noetherian reader should give $X - V$ its reduced structure and skip to the next paragraph. In the general (possibly non-noetherian) case we will have to keep in mind the requirement that V-admissible blow-ups are taken with respect to *finite type* quasi-coherent ideal sheaves (to ensure that blow-up morphisms are quasi-compact). Since $V \hookrightarrow X$ is a quasi-compact open immersion, we can use Lemma 1.3 to give the closed subset $X - V$ in X the structure of a finitely presented closed subscheme in X.

Define $T_2 = T_1 \cap (X - V)$ as a closed subscheme of X. We claim that $p(p^{-1}(T_1) \cap \overline{f(U)}) \subseteq T_2$ as sets (for now, $\overline{f(U)}$ is just the topological closure of $f(U)$ in Y). Certainly $p(p^{-1}(T_1) \cap \overline{f(U)}) \subseteq T_1$. Since $f(U)$ is *closed* in $p^{-1}(V)$ by the quasi-domination hypothesis on f, we have

$$p(p^{-1}(T_1) \cap \overline{f(U)}) \cap V = p(p^{-1}(T_1) \cap \overline{f(U)} \cap p^{-1}(V)) = p(p^{-1}(T_1) \cap f(U)),$$

which lies inside of $T_1 \cap U = \emptyset$. We conclude that $p^{-1}(T_1) \cap \overline{f(U)} \subseteq p^{-1}(T_2)$ as closed subsets in Y. Now give $\overline{f(U)}$ its canonical scheme structure as the scheme-theoretic image of the quasi-compact quasi-separated map $f : U \to Y$. Our next step is to thicken the scheme structure on $X - V$ (and hence on T_2) so that the scheme-theoretic inclusion $p_1^{-1}(T_1) \cap \overline{f(U)} \subseteq p^{-1}(T_2)$ in Y is an inclusion of closed subschemes. This is possible because $\mathscr{I}_{T_2} = \mathscr{I}_{T_1} + \mathscr{I}_{X-V}$ with $\mathscr{I}_{X-V}$ of finite type and we have the desired inclusion as closed subsets of Y. More precisely, after making a suitable nilpotent thickening of the closed subscheme structure on the closed subset $X - V$, the definition of $\mathscr{I}_{T_2}$ is correspondingly altered to give a new T_2 that works in this way.

If $\mathscr{I}_1$ is of finite type, then since $p^{-1}(T_1) \cap \overline{f(U)} \subseteq p^{-1}(T_2)$ as closed subschemes of Y we can use Lemma 1.5 to get a V-admissible blow-up of X so that after base change by this blow-up the intersection $p(\overline{f(U)}) \cap \overline{V - U}$ is empty. The noetherian reader should skip to the next paragraph. In general, to circumvent the possibility that $\mathscr{I}_1$ may not be of finite type, we use a standard limit argument as follows. Since $\mathscr{I}_1|_U = \mathscr{O}_X|_U$, we can write $\mathscr{I}_1 = \varinjlim \mathscr{I}_{1,\alpha}$, with $\mathscr{I}_{1,\alpha}$ ranging through the finite type quasi-coherent subideal sheaves in $\mathscr{I}_1$ on X satisfying $\mathscr{I}_{1,\alpha}|_U = \mathscr{O}_X|_U$. Thus, $\mathscr{I}_{T_1} = \mathscr{I}_1^2 = \varinjlim \mathscr{I}_{1,\alpha}^2$. Since $\mathscr{I}_{X-V} \cdot \mathscr{O}_Y \subseteq \mathscr{I}_{T_2} \cdot \mathscr{O}_Y \subseteq \mathscr{I}_{T_1} \cdot \mathscr{O}_Y + \mathscr{I}_{\overline{f(U)}}$ and $\mathscr{I}_{X-V}$ is of finite type, we see that for large α,

$$\mathscr{I}_{X-V} \cdot \mathscr{O}_Y \subseteq \mathscr{I}_{1,\alpha}^2 \cdot \mathscr{O}_Y + \mathscr{I}_{\overline{f(U)}}.$$

Defining $T_{1,\alpha} = V(\mathscr{I}_{1,\alpha}^2)$ and $T_{2,\alpha} = T_{1,\alpha} \cap (X - V)$, we have $\overline{V - U} \subseteq T_{1,\alpha} \subseteq X - U$ as sets and

$$p^{-1}(T_{1,\alpha}) \cap \overline{f(U)} \subseteq p^{-1}(T_{2,\alpha})$$

as closed subschemes of X. By Lemma 1.5, if we make a base change by the V-admissible blow-up of X along

$$\mathscr{I}_{1,\alpha} + \mathscr{I}_{T_{2,\alpha}} = \mathscr{I}_{1,\alpha} + \mathscr{I}_{X-V}$$

then we get to the case $p(\overline{f(U)}) \cap \overline{V - U} = \emptyset$, as desired.

Define $X' = X - \overline{V - U}$, an open subscheme in X. Clearly $p(\overline{f(U)}) \subseteq X'$. Since $V - U$ is closed in V, we also see that $X' \cap V = V - (V \cap \overline{V - U})$ is equal to $V - (V \cap (V - U)) = U$. This latter condition says that when working over X' we are in the setting of Case 2 provided that X' is quasi-compact. This property holds in the noetherian case, so the noetherian reader should skip ahead to the next paragraph. In general, express $\mathscr{I}_{\overline{V-U}}$ as a direct limit of finite type quasi-coherent subideal sheaves to replace X' with a suitable quasi-compact open subscheme while preserving the conditions $p(\overline{f(U)}) \subseteq X'$ and $X' \cap V = U$. (This can be done because $\overline{f(U)}$ is quasi-compact in Y and both U and V are quasi-compact.)

Give $\overline{f(U)}$ its canonical closed subscheme structure in $p^{-1}(X')$ (and thus in Y) as the scheme-theoretic image of the quasi-compact, quasi-separated map $f : U \to p^{-1}(X')$. By Case 2 (with $\overline{f(U)} \to X'$ in the role of $Y \to X$), there is a finite type quasi-coherent ideal sheaf $\mathscr{I}'$ on X' with $V(\mathscr{I}') \subseteq X' - U = X' - (X' \cap V) \subseteq X - V$ as sets and such that $\widetilde{X}' = \mathrm{Bl}_{\mathscr{I}'}(X')$ quasi-dominates $\overline{f(U)}$ as an X'-scheme. Let $f' : W' \to \overline{f(U)}$ be the associated X'-morphism extending f, with $W' \subseteq \widetilde{X}'$ an open subscheme containing U and graph subscheme

$$\Gamma_{f'} \subseteq W' \times_{X'} \overline{f(U)} \subseteq \widetilde{X}' \times_{X'} \overline{f(U)}$$

that is closed. As usual, such a W' is automatically dense in the U-admissible blow-up $\widetilde{X}'$ since it contains the dense open U.

We can construct a quasi-coherent finite type ideal sheaf $\mathscr{I}$ on X with $\mathscr{I}|_{X'} = \mathscr{I}'$ and $V(\mathscr{I})$ disjoint from V. Indeed, in the noetherian case we take $\mathscr{I}$ to be the coherent ideal sheaf defining the schematic closure of $V(\mathscr{I}')$ in X. In general, shrink this $\mathscr{I}$ by a direct limit argument as for the $\overline{\mathscr{I}}_i$'s in Case 2. Define the V-admissible blow-up $\widetilde{X} = \mathrm{Bl}_{\mathscr{I}}(X)$. We may view $\widetilde{X}'$ as an open subscheme in $\widetilde{X}$ and U as a dense open in $\widetilde{X}$ (perhaps not schematically dense if $U \neq V$), so W' is a dense open in $\widetilde{X}$. We want to check that $\Gamma_{f'} \subseteq \widetilde{X} \times_X Y$ is closed. But $\widetilde{X} \times_X \overline{f(U)} \subseteq \widetilde{X} \times_X Y$ is closed, so it suffices to observe that

$$\widetilde{X}' \times_{X'} \overline{f(U)} \simeq \widetilde{X} \times_X \overline{f(U)},$$

since $p(\overline{f(U)}) \subseteq X'$.

General Case: Let $\{Y_i\}$ be a finite open affine covering of Y, chosen small enough so that each $Y_i \hookrightarrow Y \xrightarrow{p} X$ factors through an open affine subscheme in X. Thus, each Y_i is quasi-affine over X. Define $U_i = f^{-1}(Y_i) \subseteq U \subseteq V$, $X_i = \overline{U_i}$ (the scheme-theoretic image of the quasi-compact open immersion $U_i \to X$), and $V_i = X_i \cap V$.

Consider the commutative diagram

$$\begin{array}{ccc} U_i & \longrightarrow & V_i \\ {\scriptstyle f_i}\downarrow & & \downarrow \\ Y_i \cap p^{-1}(X_i) & \longrightarrow & X_i \end{array}$$

Since $p_i : Y_i \cap p^{-1}(X_i) \to X_i$ is quasi-affine (because $Y_i \to X$ is) and U_i is dense in X_i, the hypotheses of Case 3 are satisfied for p_i, $U_i \subseteq V_i$, and f_i. Observe that even if $U = V$ at the start, it is possible that V_i is strictly larger than U_i. Thus, it is essential that Case 3 allows "$U \neq V$" (i.e., Case 2 is not adequate). By Case 3, there exists a finite type quasi-coherent ideal sheaf $\mathscr{I}_i$ on X_i such that

$$V(\mathscr{I}_i) \subseteq X_i - V_i = (X - V) \cap X_i$$

as sets and such that the V_i-admissible blow-up $\widetilde{X}_i = \mathrm{Bl}_{\mathscr{I}_i}(X_i)$ quasi-dominates $Y_i \cap p^{-1}(X_i)$ over X_i in a manner extending f_i.

We next claim that there exists a finite type quasi-coherent ideal sheaf $\overline{\mathscr{I}}_i$ with $V(\overline{\mathscr{I}}_i)$ disjoint from V and with $V(\overline{\mathscr{I}}_i) \cap X_i = V(\mathscr{I}_i)$ as closed subschemes of X_i; i.e., the subsheaf $(\overline{\mathscr{I}}_i + \mathscr{I}_{X_i})/\mathscr{I}_{X_i} \subseteq \mathscr{O}_X/\mathscr{I}_{X_i} \simeq \mathscr{O}_{X_i}$ is equal to $\mathscr{I}_i$. In the noetherian case, take $\overline{\mathscr{I}}_i$ to be the coherent preimage of $\mathscr{I}_i \subseteq \mathscr{O}_{X_i}$ under the surjection $\mathscr{O}_X \twoheadrightarrow \mathscr{O}_{X_i}$. The noetherian reader should now skip to the next paragraph. In the general case, we begin by canonically writing $\mathscr{I}_i = \mathscr{K}_i/\mathscr{I}_{X_i}$ with $\mathscr{K}_i \subseteq \mathscr{O}_X$ a quasi-coherent ideal sheaf containing $\mathscr{I}_{X_i}$. Note that $\mathscr{K}_i|_V = \mathscr{O}_X|_V$ because

$$(\mathscr{O}_X/\mathscr{K}_i)|_V = (\mathscr{O}_{X_i}/\mathscr{I}_i)|_V = 0$$

(since $V(\mathscr{I}_i) \cap V = \emptyset$). Thus, we have $\mathscr{K}_i = \varinjlim \mathscr{K}_{i,\alpha}$ with $\mathscr{K}_{i,\alpha} \subseteq \mathscr{K}_i$ running through the finite type quasi-coherent subideal sheaves in $\mathscr{K}_i$ which satisfy $\mathscr{K}_{i,\alpha}|_V = \mathscr{O}_X|_V$. Since $(\mathscr{K}_{i,\alpha} + \mathscr{I}_{X_i})/\mathscr{I}_{X_i}$ is a quasi-coherent $\mathscr{O}_{X_i}$-ideal sheaf inside of $\mathscr{I}_i$ and these have direct limit $\mathscr{I}_i$, the fact that $\mathscr{I}_i$ is of finite type and X_i is quasi-compact implies that we can take $\overline{\mathscr{I}}_i = \mathscr{K}_{i,\alpha}$ for suitably large α.

Define $\mathscr{I} = \prod \overline{\mathscr{I}}_i$, so $V(\mathscr{I})$ is disjoint from V. Since $\mathscr{I}$ is of finite type, $q : \widetilde{X} = \mathrm{Bl}_{\mathscr{I}}(X) \to X$ is a V-admissible blow-up. Letting $\iota : U \hookrightarrow \widetilde{X}$ be the canonical open immersion, define $F = (\iota, f) : U \to \widetilde{X} \times_Y X$ to be the quasi-compact immersion induced by f. Let $\overline{F(U)}$ denote the scheme-theoretic image of F (in particular, the underlying space of $\overline{F(U)}$ is the topological closure of $F(U)$). We claim that $\overline{F(U)} \to \widetilde{X}$ is quasi-finite. Since this map is trivially of finite type, it remains to check that the fibers are finite as sets. We will prove this by exhibiting $\overline{F(U)}$ as a subset of a finite union of subsets of $Y \times_X \widetilde{X}$ that each map injectively to $\widetilde{X}$.

Recall that by the definition of $\mathscr{I}_i$ in terms of quasi-dominations via our preceding applications of Case 3, there is an open subscheme $\widetilde{V}_i \subseteq \widetilde{X}_i = \mathrm{Bl}_{\mathscr{I}_i}(X_i)$ (not necessarily a blow-up of an open in X_i) that contains U_i and for which there exists an $\widetilde{X}_i$-section $F_i : \widetilde{V}_i \to \widetilde{X}_i \times_{X_i} (Y_i \cap p^{-1}(X_i))$ that extends the graph *morphism* Γ_{f_i} and has a closed image (and so is a closed immersion). We will use this later.

We now relate some of the different blow-ups. First, using the notion of strict transform as reviewed in §1, we can naturally view the U_i-admissible blow-up $\widetilde{X}_i = \mathrm{Bl}_{\mathscr{I}_i}(X_i)$ as a closed subscheme of $\mathrm{Bl}_{\overline{\mathscr{I}}_i}(X)$ over the closed immersion $X_i \hookrightarrow X$. Thus, we can construct the commutative diagram

$$\begin{array}{ccc} X'_i & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ \widetilde{X}_i & \longrightarrow & \mathrm{Bl}_{\overline{\mathscr{I}}_i}(X) \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & X \end{array}$$

in which the top square (with right side built via (1.2)) is cartesian, the composite along the right side is q, and all other maps are as described above. In particular, $X'_i \to \widetilde{X}_i$ is an isomorphism over the open subscheme U_i (viewed in either X_i or X) since $U_i \subseteq V$ in X.

Applying the base change by $X'_i \to \widetilde{X}_i$ to the closed immersion of $\widetilde{X}_i$-schemes

$$F_i : \widetilde{V}_i \hookrightarrow \widetilde{X}_i \times_{X_i} (Y_i \cap p^{-1}(X_i))$$

with open $\widetilde{V}_i \subseteq \widetilde{X}_i$ containing U_i gives rise to an open subscheme $U'_i \subseteq X'_i$ containing $q^{-1}(U_i) \simeq U_i$ and a closed immersion

$$\varphi_i : U'_i \hookrightarrow X'_i \times_{X_i} (Y_i \cap p^{-1}(X_i))$$

of X'_i-schemes with $\varphi_i(U'_i)$ containing the topological closure $\overline{F(U_i)}$ of $F(U_i)$ inside of $X'_i \times_{X_i} (Y_i \cap p^{-1}(X_i))$. The topological closure of $F(U)$ in $\widetilde{X} \times_X Y$, which is exactly the underlying space of $\overline{F(U)}$, has the property that it meets $\widetilde{X} \times_X Y_i$ in the relative closure of the subset $F(U) \cap (\widetilde{X} \times_X Y) = F(U_i)$ since $Y_i = f^{-1}(U_i)$.

Using the closed immersion

$$X_i' \times_{X_i} (Y_i \cap p^{-1}(X_i)) \simeq X_i' \times_X Y_i \hookrightarrow \widetilde{X} \times_X Y_i,$$

we conclude that $\overline{F(U)} \cap (\widetilde{X} \times_X Y_i) \subseteq \varphi_i(U_i')$ as sets, so $\overline{F(U)} \subseteq \bigcup \varphi_i(U_i')$. But the map $\varphi_i(U_i') \hookrightarrow X_i' \hookrightarrow \widetilde{X}$ is set-theoretically injective. Therefore, the map $\overline{F(U)} \to \widetilde{X}$ is quasi-finite. This map is also separated since $Y \to X$ is separated.

The importance of this map being quasi-finite and separated is that it implies that the morphism $\overline{F(U)} \to \widetilde{X}$ is quasi-affine, due to Zariski's Main Theorem ([EGA, IV$_4$, 18.12.12] in the general case). Combining this quasi-affineness and Lemma 1.2, we may apply the base change by the V-admissible blow-up $\widetilde{X} \to X$ to reduce to the case in which the scheme-theoretic image $\overline{f(U)}$ of U in Y is quasi-affine over X. Since we can always replace Y by $\overline{f(U)}$, we are reduced to the case in which $Y \to X$ is quasi-affine. But this was Case 3. ■

We can use Theorem 2.4 to prove a refined version of Chow's Lemma, somewhat generalizing the version given in [EGA, II, 5.6.1] even in the noetherian case. Recall that we define projectivity and quasi-projectivity for morphisms as in [EGA, II, 5.3, 5.5].

Corollary 2.6. [D, Cor. 1.4] (*Chow*) *Let S be quasi-compact and quasi-separated, $f : X \to S$ a separated and finite type map. Then for any quasi-compact dense open subscheme $U \subseteq X$ which is quasi-projective over S, there exists a diagram of S-schemes*

$$X \xleftarrow{q} X' \stackrel{j}{\hookrightarrow} \overline{X}$$

with q a U-admissible blow-up, $\overline{X} \to S$ projective, and j a dense open immersion (necessarily quasi-compact). In particular, q is projective and surjective, $q^{-1}(U) \simeq U$, and $U \simeq q^{-1}(U) \hookrightarrow X'$ is quasi-compact.

This refines the traditional formulation of Chow's Lemma not only by eliminating noetherian hypothesis in a stronger manner than in [EGA, II, 5.6.1], but also by controlling the center of the blow-up.

Proof. Since $U \to S$ is quasi-projective, there exists a quasi-compact open immersion $U \hookrightarrow Y$ with $Y \to S$ projective. Let U^* denote the scheme-theoretic image of U in Y, so $U^* \to S$ is projective and $U \to U^*$ is a quasi-compact open immersion which is schematically dense.

Because $i : U \to X$ is a quasi-domination of U over X, by Theorem 2.4 we can find a U-admissible blow-up U^{**} of U^* such that we have a commutative diagram of S-schemes

(2.2)
$$\begin{array}{ccccc} U & \longrightarrow & V & \xrightarrow{h} & U^{**} \\ \| & & \downarrow{\scriptstyle \varphi} & & \\ U & \xrightarrow[i]{} & X & & \end{array}$$

with $h : V \to U^{**}$ a quasi-compact open immersion and $\Gamma_\varphi \subseteq V \times_S X \subseteq U^{**} \times_S X$ a closed subscheme. In particular, φ factors as

$$V \simeq \Gamma_\varphi \hookrightarrow U^{**} \times_S X \xrightarrow{p_2} X,$$

so φ is *proper* and V is quasi-compact. Note also that $U \subseteq U^{**}$ is a schematically dense open immersion, so U is schematically dense in V. Also, V is S-separated since X is S-separated by hypothesis.

We now need to prove an auxiliary lemma before continuing with the proof. This lemma, whose formulation is a bit long, will be useful later as well.

Lemma 2.7. [D, Lemme 1.5] *Let S be a quasi-compact, quasi-separated scheme. Consider a commutative diagram of separated finite type S-schemes*

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & Y \\ \| & & \downarrow{\scriptstyle \varphi} & & \\ U & \longrightarrow & X & & \end{array}$$

with all horizontal maps open immersions (necessarily quasi-compact). We assume that U is dense in X and Y, and that U is schematically dense in V. Suppose that φ is a quasi-domination of Y over X.

Let $\mathscr{I}$ be a quasi-coherent finite type ideal sheaf on X with $V(\mathscr{I})$ disjoint from U, and for which the U-admissible blow-up $\widetilde{X} = \mathrm{Bl}_{\mathscr{I}}(X)$ is quasi-dominant over V in a manner extending the map $U \hookrightarrow V$. (Such an $\mathscr{I}$ is provided by Theorem 2.4, applied to $U = U \subseteq X$ and the quasi-domination $U \hookrightarrow V$ of U over V). Let $\psi : W \to V$ be the map corresponding to this quasi-domination, where $j : W \hookrightarrow \widetilde{X}$ is an open subscheme containing U. Finally, suppose that the quasi-compact open immersion $U \hookrightarrow \widetilde{X}$ (which necessarily has dense image) is schematically dense.

Under these conditions, the associated diagram

$$\begin{array}{ccccc} U & \longrightarrow & W & \xrightarrow{j} & \widetilde{X} \\ & \searrow & \downarrow{\scriptstyle \psi} & & \downarrow \\ & & V & \xrightarrow[\varphi]{} & X \end{array}$$

has the following properties:
(i) W is V-isomorphic to $\mathrm{Bl}_{\mathscr{I} \cdot \mathscr{O}_V}(V)$ (so ψ is proper),
(ii) The graph $\Gamma_\psi \subseteq W \times_S V \subseteq \widetilde{X} \times_S Y$ is closed.

Proof. Let $\widetilde{V} = \mathrm{Bl}_{\mathscr{I} \cdot \mathscr{O}_V}(V)$. By the universal property of $\widetilde{X}$, there is a unique X-morphism $\alpha : \widetilde{V} \to \widetilde{X}$. In particular, $\alpha^{-1}(\mathscr{I} \cdot \mathscr{O}_{\widetilde{X}}) \cdot \mathscr{O}_{\widetilde{V}} = (\varphi^{-1}(\mathscr{I}) \cdot \mathscr{O}_V) \cdot \mathscr{O}_{\widetilde{V}}$. Since the diagram

$$\begin{array}{ccc} W & \xrightarrow{j} & \widetilde{X} \\ {\scriptstyle \psi}\downarrow & & \downarrow \\ V & \xrightarrow[\varphi]{} & X \end{array}$$

commutes, with j an open immersion, we also have a unique V-morphism $\beta : W \to \widetilde{V}$. In particular, $\beta^{-1}((\varphi^{-1}(\mathscr{I}) \cdot \mathscr{O}_V) \cdot \mathscr{O}_{\widetilde{V}}) \cdot \mathscr{O}_W = j^{-1}(\mathscr{I} \cdot \mathscr{O}_{\widetilde{X}}) \cdot \mathscr{O}_W$.

Since the X-morphisms $\alpha \circ \beta$ and j from W to $\widetilde{X}$ pull $\mathscr{I} \cdot \mathscr{O}_{\widetilde{X}}$ back to the same ideal sheaf (in particular, to an *invertible* ideal sheaf), these maps coincide. Thus, we have a commutative diagram

$$\begin{array}{ccccc} W & \xrightarrow{\beta} & \widetilde{V} & \xrightarrow{\alpha} & \widetilde{X} \\ \uparrow & \overset{\searrow}{\scriptstyle \psi} & \downarrow & & \downarrow \\ U & \longrightarrow & V & \xrightarrow[\varphi]{} & X \end{array}$$

in which the composites along the rows are exactly the given open immersions $U \hookrightarrow X$ and $W \xrightarrow{j} \widetilde{X}$.

Since the closed subscheme $\Gamma_\varphi \subseteq V \times_S X$ projects isomorphically to V, its pullback Γ in $V \times_S \widetilde{X}$ is proper over V (because $\widetilde{X} \to X$ is proper). Also, $U \to W$ inside of $\widetilde{X}$ is schematically dense (as $U \hookrightarrow \widetilde{X}$ is), so by definition of ψ being a quasi-domination of $\widetilde{X}$ over V extending $U \to V$ we see that the closed subscheme $\Gamma_\psi \subseteq W \times_S V \subseteq \widetilde{X} \times_S V$ is the schematic closure of the composite (quasi-compact) immersion $\Delta_{U/S} \hookrightarrow U \times_S U \hookrightarrow U \times_S V$ in $\widetilde{X} \times_S V$; see the discussion preceding Theorem 2.4. We will now use this to show that ψ is proper.

Let $\Gamma' \subseteq \widetilde{X} \times_S V$ correspond to the V-proper closed subscheme $\Gamma \subseteq V \times_S \widetilde{X}$ under the flip isomorphism $V \times_S \widetilde{X} \simeq \widetilde{X} \times_S V$. The map $\widetilde{X} \to X$ is an isomorphism over U, so clearly $\Delta_{U/S} \subseteq \widetilde{X} \times_S V$ lies in Γ'. Since Γ' is a closed subscheme in $\widetilde{X} \times_S V$, it contains the schematic closure Γ_ψ of $\Delta_{U/S}$. In particular, Γ_ψ is V-proper since Γ' is, so ψ is proper.

Now we are ready to prove (*ii*), and then we will prove (*i*). Consider the commutative diagram

$$\begin{array}{ccc} V \times_S \widetilde{X} & \longrightarrow & Y \times_S \widetilde{X} \\ \downarrow & & \downarrow \\ V \times_S X & \longrightarrow & Y \times_S X \end{array} \tag{2.3}$$

in which the horizontal maps are immersions. Since $\Gamma_\varphi \subseteq Y \times_S X$ is a closed subscheme (because φ is a quasi-domination of Y over X), if we pull this back to $Y \times_S \widetilde{X}$ we get a closed subscheme Z. By the commutativity of (2.3), $\Gamma_\varphi \subseteq V \times_S X$ is a closed subscheme whose pullback Γ to $V \times_S \widetilde{X}$, when viewed as a subscheme of $Y \times_S \widetilde{X}$, is exactly Z. In particular, Γ is closed in $Y \times_S \widetilde{X}$. But we just saw above that Γ_ψ is a closed subscheme of Γ'. This proves (*ii*).

To prove (*i*), we will show that β is an isomorphism. Since ψ is proper and $\widetilde{V} \to V$ is separated, it follows that β is proper. But $\alpha \circ \beta : W \to \widetilde{X}$ is an open immersion, so β is a categorical monomorphism. Proper monomorphisms are closed immersions (by [EGA, IV_4, 18.12.6] without noetherian hypotheses), so β is a closed immersion. However, U is schematically dense in V, so the U-admissible blow-up $\widetilde{V} \to V$ makes U schematically dense in $\widetilde{V}$. That is, the composite map

$$U \to W \xrightarrow{\beta} \widetilde{V}$$

is schematically dense, so β is schematically dense. But β is a closed immersion, so it must be an isomorphism. This completes the proof of Lemma 2.7. ■

Now we return to the proof of Corollary 2.6. We shall first reduce to the case when (2.2) satisfies all of the hypotheses in Lemma 2.7. By Lemma 1.2 (and Lemma 1.3 in the non-noetherian case), at the start of the proof of Corollary 2.6 we could have blown-up X along a finitely presented closed subscheme structure on the closed set $X - U$ so that $U \hookrightarrow X$ is not just an open immersion with dense image, but is also a schematically dense map. Thus, any further U-admissible blow-ups of X will contain U as a schematically dense open subscheme. In particular, we can apply Lemma 2.7 with $\mathscr{I}$ there provided by Theorem 2.4. The S-separatedness of X is crucial for the applicability of Lemma 2.7.

Using Lemma 2.7 (which is compatible with the notation preceding Lemma 2.7), together with the properness of $\varphi : V \to X$ and $\psi : W \to V$ in that lemma, we see that the resulting dense open immersion $j : W \hookrightarrow \widetilde{X} = \mathrm{Bl}_{\mathscr{I}}(X)$ is proper and therefore an isomorphism. Thus, $W = \widetilde{X}$. In particular, $\widetilde{X}$ is X-isomorphic to the blow-up of V along $\mathscr{I} \cdot \mathscr{O}_V$. Let $\mathscr{I}'$ be a finite type quasi-coherent ideal sheaf on U^{**} with $\mathscr{I}'|_V = \mathscr{I} \cdot \mathscr{O}_V$ and $V(\mathscr{I}')$ disjoint from U; the noetherian reader should take $\mathscr{I}'$ to define the schematic closure of $V(\mathscr{I} \cdot \mathscr{O}_V)$ in U^{**}. We have seen the argument needed to justify the existence of such an $\mathscr{I}'$ in the non-noetherian case a couple of times already (using the quasi-compactness of U) so we don't repeat the argument again.

Define $X' = \widetilde{X} = \mathrm{Bl}_{\mathscr{I}}(X)$ and $\overline{X} = \mathrm{Bl}_{\mathscr{I}'}(U^{**})$. Observe that the unique map $k : X' \simeq W = \mathrm{Bl}_{\mathscr{I} \cdot \mathscr{O}_V}(V) \hookrightarrow \overline{X}$ over the quasi-compact open immersion $h : V \to U^{**}$ is an open immersion that is the identity map between the dense open copies of U in each side, so k has dense image. The quasi-compactness of k and $U \hookrightarrow X'$ follow readily from [EGA, IV_1, 1.1.2(*v*)]. This completes the proof of Corollary 2.6 since $\overline{X} \to U^*$ is a U-admissible blow-up (by Lemma 1.2) with U^* projective over S, and a composite of projective morphisms is projective. ■

We now come to what is the most time-consuming lemma in Nagata's original treatment. Thanks to Lemma 2.7, this step will not be too difficult for us.

Theorem 2.8. [N1, Lemma 4.2], [D, Lemme 1.6] *Let S be a quasi-compact, quasi-separated scheme. Consider a diagram of finite type separated S-schemes*

$$X_2 \hookleftarrow U \hookrightarrow X_1,$$

with each $U \hookrightarrow X_i$ a (necessarily quasi-compact) open immersion with dense image. Then there exists a schematically dense open immersion $j : U \hookrightarrow X$ with X separated *of finite type over S (and so j is* *automatically quasi-compact with a dense image) such that there is a proper quasi-domination of X over each X_i extending $U \hookrightarrow X_i$.*

Proof. We begin by making the simple observation that if $f : U \to Y$ is a proper quasi-domination of X_i over some Y and if $g : Y \to Y'$ is proper then $g \circ f : U \to Y'$ is a proper quasi-domination of X_i over Y'. Therefore, by blowing up X_i along $(X_i - U)_{\mathrm{red}}$ in the noetherian case, or more generally along a finite type quasi-coherent ideal sheaf with zero locus $X_i - U$ (using Lemma 1.3 since $U \to X_i$ is quasi-compact), we can assume that U is schematically dense in each X_i. In particular, U will be schematically dense in any further U-admissible blow-ups of the X_i's.

By Theorem 2.4, we see that X_1 can be replaced with a suitable U-admissible blow-up so that we have a commutative diagram of S-schemes

(2.4)
$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & X_1 \\ & \searrow & \downarrow & & \\ & & X_2 & & \end{array}$$

in which $V \to X_1$ is an open immersion with closed graph in $X_1 \times_S X_2$, so V is *quasi-compact* (since V is the image of a closed subscheme of the quasi-compact scheme $X_1 \times_S X_2$ under the continuous projection map $X_1 \times_S X_2 \to X_1$). In particular, V is separated and *finite type* over S and U is schematically dense in V. We can apply Lemma 2.7 to (2.4) to obtain a commutative diagram of S-schemes

(2.5)
$$\begin{array}{ccccc} U & \longrightarrow & W & \longrightarrow & X_2' \\ \| & & \downarrow{\scriptstyle \psi} & & \downarrow \\ U & \longrightarrow & V & \longrightarrow & X_2 \end{array}$$

such that
(*i*) X_2' is a U-admissible blow-up of X_2 in which W is an open subscheme containing U,
(*ii*) ψ is a U-admissible blow-up along some finite type quasi-coherent ideal sheaf $\mathscr{I}$ on V,
(*iii*) $\Gamma_\psi \subseteq W \times_S V \subseteq X_2' \times_S X_1$ is a closed subscheme.

By the usual arguments, we can construct a finite type quasi-coherent ideal sheaf $\mathscr{I}'$ on X_1 such that $\mathscr{I}'|_V = \mathscr{I}$ and $V(\mathscr{I}')$ is disjoint from U. (In the noetherian case, we may take $V(\mathscr{I}')$ to define the schematic closure of $V(\mathscr{I})$ in X_1.) Let $q : X_1' \to X_1$ be the U-admissible blow-up along $\mathscr{I}'$. Thus, ψ induces an isomorphism $\widetilde{\psi}$ between W and $q^{-1}(V)$. Since the graph subscheme $\Gamma_\psi \subseteq X_2' \times_S X_1$ is *closed*, its pullback $\Gamma \subseteq X_2' \times_S X_1'$ under $1 \times q : X_2' \times_S X_1' \to X_2' \times_S X_1$ is a closed subscheme. But clearly $\Gamma = \Gamma_{\widetilde{\psi}}$, so we have an S-isomorphism between open subschemes of X_2' and X_1' with a closed graph in $X_2' \times_S X_1'$. Therefore, the gluing X of the X_i''s along this isomorphism is *separated* over S.

It is obvious that $X \to S$ is of finite type. Since the left square in (2.5) commutes, we have a natural open immersion $U \hookrightarrow X$ over S, and since U is schematically dense in each X_i', U is schematically dense in X.

We claim that the natural maps $p_i : X_i' \to X_i$ are the proper quasi-dominations of X over X_i that we want. Clearly these extend $U \hookrightarrow X_i$ and are proper. To see that the graph $\Gamma_{p_i} \subseteq X_i' \times_S X_i \subseteq X \times_S X_i$ is closed, it suffices to check that its preimage under the surjective closed map $X \times_S X_i' \twoheadrightarrow X \times_S X_i$ is closed. This preimage meets $X_i' \times_S X_i'$ in the closed subset cut out by $\Delta_{X_i'/S}$, and it meets $X_{3-i}' \times X_i'$ in the closed subscheme Γ (or its "flip", depending on whether $i = 1$ or $i = 2$). Since $X_1' \times_S X_i'$ and $X_2' \times_S X_i'$ give an open covering of $X \times_S X_i'$, we are done. ∎

Remark 2.9. Using Lemma 1.2, the above proof shows that the proper quasi-domination of X over X_i can be taken to be a U-admissible blow-up of X_i.

Theorem 2.8 is the key tool for gluing without losing separatedness over the base. For completeness (and later applications), we mention the following easy generalization of Theorem 2.8.

Corollary 2.10. *Let S be quasi-compact and quasi-separated. Consider a finite collection of dense open immersions $j_i : U \hookrightarrow X_i$ between finite type separated S-schemes. Then there exist U-admissible blow-ups*

$X_i' \to X_i$ and a separated finite type S-scheme X, together with open immersions $X_i' \hookrightarrow X$ over S, such that the X_i' cover X and the composite open immersions $U \to X_i' \to X$ are all the same.

Proof. By induction and Lemma 1.2, we may assume that our collection consists of just two dense open immersions. In this case, the result follows from Remark 2.9 and the construction in the proof of Theorem 2.8. ∎

As a special case of Corollary 2.10, if X_1 and X_2 are proper S-schemes (with S quasi-compact and quasi-separated) and there are quasi-compact dense open immersions $U \to X_j$ then there are U-admissible blow-ups X_i' of X_i that are an open cover with overlap U in a separated finite type S-scheme X. But the open immersions $X_i' \to X$ over S are visibly dense and proper, hence are isomorphisms, so X_1 and X_2 have a common U-admissible blow-up (thereby "explaining" a posteriori how they share U as a common dense open subscheme). This observation can also be deduced from the following characterization of proper birational maps that is not used in the proof of Nagata's theorem (nor is it mentioned in [D]) but follows easily from Lemma 2.7 (and Lemma 1.2) and is quite interesting for its own sake.

Theorem 2.11. *Let $f : X' \to X$ be a proper map between quasi-compact and quasi-separated schemes, and let $U \subseteq X$ be a quasi-compact dense open subscheme such that f is an isomorphism over U and makes U dense in X'. There exist U-admissible blow-ups $\widetilde{X}' \to X'$ and $\widetilde{X} \to X$ and an isomorphism $\widetilde{X}' \simeq \widetilde{X}$ over f.*

Loosely speaking, this theorem says that a proper map is birational if and only if its source and target admit a common blow-up. In particular, such a map is the "composition" of a blow-up and the "inverse" of a blow-up. Even for abstract varieties over an algebraically closed field this is not a well-known fact.

Proof. The map $U \to X'$ is quasi-compact, so there is a quasi-coherent finite type ideal sheaf $\mathscr{K}$ on X' with $V(\mathscr{K}) = X' - U$ as sets. Replacing X' with $\mathrm{Bl}_{\mathscr{K}}(X')$ loses no generality (by Lemma 1.2) and lets us assume that U is schematically dense in X'. Now apply Lemma 2.7 with $X = S$, $Y = V = X'$, and $\varphi = f$. Taking $\mathscr{I}$ as in that lemma, since f is proper it follows that the dominant open immersion $j : W = \mathrm{Bl}_{\mathscr{I}\cdot\mathscr{O}_{X'}}(X') \to \mathrm{Bl}_{\mathscr{I}}(X)$ over f is proper, hence an isomorphism. ∎

Remark 2.12. By using Lemma 1.2, it follows from two applications of Theorem 2.11 that any two proper birational maps from a common source become isomorphisms after suitable blow-ups. To be precise, let $\pi_i : X \to Y_i$ be a pair of proper birational maps between quasi-compact and quasi-separated schemes in the sense that there are dense open subschemes $U_i \subseteq Y_i$ such that $U_i' = \pi_i^{-1}(U_i)$ is dense in X and π_i is an isomorphism over U_i. Let $U = U_1' \cap U_2'$, so U is a dense open subscheme of X and can be viewed as a dense open subscheme of each Y_i (with π_i an isomorphism over U). By Theorem 2.11 we can find U-admissible blow-ups of X and Y_1 that are isomorphic over π_1, and by Lemma 1.2 we can rename these blow-ups as X and Y_1 (and replace π_2 with its composite back to the chosen U-admissible blow-up of the original X) so as to get to the case when π_1 is an isomorphism. An application of Theorem 2.11 to π_2 (and another application of Lemma 1.2) then completes the argument.

After we have proved Nagata's theorem, we will be able to easily deduce generalizations of Theorem 2.11 and Remark 2.12 in which the properness hypothesis is relaxed (and the isomorphism conclusion is weakened appropriately).

3. More Blow-up Lemmas

We now prove some further lemmas, building on §1. These will be needed for the proof of Nagata's theorem in §4. For motivational purposes, the reader may prefer to go directly to §4 and to only return to §3 when results here are cited there. More precisely, Lemma 3.1 is only used in the proof of Lemma 3.2, which in turn is crucial to get properness of the gluing construction of the desired compactification in the proof of Nagata's theorem.

The following technical-looking lemma roughly says that if $q : X_2 \to X_1$ is a birational map that is an isomorphism over a dense open $U \subseteq X_1$ and if $V \subseteq X_1$ is an open subscheme containing U such that there is a containment

$$q^{-1}(Z_1 \cap V) = q^{-1}(Z_1) \cap q^{-1}(V) \subseteq Z_2 \cap q^{-1}(V)$$

as subsets of $q^{-1}(V)$ for closed subsets $Z_j \subseteq X_j$, then after base change by a V-admissible blow-up of X_1 centered in $Z_1 - (Z_1 \cap V)$ we have the stronger containment relation $q^{-1}(\overline{Z_1 \cap V}) \subseteq Z_2$ as subsets of X_2. The actual lemma is more general in the sense that it does not impose density conditions on U or require q to be birational. However, the lemma will only be used (in the proof of Lemma 3.2) in the special case just considered.

Lemma 3.1. [D, Lemme 0.4] *Consider a commutative diagram of schemes*

$$\begin{array}{ccccc} U & \longrightarrow & V_2 & \longrightarrow & X_2 \\ \| & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ U & \longrightarrow & V_1 & \longrightarrow & X_1 \end{array}$$

in which all horizontal maps are open immersions, $q^{-1}(V_1) = V_2$, and X_1 and X_2 are quasi-compact and quasi-separated. Assume that $V_1 \hookrightarrow X_1$ is quasi-compact and q is quasi-compact and separated. Let Z_i be a closed subscheme of X_i such that the open immersion $X_i - Z_i \hookrightarrow X_i$ is quasi-compact. Define $Y_1 = Z_1 \bigcap V_1$, a closed subscheme of V_1. Assume that $p^{-1}(Y_1) \subseteq Z_2$ as sets (equivalently, $q^{-1}(Y_1) \subseteq Z_2$ as sets).

There exists a finite type quasi-coherent ideal sheaf $\mathscr{I}$ on X_1 with $V(\mathscr{I}) \subseteq Z_1 - Y_1 = Z_1 \cap (X_1 - V_1)$ as sets such that after base change by the V_1-admissible blow-up $X_1' = \mathrm{Bl}_{\mathscr{I}}(X_1) \to X_1$, we have ${q'}^{-1}(\overline{Y_1'}) \bigcap \overline{U'} \subseteq Z_2' \bigcap \overline{U'}$ as subsets of X_2'. Here, q', Y_1', etc. denote the base change of q, Y_1, etc. by $X_1' \to X_1$, and $\overline{U'}$ (resp. $\overline{Y_1'}$) is the topological closure of U' in X_2' (resp. Y_1' in X_1').

Proof. We want to apply Lemma 1.5 to the diagram

(3.1)
$$\begin{array}{ccccc} q_0^{-1}(X_1 - Z_1) & & \xrightarrow{\quad j \quad} & & X_2 - Z_2 \\ \downarrow & & & & \downarrow{\scriptstyle q_0} \\ X_1 - Z_1 & \longrightarrow & X_1 - (Z_1 - Y_1) & \longrightarrow & X_1 \end{array}$$

Here, q_0 is the map induced by q, so $q_0^{-1}(X_1 - Z_1) = X_2 - (q^{-1}(Z_1) \bigcup Z_2)$. All horizontal maps in (3.1) are quasi-compact open immersions: for the top row, use that $q : X_2 - q^{-1}(Z_1) \to X_1 - Z_1$ and $X_1 - Z_1 \to X_1$ are quasi-compact and q is separated (so $X_2 - q^{-1}(Z_1) \to X_2$ is quasi-compact); for the bottom row, use that $X_1 - (Z_1 - Y_1) = (X_1 - Z_1) \cup V_1$ with $X_1 - Z_1$ and V_1 quasi-compact opens in X_1.

Let $\mathscr{I}_1 = \mathscr{I}_{Z_1}$, $T_1 = V(\mathscr{I}_1^2)$. Observe that

$$(X_1 - (Z_1 - Y_1)) - (X_1 - Z_1) = Y_1$$

has its topological closure in X_1 lying inside of Z_1, and also that

$$X_1 - (X_1 - (Z_1 - Y_1)) = Z_1 - Y_1$$

as sets, with $Z_1 - Y_1 = Z_1 \bigcap (X_1 - V_1)$ a closed set in X_1. In order to profitably apply Lemma 1.5, we need to find a *finite type* quasi-coherent ideal sheaf $\mathscr{I}_2$ on X_1 so that $T_2 = V(\mathscr{I}_2)$ has underlying space contained in the closed set $Z_1 - Y_1$ in X_1 and

$$q_0^{-1}(T_1) \bigcap \overline{q_0^{-1}(X_1 - T_1)} \subseteq q_0^{-1}(T_2)$$

as closed subschemes (not just as closed subsets) of $X_2 - Z_2$. (The finite type condition on $\mathscr{I}_2$ is not a hypothesis in Lemma 1.5, but it is needed for how we will use the conclusion of Lemma 1.5.) Here, $\overline{q_0^{-1}(X_1 - T_1)}$ denotes the scheme-theoretic image of the scheme $q_0^{-1}(X_1 - T_1) = q_0^{-1}(X_1 - Z_1)$ under the quasi-compact immersion j. In particular, the underlying space of $\overline{q_0^{-1}(X_1 - T_1)}$ is the topological closure of $j(q_0^{-1}(X_1 - T_1))$ in $X_2 - Z_2$.

Before we construct $\mathscr{I}_2$, we first check the weaker set-theoretic statement that $q_0^{-1}(Z_1)\bigcap\overline{q_0^{-1}(X_1-Z_1)} \subseteq q_0^{-1}(Z_1-Y_1)$ as closed subsets in X_2-Z_2. This is easy, since

$$q_0^{-1}(Z_1-Y_1) = q^{-1}(Z_1-Y_1)\bigcap(X_2-Z_2) = (q^{-1}(Z_1)-p^{-1}(Y_1))\bigcap(X_2-Z_2),$$

yet $p^{-1}(Y_1)\subseteq Z_2$ by hypothesis, so $q_0^{-1}(Z_1-Y_1) = q^{-1}(Z_1)\bigcap(X_2-Z_2) = q_0^{-1}(Z_1)$, which gives what we want.

Since $Z_1 - Y_1 = (X_1 - V_1)\bigcap Z_1$ as subsets of X_1, our quasi-compactness assumptions on $V_1 \hookrightarrow X_1$ and $X_1 - Z_1 \hookrightarrow X_1$ enable us (via Lemma 1.3 in the non-noetherian case) to get a finitely presented closed subscheme structure T on $Z_1 - Y_1$ inside of X_1. The proved containment

$$q_0^{-1}(T_1)\bigcap\overline{q_0^{-1}(X_1-T_1)} \subseteq q_0^{-1}(T)$$

as sets, together with the quasi-compactness of $q_0^{-1}(T)$ (this is closed in the quasi-compact space $X_2 - Z_2$), enables us to take $T_2 = V(\mathscr{I}_2)$ to be a suitable nilpotent thickening of T.

Define $\mathscr{I} = \mathscr{I}_{T_1} + \mathscr{I}_{T_2}$, so $V(\mathscr{I})$ is supported inside $Z_1 - Y_1 \subseteq X_1 - V_1$. This is a finite type quasi-coherent ideal sheaf on X_1 since $\mathscr{I}_{T_1}$ and $\mathscr{I}_{T_2}$ are finite type, and we will show that the V_1-admissible (hence U-admissible) blow-up $X_1' = \mathrm{Bl}_{\mathscr{I}}(X_1) \to X_1$ satisfies the desired properties. By Lemma 1.5 applied to (3.1), after base change by the blow-up $X_1' \to X_1$, the closure of $(q_0^{-1}(X_1-Z_1))' \simeq q_0^{-1}(X_1-Z_1)$ in $(X_2-Z_2)' = X_2' - Z_2'$ is disjoint from ${q'}_0^{-1}(\overline{Y_1'})$ inside of $X_2'-Z_2'$. Thus, ${q'}^{-1}(\overline{Y_1'})$ meets $X_2'-Z_2'$ inside of an open $W \subseteq X_2' - Z_2'$ which lies inside of

$$(X_2'-Z_2') - (q_0^{-1}(X_1-Z_1))' = ((X_2-Z_2) - q_0^{-1}(X_1-Z_1))' = (q_0^{-1}(Z_1))' = (q^{-1}(Z_1)\bigcap(X_2-Z_2))'.$$

We want $W\bigcap\overline{U'} = \emptyset$ inside of X_2' (where $\overline{U'}$ denotes the closure of $U' \simeq U$ in the base change X_2' of X_2 by the U-admissible blow-up $X_1' \to X_1$). Since W is *open* in X_2', it suffices to check $W\bigcap U' = \emptyset$. Clearly

$$W\bigcap U' \subseteq (q^{-1}(Z_1)\bigcap(X_2-Z_2))'\bigcap U' = (q^{-1}(Z_1)\bigcap(X_2-Z_2)\bigcap U)'.$$

However, by hypothesis $p^{-1}(Y_1)\subseteq Z_2$ as sets, so

$$q^{-1}(Z_1)\bigcap(X_2-Z_2) \subseteq q^{-1}(Z_1) - p^{-1}(Y_1) = q^{-1}(Z_1)\bigcap(X_2-V_2)$$

(recall $Y_1 = Z_1\bigcap V_1$ and $q^{-1}(V_1) = V_2$), and this is disjoint from U in X_2 since $U\subseteq V_2$ in X_2. ∎

We need one more lemma (with a long proof) before we can prove the main theorem. Before we give the statement of the lemma, we need to set up some notation. Fix a quasi-compact, quasi-separated scheme S, and consider a commutative diagram of quasi-compact, separated S-schemes

$$\begin{array}{ccccc} U & \longrightarrow & X_i & \longrightarrow & \overline{X}_i \\ \| & & \downarrow{\scriptstyle q_i} & & \\ U & \longrightarrow & X & & \end{array}$$

for $1 \le i \le m$, with each horizontal map a quasi-compact open immersion and U topologically dense in both X and $\overline{X}_i$. The notation $\overline{X}_i$ is merely suggestive; we do not assume that X_i is schematically dense in $\overline{X}_i$ (though it will be so in our applications). Assume that all maps $\overline{X}_i \to S$ and q_i are proper and that $X \to S$ is of finite type. Let $\overline{U}$ denote the scheme-theoretic closure of U under the quasi-compact immersion $U \hookrightarrow \prod \overline{X}_i$, and let

$$\pi_i : \overline{U} \hookrightarrow \prod_{j=1}^{m} \overline{X}_j \to \overline{X}_i$$

be the canonical quasi-compact map (which is surjective since it is closed and U lies in the image). Here and in what follows, all products are taken over S unless otherwise specified. Finally, assume that we have finitely presented closed subschemes $Y_i \hookrightarrow X_i$ for all i, satisfying $\bigcap \pi_i^{-1}(Y_i) = \emptyset$.

Fix an X_i-admissible blow-up $\overline{X}'_i \to \overline{X}_i$ for each i, so we have quasi-compact immersions $X'_i, Y'_i \rightrightarrows \overline{X}'_i$ induced by base change, with $X'_i \simeq X_i$ and $Y'_i \simeq Y_i$. Let $\overline{Y'_i}$ denote the scheme-theoretic closure of Y'_i in $\overline{X}'_i$. We have a quasi-compact immersion

$$U' \stackrel{\text{def}}{=} U \hookrightarrow \prod \overline{X}'_i$$

induced by $U \hookrightarrow X_i \simeq X'_i \hookrightarrow \overline{X}'_i$, so we have a scheme-theoretic closure $\overline{U'}$ of U' as well as canonical (surjective) maps

$$\pi'_i : \overline{U'} \hookrightarrow \prod \overline{X}'_i \to \overline{X}'_i.$$

There is a natural map $\overline{U'} \to \overline{U}$ which is proper and extends the identity map between the canonical dense open subscheme U on each side, so $\overline{U'} \to \overline{U}$ is surjective. From this we readily see that $\bigcap (\pi'_i)^{-1}(Y'_i) = \emptyset$. In other words, our hypotheses are preserved after replacing each $\overline{X}_i$ with any X_i-admissible blow-up. Our aim is to show that for such suitably well-chosen blow-ups, an even stronger disjointness property holds.

Lemma 3.2. [D, Lemme 0.5] *With the above notation and hypotheses, we can choose X_i-admissible blow-ups $\overline{X}'_i \to \overline{X}_i$ so that $\bigcap (\pi'_i)^{-1}(\overline{Y'_i}) = \emptyset$.*

Proof. Because $X \to S$ is separated and *finite type*, Theorem 2.4 ensures the existence of an X_i-admissible blow-up $\overline{X}'_i \to \overline{X}_i$ such that there is a quasi-domination (V_i, f_i) of $\overline{X}'_i$ over X extending $q_i : X_i \to X$. Since $f_i : V_i \to X$ is an S-map between separated S-schemes it must be separated, and q_i is *proper*, so the (necessarily dense) open immersion $X_i \hookrightarrow V_i$ is proper and therefore an isomorphism. Thus, q_i is a quasi-domination of $\overline{X}'_i$ over X, so $\Gamma_{q_i} \subseteq \overline{X}'_i \times_S X$ is a closed subscheme. By Lemma 1.2, we may replace $\overline{X}_i$ by $\overline{X}'_i$ and so can suppose that the graph subscheme Γ_{q_i} is *closed* in $\overline{X}_i \times_S X$. In other words, (X_i, q_i) is a quasi-domination of $\overline{X}_i$ over X. This condition will be critical later in the proof, and this is the reason why we had to assume that $X \to S$ is of finite type and all of the q_i are proper.

Since $\pi_i^{-1}(Y_i) \to Y_i$ and $Y_i \to S$ are quasi-compact and $\overline{U} \to S$ is separated, the immersion $\pi_i^{-1}(Y_i) \hookrightarrow \overline{U}$ is quasi-compact. Let $\overline{\pi_i^{-1}(Y_i)}$ denote the scheme-theoretic closure of $\pi_i^{-1}(Y_i)$ in $\overline{U}$, and let $\mathscr{I}_i$ be the associated quasi-coherent ideal sheaf in $\mathscr{O}_{\overline{U}}$, so $\mathscr{I}_i|_{\pi_i^{-1}(X_i)} = \mathscr{I}_{\pi_i^{-1}(Y_i)}$ since Y_i is closed in X_i. Since $U \subseteq \prod X_i$ inside of $\prod \overline{X}_i$ and $Y_i \subseteq X_i$ is closed, we have

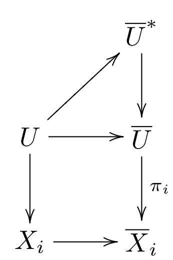
$$U \bigcap \left(\bigcap \overline{\pi_i^{-1}(Y_i)} \right) \subseteq \bigcap \pi_i^{-1}(Y_i) = \emptyset. \qquad (3.2)$$

Thus, the closed subscheme $Z_i = V(\mathscr{I}_i) \subseteq \overline{U}$ meets $\pi_i^{-1}(X_i)$ in $\pi_i^{-1}(Y_i)$ and

$$U \bigcap \left(\bigcap Z_i \right) = \emptyset,$$

so $\overline{U}^* = \mathrm{Bl}_{\cap Z_i}(\overline{U})$ is a U-admissible blow-up of $\overline{U}$ if each quasi-coherent $\mathscr{I}_i$ is of finite type. (Recall that the definition of an admissible blow-up requires that the blown-up ideal sheaf be of finite type.) The noetherian reader should skip to the next paragraph. For the general case, we write $\mathscr{I}_i = \varinjlim \mathscr{I}_{i,\alpha}$ ($\alpha \in A_i$), with $\mathscr{I}_{i,\alpha}$ running through the quasi-coherent finite type subideal sheaves in $\mathscr{I}_i$ such that $\mathscr{I}_{i,\alpha}|_{\pi_i^{-1}(X_i)} = \mathscr{I}_{\pi_i^{-1}(Y_i)}$. By (3.2), the quasi-compactness of the scheme U implies that for sufficiently large $\alpha_i \in A_i$, $1 \le i \le m$, $\mathscr{K}_i = \mathscr{I}_{i,\alpha_i}$ has the property that the finitely presented closed subscheme $V(\mathscr{K}_i) \subseteq \overline{U}$ meets $\pi_i^{-1}(X_i)$ in $\pi_i^{-1}(Y_i)$ and $U \bigcap (\bigcap V(\mathscr{K}_i)) = \emptyset$ (as in the noetherian case). We therefore rename $V(\mathscr{K}_i)$ as Z_i and redefine $\overline{U}^* = \mathrm{Bl}_{\sum \mathscr{K}_i}(\overline{U}) = \mathrm{Bl}_{\cap Z_i}(\overline{U})$, so $\overline{U}^*$ is a U-admissible blow-up of $\overline{U}$.

We have a commutative diagram



with $U \to \overline{U}^*$ an open immersion. Letting $p_i : \overline{U}^* \to \overline{U} \xrightarrow{\pi_i} \overline{X}_i$ be the canonical quasi-compact map, we get a commutative diagram

(3.3)
$$\begin{array}{ccccc} U & \longrightarrow & p_i^{-1}(X_i) & \longrightarrow & \overline{U}^* \\ \| & & \downarrow & & \downarrow{\scriptstyle p_i} \\ U & \longrightarrow & X_i & \longrightarrow & \overline{X}_i \end{array}$$

in which all horizontal maps are quasi-compact open immersions. Let $\widetilde{Z_i} \subseteq \overline{U}^*$ be the strict transform of the closed subscheme $Z_i \subseteq \overline{U}$ with respect to the U-admissible blow-up $\overline{U}^* \to \overline{U}$.

By Lemma 1.4, $\bigcap \widetilde{Z_i} = \emptyset$. The noetherian reader should define $T_i = \widetilde{Z_i} \subseteq \overline{U}^*$ and define $\overline{Y}_i$ to be the schematic closure of Y_i in $\overline{X}_i$, so $\cap T_i = \emptyset$ and $\overline{Y}_i \cap X_i = Y_i$, and skip to the next paragraph. In the general case, two problems are that $\widetilde{Z_i}$ might not be finitely presented as a closed subscheme of $\overline{U}^*$ and the schematic closure of Y_i in $\overline{X}_i$ may not be finitely presented. The usual limit argument gives *finitely presented* closed subschemes $T_i \subseteq \overline{U}^*$ with $\widetilde{Z_i} \subseteq T_i$ and $\bigcap T_i = \emptyset$. Again repeating the usual limit argument, we can find a finitely presented closed subscheme $\overline{Y}_i$ in $\overline{X}_i$ with $\overline{Y}_i \cap X_i = Y_i$. (This is a smearing-out of the schematic closure of Y_i in $\overline{X}_i$.)

Since the open immersions $\overline{U}^* - T_i \to \overline{U}^*$ and $\overline{X}_i - \overline{Y}_i \to \overline{X}_i$ are quasi-compact, we can apply Lemma 3.1 to (3.3) using $\overline{Y}_i$ and T_i as Z_1 and Z_2 in Lemma 3.1 *provided that we can choose* T_i *so that* $p_i^{-1}(Y_i) \subseteq T_i$ *as sets.* We will now show that for suitable choices of Z_i, we have $p_i^{-1}(Y_i) \subseteq \widetilde{Z_i}$ as sets. Since $Z_i \cap \pi_i^{-1}(X_i) = \pi_i^{-1}(Y_i)$, by Lemma 1.1 it suffices to show that each $\pi_i^{-1}(Y_i)$ is disjoint from $\cap Z_j$, the center of the blow-up $\overline{U}^* \to \overline{U}$. In the noetherian case (with Z_i taken to be the schematic closure of $\pi_i^{-1}(Y_i)$ in $\overline{U}$, as we may do in this case) this means that each $\pi_i^{-1}(Y_i)$ is disjoint from $\bigcap \overline{\pi_j^{-1}(Y_j)}$, and in the general case it is also sufficient to prove this same disjointeness property (as then by the usual argument with quasi-compactness and limits we can shrink each Z_j around $\overline{\pi_j^{-1}(Y_j)}$ so that $\pi_i^{-1}(Y_i) \bigcap (\cap Z_j) = \emptyset$ for all i).

Recall that the underlying space of $\overline{\pi_j^{-1}(Y_j)}$ is the topological closure of $\pi_j^{-1}(Y_j)$ in $\overline{U}$. Since Y_j is closed in X_j for all j, it is enough to show that the subset $\pi_i^{-1}(Y_i) \subseteq \overline{U} \subseteq \prod \overline{X}_j$ lies inside of the open subscheme $\prod X_j$ for each i. Indeed, this would imply that

$$\pi_i^{-1}(Y_i) \bigcap \left(\bigcap \overline{\pi_j^{-1}(Y_j)} \right) \subseteq \bigcap \pi_j^{-1}(Y_j) = \emptyset.$$

More generally, we will now show that if $\xi \in \overline{U}$ satisfies $\pi_i(\xi) \in X_i$ for some i, then $\pi_j(\xi) \in X_j$ for all j. The key point is that we have arranged that the graph $\Gamma_{q_j} \subseteq \overline{X}_j \times_S X$ of $q_j : X_j \to X$ is a closed subscheme. Using this, we will show that its pullback to a closed subscheme $\Gamma_j \subseteq \overline{U} \times_S X$ is *independent* of j. The reason for such independence is that Γ_j is (by construction) a closed subscheme in $\overline{U} \times_S X$ which contains $\Delta_{U/S}$ and maps isomorphically to an open subscheme $W_j \subseteq \overline{U}$ (just the pullback of X_j) with $U \subseteq W_j$, so Γ_j corresponds to a quasi-domination of $\overline{U}$ over X relative to $U \hookrightarrow X$. By the discussion of uniqueness of quasi-dominations in the schematically dense case at the beginning of §2, we see that the closed subscheme $\Gamma_j \subseteq \overline{U} \times_S X$ is therefore the same for all j.

Choose $\xi \in \overline{U}$ with $\pi_i(\xi) \in X_i$. Let $\eta \in \overline{U} \times_S X$ be the point $(\xi, q_i(\pi_i(\xi)))$; more precisely, this is the image ξ under the S-morphism $(1, q_i \circ \pi_i) : \overline{U} \to \overline{U} \times_S X$. Note that the image of η in $\overline{X}_i \times_S X$ under $\pi_i \times 1$ lands in Γ_{q_i}, so $\eta \in \Gamma_i = \Gamma_j$. Since $\Gamma_j \subseteq \overline{X}_j \times_S X$ lives in the subscheme $X_j \times_S X$, we see that $\pi_j(\xi) \in X_j$. This is what we needed to prove.

Now that we know we can construct the T_i's so the hypotheses of Lemma 3.1 are satisfied, we conclude that for each i there exists an X_i-admissible blow-up $\overline{X}_i' \to \overline{X}_i$ such that after base change by this blow-up we have

$$(p_i')^{-1}(\overline{Y_i'}) \cap \overline{U_i} \subseteq T_i' \cap \overline{U_i}$$

as sets, where the quasi-compact open immersion $U_i \hookrightarrow \overline{U}^* \times_{\overline{X}_i} \overline{X}'_i$ is the fiber over $U \subseteq \overline{U}^*$ (so $U_i \simeq U$), $\overline{U_i}$ is the topological closure of U_i in $\overline{U}^* \times_{\overline{X}_i} \overline{X}'_i$, and T'_i and p'_i are the base changes of T_i and p_i by the blow-up $\overline{X}'_i \to \overline{X}_i$.

Consider the canonical map $\overline{U'} \to \overline{U}$. Note that this map is proper and is compatible with the dense open immersion of U into each side, so this map is surjective. Letting Z'_i be the fiber over Z_i, we define

$$\overline{U'}^* = \mathrm{Bl}_{\cap Z'_j}(\overline{U'}).$$

This object fits into a commutative diagram

$$\begin{array}{ccc} \overline{U'}^* & \xrightarrow{r} & \overline{U}^* \\ {\scriptstyle f}\downarrow & & \downarrow \\ \overline{U'} & \longrightarrow & \overline{U} \\ {\scriptstyle \pi'_i}\downarrow & & \downarrow{\scriptstyle \pi_i} \\ \overline{X}'_i & \longrightarrow & \overline{X}_i \end{array}$$

in which r is induced by the universal propery of the blow-up map $\overline{U}^* \to \overline{U}$ and the composite map along the right column is p_i. We let $\overline{f}_i$ denote the composite map along the left column. Note that the induced map $(r, \overline{f}_i) : \overline{U'}^* \to \overline{U}^* \times_{\overline{X}_i} \overline{X}'_i$ is compatible with the open immersions of U into each side. Since U is dense in $\overline{U'}^*$, it follows that $(r, \overline{f}_i)$ has its set-theoretic image inside of $\overline{U_i}$. Hence, if we take the preimage under $(r, \overline{f}_i)$ of the containment of sets $(p'_i)^{-1}(\overline{Y'_i}) \cap \overline{U_i} \subseteq T'_i \cap \overline{U_i}$, we get $\overline{f}_i^{-1}(\overline{Y'_i}) \subseteq r^{-1}(T_i)$.

Recalling that $\bigcap T_i = \emptyset$, we obtain

$$\bigcap \overline{f}_i^{-1}(\overline{Y'_i}) = \emptyset.$$

However, the map f is proper and compatible with the dense open immersion of U into $\overline{U'}^*$ and $\overline{U'}$, so f is surjective. Using the fact that $\overline{f}_i = \pi'_i \circ f$, we therefore get

$$\bigcap (\pi'_i)^{-1}(\overline{Y'_i}) = \emptyset,$$

which is what we wanted to prove. ■

4. Nagata Compactification Theorem

Theorem 4.1. [N2, §4, Theorem 2], [D, Thm. 1.7] *Let $f : X \to S$ be separated and finite type, with S quasi-compact and quasi-separated. There exists an open immersion $j : X \hookrightarrow \overline{X}$ of S-schemes such that $\overline{X} \to S$ is proper.*

Remark 4.2. Since $X \to S$ is quasi-compact and $\overline{X} \to S$ is separated, any $j : X \hookrightarrow \overline{X}$ as in Theorem 4.1 is necessarily quasi-compact. Thus, a scheme-theoretic closure of X in $\overline{X}$ exists and if we rename this as $\overline{X}$ then we obtain a j that is also schematically dense.

Proof. Let's assume that we have a finite open covering $\{U_i\}$ of X by dense opens, with all $U_i \to S$ quasi-projective (in particular, U_i is *quasi-compact*). We will use this assumption to prove the theorem, and then will return to the verification of this assumption at the very end. By the refined version of Chow's Lemma in Corollary 2.6, there exists a commutative diagram of S-schemes

$$\begin{array}{ccccc} U_i & \longrightarrow & X_i & \xrightarrow{j_i} & \overline{X}_i \\ \| & & \downarrow{\scriptstyle q_i} & & \\ U_i & \longrightarrow & X & & \end{array}$$

with q_i proper and surjective, $q_i^{-1}(U_i) \simeq U_i$, U_i a dense open in $\overline{X}_i$ and X, j_i an open immersion, and $\overline{X}_i \to S$ a proper map. In particular, all horizontal maps are quasi-compact due to separatedness and quasi-compactness over S for all objects.

Give $X - U_i$ a finitely presented closed subscheme structure in X. (In the non-noetherian case, this rests on Lemma 1.3.) Define $Y_i = q_i^{-1}(X - U_i)$ and define $\overline{U}$ to be the scheme-theoretic closure of $U = \bigcap U_i$ under the quasi-compact immersion $U \to \prod \overline{X}_i$. Finally, define $\pi_i : \overline{U} \hookrightarrow \prod \overline{X}_j \to \overline{X}_i$. Since this is proper and the image contains the dense open U, π_i is surjective.

We first claim that $\bigcap \pi_i^{-1}(Y_i) = \emptyset$. That is, inside of $\prod \overline{X}_i$, we want $\overline{U} \bigcap (\prod Y_i) = \emptyset$. Since $\prod Y_i$ lies inside of the open subscheme $\prod X_i$, it is enough to check that $\overline{U} \bigcap \prod X_i$ does not meet $\prod Y_i$. But $\pi : \prod X_i \to X^n$ makes the n-fold diagonal subscheme $\Delta^n_{U/S} \subseteq \prod X_i$ factor through the *closed* subscheme $\pi^{-1}(\Delta^n_{X/S})$ (recall that $X \to S$ is *separated*). Thus, $\overline{U} \bigcap (\prod X_i) \subseteq \pi^{-1}(\Delta^n_{X/S})$ as subschemes of $\prod \overline{X}_i$. This yields

$$\left(\overline{U} \bigcap \prod X_i\right) \bigcap \left(\prod Y_i\right) \subseteq \pi^{-1}\left(\Delta^n_{X/S} \bigcap \prod (X - U_i)\right),$$

yet $\Delta^n_{X/S} \simeq X$ identifies $\Delta^n_{X/S} \bigcap (\prod (X - U_i))$ with $\bigcap (X - U_i) = \emptyset$.

By Lemma 3.2, replacing $\overline{X}_i$ by a suitable X_i-admissible blow-up lets us suppose that $\bigcap \pi_i^{-1}(\overline{Y_i}) = \emptyset$, with $\overline{Y_i}$ the scheme-theoretic closure of Y_i in $\overline{X}_i$ under the quasi-compact immersion j_i. The noetherian reader should skip to the next paragraph. In the general case $\overline{Y_i}$ may not be finitely presented as a closed subscheme of $\overline{X}_i$, so we need to modify $\overline{Y_i}$ slightly. By the usual limit argument, we can write $\mathscr{I}_{\overline{Y_i}} = \varinjlim \mathscr{I}_{i,\alpha}$ ($\alpha \in A_i$), with $\mathscr{I}_{i,\alpha}$ running through the finite type quasi-coherent subideal sheaves in $\mathscr{I}_{\overline{Y_i}}$ that satisfy $\mathscr{I}_{i,\alpha}|_{X_i} = \mathscr{I}_{Y_i}$. For a large α_i, we have closed subschemes $\overline{Y_i} \stackrel{\text{def}}{=} V(\mathscr{I}_{i,\alpha_i})$ in $\overline{X}_i$ that satisfy $\overline{Y_i} \bigcap X_i = Y_i$ and $\bigcap \pi^{-1}(\overline{Y_i}) = \emptyset$. The notation $\overline{Y_i}$ here is merely suggestive (though in the noetherian setting, it could be taken to be the scheme-theoretic closure of Y_i as above); we will no longer need to use the scheme-theoretic closure of Y_i in $\overline{X}_i$, so this notation should not cause confusion.

Note that $\{\overline{U} - \pi_i^{-1}(\overline{Y_i})\}$ is an open covering of $\overline{U}$. This will be used later. Let M_i be the S-scheme obtained by gluing the quasi-compact and quasi-separated (and even *finite type*) S-schemes X and $\overline{X}_i - \overline{Y}_i$ along U_i. Clearly, M_i is of finite type over S and X is dense in M_i (since U_i is dense in $\overline{X}_i$). We claim that M_i is also *separated* over S. To check this, we need to show that the diagonal subscheme $\Delta_{U_i/S} \subseteq X \times_S (\overline{X}_i - \overline{Y}_i)$ is closed (see the discussion of this point in the Introduction). Since $\overline{X}_i \to S$ is separated and q_i is proper and surjective, it suffices to show that inside of $\overline{X}_i \times_S \overline{X}_i$, we have an equality of subschemes

$$(q_i \times 1_{\overline{X}_i - \overline{Y_i}})^{-1}(\Delta_{U_i/S}) = \Delta_{\overline{X}_i/S} \bigcap (X_i \times_S (\overline{X}_i - \overline{Y_i})).$$

Combining the equality

$$\Delta_{\overline{X}_i/S} \bigcap (X_i \times_S (\overline{X}_i - \overline{Y_i})) = \Delta_{X_i/S} \bigcap (X_i \times_S ((\overline{X}_i - \overline{Y_i}) \cap X_i))$$

with

$$(\overline{X}_i - \overline{Y_i}) \bigcap X_i = X_i - \left(X_i \bigcap \overline{Y_i}\right) = X_i - Y_i = q_i^{-1}(U_i),$$

we get the result (since $q_i^{-1}(U_i) \simeq U_i$).

Next, we apply Corollary 2.10 to the collection of dense open quasi-compact immersions $X \hookrightarrow M_i$ over S. This provides X-admissible blow-ups $M_i' = \mathrm{Bl}_{\mathscr{I}_i}(M_i)$ so that X is schematically dense in each M_i' and there is a schematically dense open immersion $j : X \hookrightarrow M$, with M separated and finite type over S and covered by open immersions $M_i' \hookrightarrow M$ such that all composite maps $X \to M_i' \to M$ coincide with j. We will show that M is proper over S, thereby completing the proof (granting the existence of the initial open covering $\{U_i\}$). It suffices to construct a U-admissible blow-up U^* of the S-proper $\overline{U}$ such that U^* surjects onto the S-separated M over S.

We will need to use the commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & \overline{U} - \pi_i^{-1}(\overline{Y_i}) \\ \big\downarrow & & \big\downarrow {\scriptstyle \pi_i'} \\ U_i & \longrightarrow & M_i \end{array} \tag{4.1}$$

where π_i' on $\overline{U} - \pi_i^{-1}(\overline{Y}_i) = \pi_i^{-1}(\overline{X}_i - \overline{Y}_i)$ is defined using π_i and the open immersion of $\overline{X}_i - \overline{Y}_i$ into M_i (via the definition of M_i). Since $\overline{U} - \pi_i^{-1}(\overline{Y_i})$ is quasi-compact (as it is quasi-compact over the quasi-compact scheme $\overline{X}_i - \overline{Y_i}$), all maps in (4.1) are quasi-compact. The map across the top is schematically dense.

Let $\mathscr{K}_i$ on $\overline{U}$ be a quasi-coherent finite type ideal sheaf with

$$\mathscr{K}_i|_{\overline{U} - \pi_i^{-1}(\overline{Y_i})} = (\pi_i')^{-1}(\mathscr{I}_i) \cdot \mathscr{O}_{\overline{U} - \pi_i^{-1}(\overline{Y_i})}$$

on the quasi-compact open $\overline{U}_i - \pi_i^{-1}(\overline{Y}_i)$, so $\mathscr{K}_i|_U = \mathscr{O}_U$. Defining $U^* = \mathrm{Bl}_{\prod \mathscr{K}_i}(\overline{U})$, the natural map $q : U^* \to \overline{U}$ is a U-admissible blow-up such that the open immersion $U \to U^*$ is quasi-compact and schematically dense. Using the blow-up map q and (1.2), we get composite S-scheme maps

$$g_i : U^* - q^{-1}(\pi_i^{-1}(\overline{Y_i})) \to \mathrm{Bl}_{\mathscr{I}_i}(M_i) = M_i' \to M$$

that coincide on the common open subscheme U. By schematic density and quasi-compactness of the open immersions $U \hookrightarrow \overline{U} - \pi_i^{-1}(\overline{Y_i})$, as well as the separatedness of $M \to S$, it follows that the g_i's agree on the quasi-compact overlaps of their domains. These open domains cover U^* because $\cap \pi_i^{-1}(\overline{Y}_i) = \emptyset$, so the g_i's glue to give an S-morphism $g : U^* \to M$ that is compatible with the canonical open immersion of U into U^* and M. Thus, g has the dense open subset U in the image. But $U^* \to S$ is proper and $M \to S$ is separated, so g must be proper. Therefore the image is closed and so g is surjective. Hence, M is S-proper.

The one remaining issue is the initial construction of a finite covering of X by dense opens that are quasi-projective over S. Let $\{W_i\}_{1 \le i \le n}$ be a finite open affine covering of X, chosen so small that each $W_i \to S$ is quasi-affine (e.g., take W_i to land in an open affine in S). Since $W_i \to S$ is also of finite type, each $W_i \to S$ is quasi-projective [EGA, II, 5.3.4(i)]. For each permutation σ of $\{1, \dots, n\}$, define the open subscheme

$$U_\sigma = \bigcup_{i=1}^{n} \left(W_{\sigma(i)} - (W_{\sigma(i)} \bigcap \overline{\bigcup_{j<i} W_{\sigma(j)}}) \right).$$

It is clear that the U_σ are an open covering of X and each U_σ is dense in X. Also, the n-fold union in the definition of each U_σ is a disjoint union. By [EGA, II, 5.3.6], in order to prove the quasi-projectivity of U_σ over S, it therefore suffices to check that each of the parts of these unions is quasi-projective over S.

By [EGA, II, 5.3.4(i),(ii)], we only need to check that the open immersions

$$W_{\sigma(i)} - \left(W_{\sigma(i)} \bigcap \overline{\bigcup_{j<i} W_{\sigma(j)}} \right) \hookrightarrow W_{\sigma(i)}$$

are quasi-compact. This is obvious in the noetherian case, so for the noetherian reader the proof of Nagata's theorem is done. To handle the general case it suffices to show that the maps $X - \overline{W_i} \hookrightarrow X$ are quasi-compact for all i. If there exist finitely presented closed subscheme structures on the closed sets $\overline{W_i}$ in X, then the result is clear (and this condition is necessary as well, by Lemma 1.3). This is always the case when S is noetherian. For the general case, [D, Commentaires(c), p.14] seems to suggest working with closed subschemes in X around each $\overline{W_i}$ that are slightly larger and finitely presented. I am unable to see how to make this work because it seems that enlarging these closures may cause the U_σ to no longer be dense in X. Thus, we bypass this difficulty by giving an alternative argument to infer the general case rather formally from the settled noetherian case.

The crucial result that makes it possible to reduce to the noetherian case is a wonderful theorem of Thomason and Trobaugh that essentially says that every quasi-compact and quasi-separated scheme S "descends" to the noetherian setting. More precisely, by [TT, Thm. C.9], for any such S there exists an inverse system $\{S_i\}$ of schemes of finite type over $\mathbf{Z}$ such that the transition maps are affine and $S \simeq \varprojlim S_i$. Hence, by the standard direct limit formalism [EGA, IV$_3$, 8.8.2(*ii*), 8.10.4], *if $X \to S$ is finitely presented* then there exists an i_0 and a map $X_{i_0} \to S_{i_0}$ that is separated and of finite type such that $X \simeq X_{i_0} \times_{S_{i_0}} S$ as S-schemes. By applying Nagata's theorem in the noetherian case we get an open immersion $X_{i_0} \hookrightarrow \overline{X}_{i_0}$ into a proper S_{i_0}-scheme, and so applying base change gives an open immersion of X into the proper S-scheme $\overline{X}_{i_0} \times_{S_{i_0}} S$. This settles the case of an arbitrary S when X is finitely presented over S.

To eliminate the finite presentation restriction, we use Theorem 4.3 below to conclude that there exists a closed immersion $X \hookrightarrow X'$ over S into a finitely presented and separated S-scheme X'. By the cases already treated, there exists an open immersion $X' \hookrightarrow \overline{X}'$ into a proper S-scheme, so the composite map $k : X \to \overline{X}'$ is an immersion into a proper S-scheme. Since k is quasi-compact, it must factor as an open immersion into a closed subscheme of $\overline{X}'$; this gives the desired compactification for X over S. ■

Theorem 4.3. *Let S be a quasi-compact and quasi-separated scheme, and let X be a quasi-separated S-scheme of finite type. There exists a closed immersion $X \hookrightarrow X'$ over S into an S-scheme of finite presentation. If X is S-separated then we can choose X' to be S-separated.*

This theorem is a triviality in the noetherian case by taking $X' = X$.

Proof. Let us first check that if such an X' exists and X is S-separated then there is a finitely presented closed subscheme $Z \hookrightarrow X'$ containing X such that Z is S-separated. Let $\mathscr{I}$ be the quasi-coherent ideal sheaf for X in X', so since X' is a quasi-compact and quasi-separated scheme we may write $\mathscr{I} = \varinjlim \mathscr{I}_\alpha$ for quasi-coherent and finitely generated ideal sheaves $\mathscr{I}_\alpha \subseteq \mathscr{I}$. Let $X_\alpha \hookrightarrow X'$ be the zero-scheme of $\mathscr{I}_\alpha$, so the X_α are finitely presented over S and $X = \varprojlim X_\alpha$. We wish to prove that X_α is S-separated for large α. Since S is quasi-compact, this assertion is local on S. Hence, we may assume S is affine. The separatedness of the limit $X = \varprojlim X_\alpha$ therefore forces the X_α's to be separated for large α, by [TT, Thm. C.7].

Now it remains to find a closed immersion $X \hookrightarrow X'$ with X' finitely presented over S. If X is affine and lies over an open affine T in S then we can take $X' = \mathbf{A}^n_T$ for suitable n (this is finitely presented over S because $T \to S$ is finitely presented, as S is quasi-compact and quasi-separated). In general we can cover X by r such open opens (lying over open affines in X) and we proceed by induction on r. The case $r = 1$ was just settled, and the general case immediately reduces to the following situation: X has an open cover by two quasi-compact opens U and V (so U and V are of finite type over S) such that there exist closed immersions $U \hookrightarrow U'$ and $V \hookrightarrow V'$ with U' and V' finitely presented over S. We seek finitely presented closed subschemes in U' and V' that contain U and V and can be glued to construct the desired X'. The construction is an elaborate exercise in standard direct limit techniques, as we now explain.

Let $W = U \cap V$, so W is a quasi-compact open in both U and V. Choose quasi-compact opens $\mathscr{U} \subseteq U'$ and $\mathscr{V} \subseteq V'$ such that $\mathscr{U} \cap U = W$ and $\mathscr{V} \cap V = W$. (In particular, W is a closed subscheme in both $\mathscr{U}$ and $\mathscr{V}$.) Since U' is quasi-compact and quasi-separated, the quasi-coherent ideal $\mathscr{I}$ of U in U' has the form $\mathscr{I} = \varinjlim \mathscr{I}_\alpha$ for finitely generated quasi-coherent ideal sheaves $\mathscr{I}_\alpha \subseteq \mathscr{I}$. Thus, we have $U = \varprojlim U'_\alpha$ where $\{U'_\alpha\}$ is the inverse system of zero-schemes of the $\mathscr{I}_\alpha$'s on U'. Likewise, $W = \varprojlim(\mathscr{U} \cap U'_\alpha)$ with $\mathscr{U} \cap U'_\alpha$ closed in $\mathscr{U}$ and open in U'_α. Let $\{V'_\beta\}$ be an analogous such inverse system of finitely presented closed subschemes in V' with inverse limit V.

The inclusion $W = U \cap V \to V \to V'$ lies in

$$\mathrm{Hom}_S(W, V') = \mathrm{Hom}_S(\varprojlim(\mathscr{U} \cap U'_\alpha), V') = \varinjlim_\alpha \mathrm{Hom}_S(\mathscr{U} \cap U'_\alpha, V'), \tag{4.2}$$

where the final equality holds because V' is locally of finite presentation over S and each $\mathscr{U} \cap U'_\alpha$ is quasi-compact and quasi-separated. Hence, by replacing U' with a suitable U'_α we may suppose that there exists an S-map $\mathscr{U} \to V'$ lifting the inclusion $W \hookrightarrow V$. In fact, by applying (4.2) with V' replaced by the finitely presented S-scheme $\mathscr{V} \subseteq V'$, we can "smear out" the closed immersion $W = U \cap V = \mathscr{V} \cap V \hookrightarrow \mathscr{V}$ to a map $\phi : \mathscr{U} \to \mathscr{V}$, where $\mathscr{V}$ is the above choice of quasi-compact open in V'. (Keep in mind that since we replaced U' with a well-chosen U'_α, what we are presently calling $\mathscr{U}$ is what used to be $\mathscr{U} \cap U'_\alpha$.)

By reversing the roles of U and V, we find a large β_0 and a map $\psi : \mathscr{V} \cap V'_{\beta_0} \to \mathscr{U}$ that lifts the identity on W. The composite

$$\phi \circ \psi : \mathscr{V} \cap V'_{\beta_0} \to \mathscr{V}$$

lifts the identity on W, so the fact that $\mathscr{V}$ is *finitely presented* over S ensures that by replacing β_0 with some $\beta_1 \geq \beta_0$ (that is, composing ψ with the natural inclusion $\mathscr{V} \cap V'_{\beta_1} \to \mathscr{V} \cap V'_{\beta_0}$ for some $\beta_1 \geq \beta_0$) we can arrange that $\phi \circ \psi$ is the *canonical* closed immersion.

Now observe that for *any* finitely generated quasi-coherent ideal sheaf $\mathscr{I}$ on the quasi-compact open $\mathscr{U} \subseteq U'$ such that $W \subseteq V(\mathscr{I})$ (i.e., $\mathscr{I} \subseteq \mathscr{I}_W$ inside $\mathscr{O}_{\mathscr{U}}$), $\mathscr{I}$ extends to a finite type quasi-coherent ideal sheaf $\mathscr{I}'$ on U' such that $U \subseteq V(\mathscr{I}')$ as closed subschemes of U' (i.e., $\mathscr{I}' \subseteq \mathscr{I}_U$ inside $\mathscr{O}_{U'}$). Indeed, since $\mathscr{U} \cap U = W$ we have $\mathscr{I} \subseteq \mathscr{I}_W = \mathscr{I}_{\mathscr{U} \cap U} = \mathscr{I}_U|_{\mathscr{U}}$ inside $\mathscr{O}_{\mathscr{U}}$ with $\mathscr{I}$ of finite type, so $\mathscr{I}$ extends to a finite type quasi-coherent subsheaf $\mathscr{I}' \subseteq \mathscr{I}_U$ by [EGA, I, 9.4.7; IV_1, 1.7.7].

Let $\mathscr{V}_1 = \mathscr{V} \cap V'_{\beta_1}$ for β_1 as above. Let $\mathscr{U}_1 \hookrightarrow \mathscr{U}$ be the preimage under ϕ of the finitely presented closed subscheme $\mathscr{V}_1 \hookrightarrow \mathscr{V}$. Observe that the inclusion $W_1 := \mathscr{U}_1 \cap U \subseteq \mathscr{U} \cap U = W$ of closed subschemes of $\mathscr{U}$ is an equality because ϕ restricts to the identity map on W (and $W \subseteq V'_\beta$ for all β). Likewise, $\mathscr{V}_1 \cap V = W$ because of the inclusions $W \subseteq \mathscr{V}_1 \cap V \subseteq \mathscr{V} \cap V = W$ as subschemes of V. We can extend the ideal sheaf of the finitely presented closed subscheme $\mathscr{U}_1$ in $\mathscr{U}$ to a finite type quasi-coherent ideal sheaf on U' whose zero-scheme contains U because of the preceding paragraph. Let U'_1 be the zero scheme of this extended ideal in U', so U'_1 is a finitely presented S-scheme containing $\mathscr{U}_1$ as a quasi-compact open subscheme. (In fact, $U'_1 \cap \mathscr{U} = \mathscr{U}_1$ with $\mathscr{U}$ a quasi-compact open in U'.) Upon replacing U', $\mathscr{U}$ with U'_1, $\mathscr{U}_1$, and also V', $\mathscr{V}$ with V'_{β_1}, $\mathscr{V}_1$ (and leaving X, U, V, W unchanged!), we do not lose any generality but (by the choice of β_1) we gain the property that there exist maps

$$s : \mathscr{U} \to \mathscr{V}, \;\; t : \mathscr{V} \to \mathscr{U}$$

lifting the identity on W such that $s \circ t = \mathrm{id}_{\mathscr{V}}$. Here, s and t are respectively induced by ϕ and ψ as above (before we renamed things). If $t \circ s$ is the identity on $\mathscr{U}$ then we can glue the finitely presented S-schemes U' and V' along the inverse S-isomorphisms s and t between quasi-compact open subsets $\mathscr{U}$ and $\mathscr{V}$ to construct a scheme X' that is finitely presented (*i.e.*, quasi-separated and locally finitely presented) over S. There is then a unique S-map $X \to X'$ whose restriction over V' (resp. U') is the canonical closed immersion $V \hookrightarrow V'$ (resp. $U \hookrightarrow U'$) because $W = U \cap V$. Hence, $X \to X'$ is a closed immersion and solves the problem.

We must reduce ourselves to the special case when $t \circ s$ is the identity. Since $\mathscr{U}$ is finitely presented over S and the composite map

$$W \hookrightarrow \mathscr{U} \stackrel{s}{\to} \mathscr{V} \stackrel{t}{\to} \mathscr{U}$$

is the *canonical* inclusion, for large α_0 the composite

$$\mathscr{U} \cap U'_{\alpha_0} \hookrightarrow \mathscr{U} \stackrel{t \circ s}{\to} \mathscr{U}$$

is the canonical inclusion. Hence, if we let $\mathscr{V}_{\alpha_0} = t^{-1}(\mathscr{U} \cap U'_{\alpha_0})$ then the composite of restrictions

$$\mathscr{U} \cap U'_{\alpha_0} \stackrel{s_{\alpha_0}}{\to} \mathscr{V}_{\alpha_0} \stackrel{t_{\alpha_0}}{\to} \mathscr{U} \cap U'_{\alpha_0}$$

is the identity. Of course, $s_{\alpha_0} \circ t_{\alpha_0}$ is the identity on $\mathscr{V}_{\alpha_0}$ (because it is a restriction of $s \circ t$, which is the identity on $\mathscr{V}$). Hence, by replacing U' with U'_{α_0} and shrinking V' so that the "new" $\mathscr{V}$ is the above $\mathscr{V}_{\alpha_0}$, we find ourselves in the case where $t \circ s$ is indeed the identity, so we may glue as before to complete the proof in the general case. ■

With Nagata's theorem now proved, we get the following result that generalizes Theorem 2.11 and Remark 2.12.

Corollary 4.4. *Separated birational maps of finite type always arise from blow-ups and dense open immersions in the following two senses.*

(1) *Let $f : X \to Y$ be a separated map of finite type between quasi-compact and quasi-separated schemes, and assume that f is an isomorphism over a quasi-compact dense open subscheme $U \subseteq Y$ such that*

$f^{-1}(U)$ is dense in X. There exist U-admissible blow-ups $\widetilde{X}$ and $\widetilde{Y}$ of X and Y for which there is an open immersion $\widetilde{X} \hookrightarrow \widetilde{Y}$ over f.

(2) *Let $\pi_i : X \to Y_i$ be a pair of separated birational maps of finite type between quasi-compact quasi-separated schemes, and let $U_i \subseteq Y_i$ be a quasi-compact dense open subscheme such that π_i is an isomorphism over U_i. For the dense quasi-compact open $U = U_1 \cap U_2$ (viewed in X, and then in Y_1 and Y_2), there are U-admissible blow-ups $\widetilde{X}$, $\widetilde{Y}_1$, and $\widetilde{Y}_2$ equipped with open immersions $\widetilde{X} \hookrightarrow \widetilde{Y}_i$ over π_i for each i.*

Proof. For (1), we can apply Theorem 4.1 (and Remark 4.2) to f to get a dense open immersion j of X over Y into a proper Y-scheme $\overline{f} : \overline{X} \to Y$. The inclusion of U into $\overline{f}^{-1}(U)$ is a proper open immersion with dense image, so it is an isomorphism. In particular, $\overline{f}$ is an isomorphism over U and $\overline{f}^{-1}(U)$ is dense in $\overline{X}$. Thus, it suffices to solve the problem for $\overline{X}$ in the role of X, so we can assume that f is proper. In this case Theorem 2.11 gives the required result. For (2), the case when each π_i is proper is Remark 2.12. To handle the general case, we have to carefully apply Nagata's theorem twice. (The subtlety is due to the fact that we do not assume X and the Y_i's are finite type over a common base S.) To carry out the argument, first apply Theorem 4.1 and Remark 4.2 to π_i to get a schematically dense open immersion of X into a proper Y_i-scheme $\overline{X}_i$. The proper maps $\overline{X}_i \to Y_i$ are isomorphisms over the dense open $U \subseteq Y_i$ with dense preimage in $\overline{X}_i$, so by Theorem 2.11 there is a U-admissible blow-up $\overline{X}'_1$ of $\overline{X}_1$ that is also a U-admissible blow-up of Y_1. Let $\mathscr{I}_1$ be an ideal along which the blow-up $\overline{X}'_1$ of $\overline{X}_1$ is defined, and let $\mathscr{I}_2$ be a finite type quasi-coherent ideal on $\overline{X}_2$ whose restriction to the quasi-compact open X is $\mathscr{I}_1|_X$. We can replace $\overline{X}_i$ with its U-admissible blow-up along $\mathscr{I}_i$ to get to the case when π_1 is an open immersion. (We have implicitly used Lemma 1.2 in the non-noetherian case.) An application of Theorem 2.11 to π_2 gives a common U-admissible blow-up of $\overline{X}_2$ and Y_2. If $\mathscr{K}_2$ is an ideal along which this blow-up of $\overline{X}_2$ is defined then $\mathscr{K}_2|_X$ can be extended to a finite type quasi-coherent ideal on Y_1 since π_1 is now a quasi-compact open immersion. The desired blow-ups to complete the proof of (2) are the blow-up of Y_1 along this extended ideal, the blow-up of X along $\mathscr{K}_2|_X$, and the blow-up of Y_2 provided by our second application of Theorem 2.11. ■

Appendix A. Auxiliary elimination of noetherian hypotheses

In this appendix we give another application of Theorem 4.3 and [TT, App. C] to eliminate noetherian assumptions. The key is the following refinement of Theorem 4.3:

Theorem A.1. *Let $f : X \to S$ be a finite map of schemes with S quasi-compact and quasi-separated. There exists a closed S-immersion $j : X \to Y$ into a finite and finitely presented S-scheme Y.*

Proof. By Theorem 4.3, there is a closed S-immersion $X \hookrightarrow X'$ into a finitely presented S-scheme X'. Let $\mathscr{J}$ be the quasi-coherent ideal sheaf of X in $\mathscr{O}_{X'}$, so since X' is a quasi-compact and quasi-separated scheme we can write $\mathscr{J} = \varinjlim \mathscr{J}_i$ with $\mathscr{J}_i$ the directed system of finitely generated quasi-coherent ideal sheaves on X' contained in $\mathscr{J}$. Let X_i be the zero scheme of $\mathscr{J}_i$ in X', so the X_i's are an inverse system of finitely presented S-schemes with affine (even closed immersion) transition maps and limit X. We shall prove that X_i is S-finite for large i, and so we can take Y to be X_i for such large i. Note that if X_{i_0} is S-finite for some i_0 then X_i is S-finite for all $i \geq i_0$. Hence, it suffices to work over a finite collection of affine opens that cover S, which is to say that we can assume $S = \mathrm{Spec}(A)$ is affine. Then by S-finiteness of X we have that $X = \mathrm{Spec}(B)$ for a finite A-algebra B.

The X_i's are an inverse system of finitely presented A-schemes with affine transition maps and affine limit $\mathrm{Spec}\, B$, so by [TT, Prop. C.6] it follows that X_i is affine for large i. We restrict attention to such large i, so we can write $X_i = \mathrm{Spec}(B_i)$ with $\{B_i\}$ a directed system of finitely presented A-algebras with surjective transition maps and limit B that is A-finite. We fix algebra generators $b_1, \dots, b_n$ for some B_{i_0} over A, so each b_j has A-integral image in the limit B. Thus, for large $i \geq i_0$ and each j the image of b_j in B_i satisfies the same A-integral relation that is satisfied by the image of b_j in B. Hence, for large i we have that the quotient B_i of B_{i_0} is A-finite. ■

It is a theorem of Chevalley [EGA, II, 6.7.1] that if $X \to Y$ is a finite surjection of schemes with X affine and Y noetherian then Y is affine. The preceding theorem lets us prove the same result without noetherian restrictions:

Corollary A.2. *If $f : X \to Y$ is a finite surjection of schemes with X affine then Y is affine. In particular, if Y is a scheme such that $Y_{\rm red}$ is affine then Y is affine.*

Proof. Certainly Y is quasi-compact, and by finiteness and surjectivity of f it follows that $Y \to Y \times_{\operatorname{Spec} \mathbf{Z}} Y$ has closed image since X is a separated scheme. Hence, Y is also separated. Thus, by Theorem A.1 we may factor f through a closed immersion into a finitely presented and finite morphism $h : X' \to Y$ that is certainly surjective (since X surjects onto Y). We claim that X' can be taken to be affine. Granting this for now, we can rename X' as X so as to reduce to the case when f is finitely presented. By [TT, Thm. C.9], we may write $Y = \varprojlim Y_i$ with $\{Y_i\}$ an inverse system of finite type $\mathbf{Z}$-schemes having affine transition maps. Hence, for a large i_0 we can find a finitely presented map $f_{i_0} : X_{i_0} \to Y_{i_0}$ inducing f under the base change $Y \to Y_{i_0}$.

Define $f_i : X_i \to Y_i$ by the base change $Y_i \to Y_{i_0}$ on f_{i_0}, so $\{X_i\}$ is an inverse system of finite type $\mathbf{Z}$-schemes with affine transition maps and inverse limit X; clearly $f = \varprojlim f_i$. Since f is finite and surjective, for large i we have that f_i is finite and surjective. Since the inverse system $\{X_i\}$ of finite type $\mathbf{Z}$-schemes has affine limit X, by [TT, Prop. C.6] it follows that X_i is affine for large i. Thus, for such large i we have that $X_i \to Y_i$ is a finite surjection from an affine scheme onto a noetherian scheme. By the known noetherian case it follows that Y_i is affine for large i, so the limit $Y = \varprojlim Y_i$ is affine.

It remains to show that X' can be taken to be affine when X is affine. More generally, we claim that for a closed immersion $X \hookrightarrow X'$ of schemes with X affine and X' quasi-compact and quasi-separated, there is a finitely presented closed subscheme $Z \subseteq X'$ containing X such that Z is affine. By [TT, Thm. C.9], $X' = \varprojlim X'_\alpha$ for an inverse system $\{X'_\alpha\}$ of finite type $\mathbf{Z}$-schemes with affine transition maps. Let $Z_\alpha \subseteq X'_\alpha$ be the schematic image of the affine map $X \to X'_\alpha$. Then $\{Z_\alpha\}$ is an inverse system of $\mathbf{Z}$-schemes of finite type with affine transition maps, and $\varprojlim Z_\alpha = X$. (Note that we do not claim that the natural map $Z_\beta \to Z_\alpha \times_{X'_\alpha} X'_\beta$ is an isomorphism for $\beta \geq \alpha$; this is generally false and we do not need it.) Since the Z_α are finite type over $\mathbf{Z}$ and their limit X is affine, it follows from [TT, Prop. C.6] that Z_{α_0} is affine for some large α_0. Hence, the pullback $Z = Z_{\alpha_0} \times X'_{\alpha_0} X'$ is a closed subscheme of finite presentation in X' that contains X and is affine (as it is affine over Z_{α_0}). ■

References

[D] P. Deligne, *Le théorème de plongement de Nagata*, personal notes (unpublished).

[EGA] A. Grothendieck, *Élements de Géométrie Algébrique*, Math. Publ. IHES **4, 8, 11, 17, 20, 24, 28, 32**, 1960-7.

[SGA4] A. Grothendieck, *Théorie des Topos et Cohomologie Etale des Schémas* Tome 3, Lecture Notes in Mathematics **305**, Springer-Verlag, New York, 1973.

[GR] L. Gruson, M. Raynaud, *Critères de platitude et de projectivité*, Inventiones Math. **13**, 1971, pp. 1-89.

[H] R. Hartshorne, *Algebraic geometry*, GTM **52**, Springer-Verlag, New York, 1977.

[L] W. Lütkebohmert, *On compactification of schemes*, Manuscripta Mathematica **80**, 1993, pp. 95–111.

[N1] M. Nagata, *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto **2** (1), 1962, pp. 1-10.

[N2] M. Nagata, *A generalization of the imbedding problem*, J. Math. Kyoto **3** (1), 1963, pp. 89-102.

[T] M. Temkin, *Relative Riemann–Zariski spaces*, preprint, 2007.

[TT] R. Thomason, T. Trobaugh, "Higher algebraic K-theory of schemes and of derived categories," in *The Grothendieck Festschrift, Vol* III, pp. 247–435, Progress in Mathematics, Birkhauser, Boston, 1990.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
E-mail address: bdconrad@umich.edu