MINIMAL MODELS FOR ELLIPTIC CURVES

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1. Introduction

In the 1960's, the efforts of many mathematicians (Kodaira, Néron, Raynaud, Tate, Lichtenbaum, Shafarevich, Lipman, and Deligne-Mumford) led to a very elegant theory of preferred integral models for both (positive-genus) curves and abelian varieties. This work was largely inspired by the theory of minimal models for smooth proper algebraic surfaces over algebraically closed fields [2]. There are some very special integral models, called *minimal regular proper models* for curves and *Néron models* for abelian varieties. Excellent references on these topics are [3] and [11], as well as [4] and [1], and we will provide an overview of the main definitions and results below. Elliptic curves occupy a special place in these theories, as they straddle the worlds of both curves and abelian varieties. Thus, an elliptic curve over the fraction field K of a discrete valuation ring R has both a Néron model and a minimal regular proper model over R. Moreover, it has an abstract minimal Weierstrass model over R that is unique up to unique R-isomorphism.

It is natural to ask how the preferred models for elliptic curves are related to each other. A tricky aspect is that minimal Weierstrass models are (usually) defined in a manner that is a bit too explicit and is lacking in an abstract universal property, whereas both Néron models of abelian varieties and minimal regular proper models of smooth (positive-genus) proper curves are characterized by simple abstract universal properties. The key aspects of the story (including all necessary background on regular models of curves) are presented in [11] in complete detail, so these notes may be viewed as a complement to the discussion in [11] (I will generally refer to [1], [3], and [4] for results that are also proved in [11] because the former are the references from which I learned about these matters, before [11] was written; the reader may well find that [11] is more useful and/or more understandable than these notes).

We begin in §2 with a brief summary of the theory of Weierstrass models of elliptic curves. The main point is to formulate the theory in a manner that eliminates the appearance of Weierstrass equations; this is accomplished by using Serre duality over the residue field, generalizing the use of Riemann–Roch to free the theory of elliptic curves from the curse of Weierstrass equations. In §3, we provide an overview of the theory of integral models of smooth curves, with an emphasis on minimal regular proper models. The theory of minimal Weierstrass models is addressed in §4, where we give an abstract criterion for minimality of a Weierstrass model; this criterion (which was brought to my attention by James Parson, and is technically much simpler than my earlier viewpoint on these matters) is expressed in terms of R-rational maps. In §5 we switch to the category of abelian varieties and we present the basic definitions and existence theorem concerning Néron models, and we deduce some relations between Weierstrass models and Néron models. The basic properties of relative dualizing sheaves and arithmetic intersection theory are summarized in §6 and §7, and we apply these notions in §8 to establish some additional conceptual characterizations of minimal Weierstrass models (e.g., in terms of rational singularities). In particular, we provide a conceptual explanation for why Tate's algorithm does not require the computation of normalizations.

In these notes, a *singularity* on a curve over a field k is a point not in the k-smooth locus, rather than a point in the non-regular locus. The reason for this distinction is that there are examples of Weierstrass cubics over k that are regular and not smooth (these can only exist when k has characteristic 2 or 3; see the discussion following Theorem 5.5), and we want to consider these as singular.

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2. Weierstrass models

Let R be an arbitrary discrete valuation ring, K its fraction field, and k its residue field. Let E be an elliptic curve over K. As is well-known [14, Ch. VII, §1], there exist projective planar curves (in dehomogenized form)

$$y^2 + (a_1x + a_3)y = x^3 + a_2x^2 + a_4x + a_6 \tag{2.1}$$

that are abstractly K-isomorphic to E with the identity point corresponding to $[0,1,0]$, and such that all a_j's lie in R. We would like to give an abstract definition of such curves so that we can eliminate the appearance of equations. This provides a clarity to the theory akin to the abstract definition of an elliptic curve. We begin with an *ad hoc* explicit definition that we shall soon promote to a more abstract setting.

Definition 2.1. A *planar integral Weierstrass model* of E (over R) is a pair (W, i) where W is a closed subscheme in $\mathbf{P}^2_R$ defined by an equation as in (2.1) and i is an isomorphism $W_{/K} \simeq E$ carrying $[0,1,0]$ to the identity in $E(K)$. A *morphism* $(W,i) \to (W',i')$ between such planar models of E is an R-map $W \to W'$ over R that respects the K-fiber identifications i and i' with E.

Remark 2.2. Morphisms between planar integral Weierstrass models of an elliptic curves are not *a priori* required to be induced by a projective-linear change of coordinates on $\mathbf{P}^2_R$. We will prove that this additional property is automatically satisfied, and that all morphisms are isomorphisms.

We usually suppress the explicit mention of i. Planar integral Weierstrass models satisfy some nice properties, as we now summarize. Clearly they are R-proper, and the monicity of the leading terms in (2.1) implies that they are R-flat. The geometric theory insures that the 1-dimensional special fiber of such a W is geometrically irreducible and geometrically reduced, and is even smooth away from possibly one geometric point. A less obvious property is:

Lemma 2.3. *Let E be an elliptic curve over K, and let W be an arbitrary planar integral Weierstrass model. The scheme W is normal.*

Proof. To prove normality, we use Serre's criterion "$R_1 + S_2$" for normality (see [12, p. 183] for a discussion of Serre's homological conditions R_i and S_i, and their applications to criteria for reducedness and normality of noetherian rings). Since W is R-flat, it is automatically R-smooth away from the closed non-smooth locus in the special fiber. Thus, W is R-smooth away from possibly one geometric closed point in the special fiber. In particular, W is regular in codimension ≤ 1 (this is Serre's condition R_1). To check S_2, the R-flatness reduces us to checking condition S_1 on the special fiber. The special fiber is an integral curve, so S_1 is clear. Thus, W is normal. ∎

Definition 2.4. An *abstract integral Weierstrass model* for E (over R) is a pair (W, i) consisting of a proper flat R-scheme W and an isomorphism $i : W_{/K} \simeq E$ such that W is normal and the special fiber W_k is 1-dimensional, geometrically irreducible, and smooth at the reduction $\varepsilon(k) \in W(k)$ of the unique $\varepsilon \in W(R) = E(K)$ extending the identity; in particular, W_k is generically smooth. A *morphism* $(W,i) \to (W',i')$ is an R-map $W \to W'$ that respects i and i' on the K-fibers.

Obviously every planar integral Weierstrass model for E is an abstract integral Weierstrass model. We will relate abstract and planar integral Weierstrass models, but first let us observe that maps between abstract integral Weierstrass models satisfy a strong condition:

Example 2.5. Consider a morphism $f : W' \to W$ between abstract integral Weierstrass models of E. Such an f must be an isomorphism. Indeed, the image of f is closed (by properness) and contains the generic fiber (since f_K is an isomorphism), so f is surjective since R-flatness forces the K-fibers to be dense. It follows that the map f_k between irreducible 1-dimensional special fibers must be quasi-finite, so f is a quasi-finite and proper map. Thus, f is finite. Since W is normal and f is a birational isomorphism, we conclude that f must be an isomorphism.

Note also that R-flatness and R-separatedness ensure that there can be at most one map between abstract Weierstrass models of E. Thus, in the above setup, f is the unique isomorphism between W and W'; in this sense, we may say that isomorphic abstract integral Weierstrass models of E are uniquely isomorphic.

Here is an interesting property of abstract integral Weierstrass models:

Theorem 2.6. *Let W be an abstract integral Weierstrass model of E. Let $W^{\rm sm}$ be its R-smooth locus. The invertible sheaf $\Omega^1_{W^{\rm sm}/R}$ on $W^{\rm sm}$ is globally free, and the R-module $\mathrm{H}^0(W^{\rm sm}, \Omega^1_{W^{\rm sm}/R})$ is free of rank 1.*

Proof. The K-fiber $\Omega^1_{E/K}$ is globally free since E is an abelian variety, so let ω be a global generator of $\Omega^1_{E/K}$. Let η be the generic point of W_k. Since W_k is generically smooth and W is normal, the local ring $\mathscr{O}_{W,\eta}$ is a discrete valuation ring and a uniformizer is given by a uniformizer of W. The stalk of $\Omega^1_{W^{\rm sm}/R}$ at η is a free module of rank 1 over $\mathscr{O}_{W,\eta}$, and so we may uniquely scale ω by a power of a uniformizer of W such that it is a generator of this stalk. Thus, on the normal connected noetherian scheme $W^{\rm sm}$, the line bundle $\Omega^1_{W^{\rm sm}/R}$ has ω as a section defined in codimension ≤ 1. Since any normal noetherian domain is the intersection of its localizations at height 1 primes, it follows that ω uniquely extends to a global section of $\Omega^1_{W^{\rm sm}/R}$ that is moreover a global generator.

It remains to show that $\mathscr{O}_{W^{\rm sm}}$ has R as its ring of global sections. Since W is a connected normal noetherian scheme and $W^{\rm sm}$ has complement of codimension ≥ 2 in W, we conclude that a global section of $\mathscr{O}_{W^{\rm sm}}$ uniquely extends to a global section of $\mathscr{O}_W$. Thus, we just have to prove $\mathrm{H}^0(W, \mathscr{O}_W)$ is equal to R. Since this R-algebra is finite and flat, it is enough to check its rank is 1, and this follows from the equality $K = \mathrm{H}^0(E, \mathscr{O}_E)$. ■

Definition 2.7. A global generator of $\Omega^1_{W^{\rm sm}/R}$ is called an *invariant differential* on $W^{\rm sm}$; such elements are unique up to $R^\times$-multiple.

Now we are readily to link the planar and abstract theories.

Theorem 2.8. *Let W be an abstract integral Weierstrass model of E. There exists a planar integral Weierstrass model $W' \hookrightarrow \mathbf{P}^2_R$ of E and an R-isomorphism $W \simeq W'$ respecting K-fiber identifications with E.*

Note that an isomorphism from an abstract W onto a planar W' must carry the identity $\varepsilon \in E(K) = W(R)$ to $[0, 1, 0]$, as can be checked on K-points (where it follows from the requirements in the definition of a planar model). For a relative version of the theorem, with the base $\mathrm{Spec}\, R$ replaced by an arbitrary scheme S and with W replaced by a proper, flat, and finitely presented S-scheme having reduced and irreducible geometric fibers of arithmetic genus 1, see [7].

Proof. Since W_k is generically smooth, it satisfies Serre's condition R_0 (regularity in codimension ≤ 0). Moreover, since W is normal (and so satisfies Serre's condition S_2), R-flatness of W ensures that W_k satisfies Serre's condition S_1. Since "$R_0 + S_1$" is Serre's criterion for reducedness, we conclude that W_k is reduced (and not just generically reduced). However, a reduced and generically-smooth finite-type k-scheme must be geometrically reduced. Thus, W_k is even geometrically reduced. The section $\varepsilon \in W(R)$ lies in the smooth locus of W, as this may be checked on the special fiber (where the k-smoothness of the reduction of ε is part of the definition of an abstrat integral Weierstrass model). Thus, the coherent ideal of the section ε is invertible on W. We write $\mathscr{O}(\varepsilon)$ to denote the inverse of this ideal, and $\mathscr{O}(n\varepsilon)$ to denote its nth tensor power.

The strategy to find a planar embedding is to imitate the proof that relative elliptic curves admit Weierstrass models locally on the base. Such models will be found by using Serre duality over k, generalizing the role of the Riemann-Roch theorem in the construction of Weierstrass models for elliptic curves. Let $\omega_{W_k/k}$ be the (coherent) dualizing sheaf on W_k. We claim that $\omega_{W_k/k}$ is a torsion-free $\mathscr{O}_{W_k}$-module on the integral curve W_k. If not, then it contains a nonzero coherent subsheaf supported at a closed point. Hence, it is enough to prove that coherent sheaves with finite support do not admit nonzero maps to $\omega_{W_k/k}$. By duality, for any coherent sheaf $\mathscr{F}$ on W_k, the composite

$$\mathrm{Hom}_W(\mathscr{F}, \omega_{W_k/k}) \to \mathrm{Hom}_k(\mathrm{H}^1(W_k, \mathscr{F}), \mathrm{H}^1(W_k, \omega_{W_k/k})) \to \mathrm{Hom}_k(\mathrm{H}^1(W_k, \mathscr{F}), k)$$

(defined by the trace map $\mathrm{H}^1(W_k, \omega_{W_k/k}) \to k$) is an isomorphism. In particular, if $\mathrm{H}^1(W_k, \mathscr{F}) = 0$ then $\mathrm{Hom}_W(\mathscr{F}, \omega_{W_k/k}) = 0$. Applying this to $\mathscr{F}$ with finite support, we get the desired result.

Since $\omega_{W_k/k}$ is torsion-free on the integral curve W_k, the map $\mathscr{O}_{W_k} \to \omega_{W_k/k}$ associated to any nonzero global section of $\omega_{W_k/k}$ must be injective. Note that on the dense open k-smooth locus $U \subseteq W_k$, we have

$$\omega_{W_k/k}|_U \simeq \omega_{U/k} = \Omega^1_{U/k},$$

so $\omega_{W_k/k}$ is generically invertible on W_k. Hence, any nonzero map $\mathscr{O}_{W_k} \to \omega_{W_k/k}$ is injective and has cokernel with finite support.

The reduction $e \in W_k(k)$ of ε lies in the smooth locus of W_k, and hence has an invertible ideal sheaf. Let $\mathscr{O}(e)$ denote the inverse of this ideal sheaf, and define the twisted sheaf

$$\mathscr{F}(Ne) = \mathscr{F} \otimes_{\mathscr{O}_{W_k}} \mathscr{O}(e)^{\otimes N}$$

for any $N \in \mathbf{Z}$ and any $\mathscr{O}_{W_k}$-module $\mathscr{F}$. We claim that

$$\mathrm{H}^0(W_k, \omega_{W_k/k}(-ne)) = 0 \tag{2.2}$$

for $n > 0$. Suppose there is a nonzero global section, so we get a short exact sequence

$$0 \to \mathscr{O}_{W_k} \to \omega_{W_k/k}(-ne) \to \mathscr{F} \to 0 \tag{2.3}$$

where $\mathscr{F}$ is coherent with 0-dimensional support. By the constancy of the Euler characteristic in flat families,

$$\chi(\mathscr{O}_{W_k}) = \chi(\mathscr{O}_{W_K}) = \chi(\mathscr{O}_E) = 1 - 1 = 0,$$

so the short exact sequence (2.3) implies

$$\chi(\omega_{W_k/k}(-ne)) = \chi(\mathscr{F}) = \dim \mathrm{H}^0(W_k, \mathscr{F}) \geq 0.$$

However, by Serre duality on W_k we have a perfect duality pairing

$$\mathrm{H}^i(W_k, \mathscr{O}_{W_k}(ne)) \times \mathrm{H}^{1-i}(W_k, \omega_{W_k/k}(-ne)) \to \mathrm{H}^1(W_k, \omega_{W_k/k}) \to k,$$

and so (see [3, 9.1/1])

$$\chi(\omega_{W_k/k}(-ne)) = -\chi(\mathscr{O}(ne)) = -\deg(ne) - \chi(\mathscr{O}_{W_k}) = -n - 0 < 0,$$

a contradiction.

By duality, we conclude from (2.2) that $\mathrm{H}^1(W_k, \mathscr{O}(ne))$ vanishes for all $n > 0$. The vanishing of $\mathrm{H}^1(W_k, \mathscr{O}(ne))$ implies (by the theory of cohomology and base change) that the finite R-module $\mathrm{H}^0(W, \mathscr{O}(n\varepsilon))$ is free for all $n > 0$ and its formation commutes with base change on R; in particular, the reduction of $\mathrm{H}^0(W, \mathscr{O}(n\varepsilon))$ maps isomorphically onto $\mathrm{H}^0(W_k, \mathscr{O}(ne))$ for $n > 0$. Since

$$h^0(\mathscr{O}(ne)) = h^0(\mathscr{O}(ne)) - h^1(\mathscr{O}(ne)) = \chi(\mathscr{O}(ne)) = n$$

for $n > 0$, $\mathrm{H}^0(W, \mathscr{O}(n\varepsilon))$ is free of rank n for $n > 0$. For example, the map $R \to \mathrm{H}^0(W, \mathscr{O}(\varepsilon))$ between rank-1 R-modules must be an isomorphism because its reduction is an isomorphism (as this is an injection between 1-dimensional vector spaces over k).

We can now use the usual argument (as with Riemann-Roch for elliptic curves) to find a basis $\{1, x, y\}$ of $\mathrm{H}^0(W, \mathscr{O}(3\varepsilon))$ such that $\{1, x\}$ is a basis of $\mathrm{H}^0(W, \mathscr{O}(2\varepsilon))$. Moreover, since $\{1\}$, $\{1, x\}$, and $\{1, x, y\}$ reduce to bases of the corresponding H^0's on W_k, by picking a local parameter t along the section ε (i.e., a local generator of the invertible ideal of this section) we see that the t-adic expansions of x and y begin $x = v/t^2 + \dots$ and $y = v'/t^3 + \dots$ with $v, v' \in R^\times$. Hence, we may multiply x and y by units of R so that $x = 1/t^2 + \dots$ and $y = 1/t^3 + \dots$. We may now proceed in the usual way to deduce that there is a Weierstrass relation

$$y^2 + (a_1 x + a_3) y = x^3 + a_2 x^2 + a_4 x + a_6 \tag{2.4}$$

with $a_i \in R$. Let us define

$$f(x, y) = y^2 + (a_1 x + a_3) y - (x^3 + a_2 x^2 + a_4 x + a_6).$$

Since any integer ≥ 2 may be expressed in the form $2a + 3b$ with nonnegative integers a and b, we conclude that the graded domain

$$S = \bigoplus_{n \geq 0} \mathrm{H}^0(W, \mathscr{O}(n\varepsilon)) = \bigoplus_{n \geq 0} S_n$$

is generated over $S_0 = R$ over by the elements $1, x, y$ in respective degrees $1, 2, 3$. Thus, we have a surjection of domains

$$\varphi : R[x, y, w]/(w^3 f(x/w, y/w)) \twoheadrightarrow S^{(3)} \stackrel{\text{def}}{=} \bigoplus_{n \geq 0} S_{3n}$$

defined by $w \mapsto 1 \in S_3$, $x \mapsto x \in S_3$, $y \mapsto y \in S_3$. The line bundle $\mathscr{O}(\varepsilon)$ on W is ample, as this may be checked on the geometrically irreducible fibers, and hence [8, II, 4.6.3] the canonical map

$$W \to \mathrm{Proj}(S) \simeq \mathrm{Proj}(S^{(3)})$$

is an isomorphism. Thus, the map φ defines a closed immersion from proper R-flat $W = \mathrm{Proj}(S) = \mathrm{Proj}(S^{(3)})$ into the zero scheme C of (2.4) in

$$\mathrm{Proj}(\mathrm{Sym}(\mathrm{H}^0(W, \mathscr{O}(3\varepsilon))) \simeq \mathbf{P}^2_R$$

such that $\varepsilon \mapsto [0, 1, 0]$. The formation of the closed immersion $W \hookrightarrow C$ commutes with base change on R, so the R-flatness of the (finitely presented) source and target implies that $W \to C$ is an isomorphism if and only if the induced maps on fibers are isomorphisms (here we use the fibral flatness criterion). However, on fibers over points $s \in \mathrm{Spec}\, R$ it is clear that $W_s \hookrightarrow C_s$ is a closed immersion between integral proper $k(s)$-curves, and so this map must be an isomorphism (we can even deduce *a posteriori* by dimension reasons that φ must be an isomorphism, though this is irrelevant). ■

Corollary 2.9. *Let W and W' be two planar integral Weierstrass models of E. There is at most one isomorphism between W and W' as models of E, and (with respect to the given closed immersions of W and W' into $\mathbf{P}^2_R$) such an isomorphism is induced by a unique R-automorphism of $\mathbf{P}^2_R$ defined by*

$$[x, y, w] \mapsto [u^2 x + rw, u^3 y + sy + tw, w] \tag{2.5}$$

with $r, s, t \in R$ and $u \in R^\times$.

Since we already have seen that any morphism between abstract integral Weierstrass models must be an isomorphism, we conclude that the coordinate-changes "$(x, y) \mapsto (\pi^{2r} x, \pi^{3r} y)$" between planar integral Weierstrass models as in [14] are merely rational maps that are *not* globally-defined scheme morphisms for $r \neq 0$. We will analyze these rational maps in §4.

Proof. The elements $\{1, x, y\}$ constitute an R-basis of the finite free rank-3 R-module $\mathrm{H}^0(W, \mathscr{O}(3\varepsilon))$ and these sections generate the ample $\mathscr{O}(3\varepsilon)$, as we may check on the special fiber. In a similar manner, we see that the pair $\{1, x\}$ satisfies an analogous property for $\mathscr{O}(2\varepsilon)$, and that the finite free rank-1 R-module $\mathrm{H}^0(W, \mathscr{O}(\varepsilon))$ is equal to R. This all implies that the given cubic embedding of W into $\mathbf{P}^2_R$ is just a coordinatization of the canonical closed immersion $W \hookrightarrow \mathbf{P}(\mathrm{H}^0(W, \mathscr{O}(3\varepsilon)))$ using a basis $\{1, x, y\}$ of $\mathrm{H}^0(W, \mathscr{O}(3\varepsilon))$ such that x and y satisfy the following conditions:

- $\{1, x\}$ is a basis of $\mathrm{H}^0(W, \mathscr{O}(2\varepsilon))$;
- if t is a local generator of the invertible ideal cutting out the section ε in the smooth locus, then the t-adic expansions
$$x = \frac{v}{t^2} + \ldots, \quad y = \frac{v'}{t^3} + \ldots$$
have $v, v' \in R^\times$ satisfying $v^3 = v'^2$.

The pair (v, v') is well-defined up to the transformation $(v, v') \mapsto (u^2 v, u^3 v')$ for $u \in R^\times$, due to possible changes in the choice of t.

With this intrinsic description of the data that encodes a planar model, we deduce immediately (as in the classical case of elliptic curves over a field) that the possible choices (x', y') for a given W are precisely those obtained from the given (x, y) by exactly the standard invertible coordinate-changes over R as in the dehomogenization of (2.5). To choose such an (x', y') is exactly to give an isomorphism from W to a planar integral Weierstrass model W' of E; this yields the desired description of all isomorphisms among planar models. ■

It follows from this corollary that if W is an abstract integral Weierstrass model of E, then the nonzero element $\Delta = \Delta(a_1, \dots, a_6) \in R$ arising from an isomorphism of W onto a planar integral Weierstrass model depends on the planar model only up to $R^\times$-multiple. We call any such Δ a *discriminant* of W, so $\mathrm{ord}(\Delta)$ is intrinsic to W. If we choose an isomorphism of the abstract W onto a planar model, we get two pieces of data from the planar equation: a preferred element $\Delta \in R$ and a preferred invariant differential ω (as constructed from the equation in the standard manner; see Definition 2.7). For our purposes, the important consequence of this fact is that the resulting generating section

$$\Delta\omega^{\otimes 12} \in \mathrm{H}^0(E, (\Omega^1_{E/K})^{\otimes 12})$$

is *independent* of the choice of auxiliary planar equation: this follows from the standard transformation laws for discriminants and invariant differentials under change in a Weierstrass embedding of an elliptic curve as in (2.5), thereby proving:

Corollary 2.10. *Let W and W' be abstract integral Weierstrass models of E. The R-lines*

$$\mathrm{H}^0(W^{\mathrm{sm}}, \Omega^1_{W^{\mathrm{sm}}/R}), \mathrm{H}^0(W'^{\mathrm{sm}}, \Omega^1_{W'^{\mathrm{sm}}/R}) \subseteq \mathrm{H}^0(E, \Omega^1_{E/K})$$

satisfy the inclusion

$$\mathrm{H}^0(W^{\mathrm{sm}}, \Omega^1_{W^{\mathrm{sm}}/R}) \subseteq \mathrm{H}^0(W'^{\mathrm{sm}}, \Omega^1_{W'^{\mathrm{sm}}/R})$$

if and only if $\mathrm{ord}(\Delta) \geq \mathrm{ord}(\Delta')$.

We will now use the simpler terminology *Weierstrass model* (of E over R) to mean an abstract integral Weierstrass model. The preceding discussion shows that it is well-posed to define a *minimal Weierstrass model* of E to be a Weierstrass model W whose associated discriminant attains minimal normalized order among all Weierstrass models of E. In §4, we will prove that any two Weierstrass models W and W' of E satisfying

$$\mathrm{H}^0(W^{\mathrm{sm}}, \Omega^1_{W^{\mathrm{sm}}/R}) = \mathrm{H}^0(W'^{\mathrm{sm}}, \Omega^1_{W'^{\mathrm{sm}}/R})$$

are necessarily isomorphic. In particular, this gives an equation-free proof of the uniqueness of minimal Weierstrass models (proved in [14, VII, 1.3] by direct calculation with Weierstrass equations and transformation laws, though these explicit descriptions can be obtained in our approach *a posteriori* by using Corollary 2.9).

3. Minimal regular proper models of curves

A K-*curve* is a proper K-scheme C that is geometrically connected and 1-dimensional. An R-*curve* is a proper flat R-scheme $\mathscr{C}$ whose generic fiber is a K-curve. Note that the special fiber $\mathscr{C}_k$ is automatically a k-curve. Indeed, Stein factorization ensures that the special fiber must be geometrically connected, and the 1-dimensionality follows from [8, IV$_3$, 13.2.3]. We will generally consider only R-curves whose K-fiber is smooth (such as an elliptic curve).

Definition 3.1. If X is a separated K-scheme, an R-*model* of X is a pair $(\mathscr{X}, i)$ where $\mathscr{X}$ is a flat and separated R-scheme and $i : \mathscr{X}_{/K} \simeq X$ is a K-isomorphism. If $\mathscr{X}$ is a regular scheme, it is a *regular R-model* of X (and likewise for the properties of being normal, R-proper, finite type, *etc.*).

A *morphism* of R-models $(\mathscr{X}, i) \to (\mathscr{X}', i')$ is a map $f : \mathscr{X} \to \mathscr{X}'$ over R such that $i' \circ f_K = i$.

Note that an R-model of a reduced separated K-scheme is reduced (due to R-flatness), and a proper R-model of a K-curve is an R-curve. Also, a morphism between models is unique if one exists (as can be seen via a graph argument, since the separatedness hypothesis forces graph morphisms to be closed immersions). Thus, we can put a partial ordering on the category of R-models of a separated K-scheme X by saying that one model $(\mathscr{X}, i)$ *dominates* another model $(\mathscr{X}', i')$ if there exists a map of models $\mathscr{X} \to \mathscr{X}'$; this is a partial ordering with isomorphisms as "equality" because a self-map of an R-model must be the identity (so if two R-models dominate each other, then the unique domination maps between them must be mutually inverse isomorphisms). With these concepts understood, we generally omit the explicit mention of i unless there is a danger of confusion.

Remark 3.2. Beware that, by Example 2.5, the Weierstrass models of an elliptic curve are only related by domination when they are isomorphic; in particular, it is not true that minimal Weierstrass models are the minimal objects with respect to domination in the class of Weierstrass models of a fixed elliptic curve.

Example 3.3. If $(\mathscr{C}, i)$ is a proper R-model of C and Z is a closed subscheme of $\mathscr{C}$ (such as a closed point) with support in $\mathscr{C}_k$, then the blow-up $\widetilde{\mathscr{C}} = \mathrm{Bl}_Z(\mathscr{C})$ of $\mathscr{C}$ along Z is a proper R-scheme equipped with a map $\pi : \widetilde{\mathscr{C}} \to \mathscr{C}$ that is an isomorphism over the complement of Z; in particular, π_K is an isomorphism. From the construction of the blow-up it is clear that the blow-up has structure sheaf that is torsion-free over R, and so it is R-flat. Thus, $(\widetilde{\mathscr{C}}, i \circ \pi_K)$ is a proper R-model of C that dominates $(\mathscr{C}, i)$ and is rarely isomorphic to $(\mathscr{C}, i)$; e.g., blowing up a closed point in the special fiber gives a non-isomorphic model. In particular, it is very easy to make many proper R-models that dominate a given one without being isomorphic to it.

Example 3.4. Let $\mathscr{C}$ be a proper R-model of a smooth K-curve C (e.g., we can take $\mathscr{C}$ to be the closure in $\mathbf{P}^N_R$ for a closed embedding $C \hookrightarrow \mathbf{P}^N_K$). Since $\mathscr{C}$ must be reduced, it is reasonable to consider its normalization $\widetilde{\mathscr{C}} \to \mathscr{C}$. This is a finite map, though the proof of finiteness requires extending scalars to $\widehat{R}$ to reduce to the excellent case. We conclude that $\widetilde{\mathscr{C}}$ is proper and R-flat. Since the generic fiber C of $\mathscr{C}$ is already normal, it follows that the normalization $\widetilde{\mathscr{C}}$ of $\mathscr{C}$ is a proper normal R-model of C in the evident manner.

Example 3.5. Here is a more interesting construction of normal proper R-models. Let C be a smooth K-curve. Since it is proper and K-smooth, there exists a finite separable map $C \to \mathbf{P}^1_K$. Consider the composite map $C \to \mathbf{P}^1_K \hookrightarrow \mathbf{P}^1_R$. This expresses $K(C)$ as a finite separable extension of the function field $K(t)$ of the connected normal finite-type R-scheme $\mathbf{P}^1_R$. Thus, the normalization of $\mathbf{P}^1_R$ in $K(C)$ is a finite map $\mathscr{C} \to \mathbf{P}^1_R$, and its generic fiber is identified with $C \to \mathbf{P}^1_K$. In this way, $\mathscr{C}$ is a normal proper R-model of C, and it is even projective.

We have seen in Example 3.4 and Example 3.5 that normal proper R-models of smooth K-curves C are rather easy to construct, and by Lemma 2.3 we we can obtain models of this type for elliptic curves by considering Weierstrass models. Suppose we are given a normal proper R-model $\mathscr{C}$ of C. It may happen that $\mathscr{C}$ is not minimal in the sense that there exists a map $h : \mathscr{C} \to \mathscr{C}'$ to another normal proper R-model such that h is not an isomorphism. Such a map cannot be finite, since a finite birational map between normal connected noetherian schemes is an isomorphism. Hence, since h_K is an isomorphism and proper quasi-finite maps are finite, the map h_k must crush some irreducible component of $\mathscr{C}_k$ to a closed point in $\mathscr{C}'_k$. This process of contraction of special-fiber components can only continue finitely many times, and hence we eventually reach a normal proper R-model of C that dominates no others (except those to which it is isomorphic).

Definition 3.6. Let X be a proper and normal R-curve. Let Z be a union of irreducible components of X_k. A *blow-down* (or *contraction*) of Z in X is a morphism $\pi : X \to X'$ to a proper normal R-curve X' such that π sends the connected components of Z to pairwise distinct closed points and π has quasi-finite restriction to the other irreducible components of X_k.

If $\pi : X \to X'$ is a blow-down of Z, then Z is determined by π (as it is the set of positive-dimensional fibers of π) and the topological space of X' is the quotient of the topological space of X where each connected component of Z is replaced with a single point. Since normality forces $\mathscr{O}_{X'} = \pi_* \mathscr{O}_X$, we see that the pair (X, Z) uniquely determines π without needing to specify the scheme structure on Z. For normal projective surfaces over an algebraically closed field, Artin initiated the modern study of the existence problem for blow-downs (in the category of schemes, and then later algebraic spaces). It is a somewhat less subtle issue to determine if a blow-down exists in the case of arithmetic curves: it always does when Z is a proper subset of X_k and R is henselian [3, 6.7/3,4]. Since this will be important later on, we record the result here:

Proposition 3.7. *If X is a normal proper R-curve and R is henselian then X admits a blow-down $\pi : X \to X'$ of any union Z of a proper subset of the set of irreducible components of X_k.*

For our purposes, what matters is the universal property: if $f : X \to Y$ is a map of R-schemes and f sends each connected component of Z to a point, then f uniquely factors through the blow-down of Z (if the blow-down exists). Indeed, the factorization in the category of locally ringed spaces is clear, and the category of schemes is a full subcategory of the category of locally ringed spaces.

For our purposes, it is more interesting to investigate regular proper models such that irreducible components of the special fiber cannot be contracted without losing regularity:

Definition 3.8. A *minimal regular proper model* of a smooth K-curve C is a regular proper R-model $\mathscr{C}$ such that any domination map $\mathscr{C} \to \mathscr{C}'$ to another regular proper R-model of C is an isomorphism.

This concept of minimality is inspired by the theory of minimal models of smooth proper algebraic surfaces over a field. It is not obvious if C admits any regular proper model. It is also not obvious if a given regular proper R-model of C can dominate more than one minimal regular proper R-model, nor is it *a priori* evident how two minimal regular proper models of C are related to each other.

Example 3.9. Let $C = \mathbf{P}^1_K$, and let $\mathscr{C}$ and $\mathscr{C}'$ be equal to $\mathbf{P}^1_R$. We make $\mathscr{C}$ into a minimal regular proper R-model of C by taking $i : \mathscr{C}_K \simeq C$ to be the identity on $\mathbf{P}^1_K$. We make $\mathscr{C}'$ into a minimal regular proper R-model of C by taking $i' : \mathscr{C}'_K \simeq C$ to be an automorphism α in $\mathrm{PGL}(2, K) = \mathrm{Aut}_K(\mathbf{P}^1_K)$. Using the universal property of $\mathbf{P}^1$ and the calculation $\mathrm{Pic}(\mathbf{P}^1_A) = \mathbf{Z} \cdot \mathscr{O}(1)$ for any local ring A (such as $A = R$), it follows that $(\mathscr{C}', i')$ dominates $(\mathscr{C}, i)$ if and only if α is in the image of the map $\mathrm{PGL}(2, R) \to \mathrm{PGL}(2, K)$, in which case these R-models are isomorphic.

The map $\mathrm{PGL}(2, R) \to \mathrm{PGL}(2, K)$ is not surjective; e.g., elements of $\mathrm{GL}(2, K)$ with determinant of odd normalized order in $K^\times$ do not admit a $K^\times$-scaling in $\mathrm{GL}(2, R)$. Thus, there are many non-isomorphic minimal regular proper R-models of $\mathbf{P}^1_K$. This is the reason that the theory of regular proper R-models for smooth K-curves is not as clean for genus zero as it is for positive genus.

Néron investigated minimal regular proper models of elliptic curves over discrete valuation rings. The case of higher genus was taken up by Lichtenbaum and Shafarevich. Combining their work with results of Lipman concerning resolution of singularities for excellent 2-dimensional schemes, the following theorem is obtained:

Theorem 3.10 (Minimal models theorem)**.** *If C is a smooth K-curve with positive genus, then a minimal regular proper model C^{reg} exists and is unique. In particular, C^{reg} is dominated by all regular proper R-models of C.*

Remark 3.11. The theorem provides a *universal property* for the minimal regular proper model C^{reg} of C: for every regular proper R-model $\mathscr{C}$ of C, there is a unique map $\mathscr{C} \to C^{\mathrm{reg}}$ as R-models of C.

The proof of the minimal models theorem consists of two very different parts. First, it must be proved that some regular proper R-model exists. As we saw above, in Example 3.5, a normal proper (even projective) R-model $\mathscr{C}$ exists. Fix a choice of such a model. A natural strategy for making a regular proper R-model is to resolve singularities on $\mathscr{C}$. It can be proved (taking some care if R is not excellent) that there are only finitely many non-regular points on $\mathscr{C}_0 = \mathscr{C}$, and all of these are closed points in the special fiber; blow up $\mathscr{C}_0$ at each of these. The blow-up might not be normal, so we normalize it. It can be proved (taking some care if R is not excellent) that this normalization map is finite, and so it gives us another normal proper R-model $\mathscr{C}_1$ dominating the blow-up of the first normal proper R-model $\mathscr{C}_0$. We repeat the process. Lipman proved that in finitely many steps this canonical process reaches a regular proper R-model; see [1] for an exposition of the proof of Lipman's theorem (this reference assumes that R is excellent, and it imposes the condition $[k : k^p] < \infty$ when K has positive characteristic p, but these restrictions can be removed *a posteriori* by using the fact that the K-fiber is smooth).

Remark 3.12. One can refine Lipman's construction by contracting certain divisors if necessary, to reach a resolution resolution $\mathscr{C}^{\mathrm{reg}} \to \mathscr{C}$ that is dominated by all others; it is called the *minimal regular resolution.* The existence of this resolution is proved in [6, Thm. 2.2.2] and in [11, §9.3.4, Prop. 3.32]. Beware that in general Lipman's construction need *not* be the minimal resolution (i.e., some contractions can be necessary) although in the presence of rational singularities (to be defined in §8) it is minimal; see [11, §9.3.4, Rem. 3.34] and references therein. (In [11, §8.3, Exer. 3.27(a)] it appears to be asserted that Lipman's resolution is minimal, but that is an error.)

By construction, $\mathscr{C}^{\mathrm{reg}}$ is projective if $\mathscr{C}$ is projective (in fact, it is a general elementary result of Lichtenbaum that regular proper R-curves are automatically projective). The uniqueness of the minimal regular resolution guarantees, by an elementary gluing argument, that $\mathscr{C}^{\mathrm{reg}} \to \mathscr{C}$ is an isomorphism over the maximal open subscheme of $\mathscr{C}$ that is regular, so in particular of its maximal open subscheme that is R-smooth.

With a regular proper (even projective) R-model in hand, we can pass to a minimal one. There remains the key problem of uniqueness of such a minimal model when C has *positive* genus. This uniqueness rests on a systematic study of maps between regular proper R-models (especially the Factorization Theorem and Castelnuovo's criterion for contractibility of regular models); see [4] for a nice exposition of the proof of this uniqueness.

Example 3.13. If W is an integral Weierstrass model of an elliptic curve E over K, then its minimal regular resolution W^{reg} is a regular proper R-model of E, but W^{reg} is often not minimal. We will prove that W is a minimal Weierstrass model if and only if W^{reg} is the minimal regular proper model of E.

Remark 3.14. The assignment $C \rightsquigarrow C^{\mathrm{reg}}$ is not functorial on the category of smooth K-curves (with finite morphisms). Indeed, the minimal regular proper model C^{reg} is only characterized by a universal property among proper R-models of C, and not among R-curves in general. In particular, if $f : C' \to C$ is a finite map of smooth K-curves that is not an isomorphism, then ${C'}^{\mathrm{reg}}$ is not (via f) an R-model of C and so we cannot argue via universal properties of C^{reg} that f extends to a morphism of R-models ${C'}^{\mathrm{reg}} \to C^{\mathrm{reg}}$. An explicit counterexample is the projection $X_1(p) \to X_0(p)$ for $p = 11$ or $p > 13$, with $R = \mathbf{Z}_{(p)}$; see [6, §1.1].

4. A geometric characterization of minimal Weierstrass models

Let E be an elliptic curve over K, and let $\mathscr{E} = E^{\mathrm{reg}}$ be its minimal regular proper model. Let W be a Weierstrass model. In order to link up $\mathscr{E}$ and minimal Weierstrass models, we need to find an abstract criterion for a Weierstrass model of E to be a minimal Weierstrass model. This will be given in terms of R-rational maps, so we first digress to discuss the relative concept of a rational map. This requires a preliminary definition.

Definition 4.1. Let X a flat R-scheme that is of finite type and has geometrically integral fibers. An open subscheme $U \subseteq X$ is *R-dense* if $U_s \subseteq X_s$ is dense for all $s \in \operatorname{Spec} R$.

It is clear that if two R-maps $X \rightrightarrows Y$ agree on U then they coincide, and moreover the property of R-denseness is preserved by base change to another discrete valuation ring. Note also that an intersection of two R-dense opens is an R-dense open. The concept of R-denseness can be vastly generalized over arbitrary base schemes with much weaker restrictions on fibers (see [8, IV_4, §11.10, §20]).

Example 4.2. Let W be an integral Weierstrass model of an elliptic curve E over K. Any open subscheme $U \subseteq W$ that meets the special fiber is R-dense.

To define the relative notion of a rational map, let X and Y be R-schemes such that Y is R-separated and X is R-flat and of finite type over R with geometrically integral fibers. If $U, U' \subseteq X$ are R-dense opens and $f : U \to Y$ and $f' : U' \to Y$ are R-maps, let us say (U, f) and (U', f') are *equivalent* if there exists an R-dense open $V \subseteq U \cap U'$ such that $f|_V = f'|_V$. This is obviously an equivalence relation, and an *R-rational map* is an equivalence class of such pairs; we write $[(U, f)]$ to denote the equivalence class of a pair (U, f). It is clear how to define base change (on R) for R-rational maps. Composition of R-rational maps is usually not defined.

We claim that if (U, f) is equivalent to (U', f') then f and f' coincide on $U \cap U'$. To check this, we rename $U \cap U'$ as X and rename V as U to reduce to showing that R-maps $X \rightrightarrows Y$ agreeing on an R-dense open $U \subseteq X$ must be equal. Since Y is R-separated, such equality follows from the fact that the graph maps $X \rightrightarrows X \times_R Y$ are closed immersions and X is R-flat. We conclude by gluing that any R-rational map $\varphi = [(U, f)]$ is represented by a unique pair $(\widetilde{U}, \widetilde{f})$ with the property that $U' \subseteq \widetilde{U}$ for all (U', f') equivalent to (U, f). We call $\widetilde{U}$ the *domain of definition* of the R-rational map φ. By abuse of notation we shall write $\varphi : X \to Y$ to denote $\widetilde{f} : \widetilde{U} \to Y$ (since I do not know the TEX command to make a broken right arrow).

Lemma 4.3. *Let W and W' be integral Weierstrass models of E. Let $\Gamma \subseteq W \times_R W'$ be the closure of the graph of the isomorphism of K-fibers $W_K \simeq E \simeq W'_K$. If either projection $\Gamma \to W$ or $\Gamma \to W'$ is quasi-finite then both projections are isomorphisms and $W \simeq W'$ as models of E.*

If W and W' are not isomorphic, then the special fiber of Γ has exactly two irreducible components C and C' and both are geometrically irreducible. Moreover, these components may be labelled so that the surjective projections $\Gamma \to W$ and $\Gamma \to W'$ crush C' and C to respective closed points $w_0 \in W$ and $w'_0 \in W'$.

Proof. Suppose that $\Gamma \to W$ is quasi-finite. By construction, Γ is a proper R-flat model of E, so the quasi-finite projection to W must be a finite birational map and hence is an isomorphism (since W is normal). The composite

$$W \simeq \Gamma \to W'$$

is a morphism of models, and we have seen in Example 2.5 that a morphism between Weierstrass models must be an isomorphism. Thus, we may now assume that both (necessarily surjective) projections

$$\Gamma \to W, \quad \Gamma' \to W'$$

have a positive-dimensional fiber over some closed points on the special fibers of W and W', and so Γ has at least two irreducible components in its special fiber.

It remains to check that the special fiber of Γ has exactly two irreducible components after base change to a separable closure on the residue field. Since the formation of Γ commutes with the flat base change to a strict henselization, we may assume that R is henselian with separably closed residue field. In this case (or more generally, for henselian R) we shall prove that Γ has at most two irreducible components in its special fiber.

Let $W^{\rm reg} \to W$ and ${W'}^{\rm reg} \to W'$ be minimal regular resolutions, so these contain $W^{\rm sm}$ and ${W'}^{\rm sm}$ as open subschemes (as minimal regular resolutions are always isomorphisms over the R-smooth locus, and even over the open regular locus). In $W^{\rm reg} \times_R {W'}^{\rm reg}$, consider the closure of the graph of the identification of the K-fibers of $W^{\rm reg}$ and ${W'}^{\rm reg}$. If $\mathscr{W}$ is a resolution of singularities of this closure, then $\mathscr{W}$ is a regular proper model of E that dominates both $W^{\rm reg}$ and ${W'}^{\rm reg}$.

The projections

$$\mathscr{W} \to W^{\rm reg}, \quad \mathscr{W} \to {W'}^{\rm reg}$$

between regular integral models are surjective proper birational maps, and so (by the valuative criterion for properness) are isomorphisms over the generic points of the special fibers of the targets. Thus, R-dense opens in $W^{\rm sm}$ and ${W'}^{\rm sm}$ may be found inside of $\mathscr{W}$. Let Z and Z' be the irreducible components in the special fiber of $\mathscr{W}$ that arise from the special fibers of $W^{\rm sm}$ and ${W'}^{\rm sm}$. Since R is henselian, we may form the blow-down

$$\mathscr{W} \to \widetilde{\mathscr{W}}$$

that contracts all special-fiber irreducible components except for Z and Z', so $\widetilde{\mathscr{W}}$ is a normal proper model of E equipped with projections to W and W' as models of E. The resulting map

$$\widetilde{\mathscr{W}} \to W \times_R W'$$

clearly must factor through Γ and so it surjects onto Γ. We conclude that Γ_k has at most two irreducible components, as this is true for $\widetilde{W}_k$. ∎

Let W and W' be non-isomorphic Weierstrass models of E. Since W and W' are normal, and so have discrete valuation rings as local rings at the generic points of their special fibers, the valuative criterion for properness ensures that there exist R-rational maps $\phi : W \to W'$ and $\phi' : W' \to W$ extending the K-fiber identifications with E. Let $w_0 \in W$ and $w'_0 \in W'$ be the codimension-2 points such that the projections

$$\pi : \Gamma \to W, \quad \pi' : \Gamma \to W'$$

have their unique positive-dimensional fibers over w_0 and w'_0. We call w_0 and w'_0 the *fundamental points* of ϕ and ϕ', due to:

Lemma 4.4. *The domain of definition of ϕ is $W - \{w_0\}$, and the domain of definition of ϕ' is $W' - \{w'_0\}$.*

Proof. The surjective proper map $\pi^{-1}(W - \{w_0\}) \to W - \{w_0\}$ is a quasi-finite birational map between integral schemes, and $W - \{w_0\}$ is normal. It follows from Zariski's Main Theorem that this restricted projection is an isomorphism, so $W - \{w_0\}$ is naturally realized as an open subscheme in Γ and hence (by composition with $\Gamma \to W'$) admits an R-morphism to W' that extends the K-fiber identification. It follows that ϕ has domain of definition containing $W - \{w_0\}$. The domain of definition cannot be W, since a morphism between Weierstrass models is necessarily an isomorphism and we are assuming that W and W' are not isomorphic as models of E. ■

By the proof of Lemma 4.3, we see that $\phi(W_k - \{w_0\}) = \{w_0'\}$ and $\phi'(W' - \{w_0'\}) = \{w_0\}$.

Theorem 4.5. *Let W and W' be non-isomorphic Weierstrass models of E, and let $w_0 \in W$ and $w_0' \in W'$ be the resulting fundamental points. Exactly one of w_0 or w_0' is a smooth point, and if $w_0 \in W^{\rm sm}$ then w_0 reduces to the identity section in W_k and we have a containment of R-lines*

$$\mathrm{H}^0(W^{\rm sm}, \Omega^1_{W^{\rm sm}/R}) \subseteq \mathrm{H}^0(W'^{\rm sm}, \Omega^1_{W'^{\rm sm}/R}) \tag{4.1}$$

inside $\mathrm{H}^0(E, \Omega^1_{E/K})$; this containment is not an equality.

Proof. If w_0 is not smooth, then the R-rational map $\phi : W \to W'$ is defined at the identity and preserves identity sections (as this may be checked on the K-fibers), so $\{w_0'\} = \phi(W - \{w_0\})$ must be the reduction of the identity section on W'. Hence, w_0' is a smooth point. This shows that at least one of w_0 and w_0' must be in the smooth locus, and that when one of these is non-smooth then the other reduces to the identity section on the k-fiber.

By renaming if necessary, we may assume that w_0 is a smooth point. Thus, the unique morphism $W' - \{w_0'\} \to W$ extending the identity on K-fibers lands inside of $W^{\rm sm}$ (since on special fibers this map has image $\{w_0\}$). By pullback, we thereby get an inclusion

$$\mathrm{H}^0(W^{\rm sm}, \Omega^1_{W^{\rm sm}/R}) \to \mathrm{H}^0(U', \Omega^1_{U'/R}) = \mathrm{H}^0(W'^{\rm sm}, \Omega^1_{W'^{\rm sm}/R})$$

with U' the complement of $\{w_0'\}$ in $W'^{\rm sm}$ (so $U' = W'^{\rm sm}$ if w_0' is a non-smooth point); the equality of sections of Ω^1 over U' and over $W'^{\rm sm}$ follows from the fact that $\Omega^1_{W'^{\rm sm}/R}$ is invertible and $W'^{\rm sm}$ is normal with the open subset U' having complement of codimension ≥ 2.

Let us now show that the inclusion of R-lines of invariant differentials on $W^{\rm sm}$ and $W'^{\rm sm}$ cannot be an equality, and so it will follow (by running through the preceding argument with the roles of W and W' reversed) that w_0' cannot be a smooth point on W' when w_0 is smooth on W. The key is that nonzero invariant differentials on the smooth locus of a Weierstrass model are generators of Ω^1, and so an equality in (4.1) implies that the R-map

$$U' \subseteq W' - \{w_0'\} \to W^{\rm sm}$$

between smooth R-curves satisfies the vanishing condition $\Omega^1_{U'/W^{\rm sm}} = 0$. To obtain a contradiction, we pass to special fibers to conclude that the map between integral k-curves

$$U' \subseteq W_k' - \{w_0'\} \to W_k^{\rm sm}$$

is unramified. This is absurd, since an unramified morphism has discrete fibers and this map is a constant map onto a point $w_0 \in W_k^{\rm sm}$. ■

We now obtain many nice corollaries. The first corollary shows that the discriminant ideal of a Weierstrass model W of E, or equivalently its R-line of invariant differentials in $\mathrm{H}^0(E, \Omega^1_{E/K})$, determines W up to isomorphism as a model of E:

Corollary 4.6. *Let W and W' be integral Weierstrass models of E, and let Δ and Δ' be associated discriminants in R. The inclusion* (4.1) *holds if and only if* $\mathrm{ord}(\Delta) \leq \mathrm{ord}(\Delta')$*, and this inequality is an equality if and only if W and W' are isomorphic.*

If W and W' are non-isomorphic as models of E and $w_0 \in W$ and $w_0' \in W'$ are the associated fundamental points for the R-rational maps $W \to W'$ and $W' \to W$ extending K-fiber identifications, then w_0 is non-smooth if and only if $\mathrm{ord}(\Delta) \geq \mathrm{ord}(\Delta')$.

Proof. Use Corollary 2.10 and Theorem 4.5. ∎

In words, the preceding corollary says that a Weierstrass model W is "more minimal" than a Weierstrass model W' if and only if the R-rational map $W' \to W$ has non-smooth fundamental point on W', or equivalently if and only if this R-rational map contains the identity section in its domain of definition (in which case ${W'}^{\rm sm}$ is the domain of definition).

Corollary 4.7. *Let W be a Weierstrass model of E with minimal regular resolution $W^{\rm reg}$, and let $\mathscr{E}$ be a minimal regular proper model of E. The unique map $W^{\rm reg} \to \mathscr{E}$ is an isomorphism if and only if W is a minimal Weierstrass model of E.*

In particular, the minimal regular resolution of a minimal Weierstrass model is a minimal regular proper model of E.

Proof. We may make a base change to $R^{\rm sh}$, so we can assume that R is strictly henselian. Thus, we can contract special-fiber components in normal proper R-models of E, and all irreducible k-schemes are geometrically irreducible. Define X to be the contraction of $\mathscr{E}$ away from the unique irreducible component of $\mathscr{E}_k$ containing the point $e \in \mathscr{E}(R) = \mathscr{E}^{\rm sm}(R)$ that extends the identity in $E(K)$. By construction, X is a normal proper model of E with geometrically irreducible special fiber that is smooth at the reduction of the identity section; i.e., X is an abstract integral Weierstrass model of E.

Since W is likewise obtained from $W^{\rm reg}$ by contracting all irreducible components of the special fiber that do not contain the reduction of the identity section, it suffices to show that X is a minimal Weierstrass model of E. By Corollary 4.6, we just have to check that the canonical R-rational map $W \to X$ is defined on $W^{\rm sm}$. Since $W^{\rm sm}$ is naturally an open subscheme of $W^{\rm reg}$, we have a morphism

$$W^{\rm sm} \subseteq W^{\rm reg} \to \mathscr{E} \to X,$$

and so we are done. ∎

Here is a characterization of open subschemes of the smooth loci in general Weierstrass models of E:

Corollary 4.8. *Let X be a smooth separated R-scheme with K-fiber E such that X_k is connected and the identity section in $E(K)$ extends into $X(R)$. There exists an integral Weierstrass model W of E such that X is an open subscheme of $W^{\rm sm}$ as models of E.*

Proof. The problem may be checked over a covering of X by quasi-compact opens with connected special fiber, so we may assume X is quasi-compact and hence of finite type over R. Let us first show that it suffices to consider the situation after base change to $\widehat{R}$. Suppose there is a Weierstrass model W' over $\widehat{R}$ such that $X_{/\widehat{R}}$ is an open subscheme in W'. Since an integral Weierstrass model is determined up to isomorphism by its discriminant ideal, by slightly changing the coefficients of a planar Weierstrass model for W' we may find a Weierstrass model W for E over R such that X occurs as an open subscheme of W after base change to $\widehat{R}$. Let $\Gamma \subseteq X \times_R W$ be the closure of the K-fiber isomorphism $X_K \simeq E \simeq W_K$. The formation of Γ commutes with the flat base change to $\widehat{R}$, so the projection $\Gamma \to X$ is an isomorphism because it becomes an isomorphism after the *fpqc* base change to $\widehat{R}$. Hence, we get a morphism

$$X \simeq \Gamma \to W$$

that becomes an open immersion after base change to $\widehat{R}$, and so is an open immersion (necessarily landing in $W^{\rm sm}$ since X is R-smooth). This completes the reduction to the case of a complete base R. Thus, we now assume that R is complete.

By the Nagata compactification theorem, there exists an open immersion $X \hookrightarrow \overline{X}$ with $\overline{X}$ proper over R. We may replace $\overline{X}$ with the schematic closure of X, so we can assume that $\overline{X}$ is R-flat. By resolution of singularities, we may assume that $\overline{X}$ is regular. Thus, we have an open immersion of X into a regular proper model of E. The contraction W of $\overline{X}$ away from the connected component containing the smooth and irreducible X_k is a normal proper model of E such that W_k is irreducible and contains a smooth dense open X_k that has a k-rational point (the existence of such a contraction uses the condition that R is complete, and hence henselian). This forces X_k to be geometrically irreducible, so W_k is geometrically irreducible.

Thus, W is an abstract integral Weierstrass model of E. By construction the map $X \to W$ is a quasi-finite and separated birational isomorphism between integral schemes, so normality of W forces this map to be an open immersion (by Zariski's Main Theorem). ■

5. Néron models

Let us now leave the category of curves, and instead consider the category of abelian varieties. We again work with R and K as before, and we wish to study R-models of an abelian variety A over K. Néron's brilliant idea is to abandon the properness property of A in the search for a good model, and to instead pay close attention to its smoothness. It was known by Néron's time that if an abelian variety admits a proper smooth R-model $\mathscr{A}$, then the group law automatically extends to an R-group scheme structure on $\mathscr{A}$. Néron essentially ignored the problem of extending the group law and instead discovered that there are very nice smooth separated finite-type models that are often non-proper.

Definition 5.1. A *Néron model* of A is a smooth R-model $\mathscr{A}$ that satisfies the *Néron mapping property*: for any smooth R-scheme Z, the natural map

$$\mathrm{Hom}_R(Z, \mathscr{A}) \to \mathrm{Hom}_K(Z_K, A)$$

is bijection.

A very important example is $Z = \mathrm{Spec}\, R$; in this case, the Néron mapping property says that the natural map $\mathscr{A}(R) \to A(K)$ is a bijection. That is, from the viewpoint of the valuative criterion for properness on the level of K-points (but not K'-points for the fraction field of an arbitrary local extension $R \to R'$ of discrete valuation rings), $\mathscr{A}$ behaves as if it were proper. As a special case, we see that the identity point $e \in A(K)$ uniquely extends to a point $\widetilde{e} \in \mathscr{A}(R)$.

It is a tautology that the Néron model is unique up to unique isomorphism, and that it commutes with the formation of products over R; that is, if $\mathscr{A}$ and $\mathscr{B}$ are Néron models of abelian varieties A and B, then $\mathscr{A} \times_R \mathscr{B}$ is a Néron model of $A \times_K B$. Taking the special case $B = A$ and applying the Néron mapping property to the multiplication map $m : A \times_K A \to A$ on the generic fiber, we see that this map uniquely extends to an R-morphism

$$\widetilde{m} : \mathscr{A} \times_R \mathscr{A} \to \mathscr{A},$$

and likewise the inversion ι on A uniquely extends to an involution $\widetilde{\iota}$ of $\mathscr{A}$. The Néron mapping property implies that the resulting structures $(\widetilde{e}, \widetilde{m}, \widetilde{\iota})$ on $\mathscr{A}$ constitute a commutative R-group scheme structure, since the relevant identities among morphisms from smooth R-schemes to $\mathscr{A}$ may be checked on the K-fiber (due to the Néron mapping property).

The fundamental existence theorem, and the main result in [3], is:

Theorem 5.2 (Néron)**.** *The Néron model of an abelian variety exists, and it is separated and finite type over R, as well as quasi-projective.*

The quasi-projectivity is a byproduct of the construction, but it can also be proved abstractly [3, 6.4/1] that any smooth separated finite-type group scheme over a discrete valuation ring is necessarily quasi-projective.

Example 5.3. By [3, 1.2/8, 7.4/5], the Néron model $\mathscr{A}$ of an abelian variety A is proper over R (and so is an abelian scheme over R) if and only if $\mathscr{A}_k^0$ is k-proper, and moreover any abelian scheme over R is the Néron model of its generic fiber. In this case (i.e., the case of a proper Néron model) we say A has *good reduction*. For elliptic curves, this is equivalent to the naive notion of good reduction in terms of minimal Weierstrass models having unit discriminant in R.

We now turn to the special case of elliptic curves and consider how the Néron model can be related to the minimal regular proper R-model. Let E be an elliptic curve over K, and let $\mathscr{E}$ be its minimal regular proper model. By the valuative criterion for properness, the identity $e \in E(K)$ extends uniquely to a point $\widetilde{e} \in \mathscr{E}(R)$. Since $\mathscr{E}$ is regular, the R-section $\widetilde{e}$ must lie in the relation smooth locus: if $\mathscr{O}$ is the complete local ring of $\mathscr{E}$ at the closed point of $\widetilde{e}$, then A is a 2-dimensional regular local $\widehat{R}$-algebra, and there is a canonical section $\mathscr{O} \to \widehat{R}$ defined by $\widetilde{e}$; the kernel is a height-1 prime that must be principal, and upon choosing a generator t we obtain a map $\widehat{R}[\![t]\!] \to \mathscr{O}$ that is surjective (by completeness) and hence injective (by dimension reasons). This description of the complete local ring says exactly that the section $\widetilde{e}$ lies in the R-smooth locus of $\mathscr{E}$. We conclude that the Zariski-open R-smooth locus $\mathscr{E}^{\rm sm}$ in $\mathscr{E}$ not only contains the entire K-smooth generic fiber E, but it also meets the special fiber (at $\widetilde{e}(k)$); it can happen that (the possibly reducible) $\mathscr{E}_k$ is not generically reduced, and so the non-smooth locus in the special fiber $\mathscr{E}_k$ may contain some generic points of $\mathscr{E}_k$ (i.e., the non-smooth locus might not be 0-dimensional).

By the Néron mapping property of the Néron model $N(E)$, there is a unique map of smooth R-models $\mathscr{E}^{\rm sm} \to N(E)$. On the other hand, if W is a Weierstrass model of E then its minimal regular resolution $W^{\rm reg}$ is a regular proper model of E. By the universal property of the minimal regular proper model $\mathscr{E}$ of E, $W^{\rm reg}$ dominates $\mathscr{E}$. Thus, we arrive at two canonical maps

$$W^{\rm reg} \to \mathscr{E},\ \ \mathscr{E}^{\rm sm} \to N(E)$$

as R-models of E. This canonical procedure links the Néron model of E as an abelian variety, the minimal regular proper model of E as a smooth K-curve, and the Weierstrass models of E.

One of the motivating questions we posed at the start was the problem of the relationship between $N(E)$ and $\mathscr{E}^{\rm sm}$. The precise link is the following result:

Theorem 5.4. *The canonical map $\mathscr{E}^{\rm sm} \to N(E)$ is an isomorphism. In particular, if W is a minimal Weierstrass model then $N(E)$ is the R-smooth locus on the minimal regular resolution of W.*

Proof. See [3, 1.5/1] for a proof when R is strictly henselian, a situation to which the general case can be reduced. ∎

We will not require this theorem, but instead shall prove the following result that concerns identity components:

Theorem 5.5. *Let E be an elliptic curve over K, $N(E)$ its Néron model, and W a minimal Weierstrass model. Let $N(E)^0$ be the open R-subgroup of $N(E)$ obtained by removing non-identity components in the special fiber. The canonical map*

$$W^{\rm sm} \to N(E)$$

factors through $N(E)^0$ and is an isomorphism of $W^{\rm sm}$ onto $N(E)^0$.

The Néron model $N(E)$ is proper (i.e., E has good reduction in the sense of abelian schemes) if and only if W_k is smooth, in which case $N(E) = W$, and for an algebraic closure $\overline{k}$ of k we have $N(E)^0_{\overline{k}} \simeq \mathbf{G}_a$ (resp. $N(E)^0_{\overline{k}} \simeq \mathbf{G}_m$) and only if the geometric singularity on W_k is cuspidal (resp. nodal).

The final part of this theorem says that the *ad hoc* definitions of good, additive, and multiplicative reduction in the sense of [14] coincide with the intrinsic statements that the identity component of the special fiber of the Néron model is geometrically an elliptic curve, an additive group, and a multiplicative group respectively.

Proof. If $N(E)$ is proper then it is an elliptic curve, and so is clearly a minimal Weierstrass model of E. Thus, in such cases $N(E) = W$ and W_k is smooth. Conversely, if W_k is smooth then W is an elliptic curve over R with K-fiber E, and so E has good reduction. This settles the relationship between $N(E)$ and W in cases with good reduction or smooth Weierstrass models.

In the general case, the map $W^{\rm sm} \to N(E)$ clearly factors through $N(E)^0$, so we get a map $W^{\rm sm} \to N(E)^0$ that we want to be an isomorphism. By Corollary 4.6 and Corollary 4.8, there is an integral Weierstrass model W' of E and an open immersion $N(E)^0 \hookrightarrow W'^{\rm sm}$ as models of E. The composite morphism $W^{\rm sm} \to W'^{\rm sm}$ represents an R-rational map $\phi : W \to W'$. We claim that W' must be minimal, so ϕ is an isomorphism and we therefore have morphisms

$$W^{\rm sm} \to N(E)^0,\ \ N(E)^0 \to W^{\rm sm}$$

that are mutually inverse (as this may be checked on K-fibers), completing the proof that $N(E)^0 = W^{\rm sm}$.

Let us suppose that W and W' are not isomorphic, so by minimality of W we must have $\mathrm{ord}(\Delta) < \mathrm{ord}(\Delta')$. By Corollary 4.6, the fundamental point of ϕ on W has to be smooth. This contradicts the fact that ϕ has domain of definition W^{sm}.

For the second part of the theorem, it remains to consider the case when W is not smooth. By what has just been proved, we know $N(E)^0_k = W^{\mathrm{sm}}_k$. However, we know from the geometric theory of singular Weierstrass curves that the $\overline{k}$-fiber $W^{\mathrm{sm}}_{\overline{k}}$ is an affine curve whose smooth compactification has one point (resp. two points) at infinity if and only if the singularity on $W_{\overline{k}}$ is cuspidal (resp. nodal). The smooth $\overline{k}$-group $N(E)^0_{\overline{k}}$ is therefore a smooth 1-dimensional connected affine algebraic group over $\overline{k}$ that has one point (resp. two points) at infinity in its smooth compactification if and only if the singularity on $W_{\overline{k}}$ is cuspidal (resp. nodal). However, it is a basic fact in the theory of algebraic groups that a smooth 1-dimensional connected affine algebraic group over an algebraically closed field is isomorphic to exactly one of $\mathbf{G}_a$ or $\mathbf{G}_m$, and these cases can be distinguished geometrically based on the number of points at infinity on their smooth compactifications (namely, one and two points respectively). ■

Let us clarify several aspects of Theorem 5.5. Let W be a minimal Weierstrass model of E. The geometric theory (as in [14]) ensures that W_k has a unique geometric singularity, say with $\xi \in W_k$ the associated closed point, and so if $k(\xi)/k$ is separable (e.g., if k is perfect) then ξ must be k-rational. However, it can happen that this point is not k-rational. Let us analyze the possibilities.

If $f(x,y)$ is an affine Weierstrass equation for W_k and k does not have characteristic 2, then $\partial f/\partial y = 2y + a_1x + a_3$ is a nonzero linear form, so its intersection with the cubic curve is a degree-3 k-finite closed subscheme in $\mathbf{P}^2_k$ that contains ξ. This implies $[k(\xi) : k] \leq 3$, so in residue characteristics not equal to 2 or 3 it is automatic that $k(\xi)/k$ is separable, and hence ξ is k-rational. In residue characteristic 3, since we may put the Weierstrass equation over k in the form $y^2 = h(x)$ it is clear that the geometric singularity at ξ must be a cusp (and hence additive type) if ξ is not k-rational. In residue characteristic 2, the normal forms (as in [14, App. A, Prop. 1.1]) show that if ξ is not k-rational then again the geometric singularity must be a cusp. Hence, the only possibility for a singularity on W_k that is not k-rational is the case of additive-type with k of characteristic 2 or 3. Let us show that both possibilities may happen for special fibers of minimal Weierstrass models.

Consider $p = 2$ or 3, and suppose k is a non-perfect field of characteristic p, with $a \in k$ not a pth power. The Weierstrass cubics $y^2 = x^3 - a$ (for $p = 3$) and $y^2 = x(x^2 - a)$ (for $p = 2$) are regular k-curves that are not smooth (the geometric singularity is at $(a^{1/p}, 0)$). Conversely, it is clear (using the normal forms in [14, App. A, Prop. 1.1] in the case of characteristic 2) that this construction gives all regular non-smooth Weierstrass cubics (up to isomorphism): the key is that a k-rational point on a regular k-curve must be a k-smooth point. We conclude that when the unique singularity is not k-rational, the Weierstrass cubic *must* be regular.

Any lift of such a cubic to a Weierstrass model over R with smooth generic fiber E provides an example of an elliptic curve E with a Weierstrass model W that is regular but has a singularity that is not k-rational (and in fact gives all examples up to isomorphism). Such a regular proper R-model W is obviously minimal since the special fiber is irreducible, so it is the minimal regular proper model of E. Theorem 4.7 implies that such a W must be a minimal Weierstrass model of its generic fiber E. In these cases, $N(E)_k = W^{\mathrm{sm}}_k$ is a smooth affine group that is geometrically isomorphic to $\mathbf{G}_a$, but it is not k-isomorphic to $\mathbf{G}_a$ (since the unique point at infinity on the regular compactification W_k of $N(E)_k$ is not k-rational, in contrast to the case of the curve $\mathbf{G}_a$ over k). In particular, in cases of multiplicative reduction it is automatic that the singularity on W_k is k-rational. Thus, W_k cannot be regular when there is multiplicative reduction.

When there is additive reduction and the singularity on the special fiber of the minimal Weierstrass model is k-rational, then the special fiber cannot be regular at the singularity. It can then be deduced by geometric methods that $N(E)^0_k \simeq \mathbf{G}_a$ over k. Comparison with the geometric situation over $\overline{k}$ shows that when there is multiplicative reduction then the normalization $\widetilde{W}_k$ of W_k must be smooth with genus zero (and even a projective line, since the singularity provides a k-rational point). In such cases, a geometric analysis of $\widetilde{W}_k$ shows that the local expansion of the equation of W_k at the k-rational singularity is a product of linear forms if and only if the torus $N(E)^0_k$ is isomorphic to $\mathbf{G}_m$. That is, the concept of split multiplicative reduction in the sense of [14] is equivalent to split toric reduction in the sense of Néron models (i.e., $N(E)^0_k$ is a k-split torus).

6. Review of dualizing sheaves

Let us now review the construction of the relative dualizing sheaf for a flat map $f : Y \to S$ that is locally of finite presentation and has Cohen-Macaulay fibers with a fixed pure dimension d; such a map is called a *Cohen-Macaulay morphism* with pure relative dimension d, and if S is a Cohen-Macaulay scheme (such as a regular scheme) then a locally finite-type S-scheme Y is Cohen-Macaulay over S if and only if Y is Cohen-Macaulay. Cover Y by opens U_i such that there exist closed immersions of U_i into a smooth S-scheme Z_i whose non-empty fibers over S have a constant dimension N_i (e.g., we may take the U_i's to be open affines lying over open affines V_i in S, and take Z_i to be an affine space over V_i). With a covering chosen, the finite-type quasi-coherent sheaf $\mathscr{E}xt^{N_i-d}_{Z_i}(\mathscr{O}_{U_i}, \Omega^{N_i}_{Z_i/S})$ naturally lives on U_i, and (after a rather non-trivial argument) these may be globally glued to a *relative dualizing sheaf* $\omega_{Y/S}$ on Y that is independent of all choices and is naturally compatible with base change on S and étale localization on Y. In particular, this sheaf is of Zariski-local nature on Y.

Example 6.1. If Y is S-smooth with pure relative dimension d, then there is a canonical isomorphism $\omega_{Y/S} \simeq \Omega^d_{Y/S}$. Note that in such cases, $\omega_{Y/S}$ is invertible. In general, if Y is a CM S-scheme whose fibers over S have pure dimension d, then the fibers of Y over S are Gorenstein if and only if $\omega_{Y/S}$ is invertible. If S is Gorenstein, then the fibers are Gorenstein if and only if Y is Gorenstein.

Example 6.2. If there is a closed immersion $Y \hookrightarrow Z$ into a smooth S-scheme Z with fibers of pure dimension N, such as happens for Weierstrass models of elliptic curves over $S = \operatorname{Spec} R$ (with $Z = \mathbf{P}^2_R$ and $N = 2$), then

$$\omega_{Y/S} = \mathscr{E}xt^{N-d}_Z(\mathscr{O}_Y, \Omega^N_{Z/S}).$$

The R-models $\mathscr{E}$ and W^{reg} of E are certainly Gorenstein, as they are even regular, and the R-model W of E is also Gorenstein because it is an R-flat hypersurface in a smooth R-scheme (here we use a Weierstrass equation that puts W in $\mathbf{P}^2_R$). More generally, let us consider normal proper R-models X of E that are Gorenstein, so $\omega_{X/R}$ is invertible. The normality of X and invertibility of $\omega_{X/R}$ imply that any section of $\omega_{X/R}$ that is defined in codimension ≤ 1 extends uniquely to a global section, and such an extended section is a global generator if and only if it is so in codimension ≤ 1; this is due to the fact that any normal noetherian domain is the intersection of its localizations at height-1 primes.

Now consider the special case when $X = W$ is an arbitrary Weierstrass model of E. Let $W^{\mathrm{reg}} \to W$ be the minimal regular resolution of W. Since W^{reg} is thereby viewed as a regular proper R-model of E, by the universal property of $\mathscr{E}$ there is a unique map of R-models $p : W^{\mathrm{reg}} \to \mathscr{E}$. As in [1, (1.6)], the general machinery of Grothendieck duality provides a canonical map

$$p_*\omega_{W^{\mathrm{reg}}/R} \to \omega_{\mathscr{E}/R} \tag{6.1}$$

that uniquely extends the canonical isomorphism on the K-fibers. This map is called the *trace map* and it is injective (as may be checked over K). A key fact [1, (3.3)] is that the trace map fits into a short exact sequence

$$0 \to p_*\omega_{W^{\mathrm{reg}}/R} \to \omega_{\mathscr{E}/R} \to \mathscr{E}xt^2_X(\mathrm{R}^1p_*\mathscr{O}, \omega_{\mathscr{E}/R}) \to 0.$$

We claim that $\mathrm{R}^1p_*\mathscr{O} = 0$, and hence the natural trace map $p_*\omega_{W^{\mathrm{reg}}/R} \to \omega_{\mathscr{E}/R}$ is an isomorphism.

Granting this vanishing for a moment, we get an equality of R-lines

$$\mathrm{H}^0(W^{\mathrm{reg}}, \omega_{W^{\mathrm{reg}}/R}) = \mathrm{H}^0(\mathscr{E}, \omega_{\mathscr{E}/R}) \tag{6.2}$$

in the 1-dimensional K-vector space $\mathrm{H}^0(E, \Omega^1_{E/K})$. However, we also may restrict any global section of $\omega_{W^{\mathrm{reg}}/R}$ to a section over the canonical copy of the R-smooth locus W^{sm} of W as a Zariski-open in the minimal regular resolution W^{reg} of W, and in this way we get a restriction map

$$\mathrm{H}^0(W^{\mathrm{reg}}, \omega_{W^{\mathrm{reg}}/R}) \to \mathrm{H}^0(W^{\mathrm{sm}}, \omega_{W^{\mathrm{sm}}/R}) = \mathrm{H}^0(W, \omega_{W/R}), \tag{6.3}$$

where the final equality follows from the fact that $\omega_{W/R}$ is an invertible sheaf on the normal W and that the open $W^{\rm sm}$ in W has complement that is supported in codimension ≥ 2.

To summarize, if $\mathrm{R}^1 p_* \mathscr{O} = 0$ then we get an inclusion of R-lines

$$\mathrm{H}^0(\mathscr{E}, \omega_{\mathscr{E}/R}) = \mathrm{H}^0(W^{\rm reg}, \omega_{W^{\rm reg}/R}) \subseteq \mathrm{H}^0(W, \omega_{W/R})$$

in the 1-dimensional K-vector space $\mathrm{H}^0(E, \Omega^1_{E/K})$. We still need to prove:

Lemma 6.3. *The sheaf* $\mathrm{R}^1 p_* \mathscr{O}$ *vanishes.*

Proof. This is an elementary special case of the general theory of rational singularities [10], since $\mathscr{E}$ is regular, but we can avoid the general theory by giving a direct proof (using an argument of Lipman that occurs at the start of the general theory).

By the factorization theorem for regular proper R-curves [4], the map p factors as a composite of blow-ups at closed points in the special fiber. Thus, we shall consider a proper birational map $p : Y' \to Y$ between connected regular 2-dimensional schemes such that p is a composite of n blow-ups at codimension-2 points (necessarily closed) for some $n \geq 0$, and we will prove that $\mathrm{R}^1 p_* \mathscr{O}_{Y'} = 0$. Such blowing up preserves the property of being a connected regular 2-dimensional scheme, so we may may induct on n. The case $n = 0$ is trivial. The general problem is local on Y, so we may assume Y is local. It is equivalent to prove $\mathrm{H}^1(Y', \mathscr{O}_{Y'}) = 0$. Consider a factorization

$$Y' \stackrel{p'}{\to} \widetilde{Y} \stackrel{\widetilde{p}}{\to} Y$$

of p, where p' is a single blow-up and $\widetilde{p}$ is a composite of $n-1$ blow-ups at codimension-2 points.

The Leray spectral sequence for p' provides an exact sequence of low-degree terms

$$0 \to \mathrm{H}^1(\widetilde{Y}, p'_* \mathscr{O}_{Y'}) \to \mathrm{H}^1(Y', \mathscr{O}_{Y'}) \to \mathrm{H}^0(\widetilde{Y}, \mathrm{R}^1 p'_* \mathscr{O}_{Y'}).$$

Since p' is a proper birational map between normal irreducible schemes, $p'_* \mathscr{O}_{Y'} = \mathscr{O}_{\widetilde{Y}}$. Thus, the first term vanishes (by induction). To prove the vanishing of the middle term, we are reduced to proving the vanishing of the final term. Thus, it is enough to prove the vanishing of $\mathrm{R}^1 p'_* \mathscr{O}_{Y'}$. This brings us to the case $n = 1$.

The coherent sheaf $\mathrm{R}^1 p'_* \mathscr{O}_{Y'}$ is supported at the codimension-2 blow-up point on $\widetilde{Y}$. We may localize at this point and rename the local base as Y to get to the following situation: the base $Y = \operatorname{Spec} A$ is a local 2-dimensional regular scheme, and $p : Y' \to Y$ is the blow-up of the closed point. We want to prove that $\mathrm{H}^1(Y', \mathscr{O}_{Y'}) = 0$. We use an elementary Čech-theory calculation. Let $\{a, a'\}$ be generators of $\mathfrak{m}_A$, so the blow-up Y' is covered by two affine opens $U = \operatorname{Spec} A[a/a']$ and $U' = \operatorname{Spec} A[a'/a]$. Let $\mathfrak{U}$ be the ordered open cover $\{U, U'\}$, so $\mathrm{H}^1(Y', \mathscr{O}_{Y'}) = \mathrm{H}^1(\mathfrak{U}, \mathscr{O}_{Y'})$. The Čech complex consists of terms in degrees 0 and 1, and it must be proved that every Čech 1-cocycle is a 1-coboundary. A typical cocycle has the form

$$\sum_{i,j \geq 0} \alpha_{ij} \left(\frac{a}{a'}\right)^i \left(\frac{a'}{a}\right)^j = \sum_{i \geq j} \alpha_{ij} \left(\frac{a}{a'}\right)^{i-j} - \sum_{i<j} (-\alpha_{ij}) \left(\frac{a'}{a}\right)^{j-i}$$

with $\alpha_{ij} \in A$. The right side is obviously a 1-coboundary, so we are done. ■

It is natural to expect that the inclusion

$$\mathrm{H}^0(\mathscr{E}, \omega_{\mathscr{E}/R}) \subseteq \mathrm{H}^0(W, \omega_{W/R})$$

is an equality if (and hence only if) W is a minimal Weierstrass model of E. This criterion for Weierstrass minimality will be shown in Theorem 8.1.

7. Intersection theory on arithmetic surfaces

Let us now formulate the basic definitions and theorems in intersection theory on arithmetic surfaces; see [4] and [11, §9] for further details (with complete proofs) on what we say below. The theory is largely motivated by intersection theory on smooth proper algebraic surfaces. For proper regular flat curves over a discrete valuation ring, the intersection theory is generally restricted to irreducible components of the special fiber because these components are proper curves over a field (whereas other codimension-1 irreducible subschemes are not proper over a field). We shall work with an arbitrary regular proper R-curve $\mathscr{X}$ whose generic fiber is smooth and geometrically connected.

By regularity, all irreducible codimension-1 subschemes of $\mathscr{X}$ have an invertible ideal sheaf. More precisely (as in Hartshorne's *Algebraic Geometry*) every line bundle $\mathscr{L}$ on $\mathscr{X}$ has the form $\mathscr{O}_{\mathscr{X}}(D)$ for some Weil divisor D on $\mathscr{X}$, and $\mathscr{O}_{\mathscr{X}}(D) \simeq \mathscr{O}_{\mathscr{X}}(D')$ if and only if $D - D' = \mathrm{div}(f)$ for some $f \in K(\mathscr{X})^\times$, where

$$\mathrm{div}(f) \stackrel{\mathrm{def}}{=} \sum_F \mathrm{ord}_{\eta_F}(f) \cdot F \in \mathrm{Div}(\mathscr{X})$$

with $\mathrm{ord}_{\eta_F} : K(\mathscr{X})^\times \to \mathbf{Z}$ the normalized discrete valuation defined by the local ring $\mathscr{O}_{\mathscr{X},\eta_F}$ at the codimension-1 generic point η_F of F.

If F is an irreducible component of $\mathscr{X}_k$ and we give F its reduced structure, then F is a proper integral curve over k, so $k_F \stackrel{\mathrm{def}}{=} \mathrm{H}^0(F, \mathscr{O}_F)$ is a domain that if k-finite. Thus, k_F is a field and F has a natural structure of proper integral curve over k_F; by Stein factorization, F is geometrically connected over k_F (though it might not be geometrically reduced or geometrically irreducible over k_F). It may happen that $[k_F : k] > 1$; the case of algebraically closed k (as in [15]) is very much simplified by the fact that necessarily $k_F = k$ for all F. Since we wish to avoid restrictions (such as perfectness) on k, we need to keep track of the possibility that $[k_F : k]$ might be larger than 1.

For any line bundle $\mathscr{L}$ on $\mathscr{X}$ and any F as above, for any field k' between k and k_F we define

$$i_{k'}(F, \mathscr{L}) = \deg_{k'}(\mathscr{L}|_F),$$

where the k'-degree of a line bundle $\mathscr{N}$ on a proper curve C over k' is defined to be the degree-1 coefficient of the Hilbert polynomial

$$n \mapsto \chi_{k'}(C, \mathscr{N}^{\otimes n}) = \dim_{k'} \mathrm{H}^0(C, \mathscr{N}^{\otimes n}) - \dim_{k'} \mathrm{H}^1(C, \mathscr{N}^{\otimes n}) = \deg_{k'}(\mathscr{N})n + \chi_{k'}(\mathscr{O}_C).$$

See [3, 9.1/1] for a thorough treatment of this degree-function for arbitrary proper reduced curves over a field. Note that $i_{k'}(F, \mathscr{L}) = [k_F : k'] i_{k_F}(F, \mathscr{L})$. Whereas i_{k_F} is convenient when we wish to study a fixed F, when we are considering all irreducible components of the special fiber it is more convenient to work with $i_k(F, \mathscr{L})$.

Definition 7.1. Let D be a Weil divisor on $\mathscr{X}$, and let F be an irreducible component of $\mathscr{X}_x$. For any field k' between k and k_F,

$$i_{k'}(F, D) \stackrel{\mathrm{def}}{=} i_{k'}(F, \mathscr{O}_{\mathscr{X}}(D)) = \deg_{k'}(\mathscr{O}_{\mathscr{X}}(D)|_F).$$

Let $\mathrm{Div}(\mathscr{X}_k)$ be the free abelian group generated by the irreducible components of $\mathscr{X}_k$.

Definition 7.2. For a Weil divisor D on $\mathscr{X}$, $i_{k'}(\cdot, D) : \mathrm{Div}(\mathscr{X}_k) \to \mathbf{Z}$ is defined via extension-by-linearity in the first variable.

Example 7.3. If F and F' are distinct irreducible components of $\mathscr{X}_k$, then $i_k(F, F') \geq 0$. In fact, $i_k(F, F')$ is the k-length of the scheme-theoretic intersection $F \cap F'$ when F and F' are given the reduced structure as closed subschemes of $\mathscr{X}$.

Since $D \mapsto \mathscr{O}_{\mathscr{X}}(D)$ carries sums to tensor products and $\deg_{k'} : \mathrm{Pic}(F) \to \mathbf{Z}$ carries tensor products to sums, we see that $i_{k'}(F, \cdot)$ is additive; clearly this operation kills principal Weil divisors (*i.e.*, those of the form $\mathrm{div}(f)$) because such divisors gives rise to the trivial line bundle. Thus, $i_k(\cdot, \cdot)$ kills principal divisors in the second variable. It is a fundamental fact (see [4] or [11]) that the restriction

$$i_k(\cdot, \cdot) : \mathrm{Div}(\mathscr{X}_k) \times \mathrm{Div}(\mathscr{X}_k) \to \mathbf{Z}$$

is symmetric, and hence $i_k(D, D') = 0$ for Weil divisors D and D' supported in $\mathscr{X}_k$ if either of D or D' is principal.

The symmetric bilinear form

$$\mathrm{Div}(\mathscr{X}_k) \times \mathrm{Div}(\mathscr{X}_k) \to \mathbf{Z}$$

defined by $(D', D) \mapsto i_k(D', D)$ is called the *intersection pairing*, and we shall write $D'.D$ to denote $i_k(D', D)$. When $D' = F$ is an irreducible component of $\mathscr{X}_k$, we must be careful to distinguish $F.D$ from $i_{k_F}(F, D)$; the relationship is

$$F.D = [k_F : k] i_{k_F}(F, D).$$

In particular, the map $D \mapsto F.D$ takes values in $[k_F : k]\mathbf{Z}$.

Example 7.4. Let π be a uniformizer of R. The zero-scheme of π on $\mathscr{X}$ is the subscheme $\mathscr{X}_k$ defined by the principal ideal $\mathfrak{m}_R \mathscr{O}_{\mathscr{X}}$; this is independent of π. For each irreducible component F of the special fiber, let $n_F = \mathrm{ord}_{\eta_F}(\pi)$, so the principal divisor

$$[\mathscr{X}_k] \stackrel{\mathrm{def}}{=} \mathrm{div}(\pi) = \sum_F n_F F$$

has support equal to $\mathscr{X}_k$. All elements of the subgroup $\mathbf{Z}[\mathscr{X}_k]$ are principal divisors on $\mathscr{X}$ with support in $\mathscr{X}_k$, and this exhausts all principal Weil divisors on $\mathscr{X}$ with support in the special fiber. Indeed, if D is such a Weil divisor and $D = \mathrm{div}(f)$ for $f \in K(\mathscr{X})^\times$, then as a rational function on the smooth and geometrically connected proper K-curve $\mathscr{X}_K$ we see that f has vanishing divisor. Hence, $f \in \mathrm{H}^0(\mathscr{X}_K, \mathscr{O}) = K$ and $f \neq 0$, so $f \in K^\times$. Multiplying f by a suitable power of a uniformizer changes D modulo $\mathbf{Z}[\mathscr{X}_k]$, and so reduces us to the case $f \in R^\times$. Such f have vanishing Weil divisor on $\mathscr{X}$.

Let $\mathrm{Div}_0(\mathscr{X}_k) = \mathrm{Div}(\mathscr{X}_k)/\mathbf{Z}[\mathscr{X}_k]$; this is a finite $\mathbf{Z}$-module, and it is torsion-free if and only if the multiplicities for $[\mathscr{X}_k]$ have gcd equal to 1. For example, if $\mathscr{X}_k$ has non-empty reduced locus (as is the case if $\mathscr{X}(R) = \mathscr{X}_K(K)$ is non-empty, such as any regular resolution of the Weierstrass model of an elliptic curve) then $[\mathscr{X}_k]$ even has a coefficient equal to 1. Clearly the intersection pairing factors through a symmetric bilinear form

$$(\cdot, \cdot) : \mathrm{Div}_0(\mathscr{X}_k) \times \mathrm{Div}_0(\mathscr{X}_k) \to \mathbf{Z}.$$

A very fundamental fact is:

Theorem 7.5. *The pairing $(\cdot,\cdot)$ is* negative-definite *modulo torsion. In particular, if D is a Weil divisor on $\mathscr{X}$ that is supported in $\mathscr{X}_k$ then D is a $\mathbf{Q}$-multiple of $[\mathscr{X}_k]$ if and only if $D.(\cdot)$ is identically zero.*

Let us analyze intersection theory against the invertible dualizing sheaf $\omega = \omega_{\mathscr{X}/R}$. Let F be an irreducible component of $\mathscr{X}_k$. We would like to compute $i_k(F, \omega) = \deg_k(\omega|_F)$. Since F is a reduced curve over k, it is Cohen-Macaulay and hence has a relative dualizing sheaf $\omega_{F/k}$. We would like to compute $\omega_{F/k}$ in terms of ω. Consider the commutative diagram

$$\begin{array}{ccc} F & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, k & \longrightarrow & \mathrm{Spec}\, R \end{array}$$

The bottom and top rows are defined by locally principal ideals whose generators are nowhere zero-divisors. By using the general formalism of Grothendieck duality (as in [5, Ch. 2–3]), this diagram can be used to construct a canonical isomorphism

$$\mathscr{E}xt^1_{\mathscr{X}}(\mathscr{O}_F, \omega) \simeq \mathrm{Ext}^1_R(k, R) \otimes_k \omega_{F/k} \simeq (\mathfrak{m}^{-1}/R) \otimes_k \omega_{F/k}.$$

The 2-term vector-bundle resolution

$$0 \to \mathscr{O}_{\mathscr{X}}(-F) \to \mathscr{O}_{\mathscr{X}} \to \mathscr{O}_F \to 0$$

provides an isomorphism

$$\mathscr{E}xt^1_{\mathscr{X}}(\mathscr{O}_F, \omega) \simeq \omega(F)|_F,$$

where $\mathscr{L}(F) \stackrel{\mathrm{def}}{=} \mathscr{L} \otimes \mathscr{O}_{\mathscr{X}}(F)$. Thus, we obtain

$$\omega_{F/k} = (\mathfrak{m}/\mathfrak{m}^2) \otimes_k \omega(F)|_F \simeq (\omega|_F) \otimes_{\mathscr{O}_F} \mathscr{O}_{\mathscr{X}}(F)|_F.$$

We therefore may compute

$$i_k(F, \omega) = \deg_k(\omega|_F) = \deg_k(\omega_{F/k}) - \deg_k(\mathscr{O}(F)|_F) = \deg_k \omega_{F/k} - (F.F).$$

For any line bundle $\mathscr{L}$ on F, we have (as in [3, 9.1/1])

$$\deg_k \mathscr{L} = \chi_k(\mathscr{L}) - \chi_k(\mathscr{O}_F),$$

so

$$\deg \omega_{F/k} = \chi_k(\omega_{F/k}) - \chi_k(\mathscr{O}_F) = -2\chi_k(\mathscr{O}_F)$$

since $\mathrm{H}^i(F, \omega_{F/k})$ is Serre-dual to $\mathrm{H}^{1-i}(F, \mathscr{O}_F)$. Thus, we obtain the *adjunction formula*

$$F.F + i_k(F, \omega) = -2\chi_k(\mathscr{O}_F).$$

All terms in the adjunction formula are divisible by $[k_F : k]$, so dividing gives

$$i_{k_F}(F, F) + i_{k_F}(F, \omega) = -2\chi_{k_F}(\mathscr{O}_F) = -2 + 2p_a(F) \geq -2,$$

where $p_a(F) = \dim_{k_F} \mathrm{H}^1(F, \mathscr{O}_F) \geq 0$. Thus, if $F.F < 0$ and $i_k(F, \omega) < 0$ then necessarily $i_{k_F}(F, F) = -1$ and $p_a(F) = 0$. Castelnuovo's criterion [4, Thm. 3.1] says that if $i_{k_F}(F, F) = -1$ and $p_a(F) = 0$, then F can be contracted without losing regularity. Thus, if $\mathscr{X}$ is a minimal regular proper model of its generic fiber then every irreducible component F of $\mathscr{X}_k$ must satisfy either $F.F \geq 0$ or $i_k(F, \omega) \geq 0$. However, we also have

$$0 = F.[\mathscr{X}_k] = n_F(F.F) + \sum_{F' \neq F} n_{F'}(F.F')$$

with coefficients $n_F, n_{F'} > 0$, so if $\mathscr{X}_k$ is reducible then Example 7.3 forces $F.F < 0$. This essentially proves:

Lemma 7.6. *If $\mathscr{X}$ is a minimal regular proper model of its generic fiber $\mathscr{X}_K$, and either $\mathscr{X}_k$ is reducible or $\mathscr{X}_K$ has positive genus, then $i_k(F, \omega_{\mathscr{X}/R}) \geq 0$ for all irreducible components F of $\mathscr{X}_k$.*

Proof. The preceding discussion settles the case when $\mathscr{X}_k$ is reducible, so it remains to consider the case when $\mathscr{X}_k$ is irreducible. In this case, $[\mathscr{X}_k] = nF$ for some positive integer n (where F is the reduced special fiber), so

$$0 = [\mathscr{X}_k].[\mathscr{X}_k] = n^2(F.F),$$

and hence $F.F = 0$. The adjunction formula therefore gives

$$i_{k_F}(F, \omega_{\mathscr{X}/R}) = -2\chi_{k_F}(\mathscr{O}_F) = -2 + 2\dim_{k_F} \mathrm{H}^1(F, \mathscr{O}_F).$$

We have to rule out the possibility that $\mathrm{H}^1(F, \mathscr{O}_F) = 0$.

For any positive integer e,

$$\chi_k(\mathscr{O}(-eF)|_F) = \deg_k(\mathscr{O}(-eF)|_F) + \chi_k(\mathscr{O}_F) = -e(F.F) + \chi_k(\mathscr{O}_F) = \chi_k(\mathscr{O}_F) = [k_F : k], \tag{7.1}$$

where the final equality uses the vanishing of $\mathrm{H}^1(F, \mathscr{O}_F)$. If $\mathscr{I}_F$ denotes the invertible coherent ideal sheaf of F in $\mathscr{X}$, then we have an exact sequence

$$0 \to (\mathscr{I}_F/\mathscr{I}_F^2)^{\otimes e} \to \mathscr{O}_{eF} \to \mathscr{O}_{(e-1)F} \to 0$$

for any $e > 1$. Provided $e \leq n$, all of these terms are supported in the closed subscheme $\mathscr{X}_k$. The term on the left is $\mathscr{O}(-eF)|_F$, so taking Euler characteristics and using (7.1) gives

$$\chi_k(\mathscr{O}_{eF}) = \chi_k(\mathscr{O}_{(e-1)F}) + \chi_k(\mathscr{O}(-eF)|_F) = \chi_k(\mathscr{O}_{(e-1)F}) + [k_F : k]$$

for $e \leq n$. By induction, $\chi_k(\mathscr{O}_{eF}) = e[k_F : k]$ for $e \leq n$. Taking $e = n$, $\chi_k(\mathscr{O}_{\mathscr{X}_k}) = n[k_F : k]$. However, constancy of Euler characteristic in flat families implies

$$\chi_k(\mathscr{O}_{\mathscr{X}_k}) = \chi_K(\mathscr{O}_{\mathscr{X}_K}) = 1 - p_a(\mathscr{X}_K),$$

so $p_a(\mathscr{X}_K) = 1 - n[k_F : k] \leq 0$, and hence $\mathscr{X}_K$ must have genus zero. ∎

Example 7.7. In the setup of Lemma 7.6, suppose that $X := \mathscr{X}_K$ has genus 1 (but perhaps no K-rational points) and that the multiplicities in $[\mathscr{X}_k]$ have greatest common divisor equal to 1, such as happens when $\mathscr{X}_K(K) \neq \emptyset$. In this case we claim that $\omega = \omega_{\mathscr{X}/R}$ is a trivial line bundle (so in particular $i_k(D, \omega_k) = 0$ for all divisors D supported in the closed fiber).

To prove this triviality, we first note that the K-fiber $\omega_K = \Omega^1_{X/K}$ is trivial. Indeed, more specifically we claim that the canonical map $\mathscr{O}_X \otimes_K \mathrm{H}^0(X, \omega_K) \to \omega_K$ is an isomorphism, and this may be checked over a finite extension where X acquires a rational point (and so becomes an elliptic curve). Thus, we may pick a nonzero section η of ω_K; this is necessarily a trivializing section of ω_K, as we may check over a finite extension (where X becomes an elliptic curve). Multiplying η by a sufficiently high power of a uniformizer of R lets us assume that η is a nonzero section of the line bundle ω over the entire regular surface $\mathscr{X}$. This gives rise to an injection of invertible sheaves

$$\mathscr{O}_{\mathscr{X}} \to \omega$$

that is an isomorphism on X. The dual map

$$\omega^{-1} \to \mathscr{O}_{\mathscr{X}}$$

is an isomorphism onto an invertible ideal sheaf $\mathscr{I}$ whose associated zero-scheme is supported in $\mathscr{X}_k$. If D denotes the associated Weil divisor with support in $\mathscr{X}_k$ then we have $\omega \simeq \mathscr{O}_{\mathscr{X}}(D)$ with D an effective Weil divisor having support in $\mathscr{X}_k$. Let us write $D = \sum e_F F$ as a sum over the irreducible components F of $\mathscr{X}_k$, with $e_F \geq 0$ for all F.

By Lemma 7.6 (here is where we use the *minimality* of the regular proper model $\mathscr{X}$ of X), $F.D \geq 0$ for all irreducible divisors F of $\mathscr{X}_k$. Thus,

$$D.D = \sum e_F(F.D) \geq 0.$$

By Theorem 7.5, the intersection pairing on $\mathrm{Div}_0(\mathscr{X}_k)$ is negative-definite modulo torsion, so D represents a torsion class in $\mathrm{Div}_0(\mathscr{X}_k)$. But $[\mathscr{X}_k]$ has multiplicities with greatest common divisor 1, so $\mathrm{Div}_0(\mathscr{X}_k)$ is torsion-free. Hence, the image of D in $\mathrm{Div}_0(\mathscr{X}_k)$ vanishes. That is, D is an integral multiple of the principal divisor $[\mathscr{X}_k]$, so $\omega = \mathscr{O}(D)$ is a trivial line bundle as desired.

Example 7.8. The preceding example can be pushed a bit further. Indeed, using an argument of deJong [9, Lemma 9.1], we claim that the artin ring $\mathrm{H}^0(\mathscr{X}_k, \mathscr{O}_{\mathscr{X}_k})$ is a purely inseparable extension field of k. Granting this for a moment, if $\mathscr{X}_k$ has any closed points with residue field separable over k then by evaluating into such a residue field we get $\mathrm{H}^0(\mathscr{X}_k, \mathscr{O}_{\mathscr{X}_k}) = k$. The existence of such residue fields on the closed fiber holds if k is perfect (the case considered by deJong), or if $\mathscr{X}_k^{\mathrm{sm}} \neq \emptyset$, with this latter case happening whenever $\mathscr{X}_K$ acquires a rational point over a finite unramified extension of $\widehat{K}$ (since formation of the minimal regular proper model commutes with base change to a completion or étale-local extension of R). In particular, under this extra assumption it follows from the theory of cohomology and base change that the structure map $f : \mathscr{X} \to S = \mathrm{Spec}\, R$ is cohomologically flat in degree 0; that is, the equality $\mathscr{O}_S = f_*(\mathscr{O}_{\mathscr{X}})$ holds and it persists after any base change.

Before we give deJong's argument, we first note that it suffices to prove that $\mathrm{H}^0(\mathscr{X}_k, \mathscr{O}_{\mathscr{X}_k})$ is reduced. Indeed, if this artin ring is reduced then it is a finite product of finite extension fields of k, yet $\mathscr{X}_k$ is geometrically connected over k (due to $\mathscr{X} \to \mathrm{Spec}\, R$ being its own Stein factorization) and so is a single field k'/k. If k'/k were not purely inseparable then there is a nontrivial separable subextension k'_0/k and so $\mathscr{X}_k \otimes_k k'_0$ is disconnected. (It maps onto $\mathrm{Spec}(k'_0 \otimes_k k'_0)$.) But k'_0 can be realized as the residue field of a (finite étale) local extension $R \to R'_0$, and so $\mathscr{X} \otimes_R R'_0$ would have disconnected closed fiber. By flat base change the structure map $\mathscr{X} \otimes_R R'_0 \to \mathrm{Spec}(R'_0)$ is its own Stein factorization, so its closed fiber must be connected (contradiction).

Let us now explain why $\mathscr{X}_k$ has no nonzero nilpotent global functions. Let f be a nilpotent global function on $\mathscr{X}_k$, and assume f is nonzero. For each irreducible component F of the closed fiber, define $\mathrm{ord}_F(f) > 0$ to be the length of the nonzero quotient $\mathscr{O}_{\mathscr{X}_k,\xi_F}/(f)$ of the discrete valuation ring $\mathscr{O}_{\mathscr{X},\xi_F}$, with ξ_F the generic point of F. Let $D = \sum_F \mathrm{ord}_F(f)F$ considered as a closed subscheme of $\mathscr{X}_k$ on the regular surface $\mathscr{X}$. It is clear that $f|_D = 0$, so our problem is to prove the equality of Cartier (or Weil) divisors $D = [\mathscr{X}_k]$, which is to say that the inequality $\mathrm{ord}_F(f) \leq \mathrm{ord}_{\xi_F}([\mathscr{X}_k])$ for each F is an equality. Local lifts of f on $\mathscr{X}$ are local sections of the invertible ideal sheaf $\mathscr{O}_{\mathscr{X}}(-D)$ and are unique up to adding a local section of the subsheaf $\mathscr{O}_{\mathscr{X}}(-[\mathscr{X}_k])$, so f gives rise to a canonical section s of the invertible sheaf $\mathscr{O}_{\mathscr{X}_k}(-D)$ on $\mathscr{X}_k$ (generally not an ideal sheaf), with s a local generator at the generic points of $\mathscr{X}_k$.

Now assume $D < [\mathscr{X}_k]$, and we shall get a contradiction. Certainly $D' = [\mathscr{X}_k] - D$ is a non-empty effective Cartier divisor that is a closed subscheme of $\mathscr{X}_k$, so pullback to D' gives a section $s|_{D'}$ of $\mathscr{L} = \mathscr{O}_{D'}(-D)$

that is a generator near all generic points of D'. Hence, $\mathscr{L}^{\otimes n}$ has the nonzero section $s|_{D'}^{\otimes n}$ for all $n \geq 0$, so $\deg_{D'} \mathscr{L} \geq 0$. But this degree is

$$i_k(D', -D) = i_k([\mathscr{X}_k] - D, -D) = i_k(-D, -D) = i_k(D, D) \leq 0,$$

so we get $i_k(D, D) = 0$. Hence, D must be an integral multiple of $[\mathscr{X}_k]$, yet $0 < D < [\mathscr{X}_k]$. This is a contradiction.

8. More minimality criteria

Let W be an abstract integral Weierstrass model of E, so there is a unique morphism $W^{\mathrm{reg}} \to \mathscr{E}$ to the minimal regular proper model of E. By our work in §4, this gives rise to an inclusion

$$\mathrm{H}^0(\mathscr{E}, \omega_{\mathscr{E}/R}) = \mathrm{H}^0(W^{\mathrm{reg}}, \omega_{W^{\mathrm{reg}}/R}) \subseteq \mathrm{H}^0(W, \omega_{W/R}) \tag{8.1}$$

inside of $\mathrm{H}^0(E, \Omega^1_{E/K})$.

Theorem 8.1. *The inclusion* (8.1) *is an equality if and only if W is minimal.*

This theorem provides another criterion for minimality of an arbitrary Weierstrass model of E.

Proof. It suffices to prove that equality holds when W is minimal, so we now assume that W is minimal. In particular, $\mathscr{E} = W^{\mathrm{reg}}$ is the minimal regular resolution of W and there is a morphism of models

$$\pi : \mathscr{E} \to W.$$

Let $\mathscr{W} = W^{\mathrm{reg}}$, so (8.1) may be written as an inclusion

$$\mathrm{H}^0(\mathscr{W}, \omega_{\mathscr{W}/R}) \subseteq \mathrm{H}^0(W, \omega_{W/R}).$$

To get an equality for minimal W, first note by Example 7.7 (with $\mathscr{X} = \mathscr{W} = \mathscr{E}$ a minimal regular proper model of its generic fiber), the invertible $\omega_{\mathscr{W}/R}$ is globally free. Let η denote a global generator of $\omega_{\mathscr{W}/R}$, so restricting η to the canonical copy of the smooth locus W^{sm} in the smooth locus of $\mathscr{W}$ defines a generating section of $\omega_{W/R}$ over W^{sm}; this extends to a generating section of $\omega_{W/R}$ since $W - W^{\mathrm{sm}}$ has codimension ≥ 2 in the normal scheme W. In other words, such an η provides an inclusion

$$\mathrm{H}^0(W, \omega_{W/R}) \subseteq \mathrm{H}^0(\mathscr{W}, \omega_{\mathscr{W}/R}) \tag{8.2}$$

inside of $\mathrm{H}^0(E, \Omega^1_{E/K})$, and this is reverse to the inclusion that we already know must hold. Thus, we get the desired equality of R-lines. ■

Remark 8.2. We claim that if W is not minimal then $\omega_{W^{\mathrm{reg}}/R}$ is *not* globally free. Indeed, since $\omega_{\mathscr{E}/R}$ is globally free, the preceding argument proves that

$$\mathrm{H}^0(W, \omega_{W/R}) = \mathrm{H}^0(W^{\mathrm{reg}}, \omega_{W^{\mathrm{reg}}/R}) = \mathrm{H}^0(\mathscr{E}, \omega_{\mathscr{E}/R})$$

inside of $\mathrm{H}^0(E, \Omega^1_{E/K})$, but in §4 we also saw that all regular proper models of E have the same R-line of global sections for their relative dualizing sheaf. Since the R-line $\mathrm{H}^0(W, \omega_{W/R}) = \mathrm{H}^0(W^{\mathrm{sm}}, \Omega^1_{W^{\mathrm{sm}}/R})$ in $\mathrm{H}^0(E, \Omega^1_{E/K})$ determines W up to isomorphism as a model of E, we get a contradiction if W is not a minimal Weierstrass model.

Let us record an interesting consequence of the global freeness of $\omega_{\mathscr{E}/R}$:

Corollary 8.3. *Let W be an integral Weierstrass model of E over R. Let $\pi : W^{\mathrm{reg}} \to W$ be the minimal regular resolution. The sheaf $\mathrm{R}^1\pi_*\mathscr{O}$ vanishes if and only if W is a minimal Weierstrass model.*

Proof. If W is regular then π is an isomorphism and W is a minimal regular proper model of E. Thus, we may assume W is not regular; let $x_0 \in W_k$ be the unique point outside of the k-smooth locus, so

$$\pi : \pi^{-1}(W - \{x_0\}) \to W - \{x_0\}$$

is an isomorphism. Thus, $\mathrm{R}^1\pi_*\mathscr{O}$ is a coherent sheaf supported at x_0, and so its stalk at x_0 is a finite-length module over $\mathscr{O}_{W,x_0}$. By [1, (3.3)], the general duality machinery for proper birational maps between 2-dimensional normal noetherian schemes provides a short exact sequence

$$0 \to \pi_*\omega_{W^{\mathrm{reg}}/R} \to \omega_{W/R} \to \mathscr{E}xt^2_W(\mathrm{R}^1\pi_*\mathscr{O}, \omega_{W/R}) \to 0, \tag{8.3}$$

where the first map has K-fiber equal to the identity on $\Omega^1_{E/K}$. The compatibility of local and global duality for the normal surface W ensures that $M \rightsquigarrow \mathrm{Ext}^2_{\mathscr{O}_{W,x}}(M, \omega_{W/R,x})$ is a self-duality on the category of finite-length $\mathscr{O}_{W,x}$-modules for every closed point $x \in W_k$. Thus, the vanishing of $\mathrm{R}^1\pi_*\mathscr{O}$ is equivalent to the surjectivity of $\pi_*\omega_{W^{\mathrm{reg}}/R} \to \omega_{W/R}$.

If $\mathrm{R}^1\pi_*\mathscr{O} = 0$, then by (8.3) we have

$$\mathrm{H}^0(W^{\mathrm{reg}}, \omega_{W^{\mathrm{reg}}/R}) = \mathrm{H}^0(W, \omega_{W/R})$$

inside of $\mathrm{H}^0(E, \Omega^1_{E/K})$, and we have already seen that such an equality with global sections of the dualizing sheaf on a regular proper model forces W to be minimal. Conversely, if W is minimal then $W^{\mathrm{reg}} = \mathscr{E}$ has globally-free relative dualizing sheaf $\omega_{W^{\mathrm{reg}}/R}$. Let η be a generating section. Since $\pi_*\omega_{W^{\mathrm{reg}}/R} \to \omega_{W/R}$ has K-fiber equal to the identity on $\Omega^1_{E/K}$, the image of η may be computed by restricting η to the canonical copy of W^{sm} inside of W^{reg} and then using the isomorphism $\omega_{W/R}(W) \simeq \omega_{W/R}(W^{\mathrm{sm}})$; this is the process that led to (8.2). Thus, the section $\pi_*\eta$ of $\pi_*\omega_{W^{\mathrm{reg}}/R}$ maps to a generator of the line bundle $\omega_{W/R}$ (as its image in $\omega_{W/R}(W)$ restricts to a generator on the complement of a codimension-2 point in the normal W). This forces $\pi_*\omega_{W^{\mathrm{reg}}/R} \hookrightarrow \omega_{W/R}$ to be surjective, and so (8.3) and the global-local duality compatibility imply that $\mathrm{R}^1\pi_*\mathscr{O}$ vanishes. ■

In general, if X is a normal proper R-curve with smooth and geometrically connected generic fiber then we say it has *rational singularities* if there is a proper birational map $\pi : X' \to X$ with regular X' such that $\mathrm{R}^1\pi_*\mathscr{O}_{X'}$ vanishes; by [1, Prop. 3.2], this condition is independent of the choice of X' (and so is satisfied for regular X) and it is preserved under blow-up at a codimension-2 point. More importantly, by [1, Thm. 4.9], the blow-up of X along its (reduced) non-regular locus is *automatically* normal when X has rational singularities. Thus, Corollary 8.3 implies another geometric characterization of minimal Weierstrass models:

Corollary 8.4. *An integral Weierstrass model W of an elliptic curve $E_{/K}$ has rational singularities if and only if W is a minimal Weierstrass model.*

We have seen that for such minimal W the minimal regular resolution W^{reg} coincides with the minimal regular proper model $\mathscr{E}$ of $E_{/K}$. We noted above that Lipman's resolution process applied to such W would reach a resolution of singularities by successive blow-up at non-regular points *without ever needing to normalize* (i.e., the blow-ups are automatically normal), but it isn't apparent if Lipman's resolution would recover $\mathscr{E}$ or perhaps require some contractions to reach $\mathscr{E}$ (see Remark 3.12).

But it turns out that for normal proper R-curves which have only rational singularities, which for Weierstrass models W is the case precisely when W is minimal (by Corollary 8.4), Lipman's resolution *is* the minimal regular resolution; see [11, §9.3.4, Rem. 3.34] and [11, §9.4, Exer. 4.7(c)]. This heuristically explains why Tate was able to construct his algorithm for computing $\mathscr{E}$ by starting with a minimal W and applying well-chosen explicit blow-ups without ever having to compute normalizations; Tate's algorithm also uses some blow-ups along codimension-1 subschemes, so it is more artful than Lipman's general procedure. If it had been necessary to normalize after each blow-up, then it is hard to imagine that Tate's concrete algorithm could have been created (as computing normalizations is notoriously difficult).

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