MAIN THEOREM OF COMPLEX MULTIPLICATION

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In [S, Ch. IV, §18] the Main Theorem of complex multiplication is proved in a manner that uses some adelic formalism. However, [S] uses a framework for algebraic geometry that has long been abandoned, so many of the beautiful ideas there are somewhat shrouded in mystery for the reader who is unfamiliar with the pre-Grothendieck approaches to algebraic geometry and abelian varieties. The aim of the Main Theorem is as follows. Let (A, i) be an abelian variety over $\overline{\mathbf{Q}}$ with CM type (K, Φ), and let $\phi : A \to A^\vee$ be a K-linear polarization (where $A^\vee$ has the K-action $i^\vee(c) = i(c^*)^\vee$, with c^* denoting complex conjugation on the CM field K, so $(A^\vee, i^\vee)$ also has CM type (K, Φ)); such a ϕ exists by the complex-analytic theory. Let $K^* \subseteq \overline{\mathbf{Q}}$ be the reflex field, and pick $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$ and a finite idele $s \in \mathbf{A}_{K^*,\mathrm{f}}^\times$ that maps to $\sigma|_{(K^*)^{\mathrm{ab}}}$ under the global Artin map. We seek to describe the $\mathbf{A}_{K,\mathrm{f}}$-linear isomorphism $\sigma : \mathrm{V}_\mathrm{f}(A) \simeq \mathrm{V}_\mathrm{f}(A^\sigma)$ and the Weil pairing e_{ϕ^σ} on $\mathrm{V}_\mathrm{f}(A^\sigma)$ in terms of adelic operations on $\mathrm{V}_\mathrm{f}(A)$ and e_ϕ using s. We also wish to reconstruct (A^σ, i^σ) from (A, i) by an adelic procedure that does not use Galois automorphisms.

In these notes, we give a complete proof of the Main Theorem of complex multiplication by using the language of schemes and adelic points of algebraic groups as well as results proved earlier in the seminar. We avoid the intervention of analytic uniformizations of abelian varieties in our initial "algebraic" statement of the Main Theorem, working throughout over $\overline{\mathbf{Q}}$ and emphasizing the Galois-theoretic formulation that arises in Deligne's axiomatic definition of canonical models. By working over $\overline{\mathbf{Q}}$ we avoid far-out things like the automorphism group of $\mathbf{C}$. We also use our initial algebraic version of the Main Theorem over $\overline{\mathbf{Q}}$ to recover a "coordinate-free" version of the traditional formulation of the Main Theorem in terms of analytic uniformizations and Riemann forms of certain polarizations. Complex-analytic methods certainly have their place in the theory, but we prefer to minimize their appearance and keep proofs as algebraic as possible. The reason that we can achieve this is because of our systematic use of Serre's tensor construction [X], especially in its relative incarnation over base schemes with possibly mixed characteristic.

The central ideas in the proof we give for the Main Theorem are all due to Shimura and Taniyama, even though it will frequently be apparent to the reader that we are using a mathematical style very different from that employed by Shimura and Taniyama. Specializing our largely algebraic arguments to the 1-dimensional case gives a proof for elliptic curves that exhibits a different flavor from the traditional one (as in [A1]); moreover, some technical issues simplify tremendously in the 1-dimensional case because elliptic curves have unique polarizations of each positive square degree.

We assume that the reader is familiar with CM abelian varieties [L], polarizations [C1], CM types [C2, §1], reflex fields, Weil restriction [R], reflex norms [K], the Serre tensor construction [X], and other background that has been developed in the seminar.

In §1 we develop some further background concepts that are required in the proof of the Main Theorem, and in §2 we give the setup for and statement of the Main Theorem of complex multiplication. Before delving into the proof, in §3 we work out a few consequences of the Main Theorem. The proof of the Main Theorem occupies §4–§5, Finally, in §6 we use the "algebraic" version of the Main Theorem over $\overline{\mathbf{Q}}$ to deduce the traditional version in terms of analytic uniformizations and we also translate analytic refinements back into algebraic language (by using a variant on the Serre tensor construction in the absence of CM by the maximal order) and give purely algebraic proofs of the latter.

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Notation. We fix an algebraic closure $\mathbf{C}$ of $\mathbf{R}$, endowed with its unique absolute value extending the one on $\mathbf{R}$, and we let $\overline{\mathbf{Q}} \subseteq \mathbf{C}$ be the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$. The kernel of $\exp : \mathbf{C} \to \mathbf{C}^\times$ is denoted $\mathbf{Z}(1)$, and we write $M(1)$ to denote $\mathbf{Z}(1) \otimes_{\mathbf{Z}} M$ for any $\mathbf{Z}$-module M.

For a scheme X over a field K and an automorphism $\sigma : K \to K$, X^σ denotes the K-scheme $K \otimes_{\sigma,K} X$ obtained by base change. The same notation is used for K-morphisms. We indulge in one serious abuse of notation: if $\sigma : K \simeq K$ is an automorphism of K that restricts to an automorphism σ_0 of a subfield K_0 (the case of most interest being $K = \overline{\mathbf{Q}}$ and K_0 a number field) and if Z is a K-scheme for which a K_0-descent Z_0 is specified, we write Z_0^σ to denote the K_0-scheme $Z_0^{\sigma_0}$ descending the K-scheme Z^σ. Hopefully this will not cause confusion.

We write $\mathbf{A}_{\mathrm{f}}$ to denote the topological ring of finite adeles of $\mathbf{Q}$ (i.e., $\mathbf{Q}\otimes\widehat{\mathbf{Z}}$), and $\mathbf{A}_{K,\mathrm{f}} \simeq K\otimes_{\mathbf{Q}}\mathbf{A}_{\mathrm{f}}$ to denote the topological ring of finite adeles of a number field K. As is traditional in number theory, we normalize the Artin map of class field theory to carry local uniformizers to arithmetic Frobenius elements. (This is opposite the convention in algebraic geometry.) For a CM field K, a CM abelian variety $(A, i : K \to \mathrm{End}_k^0(A))$ over a field k is *principal* if the CM order $i^{-1}(\mathrm{End}_k(A))$ is equal to $\mathscr{O}_K$. If A is an abelian variety over a field k of characteristic 0 and $\overline{k}/k$ is an algebraic closure then $\mathrm{T}_{\mathrm{f}}(A)$ denotes the "total Tate module" $\varprojlim A[n](\overline{k}) \simeq \prod_\ell \mathrm{T}_\ell(A)$ and $\mathrm{V}_{\mathrm{f}}(A)$ denotes the finite free $\mathbf{A}_{\mathrm{f}}$-module $\mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{T}_{\mathrm{f}}(A)$. The *Weil pairing* for A is denoted $\langle\cdot,\cdot\rangle_A : \mathrm{V}_{\mathrm{f}}(A) \times \mathrm{V}_{\mathrm{f}}(A^\vee) \to \mathbf{A}_{\mathrm{f}}(1)$.

1. Q-polarizations

Let K be a CM field, with maximal totally real subfield K_0 of degree g over $\mathbf{Q}$. We write $c \mapsto c^*$ to denote complex conjugation on K. Let $\Phi \subseteq \mathrm{Hom}(K, \overline{\mathbf{Q}})$ be a CM type; i.e., a set of representatives for the quotient of $\mathrm{Hom}(K, \overline{\mathbf{Q}})$ by the free action of complex conjugation on K. Clearly Φ has size g and $[K : \mathbf{Q}] = 2g$. Finally, let $K^* \subseteq \overline{\mathbf{Q}}$ be the reflex field of (K, Φ); this is the fixed field for the open subgroup of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ consisting of those elements σ such that $\sigma\Phi = \Phi$. (That is, composition with σ permutes the set of embeddings Φ.)

We shall be interested in studying pairs (A, i) where A is an abelian variety over $\overline{\mathbf{Q}}$ and $i : K \to \mathrm{End}^0_{\overline{\mathbf{Q}}}(A)$ is a ring homomorphism making A an abelian variety of CM type (K, Φ). That is, the $\overline{\mathbf{Q}}$-linear action of K on the tangent space of A has g eigenlines on which K acts through the mappings $\varphi : K \to \overline{\mathbf{Q}}$ given by the g elements $\varphi \in \Phi$. The dual $A^\vee$ is endowed with the K-action $i^\vee : K \to \mathrm{End}^0_{\overline{\mathbf{Q}}}(A^\vee)$ defined by $i^\vee(c) = i(c^*)^\vee$, so by the complex-analytic theory the CM type for $(A^\vee, i^\vee)$ is Φ.

Since $\overline{\mathbf{Q}}$ is algebraically closed, so $\mathrm{Hom}_{\overline{\mathbf{Q}}}(X, Y) = \mathrm{Hom}_{\mathbf{C}}(X_{\mathbf{C}}, Y_{\mathbf{C}})$ for any abelian varieties X and Y over $\overline{\mathbf{Q}}$, a polarization of A over $\overline{\mathbf{Q}}$ is the same as a polarization of $A_{\mathbf{C}}$ over $\mathbf{C}$, and hence (in terms of the analytic theory of polarizations) it is encoded as a skew-symmetric bilinear pairing

$$\psi_{\mathbf{Z}} : \mathrm{H}_1(A(\mathbf{C}), \mathbf{Z}) \times \mathrm{H}_1(A(\mathbf{C}), \mathbf{Z}) \to \mathbf{Z}(1)$$

such that with respect to the complex structure induced by the $\mathbf{R}$-linear isomorphism $\mathrm{H}_1(A(\mathbf{C}), \mathbf{R}) \simeq \mathrm{T}_0(A(\mathbf{C}))$ (defined by $\sigma \mapsto \int_\sigma$) the $\mathbf{R}$-linear extension $\psi_{\mathbf{R}}$ satisfies $\psi_{\mathbf{R}}(cx, y) = \psi_{\mathbf{R}}(x, \overline{c}y)$ for all $c \in \mathbf{C}$ and the resulting $\mathbf{R}$-valued symmetric bilinear form $(2\pi\sqrt{-1})^{-1}\psi_{\mathbf{R}}(\sqrt{-1} \cdot x, y)$ is positive-definite. In terms of the algebraic theory this $\psi_{\mathbf{Z}}$ may be encoded as a symmetric isogeny $\phi : A \to A^\vee$ such that $(1, \phi)^*(\mathscr{P})$ is ample on A, where $\mathscr{P}$ is the Poincaré bundle over $A \times A^\vee$. For our purposes it is inconvenient to impose "integrality" conditions, so we make the:

Definition 1.1. A $\mathbf{Q}$-*polarization* of an abelian variety over a field is a positive rational multiple of a polarization.

For an abelian variety A over $\overline{\mathbf{Q}}$, a $\mathbf{Q}$-polarization of A is a positive rational multiple of a pairing $\psi_{\mathbf{Z}}$ as above or it is a mapping $\phi : A \to A^\vee$ *in the isogeny category* of abelian varieties over $\overline{\mathbf{Q}}$ such that ϕ admits a positive integral multiple that is a polarization.

We initially seek to construct $\mathbf{Q}$-polarizations on A such that the associated Rosati involution on $\mathrm{End}^0_{\overline{\mathbf{Q}}}(A)$ restricts to complex conjugation on K, or equivalently such that the associated symmetric isogeny $\phi : A \to A^\vee$ is K-linear. Such a $\mathbf{Q}$-polarization is called an K-*linear* $\mathbf{Q}$-*polarization*. (Since $(A^\vee, i^\vee)$ has type (K, Φ) there certainly exist K-linear isogenies $A \to A^\vee$, but the existence of such a map that is a $\mathbf{Q}$-polarization is a stronger assertion.) Another formulation of the K-linearity condition on a $\mathbf{Q}$-polarization $\phi : A \to A^\vee$ is this: under the associated skew-symmetric Weil self-pairing

$$\langle \cdot, \cdot \rangle_{\phi,\mathbf{Q}} : \mathrm{H}_1(A(\mathbf{C}), \mathbf{Q}) \times \mathrm{H}_1(A(\mathbf{C}), \mathbf{Q}) \to \mathbf{Q}(1), \tag{1.1}$$

for any $c \in K$ the adjoint to $\mathrm{H}_1(i(c))$ is $\mathrm{H}_1(i(c^*))$. In a special case, this property is automatic:

Example 1.2. Suppose A is simple, so $A_{\mathbf{C}}$ is simple and hence by the complex-analytic theory the endomorphism ring $\mathrm{End}^0_{\overline{\mathbf{Q}}}(A) = \mathrm{End}^0_{\mathbf{C}}(A_{\mathbf{C}})$ is a CM field of degree $2g$. Thus, $i : K \to \mathrm{End}^0_{\overline{\mathbf{Q}}}(A)$ is an isomorphism and so the Rosati involution for a polarization of A must be some $\sigma \in \mathrm{Aut}(K/\mathbf{Q})$ such that $\sigma^2 = 1$. The positivity of the Rosati involution implies that the quadratic form $K \to \mathbf{Q}$ defined by $\alpha \mapsto \mathrm{Tr}_{K/\mathbf{Q}}(\alpha\sigma(\alpha))$ is positive-definite. By [Ca, Lemma 5.6], since K is a CM field this positivity condition forces σ to be complex conjugation.

In the general (possibly non-simple) case, there is an existence result:

Lemma 1.3. *There exist $\mathbf{Q}$-polarizations of A whose associated Rosati involution on $\mathrm{End}^0_{\overline{\mathbf{Q}}}(A)$ restricts to complex conjugation on $i(K)$. The set of elements in $\mathrm{Hom}^0_{\overline{\mathbf{Q}}}(A, A^\vee)$ associated to such $\mathbf{Q}$-polarizations is a principal homogeneous space for the action on A (in the isogeny category) by the subgroup of totally positive $\alpha \in K_0^\times$.*

Proof. Extending scalars from $\overline{\mathbf{Q}}$ to $\mathbf{C}$ does not introduce new maps between abelian varieties, and does not affect whether or not a given map from an abelian variety to its dual is a polarization. Hence, by GAGA we may instead consider the situation over $\mathbf{C}$ after analytification. The problem only depends on A up to K-linear isogeny. Thus, we can assume $A(\mathbf{C}) \simeq (K_{\mathbf{R}})_\Phi/\mathscr{O}_K$ as complex tori with CM type (K, Φ). In [L, p. 3] there is given an explicit Riemann form constructed on this complex torus, and for this Riemann form the adjoint for the K-action is seen (by inspection) to be given by complex conjugation. This settles the existence aspect.

As for the extent of non-uniqueness, since K is its own centralizer in $\mathrm{End}^0_{\overline{\mathbf{Q}}}(A)$ it follows that any two K-linear isogenies from A to $A^\vee$ (in the isogeny category) are related through the K-action on A. Hence, given a single K-linear $\mathbf{Q}$-polarization $\phi : A \to A^\vee$, we just have to work out the condition on $c \in K^\times$ so that $\phi \circ i(c)$ is a $\mathbf{Q}$-polarization. Since ϕ is a symmetric K-linear isogeny, the symmetry condition on $\phi \circ i(c)$ says exactly that $i(c) = i(c^*)$ in $\mathrm{End}^0_{\overline{\mathbf{Q}}}(A)$; i.e., $c \in K_0$. In terms of the Hermitian form given in [L, p. 3], the positivity condition on $\phi \circ i(c)$ is that the action of $i(c)$ on the first variable preserves the positive-definiteness property. This translates into the condition that for each $\varphi \in \Phi$ the totally real algebraic number $\varphi(c) \in \overline{\mathbf{Q}} \subseteq \mathbf{C}$ is *positive*. This is equivalent to the condition that $c \in K_0$ is totally positive, since $\Phi|_{K_0}$ is the set of all embeddings of K_0 into $\mathbf{R}$ (as Φ is a CM type on K). ■

Lemma 1.4. *Let (A_0, i_0) and (A'_0, i'_0) be abelian varieties with CM type (K, Φ) over a number field L. Assume L is so large that $\mathrm{Hom}^0_L((A_0, i_0), (A'_0, i'_0))$ is nonzero. If $\mathfrak{P}$ is a prime of good reduction for A_0 and A'_0 then the injective reduction mapping*

$$\mathrm{Hom}^0_L((A_0, i_0), (A'_0, i'_0)) \to \mathrm{Hom}^0_{\kappa(\mathfrak{P})}((\overline{A}_0, \bar{i}_0), (\overline{A}'_0, \bar{i}'_0))$$

is bijective.

Proof. Since the abelian varieties all have dimension g with $[K : \mathbf{Q}] = 2g$, the source and target Hom^0's are each at most 1-dimensional over K. The reduction mapping is K-linear, so the assumption of non-vanishing in characteristic 0 gives the result. ■

Consider triples (A, i, ϕ) where (A, i) is a CM abelian variety of type (K, Φ) over $\overline{\mathbf{Q}}$ and $\phi : A \to A^\vee$ is a $\mathbf{Q}$-polarization of A that is K-linear (in the sense of Lemma 1.3). If $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$ (where K^* is the reflex field for (K, Φ)) then the base change $\sigma : \overline{\mathbf{Q}} \simeq \overline{\mathbf{Q}}$ gives another triple $(A^\sigma, i^\sigma, \phi^\sigma)$ with $i^\sigma(c) = i(c)^\sigma \in \mathrm{End}^0_{\overline{\mathbf{Q}}}(A^\sigma)$. The tangent spaces t_A and t_{A^σ} of A and A^σ at their respective identity elements satisfy $t_{A^\sigma} \simeq \sigma^*(t_A)$ as $K \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$-modules, so (A^σ, i^σ) has CM type $\sigma\Phi = \Phi$. Hence, by the complex-analytic theory as in [L, §1] we know that (A, i) and (A^σ, i^σ) are isogenous over $\overline{\mathbf{Q}}$; that is, there exist K-linear isogenies $A \to A^\sigma$. A fundamental fact is that such isogenies can be found that respect K-linear $\mathbf{Q}$-polarizations up to rational homothety:

Theorem 1.5. *The triples (A, i, ϕ) and $(A^\sigma, i^\sigma, \phi^\sigma)$ are isogenous up to rational homethety. More precisely, if $\phi : A \to A^\vee$ is a K-linear $\mathbf{Q}$-polarization then there is a K-linear isogeny $\xi : A \to A^\sigma$ such that $\xi^\vee \circ \phi^\sigma \circ \xi = q_{\phi,\xi,\sigma} \cdot \phi$ with $q_{\phi,\xi,\sigma} \in \mathbf{Q}^\times_{>0}$, and necessarily $q_{\phi,\xi,\sigma} = \deg(\xi)^{1/g}$.*

The notion of degree for maps in the isogeny category is defined by $\mathbf{Z}$-homogeneity of degree for morphisms of abelian varieties.

Proof. Fix a choice of ϕ. Once we find the desired ξ then computing the degree of both sides will then give $q_{\phi,\xi,\sigma}^{2g} = \deg(\xi)\deg(\xi^\vee) = \deg(\xi)^2$, so $q_{\phi,\xi,\sigma} = \deg(\xi)^{1/g}$.

Let L/K^* be a finite Galois subextension of $\overline{\mathbf{Q}}$ that is a field of definition for (A, i, ϕ) in the sense that: (i) A descends to an abelian variety A_0 over L, (ii) the K-action on A (in the isogeny category over $\overline{\mathbf{Q}}$) descends to a ring homomorphism $i_0 : K \to \mathrm{End}^0_L(A_0)$, and (iii) there is a K-linear $\mathbf{Q}$-polarization $\phi_0 : A_0 \to A_0^\vee$ over L whose $\overline{\mathbf{Q}}$-scalar extension is ϕ. We also take L so big that

$$\mathrm{Hom}_L((A_0, i_0), (A_0^\sigma, i_0^\sigma)) = \mathrm{Hom}_{\overline{\mathbf{Q}}}((A, i), (A^\sigma, i^\sigma)). \tag{1.2}$$

Choose a prime $\mathfrak{P}$ of L such that $\mathfrak{P}$ is unramified over its contraction $\mathfrak{p}$ in K^*, $\mathfrak{P}$ is a prime of good reduction for A_0, and $\sigma|_L = (\frac{L/K^*}{\mathfrak{P}})$. In particular, since σ acting on $\mathscr{O}_L$ fixes $\mathfrak{P}$ it follows that $\mathfrak{P}$ is also a prime of good reduction for A_0^σ. Let $\mathscr{A}_0$ denote the Néron model $\mathscr{A}_0$ for A_0 over $\mathscr{O}_{L,\mathfrak{P}}$, so this is an abelian scheme. Viewing σ as an automorphism of $\mathscr{O}_{L,\mathfrak{P}}$, the abelian scheme $\mathscr{A}_0^\sigma$ is the Néron model of its generic fiber A_0^σ, and likewise the dual abelian scheme $\mathscr{A}_0^\vee$ is the Néron model of its generic fiber $A_0^\vee$. By the Néron mapping property for abelian schemes (i.e., the valuative criterion for properness and Weil's extension lemma), ϕ_0 extends uniquely to a mapping of abelian schemes $\mathscr{A}_0 \to \mathscr{A}_0^\vee$. For any abelian scheme $X \to S$ over a scheme S and any morphism of abelian schemes $f : X \to X^\vee$, the set of $s \in S$ such that f_s is a polarization on X_s is both open and closed in S. Hence, the map $\mathscr{A}_0 \to \mathscr{A}_0^\vee$ extending ϕ_0 is a polarization on the closed fibers too. (Since we are free to choose among infinitely many $\mathfrak{P}$, we can arrange for the reduction of ϕ_0 modulo $\mathfrak{P}$ to be a polarization by "denominator-chasing", thereby avoiding some of the preceding technical issues with Néron models and abelian schemes.)

Let $q = q_\mathfrak{p}$ be the size of the residue field at $\mathfrak{p}$, so σ induces the qth-power map on the residue field $\kappa(\mathfrak{P})$ at $\mathfrak{P}$. Because we chose L so large that (1.2) holds, it follows from Lemma 1.4 that there exists a (unique) K-linear isogeny $\xi : A_0 \to A_0^\sigma$ in the isogeny category over L such that its reduction $\overline{\xi} : \overline{A}_0 \to \overline{A}_0^{(q)}$ over $\kappa(\mathfrak{P})$ is the relative q-Frobenius map $\mathrm{Fr}_{\overline{A}_0/\kappa(\mathfrak{P}),q}$, where $\overline{A}_0$ denotes the closed fiber of $\mathscr{A}_0$. Of course, ξ depends on the choice of $\mathfrak{P}$.

We claim that ξ works in the theorem, with some rational multiplier q. That is, we claim $\xi^\vee \circ \phi_0^\sigma \circ \xi = q\phi_0$. To check this equality as maps in the isogeny category of abelian varieties over L with good reduction at $\mathfrak{P}$, it is equivalent to check the corresponding equality on reductions over $\kappa(\mathfrak{P})$. Recall that the reduction of ϕ_0 is a $\mathbf{Q}$-polarization on $\overline{A}_0$. Hence, it suffices to prove more generally that if X is an abelian variety over a field k of characteristic $p > 0$ and if q is a power of p then for any $\mathbf{Q}$-polarization $\phi : X \to X^\vee$ there is an equality

$$\mathrm{Fr}^\vee_{X/k,q} \circ \phi^{(q)} \circ \mathrm{Fr}_{X/k,q} = q\phi.$$

We may extend scalars so that k is algebraically closed, and we may multiply ϕ by a sufficiently divisible nonzero integer so that it is a polarization. Hence, $\phi = \phi_\mathscr{L}$ for an ample line bundle $\mathscr{L}$ on A. Since the formation of $\phi_\mathscr{L}$ is compatible with base change, we have $\phi^{(q)} = \phi_{\mathscr{L}^{(q)}}$ with $\mathscr{L}^{(q)}$ denoting the pullback of $\mathscr{L}$ along the projection $A^{(q)} \to A$ (or equivalently, along the base change $\mathrm{Spec}\, k \to \mathrm{Spec}\, k$ given by the qth-power map on k). By the functorial properties of the "Mumford construction" $\mathscr{L} \rightsquigarrow \phi_\mathscr{L}$ [Ca, §3], $\mathrm{Fr}^\vee_{X/k,q} \circ \phi^{(q)} \circ \mathrm{Fr}_{X/k,q} = \phi_{\mathrm{Fr}^*_{X/k,q}(\mathscr{L}^{(q)})}$.

Since $q\phi_\mathscr{L} = \phi_{\mathscr{L}^{\otimes q}}$, it now suffices to prove $\mathrm{Fr}^*_{X/k,q}(\mathscr{L}^{(q)}) \simeq \mathscr{L}^{\otimes q}$ for any invertible sheaf $\mathscr{L}$ on X, where X is *any* $\mathbf{F}_p$-scheme. The composite of the relative q-Frobenius $\mathrm{Fr}_{X/k,q} : X \to X^{(q)}$ and the projection

$X^{(q)} \to X$ is the absolute q-Frobenius map $\mathrm{Fr}_{X,q} : X \to X$ that is the identity on topological spaces and the qth-power map on structure sheaves. Hence, $\mathrm{Fr}^*_{X/k,q}(\mathscr{L}^{(q)}) \simeq \mathrm{Fr}^*_{X,q}(\mathscr{L})$. Our problem is therefore to prove $\mathrm{Fr}_{X,q}(\mathscr{L}) \simeq \mathscr{L}^{\otimes q}$ for any invertible sheaf $\mathscr{L}$ on X. Since $\mathrm{Fr}_{X,q}$ is the identity on X and the qth-power map on $\mathscr{O}_X$, we conclude the proof by a calculation on $\mathscr{O}_X^\times$-valued Čech 1-cocycles (for a trivialization of $\mathscr{L}$ over X). ■

In the language of the bilinear pairings (1.1), Theorem 1.5 says

$$\langle \mathrm{H}_1(\xi)(\cdot), \mathrm{H}_1(\xi)(\cdot)\rangle_{\phi^\sigma,\mathbf{Q}} = \deg(\xi)^{1/g} \cdot \langle \cdot, \cdot \rangle_{\phi,\mathbf{Q}}$$

and in particular we may choose a K-linear isomorphism $\varphi_\sigma : A^\sigma \to A$ in the isogeny category over $\overline{\mathbf{Q}}$ such that φ_σ carries ϕ back to ϕ^σ up to $\mathbf{Q}^\times_{>0}$-multiple. This isogeny is unique up to the action on A by elements $c \in K^\times$ such that simultaneous multiplication by c on both factors of $\mathrm{H}_1(A(\mathbf{C}), \mathbf{Q})$ preserves the $\mathbf{Q}$-polarization ϕ up to $\mathbf{Q}^\times_{>0}$-multiple. That is, for a K-linear $\mathbf{Q}$-polarization $\phi : A \to A^\vee$ the condition on c is that $i(c)^\vee \circ \phi \circ i(c)$ is a $\mathbf{Q}^\times_{>0}$-multiple of ϕ. But $i(c)^\vee = i^\vee(c^*)$ and ϕ is K-linear, so the condition on c is that $\phi \circ i(cc^*) = q\phi = \phi \circ i(q)$ for some $q \in \mathbf{Q}^\times_{>0}$. Equivalently, $\mathrm{N}_{K/K_0}(c) = q \in \mathbf{Q}^\times$ (such a norm is necessarily positive). We conclude:

Corollary 1.6. *The K-linear isogeny φ_σ is unique up to the action of $T(\mathbf{Q}) \subseteq K^\times$ on A, where T is the torus*

$$T = \ker(\mathrm{Res}_{K/\mathbf{Q}}(\mathbf{G}_m) \overset{\mathrm{N}_{K/K_0}}{\twoheadrightarrow} \mathrm{Res}_{K_0/\mathbf{Q}}(\mathbf{G}_m)/\mathbf{G}_m).$$

This kernel was proved to be connected, and hence a torus, in [R, §4].

Remark 1.7. Since the choice of ϕ is unique up to the action on A by totally positive elements of K_0, by Lemma 1.3, and the action by these elements commutes with the action by $T(\mathbf{Q}) \subseteq K^\times$, we conclude that the "defining condition" on φ_σ is *independent* of the choice of ϕ. That is, such a φ_σ carries ϕ' to a positive rational multiple of ${\phi'}^\sigma$ for *every* K-linear $\mathbf{Q}$-polarization ϕ' of A.

Remark 1.8. The preceding argument applies adelically as well, so the elements of $\mathrm{Aut}_{\mathbf{A}_{K,\mathrm{f}}}(\mathrm{V}_\mathrm{f}(A)) \simeq \mathbf{A}^\times_{K,\mathrm{f}}$ preserving the $\mathbf{A}^\times_\mathrm{f}$-homothety class of the $\mathbf{A}_\mathrm{f}$-bilinear skew-symmetric Weil self-pairing

$$\langle \cdot, \cdot \rangle_\phi : \mathrm{V}_\mathrm{f}(A) \times \mathrm{V}_\mathrm{f}(A) \to \mathbf{A}_\mathrm{f}(1)$$

are precisely the elements of $T(\mathbf{A}_\mathrm{f})$.

2. Algebraic formulation of the Main Theorem

We retain the same setup as above: we fix the triple (A, i, ϕ) over $\overline{\mathbf{Q}}$ with (A, i) of type (K, Φ) and ϕ a K-linear $\mathbf{Q}$-polarization of A. We also choose $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$ and we pick a K-linear isogeny $\varphi_\sigma : A^\sigma \to A$ carrying ϕ back to ϕ^σ up to $\mathbf{Q}^\times$-multiple (determined by the degree of φ_σ); this condition on φ_σ is *independent* of the choice of ϕ (by Remark 1.7). Consider the $\mathbf{A}_{K,\mathrm{f}}$-linear isomorphism

$$\mathrm{V}_\mathrm{f}(A) \overset{[\sigma]}{\simeq} \mathrm{V}_\mathrm{f}(A^\sigma) \overset{\mathrm{V}_\mathrm{f}(\varphi_\sigma)}{\simeq} \mathrm{V}_\mathrm{f}(A). \tag{2.1}$$

The map $[\sigma]$ in (2.1) is the standard Galois-action on $\overline{\mathbf{Q}}$-points, and so in terms of $\mathbf{A}_\mathrm{f}(1)$-valued Weil self-pairings it carries the self-pairing associated to ϕ^σ to the self-pairing associated to ϕ up to the total cyclotomic character $\chi(\sigma) \in \mathbf{A}^\times_\mathrm{f}$ giving the action of σ on $\mathbf{A}_\mathrm{f}(1)$. The second step in (2.1) is equivariant for the self-pairings defined by ϕ^σ and ϕ up to positive rational multiple, due to the defining condition on φ_σ. Hence, (2.1) is an $\mathbf{A}_{K,\mathrm{f}}$-linear automorphism of the free rank-1 $\mathbf{A}_{K,\mathrm{f}}$-module $\mathrm{V}_\mathrm{f}(A)$ preserving the self-pairing associated to ϕ up to the $\mathbf{A}^\times_\mathrm{f}$-multiplier $\chi(\sigma)(\deg \varphi_\sigma)^{1/g}$. The composite (2.1) is multiplication by an element $\mu_{\sigma,\varphi_\sigma} \in \mathbf{A}^\times_{K,\mathrm{f}} = \mathrm{Res}_{K/\mathbf{Q}}(\mathbf{G}_m)(\mathbf{A}_\mathrm{f})$ that therefore preserves ϕ up to an idelic multiple and so lies in $T(\mathbf{A}_\mathrm{f})$ (by Remark 1.8).

The $T(\mathbf{Q})$-ambiguity in the choice of φ_σ implies that the $T(\mathbf{Q})$-congruence class of the multiplier $\mu_{\sigma,\varphi_\sigma} \in T(\mathbf{A}_\mathrm{f})$ does not depend on the choice of φ_σ, and by Lemma 1.3 it does not depend on the choice of ϕ. Since the pair (A, i) of type (K, Φ) is unique up to K-linear isogeny, it follows that the choice of (A, i) does not affect the element we have just built in $T(\mathbf{Q})\backslash T(\mathbf{A}_{\mathrm{f}})$. Hence, we get a map of sets

$$\mathrm{Gal}(\overline{\mathbf{Q}}/K^*) \to T(\mathbf{Q})\backslash T(\mathbf{A}_{\mathrm{f}}) \tag{2.2}$$

by sending $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$ to the common $T(\mathbf{Q})$-congruence class of elements $\mu_{\sigma,\varphi_\sigma} \in T(\mathbf{A}_{\mathrm{f}})$ for which the composite map in (2.1) is multiplication by $\mu_{\sigma,\varphi_\sigma}$ with φ_σ as above, and the map (2.2) only depends on (K, Φ) rather than on (A, i).

Lemma 2.1. *The map* (2.2) *is a continuous group homomorphism.*

Proof. We first check that it is a group homomorphism. Choose $\sigma, \sigma' \in \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$, and pick φ_σ and $\varphi_{\sigma'}$. We need to express the isomorphism

$$\mathrm{V}_{\mathrm{f}}(A) \overset{[\sigma]}{\simeq} \mathrm{V}_{\mathrm{f}}(A^\sigma) \overset{\mathrm{V}_{\mathrm{f}}(\varphi_\sigma)}{\simeq} \mathrm{V}_{\mathrm{f}}(A) \overset{[\sigma']}{\simeq} \mathrm{V}_{\mathrm{f}}(A^{\sigma'}) \overset{\mathrm{V}_{\mathrm{f}}(\varphi_{\sigma'})}{\simeq} \mathrm{V}_{\mathrm{f}}(A)$$

as $\varphi_{\sigma'\sigma} \circ [\sigma'\sigma]$ for a suitable choice of isogeny $\varphi_{\sigma'\sigma}$. The isogeny $A^{\sigma'\sigma} \simeq (A^\sigma)^{\sigma'} \xrightarrow{(\varphi_\sigma)^{\sigma'}} A^{\sigma'} \xrightarrow{\varphi_{\sigma'}} A$ carries ϕ back to $\phi^{\sigma'\sigma}$ up to positive rational multiple and is K-linear, so we may take this composite isogeny as our *choice* of $\varphi_{\sigma'\sigma}$. Thus, our problem is to prove the commutativity of the outside edge of the diagram

$$\begin{tikzcd}
\mathrm{V}_{\mathrm{f}}(A) \arrow[r, "{[\sigma]}", "\simeq"'] \arrow[dr, "{[\sigma'\sigma]}"'] & \mathrm{V}_{\mathrm{f}}(A^\sigma) \arrow[r, "\varphi_\sigma"] \arrow[d, "{[\sigma']}"'] & \mathrm{V}_{\mathrm{f}}(A) \arrow[d, "{[\sigma']}"] \\
& \mathrm{V}_{\mathrm{f}}(A^{\sigma'\sigma}) \arrow[r, "(\varphi_\sigma)^{\sigma'}"] \arrow[dr, "\varphi_{\sigma'}\circ(\varphi_\sigma)^{\sigma'}"'] & \mathrm{V}_{\mathrm{f}}(A^{\sigma'}) \arrow[d, "\varphi_{\sigma'}"] \\
& & \mathrm{V}_{\mathrm{f}}(A)
\end{tikzcd}$$

Since the left and bottom triangles clearly commute, it is therefore enough to prove that the inner square commutes. This amounts to the obvious identity $f^\tau(\tau(x')) = \tau(f(x'))$ for any $\overline{\mathbf{Q}}$-map $f : X' \to X$ between $\overline{\mathbf{Q}}$-schemes and any $x' \in X'(\overline{\mathbf{Q}})$.

We have verified the group homomorphism condition, so to check continuity it is enough to do so near the identity. Thus, we can restrict attention to $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/L)$ for a number field $L \subseteq \overline{\mathbf{Q}}$ over which (A, i, ϕ) is "defined" (any such L must contain K^*). Let (A_0, i_0, ϕ_0) be a descent of (A, i, ϕ) to such an L. This choice of descent naturally identifies A^σ with A carrying i^σ to i and ϕ^σ to ϕ. Taking φ_σ to be this canonical identification $A^\sigma \simeq A$ associated to $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/L)$, the element $\mu_{\sigma,\varphi_\sigma} \in T(\mathbf{A}_{\mathrm{f}}) \subseteq \mathbf{A}_{K,\mathrm{f}}^\times$ is the multiplier for the $\mathbf{A}_{K,\mathrm{f}}$-linear action by $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/L)$ on $\mathrm{V}_{\mathrm{f}}(A_0)$. Thus, the continuity of (2.1) on the open subgroup $\mathrm{Gal}(\overline{\mathbf{Q}}/L) \subseteq \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$ follows from the continuity of the Galois representation on the total Tate module of any abelian variety over a field. ■

Lemma 2.2. *The quotient space* $T(\mathbf{Q})\backslash T(\mathbf{A}_{\mathrm{f}})$ *is Hausdorff.*

Proof. We shall prove that $T(\mathbf{Q})$ is a discrete subgroup of $T(\mathbf{A}_{\mathrm{f}})$. Since $T(\mathbf{A}_{\mathrm{f}})$ is a second countable Hausdorff space, we can test discreteness by studying sequences that converge to the identity. Consider the short exact sequence

$$1 \to \mathbf{G}_m \to T \to \overline{T} \to 1.$$

Granting for a moment that $\overline{T}(\mathbf{Q})$ is discrete in $\overline{T}(\mathbf{A}_{\mathrm{f}})$, it follows that a sequence in $T(\mathbf{Q})$ tending to 1 in $T(\mathbf{A}_{\mathrm{f}})$ has image in $\overline{T}(\mathbf{Q})$ that stabilizes at 1. Such a sequence is eventually a sequence in $\mathbf{G}_m(\mathbf{Q}) = \mathbf{Q}^\times$ tending to 1 in $T(\mathbf{A}_{\mathrm{f}})$ and hence in $\mathbf{G}_m(\mathbf{A}_{\mathrm{f}}) = \mathbf{A}_{\mathrm{f}}^\times$. But $\mathbf{Q}^\times$ is discrete in $\mathbf{A}_{\mathrm{f}}^\times$ (as it meets the compact open subgroup $\widehat{\mathbf{Z}}^\times$ in $\{\pm 1\}$), so we would be done.

It remains to prove that $\overline{T}(\mathbf{Q})$ is discrete in $\overline{T}(\mathbf{A}_{\mathrm{f}})$. By [R, Thm. 3.1], it is enough to prove that $\overline{T}(\mathbf{R})$ is compact. (Note that $T(\mathbf{R})$ is not compact.) By Hilbert's Theorem 90 we have $\mathrm{H}^1(\mathbf{C}/\mathbf{R}, \mathbf{G}_m) = 1$, so the natural map $T(\mathbf{R})/\mathbf{R}^\times \to \overline{T}(\mathbf{R})$ is a topological group isomorphism. We shall therefore prove that $T(\mathbf{R})/\mathbf{R}^\times$ is compact. By definition of T, the group $T(\mathbf{R})$ is the closed subgroup of elements

$$(z_1, \ldots, z_g) \in \prod_{K_0 \hookrightarrow \mathbf{R}} (K \otimes_{K_0} \mathbf{R})^\times$$

whose image in $(\prod_{K_0 \hookrightarrow \mathbf{R}} \mathbf{R})^\times$ under N_{K/K_0} lies in $\mathbf{R}^\times$. This is identified with the subgroup of points $(z_1, \ldots, z_g) \in (\mathbf{C}^\times)^g$ such that $|z_j|$ is independent of j, and by working modulo the diagonally embedded subgroup $\mathbf{R}^\times$ we conclude that $T(\mathbf{R})/\mathbf{R}^\times$ is a quotient of $(S^1)^g$ and so it is compact. ∎

By Lemma 2.2, the continuous group homomorphism $\mathrm{Gal}(\overline{\mathbf{Q}}/K^*) \to T(\mathbf{Q})\backslash T(\mathbf{A}_{\mathrm{f}})$ to an abelian target must factor through the *topological* abelianization of $\mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$. The continuous Artin map $(K^*)^\times\backslash\mathbf{A}_{K^*}^\times \to \mathrm{Gal}((K^*)^{\mathrm{ab}}/K^*)$ is a surjection and it identifies the abelianized Galois group with the topological group quotient of $(K^*)^\times\backslash\mathbf{A}_{K^*}^\times$ by the closure in $(K^*)^\times\backslash\mathbf{A}_{K^*}^\times$ of the image of the identity component of the archimedean part $(\mathbf{R} \otimes_{\mathbf{Q}} K^*)^\times$ [AT, Ch. IX]. Since the reflex field K^* is a CM field and hence has no real places, $(\mathbf{R} \otimes_{\mathbf{Q}} K^*)^\times \simeq \prod_{w|\infty} (K_w^*)^\times$ is connected. The infinite divisibility of $(K_w^*)^\times$ for each $w|\infty$ on K^* therefore implies that the restricted Artin map $(K^*)^\times\backslash\mathbf{A}_{K^*,\mathrm{f}}^\times \to \mathrm{Gal}((K^*)^{\mathrm{ab}}/K^*)$ is surjective and identifies the abelianized Galois group with the maximal Hausdorff group quotient of $(K^*)^\times\backslash\mathbf{A}_{K^*,\mathrm{f}}^\times$ (i.e., the quotient by the closure of the identity point). Hence, we have built a natural continuous composite mapping of topological groups

$$(K^*)^\times\backslash\mathbf{A}_{K^*,\mathrm{f}}^\times \twoheadrightarrow \mathrm{Gal}((K^*)^{\mathrm{ab}}/K^*) \to T(\mathbf{Q})\backslash T(\mathbf{A}_{\mathrm{f}}) \tag{2.3}$$

that only depends on the CM type (K, Φ) and not on the paritcular triple (A, i, ϕ) used to build it in the first place. It is therefore natural to demand a direct description of (2.3) in terms of (K, Φ).

Remark 2.3. Note that *any* continuous group homomorphism

$$h : (K^*)^\times\backslash\mathbf{A}_{K^*,\mathrm{f}}^\times \to T(\mathbf{Q})\backslash T(\mathbf{A}_{\mathrm{f}})$$

must factor continuously through the surjective Artin quotient map $(K^*)^\times\backslash\mathbf{A}_{K^*,\mathrm{f}}^\times \twoheadrightarrow \mathrm{Gal}((K^*)^{\mathrm{ab}}/K^*)$. Indeed, h is a mapping to a Hausdorff target group $T(\mathbf{Q})\backslash T(\mathbf{A}_{\mathrm{f}})$, and so it factors continuously through the maximal Hausdorff group quotient of the source.

We may rewrite the composite map (2.3) in the form

$$\mathrm{Res}_{K^*/\mathbf{Q}}(\mathbf{G}_m)(\mathbf{Q})\backslash\mathrm{Res}_{K^*/\mathbf{Q}}(\mathbf{G}_m)(\mathbf{A}_{\mathrm{f}}) \to T(\mathbf{Q})\backslash T(\mathbf{A}_{\mathrm{f}}).$$

Now recall from [K, §2] that to the CM type (K, Φ) we associated the reflex norm $\mathrm{N}_\Phi : \mathrm{Res}_{K^*/\mathbf{Q}}(\mathbf{G}_m) \to T$. Hence, it is reasonable to ask if the maps induced by the reflex norm on finite-adelic and rational points have any relation to the composite mapping (2.3). This is the content of the following Galois-theoretic formulation of the Main Theorem of complex multiplication:

Theorem 2.4 (Main Theorem of CM; algebraic form)**.** *The mapping* (2.3) *is induced by* $s \mapsto \mathrm{N}_\Phi(s)^{-1} \in T(\mathbf{A}_{\mathrm{f}}) \subseteq \mathbf{A}_{K,\mathrm{f}}^\times$ *for* $s \in \mathbf{A}_{K^*,\mathrm{f}}^\times$.

There is a further aspect to the Main Theorem: giving an adelic description of $(A^\sigma, i^\sigma, \phi^\sigma)$ in terms of (A, i, ϕ). We shall address this aspect in Theorem 5.2 in the case of CM order $\mathscr{O}_K$ and in Theorem 6.8 in the general case.

Remark 2.5. The reason $\mathrm{N}_\Phi(s)^{-1}$ rather than $\mathrm{N}_\Phi(s)$ appears in the statement of the Main Theorem is due to our convention for defining the Artin map: it carries local uniformizers to arithmetic Frobenius elements. (See above Remark 4.3.) If we were to use the algebraic geometry convention that the Artin map carries local uniformizers to geometric Frobenius elements then the identification of $\mathrm{Gal}((K^*)^{\mathrm{ab}}/K^*)$ with the maximal Hausdorff group quotient of $(K^*)^\times\backslash\mathbf{A}_{K^*,\mathrm{f}}^\times$ would be modified by inversion and so the formula in the Main Theorem would lose the inversion.

3. Some applications

Before we take up the proof of the Main Theorem, we deduce some consequences. Pick $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$ and choose $s \in \mathbf{A}_{K^*,\mathrm{f}}^\times$ such that its image $(s|K^*) \in \mathrm{Gal}((K^*)^{\mathrm{ab}}/K^*)$ under the Artin map is $\sigma|_{(K^*)^{\mathrm{ab}}}$. By Remark 2.3, $\mathrm{N}_\Phi(s)^{-1} \in T(\mathbf{A}_\mathrm{f})$ has class modulo $T(\mathbf{Q})$ that only depends on $(s|K^*) = \sigma|_{(K^*)^{\mathrm{ab}}} \in \mathrm{Gal}((K^*)^{\mathrm{ab}}/K^*)$ and not on the choice its lifting s through the Artin map. By the Main Theorem, $\mathrm{N}_\Phi(s)^{-1} \in \mathbf{A}_{K,\mathrm{f}}^\times = \mathrm{Aut}_{\mathbf{A}_{K,\mathrm{f}}}(\mathrm{V}_\mathrm{f}(A))$ agrees modulo $T(\mathbf{Q})$ with the composite in (2.1) for any choice of K-linear isogeny $\varphi_\sigma : A^\sigma \to A$ carrying ϕ back to a (necessarily positive) rational multiple of ϕ^σ. Since this φ_σ is unique up to precisely the $T(\mathbf{Q})$-action on A, we may *uniquely* choose φ_σ to get the exact agreement: $\mathrm{N}_\Phi(s)^{-1} = \mathrm{V}_\mathrm{f}(\varphi_\sigma) \circ [\sigma]$.

To summarize, for all $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$ and $s \in \mathbf{A}_{K^*,\mathrm{f}}^\times$ such that $(s|K^*) = \sigma|_{(K^*)^{\mathrm{ab}}}$ there is a unique K-linear isogeny $\lambda_{\sigma,s} : (A, i) \to (A^\sigma, i^\sigma)$ such that $\mathrm{N}_\Phi(s^{-1}) \cdot \mathrm{V}_\mathrm{f}(\lambda_{\sigma,s}) = [\sigma]$, where $[\sigma] : \mathrm{V}_\mathrm{f}(A) \simeq \mathrm{V}_\mathrm{f}(A^\sigma)$ is the natural map defined by σ-action on torsion points. Moreover, by the construction this $\lambda_{\sigma,s}$ satisfies $\deg(\lambda_{\sigma,s})^{1/g} \in \mathbf{Q}^\times$ and it carries ϕ^σ back to to $\deg(\lambda_{\sigma,s})^{1/g}\phi$ for all K-linear polarizations ϕ of (A, i). Since $[\sigma]$ induces an isomorphism on underlying total Tate modules, the rational multiplier $\deg(\lambda_{\sigma,s})^{1/g}$ is easy to compute by formation of adelic degree on the identity $\mathrm{N}_\Phi(s^{-1})\mathrm{V}_\mathrm{f}(\lambda_{\sigma,s}) = [\sigma]$: it is the gth root of the generalized adelic lattice index

$$[\mathrm{T}_\mathrm{f}(A) : \mathrm{N}_\Phi(s)\mathrm{T}_\mathrm{f}(A)] = \mathrm{N}_{K/\mathbf{Q}}(\mathrm{N}_\Phi(s)) = \mathrm{N}_{K_0/\mathbf{Q}}(\mathrm{N}_\Phi(s)\mathrm{N}_{\Phi^*}(s)) = \mathrm{N}_{K_0/\mathbf{Q}}(\mathrm{N}_{K^*/\mathbf{Q}}(s)) = \mathrm{N}_{K^*/\mathbf{Q}}(s)^g$$

(the third equality follows from the fact that N_Φ factors through the subtorus $T \subseteq \mathrm{Res}_{K/\mathbf{Q}}(\mathbf{G}_m)$; see [K, Prop. 2.5]).

In other words, if we let $q_s \in \mathbf{Q}_{>0}^\times$ be the unique positive generator of the fractional $\mathbf{Q}$-ideal associated to the finite $\mathbf{Q}$-idele $\mathrm{N}_{K^*/\mathbf{Q}}(s)$ then $\lambda_{\sigma,s}$ carries ϕ^σ back to $q_s\phi$. We can describe q_s an a finite idele rather directly: $q_s = \chi(\sigma)\mathrm{N}_{K^*/\mathbf{Q}}(s)$ inside $\mathbf{A}_\mathrm{f}^\times$, where $\chi : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \widehat{\mathbf{Z}}^\times$ is the total cyclotomic character. Indeed, passing from $\mathbf{Q}$-ideles to fractional $\mathbf{Q}$-ideals gives the result provided that $\chi(\sigma)\mathrm{N}_{K^*/\mathbf{Q}}(s) \in \mathbf{Q}_{>0}^\times$ inside of $\mathbf{A}_\mathrm{f}^\times$. By functoriality of the Artin map with respect to the extension $\mathbf{Q} \to K^*$, the image of $\mathrm{N}_{K^*/\mathbf{Q}}(s)$ in $\mathrm{Gal}(\mathbf{Q}^{\mathrm{ab}}/\mathbf{Q})$ is $\sigma|_{\mathbf{Q}^{\mathrm{ab}}}$, and by our convention for the local Artin map the global Artin map for $\mathbf{Q}$ identifies $\mathrm{Gal}(\mathbf{Q}^{\mathrm{ab}}/\mathbf{Q})$ with $\widehat{\mathbf{Z}}^\times = \mathbf{Q}^\times\backslash\mathbf{A}_\mathbf{Q}^\times/\mathbf{R}_{>0}^\times = \mathbf{Q}_{>0}^\times\backslash\mathbf{A}_{\mathbf{Q},\mathrm{f}}^\times$ via the *reciprocal* of the cyclotomic character. Hence, $\mathrm{N}_{K^*/\mathbf{Q}}(s)$ and $\chi(\sigma)^{-1}$ represent the same coset in $\mathbf{Q}_{>0}^\times\backslash\mathbf{A}_{\mathbf{Q},\mathrm{f}}^\times$, as desired.

Remark 3.1. If we replace s with cs for $c \in (K^*)^\times$, then since $\mathrm{N}_\Phi(cs) = \mathrm{N}_\Phi(c)\mathrm{N}_\Phi(s)$ with $\mathrm{N}_\Phi(c) \in T(\mathbf{Q}) \subseteq K^\times$ we conclude by uniqueness that $\lambda_{\sigma,cs} = \mathrm{N}_\Phi(c)\lambda_{\sigma,s}$.

Let $L \subseteq \overline{\mathbf{Q}}$ be a number field that is a field of definition for (A, i), so $K^* \subseteq L$. We let (A_0, i_0) be a descent of (A, i) to L. The continuous group homomorphism

$$\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/L) \to \mathrm{Aut}_{\mathbf{A}_{K,\mathrm{f}}}(\mathrm{V}_\mathrm{f}(A_0)) \simeq \mathbf{A}_{K,\mathrm{f}}^\times$$

uniquely factors through some $\rho^{\mathrm{ab}} : \mathrm{Gal}(L^{\mathrm{ab}}/L) \to \mathbf{A}_{K,\mathrm{f}}^\times$. Pick $s' \in \mathbf{A}_{L,\mathrm{f}}^\times$, so $(s'|L) \in \mathrm{Gal}(L^{\mathrm{ab}}/L)$ acts on $\mathrm{V}_\mathrm{f}(A_0)$ via $\rho^{\mathrm{ab}}((s'|L))$. We want to describe this action:

Theorem 3.2. *For $s' \in \mathbf{A}_{L,\mathrm{f}}^\times$ there is a unique $\lambda_{s'} \in K^\times$ such that $\rho^{\mathrm{ab}}((s'|L)) = \mathrm{N}_\Phi(\mathrm{N}_{L/K^*}(s'))^{-1} \cdot \lambda_{s'}$ in $\mathbf{A}_{K,\mathrm{f}}^\times$. Moreover, $s' \mapsto \lambda_{s'} \in K^\times$ is continuous for the discrete topology on $K^\times$.*

Proof. By functoriality of the Artin symbol, the restriction of $(s'|L)$ to $(K^*)^{\mathrm{ab}} \subseteq L^{\mathrm{ab}}$ is $(\mathrm{N}_{L/K^*}(s')|K^*)$. By working over $\overline{\mathbf{Q}}$ we may find some ϕ as above, and upon choosing $\sigma' \in \mathrm{Gal}(\overline{\mathbf{Q}}/L) \subseteq \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$ lifting $(s'|L)$ we get that $\sigma'|_{(K^*)^{\mathrm{ab}}} = (s|K^*)$ for $s = \mathrm{N}_{L/K^*}(s')$. Using the L-structure on (A, i) provided by the identification $A \simeq \overline{\mathbf{Q}} \otimes_L A_0$, we get a canonical K-linear isomorphism $A^{\sigma'} \simeq A$ and hence a K-linear isogeny $\lambda_{\sigma',s} : A \to A^{\sigma'} \simeq A$ such that $\mathrm{N}_\Phi(s)^{-1} \cdot \mathrm{V}_\mathrm{f}(\lambda_{\sigma',s'}) = \rho(\sigma')$ as endomorphisms of $\mathrm{V}_\mathrm{f}(A)$. Since $\rho(\sigma') = \rho^{\mathrm{ab}}((s'|L))$, we conclude that $\lambda_{\sigma',s'}$ only depends on s' and not on σ'. But K is its own centralizer in $\mathrm{End}_\mathbf{Q}^0(A)$, so the K-linear $\lambda_{\sigma',s'}$ is multiplication by an element $\lambda_{s'} \in K^\times$. This completes the construction of $\lambda_{s'} \in K^\times$ such that $\rho^{\mathrm{ab}}((s'|L)) = \mathrm{N}_\Phi(\mathrm{N}_{L/K^*}(s))^{-1} \cdot \lambda_{s'}$ in $\mathbf{A}_{K,\mathrm{f}}^\times$.

It remains to prove continuity of $\lambda_{s'}$ with respect to the discrete topology on $K^\times$. That is, we want $\lambda_{s'} = 1$ for $s' \in \mathbf{A}_{L,\mathrm{f}}^\times$ sufficiently near 1. Since $\lambda_{s'} = \rho^{\mathrm{ab}}((s'|L))\mathrm{N}_\Phi(\mathrm{N}_{L/K^*}(s')) \in \mathbf{A}_{K,\mathrm{f}}^\times$, the mapping $s' \mapsto \lambda_{s'} \in K^\times$ is continuous for the topology on $K^\times$ induced by its inclusion into the group $\mathbf{A}_{K,\mathrm{f}}^\times$ of finite K-ideles. In particular, for s' sufficiently near 1 in $\mathbf{A}_{L,\mathrm{f}}^\times$ and a fixed choice of positive integer $M \geq 3$ we have several properties: $\lambda_{s'} \in \mathscr{O}_K^\times$, $\lambda_{s'} \equiv 1 \bmod M$, and $\lambda_{s'}$ lies in the CM order $\mathscr{O}_K \cap \mathrm{End}_{\overline{\mathbf{Q}}}(A)$. Hence, for such s' the element $\lambda_{s'}$ is an automorphism of the abelian variety A acting trivially on the M-torsion. If we can prove that $\lambda_{s'}$ has finite order then since $M \geq 3$ such an automorphism must be the identity and so we will be done. Pick a K-linear $\mathbf{Q}$-polarization ϕ for (A, i). Taking $s' \in \mathbf{A}_{L,\mathrm{f}}^\times$ sufficiently near 1, we can arrange that $(s'|L) \in \mathrm{Gal}(L^{\mathrm{ab}}/L)$ has a lift to $\mathrm{Gal}(\overline{\mathbf{Q}}/L)$ that acts trivially on a number field of definition for ϕ over L (after applying a base change to (A_0, i_0)). It therefore follows from the construction of $\lambda_{s'}$ that for such s' the element $\lambda_{s'} \in K^\times$ acts as an automorphism of A preserving ϕ up to a positive rational multiple. Degree considerations force this positive rational multiplier to be 1, and by [C1, Remark 3.5], the automorphism group of a $\mathbf{Q}$-polarized abelian variety is finite. Hence, $\lambda_{s'}$ has finite order as desired. ∎

We next use the Main Theorem (or rather, its consequence in Theorem 3.2) to deduce some results concerning L-functions. We now eliminate the mention of $\overline{\mathbf{Q}}$ and work with an "abstract" number field as the base field. Let A be an abelian variety of dimension g over a number field L and let $i : K \to \mathrm{End}_L^0(A)$ be a CM structure on A over L. The extension $L(A_{\mathrm{tor}})/L$ is abelian, so it uniquely embeds into L^{ab} over L. The field L has no real places because it contains a CM field: upon embedding L into $\overline{\mathbf{Q}}$ to define a CM type Φ for (A, i), we know that L contains the reflex field K^* for (K, Φ) and K^* is a CM field. (The reflex subfield in L depends on the $\overline{\mathbf{Q}}$-embedding of L in general.) Hence, $L_\infty^\times = (\mathbf{R} \otimes_{\mathbf{Q}} L)^\times$ is connected and infinitely divisible. The associated *CM character* is the map

$$\alpha_{(A,i)/L} : \mathbf{A}_L^\times \twoheadrightarrow \mathrm{Gal}(L^{\mathrm{ab}}/L) \to \mathbf{A}_{K,\mathrm{f}}^\times$$

defined by the Galois action on torsion of A. The image of $L_\infty^\times$ in $\mathrm{Gal}(L^{\mathrm{ab}}/L)$ is trivial, so $\alpha_{(A,i)/L}(s)$ only depends on the finite component $s_{\mathrm{f}} \in \mathbf{A}_{L,\mathrm{f}}^\times$ for any $s \in \mathbf{A}_L^\times$.

Recall from [K, Prop. 2.5] that the reflex norm $\mathrm{N}_\Phi : \mathrm{Res}_{K^*/\mathbf{Q}}(\mathbf{G}_m) \to T \subseteq \mathrm{Res}_{K/\mathbf{Q}}(\mathbf{G}_m)$ may be defined as the K-determinant of the $(K^*)^\times$-action on the $K \otimes_{\mathbf{Q}} K^*$-module t_Φ that descends the $K \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$-module $\prod_{\varphi \in \Phi} \overline{\mathbf{Q}}$ (as in [C2, §1]). The specification of the CM type Φ and the reflex field $K^* \subseteq L$ depend on a choice of embedding of L into $\overline{\mathbf{Q}}$, but the composite map

$$\mathrm{N}_{\Phi_L} : \mathrm{Res}_{L/\mathbf{Q}}(\mathbf{G}_m) \stackrel{\mathrm{N}_{L/K^*}}{\to} \mathrm{Res}_{K^*/\mathbf{Q}}(\mathbf{G}_m) \stackrel{\mathrm{N}_\Phi}{\to} \mathrm{Res}_{K/\mathbf{Q}}(\mathbf{G}_m)$$

is *independent* of the embedding of L into $\overline{\mathbf{Q}}$. The content here is the identity $\mathrm{N}_{\tau\Phi}(\tau s) = \mathrm{N}_\Phi(s)$ in $\mathbf{A}_K^\times$ for $s \in \mathbf{A}_{K^*}^\times$ and $\tau \in \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$, and by Zariski-density it suffices to prove

$$\det_K(\tau(x) : t_{\tau\Phi} \simeq t_{\tau\Phi}) = \det_K(x : t_\Phi \simeq t_\Phi)$$

in K for all $x \in K^*$.

The natural map $\langle\tau\rangle : \prod_{\varphi \in \Phi} \overline{\mathbf{Q}} \simeq \prod_{\psi \in \tau\Phi} \overline{\mathbf{Q}}$ defined by $(x_\varphi) \mapsto (\tau(x_{\tau^{-1}\psi}))$ carries the "descent data" action $(x_\varphi) \mapsto (g(x_{g^{-1}\varphi}))$ by $g \in \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$ over to the action by $\tau g \tau^{-1} \in \mathrm{Gal}(\overline{\mathbf{Q}}/\tau(K^*))$ and respects the K-actions (but carries multiplication by $y \in K^*$ over to multiplication by $\tau(y) \in \tau(K^*)$), so the isomorphism $\langle\tau\rangle$ descends to an isomorphism $t_\Phi \simeq t_{\tau\Phi}$ linear over the ring isomorphism $1 \otimes \tau : K \otimes_{\mathbf{Q}} K^* \simeq K \otimes_{\mathbf{Q}} \tau(K^*)$. The equality of K-determinants for the x-action on t_Φ and the $\tau(x)$-action on $t_{\tau\Phi}$ therefore drops out (for any $x \in K^*$).

Remark 3.3. By Theorem 3.2, for each $s \in \mathbf{A}_L^\times$ there is a unique $\lambda_s \in K^\times$ depending only on s_{f} such that

$$\alpha_{(A,i)/L}(s) = \lambda_s \mathrm{N}_{\Phi_L}(s_{\mathrm{f}})^{-1},$$

and we have $\lambda_s = 1$ if s_{f} is sufficiently close to 1. Since $\alpha_{(A,i)/L}$ kills $L^\times$, it also follows that for $c \in L^\times$, $\lambda_c = \mathrm{N}_{\Phi_L}(c)$ in $K^\times$. Consideration of quadratic twists shows that $\alpha_{(A,i)/L}$ generally depends on (A, i) over L and not just on its $\overline{\mathbf{Q}}$-fiber (i.e., not just on (K, Φ)).

Remark 3.4. Two elementary properties of the construction $s \mapsto \lambda_s$ are that for $s \in \mathbf{A}_L^\times$, $[\mathrm{N}_{\Phi_L}(s_{\mathrm{f}})]_K = \lambda_s \mathscr{O}_K$ and $\lambda_s \lambda_s^* = \mathrm{N}_{L/\mathbf{Q}}(s_{\mathrm{f}}) \alpha_{(A,i)/L}(s) \alpha_{(A,i)/L}^*(s)$ in $\mathbf{A}_{\mathrm{f}}^\times$. The second identity follows from picking an embedding of L into $\overline{\mathbf{Q}}$ (so as to get a CM type Φ) and using the identity $\mathrm{N}_\Phi \cdot \mathrm{N}_{\Phi^*} = \mathrm{N}_{K^*/\mathbf{Q}}$ and transitivity of norms. To prove the first identity, first note that the automorphism $\mathrm{V}_{\mathrm{f}}(\lambda_s \cdot \mathrm{N}_{\Phi_L}(s_{\mathrm{f}}^{-1})) = [\sigma]$ of $\mathrm{V}_{\mathrm{f}}(A)$ is an automorphism of the total Tate module $\mathrm{T}_{\mathrm{f}}(A)$ and hence is an automorphism of the free rank-1 module it generates over the ring $\mathscr{O}_K^\wedge = \prod_{v \nmid \infty} \mathscr{O}_{K_v}$ of integral K-adeles. Hence, the fractional K-ideals $\lambda_s \mathscr{O}_K$ and $[\mathrm{N}_{\Phi_L}(s_{\mathrm{f}})]_K$ have the same ord's at all finite places of K, so these ideals coincide.

Theorem 3.5. *Let (A, i) be a CM abelian variety over a number field L. Let $\lambda : \mathbf{A}_L^\times \to K^\times$ be the character $s \mapsto \lambda_s$ that is trivial on $L_\infty^\times$ and continuous for the discrete topology on K. Pick a prime $\mathfrak{P}$ of L.*

1. *The abelian variety A has good reduction at $\mathfrak{P}$ if and only if $\lambda_{\mathfrak{P}} = \lambda|_{L_{\mathfrak{P}}^\times}$ is trivial on $\mathscr{O}_{L_{\mathfrak{P}}}^\times$.*
2. *For primes $\mathfrak{P}$ of good reduction, $\lambda_{\mathfrak{P}}(\pi_{\mathfrak{P}}) \in K^\times$ lies in $\mathscr{O}_K$ and in the isogeny category over $\kappa(\mathfrak{P})$ it acts on the reduction $\overline{A}_{/\kappa(\mathfrak{P})}$ as the $q_{\mathfrak{P}}$-Frobenius endomorphism (with $\pi_{\mathfrak{P}}$ a uniformizer at $\mathfrak{P}$).*

Proof. To check good reduction at $\mathfrak{P}$, we choose a rational prime ℓ distinct from the residue characteristic of $\mathfrak{P}$ and we need to determine if the action of an inertia group $I_{\mathfrak{P}}$ at $\mathfrak{P}$ is trivial on $\mathrm{V}_\ell(A)$. The image of $I_{\mathfrak{P}}$ in $\mathrm{Gal}(L^{\mathrm{ab}}/L)$ is the image of $\mathscr{O}_{L_{\mathfrak{P}}}^\times$ under the Artin map, so it comes from ideles $s \in \mathbf{A}_L^\times$ with trivial ℓ-part. Hence, the formula

$$\alpha_{(A,i)/L}(s) = \lambda_s \mathrm{N}_{\Phi_L}(s_{\mathrm{f}})^{-1} \tag{3.1}$$

implies that $\alpha_{(A,i)/L}(s)$ has trivial ℓ-part for all $s \in \iota_{\mathfrak{P}}(\mathscr{O}_{L_{\mathfrak{P}}}^\times)$ if and only if $\lambda_s \in K^\times$ viewed in $K_\ell^\times$ is trivial for all $s \in \iota_{\mathfrak{P}}(\mathscr{O}_{L_{\mathfrak{P}}}^\times)$. That is, A has good reduction at $\mathfrak{P}$ if and only if $\lambda_{\mathfrak{P}}$ is trivial on $\mathscr{O}_{K_{\mathfrak{P}}^*}^\times$.

Now choose $\mathfrak{P}$ that does have good reduction, and pick ℓ as above. The preceding calculation shows that for a local uniformizer $\pi_{\mathfrak{P}}$ the Frobenius action at $\mathfrak{P}$ on $\mathrm{V}_\ell(A)$ is equal to the action by $\lambda_{\mathfrak{P}}(\pi_{\mathfrak{P}}) \in K^\times$. Passing to the reduction $\overline{A}$ at $\mathfrak{P}$, it follows that the action by $\lambda_{\mathfrak{P}}(\pi_{\mathfrak{P}})$ on $\mathrm{V}_\ell(\overline{A})$ agrees with the action by the $q_{\mathfrak{P}}$-Frobenius endomorphism, and so as elements of $\mathrm{End}^0_{\kappa(\mathfrak{P})}(\overline{A}_0)$ the element $\lambda_{\mathfrak{P}}(\pi_{\mathfrak{P}}) \in K^\times$ coincides with this Frobenius endomorphism. In particular, it is a genuine endomorphism of $\overline{A}$ (not just in the isogeny category) and so is integral over $\mathbf{Z}$. Thus, as an element of K it lies in $\mathscr{O}_K$. ∎

For any embedding $\tau : K \hookrightarrow \mathbf{C}^\times$, let $\lambda^\tau = \tau \circ \lambda$ with λ as in Theorem 3.5. Define $\alpha_\infty = \mathrm{N}_{\Phi_L,\infty}^{-1} \cdot \lambda : \mathbf{A}_L^\times \to K_\infty^\times$, where $\mathrm{N}_{\Phi_L,\infty}$ is the composite of $\mathrm{N}_{\Phi_L} : \mathbf{A}_L^\times \to \mathbf{A}_K^\times$ and the projection $\mathbf{A}_K^\times \twoheadrightarrow K_\infty^\times$. In particular, $\mathrm{N}_{\Phi_L,\infty}$ kills $\mathbf{A}_{L,\mathrm{f}}^\times$. Clearly α_∞ is continuous, and it kills $L^\times$ due to Remark 3.3. Hence, for each $\tau : K \hookrightarrow \mathbf{C}$ corresponding to an archimedean place v of K, the composite

$$\alpha^\tau : \mathbf{A}_L^\times \overset{\alpha_\infty}{\to} K_\infty^\times \twoheadrightarrow K_v^\times \overset{\tau}{\simeq} \mathbf{C}^\times$$

is a Hecke character and the restrictions $\alpha_{\mathfrak{P}}^\tau$ and $\lambda_{\mathfrak{P}}^\tau$ of α^τ and λ^τ to $L_{\mathfrak{P}}^\times \subseteq \mathbf{A}_L^\times$ coincide for all primes $\mathfrak{P}$ of L.

Remark 3.6. The restriction of the Hecke character α^τ to $L_\infty^\times$ is the continuous extension of the "algebraic" map $L^\times \overset{\mathrm{N}_{\Phi_L}}{\to} K^\times \overset{\tau}{\to} \mathbf{C}^\times$. More specifically, the $\mathbf{C}^\times$-valued α^τ (or, better, the $K^\times$-valued λ) is an algebraic Hecke character. Let us make this explicit. Let $\mathfrak{f}$ be an integral nonzero ideal of $\mathscr{O}_L$ such that $\lambda : \mathbf{A}_L^\times \to K^\times$ is trivial on the open subgroup $U_{\mathfrak{f}} \subseteq \mathbf{A}_{L,\mathrm{f}}^\times$ of finite ideles congruent to 1 modulo $\mathfrak{f}$, so the group $I(\mathfrak{f})$ of fractional ideals of L relatively prime to $\mathfrak{f}$ is naturally a quotient of $\mathbf{A}_{L,\mathrm{f}}^\times / U_{\mathfrak{f}}$. Hence, α^τ induces a well-defined homomorphism $[\alpha^\tau] : I(\mathfrak{f}) \to \tau(K)^\times \subseteq \mathbf{C}^\times$ that sends a prime $\mathfrak{P}$ to $\alpha_{\mathfrak{P}}^\tau(\pi_{\mathfrak{P}}) = \tau(\lambda_{\mathfrak{P}}(\pi_{\mathfrak{P}}))$ for any local uniformizer $\pi_{\mathfrak{P}}$ at $\mathfrak{P}$. On the subgroup $P(\mathfrak{f}) \subseteq I(\mathfrak{f})$ of principal fractional ideals of the form $x\mathscr{O}_L$ with $x \in L^\times$ satisfying $x \equiv 1 \bmod \mathfrak{f}$, we have

$$[\alpha^\tau](x\mathscr{O}_L) = \alpha^\tau(x_{\mathrm{f}}) = \alpha^\tau(x_\infty^{-1}) = \tau(\mathrm{N}_{\Phi_L,\infty}(x)).$$

For each embedding $\sigma : K^* \hookrightarrow \overline{\mathbf{Q}}$, pick $\widetilde{\sigma} \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ lifting σ via the canonical inclusion of K^* into $\overline{\mathbf{Q}}$ given in the definition of the reflex field K^*. Let $\Sigma(\Phi, \tau)$ be the set of $\sigma \in \mathrm{Hom}(K^*, \overline{\mathbf{Q}})$ such that $\tau = \widetilde{\sigma} \circ \varphi$

for some (necessarily unique) $\varphi \in \Phi$. Using [K, Remark 2.7], for $x\mathscr{O}_L \in P(\mathfrak{f})$ with $x \equiv 1 \bmod \mathfrak{f}$ we have

$$[\alpha^\tau](x\mathscr{O}_L) = \tau(\mathrm{N}_{\Phi_L,\infty}(x)) = \prod_{\sigma \in \Sigma(\Phi,\tau)} \sigma(\mathrm{N}_{L/K^*}(x)) = \prod_{\psi: L \to \overline{\mathbf{Q}}, \psi|_{K^*} \in \Sigma(\Phi,\tau)} \psi(x)$$

and (as we noted in Remark 3.4) the set Ψ_τ of ψ's in this final product satisfies $\prod_{\psi \in \Psi_\tau} \psi(x)\overline{\psi}(x) = \mathrm{N}_{L/\mathbf{Q}}(x)$ for all $x \in L^\times$. Thus, by Artin's lemma on linear independence of characters, Ψ_τ is a set of representatives for the archimedean places of L.

For each τ, Theorem 3.5(1) says that A has good reduction at $\mathfrak{P}$ if and only if α^τ is unramified at $\mathfrak{P}$. By Theorem 3.5(2) and the Riemann Hypothesis for abelian varieties over finite fields [Ca, §5], the Hecke character $\|\cdot\|_L^{-1/2}\alpha^\tau$ takes values in the unit circle. This unitary character is non-trivial, for otherwise by working with degree-1 primes of L it would follow from the Riemann Hypothesis for abelian varieties that K contains square roots of infinitely many odd rational primes, an absurdity. (It can happen for CM elliptic curves that this non-trivial unitary character has finite order, and even order 2.) Hence, the Euler products that define each $\mathrm{L}(s, \alpha^\tau)$ are absolutely and uniformly convergent in half-planes $\mathrm{Re}(s) \geq 3/2 + \varepsilon$ for all $\varepsilon > 0$, and extend to holomorphic functions on $\mathbf{C}$. These Hecke L-functions compute the L-function of A:

Theorem 3.7. *For $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 3/2$,*

$$\mathrm{L}(s, A_{/L}) = \prod_{\tau: K \to \mathbf{C}} \mathrm{L}(s, \alpha^\tau).$$

In particular, the L*-function for A over L has an analytic continuation to $\mathbf{C}$.*

Note that in the product in the theorem, we do repeat conjugate pairs of embeddings.

Proof. We compare Euler factors at good and bad primes separately. Let $\mathfrak{P}$ be a prime of good reduction for A, and pick ℓ distinct from the residue characteristic of $\mathfrak{P}$. Let $\lambda(\mathfrak{P}) \in K^\times$ be the common value of λ on local ideles coming from uniformizers at $\mathfrak{P}$, and likewise for λ^τ. By Theorem 3.5, we get

$$\det_{\mathbf{Q}_\ell}(1 - \lambda(\mathfrak{P})X) = \mathrm{N}_{K_\ell/\mathbf{Q}_\ell}(1 - \lambda(\mathfrak{P})X) = \mathrm{N}_{K/\mathbf{Q}}(1 - \lambda(\mathfrak{P})X) = \mathrm{N}_{\mathbf{C}\otimes_{\mathbf{Q}}K/K}(1 - (1 \otimes \lambda(\mathfrak{P}))X),$$

and under the natural decomposition $\mathbf{C} \otimes_{\mathbf{Q}} K \simeq \prod_\tau \mathbf{C}$ we get that this norm is $\prod_\tau (1 - \lambda^\tau(\mathfrak{P})X)$. This is the product of the Euler factors at $\mathfrak{P}$ for the right side of the proposed identity because α^τ and λ^τ have the same value on any idele with trivial archimedean component.

Now we turn to the bad primes for A. At such primes we know by Theorem 3.5(1) that λ is non-trivial on $\mathscr{O}_{L_\mathfrak{P}}^\times$ and hence all Hecke characters α^τ are ramified at $\mathfrak{P}$. Thus, the Euler factor at $\mathfrak{P}$ for the L-function of each of the α^τ's is trivial. We therefore have to prove that for ℓ as above, the abelian ℓ-adic representation $\mathrm{V}_\ell(A)$ for $\mathrm{Gal}(\overline{\mathbf{Q}}/L)$ has vanishing subspace of inertial invariants at $\mathfrak{P}$. Since $\lambda(\mathscr{O}_{L_\mathfrak{P}}^\times) \subseteq K^\times$ is nontrivial, it contains some element $x \in K^\times - \{1\}$. But $\lambda(\mathscr{O}_{L_\mathfrak{P}}^\times)$ viewed in $K_\ell^\times$ is the image of the ℓ-part of $\alpha_{(A,i)/L}$ on $\mathscr{O}_{L_\mathfrak{P}}^\times$ (due to (3.1)), so the subspace of inertial invariants at $\mathfrak{P}$ is contained in the subspace of $\mathrm{V}_\ell(A)$ killed by $x - 1 \in K^\times$. Hence, this subspace is 0. ■

Theorem 3.7 can be generalized to the case of L-functions of abelian varieties that are merely potentially CM (i.e., acquire a CM structure over an extension of the base field). Note that if there is a CM structure over the base field then a CM reflex field lies in the base field, so a potentially CM abelian variety over a real number field (such as $\mathbf{Q}$) cannot have a CM structure over the base field. These matters will be taken up in [A2] in the absolutely simple case (as well as in a slightly more general case).

In the case that the complex multiplication is defined over the base field L, we wish to record how $\lambda = \lambda_{(A,i)/L}$ behaves with respect to the action by $\mathrm{Aut}(K/\mathbf{Q})$.

Theorem 3.8. *For $\gamma \in \mathrm{Aut}(K/\mathbf{Q})$, $\lambda_{(A,i\circ\gamma^{-1})/L} = \gamma \circ \lambda_{(A,i)/L}$.*

In some references one also sees a discussion of behavior with respect to the action by $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ when L has been embedded into $\overline{\mathbf{Q}}$, namely $\lambda_{(A^\tau,i^\tau)/\tau(L)}(s^\tau) = \lambda_{(A,i)/L}(s)$. This is our earlier observation that the character $\lambda_{(A,i)/L} : \mathbf{A}_L^\times \to K^\times$ is intrinsic for an "abstract" L not embedded into $\overline{\mathbf{Q}}$.

Proof. Pick an embedding of L into $\overline{\mathbf{Q}}$, so we get a CM type Φ for (A,i) and a reflex field $K^* \subseteq L$. In this way, we canonically identify L^{ab} with a subfield of $\overline{\mathbf{Q}}$ over L. Pick $s \in \mathbf{A}_L^\times$ and $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/L)$ with $\sigma|_{L^{\mathrm{ab}}} = (s|L)$. We need to prove

$$\mathrm{V_f}(\lambda_{(A,i\circ\gamma^{-1})/L}(s)) \stackrel{?}{=} \mathrm{V_f}(\gamma(\lambda_{(A,i)/L}(s))) = \mathrm{V_f}(\gamma)(\mathrm{V_f}(\lambda_{(A,i)/L}(s)))$$

in $\mathbf{A}_{K,\mathrm{f}}^\times$ when the left and right sides are respectively viewed as self-maps of $\mathrm{V_f}(A)$ with $\mathbf{A}_{K,\mathrm{f}}$-module structures defined by $i\circ\gamma^{-1}$ and i respectively.

In other words, for $\mathrm{N}_{\Phi_L} \stackrel{\mathrm{def}}{=} \mathrm{N}_\Phi \circ \mathrm{N}_{L/K^*}$ and $\mathrm{N}_{(\Phi\circ\gamma)_L} \stackrel{\mathrm{def}}{=} \mathrm{N}_{\Phi\circ\gamma} \circ \mathrm{N}_{L/K^*}$ we need to prove

$$\mathrm{N}_{(\Phi\circ\gamma)_L}(s)\cdot(\mathrm{V_f}(i)\circ\gamma^{-1})^{-1}([\sigma]) \stackrel{?}{=} \mathrm{V_f}(\gamma)(\mathrm{N}_{\Phi_L}(s)\cdot \mathrm{V_f}(i)^{-1}([\sigma])) = \mathrm{V_f}(\gamma)(\mathrm{N}_{\Phi_L}(s))\cdot \mathrm{V_f}(\gamma)(\mathrm{V_f}(i)^{-1}([\sigma]))$$

(where $\mathrm{V_f}(i) : \mathbf{A}_{K,\mathrm{f}} \simeq \mathrm{End}_{\mathbf{A}_{K,\mathrm{f}}}(\mathrm{V_f}(A))$ is the natural isomorphism). The second factors on the left and right sides clearly agree, so we just have to show $\gamma^{-1}(\mathrm{N}_{(\Phi\circ\gamma)_L}(s)) = \mathrm{N}_\Phi(s)$ in $\mathbf{A}_{K,\mathrm{f}}^\times$ for all $s \in \mathbf{A}_L^\times$. Thus, we just have to prove that the isomorphism $\gamma : \mathbf{A}_K^\times \simeq \mathbf{A}_K^\times$ carries $\mathrm{N}_{\Phi\circ\gamma}$ to N_Φ as maps from $\mathbf{A}_{K^*}^\times$.

In view of the algebraic definition of the reflex norm as an K-determinant, the problem reduces to constructing an $K\otimes_{\mathbf{Q}} K^*$-module isomorphism $K \otimes_{\gamma^{-1},K} t_{\Phi\circ\gamma^{-1}} \simeq t_\Phi$. We apply the extension of scalars $K^* \to \overline{\mathbf{Q}}$ and just have to compare dimensions of isotypic factors for the K-action. By definition via descent,

$$K\otimes_{\gamma^{-1},K}(t_\Phi \otimes_{K^*} \overline{\mathbf{Q}}) \simeq K\otimes_{\gamma^{-1},K} \prod_{\psi\in\Phi\circ\gamma^{-1}} \overline{\mathbf{Q}}$$

is $\overline{\mathbf{Q}}$-linearly isomorphic to a product of copies of $\overline{\mathbf{Q}}$ indexed by $\psi \in \Phi\circ\gamma^{-1}$ such that K acts on the ψ-th copy via $\psi\circ\gamma$. Hence, the isotypic pieces are 1-dimensional over $\overline{\mathbf{Q}}$ and the eigencharacters are the elements of the set $(\Phi\circ\gamma^{-1})\circ\gamma = \Phi$. ■

4. Hom-modules and fractional ideals

We now begin the proof of the Main Theorem. By Remark 1.7 the choice of ϕ for (A,i) does not matter, and any two pairs (A,i) and (A',i') of type (K,Φ) over $\overline{\mathbf{Q}}$ are isogenous, so in fact the choice of triple (A,i,ϕ) of type (K,Φ) does not matter. Thus, we may and do take (A,i) to be principal. This opens the door to applying Serre's tensor construction and related results from [X]. Unlike adelic multiplication operations in the isogeny category that we shall develop in §6 when the CM order is not maximal, the Serre tensor construction is applicable at torsion levels. This is why we have passed to the principal case in the proof of the Main Theorem.

We pick $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$. Choose a finite Galois extension L/K^* inside of $\overline{\mathbf{Q}}$ such that (A,i) descends to a pair (A_0,i_0) over L and such that the K-linear $\mathbf{Q}$-polarization ϕ of A descends to an K-linear $\mathbf{Q}$-polarization ϕ_0 of A_0. (We will have no need for ϕ_0 until after (5.3), so the reader may safely forget about it until then.) Make L big enough so that it splits K over $\mathbf{Q}$ (i.e., it contains the Galois closure of K in $\overline{\mathbf{Q}}$) and so that

$$\mathrm{Hom}_L((A_0,i_0),(A_0^\sigma,i_0^\sigma)) = \mathrm{Hom}_{\overline{\mathbf{Q}}}((A,i),(A^\sigma,i^\sigma)).$$

In particular, this module of K-linear mappings over L is an invertible $\mathscr{O}_K$-module. We write $\mathfrak{a}_\sigma$ to denote this abstract invertible $\mathscr{O}_K$-module. It will soon be naturally identified with a fractional K-ideal (once we choose a suitable auxiliary prime of L).

Let N be the product of all primes ℓ of $\mathbf{Q}$ that arise as the residue characteristic of a prime factor of $\mathrm{disc}(L/\mathbf{Q})$ or a prime of bad reduction for A_0. (Note that if ℓ is ramified in K then ℓ is ramified in L and hence divides N, as L is assumed to split K over $\mathbf{Q}$.) We will be interested in primes $\mathfrak{P}$ of $\mathscr{O}_L[1/N]$ such that $\sigma|_L = (\frac{L/K^*}{\mathfrak{P}})$. Before we pick such a $\mathfrak{P}$, we record an important feature of working over $\mathbf{Z}[1/N]$: the $K\otimes_{\mathbf{Q}} K^*$-module t_Φ that arises in the definition of the reflex norm (descending the $K\otimes_{\mathbf{Q}}\overline{\mathbf{Q}}$-module $\prod_{\varphi\in\Phi}\overline{\mathbf{Q}}$) has a natural integral structure over $\mathbf{Z}[1/N]$. To be precise, first note that L contains all embeddings of K into $\overline{\mathbf{Q}}$ (due to the "largeness" of the choice of L), so we may first descend t_Φ to the $K\otimes_{\mathbf{Q}} L$-module $\prod_{\varphi\in\Phi} L$. Since the generically Galois extension $\mathscr{O}_{K^*}[1/N] \to \mathscr{O}_L[1/N]$ is finite étale, we may use étale descent in place of Galois descent to bring the $\mathscr{O}_K[1/N]\otimes_{\mathbf{Z}[1/N]}\mathscr{O}_L[1/N]$-module $\prod_{\varphi\in\Phi}\mathscr{O}_L[1/N]$ down to an $\mathscr{O}_K[1/N]\otimes_{\mathbf{Z}[1/N]}\mathscr{O}_{K^*}[1/N]$-module $t_{\Phi,\mathbf{Z}[1/N]}$ such that $\mathbf{Q}\otimes_{\mathbf{Z}[1/N]} t_{\Phi,\mathbf{Z}[1/N]} \simeq t_\Phi$ as $K\otimes_{\mathbf{Q}} L$-modules. In particular, $t_{\Phi,\mathbf{Z}[1/N]}$ is a finite locally free module over each of $\mathscr{O}_K[1/N]$ and $\mathscr{O}_{K^*}[1/N]$. Thus, the reflex norm N_Φ between $\mathbf{Q}$-tori extends to a surjective mapping between smooth $\mathbf{Z}[1/N]$-tori

$$\mathrm{N}_{\Phi,\mathbf{Z}[1/N]} : \mathrm{Res}_{\mathscr{O}_{K^*}[1/N]/\mathbf{Z}[1/N]}(\mathbf{G}_m) \to \mathrm{Res}_{\mathscr{O}_K[1/N]/\mathbf{Z}[1/N]}(\mathbf{G}_m) \tag{4.1}$$

defined on R-points (for a $\mathbf{Z}[1/N]$-algebra R) by

$$x \mapsto \det_{\mathscr{O}_K[1/N]\otimes_{\mathbf{Z}[1/N]}R}(x : t_{\Phi,\mathbf{Z}[1/N]} \otimes_{\mathbf{Z}[1/N]} R \simeq t_{\Phi,\mathbf{Z}[1/N]} \otimes_{\mathbf{Z}[1/N]} R) \in (\mathscr{O}_K[1/N] \otimes_{\mathbf{Z}[1/N]} R)^\times$$

for $x \in (\mathscr{O}_{K^*}[1/N] \otimes_{\mathbf{Z}[1/N]} R)^\times$. (The map $\mathrm{N}_{\Phi,\mathbf{Z}[1/N]}$ factors through the closed subtorus

$$T_{\mathbf{Z}[1/N]} = \ker(\mathrm{Res}_{\mathscr{O}_K[1/N]/\mathbf{Z}[1/N]}(\mathbf{G}_m) \overset{\mathrm{N}_{\mathscr{O}_K[1/N]/\mathscr{O}_{K_0}[1/N]}}{\twoheadrightarrow} \mathrm{Res}_{\mathscr{O}_{K_0}[1/N]/\mathbf{Z}[1/N]}(\mathbf{G}_m)/\mathbf{G}_m)$$

because it does so on the generic fibers over $\mathbf{Q}$, but we do not need to use $T_{\mathbf{Z}[1/N]}$ and so we will not address it any further.)

It follows that $\mathrm{N}_\Phi : \mathbf{A}_{K^*,\mathrm{f}}^\times \to \mathbf{A}_{K,\mathrm{f}}^\times$ carries $\prod_{w\nmid N} \mathscr{O}_{K^*_w}^\times$ into $\prod_{v\nmid N} \mathscr{O}_{K_v}^\times$ and also respects *integrality* at the collection of places over any rational prime $\ell \nmid N$. Thus, for any place w of K^* away from N and any local uniformizer π_w of K^*_w, if we let $\iota_w : (K^*_w)^\times \to \mathbf{A}_{K^*,\mathrm{f}}^\times$ be the natural inclusion map then $\mathrm{N}_\Phi(\iota_w(\pi_w)) \in \mathbf{A}_{K,\mathrm{f}}^\times$ has local component 1 away all places v of K not over p and is *integral* at all places v of K over p. Moreover, the idele $\mathrm{N}_\Phi(\iota_w(\pi_w))$ changes by a unit multiple in all components at places of K over p when π_w is replaced with an $\mathscr{O}_{K^*_w}^\times$-multiple. Hence, the reflex norm induces a well-defined homomorphism on fractional ideal groups away from N: if $\mathfrak{p}$ is a prime ideal of K^* away from N then we define the fractional K-ideal

$$\mathrm{N}_\Phi(\mathfrak{p}) \overset{\mathrm{def}}{=} \prod_{\mathfrak{q}|p\mathscr{O}_K} \mathfrak{q}^{\mathrm{ord}_\mathfrak{q}(\mathrm{N}_\Phi(\iota_\mathfrak{p}(\pi_\mathfrak{p})))} \tag{4.2}$$

for any uniformizer $\pi_\mathfrak{p}$ of $\mathscr{O}_{K^*_\mathfrak{p}}$.

Now pick a prime $\mathfrak{P}$ of L over a rational prime $p \nmid N$ such that $\sigma|_L = (\frac{L/K^*}{\mathfrak{P}})$. Let $\mathfrak{p}$ be the prime of K^* below $\mathfrak{P}$. As we saw in the proof of Theorem 1.5 (via [X]), since L is "big enough" (with respect to the finitely generated $\mathbf{Z}$-module of K-linear maps from A to A^σ) there is an unique K-linear morphism $\xi_{\sigma,\mathfrak{P}} : A_0 \to A_0^\sigma$ over L whose reduction is the relative q-Frobenius morphism

$$\mathrm{Fr}_{\overline{A}_0/\kappa(\mathfrak{P}),q} : \overline{A}_0 \to \overline{A}_0^{(q)},$$

where $q = \#\kappa(\mathfrak{p})$. The nonzero element

$$\xi_{\sigma,\mathfrak{P}} \in \mathfrak{a}_\sigma \overset{\mathrm{def}}{=} \mathrm{Hom}_L((A_0, i_0), (A_0^\sigma, i_0^\sigma))$$

endows the 1-dimensional K-vector space $K \otimes_{\mathscr{O}_K} \mathfrak{a}_\sigma$ with a distinguished element and so identifies $\mathfrak{a}_\sigma$ with a fractional K-ideal $\mathfrak{a}_{\sigma,\mathfrak{P}}$ that contains $\mathscr{O}_K$ and depends on $\mathfrak{P}$. Equivalently, composition with $\xi_{\sigma,\mathfrak{P}}$ defines an $\mathscr{O}_K$-linear embedding of invertible $\mathscr{O}_K$-modules

$$\mathscr{O}_K = \mathrm{Hom}_L((A_0, i_0), (A_0, i_0)) \overset{\xi_{\sigma,\mathfrak{P}}\circ(\cdot)}{\to} \mathrm{Hom}_L((A_0, i_0), (A_0^\sigma, i_0^\sigma)) = \mathfrak{a}_\sigma.$$

Of course, if we change the choice of the prime $\mathfrak{P}$ of $\mathscr{O}_L[1/N]$ such that $(\frac{L/K^*}{\mathfrak{P}}) = \sigma|_L$ then this inverse-integral fractional K-ideal $\mathfrak{a}_{\sigma,\mathfrak{P}}$ is replaced with another such ideal.

Note that the fractional ideal $\mathfrak{a}_{\sigma,\mathfrak{P}}$ has nothing to do with ϕ. Rather important for us is the fact that it is given by a simple formula that only depends on (K, Φ) and $\mathfrak{P}$:

Theorem 4.1. *With notation as above,* $\mathfrak{a}_{\sigma,\mathfrak{P}} = \mathrm{N}_\Phi(\mathfrak{p})^{-1}$ *with* $\mathfrak{p} = \mathfrak{P} \cap K^*$.

Proof. Since $\xi_{\sigma,\mathfrak{P}}$ has p-power degree, $\mathscr{O}_K$ has p-power index in $\mathfrak{a}_{\sigma,\mathfrak{P}}$. Hence, the fractional ideal $\mathfrak{a}_{\sigma,\mathfrak{P}}$ is a unit away from p. By construction, the same holds for $\mathrm{N}_\Phi(\mathfrak{p})$. It is therefore enough to fix a place v of K over p and to compare ord_v's for $\mathfrak{a}_{\sigma,\mathfrak{P}}$ and $\mathrm{N}_\Phi(\mathfrak{p})^{-1}$. Rather than make such a comparison directly, it will be convenient to compare after raising both ideals to the $f(\mathfrak{P}|\mathfrak{p})$th-power. To make effective use of this, we need to describe $\mathfrak{a}_{\sigma,\mathfrak{P}}^n$ for positive integers n.

First of all, note that for any positive integer n, all K-linear maps $A \to A^{\sigma^n}$ descend to K-linear maps $A_0 \to A_0^{\sigma^n}$. This says that the $\mathrm{Gal}(\overline{\mathbf{Q}}/L)$-action on $\mathrm{Hom}_{\overline{\mathbf{Q}}}(A, A^{\sigma^n})$ (via the L-structures A_0 and $A_0^{\sigma^n}$) is trivial on the subgroup of K-linear maps, and it suffices to check this on the level of the isogeny category. The subspace of K-linear maps in the isogeny category is 1-dimensional as a K-vector space, and the K-action on A over $\overline{\mathbf{Q}}$ descends to A_0 over L, so it suffices to exhibit a single nonzero K-linear mapping $A_0 \to A_0^{\sigma^n}$. There is a K-linear isogeny $h_0 : A_0 \to A_0^{\sigma}$ (e.g., $\xi_{\sigma,\mathfrak{P}}$), so the composite $h_0^{\sigma^{n-1}} \circ \cdots \circ h_0^{\sigma} \circ h_0$ does the job. We conclude that for all $n \geq 1$

$$\mathfrak{a}_{\sigma^n} \stackrel{\mathrm{def}}{=} \mathrm{Hom}_L((A_0, i_0), (A_0^{\sigma^n}, i_0^{\sigma^n}))$$

is an invertible $\mathscr{O}_K$-module equal to the module of such mappings over $\overline{\mathbf{Q}}$, and so by the complex-analytic theory of the Serre construction the natural evaluation mapping $\mathfrak{a}_{\sigma^n} \otimes_{\mathscr{O}_K} A_0 \to A_0^{\sigma^n}$ over L is an isomorphism because it becomes an isomorphism after analytification of the $\mathbf{C}$-fiber.

Next, observe that the Hom-module $\mathfrak{a}_{\sigma^n}$ contains a unique element $\xi_{\sigma,n,\mathfrak{P}}$ that lifts the K-linear relative q^n-Frobenius morphism $\overline{A}_0 \to \overline{A}_0^{(q^n)}$ over $\kappa(\mathfrak{P})$. This identifies $\mathfrak{a}_{\sigma^n}$ with a fractional K-ideal $\mathfrak{a}_{\sigma,n,\mathfrak{P}}$. For example, $\mathfrak{a}_{\sigma,1,\mathfrak{P}} = \mathfrak{a}_{\sigma,\mathfrak{P}}$. What is the ideal $\mathfrak{a}_{\sigma,n,\mathfrak{P}}$ in general? Since $\mathrm{Fr}_{\overline{A}_0/\kappa(\mathfrak{P}),q^n}$ is roughly an n-fold composite of $\mathrm{Fr}_{\overline{A}_0/\kappa(\mathfrak{P}),q}$'s, the following answer is not a surprise:

Lemma 4.2. *For all $n \geq 1$, $\mathfrak{a}_{\sigma,n,\mathfrak{P}} = \mathfrak{a}_{\sigma,\mathfrak{P}}^n$.*

Proof. Since all K-linear maps between A and A^{σ} are defined over L on the descents A_0 and A_0^{σ}, we have a canonical evaluation morphism

$$\mathfrak{a}_{\sigma} \otimes_{\mathscr{O}_K} A_0 \to A_0^{\sigma}$$

over L and it is an isomorphism. Hence, for any $\tau \in \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$, applying the base change $\tau : L \simeq L$ and using the base-change compatibility of Serre's construction yields a natural K-linear isomorphism

$$\mathfrak{a}_{\sigma} \otimes_{\mathscr{O}_K} A_0^{\tau} \simeq (\mathfrak{a}_{\sigma} \otimes_{\mathscr{O}_K} A_0)^{\tau} \simeq (A_0^{\sigma})^{\tau} \simeq A_0^{\tau\sigma} \tag{4.3}$$

over L, carrying $1 \otimes i_0^{\tau}$ to $i_0^{\tau\sigma}$.

By the definition of higher relative Frobenius maps, for any $\tau = \sigma^n$ with $n \in \mathbf{Z}^+$ the composite $\mathscr{O}_K$-linear module isomorphism

$$\begin{aligned}
\mathfrak{a}_{\tau\sigma} \stackrel{\mathrm{def}}{=} \mathrm{Hom}_L((A_0, i_0), (A_0^{\tau\sigma}, i_0^{\tau\sigma})) &\stackrel{(4.3)}{\simeq} \mathrm{Hom}_L((A_0, i_0), (\mathfrak{a}_{\sigma} \otimes_{\mathscr{O}_K} A_0^{\tau}, 1 \otimes i_0^{\tau})) \\
&\simeq \mathrm{Hom}_L((A_0, i_0), (\mathfrak{a}_{\sigma} \otimes_{\mathscr{O}_K} \mathfrak{a}_{\tau} \otimes_{\mathscr{O}_K} A, 1 \otimes 1 \otimes i_0)) \\
&\simeq \mathfrak{a}_{\sigma} \otimes_{\mathscr{O}_K} \mathfrak{a}_{\tau} \otimes_{\mathscr{O}_K} \mathrm{Hom}_L((A_0, i_0), (A_0, i_0)) \\
&\simeq \mathfrak{a}_{\sigma} \otimes_{\mathscr{O}_K} \mathfrak{a}_{\tau}
\end{aligned}$$

carries $\xi_{\sigma,n+1,\mathfrak{P}}$ to $\xi_{\sigma,\mathfrak{P}} \otimes \xi_{\sigma,n,\mathfrak{P}}$. (The reader should check this assertion!) In terms of fractional ideals resting on the distinguished elements $\xi_{\sigma,m,\mathfrak{P}}$ for $m \in \mathbf{Z}^+$, this says $\mathfrak{a}_{\sigma,n+1,\mathfrak{P}} = \mathfrak{a}_{\sigma,\mathfrak{P}} \mathfrak{a}_{\sigma,n,\mathfrak{P}}$. Hence, by induction on $n \geq 1$ we conclude that $\mathfrak{a}_{\sigma,n,\mathfrak{P}} = \mathfrak{a}_{\sigma,\mathfrak{P}}^n$ for all $n \geq 1$. ■

As a special case of Lemma 4.2, since $\sigma|_L$ has order $f(\mathfrak{P}|\mathfrak{p})$, the fractional ideal $\mathfrak{a}_{\sigma,\mathfrak{P}}^{f(\mathfrak{P}|\mathfrak{p})}$ is associated to the invertible $\mathscr{O}_K$-module $\mathrm{End}_L(A_0, i_0) = \mathscr{O}_K$ endowed with the distinguished element $\pi_0 = \xi_{\sigma,f(\mathfrak{P}|\mathfrak{p}),\mathfrak{P}}$ that lifts the Frobenius $q_{\mathfrak{P}}$-endomorphism of the abelian variety $\overline{A}_0$ over the finite field $\kappa(\mathfrak{P})$ with size $q_{\mathfrak{P}} = q_{\mathfrak{p}}^{f(\mathfrak{P}|\mathfrak{p})}$. Hence, we conclude that $\mathfrak{a}_{\sigma,\mathfrak{P}}^{f(\mathfrak{P}|\mathfrak{p})} = \pi_0^{-1}\mathscr{O}_K$, so for any place v of K over p we have

$$\mathrm{ord}_v(\mathfrak{a}_{\sigma,\mathfrak{P}}^{f(\mathfrak{P}|\mathfrak{p})}) = -\mathrm{ord}_v(\pi_0).$$

Now recall the *Shimura-Taniyama formula* (proved in [C2]):

$$\mathrm{ord}_v(\pi_0) = \mathrm{ord}_v(q_{\mathfrak{P}}) \cdot \frac{\#\Phi_v}{[K_v : \mathbf{Q}_p]}$$

with $\Phi_v \subseteq \Phi \subseteq \mathrm{Hom}_{\mathbf{Q}}(K, L)$ equal to the subset of elements $\varphi \in \Phi$ such that $\varphi : K \to L$ carries the $\mathfrak{P}$-adic place back to v. Since

$$\frac{\mathrm{ord}_v(q_{\mathfrak{P}})}{[K_v : \mathbf{Q}_p]} = \frac{f(\mathfrak{P}|p)\,\mathrm{ord}_v(p)}{[K_v : \mathbf{Q}_p]} = \frac{f(\mathfrak{P}|p)}{f(v|p)},$$

the problem $\mathrm{ord}_v(\mathfrak{a}_{\sigma,\mathfrak{P}}) \stackrel{?}{=} \mathrm{ord}_v(\mathrm{N}_\Phi(\mathfrak{p})^{-1})$ may be rephrased as

$$-\frac{f(\mathfrak{P}|p)}{f(v|p)} \cdot \#\Phi_v \stackrel{?}{=} \mathrm{ord}_v(\mathrm{N}_\Phi(\mathfrak{p})^{-f(\mathfrak{P}|\mathfrak{p})}),$$

or equivalently as

$$f(\mathfrak{p}|p)\#\Phi_v \stackrel{?}{=} f(v|p)\,\mathrm{ord}_v(\mathrm{N}_\Phi(\mathfrak{p})). \tag{4.4}$$

We will analyze the right side and eventually transform it into the left side.

Choose $\pi_\mathfrak{p} \in \mathscr{O}_{K^*}$ that has order 1 at $\mathfrak{p}$ and order 0 at all other places of K^* over p, so by (4.2) the fractional K-ideal $\mathrm{N}_\Phi(\mathfrak{p})$ is the "p-part" of the principal fractional ideal generated by $\det_K(\pi_\mathfrak{p} : t_\Phi \simeq t_\Phi) \in K^\times$. Hence, passing to the v-part of this K-determinant,

$$f(v|p)\,\mathrm{ord}_v(\mathrm{N}_\Phi(\mathfrak{p})) = f(v|p)\,\mathrm{ord}_v(\det_{K_v}(\pi_\mathfrak{p} : K_v \otimes_K t_\Phi \simeq K_v \otimes_K t_\Phi)).$$

For any $x \in K_v$, $f(v|p)\,\mathrm{ord}_v(x) = \mathrm{ord}_p(\mathrm{N}_{K_v/\mathbf{Q}_p}(x))$ since p is unramified in K. Also, for any K_v-linear endomorphism $\mu : V \to V$ of a finite-dimensional K_v-vector space V we have $\mathrm{N}_{K_v/\mathbf{Q}_p}(\det_{K_v}(\mu)) = \det_{\mathbf{Q}_p}(\mu_{\mathbf{Q}_p})$ with $\mu_{\mathbf{Q}_p}$ denoting the underlying $\mathbf{Q}_p$-linear endomorphism. Taking μ to be multiplication by $\pi_\mathfrak{p}$ on $K_v \otimes_K t_\Phi$,

$$f(v|p)\,\mathrm{ord}_v(\mathrm{N}_\Phi(\mathfrak{p})) = \mathrm{ord}_p(\det_{\mathbf{Q}_p}(\pi_\mathfrak{p} : K_v \otimes_K t_\Phi \simeq K_v \otimes_K t_\Phi)).$$

Recall that t_Φ as an $K \otimes_\mathbf{Q} K^*$-module is identified as the "generic fiber" of an $\mathscr{O}_K[1/N] \otimes_{\mathbf{Z}[1/N]} \mathscr{O}_{K^*}[1/N]$-module $t_{\Phi,\mathbf{Z}[1/N]}$. Extending scalars by $\mathbf{Z}[1/N] \to \mathbf{Z}_p$ and using the decomposition $K_v \otimes_K (K \otimes_\mathbf{Q} K^*) \simeq \prod_{w|p\mathscr{O}_{K^*}} K_v \otimes_{\mathbf{Q}_p} K^*_w$ (and the analogous one with the Dedekind rings $\mathscr{O}_{K_v} \otimes_{\mathbf{Z}_p} \mathscr{O}_{K^*_w}$), we get a decomposition of $K_v \otimes_K t_\Phi$ into a product of w-parts over all $w|p\mathscr{O}_{K^*}$. The $\pi_\mathfrak{p}$-action respects the integral structure on this module decomposition. Thus, this action has integral unit $\mathbf{Z}_p$-determinant on w-factors for all $w \neq \mathfrak{p}$ since $\pi_\mathfrak{p} \in \mathscr{O}^\times_{K^*_w}$ for all $w|p\mathscr{O}_{K^*}$ with $w \neq \mathfrak{p}$, so $f(v|p)\,\mathrm{ord}_v(\mathrm{N}_\Phi(\mathfrak{p}))$ is equal to ord_p of the $\mathbf{Q}_p$-determinant of the $\pi_\mathfrak{p}$-action on the $\mathfrak{p}$-part of $K_v \otimes_K t_\Phi$. This latter action is $K^*_\mathfrak{p}$-linear, and for any $K^*_\mathfrak{p}$-linear endomorphism μ of a finite-dimensional $K^*_\mathfrak{p}$-vector space V we have

$$\mathrm{ord}_p(\det_{\mathbf{Q}_p}(\mu_{\mathbf{Q}_p})) = f(\mathfrak{p}|p)\,\mathrm{ord}_\mathfrak{p}(\det_{K^*_\mathfrak{p}}(\mu)),$$

where $\mu_{\mathbf{Q}_p}$ is the underlying $\mathbf{Q}_p$-linear endomorphism.

We conclude that $f(v|p)\,\mathrm{ord}_v(\mathrm{N}_\Phi(\mathfrak{p}))$ is equal to $f(\mathfrak{p}|p)\,\mathrm{ord}_\mathfrak{p}(\det_{K^*_\mathfrak{p}}(\mu))$ with μ equal to scalar multiplication by $\pi_\mathfrak{p} \in (K^*_\mathfrak{p})^\times$ on the $\mathfrak{p}$-part of $K_v \otimes_K t_\Phi$. The $K^*_\mathfrak{p}$-determinant of such a scalar action is just $\pi_\mathfrak{p}^{d_\mathfrak{p}}$, where $d_\mathfrak{p}$ is the $K^*_\mathfrak{p}$-dimension of the $\mathfrak{p}$-part of $K_v \otimes_K t_\Phi$. Hence

$$f(v|p)\,\mathrm{ord}_v(\mathrm{N}_\Phi(\mathfrak{p})) = f(\mathfrak{p}|p)\,\mathrm{ord}_\mathfrak{p}(\pi_\mathfrak{p}^{d_\mathfrak{p}}) = f(\mathfrak{p}|p)d_\mathfrak{p}.$$

It follows that (4.4) is equivalent to the assertion $\#\Phi_v = d_\mathfrak{p}$. Under the extension of scalars $K^* \to L$ applied to t_Φ, the $\mathfrak{p}$-part of the $K_v \otimes_\mathbf{Q} K^*$-module $K_v \otimes_K t_\Phi$ is carried by the base change $K^*_\mathfrak{p} \to L_\mathfrak{P}$ over to the $\mathfrak{P}$-part of the $K_v \otimes_\mathbf{Q} L$-module $K_v \otimes_K (t_\Phi \otimes_{K^*} L)$. Thus, this $\mathfrak{P}$-part has $L_\mathfrak{P}$-dimension $d_\mathfrak{p}$. But by *construction* of the $K \otimes_\mathbf{Q} L$-module t_Φ via Galois descent, $t_\Phi \otimes_{K^*} L \simeq \prod_{\varphi \in \Phi} L$ as $K \otimes_\mathbf{Q} L$-modules. Hence, our problem is to prove that the $\mathfrak{P}$-part of the module $P_v = K_v \otimes_K (\prod_{\varphi\in\Phi} L)$ over the ring $K_v \otimes_\mathbf{Q} L \simeq \prod_{w'|p} K_v \otimes_{\mathbf{Q}_p} L_{w'}$ has $L_\mathfrak{P}$-dimension $\#\Phi_v$.

The module P_v decomposes as $\prod_{\varphi\in\Phi}(K_v \otimes_{K,\varphi} L)$, so it suffices to show that $P_{v,\varphi} \stackrel{\mathrm{def}}{=} K_v \otimes_{K,\varphi} L$ has vanishing $\mathfrak{P}$-part if $\varphi \notin \Phi_v$ and that it has $\mathfrak{P}$-part with $L_\mathfrak{P}$-dimension 1 for $\varphi \in \Phi_v$. By standard facts concerning completions of global fields, $K_v \otimes_{K,\varphi} L \simeq \prod_{w'\in\Sigma} L_{w'}$ where Σ is the set of places on L lifting v via φ. Thus, the $\mathfrak{P}$-part of $P_{v,\varphi}$ is at most 1-dimensional over $L_\mathfrak{P}$, and it is 1-dimensional precisely when φ pulls the $\mathfrak{P}$-adic place back to v. That is, the set of φ for which there is a nonzero $\mathfrak{P}$-part in $P_{v,\varphi}$ is precisely Φ_v. ∎

Applying functoriality of the Serre tensor construction with respect to the natural $\mathscr{O}_K$-linear inclusion $\mathscr{O}_K \to \mathrm{N}_\Phi(\mathfrak{p})^{-1}$, we may restate Theorem 4.1 in the following more convenient manner: for any prime $\mathfrak{P}$ of L over $p \nmid N$ such that $\sigma|_L = (\frac{L/K^*}{\mathfrak{P}})$, there is a unique $\mathscr{O}_K$-linear isomorphism

$$\theta_{\sigma,\mathfrak{P}} : \mathrm{N}_\Phi(\mathfrak{p})^{-1} \otimes_{\mathscr{O}_K} A \simeq A_0^\sigma$$

of abelian varieties over L such that the diagram

$$\begin{tikzcd} A_0 \arrow[r, "\xi_{\sigma,\mathfrak{P}}"] \arrow[dr] & A_0^\sigma \\ & \mathrm{N}_\Phi(\mathfrak{p})^{-1} \otimes_{\mathscr{O}_K} A_0 \arrow[u, "\theta_{\sigma,\mathfrak{P}}"', "\simeq"] \end{tikzcd} \tag{4.5}$$

commutes, where the top map is the lifting of the relative $q_{\mathfrak{p}}$-Frobenius for $\overline{A}_{0/\kappa(\mathfrak{P})}$ and the diagonal map is induced by the inclusion of $\mathscr{O}_K$ into $\mathrm{N}_\Phi(\mathfrak{p})^{-1}$. The isomorphism $\theta_{\sigma,\mathfrak{P}}$ depends on the choice of $\mathfrak{P}$.

The isomorphism $\theta_{\sigma,\mathfrak{P}}$ uses a non-canonical L-descent (A_0, i_0) of (A, i) as well as a non-canonical choice of $\mathfrak{P}$. Our task in §5 will be to improve this situation by constructing a *canonical* K-linear isomorphism

$$[\mathrm{N}_\Phi(s)^{-1}]_K \otimes_{\mathscr{O}_K} A \simeq A^\sigma \tag{4.6}$$

of abelian varieties over $\overline{\mathbf{Q}}$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$ and $s \in \mathbf{A}_{K^*,\mathrm{f}}^\times$ such that $(s|K^*) = \sigma|_{(K^*)^{\mathrm{ab}}}$ (with (A, i) having CM order $\mathscr{O}_K$), where $[\cdot]_K$ denotes the fractional K-ideal associated to a finite K-idele. The maps $\theta_{\sigma,\mathfrak{P}}$ generally do *not* descend the canonical isomorphism in (4.6) to be constructed in §5; rather, $\theta_{\sigma,\mathfrak{P}}$ over $\overline{\mathbf{Q}}$ will be related to (4.6) via the action by an element of $\mathrm{N}_\Phi((K^*)^\times) \subseteq T(\mathbf{Q}) \subseteq K^\times$. One should consider $\theta_{\sigma,\mathfrak{P}}$ as an approximation to (4.6) "at level L" since the Artin map $\mathbf{A}_{K^*,\mathrm{f}}^\times \twoheadrightarrow \mathrm{Gal}(L/K^*)$ carries the idele $s_{\mathfrak{p}} = \iota_{\mathfrak{p}}(\pi_{\mathfrak{p}})$ to $\sigma|_L$ (due to our convention on local uniformizers and arithmetic Frobenius elements under the Artin map!) and $[\mathrm{N}_\Phi(s_{\mathfrak{p}})]_K = \mathrm{N}_\Phi(\mathfrak{p})$.

Remark 4.3. The most important applications of the Main Theorem are to abelian varieties over number fields. However, as the reader will see, the situation is very much simplified by first proving the Main Theorem in the form stated over $\overline{\mathbf{Q}}$ (or more traditionally, over $\mathbf{C}$) and only later deducing arithmetic consequences over number fields. The main way in which this simplification manifests itself it is in the task of making the preceding construction become canonical by passing up to $\overline{\mathbf{Q}}$. More specifically, to eliminate the intervention of $\mathfrak{P}$ we will have to apply certain procedures involving passage to a field that splits $A_0[M]$ for several relatively prime $M \geq 3$. By working over $\overline{\mathbf{Q}}$ from the outset we thereby avoid the unpleasant task of having to increase the base field at several places in the middle of an argument.

5. Polarizations and torsion

In §4 the number field $L \subseteq \overline{\mathbf{Q}}$ was a rather general "sufficiently big" finite Galois extension of K^*, and we now need a few more "largeness" conditions on L. Choose a fixed auxiliary integer $M \geq 1$ (later we will take $M \geq 3$) and impose the additional "largeness" condition on L that the finite étale L-group $A_0[M]$ is constant. Thus, $A_0^\sigma[M]$ is constant, as is $(\mathfrak{a} \otimes_{\mathscr{O}_K} A_0)[M] \simeq \mathfrak{a} \otimes_{\mathscr{O}_K} (A_0[M])$ for any fractional ideal $\mathfrak{a}$ of K. Recall that the choice of the prime $\mathfrak{P}$ of L was controlled by $\sigma|_L$ (via the condition $(\frac{L/K^*}{\mathfrak{P}}) = \sigma|_L$) and by the geometry of A_0 and the arithmetic of L and K (since we required the residue characteristic p of $\mathfrak{P}$ to not divide the product N of the residue characteristics of the primes of bad reduction for A_0 and the ramified primes for L and K over $\mathbf{Q}$). We impose the further condition on the residue characteristic p of $\mathfrak{P}$ that $p \nmid M$ (in addition to the condition $p \nmid N$).

The natural map $A_0 \to \mathrm{N}_\Phi(\mathfrak{p})^{-1} \otimes_{\mathscr{O}_K} A_0$ has p-power degree and so induces an isomorphism on M-torsion. Hence, we get a diagram on L-points

$$\begin{tikzcd} A_0[M](L) \arrow[r, "{[\sigma]}"] \arrow[dr, "\simeq"'] & A_0^\sigma[M](L) \\ & (\mathrm{N}_\Phi(\mathfrak{p})^{-1} \otimes_{\mathscr{O}_K} A_0)[M](L) \arrow[u, "\theta_{\sigma,\mathfrak{P}}"', "\simeq"] \end{tikzcd} \tag{5.1}$$

that is the same as the associated diagram on $\overline{\mathbf{Q}}$-points (due to the constancy property of these M-torsion groups over $\mathrm{Spec}\, L$). The top map in this diagram is Galois-theoretic, and the other two sides arise from K-linear morphisms of abelian varieties over L. We claim that the diagram (5.1) commutes. Since $p \nmid M$ and the three abelian varieties involved all have good reduction at $\mathfrak{P}$, we may identify these L-point groups with the corresponding $\kappa(\mathfrak{P})$-point groups in the reductions of the abelian-scheme Néron models. The diagram (5.1) viewed on $\kappa(\mathfrak{P})$-points is the same as the diagram induced on M-torsion by the reduction of (4.5) because the morphism $\xi_{\sigma,\mathfrak{P}}$ reduces to the morphism $\mathrm{Fr}_{\overline{A}_0/\kappa(\mathfrak{P}),q}$ whose effect on $\kappa(\mathfrak{P})$-points is *exactly* the action by the arithmetic Frobenius element in $\mathrm{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p}))$ on such points. (Here, we use that the Galois-theoretic arithmetic Frobenius automorphism is given by the algebraic formula $t \mapsto t^{q_{\mathfrak{p}}}$.) This proves the commutativity of (5.1) as a consequence of the commutativity of (4.5).

To go further, we need to impose one final "largeness" condition on L/K^* before choosing $\mathfrak{P}$: the extension L/K^* must contain a certain class field for K^* to now be described. Consider the reflex norm mapping $\mathrm{N}_\Phi : \mathbf{A}_{K^*,\mathrm{f}}^\times \to \mathbf{A}_{K,\mathrm{f}}^\times$ on $\mathbf{A}_\mathrm{f}$-points. We have seen earlier that for rational primes $\ell \nmid N$, the induced mapping on $\mathbf{Q}_\ell$-points

$$\mathrm{N}_{\Phi,\ell} : \prod_{w|\ell} (K_w^*)^\times = (K^* \otimes_{\mathbf{Q}} \mathbf{Q}_\ell)^\times \to (K \otimes_{\mathbf{Q}} \mathbf{Q}_\ell)^\times = \prod_{v|\ell} K_v^\times$$

carries $\mathscr{O}_{K_\ell^*}^\times$ into $\mathscr{O}_{K_\ell}^\times$ (where $K_\ell^* = \mathbf{Q}_\ell \otimes_{\mathbf{Q}} K^*$, $\mathscr{O}_{K_\ell^*} = \mathbf{Z}_\ell \otimes_{\mathbf{Z}} \mathscr{O}_{K^*} \simeq \prod_{w|\ell} \mathscr{O}_{K_w^*}^\times$, and similarly for K). Define $U \subseteq \mathbf{A}_{K^*,\mathrm{f}}^\times$ to be the open subgroup $\prod_\ell U_\ell$ where

$$U_\ell = \mathscr{O}_{K_\ell^*}^\times \cap \mathrm{N}_{\Phi,\ell}^{-1}(\{u \in \mathscr{O}_{K_\ell}^\times \mid u \equiv 1 \bmod M\})$$

for all rational primes ℓ (so $U_\ell = \mathscr{O}_{K_\ell^*}^\times$ for $\ell \nmid NM$). We require L to contain the class field for the open subgroup $(K^*)^\times (K_\infty^*)^\times U$, so the restricted Artin map $\mathbf{A}_{K^*,\mathrm{f}}^\times \twoheadrightarrow \mathrm{Gal}(L/K^*)^{\mathrm{ab}}$ has kernel contained in $(K^*)^\times U$.

The conditions $\sigma|_L = (\frac{L/K^*}{\mathfrak{P}})$ and $(s|K^*) = \sigma|_{(K^*)^{\mathrm{ab}}}$ force the idele $s \cdot \iota_\mathfrak{p}(\pi_\mathfrak{p})^{-1}$ to be in the open kernel of the Artin map $(K^*)^\times \backslash \mathbf{A}_{K^*,\mathrm{f}}^\times \twoheadrightarrow \mathrm{Gal}(L/K^*)^{\mathrm{ab}}$, so $s = \iota_\mathfrak{p}(\pi_\mathfrak{p}) uc$ in $\mathbf{A}_{K^*,\mathrm{f}}^\times$ where $c \in (K^*)^\times$ and $u \in U$, so $\mathrm{N}_\Phi(u) \in T(\mathbf{A}_\mathrm{f}) \subseteq \mathbf{A}_{K,\mathrm{f}}^\times$ is everywhere a local unit and satisfies $\mathrm{N}_\Phi(u) \equiv 1 \bmod M$ as an integral adele for K. (For later purposes, it is convenient to consider u as being determined by $s, \pi_\mathfrak{p}$, and c.) In particular, since $p \nmid M$ the finite K-idele $\mathrm{N}_\Phi(s^{-1}c) = \mathrm{N}_\Phi(\iota_\mathfrak{p}(\pi_\mathfrak{p})^{-1} u^{-1})$ that is possibly non-integral at p has component in $\prod_{\ell|M} K_\ell^\times$ that is an integral unit congruent to 1 modulo M.

Consider the commutative diagram of abelian varieties over L:

$$\begin{tikzcd} \mathrm{N}_\Phi(\mathfrak{p})^{-1} \otimes_{\mathscr{O}_K} A_0 \arrow[r, "\theta_{\sigma,\mathfrak{P}}", "\simeq"'] \arrow[d, "\mathrm{N}_\Phi(c)^{-1}"', "\simeq"] & A_0^\sigma \\ \mathrm{N}_\Phi(c\mathfrak{p})^{-1} \otimes_{\mathscr{O}_K} A_0 \arrow[ur, "\simeq"'] & \end{tikzcd} \tag{5.2}$$

where the diagonal mapping is defined to force commutativity; this diagonal mapping is an K-linear isomorphism. Letting $[x]_K$ denote the fraction K-ideal associated to a finite idele $x \in \mathbf{A}_{K,\mathrm{f}}^\times$, we have

$$\begin{aligned} \mathrm{N}_\Phi(c\mathfrak{p}) = \mathrm{N}_\Phi(c) \cdot \mathrm{N}_\Phi(\mathfrak{p}) &= \mathrm{N}_\Phi(c) \cdot [\mathrm{N}_\Phi(\iota_\mathfrak{p}(\pi_\mathfrak{p}))]_K \\ &= [\mathrm{N}_\Phi(c \cdot \iota_\mathfrak{p}(\pi_\mathfrak{p}))]_K \\ &= [\mathrm{N}_\Phi(su^{-1})]_K \\ &= [\mathrm{N}_\Phi(s)]_K \end{aligned}$$

since $\mathrm{N}_\Phi(u) \in \prod_w \mathscr{O}_{K_w^*}^\times \subseteq \mathbf{A}_{K^*,\mathrm{f}}^\times$. Thus, the K-linear isomorphism of abelian varieties over L given by the diagonal map in (5.2) may be expressed as

$$\theta_{\sigma,\mathfrak{P},\pi_\mathfrak{p},c,s,M} : [\mathrm{N}_\Phi(s^{-1})]_K \otimes_{\mathscr{O}_K} A_0 \simeq A_0^\sigma. \tag{5.3}$$

We shall prove that if $M \geq 3$ then the K-linear isomorphism $[\mathrm{N}_\Phi(s^{-1})]_K \otimes_{\mathscr{O}_K} A \simeq A^\sigma$ obtained by applying the base change $L \to \overline{\mathbf{Q}}$ to (5.3) only depends on $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$ and $s \in \mathbf{A}_{K^*,\mathrm{f}}^\times$ (linked by the condition $(s|K^*) = \sigma|_{(K^*)^{\mathrm{ab}}}$), and *not* on the choice of L/K^*, the choice of L-descent (A_0, i_0, ϕ_0) of (A, i, ϕ), or the choices of $\mathfrak{P}$, $\pi_\mathfrak{p}$, c, or M. We will also show that these canonical isomorphisms over $\overline{\mathbf{Q}}$ are well-behaved with respect to composition in $\mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$ and multiplication in $\mathbf{A}_{K^*,\mathrm{f}}^\times$. These matters will be settled in Theorem 5.2 and Remark 5.3.

Up to now, in §4 and §5 the K-linear $\mathbf{Q}$-polarization ϕ and its L-descent ϕ_0 on A_0 have played no role. We have arrived at the point where ϕ_0 will be used. We must endow $[\mathrm{N}_\Phi(s^{-1})]_K \otimes_{\mathscr{O}_K} A_0$ with a natural K-linear $\mathbf{Q}$-polarization $\phi_{0,s}$ over L such that $\theta_{\sigma,\mathfrak{P},\pi_\mathfrak{p},c,s,M}$ carries $\phi_{0,s}$ to a positive rational multiple of ϕ_0^σ (and the rational multiplier will be made explicit in terms of s). In view of how $\theta_{\sigma,\mathfrak{P},\pi_\mathfrak{p},c,s,M}$ is defined, and the fact that the mapping $\xi_{\sigma,\mathfrak{P}}$ in (4.5) that lifts $\mathrm{Fr}_{\overline{A}_0/\kappa(\mathfrak{P}),q_\mathfrak{p}}$ respects the $\mathbf{Q}$-polarizations ϕ_0 and ϕ_0^σ up to a positive rational multiple (as we showed in the proof of Theorem 1.5), our task comes down to proving:

Lemma 5.1. *Let (A,i) be a CM abelian variety of type (K,Φ) over $\overline{\mathbf{Q}}$ with CM order $\mathscr{O}_K$, and let ϕ be an K-linear $\mathbf{Q}$-polarization on A. There is a unique way to assign a K-linear $\mathbf{Q}$-polarization ϕ_α to $[\mathrm{N}_\Phi(\alpha)]_K \otimes_{\mathscr{O}_K} A$ over $\overline{\mathbf{Q}}$ for all $\alpha \in \mathbf{A}_{K^*,\mathrm{f}}^\times$ such that:*

1. *$\phi_1 = \phi$ via the identification $\mathscr{O}_K \otimes_{\mathscr{O}_K} A = A$,*
2. *ϕ_α only depends on $[\mathrm{N}_\Phi(\alpha)]_K$,*
3. *if $h : [\mathrm{N}_\Phi(\alpha)]_K \to [\mathrm{N}_\Phi(\beta)]_K$ is an $\mathscr{O}_K$-linear map induced by multiplication by $c \in \mathrm{N}_\Phi((K^*)^\times) \subseteq T(\mathbf{Q}) \subseteq K^\times$ then the K-linear isogeny $h \otimes 1_A : [\mathrm{N}_\Phi(\alpha)]_K \otimes_{\mathscr{O}_K} A \to [\mathrm{N}_\Phi(\beta)]_K \otimes_{\mathscr{O}_K} A$ carries ϕ_α to $q_{c,\alpha,\beta}\phi_\beta$ with $q_{c,\alpha,\beta} \in \mathbf{Q}_{>0}^\times$ the unique positive generator of the fractional $\mathbf{Q}$-ideal $[\mathrm{N}_{K^*/\mathbf{Q}}(\beta/c\alpha)]_\mathbf{Q}$.*

Moreover, $\deg \phi_\alpha = \deg \phi$ and if $L \subseteq \overline{\mathbf{Q}}$ is a subextension such that (A,i,ϕ) descends to a triple (A_0,i_0,ϕ_0) over L then each such ϕ_α uniquely descends to an K-linear $\mathbf{Q}$-polarization of A_0.

Proof. Galois descent works for $\mathbf{Q}$-polarizations because descent theory is effective for polarizations. Hence, the compatibility of the Serre construction with respect to change of the base field (such as automorphisms of $\overline{\mathbf{Q}}$) and the uniqueness aspect in the lemma imply the final part of the theorem via Galois descent. Thus, we may and do focus on the existence and uniqueness problem over $\overline{\mathbf{Q}}$. The uniqueness aspect is obvious, via parts (1) and (3).

For the existence aspect, let $q_s \in \mathbf{Q}_{>0}^\times$ be the unique positive generator of $[\mathrm{N}_{K^*/\mathbf{Q}}(s)]_\mathbf{Q}$ for each $s \in \mathbf{A}_{K^*,\mathrm{f}}^\times$. Working in the isogeny category and using the evident $\mathbf{A}_{K,\mathrm{f}}$-linear isomorphism

$$\mathrm{V}_\mathrm{f}([\mathrm{N}_\Phi(\alpha)]_K \otimes_{\mathscr{O}_K} A) = \mathrm{V}_\mathrm{f}(A)$$

carrying $\mathrm{T}_\mathrm{f}([\mathrm{N}_\Phi(\alpha)]_K \otimes_{\mathscr{O}_K} A)$ to $\mathrm{N}_\Phi(\alpha) \cdot \mathrm{T}_\mathrm{f}(A)$, there is a unique $\mathbf{Q}$-polarization ϕ_α on $[\mathrm{N}_\Phi(\alpha)]_K \otimes_{\mathscr{O}_K} A$ satisfying the identity of adelic Weil pairings $e_{\phi_\alpha} = q_\alpha^{-1} e_\phi$.

By definition of the target torus T for the reflex norm

$$\mathrm{N}_\Phi : \mathrm{Res}_{K^*/\mathbf{Q}}(\mathbf{G}_m) \to T \subseteq \mathrm{Res}_{K/\mathbf{Q}}(\mathbf{G}_m),$$

the product mapping $\mathrm{N}_\Phi \mathrm{N}_\Phi^* = \mathrm{N}_\Phi \mathrm{N}_{\Phi^*}$ factors through the subtorus $\mathbf{G}_m$ and as such is equal to $\mathrm{N}_{K^*/\mathbf{Q}}$ (due to [K, Prop. 2.5]). Thus, e_{ϕ_α} is uniquely determined by $I_\alpha = [\mathrm{N}_\Phi(\alpha)]_K$ because the product of I_α and its complex conjugate is the fractional K-ideal $q_\alpha \mathscr{O}_K$ with q_α as its unique positive rational generator. The degree of a $\mathbf{Q}$-polarization can be detected by studying the failure of perfectness and integrality on the total Tate module, so this ϕ_α clearly satisfies all of the requirements. ■

Lemma 5.1 and the preceding discussion provide K-linear isomorphisms

$$\theta_{\sigma,\mathfrak{P},\pi_\mathfrak{p},c,s,M} : [\mathrm{N}_\Phi(s)]_K^{-1} \otimes_{\mathscr{O}_K} A_0 \simeq A_0^\sigma$$

of abelian varieties over L carrying $\phi_{0,1/s}$ to a positive rational multiple of ϕ_0^σ; this rational multiple must be 1 because $\deg(\phi^\sigma) = \deg \phi = \deg \phi_{1/s}$. We shall now prove that the isomorphism obtained from $\theta_{\sigma,\mathfrak{P},\pi_\mathfrak{p},c,s,M}$ after extension of scalars to $\overline{\mathbf{Q}}$ only depends on σ and s, so it can be called $\theta_{\sigma,s}$, and that it is well-behaved with respect to composition in $\mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$ and multiplication in $\mathbf{A}_{K^*,\mathrm{f}}^\times$.

Since $\mathrm{N}_\Phi(c) \cdot \mathrm{N}_\Phi(\mathfrak{p}) = [\mathrm{N}_\Phi(s)]_K$, we can define the K-linear isogeny $\theta_{\mathfrak{p},c} : A_0 \to [\mathrm{N}_\Phi(s)]_K^{-1} \otimes_{\mathscr{O}_K} A_0$ over L to be the composite mapping

$$A_0 \longrightarrow \mathrm{N}_\Phi(\mathfrak{p})^{-1} \otimes_{\mathscr{O}_K} A_0 \xrightarrow[\simeq]{\mathrm{N}_\Phi(c)^{-1} \otimes 1} [\mathrm{N}_\Phi(s)]_K^{-1} \otimes_{\mathscr{O}_K} A_0.$$

This isogeny has p-power degree, so it induces an isomorphism on M-torsion. Since $\mathrm{N}_\Phi(sc^{-1}) = \mathrm{N}_\Phi(\iota_{\mathfrak{p}}(\pi_{\mathfrak{p}})u)$ with $p \nmid M$ and $\mathrm{N}_\Phi(u) \equiv 1 \bmod M$, it follows that on the level of geometric points the composite isomorphism

$$A_0[M] \overset{\theta_{\mathfrak{p},c}}{\simeq} ([\mathrm{N}_\Phi(s)]_K^{-1} \otimes_{\mathscr{O}_K} A_0)[M] \simeq [\mathrm{N}_\Phi(s)]_K^{-1} \otimes_{\mathscr{O}_K} A_0[M] \simeq ([\mathrm{N}_\Phi(s)^{-1}]_K/M[\mathrm{N}_\Phi(s)^{-1}]_K) \otimes_{\mathscr{O}_K/M\mathscr{O}_K} A_0[M]$$

of constant finite étale $\mathscr{O}_K/M\mathscr{O}_K$-modules over $\operatorname{Spec} L$ is induced by multiplication by the idele $\mathrm{N}_\Phi(s)^{-1}$ that is naturally a representative of a basis for the invertible $\mathscr{O}_K/M\mathscr{O}_K$-module $[\mathrm{N}_\Phi(s)^{-1}]_K/M[\mathrm{N}_\Phi(s)^{-1}]_K$.

Consider induced maps on $\overline{\mathbf{Q}}$-points of M-torsion subgroups:

$$\begin{tikzcd}
A_0[M](\overline{\mathbf{Q}}) \arrow[r, "{[\sigma]}", "\simeq"'] \arrow[d, "\theta_{\mathfrak{p},c}"', "\simeq"] & A_0^\sigma[M](\overline{\mathbf{Q}}) \\
([\mathrm{N}_\Phi(s)]_K^{-1} \otimes_{\mathscr{O}_K} A_0)[M](\overline{\mathbf{Q}}) \arrow[ur, "\simeq", "\theta_{\sigma,\mathfrak{P},\pi_{\mathfrak{p}},c,s,M}"'] &
\end{tikzcd}$$

Since these $\overline{\mathbf{Q}}$-point groups coincide with L-point groups, the commutativity of this diagram follows from the commutativity of (5.1) and (5.2). Via the isomorphism $\overline{\mathbf{Q}} \otimes_L A_0 \simeq A$ (carrying i_0 and ϕ_0 to i and ϕ respectively, and likewise after base change by σ), the K-linear isomorphism $\theta_{\sigma,\mathfrak{P},\pi_{\mathfrak{p}},c,s,M}$ of abelian varieties over L must therefore induce the following composite of *canonical* maps on the $\overline{\mathbf{Q}}$-points of M-torsion subgroups:

$$([\mathrm{N}_\Phi(s)]_K^{-1} \otimes_{\mathscr{O}_K} A)[M](\overline{\mathbf{Q}}) \simeq [\mathrm{N}_\Phi(s^{-1})]_K \otimes_{\mathscr{O}_K} (A[M](\overline{\mathbf{Q}})) \overset{\mathrm{N}_\Phi(s)}{\simeq} A[M](\overline{\mathbf{Q}}) \overset{[\sigma]}{\simeq} A^\sigma[M](\overline{\mathbf{Q}}). \tag{5.4}$$

The steps in this composite are *independent* of all choices (including L/K^* and the descent (A_0, i_0, ϕ_0) over L) aside from the choices of σ, s, M, and for any $M'|M$ it is clear that on M'-torsion subgroups this intrinsic map on M-torsion restricts to the analogously described map arising from $\theta_{\sigma,\mathfrak{P},\pi_{\mathfrak{p}},c,s,M'}$. In other words, $\theta_{\sigma,\mathfrak{P},\pi_{\mathfrak{p}},c,s,M}$ and $\theta_{\sigma,\mathfrak{P},\pi_{\mathfrak{p}},c,s,M'}$ agree on M'-torsion subgroups for $M'|M$ with $M' \geq 3$.

Consider the automorphism

$$\theta_{\sigma,\mathfrak{P}',\pi'_{\mathfrak{p}'},c',s,M} \circ \theta^{-1}_{\sigma,\mathfrak{P},\pi_{\mathfrak{p}},c,s,M} \in \mathrm{Aut}_L(A_0^\sigma, i_0^\sigma, \phi_0^\sigma) = \mathrm{Aut}_{\overline{\mathbf{Q}}}(A^\sigma, i^\sigma, \phi^\sigma);$$

the preservation of the polarization is due to the fact that both θ's carry ϕ to ϕ^σ. We allow for the possibility $\mathfrak{P}' = \mathfrak{P}$ and $c' = c$ but $\pi'_{\mathfrak{p}} \neq \pi_{\mathfrak{p}}$.

The preceding argument via (5.4) shows that this automorphism of A_0^σ induces the identity on M-torsion. Since the automorphism group of a $\mathbf{Q}$-polarized abelian variety is finite, this automorphism of A_0^σ has finite order. However, it acts as the identity on M-torsion, so as long as we restrict attention to $M \geq 3$ this finite-order automorphism is trivial. Thus, if $M \geq 3$ then the K-linear isomorphism

$$\overline{\mathbf{Q}} \otimes_L \theta_{\sigma,\mathfrak{P},\pi_{\mathfrak{p}},c,s,M} : [\mathrm{N}_\Phi(s)]_K^{-1} \otimes_{\mathscr{O}_K} A \simeq A^\sigma$$

of abelian varieties over $\overline{\mathbf{Q}}$ only depends on σ, s, M; we therefore denote this isomorphism $\theta_{\sigma,s,M}$.

For $M'|M$ with $M' \geq 3$, the same argument now shows that the K-linear automorphism $\theta_{\sigma,s,M} \circ \theta^{-1}_{\sigma,s,M'}$ of the abelian variety A^σ over $\overline{\mathbf{Q}}$ is trivial on M'-torsion and hence is trivial. Thus, for any $M_1, M_2 \geq 3$ we have $\theta_{\sigma,s,M_1} = \theta_{\sigma,s,M_1M_2} = \theta_{\sigma,s,M_2}$, so $\theta_{\sigma,s,M}$ is independent of $M \geq 3$. (See Remark 4.3.)

We summarize the conclusion of our efforts as follows:

Theorem 5.2. *Let (A, i) be a CM abelian variety of type (K, Φ) over $\overline{\mathbf{Q}}$, and assume (A, i) has CM order $\mathscr{O}_K$. For any $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$ and $s \in \mathbf{A}^\times_{K^*,\mathrm{f}}$ such that $(s|K^*) = \sigma|_{(K^*)^{\mathrm{ab}}}$ there is a unique K-linear isomorphism*

$$\theta_{\sigma,s} = \theta_{\sigma,s,(A,i)} : [\mathrm{N}_\Phi(s)]_K^{-1} \otimes_{\mathscr{O}_K} A \simeq A^\sigma$$

such that for all $M \geq 1$ the isomorphism $[\sigma] : A[M](\overline{\mathbf{Q}}) \simeq A^{\sigma}[M](\overline{\mathbf{Q}})$ is the composite

$$\begin{tikzcd} A[M](\overline{\mathbf{Q}}) \arrow[r, "\mathrm{N}_{\Phi}(s^{-1})", "\simeq"'] & ([\mathrm{N}_{\Phi}(s^{-1})]_K/M[\mathrm{N}_{\Phi}(s^{-1})]_K) \otimes_{\mathscr{O}_K/M\mathscr{O}_K} A[M](\overline{\mathbf{Q}}) \arrow[d, "\simeq"] \\ & ([\mathrm{N}_{\Phi}(s)]_K^{-1} \otimes_{\mathscr{O}_K} A)[M](\overline{\mathbf{Q}}) \arrow[d, "\theta_{\sigma,s}"', "\simeq"] \\ & A^{\sigma}[M](\overline{\mathbf{Q}}). \end{tikzcd} \tag{5.5}$$

If ϕ is an K-linear $\mathbf{Q}$-polarization on A then $\theta_{\sigma,s}$ carries the $\mathbf{Q}$-polarization $\phi_{1/s}$ of $[\mathrm{N}_{\Phi}(s)]_K^{-1} \otimes_{\mathscr{O}_K} A$ to the $\mathbf{Q}$-polarization ϕ^{σ} of A^{σ}. Also, for any $c \in (K^*)^{\times}$ we have $\theta_{\sigma,cs} = \mathrm{N}_{\Phi}(c)\theta_{\sigma,s}$ and the formation of $\theta_{\sigma,s,(A,i)}$ is natural in the principal CM abelian variety (A, i).

Proof. The uniqueness follows from the specification of the induced map on every torsion level (for all M). The existence is the content of the entire preceding analysis, and the proof of existence also gives the asserted behavior with respect to K-linear $\mathbf{Q}$-polarizations. The behavior with respect to relacing s with cs for $c \in (K^*)^{\times}$ comes from the uniqueness (or, less elegantly, from the construction). The naturality in (A, i) follows from the explicit canonical description (5.5) on every torsion level. ■

Remark 5.3. We briefly address the behavior of $\theta_{\sigma,s}$ with respect to composition in σ and multiplication in s. Pick $\sigma, \sigma' \in \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$ and $s, s' \in \mathbf{A}_{K^*,\mathrm{f}}^{\times}$ satisfying $(s|K^*) = \sigma|_{(K^*)^{\mathrm{ab}}}$ and $(s'|K^*) = \sigma'|_{(K^*)^{\mathrm{ab}}}$, so $(s's|K^*) = (\sigma'\sigma)|_{(K^*)^{\mathrm{ab}}}$. We claim that $\theta_{\sigma'\sigma,s's,(A,i)}$ is the composite isomorphism

$$\begin{aligned} [\mathrm{N}_{\Phi}(s's)]_K^{-1} \otimes_{\mathscr{O}_K} A &\simeq [\mathrm{N}_{\Phi}(s')]_K^{-1} \otimes_{\mathscr{O}_K} ([\mathrm{N}_{\Phi}(s)]_K^{-1} \otimes_{\mathscr{O}_K} A) \\ &\overset{1\otimes\theta_{\sigma,s,(A,i)}}{\simeq} [\mathrm{N}_{\Phi}(s')]_K^{-1} \otimes_{\mathscr{O}_K} A^{\sigma} \\ &\overset{\theta_{\sigma',s',(A^{\sigma},i^{\sigma})}}{\simeq} (A^{\sigma})^{\sigma'} \\ &\simeq A^{\sigma'\sigma}. \end{aligned}$$

This equality may be checked by comparing what happens on M-torsion for arbitrary $M \geq 3$. This is a straightforward calculation because $[\sigma] : A(\overline{\mathbf{Q}}) \simeq A^{\sigma}(\overline{\mathbf{Q}})$ is K-linear with respect to i and i^{σ}.

The descent down to models over number fields has served its purpose, and for the rest of the proof of the Main Theorem we will work exclusively over $\overline{\mathbf{Q}}$. Passing to the inverse limit on (5.5) and tensoring with $\mathbf{Q}$ gives a commutative diagram in which the composite across the top and right sides is $[\sigma]$:

$$\begin{tikzcd} \mathrm{V}_{\mathrm{f}}(A) \arrow[r, "\simeq"] \arrow[dr, "\mathrm{N}_{\Phi}(s)^{-1}"'] & {[\mathrm{N}_{\Phi}(s)]_K^{-1} \otimes_{\mathscr{O}_K} \mathrm{V}_{\mathrm{f}}(A)} \arrow[r, "\simeq"] \arrow[d, "\mathrm{mult.}"', "\simeq"] & \mathrm{V}_{\mathrm{f}}([\mathrm{N}_{\Phi}(s)]_K^{-1} \otimes_{\mathscr{O}_K} A) \arrow[d, "\simeq"', "\mathrm{V}_{\mathrm{f}}(\theta_{\sigma,s})"] \\ & \mathrm{V}_{\mathrm{f}}(A) \arrow[ur, "\simeq"] \arrow[r, "\simeq"'] & \mathrm{V}_{\mathrm{f}}(A^{\sigma}) \end{tikzcd} \tag{5.6}$$

with the right diagonal and lower horizontal maps defined by commutativity. Since $\mathrm{N}_{\Phi}(s) \in T(\mathbf{A}_{\mathrm{f}}) \subseteq \mathbf{A}_{K,\mathrm{f}}^{\times}$, the multiplication map by $\mathrm{N}_{\Phi}(s)^{-1}$ respects the $\mathbf{A}_{\mathrm{f}}^{\times}$-homothety class of the self-pairing induced by ϕ.

Suppose for a moment that the diagonal mapping $\mathrm{V}_{\mathrm{f}}(A) \to \mathrm{V}_{\mathrm{f}}([\mathrm{N}_{\Phi}(s)]_K^{-1} \otimes_{\mathscr{O}_K} A)$ in (5.6) is induced by a K-linear isogeny $\psi_s \in \mathrm{Hom}^0_{\overline{\mathbf{Q}}}(A, [\mathrm{N}_{\Phi}(s)]_K^{-1} \otimes_{\mathscr{O}_K} A)$ *that respects* the K-linear $\mathbf{Q}$-polarizations ϕ and $\phi_{1/s}$ up to a (necessarily positive) rational multiple. The K-linear isogeny

$$\lambda_{\sigma,s} = \theta_{\sigma,s} \circ \psi_s \in \mathrm{Hom}^0_{\overline{\mathbf{Q}}}(A, A^{\sigma})$$

therefore carries ϕ to ϕ^{σ} up to a positive rational multiple and satisfies

$$\mathrm{V}_{\mathrm{f}}(\lambda_{\sigma,s}) \cdot \mathrm{N}_{\Phi}(s)^{-1} = [\sigma]$$

in $\mathrm{Hom}_{\mathbf{A}_{K,\mathrm{f}}}(\mathrm{V}_{\mathrm{f}}(A), \mathrm{V}_{\mathrm{f}}(A^\sigma))$, completing the proof of the Main Theorem! (Note that this map $\lambda_{\sigma,s}$ coincides with the one introduced with the same notation in the initial part of §3, where we saw that $\lambda_{\sigma,s}$ carries ϕ^σ back to $q_s\phi$ with q_s the unique positive generator of $[\mathrm{N}_{K^*/\mathbf{Q}}(s)]_{\mathbf{Q}}$.)

To construct ψ_s, first recall that for any $c \in (K^*)^\times$ the multiplier $\mathrm{N}_\Phi(c) \in T(\mathbf{Q}) \subseteq K^\times$ acting on A (in the isogeny category) preserves ϕ up to a rational multiple. Since $\theta_{\sigma,cs} = \mathrm{N}_\Phi(c)\theta_{\sigma,s}$, it follows that we only need to construct ψ_{cs} for some $c \in (K^*)^\times$ (and then $\mathrm{N}_\Phi(c)\cdot\psi_{cs}$ serves as ψ_s). Choosing $c \in \mathscr{O}_{K^*} - \{0\}$ that is sufficiently divisible (depending on s), we can replace s with cs to reduce to the case when $\mathrm{N}_\Phi(cs) = \mathrm{N}_\Phi(c)\mathrm{N}_\Phi(s) \in \mathbf{A}_{K,\mathrm{f}}^\times$ is everywhere integral as an adele. Hence, $\mathscr{O}_K \subseteq [\mathrm{N}_\Phi(s)]_K^{-1}$. In this case the canonical mapping

$$\mathrm{V}_{\mathrm{f}}(A) \to [\mathrm{N}_\Phi(s)]_K^{-1} \otimes_{\mathscr{O}_K} \mathrm{V}_{\mathrm{f}}(A)$$

is inverse to the vertical multiplication mapping in (5.6), so the map $A \to [\mathrm{N}_\Phi(s)]_K^{-1} \otimes_{\mathscr{O}_K} A$ induced by the canonical inclusion $\mathscr{O}_K \to [\mathrm{N}_\Phi(s)]_K^{-1}$ may be taken to be ψ_s. (The compatibility of this map with K-linear $\mathbf{Q}$-polarizations follows from property (3) in Lemma 5.1, taking $\alpha = 1$ and $\beta = 1/s$.) The Main Theorem of complex multiplication is now proved.

Remark 5.4. Without recourse to replacing s with cs for a sufficiently divisible nonzero $c \in \mathscr{O}_K$ as in the preceding proof, the map ψ_s can always be described by the same recipe as at the end of the proof: it is the element in

$$\mathrm{Hom}^0_K(A, [\mathrm{N}_\Phi(s)]_K^{-1} \otimes_{\mathscr{O}_K} A) \simeq [\mathrm{N}_\Phi(s)]_K^{-1} \otimes_{\mathscr{O}_K} \mathrm{Hom}^0_K(A, A) \simeq K$$

corresponding to 1 (where the second isomorphism is induced by multiplication and the first isomorphism is [X, Lemma 1.4]).

6. Analytic version of Main Theorem and generalized Serre construction

We conclude these notes by explaining how to express the Main Theorem in terms of adelic operations and analytic uniformizations. This recovers the traditional statement of the Main Theorem, except that we formulate things in more intrinsic terms; for example, our formulation of the analytic version of the Main Theorem omits all mention of bases of tangent spaces, in contrast with [S, 18.6]. We also explain how to translate such adelic operations into purely algebraic language by generalizing the Serre tensor construction within the framework of certain abelian varieties of CM type over a field, and this provides a purely algebraic generalization of Theorem 5.2 to the case of any CM order.

Let (A, i) be a CM abelian variety over $\mathbf{C}$ with CM type (K, Φ), for $\Phi \subseteq \mathrm{Hom}(K, \mathbf{C}) = \mathrm{Hom}(K, \overline{\mathbf{Q}})$. The pair (A, i) and any morphism among such pairs uniquely and functorially descends to the subfield $\overline{\mathbf{Q}}$, so to avoid far-out things such as $\mathrm{Aut}(\mathbf{C}/\mathbf{Q})$ we work with such abelian varieties over $\overline{\mathbf{Q}}$.

Consider the canonical analytic exponential uniformization $V/\Lambda \simeq A(\mathbf{C})$. Clearly Λ is a module for the CM order $i^{-1}(\mathrm{End}(A)) \subseteq \mathscr{O}_K$, and $A(\mathbf{C})_{\mathrm{tor}} = \Lambda_{\mathbf{Q}}/\Lambda$. Hence, $\Lambda_{\mathbf{Q}}$ is a 1-dimensional K-vector space and the $\mathbf{R}$-vector space $\mathbf{R}\otimes_{\mathbf{Q}}\Lambda_{\mathbf{Q}} \simeq V$ with its complex structure is isomorphic to $(\mathbf{R}\otimes_{\mathbf{Q}}K)_\Phi \simeq \prod_{\varphi\in\Phi}\mathbf{C}_\varphi = \mathbf{C}^g$ as a $\mathbf{C}\otimes_{\mathbf{Q}}K$-module (where K acts on $\mathbf{C}_\varphi = \mathbf{C}$ through φ). Traditionally, an identification of V with $(\mathbf{R}\otimes_{\mathbf{Q}}K)_\Phi$ as $\mathbf{C}\otimes_{\mathbf{Q}} K$-modules is chosen in the analytic formulation of the Main Theorem, apparently because this identifies the "abstract" Λ with something more concrete, namely a $\mathbf{Z}$-lattice in $(\mathbf{R}\otimes_{\mathbf{Q}} K)_\Phi = \mathbf{C}^g$ that is stable under some order of $\mathscr{O}_K$. However, we do not understand what benefit is obtained in this way, and so we avoid such a choice. Given $\sigma \in \mathrm{Aut}(\overline{\mathbf{Q}}/K^*)$, our goal is to describe an analytic uniformization of $A^\sigma(\mathbf{C})$ in terms of the canonical one for A such that the mapping $A(\mathbf{C})_{\mathrm{tor}} = A(\overline{\mathbf{Q}})_{\mathrm{tor}} \overset{\sigma}{\simeq} A^\sigma(\overline{\mathbf{Q}})_{\mathrm{tor}} = A^\sigma(\mathbf{C})_{\mathrm{tor}}$ is easily described via an adelic operation on lattices arising from these analytic uniformizations.

We need to first discuss adelic operations on $\Lambda_{\mathbf{Q}}/\Lambda$. Somewhat more abstractly, let W be a finite-dimensional nonzero K-vector space and let $\Lambda \subseteq W$ be a $\mathbf{Z}$-lattice that is stable under some order $\mathscr{O}'$ of $\mathscr{O}_K$. We call such a lattice Λ an *order lattice* in W. There are many orders $\mathscr{O}'$ that preserve Λ, but there is a unique largest one, namely $\mathrm{End}_{\mathbf{Z}}(\Lambda)\cap K \subseteq \mathscr{O}_K$ inside of $\mathrm{End}_{\mathbf{Q}}(W)$, and it is called the *endomorphism order* for Λ in K. The quotient W/Λ is a torsion $\mathbf{Z}$-module, so it is a torsion $\mathscr{O}'$-module. Under contraction each maximal ideal of $\mathscr{O}_K$ gives rise to one of $\mathscr{O}'$ (though several on $\mathscr{O}_K$ may give rise to the same on $\mathscr{O}'$), and for each maximal ideal v' of $\mathscr{O}'$ we write $\mathscr{O}'_{v'}$ to denote the corresponding completion of $\mathscr{O}'$, so this completion is a *local* order in the semi-local product $\prod_{v|v'} \mathscr{O}_v$ of the local rings for K at places over v'. In particular, $\mathscr{O}'_{v'}$ has total ring of fractions $K_{v'} \stackrel{\text{def}}{=} \prod_{v|v'} K_v$. For all but finitely many v on K we have that v is the only prime of $\mathscr{O}_K$ over its contraction v' on $\mathscr{O}'$ and then $\mathscr{O}'_{v'} = \mathscr{O}_{K_v}$ inside of K_v. There is a canonical isomorphism

$$W/\Lambda \simeq \bigoplus_{v'} (W_{v'}/\Lambda_{v'})$$

with $W_{v'} = \prod_{v|v'}(K_v \otimes_K W)$ a free module of rank 1 over $K_{v'} = \prod_{v|v'} K_v$ and $\Lambda_{v'} = \mathscr{O}'_{v'} \otimes_{\mathscr{O}'} \Lambda$; concretely, $W_{v'}/\Lambda_{v'}$ is identified with the $\mathfrak{m}'_{v'}$-power torsion submodule of W/Λ. It is easy to check that $\Lambda_{v'} \subseteq W_{v'}$ is the closure of Λ in $W_{v'}$, and that if we shrink the order $\mathscr{O}'$ to some $\mathscr{O}'' \subseteq \mathscr{O}'$ then for each maximal ideal v'' of $\mathscr{O}''$ we have $W_{v''} = \prod_{v'|v''} W_{v'}$ and $\Lambda_{v'} = \prod_{v'|v''} \Lambda_{v'}$. In this sense, the description of the primary decomposition of W/Λ is well-behaved with respect to change in the choice of order $\mathscr{O}'$ in K preserving Λ (as there is a unique maximal such $\mathscr{O}'$ containing all others).

Lemma 6.1. *Let $\mathscr{O}'$ be the endomorphism order for Λ in K. For any $s \in \mathbf{A}_{K,\mathrm{f}}^{\times}$, there is a unique order lattice $s\Lambda$ in W with endomorphism order $\mathscr{O}'$ such that $(s\Lambda)_{v'} = s_{v'}\Lambda_{v'}$ inside of $W_{v'}$ for all v' on $\mathscr{O}'$, where $s_{v'} \in \prod_{v|v'} s_v$ denotes the v'-part of the idele s, and its endomorphism order in K is equal to that of Λ.*

There is a unique $\mathscr{O}'$-linear isomorphism $W/\Lambda \simeq W/(s\Lambda)$ such that on v'-factors it is the mapping $W_{v'}/\Lambda_{v'} \simeq W_{v'}/(s_{v'}\Lambda_{v'})$ induced by multiplication by $s_{v'}$ on $W_{v'}$.

If $s, s' \in \mathbf{A}_{K,\mathrm{f}}^{\times}$ then $s'(s\Lambda) = (s's)\Lambda$ inside of W.

A quick proof of Lemma 6.1 can be given by working with $\mathbf{A}_{\mathbf{Q},\mathrm{f}}$-modules and $\widehat{\mathbf{Z}}$-algebras (such as $\mathscr{O}'^{\wedge}$), but we prefer to give a proof that is "intrinsic" to the order $\mathscr{O}'$ and avoids the crutch of the subring $\mathbf{Z}$ over which $\mathscr{O}'$ is finite and flat.

Proof. For all but finitely many v', the endomorphism order $\mathscr{O}'$ for Λ is maximal at v' and $s_{v'} \in \mathscr{O}_{K_{v'}}^{\times}$. Hence, by multiplying Λ by a suitable divisible nonzero element of $\mathscr{O}_K$ we get an $\mathscr{O}_K$-lattice $\Lambda' \subseteq \Lambda$ such that $\prod_{v|v'} \Lambda'_v \subseteq s_{v'}\Lambda_{v'}$ for all v'. Moreover, for all but finitely many v' we have $\prod_{v|v'} \Lambda'_v = s_{v'}\Lambda_{v'} = \Lambda_{v'}$ inside of $W_{v'}$. Our construction problem therefore takes place inside of the $\mathscr{O}_K$-module W/Λ', and we need to look inside of the I-torsion submodule for a suitably divisible nonzero ideal I of $\mathscr{O}_K$. Such a torsion submodule is of finite length over $\mathscr{O}_K$, and so decomposing along the finitely many primes in its support gives a solution to our $\mathscr{O}'$-module existence problem via finitely many local constructions. The uniqueness is seen in the same way.

The uniqueness implies $s'(s\Lambda) = (s's)\Lambda$. By working locally, we see that the endomorphism orders for Λ and $s\Lambda$ coincide. The existence and uniqueness of the desired isomorphism $W/\Lambda \simeq W/(s\Lambda)$ is proved by using the decomposition into v'-components for each v'. ∎

Example 6.2. If $\Lambda \subseteq W$ is an $\mathscr{O}_K$-submodule then $s\Lambda = [s]_K\Lambda$, where $[s]_K$ is the fractional K-ideal associated to $s \in \mathbf{A}_{K,\mathrm{f}}^{\times}$.

We can now prove the Main Theorem in its analytic guise as originally stated by Shimura and Taniyama in a coordinatized manner (but we avoid their "coordinates" on tangent spaces). There are two parts to this theorem: the first gives a description of analytic uniformizations for a Galois twist preserving the reflex field and the second describes the Riemann form of a Galois twist of a K-linear polarization (providing an analytic version of the observation in §3 that $\lambda_{\sigma,s}$ carries ϕ^{σ} back to $q_s\phi$ with $q_s \in \mathbf{Q}_{>0}^{\times}$ the unique positive generator of $[\mathrm{N}_{K^*/\mathbf{Q}}(s)]_{\mathbf{Q}}$).

Theorem 6.3 (Main Theorem of CM; analytic form)**.** *Pick $\sigma \in \mathrm{Aut}(\overline{\mathbf{Q}}/K^*)$ and $s \in \mathbf{A}_{K^*,\mathrm{f}}^{\times}$ such that $(s|K^*) = \sigma|_{(K^*)^{\mathrm{ab}}}$ in $\mathrm{Gal}((K^*)^{\mathrm{ab}}/K^*)$. Let ϕ be a K-linear $\mathbf{Q}$-polarization of A. There is a unique $\mathbf{C} \otimes_{\mathbf{Q}} K$-linear identification of $V = \mathrm{T}_0(A(\mathbf{C}))$ with $\mathrm{T}_0(A^{\sigma}(\mathbf{C}))$ under which the canonical analytic uniformization of*

$A^\sigma(\mathbf{C})$ is identified with $V/\mathrm{N}_\Phi(1/s)\Lambda$ and the diagram

$$\begin{tikzcd}
\Lambda_{\mathbf{Q}}/\Lambda \arrow[r, "\simeq"] \arrow[d, "\mathrm{N}_\Phi(1/s)"', "\simeq"] & A(\mathbf{C})_{\mathrm{tor}} & A(\overline{\mathbf{Q}})_{\mathrm{tor}} \arrow[l, "\simeq"'] \arrow[d, "\simeq"', "{[\sigma]}"] \\
\Lambda_{\mathbf{Q}}/\mathrm{N}_\Phi(1/s)\Lambda \arrow[r, "\simeq"'] & A^\sigma(\mathbf{C})_{\mathrm{tor}} & A^\sigma(\overline{\mathbf{Q}})_{\mathrm{tor}} \arrow[l, "\simeq"']
\end{tikzcd}$$

commutes. Under the identification of $\mathrm{H}_1(A^\sigma(\mathbf{C}),\mathbf{Q})$ with $\Lambda_{\mathbf{Q}}$, the $\mathbf{Q}(1)$-valued Riemann form Ψ_{ϕ^σ} is identified with $q_s\Psi_\phi$, where q_s is the unique positive generator of the fractional $\mathbf{Q}$-ideal $[\mathrm{N}_{K^*/\mathbf{Q}}(s)]_{\mathbf{Q}}$.

See Remark 2.5 for the reason $1/s$ rather than s intervenes in this diagram.

Proof. Any two such isomorphisms $V \simeq \mathrm{T}_0(A^\sigma(\mathbf{C}))$ are related through the action of an automorphism of V that preserves Λ and induces on $V/\Lambda \simeq A(\mathbf{C})$ an analytic self-map that is the identity on all torsion points and so is the identity. This establishes uniqueness.

For existence, we use the algebraic form of the Main Theorem over $\overline{\mathbf{Q}}$ in Theorem 2.4. As we saw at the start of §3, this provides a (necessarily unique) K-linear isomorphism in the isogeny category $\lambda_{\sigma,s} \in \mathrm{Hom}^0_{\overline{\mathbf{Q}}}(A, A^\sigma)$ (possibly not a genuine morphism of abelian varieties) such that the diagram

$$\begin{tikzcd}
\mathrm{V}_{\mathrm{f}}(A) \arrow[r, "{[\sigma]}", "\simeq"'] \arrow[d, "\mathrm{N}_\Phi(1/s)"', "\simeq"] & \mathrm{V}_{\mathrm{f}}(A^\sigma) \\
\mathrm{V}_{\mathrm{f}}(A) \arrow[ur, "\mathrm{V}_{\mathrm{f}}(\lambda_{\sigma,s})"'] &
\end{tikzcd} \tag{6.1}$$

commutes and $\lambda_{\sigma,s}$ carries ϕ^σ back to $q_s\phi$.

Letting V_σ/Λ_σ be the canonical analytic uniformization of $A^\sigma(\mathbf{C})$, there is a well-defined K-linear map $\mathrm{H}_1(\lambda_{\sigma,s}) : \Lambda_{\mathbf{Q}} \simeq (\Lambda_\sigma)_{\mathbf{Q}}$ on rational homology lattices. The $\mathbf{R}$-scalar extension of this $\mathbf{Q}$-vector space isomorphism is the $\mathbf{C}$-linear isomorphism $V \simeq V_\sigma$ induced by $\lambda_{\sigma,s}$ on tangent spaces at the identity. I claim that $\mathrm{H}_1(\lambda_{\sigma,s})$ carries $\mathrm{N}_\Phi(1/s)\Lambda$ isomorphically onto Λ_σ. It suffices to check this inside of the $\mathbf{A}_{\mathrm{f}}$-modules obtained through extension of scalars, but the above commutative diagram of V_{f}'s can be rewritten as a commutative diagram

$$\begin{tikzcd}
\mathbf{A}_{\mathrm{f}} \otimes_{\mathbf{Z}} \Lambda \arrow[r, "{[\sigma]_{\mathbf{A}_{\mathrm{f}}}}", "\simeq"'] \arrow[d, "\mathrm{N}_\Phi(1/s)"'] & \mathbf{A}_{\mathrm{f}} \otimes_{\mathbf{Z}} \Lambda_\sigma \\
\mathbf{A}_{\mathrm{f}} \otimes_{\mathbf{Z}} \Lambda \arrow[ur, "\mathrm{H}_1(\lambda_{\sigma,s})"'] &
\end{tikzcd} \tag{6.2}$$

where $[\sigma]_{\mathbf{A}_{\mathrm{f}}}$ is obtained by applying the scalar extension $\widehat{\mathbf{Z}} \to \mathbf{A}_{\mathrm{f}}$ to the isomorphism

$$\widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} \Lambda = \mathrm{T}_{\mathrm{f}}(A) \overset{[\sigma]}{\simeq} \mathrm{T}_{\mathrm{f}}(A^\sigma) = \widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} \Lambda_\sigma$$

of finite free $\widehat{\mathbf{Z}}$-modules. Hence, the image $\widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} (\mathrm{N}_\Phi(1/s)\Lambda)$ of $\widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} \Lambda$ along the left side of (6.2) (check this really is the image!) is carried by the $\mathbf{A}_{\mathrm{f}}$-module map $\mathbf{A}_{\mathrm{f}} \otimes_{\mathbf{Q}} \mathrm{H}_1(\lambda_{\sigma,s})$ isomorphically onto $\widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} \Lambda_\sigma$. This confirms that $\mathrm{H}_1(\lambda_{\sigma,s})$ carries $\mathrm{N}_\Phi(1/s)\Lambda$ onto Λ_σ, since $\mathbf{Z} \to \widehat{\mathbf{Z}}$ is faithfully flat.

In this way, we get an analytic isomorphism $V/\mathrm{N}_\Phi(1/s)\Lambda \simeq V_\sigma/\Lambda_\sigma = A^\sigma(\mathbf{C})$. This identifies $A^\sigma(\mathbf{C})_{\mathrm{tor}}$ with $\Lambda_{\mathbf{Q}}/\mathrm{N}_\Phi(1/s)\Lambda$. Due to how this identification has been *constructed*, when it is used in conjunction with the identification $\Lambda_{\mathbf{Q}}/\Lambda \simeq A(\mathbf{C})_{\mathrm{tor}}$ the map $A(\mathbf{C})_{\mathrm{tor}} = A(\overline{\mathbf{Q}})_{\mathrm{tor}} \overset{[\sigma]}{\simeq} A^\sigma(\overline{\mathbf{Q}})_{\mathrm{tor}} = A^\sigma(\mathbf{C})_{\mathrm{tor}}$ is identified with the isomorphism $\Lambda_{\mathbf{Q}}/\Lambda \simeq \Lambda_{\mathbf{Q}}/\mathrm{N}_\Phi(1/s)\Lambda$ induced by multiplication by $\mathrm{N}_\Phi(1/s)$ on primary components (as in Lemma 6.1). ■

We leave it as an exercise for the reader to check that this analytic theorem implies Theorem 6.8, the algebraic version over $\overline{\mathbf{Q}}$ that we have called the Main Theorem in these notes. (Briefly, one runs the preceding proof in reverse to see how to reconstruct $\lambda_{\sigma,s} \in \mathrm{Hom}^0_{\overline{\mathbf{Q}}}(A, A^\sigma)$ from the specification of how to identify $V/\mathrm{N}_\Phi(1/s)\Lambda$ with V_σ/Λ_σ.)

Remark 6.4. For a principal CM abelian variety (A, i) of type (K, Φ) over $\overline{\mathbf{Q}}$, Theorem 6.3 and Example 6.2 provide an K-linear analytic isomorphism

$$A^\sigma(\mathbf{C}) \simeq V/\mathrm{N}_\Phi(1/s)\Lambda \simeq [\mathrm{N}_\Phi(1/s)]_K \otimes_{\mathscr{O}_K} (V/\Lambda) \simeq ([\mathrm{N}_\Phi(1/s)]_K \otimes_{\mathscr{O}_K} A)(\mathbf{C}),$$

and by GAGA this is induced by an K-linear isomorphism $A^\sigma \simeq [\mathrm{N}_\Phi(1/s)]_K \otimes_{\mathscr{O}_K} A$ over $\overline{\mathbf{Q}}$. This isomorphism is precisely the isomorphism $\theta_{\sigma,s}$ that is uniquely characterized in Theorem 5.2. In this sense, Theorem 6.3 may be considered to be a generalization of Theorem 5.2 to the non-principal case.

The interested reader may check as an exercise that the identification of Ψ_{ϕ^σ} with $q_s\Psi_\phi$ in Theorem 6.3 is equivalent to the description of Ψ_{ϕ^σ} in terms of Ψ_ϕ given in [S, 18.6] using the classical description of Riemann forms of K-linear $\mathbf{Q}$-polarizations via 4-tuples $(K, \Phi, \mathfrak{a}, \zeta)$ (resting on choosing bases for tangent spaces as rank-1 free modules over $(\mathbf{R} \otimes_{\mathbf{Q}} K)_\Phi$).

Let us finish the discussion of adelic operations by using a generalization the Serre tensor construction to give an algebraic formulation and proof of the adelic multiplication formalism in Theorem 6.3.

Let (A, i) and (A', i') be CM abelian varieties over a field k of characteristic 0, with CM field K. (Much of the discussion that follows can be carried out over rather general base schemes at the expense of more technical language.) Thus, $\mathrm{Hom}_k((A', i'), (A, i))$ is a lattice in the K-vector space $\mathrm{Hom}^0_k((A', i'), (A, i))$ that has dimension ≤ 1. Hence, for any finite idele $s \in \mathbf{A}^\times_{K,\mathrm{f}}$ another such lattice is $s\,\mathrm{Hom}_k((A', i'), (A, i))$. For a fixed (A, i), consider the following functor $F = F_{(A,i)}$ from such pairs (A', i') to the category of order lattices in finite-dimensional K-vector spaces: $F(A', i') = s\,\mathrm{Hom}_k((A', i'), (A, i))$. In the case $k = \mathbf{C}$, if $A = V/\Lambda$ and $A' = V'/\Lambda'$ then working locally over $\mathbf{Z}$ shows that $F(A', i') = \mathrm{Hom}_k((V'/\Lambda', i'), (V/s\Lambda, i_s))$ inside of $\mathrm{Hom}_K(\Lambda'_{\mathbf{Q}}, \Lambda_{\mathbf{Q}})$, with i_s denoting the action map for (an order in) K on $s\Lambda$. Thus, an algebraic version of the adelic operation on lattices is provided by:

Lemma 6.5. *Let (A, i) over k be as above, and work in the category of K-linear maps over k in what follows. The functor $F_{(A,i)}$ is represented by a pair (sA, i_s) and there is a canonical K-linear equality*

$$\mathrm{Hom}_k(A', sA) = \mathrm{Hom}^0_k(A', A) \cap \mathrm{Hom}_k(\mathrm{T_f}(A'), s\mathrm{T_f}(A))$$

inside of $\mathrm{Hom}_k(\mathrm{V_f}(A'), \mathrm{V_f}(A))$.

In particular, there is a canonical isomorphism $(sA)^\vee \simeq s^{-1}A^\vee$ and the formation of the representing object sA commutes with extension of the base field k'/k in the sense that the canonical map $(sA)_{k'} \to s \cdot A_{k'}$ is an isomorphism.

Proof. It is harmless to multiply s by a sufficiently divisible nonzero integer so that it lies in the profinite completion $\mathscr{O}^\wedge$ of a common order $\mathscr{O} \subseteq \mathscr{O}_K$ acting on A and A'. Hence, suppressing explicit mention of the K-action maps, $s\,\mathrm{Hom}_k(A', A) \subseteq \mathrm{Hom}_k(A', A)$ is given by local conditions: its ℓ-adic completion is $s_\ell(\mathbf{Z}_\ell \otimes_{\mathbf{Z}} \mathrm{Hom}_k(A', A))$ for all primes ℓ (with $s_\ell \in (\mathbf{Q}_\ell \otimes_{\mathbf{Q}} K)^\times$). Under the injection of $\mathrm{Hom}_k(A', A)$ into the $\mathscr{O}_\ell$-module $\mathrm{Hom}_k(A'[\ell^\infty], A[\ell^\infty])$ such elements are precisely those that kill the kernel of the isogeny s_ℓ on $A'[\ell^\infty]$. By $\mathscr{O}_\ell$-linearity of the maps under consideration, it is equivalent to require that the dual map of ℓ-divisible groups kills the kernel of s_ℓ acting on $A^\vee[\ell^\infty]$ through duality (without the intervention of complex conjugation on K).

Hence, if we let $G_\ell \subseteq A^\vee[\ell^\infty]$ be the k-finite kernel of the isogeny given by the dual action of s_ℓ then $G_\ell = 0$ for all but finitely many ℓ and so there is a unique k-finite subgroup G in $A^\vee$ whose ℓ-component is G_ℓ for all ℓ. Dualizing again, $F_{(A,i)}$ consists of those $\mathscr{O}$-linear maps $A' \to A$ over k that lift (necessarily uniquely) through the isogeny $(A^\vee/G)^\vee \to A$. That is, $(A^\vee/G)^\vee$ represents the functor. It is clear from this construction that the other properties hold. ■

The proof of the lemma also provides a canonical $\mathbf{A}_{K,\mathrm{f}}$-linear k-isomorphism $\mathrm{V_f}(sA) = \mathrm{V_f}(A)$ within which $\mathrm{T_f}(sA)$ goes over to $s\mathrm{T_f}(A)$, and this respects duality and extension of the base field. There is likewise a unique K-linear k-isomorphism $s'(sA) \simeq (s's)A$ compatible with the total Tate module description (so these isomorphisms are associative with respect to any three ideles). By the construction, in the case $k = \mathbf{C}$ these Tate-module descriptions are the $\widehat{\mathbf{Z}}$-scalar extensions arising from unique analytic isomorphisms $V/(s\Lambda) \simeq s \cdot (V/\Lambda)$. In this sense, the formation of sA is an algebraic substitute for adelic operations on uniformization lattices and so (via the Lefschetz principle) it follows that the CM type of sA coincides with that of A.

Example 6.6. By [X, Lemma 1.4], in the case of CM order $\mathscr{O}_K$ there are canonical K-linear k-isomorphisms $sA \simeq [s]_K \otimes_{\mathscr{O}_K} A$ that respect iteration of the idelic multiplication and Serre tensor operations. In the case $s = \mathrm{N}_\Phi(1/s')$ for $s' \in \mathbf{A}_{K^*,\mathrm{f}}^\times$, the resulting identification $\mathrm{V}_\mathrm{f}(A) = \mathrm{V}_\mathrm{f}(sA) \simeq \mathrm{V}_\mathrm{f}([s]_K \otimes_{\mathscr{O}_K} A)$ is $\mathrm{V}_\mathrm{f}(\psi_{s'})$ for $\psi_{s'}$ as in §5, due to Remark 5.4.

The following lemma is obvious, and by the *proof* of Lemma 5.1 it recovers the construction in Theorem 5.2 in the principal case:

Lemma 6.7. *Let ϕ be a K-linear polarization on a pair (A, i) over $\overline{\mathbf{Q}}$ with CM type (K, Φ). For all $s \in \mathbf{A}_{K^*,\mathrm{f}}^\times$ there is a unique K-linear polarization ϕ_s on $\mathrm{N}_\Phi(s)A$ such that the $\mathbf{A}_{K,\mathrm{f}}$-linear identification $\mathrm{V}_\mathrm{f}(A) = \mathrm{V}_\mathrm{f}(\mathrm{N}_\Phi(s)A)$ carries e_{ϕ_s} to $q_s^{-1}e_\phi$ as total Weil pairings, where $q_s \in \mathbf{Q}_{>0}^\times$ is the unique positive generator of $[\mathrm{N}_{K^*/\mathbf{Q}}(s)]_\mathbf{Q}$.*

The algebraic version of Theorem 6.3 is:

Theorem 6.8. *Let (A, i) be a CM abelian variety over $\overline{\mathbf{Q}}$ with CM type (K, Φ). Choose $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/K^*)$ and $s \in \mathbf{A}_{K^*,\mathrm{f}}^\times$ mapping to $\sigma|_{(K^*)^{\mathrm{ab}}}$ under the Artin map. There is a unique K-linear isomorphism $\theta_{\sigma,s}$: $\mathrm{N}_\Phi(1/s) \cdot A \simeq A^\sigma$ with respect to which the composite map*

$$\mathrm{T}_\mathrm{f}(A) \stackrel{\sigma}{\simeq} \mathrm{T}_\mathrm{f}(A^\sigma) \stackrel{\theta_{\sigma,s}^{-1}}{\simeq} \mathrm{T}_\mathrm{f}(\mathrm{N}_\Phi(1/s) \cdot A) = \mathrm{N}_\Phi(1/s)\mathrm{T}_\mathrm{f}(A)$$

is multiplication by $\mathrm{N}_\Phi(1/s) \in \mathbf{A}_{K,\mathrm{f}}^\times$.

Moreover, for any K-linear polarization ϕ the isomorphism $\theta_{\sigma,s}$ carries ϕ^σ to $\phi_{1/s}$.

The analogue of Remark 5.3 carries over by essentially the same argument (chasing actions on V_f's rather than at torsion levels).

Proof. This can be deduced from the analytic version in Theorem 6.3, but let us instead give a purely algebraic proof. In the special case that A has CM order $\mathscr{O}_K$, the theorem is exactly Theorem 5.2. In the general case, uniqueness for $\theta_{\sigma,s}$ is clear. For existence, first observe that if we construct $\theta_{\sigma,s}$ merely in the isogeny category subject to the condition that composing $\mathrm{V}_\mathrm{f}(\theta_{\sigma,s}^{-1})$ and $[\sigma]$ is multiplication by $\mathrm{N}_\Phi(1/s)$ on $\mathrm{V}_\mathrm{f}(A)$ then consideration with total Tate modules forces $\theta_{\sigma,s}$ to be a genuine isomorphism of abelian varieties. Hence, it is enough to carry out the existence proof in the isogeny category. Moreover, if $A' \to A$ is a K-linear isomorphism in the isogeny category then it is equivalent to solve the existence problem for A or for A' (by functoriality of the idelic multiplication operation on abelian varieties with a fixed CM type). But every A of type (K, Φ) over $\overline{\mathbf{Q}}$ is K-linearly isogenous to one with CM order $\mathscr{O}_K$. Hence, we are done. ■

Theorem 6.8 accomplishes something remarkable: without Galois theory or complex analysis, it gives a purely adelic construction of $(A^\sigma, i^*, \phi^\sigma)$ from (A, i, ϕ): this σ-twisted triple is $(\mathrm{N}_\Phi(1/s) \cdot A, i_{\mathrm{N}_\Phi(1/s)}, \phi_{1/s})$ with the canonical $\mathbf{A}_{K,\mathrm{f}}$-linear identification $\mathrm{V}_\mathrm{f}(\mathrm{N}_\Phi(1/s)A) = \mathrm{V}_\mathrm{f}(A)$ carrying $e_{\phi_{1/s}}$ to $q_s e_\phi$, and the K-linear isomorphism $\theta_{\sigma,s} : \mathrm{N}_\Phi(1/s)A \simeq A^\sigma$ is uniquely characterized by the fact that it fits into the commutative diagram

$$\begin{tikzcd} \mathrm{V}_\mathrm{f}(A) \arrow[r, "{[\sigma]}", "\simeq"'] \arrow[d, "\mathrm{N}_\Phi(1/s)"', "\simeq"] & \mathrm{V}_\mathrm{f}(A^\sigma) \\ \mathrm{V}_\mathrm{f}(A) \arrow[r, "\simeq"'] & \mathrm{V}_\mathrm{f}(\mathrm{N}_\Phi(1/s)A) \arrow[u, "\simeq", "\mathrm{V}_\mathrm{f}(\theta_{\sigma,s})"'] \end{tikzcd}$$

with the bottom side given by the canonical identification (that in turn arises from a unique K-linear isogeny $\psi_s : A \to \mathrm{N}_\Phi(1/s)A)$.

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