

# REDUCTIVE GROUP SCHEMES (SGA3 SUMMER SCHOOL, 2011)

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## INTRODUCTION

The aim of these notes is to develop the theory of reductive group schemes, incorporating some simplifications into the methods of [SGA3]. We assume the reader is familiar with the basic structure theory over an algebraically closed field  $k$  (as developed in [Bo91], [Hum87], [Spr]), but a review is given in §1 in order to fix our terminology, set everything in the framework of  $k$ -schemes (rather than classical varieties), and provide a convenient reference for the subsequent scheme-theoretic developments.

We give complete proofs of the main results in the theory of reductive groups over any non-empty scheme (conjugacy theorems, scheme of maximal tori, construction of root groups and root datum, structure of open cell, parameterization of parabolics, schemes of Borel and parabolic subgroups, Existence and Isomorphism theorems, existence of automorphism scheme), apart from some calculations with low-rank root systems. We do *not* assume the Existence and Isomorphism Theorems over a general algebraically closed field  $k$  because the scheme-theoretic approach proves these over any non-empty base scheme granting only the Existence Theorem over  $k = \mathbf{C}$  (which we prove in Appendix D). Although the structure theory of connected reductive groups is developed over a general field in [Bo91] and [Spr], we take on faith only the case of an algebraically closed ground field (as is done in [SGA3] via its reliance on [BIBLE]). Some results are proved here in less generality than in [SGA3], since our aim is to reach the structure theory of reductive group schemes as quickly as possible.

**Background.** We assume familiarity with smooth and étale morphisms (e.g., functorial criteria for each; cf. [BLR, 2.1–2.2]), faithfully flat descent (developed in [BLR, 6.1–6.2]), and the functorial approach to group schemes (e.g., scheme-theoretic kernels, intersections of closed subgroup schemes, and quotient morphisms via sheaves for a Grothendieck topology). We also use Cartier duality for finite flat commutative group schemes (of finite presentation).

In the arguments after §1 we use results from the theory of multiplicative type group schemes in [SGA3, IX, X]. This material is covered in Appendix B, building on Oesterlé’s lectures [Oes]. We use a more restricted notion of “multiplicative type” than in [SGA3] and [Oes]. In these notes, by definition *multiplicative type groups are required to be of finite type*

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*Date:* December 31, 2011.

This work was supported by NSF grant DMS-0917686. I am grateful to A. Auel, O. Gabber, S. Garibaldi, B. Gross, L. Illusie, J. Parson, P. Polo, G. Prasad, J-P. Serre, A. Venkatesh, and J-K. Yu for helpful suggestions.

over the base, and are required to split *fppf-locally* on the base. That is, they are fppf-local forms of diagonalizable groups of finite type. The fpqc topology is used instead of the fppf topology in the definition in [SGA3, IX, 1.1] and [Oes] (allowing fpqc groups that are not of finite type). Our restriction to the fppf topology is harmless, as we explain in Appendix B (due to the “finite type” requirement that we impose). We often use the fundamental fact that multiplicative type groups are necessarily split étale-locally on the base. It is inconvenient to include this in the *definition* of “multiplicative type”, so a proof of étale-local splitting is given in Proposition B.3.4 (as a mild variant of the proof of an analogous result in [SGA3, IX, 4.5]). The equivalence with fpqc-local triviality (under a finite-type hypothesis on the group) is proved in Corollary B.4.2(1) but is never used.

We require the notion of Lie algebra for a group scheme, including non-smooth group schemes. A reference that covers what we need (and beyond) is [CGP, A.7]. One of the basic properties we use for the Lie functor is that it commutes with fiber products of groups (and so is left exact), as one sees by consideration of points valued in the dual numbers over the base.

For a reader who is unfamiliar with the theory of algebraic spaces, in a few places it will be necessary to accept on faith that the theory of algebraic spaces provides a useful mechanism to equip certain set-valued functors with enough “geometric structure” that it makes sense to carry over concepts in the theory of schemes (e.g., properness, flatness, quasi-finiteness, etc.) to such functors; an excellent reference for this is [Knut]. All algebraic spaces we use will almost immediately be proved to be schemes, so our use of algebraic spaces will be similar to the use of distributions to provide function solutions to elliptic partial differential equations (i.e., they appear in the middle of a construction, the end result of which is an object of a more familiar type that we prefer to use).

**Theory over a field.** Many arithmetic applications (as well as applications over  $\mathbf{R}$ ) require a structure theory for connected reductive groups over general fields  $k$  without assuming the existence of a split (geometrically) maximal  $k$ -torus. Such a structure theory is due to Borel and Tits (see [BoTi] and [Bo91, §20–24]). In [SGA3, XXVI, 6.16–6.18, §7] it is partially generalized to reductive groups over non-empty connected semi-local schemes (and see [PS] for interesting recent work in this direction). We will not discuss this important topic.

However, there is a fundamental result at the foundation of the Borel–Tits structure theory for which we do provide a proof (in Appendix A): Grothendieck’s theorem [SGA3, XIV, 1.1] (later proved in more elementary terms by Borel and Springer) that any smooth affine group  $G$  over a field  $k$  admits a  $k$ -torus  $T \subset G$  such that  $T_{\bar{k}}$  is maximal in  $G_{\bar{k}}$ . This existence result implies that for *every*  $k$ -torus  $T'$  in  $G$  not contained in a strictly larger  $k$ -torus and for every extension field  $K/k$ ,  $T'_K$  is not contained in a strictly larger  $K$ -torus of  $G_K$  (see Remark A.1.2 for a proof). In particular, the concept of “maximal  $k$ -torus” in a smooth affine group over  $k$  is insensitive to ground field extension. (We never use this except in the elementary case  $k = \bar{k}$ .)

Since Grothendieck’s existence theorem for maximal tori over arbitrary fields is not needed in the development of the theory of reductive groups over general schemes, we have relegated

our discussion of his theorem to an appendix (which provides a scheme-theoretic version of the Borel–Springer proof of Grothendieck’s theorem). What matters for our purposes is the existence of a “geometrically maximal” torus defined over a finite *separable* extension of the ground field. We present Grothendieck’s construction of such a torus, using the existence and smoothness properties of a “scheme of maximal tori”. The proof of Grothendieck’s finer result that such a torus exists over the ground field uses a detailed study of Lie algebras. It can also be deduced from the deeper result that the scheme of maximal tori is a rational variety [SGA3, XIV, 6.1] (coupled with special arguments for finite ground fields).

**What’s new?** We take advantage of three post-1963 developments to streamline or simplify some of the original proofs:

(i) (Artin approximation). In the study of lifting problems, infinitesimal methods allow one to build liftings over the *completion* of a local noetherian ring (when starting with an algebro-geometric object over the residue field). The *Artin approximation theorem* [BLR, 3.6/16] provides a method to use such lifts over a “formal” neighborhood of a point to construct lifts over an étale neighborhood of a point. This solves global problems over an étale cover, a dramatic improvement on having solutions only in formal neighborhoods of points. We will use this reasoning to study of liftings of tori in smooth affine groups.

(ii) (algebraic spaces). A representability result of Murre [Mur, §3, Thm. 2, Cor. 2] is used in [SGA3] to build certain quotients by flat equivalence relations (see [SGA3, XVI, §2]). This underlies Grothendieck’s quotient constructions in [SGA3, XVI, 2.4]. Murre’s result is a precursor to Artin’s criteria for a functor on schemes to be an algebraic space. The work of Artin ([Ar69], [Ar74]) and Knutson [Knut] on algebraic spaces provides an ideal framework for a geometric theory of quotients by flat equivalence relations in algebraic geometry (and includes a sufficient criterion for an algebraic space to be a scheme), so we use algebraic spaces in place of Murre’s result.

(iii) (dynamic method). There is a “dynamic” approach to describing parabolic subgroups as well as their unipotent radicals and Levi factors in connected reductive groups over an algebraically closed field. This method involves the limiting behavior along orbits under the conjugation action of a 1-parameter subgroup; it is a standard tool in the classical setting (see [Bo91, 13.8(1),(2)], [Spr, 8.4.5]) and also arises in Mumford’s GIT.

The relative version of the dynamic viewpoint for group schemes over an arbitrary ring seems to have been missed in [SGA3], and was introduced in [CGP, 2.1] for the study of pseudo-reductive groups over imperfect fields. It leads to simplifications in several places. For instance, the arguments in [SGA3, XX] to construct closed root groups in split reductive group schemes, as well as to classify the split semisimple-rank 1 case over a general scheme, rest on elaborate computations. The relative dynamic method, which requires only elementary arguments to set up, yields an approach to these matters that eliminates the need for most of the computations in [SGA3, XX]. We will review what we need from the dynamic method, giving references to [CGP, 2.1] for the proofs.

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## 1. REVIEW OF THE CLASSICAL THEORY

We fix an algebraically closed field  $k$  and consider smooth affine group schemes over  $k$  (not necessarily connected). Such groups are called *linear algebraic groups* over  $k$  because they are precisely the smooth  $k$ -groups  $G$  of finite type for which there exists a  $k$ -homomorphism

$$j : G \hookrightarrow \mathrm{GL}(V) \simeq \mathrm{GL}_n$$

that is a closed immersion [Bo91, 1.10]. Although we will be working in the classical setting, we take the viewpoint of schemes rather than classical varieties. For instance, we do not use any “universal domain” as in [Bo91], and a linear algebraic group over  $k$  is a scheme rather than a set of  $k$ -points equipped with extra structure. Likewise, the *kernel* of a  $k$ -homomorphism always means the scheme-theoretic kernel (e.g.,  $\ker(\mathrm{SL}_n \rightarrow \mathrm{PGL}_n)$ ) is identified with the  $k$ -group scheme  $\mu_n = \ker(t^n : \mathbf{G}_m \rightarrow \mathbf{G}_m)$  even if  $\mathrm{char}(k)|n$ , and an intersection of closed linear algebraic  $k$ -subgroups of a linear algebraic  $k$ -group always means scheme-theoretic intersection (which may be non-smooth when  $\mathrm{char}(k) > 0$ ; e.g., the central  $\mathbf{G}_m$  in  $\mathrm{GL}_n$  meets  $\mathrm{SL}_n$  in precisely  $\mu_n$ , even if  $\mathrm{char}(k)|n$ ).

We do not require linear algebraic groups to be connected because there are interesting disconnected examples (such as orthogonal groups) and group-theoretic operations (centralizers, intersection of subgroups, kernels, etc.) can lead to disconnected (possibly non-smooth) groups. The identity component  $G^0$  of any linear algebraic  $k$ -group is irreducible (as for any smooth connected non-empty  $k$ -scheme).

The basic structure theory of linear algebraic groups is developed in Borel’s book [Bo91] and Springer’s book [Spr], as well as Chevalley’s [BIBLE]. In this section we summarize some of the highlights of this theory, as a review of classical notions and results that are needed in the development of the theory over a general base scheme in [SGA3]. We also discuss some aspects of the representation theory. It is assumed that the reader has prior experience with the classical theory.

Before we get started, it will be helpful to check the consistency between the notions of  $G/H$  in the classical and scheme-theoretic theories, with  $H$  a linear algebraic subgroup of a linear algebraic group  $G$ . This goes as follows. Classically, a smooth quasi-projective quotient  $G/H$  is built so that  $q : G \rightarrow G/H$  identifies  $(G/H)(k)$  with  $G(k)/H(k)$  and the map  $\mathrm{Tan}_e(q)$  is the natural surjection  $\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{h}$  [Bo91, 6.7, 6.8]. In the scheme-theoretic approach the “quotient” morphism  $G \rightarrow Q$  by the right  $H$ -action on  $G$  is uniquely characterized (if it exists!) by the condition that it represents the fppf-sheafification of the functor  $S \rightsquigarrow G(S)/H(S)$  on the category of  $k$ -schemes. We claim that the classical quotient construction satisfies this property, so it is the desired quotient in the sense of [SGA3].

To compare these concepts, first observe (via  $G(k)$ -translations) that  $q$  is surjective on tangent spaces at all  $g \in G(k)$ , so  $q$  is a smooth morphism (“submersion theorem”). The agreement with the scheme-theoretic notion of quotient amounts to saying that  $q$  is the quotient of  $G$  modulo the flat equivalence relation  $R := G \times H \hookrightarrow G \times G$  defined by  $(g, h) \mapsto (g, gh)$ . The map  $q$  is fppf, so by descent theory  $q$  is the quotient of  $G$  modulo the flat equivalence relation  $R' = G \times_{G/H} G \rightarrow G \times G$ . The relation  $R'$  is a closed subscheme

that is smooth (since  $q$  is smooth), and the same clearly holds for  $R$ . Hence, we just have to show that  $R = R'$  as subschemes of  $G \times G$ , so by smoothness it is equivalent to show  $R(k) = R'(k)$  inside  $G(k) \times G(k)$ . Since  $G(k)/H(k) \rightarrow (G/H)(k)$  is a bijection, the desired equality is clear.

**1.1. Solvable groups and reductive groups.** By the closed orbit lemma [Bo91, 1.8], if  $f : G' \rightarrow G$  is a  $k$ -homomorphism between linear algebraic  $k$ -groups then the image  $f(G')$  is a smooth closed  $k$ -subgroup of  $G$ . In particular, since  $G' \rightarrow f(G')$  is a surjective map between smooth  $k$ -schemes, by homogeneity using translation by  $k$ -points over  $k = \bar{k}$  it follows that generic flatness propagates everywhere, so  $G' \rightarrow f(G')$  is faithfully flat. Thus, descent theory gives that  $G' / (\ker f) \simeq f(G')$ ; i.e.,  $f(G')$  represents the quotient sheaf  $G' / (\ker f)$  for the fppf topology, where  $\ker f$  is the scheme-theoretic kernel. If  $\mathbf{P}$  is a property of scheme morphisms that is fppf local on the base (e.g., being finite, smooth, or an isomorphism) then the fppf map  $G' \rightarrow G' / (\ker f)$  satisfies  $\mathbf{P}$  if and only if  $\ker f \rightarrow \operatorname{Spec} k$  does, since  $G' \times_{G' / (\ker f)} G' \simeq \ker f \times G'$  via  $(g'_1, g'_2) \mapsto (g'_1 g'^{-1}_2, g'_2)$ . In particular, since  $\ker f$  is finite as a  $k$ -scheme if and only if  $(\ker f)(k)$  is finite (as  $k = \bar{k}$ ), we get:

**Proposition 1.1.1.** *If  $(\ker f)(k)$  is finite then  $f$  is finite flat onto its smooth closed image, and if  $\ker f = 1$  then  $f$  is a closed immersion. If  $f$  is surjective and  $\ker f = 1$  then  $f$  is an isomorphism.*

**Example 1.1.2.** The natural  $k$ -homomorphism  $\pi : \operatorname{SL}_n \rightarrow \operatorname{PGL}_n$  between finite type  $k$ -schemes is surjective on  $k$ -points, hence surjective as a scheme map, and the scheme-theoretic kernel  $\ker \pi = \mu_n$  is a finite  $k$ -scheme of degree  $n$ , so  $\pi$  is finite flat of degree  $n$ . If  $n = p = \operatorname{char}(k) > 0$  then  $\ker \pi$  is infinitesimal,  $\pi$  is purely inseparable, and  $\pi$  is bijective on  $k$ -points but is not an isomorphism.

It follows from Proposition 1.1.1 that if a linear algebraic group  $G'$  is a  $k$ -subgroup of another such group  $G$  in the sense that there is given a monic homomorphism  $G' \rightarrow G$  then  $G'$  is a closed  $k$ -subgroup of  $G$ . In particular, if a linear representation  $G \rightarrow \operatorname{GL}(V)$  is *faithful* in the sense that it has trivial (scheme-theoretic) kernel then it is necessarily a closed immersion.

**Example 1.1.3.** The action of  $\operatorname{SL}_n$  on the vector space  $V = \operatorname{Mat}_n$  of  $n \times n$  matrices via conjugation defines a linear representation  $\rho : \operatorname{SL}_n \rightarrow \operatorname{GL}(V)$  whose kernel is  $\mu_n$ . In particular, if  $n = \operatorname{char}(k) = p > 0$  then this representation is injective on  $k$ -points but it is not faithful in the scheme-theoretic sense (i.e.,  $\ker \rho \neq 1$ ).

The closed immersion property for monic homomorphisms between linear algebraic  $k$ -groups is a wonderful feature of the theory over fields; it is not true for smooth affine groups over rings (see Example 3.1.2) and criteria to ensure it over rings can lie quite deep (see Theorem 5.3.5).

We now recall notions related to Jordan decomposition for a linear algebraic group  $G$  (see [Bo91, 4.4] for proofs). Fix a faithful linear representation  $j : G \hookrightarrow \operatorname{GL}(V)$ . For any

$g \in G(k)$ , we say that  $g$  is *semisimple* if the linear endomorphism  $j(g)$  of  $V$  is semisimple in the sense of linear algebra; i.e.,  $j(g)$  is diagonalizable. Likewise, we say that  $g$  is *unipotent* if  $j(g)$  is unipotent as a linear endomorphism of  $V$ . These properties are independent of the choice of  $j$  and are preserved under any  $k$ -homomorphism  $f : G \rightarrow H$  to another linear algebraic  $k$ -group; that is, if  $g \in G(k)$  is semisimple (resp. unipotent) then so is  $f(g) \in H(k)$ .

In general for any  $g \in G(k)$  there are unique commuting elements  $g_{\text{ss}}, g_{\text{u}} \in G(k)$  such that  $g_{\text{ss}}$  is semisimple,  $g_{\text{u}}$  is unipotent, and  $g = g_{\text{ss}}g_{\text{u}} = g_{\text{u}}g_{\text{ss}}$ . We call these the *semisimple part* and *unipotent part* of  $g$  respectively, and refer to these product expressions for  $g$  as its *Jordan decomposition*. The existence of this Jordan decomposition demonstrates an advantage of linear algebraic groups over Lie algebras: the former is defined and studied in a characteristic-free way (over algebraically closed fields), whereas the development of the latter is entirely different in characteristic 0 and in positive characteristic (see [Sel, V.7.2] and [Hum98]).

The formation of the Jordan decomposition is functorial in the sense that if  $f : G \rightarrow H$  is a  $k$ -homomorphism to another linear algebraic  $k$ -group then

$$f(g_{\text{ss}}) = f(g)_{\text{ss}}, \quad f(g_{\text{u}}) = f(g)_{\text{u}}.$$

In particular, if  $H = \text{GL}(W)$  for a finite-dimensional vector space  $W$  then  $f(g)$  and  $f(g_{\text{ss}})$  have the same characteristic polynomial. This is a very useful fact.

**Definition 1.1.4.** A linear algebraic group  $G$  is *solvable* if  $G(k)$  is solvable, and is *unipotent* if  $g = g_{\text{u}}$  for all  $g \in G(k)$ .

**Remark 1.1.5.** A more elegant definition of unipotence, avoiding the crutch of  $\text{GL}_n$ -embeddings, is given in [SGA3, XVII, 1.3]; its equivalence with the above definition is given in [SGA3, XVII, 2.1].

By the Lie–Kochin theorem [Bo91, 10.5], if  $G$  is solvable and connected then every linear representation  $G \rightarrow \text{GL}_N$  can be conjugated to have image inside the upper triangular subgroup  $B_N$ . Also, *without* a connectedness hypothesis,  $G$  is unipotent precisely when it occurs as a closed subgroup of the strictly upper-triangular subgroup  $U_n \subset \text{GL}_n$  for some  $n$  [Bo91, 4.8]. In particular, every unipotent linear algebraic group is solvable.

**Example 1.1.6.** The group  $\mathbf{G}_a$  is unipotent, due to the faithful representation  $x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ .

The strictly upper triangular subgroup  $U_n$  of  $\text{GL}_n$  admits a composition series whose successive quotients are  $\mathbf{G}_a$ , so every unipotent linear algebraic group  $U$  admits a composition series whose successive quotients are  $\mathbf{G}_a$  or a finite étale subgroup of  $\mathbf{G}_a$ . When  $U$  is connected, we may arrange the composition series so that the successive quotients are all equal to  $\mathbf{G}_a$  [Bo91, 10.6(2), 10.9].

Functoriality of Jordan decomposition implies that every image or smooth closed subgroup of a unipotent linear algebraic group is again unipotent, and likewise that if a linear algebraic group  $G$  contains a unipotent normal linear algebraic subgroup  $U$  such that  $G/U$  is also unipotent then  $G$  is unipotent. Since  $\mathbf{G}_a$  has no nontrivial finite subgroups when  $\text{char}(k) = 0$  and its finite étale subgroups are  $p$ -groups when  $\text{char}(k) = p > 0$ , it follows from the

unipotence of the finite étale component group  $U/U^0$  of a unipotent linear algebraic group  $U$  that unipotent linear algebraic  $k$ -groups are connected when  $\text{char}(k) = 0$  and have component group that is a  $p$ -group when  $\text{char}(k) = p > 0$ .

A key result at the start of the theory of linear algebraic groups is [Bo91, 11.1, 11.2]:

**Theorem 1.1.7.** *Let  $G$  be a linear algebraic group over  $k$ . The maximal connected solvable linear algebraic  $k$ -subgroups of  $G$  are all  $G(k)$ -conjugate to each other, and these are precisely the connected solvable linear algebraic subgroups  $B$  such that the quasi-projective quotient scheme  $G/B$  is projective.*

A *Borel subgroup* of  $G$  is a maximal connected solvable linear algebraic subgroup of  $G$ . A *parabolic subgroup* of  $G$  is a linear algebraic subgroup  $P \subset G$  such that  $G/P$  is projective. All Borel subgroups are parabolic, by Theorem 1.1.7, so any linear algebraic subgroup containing a Borel subgroup is also parabolic. In fact, the parabolic subgroups are precisely the subgroups that contain Borel subgroups [Bo91, 11.2, Cor.]. Note that the definition of parabolicity does not require connectedness. The following result is fundamental (see [Bo91, 11.16] for a proof):

**Theorem 1.1.8** (Chevalley). *The parabolic subgroups  $P$  of any connected linear algebraic group  $G$  are connected, and  $N_{G(k)}(P) = P(k)$ .*

In Corollary 5.2.8 this will be improved to a scheme-theoretic equality  $N_G(P) = P$  when  $G$  is connected reductive.

**Example 1.1.9.** For  $G = \text{SL}_n$ , the subgroup  $B$  of upper triangular matrices is clearly solvable, and under the natural transitive  $G$ -action on the complete variety  $F$  of full flags in  $k^n$  the smooth group  $B$  is the stabilizer scheme of the standard flag  $F_0$ . Thus, the orbit map  $G \rightarrow F$  through  $F_0$  is faithfully flat (even smooth) and induces a scheme isomorphism  $G/B \simeq F$ , so  $B$  is also parabolic, hence a Borel subgroup.

Some obvious parabolic subgroups of  $G$  containing  $B$  are labelled by ordered partitions  $\vec{a} = (a_1, \dots, a_r)$  of  $n$  into non-empty parts (i.e., all  $a_i > 0$ ,  $\sum a_i = n$ ): we associate to  $\vec{a}$  the subgroup  $P_{\vec{a}}$  consisting of points of  $\text{SL}_n$  that preserve each of the subspaces  $V_j$  spanned by the first  $b_j := a_1 + \dots + a_j$  standard basis vectors. (Note that  $\vec{a} \mapsto \{b_1, \dots, b_{r-1}\}$  is a bijection from the set of such  $\vec{a}$  onto the set of subsets of  $\{1, \dots, n-1\}$ .) These  $2^{n-1}$  parabolic subgroups are the only parabolic subgroups containing  $B$ , so they represent (without repetition; see Corollary 1.4.8) the conjugacy classes of parabolic subgroups of  $G$ . (In Example 1.4.7 this example is addressed more fully.)

Let  $G$  be a linear algebraic group over  $k$ . If  $U, U' \subset G$  are connected *normal* unipotent linear algebraic subgroups of  $G$  then the normal closed subgroup  $U \cdot U'$  that they generate is unipotent (as it is a quotient of  $U \times U'$ ). Hence, by dimension considerations there exists a unique maximal connected unipotent normal linear algebraic subgroup  $\mathcal{R}_u(G) \subset G$ , called the *unipotent radical* of  $G$ . In a similar manner, there is a maximal connected solvable normal linear algebraic subgroup  $\mathcal{R}(G)$ , called the *radical* of  $G$ .

If  $H$  is a normal linear algebraic subgroup of  $G$  then  $\mathcal{R}_u(H) = (H \cap \mathcal{R}_u(G))_{\text{red}}^0$ , and similarly for radicals. (Note that the formation of the underlying reduced scheme is a local operation for the Zariski topology, so the formation of identity component and underlying reduced scheme of a finite type  $k$ -group scheme commute. In particular, there is no ambiguity in notation such as  $\mathcal{G}_{\text{red}}^0$  for such a group scheme  $\mathcal{G}$ .) These notions also behave well with respect to quotients: if  $\pi : G \twoheadrightarrow G'$  is a surjective homomorphism between linear algebraic groups then  $\mathcal{R}_u(G') = \pi(\mathcal{R}_u(G))$  and likewise for radicals. However, the proof of this result for quotient maps is nontrivial when  $\text{char}(k) > 0$  (if  $\ker \pi$  is not smooth); see [Bo91, 14.11] for an argument that works regardless of the smoothness properties of  $\pi$ .

**Definition 1.1.10.** A *reductive*  $k$ -group is a linear algebraic  $k$ -group  $G$  such that  $\mathcal{R}_u(G) = 1$  (i.e.,  $G$  contains no nontrivial unipotent normal connected linear algebraic  $k$ -subgroup). A *semisimple*  $k$ -group is a linear algebraic  $k$ -group  $G$  such that  $\mathcal{R}(G) = 1$ .

It is immediate from the good behavior of radicals and unipotent radicals with respect to normal subgroups and quotients that semisimplicity and reductivity are inherited by normal linear algebraic subgroups and images of homomorphisms.

**Example 1.1.11.** A basic example of a connected reductive group is the open unit group  $\text{GL}(V)$  in the affine space of linear endomorphisms of  $V$ . (This recovers  $\mathbf{G}_m$  when  $\dim V = 1$ .) Examples of connected semisimple groups are: the smooth irreducible hypersurface  $\text{SL}(V)$  in  $\text{GL}(V)$  (which recovers  $\text{SL}_2$  when  $\dim V = 2$ ), its quotient  $\text{PGL}(V)$ , and symplectic groups  $\text{Sp}(V, \psi)$  for non-degenerate alternating forms  $\psi$  on a nonzero finite-dimensional vector space  $V$ . (The connectedness of symplectic groups can be proved via induction on  $\dim V$  via a fibration argument, and the smoothness can be proved by the infinitesimal criterion.)

Special orthogonal groups  $\text{SO}(V, q)$  associated to non-degenerate quadratic spaces  $(V, q)$  of dimension  $\geq 3$  are also connected semisimple, but special care is needed to give a characteristic-free development of such groups that works well in characteristic 2 without parity restrictions on  $\dim V$ ; see Exercise 1.6.9. Smoothness and connectedness of special orthogonal groups are proved in Appendix C.

For proofs of reductivity of these groups, see Exercises 1.6.10(i) and 1.6.15.

**Remark 1.1.12.** There is no universal convention as to whether or not reductive groups should be required to be connected. If  $G$  is connected and reductive then the centralizer in  $G$  of any torus is connected and reductive (see Theorem 1.1.18(3)), so for arguments with torus centralizers there is no harm in requiring connectedness. A more subtle case is the centralizer  $Z_G(g)$  for connected reductive  $G$  and semisimple  $g \in G(k)$ . (The  $k$ -group  $Z_G(g)$  is smooth even when defined as a fiber product, to represent a “centralizer functor”; see [Bo91, 9.2, Cor.] and Exercise 1.6.8(iii)). Such centralizers appear in orbital integrals in the trace formula for automorphic forms. In general  $Z_G(g)^0$  is reductive (see Theorem 1.1.18(3)) but  $Z_G(g)$  may be disconnected (e.g.,  $G = \text{PGL}_2$  and  $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  with  $\text{char}(k) \neq 2$ ; for  $\text{SL}_2$  the analogous centralizer is connected). Whether or not to prefer saying that  $Z_G(g)$  is reductive or that  $Z_G(g)^0$  is reductive (for semisimple  $g \in G(k)$ ) is a matter of taste.

In [Bo91] and [Spr], as well as in these notes, reductive groups over a field are not assumed to be connected. This is a contrast with [SGA3, XIX, 2.7], and it may seem to present a slightly inconsistency with the general definition over schemes, but no real confusion should arise. For reductive groups over schemes, the usefulness of a fibral connectedness condition is more compelling than in the theory over a field (see Example 3.1.2).

**Remark 1.1.13.** For linear algebraic groups over an algebraically closed field of characteristic 0, the property of reductivity for the identity component is equivalent to the semisimplicity of all linear representations, but this fails in positive characteristic; see Exercise 1.6.10.

In the context of connected solvable linear algebraic groups, the opposite extreme from the unipotent groups is the following class of groups:

**Definition 1.1.14.** A *torus* over the algebraically closed field  $k$  is a  $k$ -group  $T$  that is isomorphic to a power  $(\mathbf{G}_m)^r$  for some  $r \geq 0$ .

In view of the general structure of connected solvable linear algebraic groups (as  $T \rtimes U$  for a torus  $T$  and unipotent radical  $U$  [Bo91, 10.6(4)]), we see that the solvable connected reductive groups are precisely the tori.

**Example 1.1.15.** Let  $G$  be connected reductive. The solvable connected  $\mathcal{R}(G)$  is a torus (as it must be reductive). Since the automorphism scheme  $\text{Aut}_{T/k}$  of a  $k$ -torus  $T$  is a constant group (see the proof of Theorem 1.1.18), a normal torus  $T'$  in any *connected* linear algebraic group  $G'$  must be central (as the  $k$ -homomorphism  $G' \rightarrow \text{Aut}_{T'/k}$  giving the conjugation action has to be trivial). Hence,  $Z := \mathcal{R}(G)$  is a central torus such that  $G/Z$  is semisimple. In other words, every connected reductive group is a central extension of a connected semisimple group by a torus. For example, if  $G = \text{GL}(V)$  then  $G/Z = \text{PGL}(V)$ .

Deeper structure theory of reductive groups [Bo91, 14.2] ensures that the derived group  $\mathcal{D}(G)$  is semisimple and that the commutative reductive quotient  $G/\mathcal{D}(G)$  (which must be a torus) is an isogenous quotient of  $Z$ . In other words,  $G$  is also canonically an extension of a torus by a connected semisimple group. For  $G = \text{GL}(V)$  this is the exact sequence

$$1 \rightarrow \text{SL}(V) \rightarrow \text{GL}(V) \xrightarrow{\det} \mathbf{G}_m \rightarrow 1,$$

and the isogeny of tori  $Z \rightarrow G/\mathcal{D}(G)$  in this case is identified with the endomorphism  $t \mapsto t^n$  of  $\mathbf{G}_m$  for  $n = \dim V$ .

The natural homomorphism  $Z \times \mathcal{D}(G) \rightarrow G$  is an isogeny for any connected reductive  $k$ -group  $G$ . Because of this, for many (but not all!) problems in the classical theory of reductive groups one can reduce to a separate treatment of tori and semisimple groups. Note also that the semisimple  $\mathcal{D}(G)$  must in fact be equal to its own derived group. Indeed, the quotient  $\mathcal{D}(G)/\mathcal{D}^2(G)$  is connected, semisimple (inherited from  $\mathcal{D}(G)$ ), and commutative, hence trivial.

A fundamental result (whose proof requires some input from the theory of Borel subgroups) is the fact that a connected linear algebraic group  $G$  over  $k = \bar{k}$  is a torus when all  $k$ -points

are semisimple [Bo91, 11.5(1)]. If instead  $G$  contains no nontrivial tori then it must be unipotent [Bo91, 11.5(2)]. This is very useful: it implies that a general connected linear algebraic  $k$ -group  $G$  either admits a strictly upper triangular faithful representation or it contains a nontrivial  $k$ -torus. (This dichotomy is also true without assuming  $k = \bar{k}$ , but requires Grothendieck's deep result on the existence of geometrically maximal  $k$ -tori, proved in Appendix A.) Thus, if  $\dim G > 1$  then  $G$  contains either  $\mathbf{G}_a$  or  $\mathbf{G}_m$  as a proper  $k$ -subgroup (since  $k = \bar{k}$ ).

A very effective way to work with tori is by means of some associated lattices. To be precise, since  $\text{End}(\mathbf{G}_m) = \mathbf{Z}$  via the endomorphisms  $t \mapsto t^n$ , for any torus  $T$  the commutative groups

$$X(T) = \text{Hom}_{k\text{-gp}}(T, \mathbf{G}_m), \quad X_*(T) = \text{Hom}_{k\text{-gp}}(\mathbf{G}_m, T)$$

are finite free  $\mathbf{Z}$ -modules of rank  $\dim T$  that are respectively contravariant and covariant in  $T$ . (Elements of  $X_*(T)$  are called *cocharacters* of  $T$ .) Beware that it is a standard convention to use additive notation when discussing elements of the character and cocharacter lattices of  $T$ . For example, if  $a, b : T \rightrightarrows \mathbf{G}_m$  are two characters then  $a + b$  denotes  $t \mapsto a(t)b(t)$  and  $-a$  denotes  $t \mapsto 1/a(t)$  (and  $0$  denotes the trivial character). For this reason,  $a(t)$  is often denoted as  $t^a$ .

Evaluation defines a perfect duality of lattices

$$\langle \cdot, \cdot \rangle : X(T) \times X_*(T) \rightarrow \text{End}(\mathbf{G}_m) = \mathbf{Z}$$

via  $\langle \lambda, a \rangle = a \circ \lambda$ . (Here, by ‘‘perfect’’ we mean that the  $\mathbf{Z}$ -bilinear form identifies each lattice with the  $\mathbf{Z}$ -dual of the other. In terms of matrices relative to a  $\mathbf{Z}$ -basis, it means that the matrix of the bilinear form has determinant in  $\mathbf{Z}^\times = \{\pm 1\}$ .) In terms of the cocharacter group  $X_*(T)$  we have

$$X_*(T) \otimes_{\mathbf{Z}} k^\times \simeq T(k)$$

via  $\lambda \otimes c \mapsto \lambda(c)$ . In fact the algebraic group  $T$  (and not just its group of  $k$ -points) can be reconstructed from its cocharacter group, by considering the functor  $X_*(T) \otimes_{\mathbf{Z}} \mathbf{G}_m$ . More specifically, the contravariant functors

$$T \rightsquigarrow X(T), \quad M \rightsquigarrow \text{Hom}_{k\text{-gp}}(M_k, \mathbf{G}_m)$$

(where  $M_k$  is the constant  $k$ -group associated to  $M$ ) are inverse anti-equivalences between the categories of tori and finite free  $\mathbf{Z}$ -modules.

A fundamental fact about tori is that their linear representations are completely reducible in *any* characteristic. This can be expressed in the following canonical form.

**Proposition 1.1.16.** *Let  $T$  be a torus,  $M = X(T)$  its character group, and  $V$  a finite-dimensional linear representation of  $T$  over  $k$ . For each  $a \in M$ , let  $V_a$  be the subspace of  $v \in V$  such that  $t.v = a(t)v$  for all  $t \in T(k)$ . The natural  $T$ -equivariant map  $\bigoplus_{a \in M} V_a \rightarrow V$  is an isomorphism.*

*In this way, the category of linear representations of  $T$  is equivalent to the category of  $M$ -graded  $k$ -vector spaces.*

We call  $V_a$  the *a-weight space* for  $T$  acting on  $V$ . Note that these vanish for all but finitely many  $a$ ; the *weights* for the  $T$ -action on  $V$  are the  $a$  such that  $V_a \neq 0$ . For the dual representation space  $V^*$ , we have  $(V^*)_a = (V_{-a})^*$  for all  $a \in M$ . Hence, if  $V$  is a self-dual representation of  $T$  then its set of weights is stable under negation.

**Example 1.1.17.** For  $T = \mathbf{G}_m$  we have  $X(T) = \text{End}(\mathbf{G}_m) = \mathbf{Z}$ , so a  $k$ -homomorphism  $\mathbf{G}_m \rightarrow \text{GL}(V)$  is the “same” as a  $k$ -linear  $\mathbf{Z}$ -grading  $V = \bigoplus_{n \in \mathbf{Z}} V(n)$ , with  $t \in \mathbf{G}_m = T$  acting on  $V(n)$  via  $t^n$ -scaling. (There is an analogous result for linear  $\mathbf{G}_m^r$ -actions using  $\mathbf{Z}^r$ -gradings, as well as for linear  $\mu_n$ -actions using  $\mathbf{Z}/n\mathbf{Z}$ -gradings. A common generalization for linear representations of any diagonalizable  $k$ -group  $D_k(M)$  is expressed in terms of  $M$ -gradings, but we will not need it here. See [CGP, A.8.1–A.8.9].)

Choose  $v \in V$ , and write  $v = \sum_{n \in \mathbf{Z}} v_n$  with  $v_n \in V(n)$ , so  $t.v = \sum_{n \in \mathbf{Z}} t^n v_n$ . Viewing  $V$  as an affine space over  $k$ , clearly the orbit map  $\mathbf{G}_m \rightarrow V$  defined by  $t \mapsto t.v$  extends to a  $k$ -scheme morphism  $\mathbf{A}^1 \rightarrow V$  if and only if  $v \in V_{\geq 0} := \bigoplus_{n \geq 0} V(n)$ . For any such  $v$ , we define  $\lim_{t \rightarrow 0} t.v$  to be the image of  $0 \in \mathbf{A}^1(k) = k$  under the extension  $\mathbf{A}^1 \rightarrow V$  of the orbit map. This limiting value is  $v_0$ , so the space of  $v \in V$  for which  $\lim_{t \rightarrow 0} t.v$  exists and vanishes is  $V_{>0} := \bigoplus_{n > 0} V(n)$ .

By dimension considerations, any torus in a linear algebraic  $k$ -group is contained in a maximal such torus. Here are some important properties of tori in linear algebraic groups:

**Theorem 1.1.18.** *Let  $G$  be a linear algebraic group over  $k$ .*

- (1) *For any torus  $T'$  in  $G$ , the Zariski-closed centralizer  $Z_G(T')$  has finite index in the Zariski-closed normalizer  $N_G(T')$ , and if  $G$  is connected then  $Z_G(T')$  is connected.*
- (2) *All maximal tori  $T$  in  $G$  are  $G(k)$ -conjugate.*
- (3) *Assume  $G$  is connected reductive. The centralizer  $Z_G(T')$  is connected reductive for any torus  $T'$  in  $G$ , and if  $T$  is a maximal torus in  $G$  then  $Z_G(T) = T$ . If  $g \in G(k)$  is semisimple then  $Z_G(g)^0$  is reductive.*

In this theorem we take  $N_G(T')$  and  $Z_G(T')$  to have the classical meaning, as smooth closed subgroup schemes corresponding to  $N_{G(k)}(T')$  and  $Z_{G(k)}(T')$  respectively. (This coincides with the functorial viewpoint on normalizers and centralizers that is addressed in Proposition 2.1.2, Definition 2.2.1, Lemma 2.2.4, and Exercise 2.4.4.)

*Proof.* Since the endomorphism functor  $R \rightsquigarrow \text{End}_{R\text{-gp}}(\mathbf{G}_m)$  on  $k$ -algebras is represented by the constant  $k$ -group  $\mathbf{Z}$  [Oes, I, §5.2], the endomorphism functor of  $\mathbf{G}_m^r$  is represented by the constant  $k$ -group  $\text{Mat}_r(\mathbf{Z})$ . Thus, the automorphism functor of  $T' \simeq \mathbf{G}_m^r$  is represented by the locally finite type constant  $k$ -group  $\text{GL}_r(\mathbf{Z})$ . In particular, all quasi-compact  $k$ -subschemes of this automorphism scheme are finite and closed. Since  $N_G(T')$  is finite type, its image in the automorphism scheme must therefore be finite. But the kernel of the action of  $N_G(T')$  on  $T'$  has underlying space  $Z_G(T')$  (in fact this kernel is smooth, but we do not need it here), so the finite-index claim in (1) is proved. The connectedness in (1) is [Bo91, 11.12]. The conjugacy of maximal tori is [Bo91, 11.3(1)]. The assertions in (3) concerning torus

centralizers are part of [Bo91, 13.17, Cor. 2], and the reductivity of  $Z_G(g)^0$  for semisimple  $g \in G(k)$  is [Bo91, 13.19]  $\square$

**Remark 1.1.19.** For  $G = \mathrm{GL}_n$ , every element can be conjugated into an upper triangular form and every semisimple element can be diagonalized. In other words,  $G(k)$  is covered by the subgroups  $B(k)$  as  $B$  varies through the Borel subgroups, and the subset of semisimple elements in  $G(k)$  is covered by the subgroups  $T(k)$  as  $T$  varies through the maximal tori. These properties are valid for every connected linear algebraic group  $G$ . Indeed, for solvable  $G$  the result can be deduced from the structure of connected solvable groups (see [Bo91, 10.6(5)]), and so by the general conjugacy of Borel subgroups and maximal tori it suffices to show that all elements of  $G(k)$  lie in a Borel subgroup. See [Bo91, 11.10] for this result.

In our later work with reductive groups over schemes we will define scheme-theoretic notions of centralizer and normalizer by closed subgroup schemes of a linear algebraic group, and in the case of a torus  $T'$  in a linear algebraic group  $G$  over  $k$  we will show that the scheme-theoretic notions of  $Z_G(T')$  and  $N_G(T')$  are smooth (and so coincide with their classical counterparts); see Proposition 2.1.2 and Lemma 2.2.4. The quotient  $N_G(T')/Z_G(T')$  will then be a finite étale  $k$ -group, denoted as  $W_G(T')$ , and it is the constant group associated to its group of  $k$ -points, which is  $N_{G(k)}(T')/Z_{G(k)}(T')$ . This latter group is the *Weyl group* associated to  $(G, T')$  in the classical theory, and it agrees with the group of  $k$ -points of the étale quotient construction that will be used in the scheme-theoretic theory. The Weyl group  $W_G(T')$  is especially important when  $T'$  is a maximal torus  $T$  in a connected reductive  $k$ -group  $G$ , in which case it is  $N_G(T)/T$ .

For a linear algebraic group  $G$ , the centralizers  $Z_G(T)$  of the maximal tori are called the *Cartan subgroups*. These are visibly all  $G(k)$ -conjugate to each other; their common dimension is called the *nilpotent rank* of  $G$ , and the common dimension of the maximal tori is called the *reductive rank* of  $G$ . In the special case of connected reductive  $G$  the Cartan subgroups are precisely the maximal tori, and their common dimension is then simply called the *rank* of  $G$ .

**1.2. Roots and coroots.** Let  $G$  be a connected reductive  $k$ -group,  $T$  a maximal torus in  $G$ , and  $\mathfrak{g}$  and  $\mathfrak{t}$  the respective Lie algebras. In contrast with characteristic zero, the adjoint action of  $G$  on  $\mathfrak{g}$  can sometimes fail to be semisimple when  $\mathrm{char}(k) > 0$ . For instance, if  $\mathrm{char}(k) = p > 0$  and  $G = \mathrm{SL}_p$  then  $\mathfrak{g} = \mathfrak{sl}_p = \mathfrak{gl}_p^{\mathrm{Tr}=0}$  (the kernel of the trace map) and this contains the diagonal scalar subspace of  $\mathfrak{gl}_p$  on which the adjoint action of  $G$  is trivial. This line admits no  $G$ -equivariant complement (see Exercise 1.6.4).

However, the  $T$ -action on  $\mathfrak{g}$  is completely reducible, as for any linear representation of a torus. When  $\mathfrak{g}$  is equipped with this extra structure then it becomes a useful invariant of  $G$  in any characteristic. To see this, consider the weight space decomposition

$$(1.2.1) \quad \mathfrak{g} = \mathfrak{g}_0 \oplus \left( \bigoplus_{a \in \Phi} \mathfrak{g}_a \right)$$

for a finite subset  $\Phi = \Phi(G, T) \subset X(T) - \{0\}$ . The following properties hold: the subspace  $\mathfrak{g}_0 = \mathfrak{g}^T$  coincides with  $\mathfrak{t}$ , for each  $a \in \Phi$  the weight space  $\mathfrak{g}_a$  is 1-dimensional, and  $\Phi$  is

stable under negation in  $X(T)$  [Bo91, 13.18(1),(4a),(4b)]. Beware that even though the set of  $T$ -weights on  $\mathfrak{g}$  is stable under negation, just like self-dual representations of  $G$ , when  $\text{char}(k) > 0$  the  $G$ -representation space  $\mathfrak{g}$  can fail to be self-dual; see Exercise 1.6.4.

Letting  $r = \dim T$  denote the rank of  $G$ , the 1-dimensionality of the weight spaces  $\mathfrak{g}_a$  for the nontrivial  $T$ -weights on  $\mathfrak{g}$  implies that the characteristic polynomial of the  $T$ -action on  $\mathfrak{g}$  is

$$\det(xI - \text{Ad}_G(t)|_{\mathfrak{g}}) = (x - 1)^r \prod_{a \in \Phi} (x - a(t))$$

for  $t \in T(k)$ . For this reason, the elements of  $\Phi$  are called the *roots* of  $(G, T)$ , and the corresponding 1-dimensional weight spaces  $\mathfrak{g}_a$  are called the *root spaces* in  $\mathfrak{g}$ .

**Example 1.2.1.** Take  $G = \text{GL}(V) = \text{GL}_n$  with  $n > 0$  and let  $T = \mathbf{G}_m^n$  be the torus consisting of diagonal matrices

$$t = \text{diag}(c_1, \dots, c_n).$$

It is easy to check that  $T(k) = Z_{G(k)}(T)$ , so  $T$  is a maximal torus. We have

$$X(T) = \bigoplus \mathbf{Z}e_i, \quad X_*(T) = \bigoplus \mathbf{Z}e_i^\vee$$

where  $e_i(\text{diag}(c_1, \dots, c_n)) = c_i$  and  $e_i^\vee(c) = \text{diag}(1, \dots, c, \dots, 1)$  (with  $c$  as the  $i$ th diagonal entry, all others being 1). The normalizer  $N_{G(k)}(T)$  is the group of invertible monomial matrices (i.e., one nonzero entry in each row and in each column) and  $N_{G(k)}(T)/T(k) = S_n$  is represented by the group of  $n \times n$  permutation matrices (relative to the standard basis).

The Lie algebra  $\mathfrak{g}$  is  $\text{End}(V) = \text{Mat}_n(k)$ , and the Lie subalgebra  $\mathfrak{t}$  is the subspace of diagonal matrices. The roots are the characters

$$a_{ij}(\text{diag}(c_1, \dots, c_n)) = c_i/c_j$$

for  $1 \leq i \neq j \leq n$ ; in other words,  $a_{ij} = e_i - e_j$ . The corresponding root space  $\mathfrak{g}_{a_{ij}} \subset \text{Mat}_n(k)$  consists of matrices with vanishing entries away from the  $ij$ -entry.

**Example 1.2.2.** For  $G = \text{SL}_2$ , the diagonal torus  $T = \{\text{diag}(c, 1/c)\}$  is maximal and is usually identified with  $\mathbf{G}_m$  via  $\lambda : c \mapsto \text{diag}(c, 1/c)$ . The Lie algebra  $\mathfrak{g}$  is the space  $\mathfrak{sl}_2 = \mathfrak{gl}_2^{\text{Tr}=0}$  of traceless  $2 \times 2$  matrices over  $k$ , in which a basis is given by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The line  $kH$  coincides with  $\mathfrak{t}$ , and the lines  $kE$  and  $kF$  are the root spaces. Explicitly,

$$\text{Ad}_G(\lambda(c))(E) = \lambda(c)E\lambda(c)^{-1} = c^2E, \quad \text{Ad}_G(\lambda(c))(F) = \lambda(c)F\lambda(c)^{-1} = c^{-2}F,$$

so the roots in  $X(T) = X(\mathbf{G}_m) = \mathbf{Z}$  are  $\pm 2$  with corresponding root spaces  $\mathfrak{g}_2 = kE$  and  $\mathfrak{g}_{-2} = kF$ . Note that  $H$  is central in  $\mathfrak{sl}_2$  when  $\text{char}(k) = 2$  (and the central  $\mu_2$  in  $\text{SL}_2$  has Lie algebra  $\text{Lie}(\mu_2) = \mathfrak{t} = kH$  when  $\text{char}(k) = 2$ ).

For each  $a \in \Phi$ , there exists a unique subgroup  $U_a \simeq \mathbf{G}_a$  normalized by  $T$  such that  $\text{Lie}(U_a) = \mathfrak{g}_a$  [Bo91, 13.18(4d)]. This is the *root group* associated to  $a$ .

**Example 1.2.3.** For  $G = \mathrm{SL}_2$  and  $T = D$  the diagonal torus, the root groups are the strictly upper and lower triangular unipotent subgroups  $U^\pm$ . The same holds for  $G = \mathrm{PGL}_2$  and the diagonal torus  $T = \overline{D}$ , using the strictly upper and lower triangular unipotent subgroups  $\overline{U}^\pm$ .

The following lemma will turn out to be a generalization of the fact that  $\mathrm{SL}_2$  is generated by the root groups  $U^\pm$ .

**Lemma 1.2.4.** *Let  $T_a = (\ker a)_{\mathrm{red}}^0$  be the unique codimension-1 torus in  $T$  killed by  $a \in \Phi$ . The root groups  $U_a$  and  $U_{-a}$  generate  $G_a := \mathcal{D}(Z_G(T_a))$ , a closed subgroup isogenous to  $\mathrm{PGL}_2$ .*

*Proof.* Fix an isomorphism  $u_{\pm a} : \mathbf{G}_a \simeq U_{\pm a}$ , so  $tu_{\pm a}(x)t^{-1} = u_{\pm a}(a(t)^{\pm 1}x)$  for  $t \in T$ , by consideration of the Lie algebra and Exercise 1.6.2(iv). Hence,  $T_a$  must centralize  $U_{\pm a}$ , so these root groups lie in  $Z_G(T_a)$ . The group  $Z_G(T_a)$  is a connected reductive subgroup of  $G$  in which the maximal torus  $T$  contains the central subtorus  $T_a$  of codimension 1. The Lie algebra  $\mathrm{Lie}(Z_G(T_a))$  is equal to  $\mathfrak{g}^{T_a}$  [Bo91, 9.4], and this in turn is equal to  $\mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$  [Bo91, 13.18(4a)]. In particular,  $T$  is noncentral in  $Z_G(T_a)$ , so  $T_a$  is the maximal central torus in  $Z_G(T_a)$  (and hence it coincides with  $\mathcal{R}(Z_G(T_a))$ ). We conclude that the quotient  $Z_G(T_a)/T_a$  is semisimple with a 1-dimensional maximal torus. Equivalently, the isogenous  $G_a$  is semisimple with a 1-dimensional maximal torus  $T'_a = (T \cap G_a)_{\mathrm{red}}^0$  (an isogeny-complement to  $T_a$  in  $T$ ); since  $T$  is maximal in  $Z_G(T_a)$ , the maximality of  $T'_a$  in  $G_a$  is a special case of Exercise 1.6.11.

Since the only nontrivial  $T$ -weights on  $\mathrm{Lie}(Z_G(T_a))$  are  $\pm a$ , the semisimple  $G_a$  must have Lie algebra  $\mathfrak{t}'_a \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$ , with  $\mathfrak{t}'_a := \mathrm{Lie}(T'_a)$ . Thus,  $G_a$  is 3-dimensional. By [Bo91, 13.13(5)], there exists an isogeny  $G_a \rightarrow \mathrm{PGL}_2$ , and by conjugacy of maximal tori it can be arranged to carry  $T'_a$  onto the diagonal torus  $\overline{D}$ . Thus, this isogeny carries  $U_{\pm a}$  onto the root groups  $\overline{U}^\pm$  for  $\overline{D}$ . But the pair of subgroups  $\overline{U}^\pm$  visibly generates  $\mathrm{PGL}_2$  (since the subgroups  $U^\pm$  generate  $\mathrm{SL}_2$ ), so we conclude that indeed  $U_{\pm a}$  generate  $G_a$ .  $\square$

We wish to introduce coroots: to each root  $a \in \Phi$  we will attach a canonical nontrivial cocharacter  $a^\vee : \mathbf{G}_m \rightarrow T$  that generalizes the cocharacter  $c \mapsto \mathrm{diag}(c, 1/c)$  in  $\mathrm{SL}_2$  attached to the root  $\mathrm{diag}(c, 1/c) \mapsto c^2$  for  $D$ . The definition of coroots rests on the classification of semisimple  $k$ -groups of rank 1. This classification is the assertion that any such group  $G$  is isomorphic to either  $\mathrm{SL}_2$  or  $\mathrm{PGL}_2$ . In other words, there exists an isogeny  $\mathrm{SL}_2 \rightarrow G$  whose kernel is contained in the central  $\mu_2$ . When combined with Lemma 1.2.4, this provides interesting homomorphisms from  $\mathrm{SL}_2$  into nontrivial connected semisimple groups. Here is the statement in the form that we will need.

**Theorem 1.2.5.** *For each  $a \in \Phi(G, T)$ , there exists a homomorphism  $\varphi_a : \mathrm{SL}_2 \rightarrow G$  such that  $\varphi_a$  carries the diagonal torus  $D$  into  $T$  and carries the strictly upper triangular and strictly lower triangular unipotent subgroups  $U^\pm$  isomorphically onto the respective root groups  $U_{\pm a}$ .*

Such a homomorphism  $\varphi_a$  is an isogeny onto  $G_a$  with  $\ker \varphi_a \subseteq \mu_2$ , it is unique up to  $T(k)$ -conjugation on  $G$ , and it carries the standard Weyl element  $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  to an element  $n_a \in N_{G(k)}(T) - T(k)$ .

*Proof.* The first step is to justify replacing  $(G, T)$  with  $(G_a, T'_a)$ , where  $T'_a = (T \cap G_a)_{\text{red}}^0$  is the maximal torus of  $G_a$  that is an isogeny complement to  $T_a$  in  $T$ . Assume there exists a homomorphism  $\varphi_a$  with the desired properties on  $D$  and  $U^\pm$ . We shall prove that  $\varphi_a$  must land in  $G_a$  and that  $\varphi_a(T) \subset T'_a$ . By Lemma 1.2.4 we know  $U_{\pm a} \subset G_a$ , so the classical fact that  $U^\pm(k)$  generate  $\text{SL}_2(k)$  implies that  $\varphi_a$  must land inside  $G_a$ . Likewise,  $\varphi_a(D)$  must be contained in  $(T \cap G_a)_{\text{red}}^0 = T'_a$ . Thus, we can replace  $(G, T)$  with  $(G_a, T'_a)$ .

Now  $G$  is semisimple with a 1-dimensional maximal torus  $T$  and  $\Phi(G, T) = \{\pm a\}$ . In particular,  $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$  is 3-dimensional, so  $\dim G = 3$ . The key point is to show that  $G$  is isomorphic to either  $\text{SL}_2$  or  $\text{PGL}_2$ . This is not proved in [Bo91] (and correspondingly, coroots are not discussed in [Bo91]), so we provide a proof below.

**Step 1.** We first show that  $G$  admits an isogeny onto  $\text{PGL}_2$  with scheme-theoretic kernel  $Z$  that is isomorphic to 1 or  $\mu_2$  and is scheme-theoretically central in  $G$  (i.e.,  $Z(R)$  is central in  $G(R)$  for every  $k$ -algebra  $R$ ). Note that the case  $\text{char}(k) = 2$  is “non-classical” since  $\mu_2$  is non-reduced for such  $k$ , but our arguments will be characteristic-free.

By dimension considerations,  $B := T \times U_a$  is a Borel subgroup of  $G$  containing  $T$ , and another is  $B' = T \times U_{-a}$ . The multiplication map

$$(1.2.2) \quad \mu : U_{-a} \times B = U_{-a} \times T \times U_a \rightarrow G$$

is étale at the identity, and hence (by left  $U_{-a}(k)$ -translation and right  $B(k)$ -translation) is étale everywhere. Since the closed  $k$ -subgroup scheme  $U_{-a} \cap B$  is étale (due to transversality:  $\mathfrak{u}_{-a} \cap \mathfrak{b} = 0$ ) and normalized by  $T$ , yet the  $k$ -group  $U_{-a} \simeq \mathbf{G}_a$  clearly contains no nontrivial finite étale subgroups normalized by  $T$ , it follows that the étale map  $\mu$  is injective on  $k$ -points. Thus,  $\mu$  is an isomorphism onto its open image  $\Omega$ ; i.e.,  $\mu$  is an open immersion. We conclude that  $B \cap B' = T$  scheme-theoretically.

Since  $G$  is semisimple of rank 1,  $G/B \simeq \mathbf{P}^1$  [Bo91, 13.13(4)]. The left translation action of  $G$  on  $G/B$  then defines a  $k$ -homomorphism to the automorphism scheme

$$f : G \rightarrow \text{Aut}_{\mathbf{P}^1/k} = \text{PGL}_2$$

(see Exercise 1.6.3(iv),(v)) whose scheme-theoretic kernel  $K$  is a normal subgroup scheme that is contained in  $B$ . We shall now prove that  $K = \ker a$  and that this is a finite central subgroup scheme of  $G$  (so  $f$  is an isogeny with central kernel). By normality  $K$  is contained in the  $G(k)$ -conjugate  $B'$  of  $B$ , so  $K \subset B \cap B' = T$  as closed subschemes of  $G$ . The left translation action by  $T$  on  $G$  preserves the open subscheme  $\Omega = U_{-a} \times T \times U_a$  via the formula

$$t.(u_-t'u_+) = (tu_-t^{-1})(tt')u_+,$$

so the left  $T$ -action on  $G/B$  preserves the open subscheme  $\Omega/B = U_{-a} \simeq \mathbf{G}_a$  on which it acts via scaling through  $-a : T \rightarrow \mathbf{G}_m$ . Hence,  $K$  must be contained in  $\ker a$ . But the group scheme  $\ker a$  visibly centralizes the dense open subscheme  $\Omega$  in the smooth group  $G$ , so it centralizes  $G$ . Since  $\ker a \subset T \subset B$ , we obtain the reverse inclusion  $\ker a \subset K$ .

To summarize, we have built a short exact sequence of group schemes

$$(1.2.3) \quad 1 \rightarrow Z \rightarrow G \xrightarrow{f} \mathrm{PGL}_2 \rightarrow 1$$

in which  $Z = \ker a$  is a finite subgroup scheme of  $T = \mathbf{G}_m$  that is central in  $G$ . In particular,  $Z \simeq \mu_n$  for some  $n \geq 1$ . The Weyl group  $W_G(T)$  has order 2 [Bo91, 13.13(2)], so conjugation by a representative  $n \in N_{G(k)}(T)$  of the nontrivial element in  $W_G(T)$  acts on  $T = \mathbf{G}_m$  by its only nontrivial automorphism, namely inversion. But such conjugation must be trivial on the *central* subgroup scheme  $Z$  in  $G$ , so inversion on  $\mu_n$  is trivial. This forces  $n|2$ . If  $n = 1$  then  $f$  is an isomorphism, and if  $n = 2$  then the conjugation action by  $T = \mathbf{G}_m$  on  $U_{\pm a} = \mathbf{G}_a$  must be scaling by the only two characters of  $\mathbf{G}_m$  with kernel  $\mu_2$ , namely  $t \mapsto t^{\pm 2}$ .

**Step 2.** Now we relate  $G$  to  $\mathrm{SL}_2$  in case  $Z = \mu_2$ , and in general we adjust  $f$  so that it relates  $T$  and  $U_{\pm a}$  to  $\overline{D}$  and  $\overline{U}^{\pm}$ . The isogeny  $f$  must carry  $T$  onto a maximal torus of  $\mathrm{PGL}_2$ , so by composing  $f$  with a suitable conjugation we can arrange that  $f(T)$  is the diagonal torus  $\overline{D}$ . It then follows that  $T = f^{-1}(\overline{D})$  scheme-theoretically since  $Z \subset T$ .

Since the  $k$ -subgroup  $Z = \ker a$  has trivial scheme-theoretic intersection with the root groups  $U_{\pm a}$  (as even  $U_{\pm a} \cap T = 1$ ), these root groups are carried *isomorphically* by  $f$  onto their images in  $\mathrm{PGL}_2$ . By the unique characterization of root groups [Bo91, 13.18(4d)], it follows that (as unordered pairs)  $\{f(U_a), f(U_{-a})\}$  is the set of root groups  $\{\overline{U}^+, \overline{U}^-\}$  for  $\overline{D}$ , so by composing  $f$  with a standard Weyl element if necessary we can arrange that  $f(U_a) = \overline{U}^+$  and  $f(U_{-a}) = \overline{U}^-$ .

In the case  $Z = \mu_2$ , there exists a unique isomorphism  $T \simeq \mathbf{G}_m$  carrying the degree-2 quotient map  $T \rightarrow \mathbf{G}_m$  over to the map  $t^2 : \mathbf{G}_m \rightarrow \mathbf{G}_m$ . Combining this with the isomorphisms  $U_{\pm a} \simeq \overline{U}^{\pm} = \mathbf{G}_a$  and the isomorphisms  $U^{\pm} \simeq \overline{U}^{\pm}$  we identify the open subscheme  $\Omega = U_{-a} \times T \times U_a$  in  $G$  with the standard open subscheme

$$(1.2.4) \quad \mathbf{G}_a \times \mathbf{G}_m \times \mathbf{G}_a \simeq U^- \times D \times U^+ \subset \mathrm{SL}_2,$$

where the isomorphism (1.2.4) is defined by

$$(x', c, x) \mapsto \begin{pmatrix} 1 & 0 \\ x' & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Let  $V \subset \Omega \times \Omega$  be the dense open locus of points  $(\omega, \omega')$  such that  $\omega\omega' \in \Omega$  inside  $G$ . We claim that the open immersion  $j : \Omega \simeq U^- \times D \times U^+ \hookrightarrow \mathrm{SL}_2$  is a “birational homomorphism” in the sense that  $j(\omega\omega') = j(\omega)j(\omega')$  for  $(\omega, \omega') \in V$ . Since the composition of  $j$  with the canonical isogeny  $q : \mathrm{SL}_2 \rightarrow \mathrm{PGL}_2$  is a homomorphism (namely,  $f$ ), the map  $V \rightarrow \mathrm{SL}_2$  defined by  $j(\omega\omega')j(\omega')^{-1}j(\omega)^{-1}$  factors through  $\ker q = \mu_2$  and so is identically 1. Hence, indeed  $j$  is a birational homomorphism between linear algebraic  $k$ -groups, so it extends uniquely to an isomorphism of  $k$ -groups (Exercise 1.6.5)!

The map  $f : G \rightarrow \mathrm{PGL}_2$  is either an isomorphism carrying  $T$  onto  $\overline{D}$  and carrying  $U_{\pm a}$  onto  $\overline{U}^{\pm}$  or else it factors through  $q : \mathrm{SL}_2 \rightarrow \mathrm{PGL}_2$  via an isomorphism  $G \simeq \mathrm{SL}_2$  carrying  $T$  onto  $D$  and carrying  $U_{\pm a}$  onto  $U^{\pm}$ . Either way, there exists a unique homomorphism

$\varphi : \mathrm{SL}_2 \rightarrow G$  factoring  $q$  through  $f$ , and  $\varphi$  satisfies all of the desired properties to be  $\varphi_a$  except possibly uniqueness up to  $T(k)$ -conjugation (e.g.,  $\varphi(D) = T$  since  $T = f^{-1}(\overline{D})$ ).

**Step 3.** Finally, we prove the uniqueness of  $\varphi_a$  up to  $T(k)$ -conjugation. By using composition with  $f$ , this reduces to the assertion that the only homomorphisms  $\pi : \mathrm{SL}_2 \rightarrow \mathrm{PGL}_2$  carrying  $D$  into  $\overline{D}$  and  $U^\pm$  isomorphically onto  $\overline{U}^\pm$  respectively are  $\overline{D}(k)$ -conjugates of  $q$ . Since the roots for  $\mathrm{PGL}_2$  have trivial kernel whereas the roots for  $\mathrm{SL}_2$  have kernel equal to the central  $D[2] = \mu_2$ , it follows from the isomorphism condition on root groups that any such  $\pi$  must kill  $D[2]$  and so factors through  $q$ . In other words, to prove the uniqueness of  $\pi$  up to  $\overline{D}(k)$ -conjugation it suffices to treat the analogous assertion for endomorphisms  $\overline{\pi} : \mathrm{PGL}_2 \rightarrow \mathrm{PGL}_2$  that satisfy  $\overline{\pi}(\overline{D}) = \overline{D}$  and  $\overline{\pi} : \overline{U}^\pm \simeq \overline{U}^\pm$ .

Since  $\mathrm{Aut}(\mathbf{G}_a) = k^\times$  and  $\overline{\pi}$  carries  $\overline{U}^+ = \mathbf{G}_a$  isomorphically onto  $\overline{U}^+$ , by composing with a  $\overline{D}(k)$ -conjugation (which makes  $\mathrm{diag}(t, 1) \in \overline{D}(k)$  act on  $\overline{U}^+ = \mathbf{G}_a$  via  $t.x = tx$ ) we may arrange that  $\overline{\pi}$  is the *identity* map on  $\overline{U}^+$ . By hypothesis  $\overline{\pi}$  carries  $\overline{D}$  into  $\overline{D}$ , so the faithfulness of the  $\overline{D}$ -action on  $\overline{U}^+ = \mathbf{G}_a$  implies that  $\overline{\pi}$  restricts to the identity on  $\overline{D}$ . We will prove that  $\overline{\pi}$  restricts to the identity map on the dense open  $\overline{\Omega} := \overline{U}^- \times \overline{D} \times \overline{U}^+$ , thereby forcing  $\overline{\pi}$  to be the identity map.

The restriction of  $\overline{\pi}$  to the dense open direct product subscheme  $\overline{\Omega} \subset \mathrm{PGL}_2$  is visibly an automorphism of  $\overline{\Omega}$ , so  $\ker \overline{\pi}$  is étale. But an étale closed normal subgroup of a connected linear algebraic group is central, yet  $\mathrm{PGL}_2$  has no nontrivial central finite étale subgroup (as  $\overline{D}$  is its own centralizer on  $k$ -points and acts faithfully under conjugation on  $\overline{U}^+$ ), so  $\ker \overline{\pi} = 1$ . Thus,  $\overline{\pi}$  is an isomorphism. In particular,  $\overline{\pi}^{-1}(\overline{\Omega}) = \overline{\Omega}$ .

Let  $\overline{u}_\pm : \mathbf{G}_a \simeq \overline{U}^\pm$  be the standard parameterizations  $x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$  and  $x \mapsto \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ , so by calculation the product  $\overline{u}_+(x)\overline{u}_-(y)$  lies in  $\overline{\Omega} = \overline{U}^- \times \overline{D} \times \overline{U}^+$  if and only if  $1 + xy \in \mathbf{G}_m$ . The restriction  $\overline{\pi} : \overline{U}^- \simeq \overline{U}^-$  corresponds to an automorphism of  $\mathbf{G}_a$ , which is to say  $\overline{\pi}(\overline{u}_-(y)) = \overline{u}_-(cy)$  for some  $c \in k^\times$ . Thus,  $\overline{\pi}(\overline{u}_+(x)\overline{u}_-(y)) = \overline{u}_+(x)\overline{u}_-(cy)$ , so the equality  $\overline{\pi}^{-1}(\overline{\Omega}) = \overline{\Omega}$  implies that  $1 + xy \in \mathbf{G}_m$  if and only if  $1 + x \cdot cy \in \mathbf{G}_m$ . It follows that  $c = 1$ , so  $\overline{\pi}$  is the identity on  $\overline{U}^-$  and we are done.  $\square$

Using any  $\varphi_a$  as in Theorem 1.2.5, the cocharacter

$$a^\vee : \mathbf{G}_m \rightarrow D \rightarrow T$$

defined by  $a^\vee(c) = \varphi_a(\mathrm{diag}(c, 1/c))$  is unaffected by  $T(k)$ -conjugation on  $G$ , so it is intrinsic.

**Definition 1.2.6.** The *coroot* associated to  $(G, T, a)$  is the nontrivial cocharacter  $a^\vee \in X_*(T)$ . The finite set of coroots in  $X_*(T) - \{0\}$  is denoted  $\Phi^\vee$ .

Concretely,  $a^\vee$  is a parameterization (with kernel 1 or  $\mu_2$ ) of the 1-dimensional torus  $(T \cap \mathcal{D}(Z_G(T_a)))_{\mathrm{red}}^0$  that is an isogeny complement to  $T_a$  in  $T$ . The composition of any  $\varphi_a$  with transpose-inverse on  $\mathrm{SL}_2$  satisfies the requirements to be  $\varphi_{-a}$ . Since transpose-inverse acts by inversion of the diagonal torus of  $\mathrm{SL}_2$ , we conclude that  $(-a)^\vee = -a^\vee$ .

**Example 1.2.7.** Suppose  $G = \mathrm{SL}_2$  and  $T$  is the diagonal torus  $D$ . The roots for  $(G, T)$  are  $\mathrm{diag}(c, 1/c) \mapsto c^{\pm 2}$ . Let  $a$  be the root  $\mathrm{diag}(c, 1/c) \mapsto c^2$ , so  $\mathfrak{g}_a$  is the subspace of upper triangular nilpotent matrices in  $\mathfrak{sl}_2$  and  $\mathfrak{g}_{-a}$  is the subspace of lower triangular nilpotent matrices in  $\mathfrak{sl}_2$ . By Example 1.2.3, the corresponding root groups are  $U_a = U^+$  and  $U_{-a} = U^-$ , so  $\varphi_a$  can be taken to be the identity map. In particular,  $a^\vee(c) = \mathrm{diag}(c, 1/c)$ . Note that  $\langle a, a^\vee \rangle = 2$  in  $\mathrm{End}(\mathbf{G}_m) = \mathbf{Z}$  (i.e.,  $a(a^\vee(c)) = c^2$ ): this follows from the calculation

$$\begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix}^{-1} = \begin{pmatrix} 1 & c^2 x \\ 0 & 1 \end{pmatrix},$$

which implies that the adjoint action of  $a^\vee(c)$  on  $\mathfrak{g}_a = \mathrm{Lie}(U_a)$  is scaling by  $c^2$ . Observe also that in  $\mathfrak{t}$  we have

$$\mathrm{Lie}(a^\vee)(t\partial_t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H_a = [E_a, F_a]$$

(where  $t$  is the standard coordinate on  $\mathbf{G}_m \simeq D$  via the isomorphism  $c \mapsto \mathrm{diag}(c, 1/c)$ ).

For  $G = \mathrm{PGL}_2$  and  $T = \overline{D}$ , we have an isomorphism  $\mathbf{G}_m \simeq T$  via  $c \mapsto \mathrm{diag}(c, 1) \bmod \mathbf{G}_m$ . The inverse isomorphism  $\bar{a} : T \simeq \mathbf{G}_m$  is a root whose root space consists of the upper triangular nilpotent matrices in  $\mathfrak{pgl}_2 = \mathfrak{gl}_2/\mathfrak{gl}_1$ , and similarly for the root  $-\bar{a}$  using lower triangular nilpotent matrices. Thus,  $\varphi_{\bar{a}} : \mathrm{SL}_2 \rightarrow G$  can be taken to be the canonical projection. In particular,  $\bar{a}^\vee : \mathbf{G}_m \rightarrow T$  is  $c \mapsto \mathrm{diag}(c, 1/c) \bmod \mathbf{G}_m = \mathrm{diag}(c^2, 1) \bmod \mathbf{G}_m$ , so  $\langle \bar{a}, \bar{a}^\vee \rangle = 2$ .

**Remark 1.2.8.** It follows from Example 1.2.7 (for both  $\mathrm{SL}_2$  and  $\mathrm{PGL}_2$ ) that in general  $\langle a, a^\vee \rangle = 2$  for any connected reductive  $k$ -group  $G$  and maximal torus  $T \subset G$ .

**Example 1.2.9.** Let  $G = \mathrm{GL}(V) = \mathrm{GL}_n$ , and take  $T$  to be the diagonal torus  $D_n$ . For the root  $a = e_i - e_j$  ( $1 \leq i \neq j \leq n$ ), the associated codimension-1 subtorus  $T_a$  is defined by the equality  $\{c_i = c_j\}$  between  $i$ th and  $j$ th diagonal entries. The centralizer  $Z_G(T_a)$  consists of elements of  $\mathrm{GL}_n$  whose off-diagonal entries vanish away from the  $ij$  and  $ji$  positions, and the subgroup  $G_a := \mathcal{D}(Z_G(T_a))$  is  $\mathrm{SL}(ke_i \oplus ke_j) = \mathrm{SL}_2$ . In particular,  $a^\vee = e_i^\vee - e_j^\vee$ . This makes explicit that  $\langle a, a^\vee \rangle = 2$  for all  $a \in \Phi(\mathrm{GL}_n, D_n)$ .

**1.3. Root datum, root system, and classification theorem.** Let  $G$  be a connected reductive  $k$ -group, and let  $T \subset G$  be a maximal torus. In the dual lattices  $X = X(T)$  and  $X^\vee = X_*(T)$  we have defined finite subsets  $\Phi = \Phi(G, T) \subset X - \{0\}$  and  $\Phi^\vee = \Phi(G, T)^\vee \subset X^\vee - \{0\}$  and a bijection  $a \mapsto a^\vee$  between  $\Phi$  and  $\Phi^\vee$  such that (i)  $\langle a, a^\vee \rangle = 2$  for all  $a$ , (ii) for each  $a \in \Phi$ ,  $\Phi \cap \mathbf{Q}a = \{\pm a\}$  inside  $X_{\mathbf{Q}}$ .

There are additional properties satisfied by this combinatorial data. To formulate them, we introduce some reflections. For each  $a \in \Phi$  define the linear endomorphisms  $s_a : X \rightarrow X$  and  $s_{a^\vee} : X^\vee \rightarrow X^\vee$  by

$$(1.3.1) \quad s_a(x) = x - \langle x, a^\vee \rangle a, \quad s_{a^\vee}(\lambda) = \lambda - \langle a, \lambda \rangle a^\vee.$$

Since  $\langle a, a^\vee \rangle = 2$ , it is easy to check that  $s_a^2 = 1$ ,  $s_{a^\vee}^2 = 1$ ,  $s_a(a) = -a$ , and  $s_{a^\vee}(a^\vee) = -a^\vee$ , so on  $X_{\mathbf{Q}}$  and  $X_{\mathbf{Q}}^\vee$  the involutions  $s_a$  and  $s_{a^\vee}$  are reflections in the lines spanned by  $a$  and  $a^\vee$

respectively. It is also easy to check that  $s_{a^\vee}$  is dual to  $s_a$ . (Some introductory accounts of the theory of root systems impose a Euclidean structure at the outset, such as in [Hum72, III], but this is not necessary. To that end, note that we have not imposed any positive-definite quadratic form on  $X_{\mathbf{Q}}$  or  $X_{\mathbf{R}}$ .)

**Example 1.3.1.** Let  $G = \mathrm{GL}_n$  and let  $T$  be the diagonal torus. For  $1 \leq i \neq j \leq n$  and  $1 \leq h \leq n$ , we have  $s_{e_i - e_j}(e_h) = e_h$  when  $h \neq i, j$  and the reflection  $s_{e_i - e_j}$  swaps  $e_i$  and  $e_j$ . In particular,  $s_a(\Phi) = \Phi$  for all  $a \in \Phi$ .

The most basic instance of these formulas for the  $s_a$ -actions is seen for  $\mathrm{GL}_2$ , where it amounts to the fact that the standard Weyl element  $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  acts on the diagonal torus of  $\mathrm{SL}_2$  via inversion.

The importance of the reflections  $s_a$  and  $s_{a^\vee}$  in general is:

**Proposition 1.3.2.** *For all  $a \in \Phi$ ,  $s_a(\Phi) = \Phi$  and  $s_{a^\vee}(\Phi^\vee) = \Phi^\vee$ .*

*Proof.* Use the isogeny  $T'_a \times T_a \rightarrow T$  to identify  $X(T)_{\mathbf{Q}}$  with  $X(T'_a)_{\mathbf{Q}} \oplus X(T_a)_{\mathbf{Q}}$ , so  $X(T_a)_{\mathbf{Q}}$  is identified with the hyperplane spanned over  $\mathbf{Q}$  by the characters that kill  $T'_a = a^\vee(\mathbf{G}_m)$  and  $X(T'_a)_{\mathbf{Q}}$  is identified with the line spanned over  $\mathbf{Q}$  by the characters that kill  $T_a$  (i.e., it is the  $\mathbf{Q}$ -span of  $a$ ).

Let  $n_a \in G_a = \mathcal{D}(Z_G(T_a))$  be a representative in  $N_{G_a(k)}(T'_a)$  for the nontrivial element in the group  $W_{G_a}(T'_a)$  of order 2. The conjugation action by  $n_a$  on the almost direct product  $T = T'_a \cdot T_a$  is trivial on  $T_a$  and inversion on  $T'_a = a^\vee(\mathbf{G}_m)$ , so  $n_a$  acts on  $X(T)$  as an involution whose effect on  $X(T)_{\mathbf{Q}}$  negates the line spanned by  $a$  and fixes pointwise the hyperplane  $X(T_a)_{\mathbf{Q}}$ .

We conclude that  $n_a$  acts on  $X(T)$  as a reflection  $r_a$ . This reflection visibly preserves  $\Phi$  (as does any element of  $N_{G(k)}(T)$ ), and we claim that  $r_a = s_a$  (so  $s_a(\Phi) = \Phi$ ). Both of these reflections negate the same line  $\mathbf{Q}a$ , so it suffices to show that they restrict to the identity on a common hyperplane, namely  $X(T_a)_{\mathbf{Q}}$ . But this hyperplane is the annihilator of  $\mathbf{Q}a^\vee \subset X_*(T)_{\mathbf{Q}}$ , so the definition of  $s_a$  makes it clear that  $s_a$  is the identity on  $X(T_a)_{\mathbf{Q}}$ .

The equality  $r_a = s_a$  implies that the dual reflection  $s_a^\vee = s_{a^\vee}$  on  $X_*(T)$  is also induced by  $n_a$ -conjugation on  $T$ . Thus,  $s_{a^\vee}$  preserves  $\Phi^\vee$ .  $\square$

We have shown that  $(X(T), \Phi(G, T), X_*(T), \Phi(G, T)^\vee)$  satisfies the requirements in the following definition that was introduced by Demazure in [SGA3, XXI].

**Definition 1.3.3.** A *root datum* is a 4-tuple  $(X, \Phi, X^\vee, \Phi^\vee)$  consisting of a pair of finite free  $\mathbf{Z}$ -modules  $X$  and  $X^\vee$  equipped with

- a perfect duality  $\langle \cdot, \cdot \rangle : X \times X^\vee \rightarrow \mathbf{Z}$ ,
- finite subsets  $\Phi \subset X - \{0\}$  and  $\Phi^\vee \subset X^\vee - \{0\}$  stable under negation

for which there exists a bijection  $a \mapsto a^\vee$  from  $\Phi$  to  $\Phi^\vee$  such that  $\langle a, a^\vee \rangle = 2$  and the resulting reflections  $s_a : X \simeq X$  and  $s_{a^\vee} : X^\vee \simeq X^\vee$  as in (1.3.1) satisfy  $s_a(\Phi) = \Phi$  and  $s_{a^\vee}(\Phi^\vee) = \Phi^\vee$ .

If moreover  $\mathbf{Q}a \cap X = \{\pm a\}$  inside  $X_{\mathbf{Q}}$  for all  $a \in \Phi$  then the root datum is *reduced*.

**Remark 1.3.4.** The bijection  $a \mapsto a^\vee$  in the definition of a root datum is uniquely determined. For a proof, see [CGP, Lemma 3.2.4].

**Remark 1.3.5.** It is immediate from the axioms that if  $(X, \Phi, X^\vee, \Phi^\vee)$  is a root datum then so is  $(X^\vee, \Phi^\vee, X, \Phi)$ . This is called the *dual* root datum.

In the study of connected semisimple  $k$ -groups “up to central isogeny” (see Exercise 1.6.12), it is convenient to work with a coarser notion than a root datum, in which we relax the  $\mathbf{Z}$ -structure to a  $\mathbf{Q}$ -structure and remove the explicit mention of the coroots. This leads to the notion of a *root system* (which historically arose much earlier than the notion of a root datum, in the classification of semisimple Lie algebras over  $\mathbf{C}$ , and is extensively studied in [Bou2, Ch. VI]): this is a pair  $(V, \Phi)$  consisting of a finite-dimensional  $\mathbf{Q}$ -vector space  $V$  and a finite spanning set  $\Phi \subset V - \{0\}$  such that for each  $a \in \Phi$  there exists a (necessarily unique) reflection  $s_a : v \mapsto v - \lambda(v)a$  with  $\lambda \in V^*$  such that  $s_a(\Phi) = \Phi$ ,  $\lambda(a) = 2$  (so  $s_a(a) = -a$ ), and  $\lambda(\Phi) \subset \mathbf{Z}$ .

**Example 1.3.6.** If  $(X, \Phi, X^\vee, \Phi^\vee)$  is a root datum then the  $\mathbf{Q}$ -span  $V$  of  $\Phi$  in  $X_{\mathbf{Q}}$  equipped with the subset  $\Phi$  is a root system.

The difference between root systems and root data is analogous to the difference between connected semisimple  $k$ -groups considered up to central isogeny and connected reductive  $k$ -groups considered up to isomorphism: the possible failure of  $\Phi$  to span  $X_{\mathbf{Q}}$  is analogous to the possibility that a connected reductive group may have a nontrivial central torus (i.e., fail to be semisimple), and the use of  $\mathbf{Q}$ -structures rather than  $\mathbf{Z}$ -structures amounts to considering groups up to central isogeny.

More explicitly, if  $(G, T)$  is a connected reductive  $k$ -group equipped with a maximal torus  $T$ , then the saturation in  $X(T)$  of the  $\mathbf{Z}$ -span  $\mathbf{Z}\Phi$  of  $\Phi$  is  $X(T/Z)$  where  $Z$  is the maximal central torus of  $G$  (since a torus  $T'$  in  $G$  is central if and only if  $a(T') = 1$  for all  $a \in \Phi$  [Bo91, 14.2(1)]). Thus,  $\Phi$  spans  $X(T)_{\mathbf{Q}}$  if and only if  $G$  has no nontrivial central torus, which is to say that  $G$  is semisimple.

**Remark 1.3.7.** If a root datum is not reduced then the only possibilities for  $\mathbf{Q}a \cap X$  aside from  $\{\pm a\}$  are  $\{\pm a, \pm 2a\}$  (the *multiplicable* case) or  $\{\pm a, \pm a/2\}$  (the *divisible* case). Indeed, this is a property of the underlying root system, so it holds by [Bou2, VI, §1.3, Prop. 8(i)]. The study of connected reductive groups over fields  $k \neq k_s$  gives rise to non-reduced root data; e.g., this occurs for  $k = \mathbf{R}$  in the study of non-compact semisimple Lie groups, as well as in the study of special unitary groups over general  $k$  admitting a separable quadratic extension.

In general, the quotient  $X(T)/\mathbf{Z}\Phi$  is the Cartier dual  $\text{Hom}(Z_G, \mathbf{G}_m)$  of the scheme-theoretic center  $Z_G$  of  $G$  (Corollary 3.3.6(1)). A special case is that  $\mathbf{Z}\Phi$  has finite index in  $X(T)$  if and only if  $Z_G$  is finite, which is to say that there is no nontrivial torus, or in other words that  $G$  is semisimple. Here are some illustrations of the relations between the root datum and the (scheme-theoretic) center.

**Example 1.3.8.** The set  $\Phi$  spans  $X(T)$  over  $\mathbf{Z}$  if and only if the scheme-theoretic center is trivial (the “adjoint semisimple” case). For example,  $(\mathrm{PGL}_2, \overline{D})$  has roots  $\pm\bar{a}$  that are isomorphisms  $\overline{D} \simeq \mathbf{G}_m$ , so these each span  $X(\overline{D}) = \mathbf{Z}$ , whereas  $(\mathrm{SL}_2, D)$  has roots  $\pm a : D \rightarrow \mathbf{G}_m$  with kernel  $D[2] = \mu_2$ , so these each span the unique index-2 subgroup of  $X(D) = \mathbf{Z}$ . This encodes the fact that  $\mathrm{SL}_2$  has scheme-theoretic center  $\mu_2$  whereas  $\mathrm{PGL}_2$  has trivial scheme-theoretic center.

**Example 1.3.9.** The set  $\Phi(G, T)$  is empty if and only if  $G$  is a torus (i.e.,  $G = T$ ), or equivalently  $G$  is commutative. This is immediate from the weight space decomposition (1.2.1) since  $\mathfrak{g}_0 = \mathfrak{t}$ .

**Example 1.3.10.** Suppose  $G$  is semisimple. In this case we have the containment of lattices (1.3.2)

$$Q := \mathbf{Z}\Phi \subset X(T) \subset (\mathbf{Z}\Phi^\vee)^* =: P,$$

where the dual lattice  $(\mathbf{Z}\Phi^\vee)^*$  in  $X(T)_{\mathbf{Q}}$  is  $\mathbf{Z}$ -dual to the lattice  $\mathbf{Z}\Phi^\vee \subset X(T)^\vee = X(T)^*$ . Thus, the two “extreme” cases are  $X(T) = \mathbf{Z}\Phi$  and  $X(T) = (\mathbf{Z}\Phi^\vee)^*$  (i.e.,  $\mathbf{Z}\Phi^\vee = X_*(T)$ ). In the language of root systems, these cases respectively correspond to the properties that (i) the base for a positive system of roots  $\Phi(G, T)$  (see Definition 1.4.1) is a basis of the character group of  $T$ , and (ii) the base for a positive system of coroots is a basis of the cocharacter group of  $T$ . The first of these two extremes is the case of adjoint  $G$  (such as  $\mathrm{PGL}_n$ ) and the second is the case of simply connected  $G$  (such as  $\mathrm{SL}_n$ ); see Exercise 1.6.12(ii) and Corollary 3.3.6(1).

The preceding examples (and the theory over  $\mathbf{C}$ ; see Proposition D.4.1) inspire the following terminology:

**Definition 1.3.11.** A reduced root datum  $(X, \Phi, X^\vee, \Phi^\vee)$  is *semisimple* if  $\Phi$  spans  $X_{\mathbf{Q}}$  over  $\mathbf{Q}$ . In the semisimple case, it is *adjoint* if  $\mathbf{Z}\Phi = X$  and is *simply connected* if  $\mathbf{Z}\Phi^\vee = X^\vee$ .

If  $(\mathbf{Q}\Phi^\vee)^\perp$  denotes the annihilator in  $X_{\mathbf{Q}}$  of the subspace  $\mathbf{Q}\Phi^\vee \subset X_{\mathbf{Q}}^\vee$  then the natural map  $(\mathbf{Q}\Phi) \oplus (\mathbf{Q}\Phi^\vee)^\perp \rightarrow X_{\mathbf{Q}}$  is an isomorphism [Spr, Exer. 7.4.2] (mirroring the isogeny decomposition of a connected reductive group into the almost direct product of a torus and a connected semisimple group as in Example 1.1.15). Thus, the subgroup

$$W(R) = \langle s_a \mid a \in \Phi \rangle \subset \mathrm{Aut}(X)$$

is trivial on  $(\mathbf{Q}\Phi^\vee)^\perp$  and acts on  $\mathbf{Q}\Phi$  through permutations of a finite spanning set. It follows that  $W(R)$  is finite; it is called the *Weyl group* of the root datum, and is naturally isomorphic to the Weyl group of the associated root system:

**Example 1.3.12.** If  $(G, T)$  is a connected reductive  $k$ -group equipped with a maximal torus  $T$  then the equality  $Z_{G(k)}(T) = T(k)$  implies that the action of  $N_{G(k)}(T)$  on  $T$  identifies the finite group  $W_G(T)$  with a subgroup of  $\mathrm{Aut}(X(T))$ . This subgroup is exactly the Weyl group of the root datum associated to  $(G, T)$  [Bo91, 14.8]. Here is the idea of the proof. In the proof of Proposition 1.3.2 we showed that each reflection  $s_a$  in the Weyl group of the root datum  $R(G, T)$  lies in  $W_G(T)$ , so  $W(R(G, T)) \subset W_G(T)$ . The reverse containment is proved

by relating Borel subgroups to positive systems of roots in the associated root system (see Definition 1.4.1 and Proposition 1.4.4) and using that the Weyl group of any root system acts simply transitively on the set of positive systems of roots ([Bou2, VI, §1.5, Thm. 2(i); §1.6, Thm. 3]).

**Definition 1.3.13.** A root system  $(V, \Phi)$  is *non-empty* if  $\Phi \neq \emptyset$  (equivalently,  $V \neq 0$ ). The *direct product* of root systems  $(V_1, \Phi_1)$  and  $(V_2, \Phi_2)$  is  $(V_1, \Phi_1) \times (V_2, \Phi_2) = (V_1 \oplus V_2, \Phi_1 \amalg \Phi_2)$ . A root system  $(V, \Phi)$  is *irreducible* if it is non-empty and not a direct product of two non-empty root systems.

**Remark 1.3.14.** Every non-empty root system is uniquely a direct product of irreducible root systems [Bou2, VI, §1.2, Prop. 6] and there is a classification of irreducible root systems ([Bou2, VI, §4.2, Prop. 1, Thm. 3] in the reduced case, and [Bou2, VI, §1.4, Prop. 13] and [Bou2, VI, §4.14] in the non-reduced case). This is very useful in the study of root data.

Every irreducible root system  $\Phi$  admits a positive-definite  $\mathbf{Q}$ -valued quadratic form  $Q$  that is invariant under the action of the Weyl group [Bou2, VI, §1.1, Prop. 3]. This quadratic form is unique up to scaling whether we take  $V$  to be finite-dimensional over  $\mathbf{Q}$  (as we have done above) or over  $\mathbf{R}$  (as in much of the literature) [Bou2, VI, §1.2, Prop. 7]. In particular, it is intrinsic to compare ratios  $Q(a)/Q(b)$  for  $a, b \in \Phi$ . After extending scalars to  $\mathbf{R}$ , this equips irreducible root systems with a canonical inner product (up to scaling), and thereby puts the study of root systems into the framework of Euclidean geometry.

For example, suppose  $R = (X, \Phi, X^\vee, \Phi^\vee)$  is a root datum such that  $\Phi$  spans  $V := X_{\mathbf{Q}}$  and  $(V, \Phi)$  is an irreducible root system. Choose a  $W(R)$ -invariant inner product  $(\cdot | \cdot)$  on  $V_{\mathbf{R}}$ . We use this inner product to identify  $X_{\mathbf{R}}$  with its linear dual  $X_{\mathbf{R}}^\vee$ . In this way the coroot  $a^\vee$  is identified with the element  $2a/(a|a) \in X_{\mathbf{R}}$  [Bou2, VI, §1.1, Lemma 2]. This is the most convenient way for drawing pictures of low-rank irreducible root data. Also, it is intrinsic to compare *ratios* of root lengths, and in cases with distinct root lengths there are exactly two, with  $(a|a)/(b|b) \in \{2, 3\}$  when  $(a|a) > (b|b)$  [Bou2, VI, §1.4, Prop. 12].

There is an evident notion of *isomorphism* between root data (required to respect the given perfect duality between  $X$  and  $X^\vee$ ). If  $f : (G, T) \simeq (G', T')$  is an isomorphism, then it is clear that the induced isomorphisms  $X(T') \simeq X(T)$  and  $X_*(T) \simeq X_*(T')$  respect the dualities and the subsets of roots and coroots, so we get an isomorphism of root data  $R(G', T') \simeq R(G, T)$ . If we change  $f$  via composition with a  $T'(k)$ -conjugation or  $T(k)$ -conjugation then the isomorphism between the root data is unchanged. Now we can finally state a fundamental result in the classical theory over  $k = \bar{k}$ :

**Theorem 1.3.15** (Existence and Isomorphism Theorems). *The reduced root datum  $R(G, T)$  associated to a connected reductive  $k$ -group  $G$  and maximal torus  $T \subset G$  determines  $(G, T)$  uniquely up to isomorphism.*

*More precisely, for any two pairs  $(G, T)$  and  $(G', T')$ , every isomorphism  $R(G', T') \simeq R(G, T)$  arises from an isomorphism  $(G, T) \simeq (G', T')$  that is unique up to the conjugation actions of  $T'(k)$  and  $T(k)$ , and every reduced root datum is isomorphic to  $R(G, T)$  for some pair  $(G, T)$  over  $k$ .*

A remarkable aspect of this theorem is that the root datum has nothing to do with  $k$  or  $\text{char}(k)$ . There is a finer version of the theorem that also classifies isogenies in terms of a notion of “isogeny” between root data that can encode characteristic-dependent concepts (such as the Frobenius isogeny in characteristic  $p > 0$ ), but we postpone the statement and proof of this Isogeny Theorem until we discuss the Existence and Isomorphism Theorems over a general non-empty base scheme in §6. The proof of these theorems over a general base scheme will *not* require the classical version over a general algebraically closed field. It only requires the Existence Theorem for connected semisimple groups over a single algebraically closed of characteristic 0; see Appendix D.

**Example 1.3.16.** For any  $(G, T)$ , consider the root datum  $R'$  dual to  $R(G, T)$  in the sense of Remark 1.3.5. By the Existence Theorem, there exists another pair  $(G', T')$  over  $k$  unique up to isomorphism for which  $R(G', T') \simeq R'$ . Since the isomorphism in  $(G', T')$  is only ambiguous up to a  $T'(k)$ -conjugation, it is reasonable to consider the representation theory of  $(G', T')$  as a structure that is intrinsically associated to  $(G, T)$ . This is a version of *Langlands duality*.

A basic example is  $G = \text{GL}_n$ , in which case  $G' = \text{GL}_n$ . Slightly more interesting is the case  $G = \text{SL}_n$ , for which  $G' = \text{PGL}_n$ . In general, Langlands duality in the semisimple case swaps adjoint and simply connected groups (see Example 1.3.10).

**Remark 1.3.17.** Operations with root systems have analogues for connected semisimple groups. For example, the decomposition of a non-empty root system into its irreducible components corresponds to the fact that every nontrivial connected semisimple  $k$ -group  $G$  has only finitely many minimal (necessarily semisimple) nontrivial normal smooth connected subgroups  $G_i$ , each  $G_i$  is simple (in the sense of having no nontrivial proper connected normal linear algebraic subgroup), and these pairwise commute and define a central isogeny  $\prod G_i \rightarrow G$  via multiplication. See [Bo91, 14.10] for proofs based on the structure of automorphisms of semisimple groups, and see Proposition 5.1.17ff. for a simple proof based on the structure of the “open cell” (using (1.4.1)).

**1.4. Positive systems of roots and parabolic subgroups.** Let  $G$  be a connected reductive  $k$ -group and  $T$  a maximal torus. There are only finitely many parabolic subgroups  $P$  of  $G$  containing  $T$ , and in particular only finitely many Borel subgroups  $B$  of  $G$  containing  $T$ . These can be described in terms of the following combinatorial notion applied to the root system  $\Phi(G, T)$ :

**Definition 1.4.1.** Let  $(V, \Phi)$  be a non-empty root system. A *positive system of roots* is a subset  $\Phi^+ \subset \Phi$  such that  $\Phi^+ = \Phi_{\lambda > 0} := \{a \in \Phi \mid \lambda(a) > 0\}$  for some  $\lambda \in V^*$  that is non-vanishing on  $\Phi$ ; i.e.,  $\Phi^+$  is the part of  $\Phi$  lying in an open half-space of  $V_{\mathbf{R}}$  whose boundary hyperplane is disjoint from  $\Phi$ .

For any positive system of roots  $\Phi^+$  in  $\Phi$ , the subset  $\Delta \subset \Phi^+$  of elements of  $\Phi^+$  that cannot be expressed as a sum of two elements of  $\Phi^+$  turns out to be a basis of  $V$  and every element of  $\Phi$  has the form  $\sum_{a \in \Delta} m_a a$  with integers  $m_a$  that are either all  $\geq 0$  or all  $\leq 0$

[Bou2, VI, §1.6, Thm. 3]. The elements of  $\Delta$  are called the *simple positive roots* relative to  $\Phi^+$ , and any such  $\Delta$  is called a *base* for the root system  $\Phi$ .

Note that if we fix an enumeration  $\{a_i\}$  of such a  $\Delta$  and equip  $V = \bigoplus \mathbf{Q}a_i$  with the lexicographical ordering then  $\Phi^+$  consists of the elements of  $\Phi$  that are positive relative to this ordering. Conversely, for any ordered vector space structure on  $V$  the set of positive elements of  $\Phi$  turns out to be a positive system of roots [Bou2, VI, §1.7, Cor. 2] (a useful fact in the theory of “highest weight vectors” in representation theory; see §1.5).

**Example 1.4.2.** For  $G = \mathrm{GL}_n$  ( $n \geq 2$ ) and  $T = \mathbf{G}_m^n$  the diagonal torus, identify  $X(T)_{\mathbf{Q}} = X(\mathbf{G}_m)_{\mathbf{Q}}^n$  with  $\mathbf{Q}^n$  via the canonical identification  $X(\mathbf{G}_m) = \mathbf{Z}$ . In other words, to each  $\chi : \mathbf{G}_m^n \rightarrow \mathbf{G}_m$  associate the  $n$ -tuple  $(c_1, \dots, c_n) \in \mathbf{Z}^n$  for which  $\chi(t_1, \dots, t_n) = \prod t_i^{c_i}$ . In this way,  $\Phi(G, T)$  is a root system in the hyperplane  $V = \{\vec{c} \in \mathbf{Q}^n \mid \sum c_j = 0\}$ . Equip  $\mathbf{Q}^n$  with the lexicographical ordering, and  $V$  with the induced ordering. The root system  $\Phi(G, T)$  consists of the vectors  $e_i - e_j$  for  $1 \leq i \neq j \leq n$ , and the positive ones for this ordering are the differences  $e_i - e_j$  for  $i < j$ . The corresponding base  $\Delta$  consists of the roots  $e_i - e_{i+1}$  for  $1 \leq i \leq n - 1$ .

This positive system of roots is exactly the set of roots that occur as nontrivial  $T$ -weights on the Lie algebra of the Borel subgroup  $B$  of upper triangular matrices in  $G$ . See Proposition 1.4.4 for the general result of which this is a special case.

Every element of  $\Phi$  is a simple positive root for some choice of  $\Phi^+$  (equivalently, for any  $\Delta$  the  $W(\Phi)$ -orbits of the elements of  $\Delta$  cover  $\Phi$ ) [Bou2, VI, §1.5, Prop. 15], and if we fix a choice of  $\Phi^+$  and let  $\Delta$  be the corresponding base then  $W(\Phi)$  is generated by the reflections  $s_a$  for  $a \in \Delta$ . In fact,  $W(\Phi)$  has a presentation as a reflection group generated by the reflections in the simple positive roots [Bou2, VI, §1.5, Thm. 2(vii), Rem. 3, (11)]:

$$W(\Phi) = \langle \{s_a\}_{a \in \Delta} \mid (s_a s_b)^{m_{ab}} = 1 \text{ for } a, b \in \Delta \rangle$$

where  $m_{aa} = 1$  for all  $a \in \Delta$ ,  $m_{ab} = 2$  (equivalently,  $s_a s_b = s_b s_a$ ) when  $a$  and  $b$  are in distinct irreducible components of  $\Phi$  or are orthogonal in the same component, and otherwise  $m_{ab} = 3$  for non-orthogonal  $a, b \in \Delta$  with the same length and  $m_{ab} = 2 \langle a, b^\vee \rangle \langle b, a^\vee \rangle \in \{4, 6\}$  for non-orthogonal  $a, b \in \Delta$  with distinct lengths.

**Remark 1.4.3.** An important invariant of a reduced root system  $(V, \Phi)$  is its *Dynkin diagram*  $\mathrm{Dyn}(\Phi)$ , a graph with extra structure on certain edges. This intervenes in the classification of root systems, which is an ingredient in the proof of the Existence and Isomorphism Theorems (at least in low rank), and it can be defined in terms of Euclidean geometry or combinatorics with a root system (see [Bou2, VI, §4.2]). We will give both definitions. In all cases, the vertices of the graph are the elements of a base  $\Delta$  for a positive system of roots (and the simply transitive action of  $W(\Phi)$  on the set of all  $\Delta$ 's identifies vertices with certain  $W(\Phi)$ -orbits in  $\Phi$ , eliminating the dependence on the choice of  $\Delta$ ).

For the Euclidean definition we shall define  $\mathrm{Dyn}(\Phi) = \coprod \mathrm{Dyn}(\Phi_i)$  for the irreducible components  $(V_i, \Phi_i)$  of  $(V, \Phi)$ , so suppose  $\Phi$  is irreducible. Recall from Remark 1.3.14 that there exists a  $W(\Phi)$ -invariant inner product  $(\cdot | \cdot)$  on  $V_{\mathbf{R}}$  (even a  $\mathbf{Q}$ -valued  $W(\Phi)$ -invariant

positive-definite quadratic form on  $V$ ), it is unique up to scaling, and as we vary through  $a, b \in \Phi$  with  $(a|a) \geq (b|b)$ , the ratio  $(a|a)/(b|b)$  is  $\mathbf{Z}$ -valued and takes on at most two possible values. The diagram  $\text{Dyn}(\Phi)$  has as its vertices the elements of  $\Delta$ , and there exists an edge linking vertices  $a$  and  $b$  precisely when  $(a|b) \neq 0$ . If  $(a|a) > (b|b)$ , this edge is assigned multiplicity  $(a|a)/(b|b)$  and a direction pointing from  $a$  to  $b$ . The graph is always connected (for irreducible  $\Phi$ ).

In the combinatorial definition we do not need to pass to the irreducible components:  $\text{Dyn}(\Phi)$  is a graph whose vertices are the elements of  $\Delta$  and an edge joins  $a$  and  $b$  precisely when  $a + b \in \Phi$ . (Note that  $a - b \notin \Phi$  since  $a, b \in \Delta$ .) For such  $a$  and  $b$ , the multiplicity and direction of the edge joining  $a$  and  $b$  are defined in terms of the sets

$$I_{a,b} = \{j \in \mathbf{Z} \mid a + jb \in \Phi\}, \quad I_{b,a} = \{j \in \mathbf{Z} \mid b + ja \in \Phi\}$$

as follows. Since  $a, b \in \Delta$ , we have  $I_{a,b} = \mathbf{Z} \cap [0, \dots, -\langle a, b^\vee \rangle]$  and  $I_{b,a} = \mathbf{Z} \cap [0, \dots, -\langle b, a^\vee \rangle]$  [Bou2, VI, §1.3, Prop. 9]. An inspection of cases [Bou2, VI, §1.3, Rem.] shows that either both sets coincide with  $\{0, 1\}$ , in which case  $a$  and  $b$  are joined by a single undirected edge, or one of them is  $\{0, 1\}$  and the other is either  $\{0, 1, 2\}$  or  $\{0, 1, 2, 3\}$ . In these latter cases if we arrange the labels so that  $I_{a,b}$  contains 2 or 3 then  $a$  and  $b$  are joined by an edge with multiplicity  $-\langle a, b^\vee \rangle$  pointing from  $a$  to  $b$ .

Returning to the setting of a connected reductive  $k$ -group  $G$  and a maximal torus  $T$ , if  $B$  a Borel subgroup containing  $T$  then the subalgebra  $\mathfrak{b} = \text{Lie}(B) \subset \mathfrak{g}$  contains  $\mathfrak{t}$  and so has a weight space decomposition  $\mathfrak{b} = \mathfrak{t} \oplus (\bigoplus_{a \in \Phi(B,T)} \mathfrak{g}_a)$  for a subset  $\Phi(B, T) \subset \Phi(G, T)$ . Such subsets are positive systems of roots in  $\Phi(G, T)$  [Bo91, 14.1]. If  $U = \mathcal{R}_u(B)$  then for any enumeration  $\{a_i\}$  of  $\Phi(B, T)$  the multiplication map

$$(1.4.1) \quad \prod U_{a_i} \rightarrow U$$

is an isomorphism of schemes (see [Bo91, 14.4] or [CGP, 3.3.6, 3.3.7, 3.3.11]); we say that  $U$  is “directly spanned in any order” by the root groups contained in  $U$ .

Since  $W_G(T)$  acts simply transitively on the set of all  $B \supset T$  [Bo91, 13.10(2)], and  $W(\Phi)$  acts simply transitively on the set of positive systems of roots in  $\Phi$  [Bou2, VI, §1.5, Thm. 2(i)], the inclusion  $W(\Phi) \subset W_G(T)$  is forced to be an equality and we obtain:

**Proposition 1.4.4.** *The map  $B \mapsto \Phi(B, T)$  is a bijection from the set of  $B \supset T$  onto the set of positive systems of roots in  $\Phi(G, T)$ .*

The generalization of Proposition 1.4.4 to the case of parabolic subgroups containing  $T$  requires another class of distinguished subsets of a root system:

**Definition 1.4.5.** Let  $(V, \Phi)$  be a root system. A *parabolic subset* of  $\Phi$  is a subset  $\Psi \subset \Phi$  of the form  $\Psi = \Phi_{\lambda \geq 0} := \{a \in \Phi \mid \lambda(a) \geq 0\}$  for some  $\lambda \in V^*$ ; i.e.,  $\Psi$  is the part of  $\Phi$  lying in a closed half-space of  $V$  (or  $V_{\mathbf{R}}$ ).

The reason for this terminology is that it will turn out that for any parabolic subgroup  $P \subset G$  containing  $T$ , the set  $\Phi(P, T)$  of nontrivial  $T$ -weights occurring in  $\mathfrak{p} = \text{Lie}(P)$  is a

parabolic subset of  $\Phi(G, T)$ . Before we can state the precise bijective correspondence in this direction, it will be convenient to discuss some alternative formulations of the definition of parabolicity for a subset of a root system  $(V, \Phi)$ .

It is clear from the definition that any parabolic subset  $\Psi$  of  $\Phi$  satisfies the following two properties: (i)  $\Phi = \Psi \cup -\Psi$ , and (ii)  $\Psi$  is a *closed* set in  $\Phi$  (i.e., if  $a, b \in \Psi$  and  $a+b \in \Phi$  then  $a+b \in \Psi$ ). Conversely, any subset of  $\Psi$  satisfying (i) and (ii) is parabolic. The equivalence is proved in [CGP, 2.2.8] (where parabolicity is defined using (i) and (ii), as in [Bou2, Ch. IV]), and a more explicit description of parabolic sets is provided there: they are precisely the subsets  $\Phi^+ \cup [I]$ , where  $\Phi^+$  is a positive system of roots,  $I$  is a subset of the corresponding base  $\Delta$ , and  $[I]$  denotes the set of roots that are  $\mathbf{Z}$ -linear combinations of elements of  $I$ . In particular, every parabolic set contains a positive system of roots.

A given parabolic set  $\Psi \subset \Phi$  can contain more than one positive system of roots, just as a parabolic subgroup  $P$  can contain more than one Borel subgroup containing a fixed maximal torus  $T$  in  $P$ . Nonetheless, for *any* positive system of roots  $\Phi^+$  (with corresponding base  $\Delta$ ) that is contained in  $\Psi$  we have  $\Psi = \Phi^+ \cup [I]$  for a unique subset  $I \subset \Delta$  [Bou2, VI, §1.7, Lemma 3].

**Proposition 1.4.6.** *The map  $P \mapsto \Phi(P, T)$  is a bijective correspondence between the set of parabolic subgroups of  $G$  containing  $T$  and the set of parabolic subsets of  $\Phi = \Phi(G, T)$ , and the following are equivalent:  $P \subseteq P'$ ,  $\Phi(P, T) \subseteq \Phi(P', T)$ , and  $\mathfrak{p} \subseteq \mathfrak{p}'$  inside  $\mathfrak{g}$ .*

*Proof.* Since  $\mathfrak{p}$  is spanned by  $\text{Lie}(T)$  and the weight spaces  $\mathfrak{g}_c = \text{Lie}(U_c)$  for  $c \in \Phi(P, T)$ , to prove the equivalence of containment assertions it suffices to prove that if  $c \in \Phi(P, T)$  then  $U_c \subset P$  (as then  $P$  is generated by  $T$  and such  $U_c$ , by Lie algebra considerations since  $P$  is connected). The key to proving this is to reduce to the rank-1 case, as follows.

Any  $P$  contains a Borel subgroup  $B$  of  $G$  that contains  $T$  (due to the conjugacy of maximal tori in  $P$ ). For any subtorus  $S \subset T$ , the intersection  $P \cap Z_G(S) = Z_P(S)$  is smooth and connected, and it contains the subgroup  $B \cap Z_G(S) = Z_B(S)$  that is a Borel subgroup of  $Z_G(S)$  [Bo91, 11.15], so  $P \cap Z_G(S)$  is a parabolic subgroup of  $Z_G(S)$  containing  $T$ . The bijective correspondence between parabolic subgroups of a connected reductive group  $H$  and of its derived group  $\mathcal{D}(H)$  is defined by  $P' \mapsto P' \cap \mathcal{D}(H)$ , so  $P \cap \mathcal{D}(Z_G(S))$  is a parabolic subgroup of  $\mathcal{D}(Z_G(S))$  containing the maximal torus  $T \cap \mathcal{D}(Z_G(S))$ .

Taking  $S$  to be the codimension-1 subtorus  $T_c := (\ker c)_{\text{red}}^0 \subset T$ , it follows that  $P_c := P \cap \mathcal{D}(Z_G(T_c))$  is a parabolic subgroup of the 3-dimensional connected semisimple group  $\mathcal{D}(Z_G(T_c)) = \langle U_c, U_{-c} \rangle$  containing the maximal torus  $T \cap \mathcal{D}(Z_G(T_c)) = c^\vee(\mathbf{G}_m)$ . Since the root groups of  $\mathcal{D}(Z_G(T_c))$  relative to  $c^\vee(\mathbf{G}_m)$  are  $U_{\pm c}$ , we can replace  $(G, T, P)$  with  $(\mathcal{D}(Z_G(T_c)), c^\vee(\mathbf{G}_m), P_c)$  to reduce to the case when  $G$  is semisimple of rank 1. We can then choose an isomorphism from  $G$  onto either  $\text{SL}_2$  or  $\text{PGL}_2$  such that  $T$  is carried to the diagonal torus and  $U_c$  is carried to the upper triangular unipotent subgroup. There are only three parabolic subgroups containing the diagonal torus, namely the entire group and the upper and lower triangular Borel subgroups. The condition  $c \in \Phi(P, T)$  rules out the possibility of

the lower triangular Borel subgroup, and inspection of the two remaining possibilities shows that  $U_c \subset P$ .

It remains to show that every parabolic subset  $\Psi$  of  $\Phi$  arises as  $\Phi(P, T)$  for some parabolic subgroup  $P$  containing  $T$ . Since  $W(G, T)$  acts transitively on the set of positive systems of roots in  $\Phi$ , we may restrict attention to  $\Psi$  that contain  $\Phi^+ := \Phi(B, T)$  for a fixed Borel subgroup  $B$  of  $G$  containing  $T$ . Thus,  $\Psi = \Phi^+ \cup [I]$  for a unique subset  $I$  of the base  $\Delta$  of  $\Phi^+$ . Since  $\Delta$  is a basis of the character group of the adjoint torus  $T/Z_G$ , for the subtorus  $T_I = (\cap_{a \in I} \ker a)_{\text{red}}^0$  we see that  $Z_G(T_I)$  contains as its  $T$ -root groups exactly  $U_c$  for  $c \in \Phi$  killed by  $I$ . Let  $U_I$  be the smooth connected subgroup of  $\mathcal{R}_u(B)$  directly spanned in any order by the root groups  $U_b$  for  $b$  in the closed subset  $[I] \cap \Phi^+ \subseteq \Phi^+$ .

The group  $Z_G(T_I)$  normalizes  $U_I$ . Indeed,  $U_I \subset Z_{T_I}(\mathcal{R}_u(B))$ , yet this containment between smooth connected groups is an equality on Lie algebras and hence is an equality of groups, so the positive root groups in  $Z_G(T_I)$  certainly normalize  $U_I$ . Clearly  $T$  normalizes  $U_I$ , and if  $c \in -\Phi^+$  not in  $[I]$  then  $U_c$  even centralizes  $U_I$  since  $c + b$  cannot be a root for any  $b \in [I] \cap \Phi^+$ . Thus, the semi-direct product  $P_I = Z_G(T_I) \ltimes U_I$  makes sense inside  $G$ , and this smooth subgroup clearly contains  $B$  (so it is a parabolic subgroup containing  $T$ ) and satisfies  $\Phi(P_I, T) = \Phi^+ \cup [I] = \Psi$ . (This recovers the parameterization of ‘‘standard’’ parabolic subgroups of  $G$  in [Bo91, 14.18] via a different proof. Note that  $\mathcal{R}_u(P_I) = U_I$ .)  $\square$

**Example 1.4.7.** Let  $G = \text{SL}_n$  and use the upper triangular  $B$  and diagonal  $T$  with character group  $\mathbf{Z}^n/\text{diag}(\mathbf{Z})$ . Then  $\Delta = \{e_{i+1} - e_i\}_{1 \leq i \leq n-1} = \{1, \dots, n-1\}$ , and for  $I \subseteq \{1, \dots, n-1\}$  the parabolic  $P_I$  corresponds via Example 1.1.9 to the ordered partition  $\vec{a} = (a_1, \dots, a_r)$  of  $n$  into non-empty parts for which the associated subset  $\{b_j = a_1 + \dots + a_j\}_{1 \leq j < r}$  in  $\{1, \dots, n-1\}$  is the *complement* of  $I$ ; e.g.,  $P_\emptyset = B$ .

**Corollary 1.4.8.** *The number of  $G(k)$ -conjugacy classes of parabolic subgroups of  $G$  is  $2^{r_{\text{ss}}}$ , where  $r_{\text{ss}} = \text{rank}(\mathcal{D}(G))$  is the dimension of the maximal tori of  $\mathcal{D}(G)$ . (Equivalently,  $r_{\text{ss}} = \#\Delta$ , where  $\Delta$  is a base of a positive system of roots in  $\Phi = \Phi(G, T)$  for any maximal torus  $T$  of  $G$ .)*

A description of conjugacy class representatives in terms of explicit cocharacters will be given in Exercise 4.4.8.

*Proof.* We just have to show that for a fixed Borel subgroup  $B \subseteq G$  and a maximal torus  $T \subseteq B$ , parabolic subgroups  $P_I$  and  $P_J$  associated to subsets  $I$  and  $J$  of the base  $\Delta$  of  $\Phi^+ = \Phi(B, T)$  are  $G(k)$ -conjugate if and only if  $I = J$ . More conceptually, if  $P$  and  $P'$  are parabolic subgroups of  $G$  containing  $B$  and  $P' = gPg^{-1}$  for some  $g \in G(k)$  then we claim that  $P' = P$ . The Borel subgroups  $B$  and  $gBg^{-1}$  in  $P'$  are  $P'(k)$ -conjugate, so by replacing  $g$  with a suitable left  $P'(k)$ -translate we can arrange that  $gBg^{-1} = B$ . But then  $g \in N_{G(k)}(B) = B(k) \subseteq P(k)$ , so  $P' = P$ .  $\square$

**Remark 1.4.9.** Note the remarkable consequence of Proposition 1.4.6 that in arbitrary characteristic, a parabolic subgroup  $P$  is uniquely determined by its Lie algebra  $\mathfrak{p}$  as a subalgebra of  $\mathfrak{g}$  given that  $P$  is a parabolic subgroup containing a specific maximal torus  $T$ .

In fact,  $P$  is determined by  $\mathfrak{p}$  without reference to  $T$ : under the adjoint representation of  $G$  on  $\mathfrak{g}$ ,  $P(k)$  is the stabilizer in  $G(k)$  of the subspace  $\mathfrak{p}$ . This goes beyond the classical result  $P(k) = N_{G(k)}(P)$  in Theorem 1.1.8; see Exercise 2.4.8 for further discussion (and Corollary 1.4.13 for a version with Borel subgroups in the classical theory).

We call  $\Phi(B, T)$  the positive system of roots *associated to*  $B$  (and  $T$ ). For each  $B$ , the unique Borel subgroup  $B' \supset T$  for which  $\Phi(B', T) = -\Phi(B, T)$  is the *opposite Borel subgroup* (relative to  $T$ ), and its unipotent radical  $U'$  is directly spanned in any order by the negative root groups  $U_{-a}$  for  $a$  in the positive system of roots  $\Phi(B, T)$ . (Explicitly,  $B' = U' \rtimes T$ .) For example, if  $G = \mathrm{GL}_n$ ,  $T$  is the diagonal torus, and  $B$  is the Borel subgroup of upper triangular matrices then  $B'$  is the Borel subgroup of lower triangular matrices.

**Proposition 1.4.10.** *The natural multiplication map*

$$(1.4.2) \quad U' \times T \times U = U' \times B \rightarrow G$$

*is an open immersion.*

*Proof.* This map is visibly étale at the origin, corresponding to the decomposition  $\mathfrak{g} = \mathfrak{g}_{<0} \oplus \mathfrak{t} \oplus \mathfrak{g}_{>0}$  relative to an ordering on  $\mathbf{Q}\Phi$  that makes  $\Phi(B, T)$  positive. Thus, it is everywhere étale, due to  $U'(k)$ -translation on the left and  $B(k)$ -translation on the right. Now we can argue exactly as in the analysis of (1.2.2).  $\square$

The image  $\Omega$  of (1.4.2) is the *open cell* for  $(G, T, B)$ , though this terminology is also often used for the “inverse”  $U \times T \times U' \subset G$  involving multiplication in the opposite order. It is also commonly called the *big cell*. Here is a useful application of the open cell:

**Corollary 1.4.11.** *Let  $G$  be a connected reductive  $k$ -group and  $T$  a maximal torus. Then  $T$  is its own scheme-theoretic centralizer (i.e., for any  $k$ -algebra  $R$ , if  $g \in G(R)$  centralizes  $T_R$  then  $g \in T(R)$ ). In particular, every central closed subgroup scheme of  $G$  lies in  $T$ .*

*Proof.* By direct limit considerations we can assume  $R$  is finite type over  $k$ , and then (by the Krull Intersection Theorem) that  $R$  is a finite local  $k$ -algebra. Since  $T(k) = Z_{G(k)}(T)$ , a point  $g \in G(R)$  that centralizes  $T_R$  is physically supported on  $T$  and thus  $g$  factors through the open cell  $\Omega$ . Hence, we can write  $g = u'tu$  for  $u' \in U'(R)$ ,  $t \in T(R)$ , and  $u \in U(R)$ . We aim to prove that  $u' = u = 1$ .

Since  $T_R$  centralizes  $g$  and normalizes  $U_R$  and  $U'_R$ , it follows that  $u'$  and  $u$  are each centralized by  $T_R$ . Using the direct spanning of  $U$  and  $U'$  by the root groups (which are normalized by  $T$ ), it suffices to check that if  $a \in \Phi(G, T)$  and  $u \in U_a(R)$  is centralized by the  $T_R$ -action on  $(U_a)_R$  then  $u = 1$ . The character  $a : T \rightarrow \mathbf{G}_m$  is an fppf covering, so via a choice of isomorphism  $U_a \simeq \mathbf{G}_a$  we are reduced to proving that a point  $u \in \mathbf{G}_a(R)$  invariant under the  $\mathbf{G}_m$ -action on  $\mathbf{G}_a$  (over the category of  $R$ -algebras) must vanish. Since  $R$  has an algebraically closed residue field, it contains a unit  $c$  such that  $c - 1 \in R^\times$ , so the equality  $cu = u$  forces  $u = 0$ .  $\square$

The open cell also naturally arises in the description of double cosets for  $B$  in  $G$ . To make this precise, first note that by the orbit lemma (applied to  $B \times B$  acting on  $G$  through the

commuting operations of left and right multiplication by  $B$ ), each double coset  $BgB$  in  $G$  (for  $g \in G(k)$ ) is locally closed. If  $w_0 \in W(\Phi)$  is the unique longest element relative to the reflections in the simple positive roots for  $\Phi^+ = \Phi(B, T)$  (i.e.,  $w_0(\Phi^+) = -\Phi^+$ ) then for any representative  $n_0 \in N_{G(k)}(T)$  of  $w_0$  we have  $U' = n_0^{-1}Un_0$ , so  $Bn_0B = n_0\Omega$ . This is the only open double coset, due to dimension considerations applied to the explicit description of the double cosets in:

**Theorem 1.4.12** (Bruhat decomposition). *Let  $G$  be a connected reductive group,  $T$  a maximal torus, and  $B$  a Borel subgroup. For each  $w \in W := W_G(T)$ , let  $n_w \in N_G(T)(k)$  be a representative. Each locally closed double coset  $Bn_wB \subset G$  only depends on  $w$ , and*

$$G = \coprod_{w \in W} Bn_wB.$$

*If  $\Phi^+ := \Phi(B, T)$  then  $Bn_wB = U_w \times n_wB$  as  $k$ -schemes via multiplication, where  $U_w$  is a unipotent smooth connected subgroup directly spanned in any order by the positive root groups  $U_a$  for  $a$  in the set  $\Phi_w^+$  of roots in  $\Phi^+$  such that  $w^{-1}(a) \in -\Phi^+$ .*

*Proof.* Since  $T \subset B$  and  $n_w$  is well-defined modulo  $T(k)$ , it is clear that  $Bn_wB$  only depends on  $w$ . The asserted decomposition of  $G$  into locally closed sets amounts to an equality on  $k$ -points:  $G(k) = \coprod_{w \in W} B(k)n_wB(k)$ . This equality is proved in [Bo91, §14.12] (and it is a special case of a general decomposition for any abstract group equipped with a “ $BN$ -pair” [Bou2, IV, §2.1, Def. 1], which in turn was inspired by axiomatizing the argument for reductive groups; see [Bo91, 14.15]). To establish the desired description of each  $Bn_wB$  as a product scheme, since  $B = U \rtimes T$  and the  $k$ -group  $U$  is directly spanned in any order by the  $U_a$ 's for  $a \in \Phi^+$ , the equalities  $n_w^{-1}Tn_w = T$  and  $n_w^{-1}U_a n_w = U_{w^{-1}(a)}$  give that  $Bn_wB = U_w n_w B$  where the smooth closed subscheme  $U_w$  is the direct product of the root groups  $U_a$  for  $a \in \Phi_w^+$  using a fixed enumeration of  $\Phi_w^+$ .

It remains to show that  $U_w$  is a  $k$ -subgroup of  $U$  and that it is independent of the enumeration of  $\Phi_w^+$ . The subset  $\Phi_w^+$  of  $\Phi^+$  is closed (i.e., if  $a, b \in \Phi_w^+$  and  $a + b \in \Phi$  then  $a + b \in \Phi_w^+$ ), so the desired property of  $U_w$  is a special case of a general result on the direct spanning by root groups associated to any closed set of positive roots (see [Bo91, 14.5, Prop. (2)] or [CGP, 3.3.11, 3.3.13(1)]).  $\square$

**Corollary 1.4.13.** *For a Borel subgroup  $B$  of a connected reductive  $k$ -group  $G$ , the stabilizer of  $\mathfrak{b}$  under the adjoint action of  $G(k)$  on  $\mathfrak{g}$  is  $B(k)$ .*

*Proof.* Choose a maximal torus  $T$  in  $B$  and let  $\Phi^+ = \Phi(B, T)$  be the positive system of roots associated to  $B$ . Choose  $g \in G(k)$  whose adjoint action preserves  $\mathfrak{b}$ . By the Bruhat decomposition,  $g = bnb'$  for some  $b, b' \in B(k)$  and  $n \in N_{G(k)}(T)$ . It is harmless to replace  $g$  with  $b^{-1}gb'^{-1}$ , so  $g$  normalizes  $T$ . Then  $g$  represents some  $w \in W_G(T)$ , and the adjoint action of  $g$  carries  $\mathfrak{b} = \mathfrak{t} \oplus (\oplus_{a \in \Phi^+} \mathfrak{g}_a)$  to  $\mathfrak{t} \oplus (\oplus_{a' \in w(\Phi^+)} \mathfrak{g}_{a'})$ . Hence, the preservation of  $\mathfrak{b}$  forces  $w(\Phi^+) = \Phi^+$ , so  $w = 1$  and hence  $g \in T(k) \subset B(k)$ .  $\square$

In translation arguments with split reductive group schemes over a general scheme  $S$ , the following covering result on geometric fibers will be very useful.

**Corollary 1.4.14.** *For any  $(G, T, B)$ , with associated open cell  $\Omega = U' \times B$  and Weyl group  $W = W_G(T)$ ,  $G = \bigcup_{w \in W} n_w \Omega$  for any set of representatives  $\{n_w\}_{w \in W} \subset N_{G(k)}(T)$  of  $W$ .*

*Proof.* For each  $w \in W$ ,  $n_w^{-1} U_w n_w B \subset U' \times B = \Omega$ . Thus,  $B n_w B = U_w n_w B \subset n_w \Omega$ .  $\square$

**1.5. Based root datum, pinnings, and representation theory.** In this section, we will explore some consequences of the Existence and Isomorphism Theorems over  $k$ . Some of the basic concepts introduced (such as a “based root datum” and a “pinning” of a triple  $(G, T, B)$ ) will be used in the proof of the Existence and Isomorphism Theorems.

One often works with triples  $(G, T, B)$  rather than with pairs  $(G, T)$  (for a connected reductive  $k$ -group  $G$ ), and likewise when working with root data  $R = (X, \Phi, X^\vee, \Phi^\vee)$  it is convenient to choose a positive system of roots  $\Phi^+ \subset \Phi$ . By Proposition 1.4.4, the choice of  $\Phi^+$  corresponds to a choice of  $B \supset T$  when  $R$  arises from a connected reductive  $k$ -group  $G$  equipped with a maximal  $k$ -torus  $T$ . Another way to express the choice of  $\Phi^+$  is to specify its base  $\Delta$ . The set  $\Delta^\vee$  of coroots  $a^\vee$  for  $a \in \Delta$  is a base for  $\Phi^\vee$  (Exercise 1.6.16), and we call the 6-tuple  $(X, \Phi, \Delta, X^\vee, \Phi^\vee, \Delta^\vee)$  (or the triple  $(R, \Delta, \Delta^\vee)$ , or simply  $(R, \Delta)$ ) a *based root datum*.

There is an evident notion of *isomorphism* for based root data, as well as *duality*. If  $\Theta$  denotes the automorphism group of a based root datum  $(R, \Delta, \Delta^\vee)$  then

$$(1.5.1) \quad \text{Aut}(R) = W(\Phi) \rtimes \Theta$$

because the action of  $W(\Phi)$  on  $X$  is simply transitive on the set of choices of  $(\Delta, \Delta^\vee)$ . If  $\Phi$  spans  $X_{\mathbf{Q}}$  (or equivalently,  $\Delta$  is a  $\mathbf{Q}$ -basis of  $X_{\mathbf{Q}}$ ) and the root system  $(X_{\mathbf{Q}}, \Phi)$  is reduced then  $\Theta$  is naturally identified with a subgroup of the automorphism group of the Dynkin diagram  $\text{Dyn}(\Phi)$  of  $\Phi$ . The interest in  $\Theta$  is due to the fact (discussed below; see (1.5.2)) that it computes the outer automorphism group of a connected reductive  $k$ -group, so for the connected semisimple case (corresponding to the condition that  $\Phi$  spans  $X_{\mathbf{Q}}$ ) it is useful to know sufficient criteria for  $\Theta$  to exhaust the automorphism group of the Dynkin diagram:

**Proposition 1.5.1.** *Assume  $\Phi$  spans  $X_{\mathbf{Q}}$  and that  $(X_{\mathbf{Q}}, \Phi)$  is reduced. The inclusion  $\Theta \subset \text{Aut}(\text{Dyn}(\Phi))$  is an equality when the root datum is either adjoint or simply connected, as well as when  $(X_{\mathbf{Q}}, \Phi)$  is irreducible and the finite group  $(\mathbf{Z}\Phi^\vee)^*/\mathbf{Z}\Phi$  is cyclic.*

See Definition 1.3.11 for “adjoint” and “simply connected” applied to root data.

*Proof.* In the adjoint cases of Example 1.3.10, the root datum is a direct product of root data whose associated root system is irreducible and adjoint; similarly in the simply connected case. Thus, we can assume  $(X_{\mathbf{Q}}, \Phi)$  is irreducible. Hence, there exists a  $W(\Phi)$ -invariant inner product on  $X_{\mathbf{R}}$  that is unique up to scaling, and it is uniquely determined by imposing the condition that the shortest root length is 1. The resulting inner products among the roots are encoded in the Dynkin diagram, as is the resulting identification of the coroots with elements of  $X_{\mathbf{R}}$  (see Remark 1.3.14 and Remark 1.4.3).

By identifying the vertices of the Dynkin diagram with the elements of  $\Delta$ , we see that any automorphism of the Dynkin diagram uniquely extends to an automorphism of  $X_{\mathbf{R}}$  that

preserves  $\Phi$  and  $\Phi^\vee$ . It remains to check this automorphism of  $X_{\mathbf{R}}$  carries  $X$  onto itself, or equivalently (by duality) carries  $X^\vee$  onto itself. This is automatic in the adjoint and simply connected cases. It is also automatic when  $\Pi := (\mathbf{Z}\Phi^\vee)^*/\mathbf{Z}\Phi$  is cyclic, as then every automorphism of  $\Pi$  carries each subgroup of  $\Pi$  onto itself (and  $X$  is uniquely determined by the corresponding subgroup in  $\Pi$ ).  $\square$

Here is a visualization of Proposition 1.5.1 when  $R = R(G, T)$  for a connected semisimple group  $G$ . Whereas every automorphism of a connected semisimple group  $G$  lifts (uniquely) to the simply connected central cover  $\tilde{G}$ , an automorphism  $\tilde{f}$  of  $\tilde{G}$  descends to  $G$  if and only if its restriction to  $Z_{\tilde{G}}$  preserves the central subgroup  $\mu = \ker(\tilde{G} \rightarrow G)$ . If  $\tilde{f}$  is inner or  $Z_{\tilde{G}}$  is cyclic then preservation of  $\mu$  is automatic (because any automorphism of a finite cyclic group preserves every subgroup). In general,  $\Theta$  is the group of outer automorphisms of  $\tilde{G}$  that arise from lifting automorphisms of  $G$  (see (1.5.2) below).

**Example 1.5.2.** The only cases with *irreducible*  $\Phi$  when non-cyclicity occurs in Proposition 1.5.1 is type  $D_{2n}$  ( $n \geq 2$ ), with  $(\mathbf{Z}\Phi^\vee)^*/(\mathbf{Z}\Phi) = (\mathbf{Z}/2\mathbf{Z})^2$ . Let  $\tilde{G} = \text{Spin}_{4n}$  (type  $D_{2n}$ ). Choose a maximal torus  $T \subset \tilde{G}$  and label  $\text{Dyn}(\Phi(\tilde{G}, T))$  as follows:

$$(t, t') \in \mu_2 \times \mu_2 \simeq Z_{\tilde{G}} : \quad \begin{array}{ccccccc} \begin{array}{c} tt' \\ \circ \\ a_1 \end{array} & \text{---} & \begin{array}{c} 1 \\ \circ \\ a_2 \end{array} & \text{---} & \begin{array}{c} tt' \\ \circ \\ a_3 \end{array} & \text{---} & \begin{array}{c} 1 \\ \circ \\ a_4 \end{array} & \dots & \begin{array}{c} tt' \\ \circ \\ a_{2n-3} \end{array} & \text{---} & \begin{array}{c} 1 \\ \circ \\ a_{2n-2} \end{array} & \begin{array}{l} \nearrow \begin{array}{c} t \\ \circ \\ a_{2n-1} \end{array} \\ \searrow \begin{array}{c} t' \\ \circ \\ a_{2n} \end{array} \end{array} \end{array}$$

We have  $T = \prod_j a_j^\vee(\mathbf{G}_m)$ , and the center  $Z_{\tilde{G}}$  is  $\mu_2 \times \mu_2$  embedded in  $T[2]$  by diagonally mapping each  $\mu_2$  into the 2-torsion of the indicated coroot groups.

There are three intermediate groups  $G$  strictly between  $\tilde{G}$  and  $\tilde{G}/Z_{\tilde{G}}$  (one for each copy  $\mu$  of  $\mu_2$  in  $\mu_2 \times \mu_2 \subset Z_{\tilde{G}}$ ), and  $\text{Aut}(\text{Dyn}(\Phi))$  has order 2 when  $n \geq 3$  and is  $S_3 = \text{GL}_2(\mathbf{F}_2)$  when  $n = 2$ . Thus, if  $n = 2$  then each  $\mu \simeq \mu_2 \subset Z_{\tilde{G}}$  has  $\text{Aut}(\text{Dyn}(\Phi))$ -stabilizer of order 2, and if  $n \geq 3$  then only one such  $\mu$  is preserved under the order-2 group  $\text{Aut}(\text{Dyn}(\Phi))$ ; we claim that this is the non-adjoint quotient  $G = \text{SO}_{4n}$ .

More generally, noting that  $\text{Spin}_{2n}$  has center  $\mu_4$  for odd  $n \geq 3$  (e.g.,  $\text{Spin}_6 = \text{SL}_4$ ), it suffices to show that for all  $n \geq 2$ , the group  $\text{SO}_{2n}$  has  $\Theta$  of order 2, rather than of order 1. (In contrast,  $\text{SO}_{2n+1}$  for  $n \geq 1$  has  $\Theta = 1$  since  $\text{Aut}(\text{Dyn}(\mathbf{B}_n)) = 1$  for  $n \geq 1$ ; see Example C.6.2 for  $n = 1$ .) In view of (1.5.2) below, this says exactly that the outer automorphism of  $\text{SO}_{2n}$  has order 2 (rather than order 1). Where does a non-inner automorphism come from? Conjugation by  $\text{O}_{2n}$ ! Indeed, if  $g \in \text{O}_{2n}(k) - \text{SO}_{2n}(k)$  and  $g$ -conjugation on  $\text{SO}_{2n}$  is inner then by replacing  $g$  with a suitable left  $\text{SO}_{2n}(k)$ -translate we would get that  $g$  centralizes  $\text{SO}_{2n}$ . But the diagonal torus  $T$  in  $\text{SO}_{2n}$  has  $\text{O}_{2n}$ -centralizer equal to  $T$ , so this is impossible.

The Isomorphism Theorem and (1.5.1) lead to the determination of the automorphism group of a connected reductive  $k$ -group  $G$ , as follows. Inside  $\text{Aut}(G)$  there is the normal

subgroup  $G(k)/Z_{G(k)}$  of inner automorphisms; for  $g \in G(k)$  let  $c_g$  denote the inner automorphism  $x \mapsto gxg^{-1}$ . To describe the quotient group  $\text{Out}(G)$  of outer automorphisms, fix a choice of  $(B, T)$  as usual and consider an automorphism  $\varphi$  of  $G$ . By composing  $\varphi$  with a suitable inner automorphism, we can arrange that  $\varphi(B) = B$  and  $\varphi(T) = T$ . The only  $g \in G(k)$  such that  $\varphi \circ c_g$  preserves  $B$  and  $T$  are  $g \in N_{G(k)}(B) \cap N_{G(k)}(T) = B(k) \cap N_{G(k)}(T) = T(k)$ , and this is precisely the ambiguity that arises by passing from  $\varphi$  to the induced automorphism of the root datum  $R(G, T)$ . The automorphism  $\varphi$  induces an automorphism  $\bar{\varphi}$  of the based root datum associated to  $(G, T, B)$ , and by (1.5.1) the element  $\bar{\varphi} \in \Theta$  determines the outer automorphism class of  $\varphi$ . Moreover, every element of  $\Theta$  arises in this way, due to the Isomorphism Theorem. We thereby obtain a short exact sequence of abstract groups

$$(1.5.2) \quad 1 \rightarrow G(k)/Z_{G(k)} \rightarrow \text{Aut}(G) \rightarrow \Theta \rightarrow 1.$$

**Corollary 1.5.3.** *The natural map  $\text{Aut}(G, T) \rightarrow \text{Aut}(R(G, T))$  is surjective with kernel  $T(k)/Z_{G(k)}$ .*

*Proof.* The subgroup  $\text{Aut}(G, T)$  of  $\text{Aut}(G)$  meets  $G(k)/Z_{G(k)}$  in  $N_{G(k)}(T)/Z_{G(k)}$ , so the quotient group  $\text{Aut}(G, T)/(T(k)/Z_{G(k)})$  contains  $W_G(T)$  as a subgroup that is carried isomorphically onto the subgroup  $W(\Phi) \subset \text{Aut}(R(G, T))$ . The quotient of  $\text{Aut}(G, T)/(T(k)/Z_{G(k)})$  modulo its normal subgroup  $W_G(T)$  is clearly  $\text{Out}(G)$ , and likewise  $\text{Aut}(R(G, T))/W(\Phi) = \Theta$ . This is compatible with the isomorphism  $\Theta \simeq \text{Out}(G)$  defined via (1.5.2).  $\square$

Remarkably, the short exact sequence (1.5.2) splits as a semi-direct product. To formulate this, we need to introduce some more structure beyond the triple  $(G, T, B)$ .

**Definition 1.5.4.** A *pinning* of  $(G, T, B)$  is the specification of an isomorphism  $p_a : \mathbf{G}_a \simeq U_a$  for each  $a \in \Delta$ ; equivalently, it is the choice of a nonzero  $X_a \in \mathfrak{g}_a$  for each  $a \in \Delta$  (via  $\text{Lie}(p_a)(\partial_x) = X_a$ , with  $x$  the standard coordinate on  $\mathbf{G}_a$ ). The data  $(G, T, B, \{X_a\}_{a \in \Delta})$  is a *pinned* connected reductive group.

(In [Bou3, IX, §4.10, Def. 3] the analogous notion for a connected compact Lie group is called a *framing*. Kottwitz, Langlands, and Shelstad use the terminology *splitting*.) There is an evident notion of *isomorphism* for pinned connected reductive groups. Pinnings remove  $T(k)$ -conjugation ambiguity in the Isomorphism Theorem:

**Proposition 1.5.5.** *For any  $(G, T, B, \{X_a\}_{a \in \Delta})$  and  $(G', T', B', \{X'_{a'}\}_{a' \in \Delta'})$ , the natural map*

$$\begin{array}{c} \text{Isom}((G, T, B, \{X_a\}_{a \in \Delta}), (G', T', B', \{X'_{a'}\}_{a' \in \Delta'})) \\ \downarrow \\ \text{Isom}((R(G, T), \Delta, \Delta^\vee), (R(G', T'), \Delta', \Delta'^\vee)) \end{array}$$

*is bijective. In particular, if  $f$  is an automorphism of  $(G, T, B)$  that is the identity on  $T$  and on the simple positive root groups then  $f$  is the identity on  $G$ , and a choice of pinning on  $(G, T, B)$  defines a homomorphic section to the quotient map  $\text{Aut}(G) \rightarrow \text{Out}(G) \simeq \Theta$ .*

*Proof.* Since the elements of  $\Delta$  are linearly independent in  $X(T)$ , the map  $T \rightarrow \mathbf{G}_m^\Delta$  defined by  $t \mapsto (a(t))_{a \in \Delta}$  is surjective. Thus, the group of inner automorphisms by  $T(k)$  acts transitively on the set of possible pinning. Hence, the Isomorphism Theorem guarantees that if the based root data are isomorphic then there exists an isomorphism between the pinned data. We may therefore pass to the case of automorphisms.

Since the  $T(k)$ -action is trivial on the root datum, it follows from the Isomorphism Theorem that any automorphism of the based root datum arises from an automorphism of  $(G, T, B, \{X_a\}_{a \in \Delta})$ ; i.e., surjectivity is proved. It remains to prove injectivity, so it suffices to show that if  $f$  is an automorphism of  $(G, T, B)$  that is the identity on  $T$  and on the simple positive root groups then  $f$  is the identity on  $G$ .

Corollary 1.5.3 implies that  $f$  must be conjugation against some  $t \in T(k)$ , but the condition on the simple positive root groups implies that  $a(t) = 1$  for all  $a \in \Delta$ , so  $a(t) = 1$  for all  $a \in \Phi$ . It follows that  $t$  centralizes every root group, and hence an open cell, so  $t$  is central in  $G$ . That is, the conjugation action  $f$  by  $t$  is the identity map.  $\square$

We now discuss some representation theory. Fix a triple  $(G, T, B)$ , with  $G$  a connected reductive group over an algebraically closed field  $k$  of any characteristic. Let  $\Delta$  be the base of simple positive roots of  $\Phi^+ = \Phi(B, T)$  in  $\Phi = \Phi(G, T)$ , and  $W = W_G(T) = W(\Phi)$ . By [Bou2, VI, §1.5, Thm. 2(ii),(vi)],  $X(T)$  is covered by  $W$ -translates of the closed Weyl chamber

$$(1.5.3) \quad C = \{\lambda \in X(T) \mid \langle \lambda, a^\vee \rangle \geq 0 \text{ for all } a \in \Delta\}.$$

For an irreducible representation  $G \rightarrow \mathrm{GL}(V)$ , consider the  $W$ -stable finite set  $\Omega_V \subset X(T)$  of  $T$ -weights on  $V$ .

**Theorem 1.5.6** (Theorem of the highest weight: group version). *There is a unique weight  $\lambda_V \in \Omega_V \cap C$  that is highest in the sense that all weights in  $\Omega_V$  have the form*

$$\lambda_V - \sum_{a \in \Delta} n_a a$$

*with integers  $n_a \geq 0$ , and the  $\lambda$ -weight space is 1-dimensional and  $B$ -stable. For each  $\lambda \in X(T)$  there exists a unique irreducible representation  $V_\lambda$  of  $G$  with highest weight  $\lambda$ .*

*Proof.* See [Hum87, 31.2–31.4]. For the relationship between the constructions in characteristic 0 and positive characteristic, see [Jan, II, 2.2, 2.4, 2.6].  $\square$

Theorem 1.5.6 is not used later, but to prove the Existence Theorem over  $\mathbf{C}$  by analytic methods we need a link between the notions of “simply connected” in the sense of topology and in the sense of root data. This rests on a variant of Theorem 1.5.6 for Lie algebras. To state this result, let  $\mathfrak{g}$  be a semisimple Lie algebra over an algebraically closed field  $k$  of characteristic 0, and let  $\mathfrak{t}$  be a Cartan subalgebra. Fix a positive system of roots  $\Phi^+$  in  $\Phi(\mathfrak{g}, \mathfrak{t}) \subset \mathfrak{t}^* - \{0\}$ , and for each  $a \in \Phi$  define an “infinitesimal coroot”  $h_a \in \mathfrak{t}$  using  $\mathfrak{sl}_2$  similarly to how coroots are defined in reductive  $k$ -groups using  $\mathrm{SL}_2$ . (See [Ser01, VI, 3.1(iii)] for another formulation.) A linear form  $\lambda \in \mathfrak{t}^*$  is *integral* if  $\lambda(h_a) \in \mathbf{Z}$  for all  $a \in \Phi$ , and *dominant integral* if  $\lambda(h_a) \in \mathbf{Z}_{\geq 0}$  for all  $a \in \Phi^+$  (equivalently, for all simple positive  $a$ ).

**Theorem 1.5.7** (Theorem of the highest weight: Lie algebra version). *For each dominant integral  $\lambda \in \mathfrak{t}^*$ , there exists an irreducible finite-dimensional  $\mathfrak{g}$ -representation whose highest weight for  $\mathfrak{t}$  is  $\lambda$  in the sense that its  $\mathfrak{t}$ -weights have the form  $\lambda - \sum_{a \in \Delta} n_a a$  with integers  $n_a \geq 0$ . It is unique up to isomorphism if  $\text{char}(k) = 0$ , and in general such a representation exists with a 1-dimensional  $\lambda$ -weight space.*

*Proof.* See [Hum72, 20.2, 21.1–21.2] (and [Hum98]). □

The existence aspect of Theorem 1.5.7 is the key construction in the analytic proof of the Existence Theorem for connected reductive groups over  $\mathbf{C}$ . In characteristic 0, the character of the highest-weight representation  $V_\lambda$  of  $G$  in Theorem 1.5.6 is determined by its restriction to the regular semisimple locus of  $G(k)$ , a dense open locus introduced in Exercise 1.6.8. Since the character is conjugation-invariant, it is even determined by its restriction to the dense open set of elements of  $T(k)$  that are regular in  $G(k)$ . This restriction is given by:

**Theorem 1.5.8** (Weyl character formula). *Let  $G$  be a nontrivial connected semisimple group over an algebraically closed field  $k$  of characteristic 0,  $T$  a maximal torus, and  $B$  a Borel subgroup containing  $T$ . Assume that  $G$  is simply connected. Let  $W = W_G(T)$ , and let  $\varepsilon : W \rightarrow \{1, -1\}$  be the natural quadratic character carrying each simple reflection to  $-1$ .*

*For each  $\lambda$  in the Weyl chamber of  $X(T)$  corresponding to  $B$  (see (1.5.3)) and for each regular semisimple  $t \in T(k)$ , the character of  $t$  on  $V_\lambda$  is given by*

$$\text{Tr}(t|V_\lambda) = \frac{\sum_{w \in W} \varepsilon(w) t^{w(\lambda + \rho)}}{\sum_{w \in W} \varepsilon(w) t^{w\rho}},$$

where  $t^a := a(t)$  for  $a \in X(T)$  and  $\rho := (1/2) \sum_{a \in \Phi^+} a \in X(T)$ .

*Proof.* Using the weight space decomposition  $V_\lambda = \bigoplus_{\mu \in X(T)} V_\lambda(\mu)$ , the character  $\chi_\lambda(t) = \text{Tr}(t|V_\lambda)$  is equal to  $\sum_{\mu} \dim V_\lambda(\mu) t^\mu$ . Hence, the assertion of the formula is the identity

$$\left( \sum_{\mu \in X(T)} \dim V_\lambda(\mu) t^\mu \right) \left( \sum_{w \in W} \varepsilon(w) t^{w\rho} \right) = \sum_{w \in W} \varepsilon(w) t^{w(\lambda + \rho)}$$

in the group ring  $\mathbf{Z}[X(T)]$ . The  $T$ -weight spaces for a finite-dimensional  $G$ -representation  $V$  are the same as the  $\mathfrak{t}$ -weight spaces for  $V$  viewed as a  $\mathfrak{g}$ -module since  $G$  is connected and  $\text{char}(k) = 0$ . Thus, by embedding  $X(T) = \text{Hom}(T, \mathbf{G}_m)$  into  $\mathfrak{t}^*$  in the natural way, it is equivalent to prove the analogue of the Weyl character formula for  $V_\lambda$  viewed as a  $\mathfrak{g}$ -module (by identifying  $\Phi(G, T)$  with  $\Phi(\mathfrak{g}, \mathfrak{t})$ ). For a proof of the Lie algebra version, see [Bou3, VIII, 9.1, Thm. 1]. (The group version can be reduced to the case  $k = \mathbf{C}$ , in which case it can be proved using integration on a compact form of the group; see [FH, 26.2].) □

**Remark 1.5.9.** The regularity condition on  $t$  is needed in the Weyl character formula to ensure that the denominator is nonzero; see Example 1.5.10. Also, the integrality of  $\rho$  can be made explicit: it is the sum of the elements in  $X(T)$  dual to the  $\mathbf{Z}$ -basis  $\Delta^\vee$  of  $X_*(T)$

(a  $\mathbf{Z}$ -basis due to the simply connected condition on  $G$ ; see Exercise 1.6.12(ii)). This is the “dual” of the situation in Exercise 4.4.8(iii) (which treats adjoint semisimple groups).

**Example 1.5.10.** Consider  $G = \mathrm{SL}_2$  over an algebraically closed field  $k$  of characteristic 0, and let  $T$  be the diagonal torus. Identify  $\mathbf{G}_m$  with  $T$  via  $c \mapsto \mathrm{diag}(c, 1/c)$ , so  $X(T) = \mathbf{Z}$ . Take  $B$  to be the upper triangular Borel subgroup, so  $\Phi(B, T) = \{2\}$ . The Weyl chamber  $C$  is  $\mathbf{Z}_{\geq 0}$  and  $\rho = 2/2 = 1$ .

The irreducible representation of  $G$  with highest weight  $n \geq 0$  is the symmetric power  $V_n = \mathrm{Sym}^n(k^2)$ . A regular semisimple element  $t \in T(k)$  is any  $t \in k^\times$  such that  $t^2 \neq 1$ . Clearly

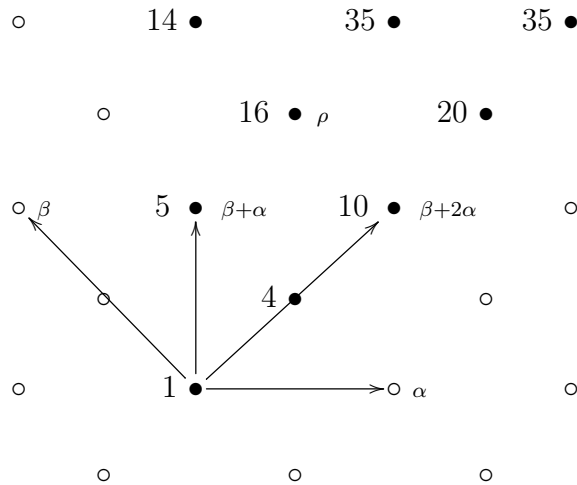
$$\mathrm{Tr}(t|V_n) = t^n + t^{n-2} + \dots + t^{-n} = \frac{t^{n+1} - t^{-(n+1)}}{t - t^{-1}},$$

and the right side is exactly the Weyl character formula in this case (likewise illustrating the need for the regularity condition on  $t$  in order that the denominator be nonzero).

If  $\mathrm{char}(k) = p > 0$  then these symmetric powers can be non-semisimple; e.g.,  $\mathrm{Sym}^p(k^2)$  is not semisimple (Exercise 1.6.10(iii)) but it has a unique irreducible quotient with highest weight  $p$ . In general, the dimensions of highest weight representations are not known when  $\mathrm{char}(k) > 0$ , nor is there a formula for the character at regular semisimple points of  $T(k)$ .

**Example 1.5.11.** Consider  $G = \mathrm{Sp}_4$  over an algebraically closed field of characteristic 0. The Dynkin diagram is  $\circ \xleftarrow{\alpha} \circ_{\beta}$ . The weight lattice  $X(T)$  is represented by dots in the

diagram below, and the arrows represent positive roots. Note that  $X(T)$  contains  $\mathbf{Z}\alpha + \mathbf{Z}\beta$  as a subgroup of index 2 (so half the sum of the positive roots,  $\rho = (3/2)(\alpha + \beta)$ , lies in  $X(T)$  as it should). The weights in the Weyl chamber (1.5.3) are denoted by black dots and labelled with the dimension of the corresponding irreducible representation.



We can describe some irreducible representations of small dimension in more familiar terms:

- The 4-dimensional representation  $V_4$  is the standard representation of  $G = \mathrm{Sp}_4$ .

- The 5-dimensional representation  $V_5$  is  $\wedge^2 V_4 / L$ , where  $L$  is the line in  $\wedge^2 V_4$  fixed by  $G$  (i.e., the line spanned by the symplectic form). This can be regarded as the standard representation of  $\mathrm{SO}_5 \simeq \mathrm{Sp}_4 / \mu_2$  (see Example C.6.5).
- The 10-dimensional representation is  $\mathrm{Sym}^2(V_4)$ .
- The 14-dimensional representation is  $\mathrm{Sym}^2(V_5) / L'$ , where  $L'$  is the line  $\mathrm{Sym}^2(V_5)^{\mathrm{SO}_5}$ .
- The 20-dimensional representation is  $\mathrm{Sym}^3(V_4)$ .

**1.6. Exercises.** The first seven exercises below require no specific background in the classical theory beyond basic definitions that we assume are familiar to the reader. However, some of those exercises are referenced in §1 to clarify features of the classical theory.

**Exercise 1.6.1.** Let  $G \subset \mathrm{GL}_3$  be the non-reductive connected solvable subgroup

$$\begin{pmatrix} t & x & z \\ 0 & 1 & y \\ 0 & 0 & t^{-1} \end{pmatrix} \subset \mathrm{GL}_3$$

in which a maximal torus is given by  $T = \mathbf{G}_m$  via  $t \mapsto \mathrm{diag}(t, 1, 1/t)$ . Consider the weight space decomposition for  $\mathfrak{g}$  under the adjoint action of  $T$ .

Show that the nontrivial character  $\chi(t) = t$  occurs with multiplicity 2, and that its inverse  $\chi(t)^{-1} = 1/t$  occurs with multiplicity 0. (This contrasts with two important features of the reductive case, namely that root spaces are 1-dimensional and that the set of roots is stable under negation in the character lattice.)

**Exercise 1.6.2.** (i) Prove that the only automorphisms of  $\mathbf{G}_a$  over a field  $F$  are the usual  $F^\times$ -scalings, and conclude that over a reduced ring  $R$  the only automorphisms of  $\mathbf{G}_a$  over  $R$  are the usual  $R^\times$ -scalings. (Reduce to the case of noetherian  $R$ .)

(ii) Prove  $\mathbf{G}_m$  represents the automorphism functor of  $\mathbf{G}_a$  on the category of  $\mathbf{Q}$ -algebras. (Reduce to the noetherian case, then induct on the nilpotence order of the nilradical.)

(iii) Let  $k$  be a field with  $\mathrm{char}(k) = p > 0$ . Prove that  $\mathbf{G}_m$  does not represent the automorphism functor of  $\mathbf{G}_a$  on  $k$ -algebras by giving an example of an automorphism of  $\mathbf{G}_a$  over the dual numbers  $k[\epsilon]$  that does *not* arise from the usual  $\mathbf{G}_m$ -action on  $\mathbf{G}_a$ .

(iv) Despite the problems in (iii), prove that an action by a torus  $T$  on  $\mathbf{G}_a$  over a field  $k$  of any characteristic is given by  $t.x = \chi(t)x$  for a homomorphism of  $k$ -groups  $\chi : T \rightarrow \mathbf{G}_m$ . (Hint: use (i) to get a homomorphism  $\chi : T(k_s) \rightarrow k_s^\times$  and then work over the function field  $K = k(T)$  of  $T$  to prove that  $\chi$  is “algebraic” over  $k_s$  and defined over  $k$ . The algebraicity is the main point.)

**Exercise 1.6.3.** Let  $\mathbf{Z}[x_{ij}]_{(\det)}$  be the degree-0 part of  $\mathbf{Z}[x_{ij}][1/\det]$  ( $f/\det^e$  with  $f$  homogeneous,  $\deg f = e \deg(\det) = en$ ). Define  $\mathrm{PGL}_n = \mathrm{Spec}(\mathbf{Z}[x_{ij}]_{(\det)}) = \{\det \neq 0\} \subset \mathbf{P}_{\mathbf{Z}}^{n^2-1}$ .

(i) Construct an injective map  $\mathrm{GL}_n(R)/R^\times \rightarrow \mathrm{PGL}_n(R)$  natural in rings  $R$ .

(ii) Prove  $\mathrm{PGL}_n$  is the Zariski-sheafification of  $S \rightsquigarrow \mathrm{GL}_n(S)/\mathrm{GL}_1(S)$  on the category of schemes, and that it has a unique  $\mathbf{Z}$ -group structure making  $\mathrm{GL}_n \rightarrow \mathrm{PGL}_n$  a homomorphism.

(iii) Prove  $\mathrm{GL}_n(R)/R^\times = \mathrm{PGL}_n(R)$  for local  $R$ , and construct a *counterexample* with  $n = 2$  for any Dedekind domain  $R$  with  $\mathrm{Pic}(R)[2] \neq 1$ .

(iv) For local  $R$  show  $\mathrm{Pic}(\mathbf{P}_R^N) = \mathbf{Z}$  (generated by  $\mathcal{O}(1)$ ) by using deformation from the residue field and the theorem on formal functions (after reducing to the case of noetherian  $R$ ). Deduce that the evident action of  $\mathrm{PGL}_n$  on  $\mathbf{P}_{\mathbf{Z}}^{n-1}$  via (ii) makes  $\mathrm{PGL}_n$  represent the automorphism functor  $S \rightsquigarrow \mathrm{Aut}_S(\mathbf{P}_S^{n-1})$  on the category of schemes. (For example, the  $\mathbf{Z}$ -group  $\mathrm{PGL}_2$  represents the automorphism functor of  $\mathbf{P}_{\mathbf{Z}}^1$ .) By using intersection theory on  $\mathbf{P}^1 \times \mathbf{P}^1$  over a field, adapt the deformation arguments to prove that  $(\mathrm{PGL}_2 \times \mathrm{PGL}_2) \rtimes (\mathbf{Z}/2\mathbf{Z})$

represents the automorphism functor of  $\mathbf{P}^1 \times \mathbf{P}^1$  in the evident manner. (Hint: prove that  $\text{Pic}_{(\mathbf{P}^1 \times \mathbf{P}^1)/\mathbf{Z}}$  is the constant  $\mathbf{Z}$ -group associated to  $\mathbf{Z} \oplus \mathbf{Z}$ .)

(v) Give a pre-Grothendieck proof (no functors, non-reduced schemes, or cohomology) that if a linear algebraic group  $G$  over a field  $k$  acts on  $\mathbf{P}_k^{n-1}$  then the resulting homomorphism  $G(k_s) \rightarrow \text{Aut}_{k_s}(\mathbf{P}_{k_s}^{n-1}) = \text{PGL}_n(k_s)$  arises from a  $k$ -homomorphism  $G \rightarrow \text{PGL}_n$ .

**Exercise 1.6.4.** Construct natural isomorphisms between  $\mathfrak{pgl}_n := \text{Lie}(\text{PGL}_n)$  and  $\mathfrak{gl}_n/\mathfrak{gl}_1$ , as well as between  $\mathfrak{sl}_n := \text{Lie}(\text{SL}_n)$  with  $\mathfrak{gl}_n^{\text{Tr}=0}$  (kernel of the trace) as Lie algebras over  $\mathbf{Z}$ . Then do the following:

(i) Construct a  $\text{GL}_n$ -equivariant duality between  $\mathfrak{sl}_n$  and  $\mathfrak{pgl}_n$  over  $\mathbf{Z}$ .

(ii) Over any field  $k$  of characteristic  $p > 0$ , prove that  $\mathfrak{sl}_p$  and  $\mathfrak{pgl}_p$  are not isomorphic as representation spaces for the diagonal torus of  $\text{SL}_p$ . In particular, neither is self-dual as an  $\text{SL}_p$ -representation space.

(iii) In the setup of (ii), prove that the central line  $\text{Lie}(\mu_p) \subset \mathfrak{sl}_p$  consisting of scalar diagonal matrices does not admit an  $\text{SL}_p$ -equivariant complement.

(iv) Note that the conclusion of (ii) holds with  $\mathfrak{sl}_p$  and  $\mathfrak{pgl}_p$  respectively replaced by  $\mathfrak{sl}_{p^e}$  and  $\mathfrak{pgl}_{p^e}$ . Verify that for  $0 < f < e$ , the central line in  $\text{Lie}(\text{SL}_{p^e}/\mu_{p^f})$  does have an  $\text{SL}_{p^e}$ -equivariant complement, namely the image of  $\mathfrak{sl}_{p^e}$ .

**Exercise 1.6.5.** Let  $G$  and  $G'$  be smooth connected groups over a field  $k$ . Let  $f : \Omega \rightarrow G'$  be a  $k$ -morphism defined on a dense open subset  $\Omega \subset G$ . Assume that  $f$  is a rational homomorphism in the sense that  $f(\omega\omega')f(\omega')^{-1}f(\omega)^{-1} : V \rightarrow G'$  is identically 1 for a dense open subset  $V \subset \Omega \times \Omega$  such that  $\omega\omega' \in \Omega$  for all  $(\omega, \omega') \in V$ . Use pre-Grothendieck arguments (i.e., no descent theory or scheme-theoretic methods) to prove that  $f$  uniquely extends to a  $k$ -homomorphism  $\tilde{f} : G \rightarrow G'$  and that if  $f$  is birational then  $\tilde{f}$  is an isomorphism.

**Exercise 1.6.6.** Let a smooth finite type  $k$ -group  $G$  act linearly on a finite-dimensional  $k$ -vector space  $V$ . Let  $\underline{V}$  denote the affine space over  $k$  whose  $A$ -points are  $V_A := V \otimes_k A$  for any  $k$ -algebra  $A$ . Define  $\underline{V}^G(A)$  to be the set of  $v \in V_A$  on which the  $A$ -group  $G_A$  acts trivially.

(i) Prove  $\underline{V}^G$  is represented by the closed subscheme associated to a  $k$ -subspace of  $V$  (denoted  $V^G$ ). Hint: use Galois descent to reduce to the case  $k = k_s$ ; prove  $V^{G(k)}$  works.

(ii) For an extension field  $K/k$ , prove  $(V_K)^{G_K} = (V^G)_K$  inside  $V_K$ .

**Exercise 1.6.7.** Let  $G$  be a connected reductive group over an algebraically closed field  $k$  of characteristic  $p > 0$ , and let  $T \subset G$  be a maximal  $k$ -torus. For any affine  $k$ -scheme  $X$  of finite type, let  $X^{(p^n)}$  denote the scalar extension by the  $p^n$ -power endomorphism of  $k$ , and define  $F_{X/k,n} : X \rightarrow X^{(p^n)}$  to be the natural  $k$ -morphism induced by the absolute  $p^n$ -Frobenius  $F_X^n : X \rightarrow X$  over the  $p^n$ -Frobenius  $F_k^n : \text{Spec } k \rightarrow \text{Spec } k$ .

(i) Prove that  $F_{X/k,n}$  is functorial in  $X$ , compatible with direct products in  $X$  (over  $k$ ), and compatible with extension of the ground field. Deduce that if  $X$  is a  $k$ -group and  $X^{(p^n)}$  is made into a  $k$ -group via scalar extension then  $F_{X/k,n}$  is a  $k$ -homomorphism.

(ii) If  $X$  is smooth of pure dimension  $d > 0$  then prove that  $F_{X/k,n}$  is finite flat of degree  $p^{dn}$ . In particular,  $F_{G/k,n} : G \rightarrow G^{(p^n)}$  is an isogeny of degree  $p^{dn}$  carrying  $T$  onto  $T^{(p^n)}$ .

(iii) Compute  $F_{G/k,n}$  for  $\mathrm{GL}(V)$  and  $\mathrm{SO}(q)$ , and prove  $\mathrm{Lie}(F_{G/k}) = 0$  for any  $G$ . In general, compute the effect on root data arising from  $F_{G/k,n} : (G, T) \rightarrow (G^{(p^n)}, T^{(p^n)})$ .

**Exercise 1.6.8.** Let  $G$  be a connected linear algebraic group over an algebraically closed field  $k$ . By Remark 1.1.19, the set of semisimple elements of  $G(k)$  is the union  $\bigcup_T T(k)$  as  $T$  varies through the maximal tori of  $G$ .

(i) Prove that for semisimple  $g \in G(k)$ ,  $Z_G(g)^0$  contains some Cartan subgroup  $Z_G(T)$  of  $G$  (these are the maximal tori when  $G$  is reductive). Deduce that  $\dim Z_G(g)$  coincides with the common dimension of the Cartan subgroups if and only if  $Z_G(g)^0$  is a Cartan subgroup.

(ii) Prove that there exist semisimple  $g \in G(k)$  such that  $Z_G(g)^0$  is a Cartan subgroup. (Hint: For a maximal torus  $T$ , consider the *finitely many* nontrivial  $T$ -weights that occur on  $\mathfrak{g}$ .) An element  $g \in G(k)$  is *regular* when  $Z_G(g_{\mathrm{ss}})^0$  is a Cartan subgroup. Using that  $g \in Z_G(g_{\mathrm{ss}})^0$  [Bo91, 11.12], deduce that  $g$  is regular if and only if it belongs to a unique Cartan subgroup. (For  $G = \mathrm{GL}(V)$ , these are precisely the  $g \in G(k)$  whose characteristic polynomial has non-zero discriminant, since  $g$  and  $g_{\mathrm{ss}}$  have the same characteristic polynomial, so this is a Zariski-dense open locus in  $\mathrm{GL}(V)$ .)

(iii) For semisimple  $g \in G(k)$  prove that  $\mathrm{Lie}(Z_G(g)) = \mathfrak{g}^{g=1}$ , and by considering the multiplicity of  $x - 1$  as a factor of the characteristic polynomial for the adjoint action of  $G$  on  $\mathfrak{g}$  prove that the regular locus of  $G(k)$  is a (non-empty) Zariski-open subset. (Hint: for any  $g \in G(k)$ ,  $\mathrm{Ad}_G(g)$  and  $\mathrm{Ad}_G(g_{\mathrm{ss}})$  have the same characteristic polynomial on  $\mathfrak{g}$ .)

(iv) Steinberg proved that all regular elements in  $G(k)$  are semisimple when  $G$  is reductive. Without using this fact, prove that for reductive  $G$  the non-empty locus of regular semisimple elements in  $G(k)$  is Zariski-open. (This is false if  $G$  is nontrivial and unipotent!) In other words, within the dense Zariski-open locus of regular elements, prove that the non-empty semisimple locus is open. (Hint: consider  $\dim Z_G(g)$  rather than  $\dim Z_G(g_{\mathrm{ss}})$ , and apply semicontinuity of fiber dimension to a “universal centralizer scheme” over  $G$ .)

**Exercise 1.6.9.** Let  $k$  be a field. To define special orthogonal groups over rings in a uniform way, we need a characteristic-free definition of non-degeneracy for a quadratic form  $q : V \rightarrow k$  on a finite-dimensional  $k$ -vector space  $V$  of dimension  $d \geq 2$ . We say  $q$  is *non-degenerate* when  $q \neq 0$  and  $(q = 0)$  is smooth in  $\mathbf{P}(V^*) := \mathrm{Proj}(\mathrm{Sym}(V^*)) \simeq \mathbf{P}_k^{d-1}$ .

(i) Let  $B_q : V \times V \rightarrow k$  be the corresponding symmetric bilinear form  $(v, v') \mapsto q(v + v') - q(v) - q(v')$ , and define  $V^\perp = \{v \in V \mid B_q(v, \cdot) = 0\}$ ; we call  $\delta_q := \dim V^\perp$  the *defect* of  $q$ . Prove that  $B_q$  uniquely factors through a non-degenerate symmetric bilinear form on  $V/V^\perp$ , and that  $B_q$  is non-degenerate precisely when the defect is 0. Also show that if  $\mathrm{char}(k) = 2$  then  $B_q$  is alternating, and deduce that  $\delta_q \equiv \dim V \pmod{2}$  for such  $k$  (so  $\delta_q \geq 1$  if  $\dim V$  is odd).

(ii) Prove that if  $\delta_q = 0$  then  $q_{\bar{k}}$  admits one of the following “standard forms”:  $\sum_{i=1}^n x_i x_{i+n}$  if  $\dim V = 2n$  ( $n \geq 1$ ), and  $x_0^2 + \sum_{i=1}^n x_i x_{i+n}$  if  $\dim V = 2n + 1$  ( $n \geq 1$ ). Do the same if  $\mathrm{char}(k) = 2$  and  $\delta_q = 1$ . (Distinguish whether or not  $q|_{V^\perp} \neq 0$ .) How about the converse?

(iii) If  $\text{char}(k) \neq 2$ , prove that  $q$  is non-degenerate if and only if  $\delta_q = 0$ . For  $\text{char}(k) = 2$ , prove that  $q$  is non-degenerate if and only if  $\delta_q \leq 1$  with  $q|_{V^\perp} \neq 0$  when  $\delta_q = 1$ . (Hint: use (ii) to simplify calculations.) This is the “classical” notion of non-degeneracy for  $q$ .

**Exercise 1.6.10.** Let  $G$  be a smooth affine group over a field  $k = \bar{k}$  (allow  $G \neq G^0$ ).

(i) If all finite-dimensional linear representations of  $G$  are completely reducible, or if there is even a single faithful semi-simple representation of  $G$ , then prove that  $G^0$  is reductive. (Hint: use Lie–Kolchin, and behavior of semisimplicity under restriction to a normal subgroup.) Deduce that  $\text{GL}(V)$  is reductive and  $\text{SL}(V)$  is semisimple.

(ii) Conversely, assume that  $G^0$  is reductive and  $\text{char}(k) = 0$ . Prove that all finite-dimensional linear representations of  $G$  are completely reducible. (Hint: prove  $\text{Lie}(\mathcal{Z}(G))$  is semisimple.)

(iii) Let  $V$  be the standard 2-dimensional representation of  $\text{SL}_2$  over  $k$ , and assume  $\text{char}(k) = p > 0$ . In  $\text{Sym}^p(V)$ , prove that the line of  $p$ th powers has no  $\text{SL}_2$ -equivariant complement (so  $\text{Sym}^p(V)$  is not semisimple as a representation, and hence  $V^{\otimes p}$  is not semisimple).

**Exercise 1.6.11.** Let  $G$  be a linear algebraic group over an algebraically closed field  $k$ ,  $N$  a normal linear algebraic subgroup (e.g.,  $N = \mathcal{Z}(G)$ ), and  $T$  a maximal torus in  $G$ . Prove that  $(T \cap N)_{\text{red}}^0$  is a maximal torus in  $N$ . (See Example 2.2.6 and Exercise 5.4.1 for smoothness and connectedness properties of  $T \cap N$ .)

**Exercise 1.6.12.** Let  $G$  be a connected semisimple group over an algebraically closed field  $k$ . The *central isogeny class* of  $G$  is the equivalence class of  $G$  generated by the relation on connected semisimple  $k$ -groups  $G'$  and  $G''$  that there exists a central isogeny  $G' \rightarrow G''$  or  $G'' \rightarrow G'$  (i.e., an isogeny whose scheme-theoretic kernel is central). Composites of central isogenies between connected reductive groups are central (due to Corollary 3.3.5 applied over  $k$ , where the proof simplifies); this is false for general smooth connected affine groups (Exercise 3.4.3(ii)).

(i) Prove that any central isogeny  $f : G' \rightarrow G''$  can be arranged to satisfy  $f(T') = T''$  and  $f(B') = B''$  for choices of Borel subgroups and maximal tori that they contain, and use the open cells to show that such an  $f$  must induce isomorphisms between root groups (false for Frobenius isogenies!). Deduce via Corollary 1.4.11, the Isomorphism Theorem, and (1.3.2) that if  $G' \simeq G''$  abstractly then  $f$  must be an isomorphism and that if  $G' \not\simeq G''$  then up to conjugacy  $f$  is the only central isogeny between  $G'$  and  $G''$  (in either direction!). In particular, a central isogeny class is *partially ordered* (when members are considered up to abstract  $k$ -isomorphism).

(ii) Using the Existence and Isomorphism Theorems over  $k$ , prove the equivalence of the following conditions on  $G$ : the only central isogenies  $G' \rightarrow G$  from connected semisimple groups are isomorphisms,  $G$  dominates all other members of its central isogeny class, and the simple positive coroots of  $(G, T)$  (relative to a choice of  $\Phi^+$ ) are a  $\mathbf{Z}$ -basis of the cocharacter group of  $T$ . Under these conditions, we say  $G$  is *simply connected* (e.g.,  $\text{Sp}_{2n}$  and  $\text{SL}_n$ ).

Likewise prove the equivalence of:  $\text{Ad}_G$  is a closed immersion,  $G$  is dominated by all members of its central isogeny class, and the simple positive roots of  $(G, T)$  (relative to a

choice of  $\Phi^+$ ) are a  $\mathbf{Z}$ -basis of the character group of  $T$ . Under these conditions we say that  $G$  is *adjoint* (e.g.,  $\mathrm{PGL}_n$  and  $\mathrm{SO}_{2n+1}$ ).

(iii) Let  $T \subset G$  be a maximal torus. For each  $a \in \Phi(G, T)$ , let  $T_a = (\ker a)_{\mathrm{red}}^0$  be the unique codimension-1 torus in  $T$  killed by  $a$ . By the structure theory underlying the *definition* of coroots,  $G_a := \mathcal{D}(Z_G(T_a))$  is either  $\mathrm{SL}_2$  or  $\mathrm{PGL}_2$  with maximal torus  $a^\vee(\mathbf{G}_m)$  having root groups  $U_a$  and  $U_{-a}$ .

If  $G$  is simply connected, prove that  $G_a = \mathrm{SL}_2$  for all  $a \in \Phi(G, T)$ . Show that the converse is false by proving that  $G_a = \mathrm{SL}_2$  for all  $a \in \Phi(G, T)$  when  $G = \mathrm{PGL}_n$  with  $n \geq 3$ .

**Exercise 1.6.13.** Let  $U$  be a nonzero finite-dimensional vector space over a field  $k$ , and  $U^*$  its dual. Define  $W = U \oplus U^*$ .

(i) Define  $\psi_0 : W \times W \rightarrow k$  by  $\psi_0((v, f), (v', f')) = f'(v) - f(v')$ . Show that  $(W, \psi_0)$  is a non-degenerate symplectic space. Let  $\mathrm{GL}(U)$  acts on  $W$  by  $g.(v, f) = (g.v, f \circ g^{-1})$ . Show that this defines a monic homomorphism (hence closed immersion) of  $k$ -groups  $\mathrm{GL}(U) \rightarrow \mathrm{GL}(W)$  and that the image lies in  $\mathrm{Sp}(W, \psi_0)$ .

(ii) Define a quadratic form  $q_0$  on  $W$  by  $q_0(v, f) = f(v)$ . Show that  $q_0$  is non-degenerate and that the image of  $\mathrm{GL}(U) \rightarrow \mathrm{GL}(W)$  also lies in  $\mathrm{SO}(W, q_0)$ .

**Exercise 1.6.14.** Let  $U = k^2$  for an algebraically closed field  $k$  and define  $W, \psi_0, q_0$  as in the previous exercise. Thus, we have inclusions  $\mathrm{SL}_2 \subset \mathrm{GL}_2 \subset \mathrm{Sp}(W, \psi_0)$  and  $\mathrm{SL}_2 \subset \mathrm{SO}(W, q_0)$ . In this exercise, you can take for granted that the groups  $\mathrm{SO}(V, q)$  and  $\mathrm{Sp}(V, \psi)$  are connected reductive groups. For such groups, we shall interpret the maps  $\varphi_a$  from Theorem 1.2.5.

(i) Let  $(V, q)$  be a non-degenerate quadratic space of dimension  $2n$  over  $k$ ,  $n \geq 2$ . Construct a maximal torus  $T$  of  $G = \mathrm{SO}(V, q)$  and compute  $\Phi(G, T)$ . For each  $a \in \Phi(G, T)$ , show that  $\varphi_a : \mathrm{SL}_2 \rightarrow G$  arises from the following construction: take an embedding  $i : (W, q_0) \hookrightarrow (V, q)$  of quadratic spaces. This gives as an embedding  $\mathrm{SO}(W, q_0) \hookrightarrow \mathrm{SO}(V, q)$  in an obvious manner and hence we get  $\mathrm{SL}_2 \hookrightarrow \mathrm{SO}(W, q_0) \hookrightarrow \mathrm{SO}(V, q)$ .

(ii) Let  $(V, q)$  be a non-degenerate quadratic space of dimension  $2n + 1$  over  $k$ ,  $n \geq 1$ . Construct a maximal torus  $T$  of  $G = \mathrm{SO}(V, q)$  and compute  $\Phi(G, T)$ . For each  $a \in \Phi(G, T)$ , show that  $\varphi_a : \mathrm{SL}_2 \rightarrow G$  arises from one of the following two constructions: (a) take an embedding  $i : (W, q_0) \hookrightarrow (V, q)$  of quadratic spaces and form  $\mathrm{SL}_2 \hookrightarrow \mathrm{SO}(W, q_0) \hookrightarrow \mathrm{SO}(V, q)$  as in (i); (b) take a 3-dimensional subspace  $V_3$  of  $V$  such that  $(V_3, q)$  is non-degenerate. Then we have  $\mathrm{SO}(V_3, q) \simeq \mathrm{PGL}_2$  (Example B.2.2) and hence a homomorphism  $\mathrm{SL}_2 \rightarrow \mathrm{SO}(V_3, q) \hookrightarrow \mathrm{SO}(V, q)$ . Case (a) is for the long roots, and (b) is for the short roots.

(iii) Let  $(V, \psi)$  be a non-degenerate symplectic space of dimension  $2n$  over  $k$ . Construct a maximal torus  $T$  of  $G = \mathrm{Sp}(V, \psi)$  and compute  $\Phi(G, T)$ . For each  $a \in \Phi(G, T)$ , show that  $\varphi_a : \mathrm{SL}_2 \rightarrow G$  arises from one of the following two constructions: (a) take an embedding  $i : (W, \psi_0) \hookrightarrow (V, \psi)$  and form  $\mathrm{SL}_2 \hookrightarrow \mathrm{Sp}(W, \psi_0) \hookrightarrow \mathrm{Sp}(V, \psi)$ ; (b) take a 2-dimensional subspace  $V_2$  of  $V$  such that  $(V_2, \psi)$  is non-degenerate. Then we have  $\mathrm{SL}_2 = \mathrm{Sp}(V_2, \psi) \hookrightarrow \mathrm{Sp}(V, \psi)$ . (Determine which of (a) or (b) corresponds to the long roots, and which corresponds to the short roots.)

**Exercise 1.6.15.** (i) Let  $G$  be a connected linear algebraic group over an algebraically closed field  $k$ , and let  $T$  be a torus of  $G$ . Let  $\mathfrak{g} = \bigoplus_{a \in X(T)} \mathfrak{g}_a$  be the weight decomposition of  $\mathfrak{g} = \text{Lie } G$  for the action of  $T$ . Let  $\Phi = \{a \in X(T) - \{0\} \mid \mathfrak{g}_a \neq 0\}$ . Assume:

- $\mathfrak{g}_0 = \text{Lie } T$  (so  $T$  is maximal as a torus of  $G$ ) and  $\dim \mathfrak{g}_a \leq 1$  for all  $a \in \Phi$ ;
- $\Phi$  is stable under negation in  $X(T)$  and if  $a, b \in \Phi$  are  $\mathbf{Z}$ -linearly dependent then  $b = \pm a$ ;
- for all  $a \in \Phi$  there exists a homomorphism  $\varphi_a : \text{SL}_2 \rightarrow G$  with finite kernel such that the image commutes with the codimension-1 torus  $T_a = (\ker a)_{\text{red}}^0 \subset T$ .

Show that  $G$  is a reductive group. (Hint: prove  $\text{Lie}(\mathcal{R}_u(G_{\bar{k}})) = 0$ . Also see Lemma 3.1.10.)

(ii) With the notation of Exercise 1.6.14, show that  $\text{SO}(V, q)$  and  $\text{Sp}(V, \psi)$  are reductive (granting that each is connected and smooth).

**Exercise 1.6.16.** Let  $\Delta$  be the base of a positive system of roots  $\Phi^+$  in a reduced root system  $(V, \Phi)$ . Consider the reduced dual root system  $(V^*, \Phi^\vee)$  arising from the coroots (where  $a^\vee$  is the linear form that computes the unique reflection  $s_a : V \simeq V$  preserving  $\Phi$  and negating  $a$ ; i.e.,  $s_a(v) = v - a^\vee(v)a$ ).

(i) Assume  $(V, \Phi)$  is irreducible and fix a  $W(\Phi)$ -invariant positive-definite quadratic form  $Q : V \rightarrow \mathbf{Q}$  on  $V$  (unique up to scaling, by Remark 1.3.14). Prove that under the resulting isomorphism  $V^* \simeq V$ ,  $a^\vee$  is identified with  $2a/Q(a)$ .

(ii) Let  $\Delta^\vee$  be the set of coroots  $a^\vee$  for  $a \in \Delta$ . Using (i), prove that  $\Delta^\vee$  is a basis of  $V^*$  and that every element of  $\Phi^\vee$  is a linear combination of  $\Delta^\vee$  with all coefficients in  $\mathbf{Q}_{\geq 0}$  or in  $\mathbf{Q}_{\leq 0}$ . Use the equivalent characterizations of bases of root systems in [SGA3, XXI, 3.1.5] to deduce that  $\Delta^\vee$  is the base of a positive system of roots in  $\Phi^\vee$ .

(iii) Assume  $(V, \Phi)$  is irreducible and chose a nonzero  $\lambda \in V^*$  such that  $\Phi_{\lambda > 0} = \Phi^+$  (so  $\langle \lambda, a \rangle \neq 0$  for all  $a \in \Phi$ ). For  $Q$  as in (i) and  $v \in V$  satisfying  $B_Q(v, \cdot) = \lambda$ , prove that  $v$  is not orthogonal to any coroot and that  $\Delta^\vee$  is a base for the positive system of roots  $\Phi_{v > 0}^\vee$ .

(iv) Using (iii), prove that if  $(V, \Phi)$  is irreducible then the Dynkin diagrams for  $\Phi$  and  $\Phi^\vee$  coincide in the simply laced case (i.e., one root length) and otherwise are related by swapping the direction of the unique multiple edge. (Hence, by the classification of irreducible reduced root systems, the diagrams coincide except that  $B_n$  and  $C_n$  are swapped for  $n \geq 3$ .)

## 2. NORMALIZERS, CENTRALIZERS, AND QUOTIENTS

Here is an example that motivates the need for a theory of reductive groups over rings. Let  $G$  be a connected reductive group over a number field  $K$  with ring of integers  $R$ , and choose a faithful representation  $G \hookrightarrow \mathrm{GL}_n$  over  $K$ . The schematic closure of  $G$  in the  $R$ -group  $\mathrm{GL}_n$  is a flat closed  $R$ -subgroup  $\mathcal{G}$  of  $\mathrm{GL}_n$ . By direct limit considerations, if  $a \in R$  is nonzero and sufficiently divisible (i.e.,  $\mathrm{Spec} R[1/a]$  is a sufficiently “small” neighborhood of the generic point in  $\mathrm{Spec} R$ ) then  $\mathcal{G}_{R[1/a]}$  is smooth over  $\mathrm{Spec} R[1/a]$  with connected fibers. Is  $\mathcal{G}_s$  reductive for all  $s \in \mathrm{Spec} R[1/a]$ , at least at the cost of making  $a$  more divisible in  $R$ , and if so then is the isomorphism type (of the associated root datum) “independent of  $s$ ”? In such cases we wish to say that  $\mathcal{G}_{R[1/a]}$  is a reductive group scheme over  $R[1/a]$ .

Also, although  $G$  may not have a Borel  $K$ -subgroup,  $G_{K_v}$  has a Borel  $K_v$ -subgroup for all but finitely many  $v$ . This assertion only involves the theory of reductive groups over fields (of characteristic 0), but its only known proof requires using the theory of reductive groups over localized rings of integers.

A theory of reductive groups over discrete valuation rings should link the theories over  $\mathbf{F}_p$  and  $\mathbf{Q}$ , as well as relate finite groups of Lie type to complex semisimple Lie groups. We also seek a conceptual understanding of the smooth affine  $\mathbf{Z}$ -groups constructed by Chevalley in [Chev61] (with  $\mathbf{Q}$ -split semisimple generic fiber). The Bruhat–Tits structure theory for  $p$ -adic groups, the study of integral models of Shimura varieties, and Galois deformation theory valued in reductive groups provide further motivation.

The main result that we are aiming for is this: over *any* non-empty scheme  $S$ , there will be analogues of the Existence and Isomorphism Theorems for connected reductive groups over algebraically closed fields. More specifically, we will show that the category of “split” reductive  $S$ -groups (equipped with a suitable notion of isomorphism) is the same as the category of root data, which in turn has nothing to do with  $S$ !

In the classical theory, normalizers and centralizers of tori are ubiquitous tools for creating interesting subquotients of a non-solvable smooth connected affine group. In the relative theory these constructions remain essential, but we need to use torsion-levels in a torus because those are *finite* flat over the base (unlike the torus). Such finiteness makes the torsion-levels very useful in proving representability results for functorial normalizers and centralizers of tori. The method of passing to torsion-levels in tori has a useful variant when considering normalizers and centralizers of smooth closed subgroups with connected fibers: passing to infinitesimal neighborhoods of the identity (which are often finite flat closed subschemes, but generally not subgroups). The construction of quotients by normalizers of subgroups will also use the consideration of “finite flat approximations”.

Our use of torsion levels in tori forces us to consider general (possibly non-smooth) groups of multiplicative type. See Appendix B for the basics of the theory of group schemes of multiplicative type, including definitions and notation (and precise references within [Oes] for proofs of some fundamental results). The reader should learn the material in Appendix B before continuing on to the discussion that follows

**2.1. Transporter schemes and Hom schemes.** For subschemes  $Y$  and  $Y'$  of an  $S$ -group scheme  $G$ , the transporter functor on  $S$ -schemes is defined to be

$$\underline{\text{Transp}}_G(Y, Y') : S' \rightsquigarrow \{g \in G(S') \mid g(Y_{S'})g^{-1} \subseteq Y'_{S'}\}.$$

In the special case  $Y' = Y$  with  $Y$  of finite presentation over  $S$ , this coincides with the normalizer functor

$$\underline{N}_G(Y) : S' \rightsquigarrow \{g \in G(S') \mid gY_{S'}g^{-1} = Y_{S'}\}$$

because a monic endomorphism of  $Y_{S'}$  given by  $g$ -conjugation is an automorphism, as for any finitely presented scheme over any base [EGA, IV<sub>4</sub>, 17.9.6]. When these functors are representable, we denote representing objects as  $\text{Transp}_G(Y, Y')$  and  $N_G(Y)$  respectively.

**Definition 2.1.1.** A finitely presented subgroup  $G'$  in  $G$  is *normal* if  $\underline{N}_G(G') = G$ , or equivalently for all  $S$ -schemes  $S'$  the conjugation on  $G_{S'}$  by any  $g \in G(S')$  carries  $G'_{S'}$  into (and hence isomorphically onto) itself.

Another formulation of this definition is that  $G'(S')$  is a normal subgroup of  $G(S')$  for every  $S$ -scheme  $S'$ . Normality of  $G'$  in  $G$  *cannot* be checked on geometric points even when  $G'$  is smooth and  $S = \text{Spec } k$  for an algebraically closed field  $k$ ; see Example 2.2.3.

**Proposition 2.1.2.** *Let  $Y$  and  $Y'$  be finitely presented closed subschemes of an affine  $S$ -group  $G$  of finite presentation. Assume  $Y$  is either a multiplicative type subgroup or is finite flat over  $S$ . The scheme  $\text{Transp}_G(Y, Y')$  exists as a finitely presented closed subscheme of  $G$ . In particular, the normalizer  $N_G(Y)$  exists as a finitely presented closed subgroup of  $G$ .*

*If  $G$  is smooth and  $Y$  and  $Y'$  are both multiplicative type subgroups of  $G$  then these subschemes are smooth.*

In [SGA3, XI, 2.4bis], this proposition is proved without affineness hypotheses on  $G$ . We restrict to the affine case because the proof is much simpler in that case and it is sufficient for our needs. In Proposition 2.1.6 we will adapt the method of proof to apply to smooth closed subschemes  $Y, Y' \subset G$  with connected fibers, but the *smoothness* of  $\text{Transp}_G(Y, Y')$  may then fail, even when  $G$  is smooth. The smoothness of  $\text{Transp}_G(H, H')$  for any  $H$  and  $H'$  of multiplicative type (when  $G$  is smooth) is remarkably useful.

*Proof.* By “standard” direct limit arguments, we may and do assume that  $S$  is noetherian. Granting the representability by a closed subscheme, let’s address the  $S$ -smoothness when  $G$  is smooth and  $Y$  and  $Y'$  respectively coincide with multiplicative type subgroups  $H$  and  $H'$  of  $G$ . We shall verify the functorial criterion for smoothness. The condition is that for an affine scheme  $S'$  over  $S$  (which we may and do take to be noetherian, or even artinian) and a closed subscheme  $S'_0$  defined by a square-zero quasi-coherent ideal on  $S'$ , any  $g_0 \in G(S'_0)$  conjugating  $H_{S'_0}$  into  $H'_{S'_0}$  admits a lift  $g \in G(S')$  conjugating  $H_{S'}$  into  $H'_{S'}$ . As usual we may rename  $S'$  as  $S$  to save some notation, and by  $S$ -smoothness of  $G$  we can lift  $g_0$  to some  $g \in G(S)$  but perhaps  $gHg^{-1}$  is not contained in  $H'$ . Nonetheless, these two subgroups of multiplicative type in  $G$  have the same reduction in  $G_0$ , so by renaming  $gHg^{-1}$  as  $H$  we may assume that  $H_0 \subseteq H'_0$  and  $g_0 = 1$ .

Since  $H'$  is multiplicative type, the multiplicative type  $S_0$ -subgroup  $H_0$  in  $H'_0$  uniquely lifts to a multiplicative type  $S$ -subgroup  $\tilde{H}$  of  $H'$  (as the uniqueness allows us to work étale-locally on  $S$ , so it suffices to consider the easy case that  $H'$  is  $S$ -split and  $H_0$  is  $S_0$ -split). The multiplicative type  $S$ -subgroups  $H$  and  $\tilde{H}$  in  $G$  have the same reduction in  $G_0$ , so by Corollary B.2.4 and [Oes, III, 3.3] there exists  $g' \in G(S)$  satisfying  $g'_0 = 1$  and  $g'Hg'^{-1} = \tilde{H} \subseteq H'$ . This completes the proof of smoothness of the transporter scheme (granting its existence).

It remains to prove that the transporter functor is represented by a closed subscheme of  $G$  when  $Y$  is either finite flat or a subgroup of multiplicative type. We first reduce the second case to the first, so suppose  $Y = H$  is a subgroup of multiplicative type. By the relative schematic density of  $\{H[n]\}_{n \geq 1}$  in  $H$  (in the sense of [EGA, IV<sub>3</sub>, 11.10.8–11.10.10]) we have the equality

$$\underline{\mathrm{Transp}}_G(H, Y') = \bigcap_{n > 0} \underline{\mathrm{Transp}}_G(H[n], Y')$$

as subfunctors of  $G$ , so it suffices to treat each of the pairs  $(H[n], Y')$  separately. Hence, we may and do assume that  $Y$  is finite and flat over  $S$ . We may also assume  $S$  is noetherian.

To build  $\underline{\mathrm{Transp}}_G(Y, Y')$  for finite flat  $Y$ , we will use Hom-schemes. Hence, we now digress to prove a lemma concerning the representability of Hom-functors in a special case.

**Lemma 2.1.3.** *Let  $S$  be a scheme,  $X \rightarrow S$  a finite flat and finitely presented map, and  $Y \rightarrow S$  an affine morphism of finite presentation. The functor on  $S$ -schemes defined by*

$$S' \rightsquigarrow \mathrm{Hom}_{S'}(X_{S'}, Y_{S'})$$

*is represented by an affine  $S$ -scheme of finite presentation.*

*Likewise, if  $G$  and  $G'$  are finitely presented  $S$ -groups with  $G$  finite flat and  $G'$  affine over  $S$  then the functor  $S' \rightsquigarrow \mathrm{Hom}_{S'\text{-gp}}(G_{S'}, G'_{S'})$  classifying group scheme homomorphisms is represented by an affine  $S$ -scheme of finite presentation.*

The representing schemes are denoted  $\underline{\mathrm{Hom}}(X, Y)$  and  $\underline{\mathrm{Hom}}_{S\text{-gp}}(G, G')$  respectively.

*Proof.* We may and do assume that  $S$  is affine, and then noetherian, say  $S = \mathrm{Spec}(A)$ , with  $X = \mathrm{Spec}(B)$  for  $B$  that is finite free as an  $A$ -module (admitting an  $A$ -basis containing 1) and  $Y = \mathrm{Spec}(C)$  for  $C$  a finitely generated  $A$ -algebra. Once the case of the functor of scheme morphisms is settled, the refinement for group schemes amounts to the formation of several fiber products (see Exercise 2.4.5). Thus, we just focus on the assertion for scheme morphisms with  $X$  and  $Y$ , avoiding any involvement of group schemes.

The basic idea of the construction of  $\underline{\mathrm{Hom}}(X, Y)$  is similar to the construction of Weil restriction of scalars  $R_{S'/S}$  for a finite flat map  $S' \rightarrow S$  in the affine noetherian case (see [BLR, 7.6]). The reason for the similarity is that  $\underline{\mathrm{Hom}}(X, Y) = R_{X/S}(X \times_S Y)$ , since for any  $S$ -scheme  $S'$  the set  $\mathrm{Hom}_S(S', \underline{\mathrm{Hom}}(X, Y))$  is identified with

$$\mathrm{Hom}_S(S' \times_S X, Y) = \mathrm{Hom}_X(S' \times_S X, X \times_S Y) = \mathrm{Hom}_S(S', R_{X/S}(X \times_S Y)).$$

Let  $\{e_1, \dots, e_r\}$  be an  $A$ -basis of  $B$  with  $e_1 = 1$ , and let

$$A[t_1, \dots, t_n]/(f_1, \dots, f_m) \simeq C$$

be a presentation of  $C$ . For any  $A$ -algebra  $A'$  and the associated scalar extensions  $B' = A' \otimes_A B$  and  $C' = A' \otimes_A C$  over  $A'$ , we identify  $\mathrm{Hom}_{A'\text{-alg}}(C', B')$  with the set of ordered  $n$ -tuples  $b' = (b'_1, \dots, b'_n) \in B'^n$  such that  $f_j(b') = 0$  in  $B'$  for all  $j$ . By expressing the  $A$ -algebra structure on the  $A$ -module  $B = \bigoplus A e_\alpha$  in terms of “structure constants” in  $A$ , the specification of  $b'$  amounts to the specification of an ordered  $nr$ -tuple in  $A'$ , and the relations  $f_j(b') = 0$  amount to a “universal” system of polynomial conditions over  $A$  on this ordered  $nr$ -tuple. These polynomial conditions define the desired representing object as a closed subscheme of an affine space over  $A$ .  $\square$

**Remark 2.1.4.** In [SGA3, XI, 4.1, 4.2], a fundamental result is proved: for any smooth affine  $S$ -group  $G$  and multiplicative type group  $H$ , the functor  $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$  classifying  $S'$ -group homomorphisms  $H_{S'} \rightarrow G_{S'}$  over  $S$ -schemes  $S'$  is represented by a smooth separated  $S$ -scheme and the functor  $\underline{\mathrm{Mult}}_{G/S}$  classifying subgroups of  $G$  of multiplicative type is likewise represented by a smooth separated  $S$ -scheme. (Beware that each of these representing schemes is generally just *locally* of finite presentation over  $S$ ; they are typically *not* quasi-compact over  $S$ .)

The construction of schemes representing  $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$  and  $\underline{\mathrm{Mult}}_{G/S}$  rests on deep representability criteria for functors in [SGA3, XI, §3] that are specially designed for such applications. These moduli schemes in turn underlie Grothendieck’s construction of quasi-affine quotients  $G/N_G(H)$  and  $G/Z_G(H)$  for multiplicative type subgroups  $H$  in  $G$ . The quasi-affineness property of these quotients [SGA3, XI, 5.11] is crucial for applications with descent theory (as it ensures effectivity of descent).

Later we will use a variant on Grothendieck’s method, requiring only the more elementary case of Hom-schemes as in Lemma 2.1.3 and establishing the existence and quasi-affineness of the quotient schemes  $G/N_G(H)$  and  $G/Z_G(H)$  for smooth  $G$  via an alternative procedure. The reason that we can succeed in this way is that we will appeal to general theorems from the theory of algebraic spaces to understand representability and geometric properties of quotient sheaves.

Returning to the proof of Proposition 2.1.2, by Lemma 2.1.3 the Hom-functors  $\underline{\mathrm{Hom}}(Y, G)$  and  $\underline{\mathrm{Hom}}(Y, Y')$  classifying scheme homomorphisms (over varying  $S$ -schemes) are represented by affine  $S$ -schemes of finite type. There is a natural  $G$ -action on  $\underline{\mathrm{Hom}}(Y, G)$  via  $G$ -conjugation on  $G$ , so the  $S$ -point  $j \in \underline{\mathrm{Hom}}(Y, G)(S) = \mathrm{Hom}(Y, G)$  corresponding to the given inclusion yields a  $G$ -orbit map

$$(2.1.1) \quad G \rightarrow \underline{\mathrm{Hom}}(Y, G)$$

over  $S$  defined by  $g \mapsto (y \mapsto gj(y)g^{-1})$ . Consider the pullback of the natural monomorphism

$$(2.1.2) \quad \underline{\mathrm{Hom}}(Y, Y') \rightarrow \underline{\mathrm{Hom}}(Y, G)$$

(defined by composition with the inclusion  $Y' \hookrightarrow G$ ) under the map (2.1.1). This pullback recovers the subfunctor  $\underline{\text{Transp}}_G(Y, Y')$ ; i.e., we have cartesian diagram of functors

$$\begin{array}{ccc} \underline{\text{Transp}}_G(Y, Y') & \longrightarrow & \underline{\text{Hom}}_S(Y, Y') \\ \downarrow & & \downarrow \\ G & \longrightarrow & \underline{\text{Hom}}_S(Y, G) \end{array}$$

This establishes the representability of the transporter functor, and to prove that it is a closed subscheme of  $G$  it suffices to prove that (2.1.2) is a closed immersion. Exactly as in the construction of these Hom-schemes in the proof of Lemma 2.1.3, the condition on an  $S'$ -scheme morphism  $Y_{S'} \rightarrow G_{S'}$  (for an  $S$ -scheme  $S'$ ) that it factors through the closed subschemes  $Y'_{S'}$  is represented by an additional system of universal Zariski-closed conditions arising from generators of the ideal of  $Y'$  in  $G$  (Zariski-locally over  $S$ ). This establishes the required closed immersion property.  $\square$

See Exercise 2.4.4 for further discussion of constructions of transporters and normalizers. As an application of Proposition 2.1.2, we have the following mild refinement on [SGA3, XI, 5.4bis]:

**Corollary 2.1.5.** *Let  $G$  be a smooth affine  $S$ -group, and  $H$  and  $H'$  a pair of subgroups of multiplicative type. For any  $s \in S$ , if there exists an extension field  $K/k(s)$  such that  $(H_s)_K$  is  $G_s(K)$ -conjugate to  $(H'_s)_K$  then there exists an étale neighborhood  $U \rightarrow S$  of  $s$  such that  $H_U$  is  $G(U)$ -conjugate to  $H'_U$ .*

*Proof.* The hypothesis is that the smooth map  $\text{Transp}_G(H, H') \rightarrow S$  hits a  $K$ -point over  $s$ , so its open image  $V$  contains  $s$ . Any surjective smooth map of schemes admits sections étale-locally on the base. Applying this to  $\text{Transp}_G(H, H') \rightarrow V$  provides an étale neighborhood  $U$  of  $(S, s)$  such that  $\text{Transp}_G(H, H')(U)$  is non-empty. This  $U$  does the job. (The second assertion is immediate since any containment between multiplicative type groups that is an equality on geometric fibers at one point is an equality over an open neighborhood, as we see by passing to an fppf or étale covering that splits both groups.)  $\square$

In our later study of parabolic subgroups  $P$  of reductive group schemes  $G$ , it will be important to establish that  $P$  is self-normalizing, which is to say that  $P$  represents  $\underline{N}_G(P)$  (a scheme-theoretic improvement on the result  $N_{G(k)}(P) = P(k)$  in the classical theory over an algebraically closed field  $k$ ). The proof of this property will rest on knowing a priori that  $\underline{N}_G(P)$  is represented by *some* finitely presented closed subscheme of  $G$ , so we need a variant on Proposition 2.1.2 that is applicable to smooth closed subschemes  $Y$  of  $G$  such that all fibers  $Y_s$  are geometrically connected (e.g.,  $Y = P$ ):

**Proposition 2.1.6.** *Let  $G$  be a smooth affine  $S$ -group and  $Y, Y' \subset G$  finitely presented closed subschemes such that  $Y$  is smooth with non-empty and geometrically connected fibers over  $S$ . The transporter functor  $\underline{\text{Transp}}_G(Y, Y')$  is represented by a finitely presented closed*

subscheme of  $G$ . In particular, the normalizer functor

$$\underline{N}_G(Y) : S' \rightsquigarrow \{g \in G(S') \mid gY_{S'}g^{-1} = Y_{S'}\}$$

is represented by a finitely presented closed subgroup of  $G$ .

The main idea in the proof of this result is to reduce to the finite flat case treated in Proposition 2.1.2. The role of relative schematic density of the torsion-level subgroups in the proof of Proposition 2.1.2 will be replaced by the relative schematic density of the collection of infinitesimal neighborhoods of a section  $y : S \rightarrow Y$  (which exists étale-locally on the base). This is easiest to understand in the classical setting, so let us first see how that case works:

**Example 2.1.7.** Let  $Y$  be a smooth, non-empty, and connected scheme over an algebraically closed field  $k$  (so  $Y$  is irreducible and reduced). Pick  $y \in Y(k)$ , and let  $Y_n$  denote the  $n$ th infinitesimal neighborhood of  $y$ ; i.e.,  $Y_n$  is the infinitesimal closed subscheme defined by the vanishing of  $\mathcal{I}_y^{n+1}$ , where  $\mathcal{I}_y$  is the ideal of  $y$  in  $\mathcal{O}_Y$ . We claim that the collection  $\{Y_n\}$  is schematically dense in  $Y$ ; i.e., if  $Z$  is a closed subscheme of  $Y$  that contains  $Y_n$  for all  $n$  then  $Z = Y$ . If  $\mathcal{J}$  is the ideal of  $Z$  in  $Y$  then the stalk  $\mathcal{J}_y$  in the local ring  $\mathcal{O}_{Y,y}$  clearly vanishes in the completion  $\mathcal{O}_{Y,y}^\wedge$ , so  $\mathcal{J}_y = 0$ . But  $Y$  is integral, so therefore  $\mathcal{J} = 0$  as desired.

Now we turn to the proof of Proposition 2.1.6.

*Proof.* The problem is étale-local on  $S$ , so we may assume that there exists  $y \in Y(S)$ . For each  $n \geq 0$ , let  $Y_n$  denote the  $n$ th infinitesimal neighborhood of  $y$  in  $Y$ ; this is the closed subscheme defined by the  $(n+1)$ th-power of the ideal of the closed immersion  $y : S \rightarrow Y$ . The proof of Proposition 2.1.2 will be carried over by using these neighborhoods in the role of the  $n$ -torsion subgroups in that earlier proof. That is, we claim that each  $Y_n$  is finite flat over  $S$  and that the collection  $\{Y_n\}$  is relatively schematically dense in  $Y$  (in the sense of [EGA, IV<sub>3</sub>, 11.10.8–11.10.10]). Once these are established, the resulting equality

$$\underline{\text{Transp}}_G(Y, Y') = \bigcap_{n \geq 0} \underline{\text{Transp}}_G(Y_n, Y')$$

as subfunctors of  $G$  and the known representability of  $\underline{\text{Transp}}_G(Y_n, Y')$  will complete the proof.

Since the formation of  $Y_n$  commutes with any base change on  $S$ , by direct limit arguments we may and do assume that  $S$  is noetherian. It suffices to solve the representability problem on the category of noetherian  $S$ -schemes, since  $Y$  and  $Y'$  are finitely presented over  $S$ . To prove that  $Y_n$  is flat over  $S$ , we can reduce to the case  $S = \text{Spec } A$  for an artin local ring  $A$  (by the local flatness criterion). Now  $y$  has a single physical point  $y_0$ , and by  $A$ -smoothness of  $Y$  near  $y$  the completion of  $\mathcal{O}_{Y,y_0}$  along  $y$  is identified with  $A[[t_1, \dots, t_d]]$  carrying the ideal of  $y$  over to the ideal  $(t_1, \dots, t_d)$ . (This is seen by using an étale map  $f : (Y, y) \rightarrow (\mathbf{A}_S^d, 0)$ .) Thus,  $Y_n \simeq A[[t_1, \dots, t_d]]/(t_1, \dots, t_d)^n$ , which is visibly  $S$ -flat. Since  $(Y_n)_{\text{red}} = S_{\text{red}}$  and  $Y_n$  is finite type over  $S$ , clearly  $Y_n$  is  $S$ -finite.

It remains to show that  $\{Y_n\}$  is relatively schematically dense in  $Y$ . By the  $S$ -flatness of the  $Y_n$ , it suffices to prove the schematic density on geometric fibers over  $S$  [EGA, IV<sub>3</sub>, 11.10.9]. The case of geometric fibers is Example 2.1.7.  $\square$

**Example 2.1.8.** In contrast with the smoothness of normalizers (and centralizers) of multiplicative type subgroups in smooth affine groups, the normalizers in Proposition 2.1.6 can even fail to be flat when  $G$  is smooth and  $Y$  is a smooth subgroup with connected fibers. In particular, normality on a fiber does not imply normality on nearby fibers, even when working with smooth groups. More specifically, a family of non-normal smooth closed subgroups can degenerate to a normal subgroup.

We give an example over  $S = \mathbf{A}^1$  (with coordinate  $t$ ) using the  $S$ -group  $G = \mathbf{G}_a \times \mathrm{SL}_2$ . Consider the closed  $S$ -subgroup  $G' = \mathbf{G}_a$  defined by the closed immersion

$$j : (u, t) \mapsto (u, \begin{pmatrix} 1 & tu \\ 0 & 1 \end{pmatrix}, t)$$

over the  $t$ -line  $S$ . The fiber map over  $t = 0$  is the inclusion of  $\mathbf{G}_a$  into the first factor of  $\mathbf{G}_a \times \mathrm{SL}_2$ , but for  $t \neq 0$  the fiber map  $j_t$  has non-normal image. Hence, the normalizer  $N_G(G')$  has fiber  $G_0$  over  $t = 0$  but its fiber over any  $t \neq 0$  has strictly smaller dimension. It follows that  $N_G(G')$  is not  $S$ -flat.

**2.2. Centralizer schemes.** In the classical theory of linear algebraic groups over an algebraically closed field  $k$ , the centralizer of a smooth closed subscheme  $Y$  of a smooth affine group  $G$  is defined by brute force: it is the reduced (hence smooth) Zariski-closed subgroup structure on  $\bigcap_{y \in Y(k)} Z_G(y)$ . This definition makes it rather unclear what the Lie algebra is when  $Y$  is a subgroup of  $G$ . The scheme-theoretic approach defines the centralizer in a more functorial way that makes it easy to identify the Lie algebra, but shifts the burden of work to proving that (in favorable circumstances) this subgroup is actually *smooth*.

**Definition 2.2.1.** Let  $G \rightarrow S$  be a separated group scheme and  $Y$  a closed subscheme. The *functorial centralizer*  $\underline{Z}_G(Y)$  on  $S$ -schemes assigns to any  $S$ -scheme  $S'$  the set of  $g \in G(S')$  such that  $g$  centralizes  $Y_{S'}$  inside  $G_{S'}$ .

A closed subgroup of  $G$  representing  $\underline{Z}_G(Y)$ , if one exists, is denoted  $Z_G(Y)$  and is called the *centralizer* of  $Y$  in  $G$ . In the special case  $Y = G$ , such a subgroup scheme is called the *center* of  $G$  (if it exists!) and is denoted  $Z_G$ .

The general existence of centralizers seems to be quite delicate, but in some special cases there are good results. One favorable situation is when  $S = \mathrm{Spec}(k)$  for a field  $k$ , in which case  $Z_G(Y)$  exists for any  $Y$ ; see Exercise 2.4.4. The case of centralizers of smooth closed subgroups over an arbitrary base is [SGA3, XI, 6.11], but for later purposes over a general base  $S$  we must allow any  $Y$  of multiplicative type (so  $Y$  may not be  $S$ -smooth) and show that in such cases the centralizer of  $Y$  is smooth when  $G$  is smooth (even if  $Y$  is not smooth!). This will be handled in Lemma 2.2.4 below.

**Definition 2.2.2.** A subgroup scheme  $G'$  in  $G$  is *central* when  $G$ -conjugation on  $G'$  is trivial on  $G'$  (equivalently,  $\underline{Z}_G(G') = G'$ ).

Another way to express the centrality condition is that for every  $S$ -scheme  $S'$  and  $g' \in G'(S')$ , the  $g'$ -conjugation action on  $G_{S'}$  is trivial.

**Example 2.2.3.** If  $G$  is a *smooth* finite type group over an algebraically closed field  $k$ , a closed subgroup scheme  $G'$  is central if  $G'$  is centralized by all elements of  $G(k)$ ; see Exercise 2.4.4(iv). This is false when  $G$  is not assumed to be smooth, even when  $G'$  is smooth. For example, in characteristic  $p > 0$  the usual semi-direct product  $G = \mathbf{G}_m \rtimes \alpha_p$  and its smooth closed subgroup  $G' = \mathbf{G}_m$  have the same geometric points, but  $G'$  is not even normal in  $G$ .

As another example of a central subgroup scheme, for any scheme  $S$  the diagonal  $\mu_n$  in the  $S$ -group  $\mathrm{SL}_n$  is central for any  $n > 1$ . See Example 3.3.7 for a stronger centrality property of  $\mu_n$  in  $\mathrm{SL}_n$ .

**Lemma 2.2.4.** *Let  $G \rightarrow S$  be a finitely presented affine  $S$ -group, and  $Y$  a finitely presented closed subscheme. Assume that  $Y$  is a subgroup of multiplicative type or that  $Y \rightarrow S$  is either smooth with non-empty connected fibers or finite locally free. Then  $Z_G(Y)$  exists, and it is smooth when  $G$  is smooth and  $Y$  is a subgroup of multiplicative type.*

For  $Y$  a subgroup of multiplicative type, this lemma is part of [SGA3, XI, 5.3].

*Proof.* We may assume  $S$  is noetherian, and for the representability assertions it suffices to restrict to functors on the category of noetherian  $S$ -schemes. First we treat the existence when  $Y$  is finite flat over  $S$ . In this case Lemma 2.1.3 provides an  $S$ -scheme  $\underline{\mathrm{Hom}}(Y, G)$  that is affine of finite type over  $S$ . The given inclusion of  $Y$  into  $G$  corresponds to an  $S$ -point  $j$  of  $\underline{\mathrm{Hom}}(Y, G)$ , and the orbit map for  $G$  through  $j$  (via the  $G$ -action on  $\underline{\mathrm{Hom}}(Y, G)$  through conjugation) pulls back  $j : S \rightarrow \underline{\mathrm{Hom}}(Y, G)$  to a closed subscheme of  $G$  representing  $Z_G(Y)$ .

When  $Y$  is a subgroup  $H$  of multiplicative type, since  $\{H[n]\}$  is relatively schematically dense in  $H$  we have  $Z_G(H) = \bigcap_{n>0} Z_G(H[n])$  as subfunctors of  $G$ . Each  $H[n]$  is finite flat, and  $\bigcap_{n>0} Z_G(H[n])$  represents  $Z_G(H)$ . Likewise, if  $Y$  is a smooth closed subscheme with connected non-empty fibers then by working étale-locally on  $S$  we can assume that  $Y \rightarrow S$  admits a section  $y$ . Then we can argue exactly as in the multiplicative type case by using the finite flat infinitesimal neighborhoods  $Y_n$  of  $y$  in the role of the torsion-levels  $H[n]$  (due to the relative schematic density of  $\{Y_n\}$  in  $Y$ , established in the proof of Proposition 2.1.6).

Finally, for smooth  $G$  and general  $H$  of multiplicative type we prove that  $Z_G(H)$  is smooth. By the functorial smoothness criterion and the generality of the base scheme, it suffices to show that if  $S$  is affine and  $S_0$  is a closed subscheme of  $S$  defined by a square-zero quasi-coherent ideal then any  $g_0 \in G_0(S_0)$  centralizing  $H_0$  lifts to some  $g \in G(S)$  that centralizes  $H$ . By smoothness of  $G$  we can pick some  $g \in G(S)$  lifting  $g_0$ , so  $gHg^{-1}$  and  $H$  are multiplicative type subgroups of  $G$  with the same reduction in  $G_0$ . By Corollary B.2.4 and [Oes, III, 3.3], there exists  $g' \in G(S)$  lifting  $1 \in G_0(S_0)$  such that  $g'$  conjugates  $gHg^{-1}$  to  $H$ , so  $g'g$  normalizes  $H$ . But over  $S_0$  this conjugation endomorphism of  $H_0$  is conjugation by  $(g'g)_0 = g_0$ , which is the trivial action on  $H_0$ . Since  $H$  is multiplicative type and  $S_0$  is defined by a nilpotent ideal on  $S$ , it follows (by “rigidity” for homomorphisms between multiplicative type groups) that  $g'g$  must centralize  $H$ , so  $g'g \in Z_G(H)(S)$  and this lifts  $g_0 \in Z_G(H)(S_0)$ .  $\square$

**Remark 2.2.5.** An important special case of Lemma 2.2.4 is  $Y = G$ , for which this lemma asserts the existence of the scheme-theoretic center  $Z_G$  of a smooth affine  $S$ -group  $G$  such that all  $G_{\bar{s}}$  are connected. (See [SGA3, XI, 6.11] for the removal of the fibral connectedness condition, taking  $G = H$  there.) In general  $Z_G$  can fail to be flat (see the end of [SGA3, XVI, §3] for an example), in which case it is not very useful.

See Exercise 2.4.6(ii) for explicit examples of Lemma 2.2.4 with  $Z_G$  of multiplicative type, and Exercises 2.4.4 and 2.4.10 for generalizations of Lemma 2.2.4 over fields. In Theorem 3.3.4 we will see that  $Z_G$  is of multiplicative type (hence flat) for any reductive group scheme  $G \rightarrow S$ .

**Example 2.2.6.** Here are two examples of the smoothness aspect of Lemma 2.2.4 in the context of smooth affine groups over a field  $k$ . Let  $G$  be such a group, and  $T$  a torus in  $G$  (not necessarily maximal). The lemma ensures that the centralizer  $Z_G(T)$  is always smooth (and it is connected when  $G$  is connected, by the classical theory over  $\bar{k}$ ). This is the scheme-theoretic proof of a fact known from the classical theory [Bo91, 9.2, Cor.]: the reduced structure on the closed subset of  $G_{\bar{k}}$  corresponding to the centralizer of  $T_{\bar{k}}$  in  $G(\bar{k})$  descends to a smooth closed  $k$ -subgroup of  $G$  (namely  $Z_G(T)$ ). As a special case, if  $G$  is connected reductive and  $T$  is a (geometrically) maximal torus then  $Z_G(T) = T$  because such an equality between smooth closed subgroups can be checked on  $\bar{k}$ -points, where it follows from the classical theory.

For another application, if  $G'$  is a smooth closed  $k$ -subgroup of  $G$  that is normalized by  $T$  (e.g., a normal  $k$ -subgroup of  $G$ ) then the scheme-theoretic intersection  $Z_G(T) \cap G'$  represents the functorial centralizer for the  $T$ -action on  $G'$ . Although  $T$  may not be a  $k$ -subgroup of  $G'$  inside  $G$ , so Lemma 2.2.4 does not literally apply to  $T$  acting on  $G'$ , there is a standard trick with semi-direct products that enables us to apply the lemma anyway to prove  $Z_G(T) \cap G'$  is smooth: form the semi-direct product  $G' \rtimes T$  in which  $T$  embeds along the second factor. Lemma 2.2.4 can be applied to this semi-direct product. Thus,  $Z_{G' \rtimes T}(T)$  is smooth, and also is connected when  $G'$  is connected. But clearly

$$Z_{G' \rtimes T}(T) = (Z_G(T) \cap G') \rtimes T$$

as  $k$ -schemes, and the left side is smooth (and connected when  $G'$  is connected). Thus, the direct factor scheme  $Z_G(T) \cap G'$  is smooth (and connected when  $G'$  is). For example, if  $N \subset G$  is any smooth closed normal subgroup then  $N \cap Z_G(T)$  is smooth (and connected when  $N$  is connected).

**Remark 2.2.7.** Our construction of centralizers and normalizers relied on passage to the case of  $Y$  that is finite flat over the base. There is an alternative approach to the construction of centralizers and normalizers in the presence of enough “module-freeness” for some coordinate rings (as algebras over a base ring); see [SGA3, VIII, §6], [CGP, A.8.10(1)], and Exercise 2.4.4.

**2.3. Some quotient constructions.** As was noted in Remark 2.1.4, for the construction of quotients in the relative setting we will bypass some of Grothendieck’s techniques in [SGA3]

in favor of the theory of algebraic spaces. This is illustrated in the following results, the first of which is a variant on [SGA3, XI, 5.3bis].

**Theorem 2.3.1.** *Let  $G \rightarrow S$  be a smooth affine group scheme, and  $H$  a subgroup of multiplicative type. The quotients  $G/Z_G(H)$  and  $G/N_G(H)$  exist as smooth quasi-affine  $S$ -schemes. Moreover, the quotient  $W_G(H) := N_G(H)/Z_G(H)$  exists as a separated and étale  $S$ -group of finite presentation.*

*In particular, if  $H$  is normal in  $G$  (i.e.,  $N_G(H) = G$ ) and the fibers  $G_s$  are connected then  $H$  is central in  $G$ .*

Before we prove this result, we briefly digress to make some remarks.

**Remark 2.3.2.** As in [EGA, II, 5.1.1], a map of schemes  $f : X \rightarrow Y$  is *quasi-affine* if, over the constituents of some affine open cover of  $Y$ , it factors as a quasi-compact open immersion into an affine scheme (so  $f$  is quasi-compact and separated). In [EGA, II, 5.1.2, 5.1.6] there are several equivalent versions of this definition. A more “practical” description is provided in [EGA, II, 5.1.9] when  $f : X \rightarrow Y$  is finite type and  $Y$  is noetherian (or more generally,  $Y$  is quasi-compact and quasi-separated): such an  $f$  is quasi-affine if and only if  $f$  factors as a quasi-compact open immersion followed by an affine map  $Y' \rightarrow Y$  of finite type.

Since fppf descent is always effective for schemes that are quasi-affine over the base, the quasi-affineness in Theorem 2.3.1 useful; e.g., it will underlie our later construction of the “scheme of maximal tori” in a reductive group scheme (even when there is no torus over the given base scheme that is maximal on all geometric fibers).

**Remark 2.3.3.** We will build the quotients in Theorem 2.3.1 via an orbit argument, as is also done in the classical setting (i.e., the case  $S = \text{Spec}(k)$  for a field  $k$ ). However, unlike the classical case, in the relative setting we do *not* have a plentiful supply of linear representations. In fact, we do not even know if every smooth affine group over the dual numbers  $k[\epsilon]$  is a closed subgroup scheme of some  $\text{GL}_n!$  (See [SGA3, VI<sub>B</sub>, 13.2, 13.5] and [SGA3, XI, 4.3] for further discussion in this direction.) Thus, the construction of quotients  $G/H$  modulo flat and finitely presented closed subgroups  $H$  is rather subtle when the base is not a field.

Generally such quotients  $G/H$  are algebraic spaces, and we do not wish to get into the issue of whether they are always schemes. In some cases we will prove that  $G/H$  is a scheme by using an orbit argument with a slightly different flavor from the scheme case: we will *not* directly build the quotient scheme as a subscheme of a scheme  $X$  on which  $G$  acts (or rather, we will identify the quotient sheaf  $G/H$  as a subfunctor of such an  $X$  and we do not try to represent the subfunctor inclusion  $G/H \hookrightarrow X$  by a subscheme of  $X$ ) because instead we will appeal to general criteria for an algebraic space to be a scheme. Our deviation from the method of construction of  $G/H$  in the classical case is forced by the fact that there is no general analogue in the relative setting of the result in the classical case that  $G$ -orbits are always smooth and locally closed when  $G$  is smooth. In Remark 2.3.5 we will prove that the orbit maps arising in the cases of interest to us in Theorem 2.3.1 are open immersions, but this is never used in what follows.

Now we turn to the proof of Theorem 2.3.1.

*Proof.* We may and do assume that  $S$  is noetherian. The relative schematic density of  $\{H[n]\}$  in  $H$  implies that  $Z_G(H) = \bigcap_{n>0} Z_G(H[n])$  and  $N_G(H) = \bigcap_{n>0} N_G(H[n])$  as closed subschemes of  $G$ . But the noetherian condition on  $G$  implies that any descending chain of closed subschemes of  $G$  stabilizes, so  $Z_G(H) = Z_G(H[n])$  and  $N_G(H) = N_G(H[n])$  for sufficiently large  $n$ . Hence, it suffices to treat each  $H[n]$  in place of  $H$ , so we may assume that  $H$  is  $S$ -finite. In particular, Lemma 2.1.3 provides the scheme  $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$  that is  $S$ -affine of finite type.

Consider the natural  $G$ -action on this Hom-scheme via composition with the conjugation action of  $G$  on itself, and the  $S$ -point corresponding to the given inclusion  $j : H \rightarrow G$ . The orbit map  $G \rightarrow \underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$  through  $j$  is right-invariant by the stabilizer scheme  $Z_G(H)$  of the  $S$ -point  $j$ , so the quotient sheaf  $G/Z_G(H)$  for the fppf (or étale) topology is naturally a subfunctor of the Hom-scheme. By a general theorem of Artin [Ar74, Cor. 6.3], for any finite type  $S$ -scheme  $X$  and equivalence relation  $R$  on  $X$  that is represented by a closed subscheme in  $X \times_S X$  and for which both projections  $R \rightrightarrows X$  are flat, the fppf quotient sheaf  $X/R$  is a separated algebraic space of finite type over  $S$ . Thus,  $G/Z_G(H)$  is such an algebraic space. The monomorphism  $G/Z_G(H) \rightarrow \underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$  over  $S$  must be separated and finite type with finite fibers, so  $G/Z_G(H)$  is separated and quasi-finite over a scheme.

By a result of Knutson [Knut, II, 6.15], an algebraic space that is quasi-finite and separated over a noetherian scheme is a scheme. (See [LMB, Thm. A.2] for a generalization without noetherian hypotheses.) Hence,  $G/Z_G(H)$  is a scheme that is moreover quasi-finite and separated over the scheme  $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ . By Zariski's Main Theorem [EGA, IV<sub>3</sub>, 8.12.6], any quasi-finite and separated map between noetherian schemes is quasi-affine, so  $G/Z_G(H)$  is quasi-affine over the  $S$ -affine  $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$  and hence it is quasi-affine over  $S$ . The quotient  $G/Z_G(H)$  is  $S$ -smooth, since  $G \rightarrow G/Z_G(H)$  is an fppf cover by the smooth  $G$ . (In Remark 2.3.5 we show that the orbit map  $G/Z_G(H) \rightarrow \underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$  is an open immersion.)

By the same reasoning, the quotient sheaves  $G/N_G(H)$  and  $W_G(H) = N_G(H)/Z_G(H)$  are smooth and separated algebraic space of finite type over  $S$ . The map  $W_G(H) \rightarrow G/Z_G(H)$  is a closed immersion, via fppf descent of the closed immersion property for its pullback  $N_G(H) \rightarrow G$  along the smooth covering  $G \rightarrow G/Z_G(H)$ , so  $W_G(H)$  is a scheme as well. To prove it is  $S$ -étale we may pass to geometric fibers over  $S$ . With  $S = \mathrm{Spec}(k)$  for an algebraically closed field  $k$ , the automorphism functor of  $H$  is represented by a disjoint union  $\mathrm{Aut}_{H/k}$  of rational points (opposite to the automorphism functor of the constant dual of  $H$ ) and  $N_G(H)/Z_G(H)$  is a finite type  $k$ -group equipped with a monic homomorphism to the étale  $k$ -group  $\mathrm{Aut}_{H/k}$ . This forces the smooth  $N_G(H)/Z_G(H)$  to be *finite*, hence étale.

Exhibiting  $G/N_G(H)$  as quasi-finite and separated over a quasi-affine  $S$ -scheme requires a new idea, as follows. We shall exhibit it as a subfunctor of a scheme that is finite type and quasi-affine over  $S$ . To do this, we replace the role of scheme  $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$  that

classifies *homomorphisms* of  $H$  into  $G$  with the scheme that classifies *subschemes* of  $G$  that are “twisted forms” of  $H$ :

**Lemma 2.3.4.** *Let  $G \rightarrow S$  be a smooth affine group scheme, and  $H \subset G$  a subgroup of multiplicative type with finite fibers. There is a quasi-affine  $S$ -scheme  $\text{Twist}_{H/G}$  of finite presentation that represents the functor  $\underline{\text{Twist}}_{H/G}$  assigning to any  $S$ -scheme  $S'$  the set of multiplicative type subgroups  $H' \subset G_{S'}$  such that  $H'_{\bar{s}'} \simeq H_{\bar{s}'}$  for all geometric points  $\bar{s}'$  of  $S'$ .*

This lemma is a special case of the deeper result [SGA3, XI, 4.1] that the functor classifying *all* multiplicative type subgroups of  $G$  is represented by a smooth and separated  $S$ -scheme (without requiring that  $G$  contains any such nontrivial subgroups over  $S$ ). Whereas Lemma 2.3.4 will be deduced from Proposition 2.1.3, which is a special case of [SGA3, XI, 4.2], in [SGA3] the logic goes the other way: [SGA3, XI, 4.2] is deduced from [SGA3, XI, 4.1].

*Proof.* We may and do assume  $S$  is noetherian. Since étale descent is effective for schemes that are quasi-affine over the base, to construct the quasi-affine  $\text{Twist}_{H/G} \rightarrow S$  representing  $\underline{\text{Twist}}_{H/G}$  we may work étale-locally on  $S$  so that  $H$  has *constant* Cartier dual  $M$ . Thus,  $\underline{\text{Aut}}_{S\text{-gp}}(H) = \Gamma_S$  for the ordinary finite group  $\Gamma = \text{Aut}(M^*)$ , where  $M^* = \text{Hom}(M, \mathbf{Q}/\mathbf{Z})$ .

The natural right action of  $\Gamma_S$  on the finite type  $S$ -affine scheme  $\underline{\text{Hom}}_{S\text{-gp}}(H, G)$  via  $\gamma.f = f \circ \gamma$  is free, so by [SGA3, V, Thm. 4.1(iv)] there exists a finite étale quotient map

$$\underline{\text{Hom}}_{S\text{-gp}}(H, G) \rightarrow Q := \underline{\text{Hom}}_{S\text{-gp}}(H, G)/\Gamma_S.$$

This quotient is  $S$ -affine and constructed using  $\Gamma$ -invariants over open affines in  $S$ .

Inside  $\underline{\text{Hom}}_{S\text{-gp}}(H, G)$ , the monicity condition on  $S'$ -homomorphisms  $H_{S'} \rightarrow G_{S'}$  is represented by an open subscheme  $V$  (by Nakayama’s Lemma applied to the kernel of the universal homomorphism: if a closed subscheme of a *finite* scheme has full-degree fiber at some point, its defining ideal vanishes nearby). Clearly  $V$  is stable under the  $\Gamma_S$ -action, and  $V/\Gamma_S$  is an open subscheme of  $Q$ . It remains to show that  $V/\Gamma_S$  represents  $\underline{\text{Twist}}_{H/G}$ . There is visibly a map  $V \rightarrow \underline{\text{Twist}}_{H/G}$  that is  $\Gamma_S$ -invariant, and the induced map  $V/\Gamma_S \rightarrow \underline{\text{Twist}}_{H/G}$  is a monomorphism because if  $j, j' : H_{S'} \rightrightarrows G_{S'}$  are  $S'$ -subgroup inclusions whose images agree as closed subschemes then  $j' = j \circ \gamma$  for some  $\gamma \in \text{Aut}_{S'}(H_{S'}) = \Gamma_S(S')$ .

It remains to prove that the map  $V \rightarrow \underline{\text{Twist}}_{H/G}$  between sheaves for the étale topology is a surjection (thereby forcing the inclusion  $V/\Gamma_S \hookrightarrow \underline{\text{Twist}}_{H/G}$  to be an equality). Pick an  $S$ -scheme  $S'$  and an  $S'$ -subgroup  $H'$  of  $G_{S'}$  such that  $H'_{\bar{s}'} \simeq H_{\bar{s}'}$  for all geometric points  $\bar{s}'$  of  $S'$ . In particular,  $H'$  has finite fibers. It suffices to find an étale cover  $S''$  of  $S'$  over which the pullbacks of  $H'$  and  $H$  become isomorphic as group schemes. Passing to an étale cover brings us to the trivial case that  $H'$  and  $H$  each have constant Cartier dual.  $\square$

To finish the proof of Theorem 2.3.1, note that the conjugation action of  $G$  on itself defines a left action of  $G$  on  $\text{Twist}_{H/G}$ , and  $N_G(H)$  is the stabilizer of the  $S$ -point of  $\text{Twist}_{H/G}$  corresponding to the given copy of  $H$  in  $G$ . Thus, the separated and finite type algebraic space  $G/N_G(H)$  over  $S$  is a subfunctor of  $\text{Twist}_{H/G}$ , so applying Knutson’s schematic

criterion proves that  $G/N_G(H)$  is a scheme that is separated and quasi-finite over the  $S$ -scheme  $\text{Twist}_{H/G}$  that we know is quasi-affine over  $S$ . Applying Zariski's Main Theorem, the monomorphism  $G/N_G(H) \hookrightarrow \text{Twist}_{H/G}$  must be quasi-affine (see Remark 2.3.5 for a proof that this monomorphism is an open immersion). Hence,  $G/N_G(H)$  is quasi-affine over  $S$  since  $\text{Twist}_{H/G}$  is.  $\square$

**Remark 2.3.5.** A common geometric difficulty when working with algebraic spaces is the intervention of monomorphisms  $j : X \rightarrow Y$  between noetherian schemes  $j$  such that it is not obvious if  $j$  is a (locally closed) immersion. In the proof of Theorem 2.3.1 we encountered two such maps, namely the orbit maps  $j : G/Z_G(H) \rightarrow \underline{\text{Hom}}_{S\text{-gp}}(H, G)$  and  $j' : G/N_G(H) \rightarrow \text{Twist}_{H/G}$  for  $H$  a *finite*  $S$ -group of multiplicative type (with  $S$  a noetherian scheme). Ignorance of the immersion property for these maps is irrelevant for our purposes. We shall now prove that these monomorphisms are open immersions.

Let  $f : X \rightarrow Y$  be a map of finite type between noetherian schemes. It is an open immersion if and only if it is an étale monomorphism [EGA, IV<sub>4</sub>, 17.9.1], or equivalently a smooth monomorphism. We apply this criterion for open immersions to the orbit maps

$$j : G/Z_G(H) \rightarrow \underline{\text{Hom}}_{S\text{-gp}}(H, G), \quad j' : G/N_G(H) \rightarrow \text{Twist}_{H/G}$$

with  $G$  and  $H$  as in Theorem 2.3.1 and  $H$  finite over  $S$  (a noetherian scheme). These maps are monomorphisms of finite type, and to verify the functorial criterion for étaleness we may assume  $S = \text{Spec } R$  for an artin local ring  $(R, \mathfrak{m})$  with algebraically closed residue field  $k$ . Let  $R_0 = R/J$  for an ideal  $J \subset R$  satisfying  $J^2 = 0$ . Consider an  $R$ -homomorphism  $f : H \rightarrow G$  (resp. an  $R$ -subgroup  $H' \subset G$  that is conjugate to  $H$  étale-locally over  $\text{Spec } R$ ). For any  $R$ -algebra  $A$  and  $g \in G(A)$ , let  $c_g$  denote conjugation on  $G_A$  by  $g$ . Assuming that  $f_0$  arises from  $(G/Z_G(H))(R_0)$  (resp.  $H'$  arises from  $(G/N_G(H))(R_0)$ ) we seek to prove that  $f$  arises from  $(G/Z_G(H))(R)$  (resp.  $H'$  arises from  $(G/N_G(H))(R)$ ).

Such an  $R$ -point is unique if it exists (as we are studying the étale property for a monomorphism), so to prove existence we may make a preliminary finite étale extension on  $R$  (as then étale descent can be applied). The quotient maps  $G \rightarrow G/Z_G(H)$  and  $G \rightarrow G/N_G(H)$  are smooth since  $Z_G(H)$  and  $N_G(H)$  are smooth. Any  $R_0$ -point lifts through a smooth surjection since the residue field is algebraically closed, so  $(G/Z_G(H))(R_0) = G(R_0)/Z_G(H)(R_0)$  and similarly for  $G/N_G(H)$ .

Consider the case of  $G/Z_G(H)$ , so  $f : H \rightarrow G$  is an  $R$ -homomorphism that lifts  $c_{g_0}|_{H_0}$  for some  $g_0 \in G(R_0)$ . By  $R$ -smoothness of  $G$ ,  $g_0$  lifts to some  $g \in G(R)$ , so  $c_g|_H : H \rightarrow G$  and  $f$  are  $R$ -homomorphisms with the same reduction. By Corollary B.3.5, we can change the choice of  $g$  lifting  $g_0$  if necessary so that  $f = c_g|_H$ .

The case of  $G/N_G(H)$  goes similarly. Indeed, by hypothesis  $H'_0 = c_{g'_0}(H_0)$  for some  $g'_0 \in G(R_0)$ , and we choose  $g' \in G(R)$  lifting  $g'_0$ , so  $c_{g'}(H)$  and  $H'$  are multiplicative type subgroups of  $G$  that lift  $H'_0$ . By the deformation theory of multiplicative type subgroups of smooth affine groups (Corollary B.2.4 and [Oes, III, 3.3]) we can change the choice of  $g'$  lifting  $g'_0$  if necessary so that  $c_{g'}(H) = H'$ .

Whereas Theorem 2.3.1 concerns quasi-affine quotients, we know from the classical theory that it is also necessary to consider quotients  $G/H$  that turn out to be projective. The criterion we will use make such quotients as schemes, and not merely as algebraic spaces, is a self-normalizer hypothesis (see Corollary 5.2.8 for an important class of examples):

**Theorem 2.3.6.** *Let  $G \rightarrow S$  be a smooth affine group scheme with connected fibers, and  $H$  a smooth closed subgroup with connected fibers such that  $H = N_G(H)$ .*

- (1) *The quotient sheaf  $G/H$  is represented by a smooth  $S$ -scheme that is quasi-projective Zariski-locally over  $S$ , and it coincides with the functor  $\underline{\text{Twist}}_{H/G}$  of closed subgroups of  $G$  that are conjugate to  $H$  étale-locally on the base.*
- (2) *Assume that the geometric fibers  $(G/H)_{\bar{s}} = G_{\bar{s}}/H_{\bar{s}}$  are projective. The morphism  $G/H \rightarrow S$  is proper and admits as a canonical  $S$ -ample line  $\det(\text{Lie}(\mathcal{H}))^*$  where  $\mathcal{H} \hookrightarrow G \times (G/H)$  is the universal étale-local conjugate of  $H$  in  $G$ . In particular,  $G/H \rightarrow S$  is projective Zariski-locally on the base.*

The following preliminary remarks should clarify aspects of Theorem 2.3.6 before we undertake the proof. Recall from Proposition 2.1.6 that under the given smoothness and connectedness hypotheses on  $H$ , the normalizer  $N_G(H)$  does exist a priori. If we do not assume that  $N_G(H) = H$  then typically  $N_G(H)$  may not be  $S$ -flat (so there would not be a useful notion of quotient  $G/N_G(H)$ ); see Example 2.1.8. Also, the notion of “ $S$ -ample” that we use in part (2) means “ample on fibers”; by [EGA, IV<sub>3</sub>, 9.6.4], for proper and finitely presented  $S$ -schemes this implies the usual notion of ampleness over affine opens in  $S$ .

Part (2) of Theorem 2.3.6 is [SGA3, XXII, 5.8.2], apart from the explicit  $S$ -ample line bundle  $G/H$  (which is borrowed from the proof of [SGA3, XVI, 2.4]). The existence of a canonical  $S$ -ample line bundle on  $G/H$  will be crucial in our later construction of the “scheme of Borel subgroups” of a reductive group scheme  $G$ . The reason is that in general  $G$  does not admit a Borel subgroup over  $S$ , so it is necessary to pass to an étale cover  $S' \rightarrow S$  in order that there exists a Borel subgroup  $B' \subset G_{S'}$ . We will then be faced with a descent problem for  $G_{S'}/B'$  relative to  $S' \rightarrow S$ . The *canonical*  $S'$ -ample line bundle on  $G_{S'}/B'$  (arising from the Lie algebra of the universal Borel subgroup in the  $G_{S'}/B'$ -group  $G_{S'} \times (G_{S'}/B')$ ) will ensure the effectivity of the descent. The fibral projectivity hypothesis in part (2) is a familiar condition in the classical theory.

**Remark 2.3.7.** In the classical theory over an algebraically closed field  $k$ , it is well-known that the quotient  $G/P$  modulo a parabolic subgroup has a canonical ample line bundle: the anti-canonical bundle  $\det(\Omega_{(G/P)/k}^1)^*$ . This is the line bundle in Theorem 2.3.6(2) when  $S = \text{Spec } k$ , up to a twist against the  $k$ -line  $\det(\text{Lie}(G))^*$ . To explain this link in the relative setting, view  $f : X = G/H \rightarrow S$  as the moduli scheme classifying closed subgroups of  $G$  that are conjugate to  $H$  étale-locally on the base. We claim that the  $S$ -ample line bundle  $\det(\text{Lie}(\mathcal{H}))^*$  arising from the universal closed  $X$ -subgroup  $j : \mathcal{H} \hookrightarrow G \times_S X$  is canonically isomorphic to  $(\det_X(\Omega_{X/S}^1))^* \otimes f^*(\det_S \text{Lie}(G))^*$ .

Let  $\mathcal{I}_j$  be the ideal defining  $j$ , so we obtain an exact sequence of vector bundles on  $\mathcal{H}$ :

$$0 \rightarrow \mathcal{I}_j/\mathcal{I}_j^2 \rightarrow j^*(\Omega_{(G \times_S X)/X}^1) \rightarrow \Omega_{\mathcal{H}/X}^1 \rightarrow 0.$$

Pulling back along the identity section  $e : X \rightarrow \mathcal{H}$  yields an exact sequence

$$0 \rightarrow e^*(\mathcal{I}_j/\mathcal{I}_j^2) \rightarrow f^*(\mathrm{Lie}(G)^*) \rightarrow \mathrm{Lie}(\mathcal{H})^* \rightarrow 0$$

of vector bundles on  $X$ . But the left term is identified with  $\Omega_{X/S}^1$  due to the cartesian square

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{j} & G \times_S X \\ \pi \downarrow & & \downarrow a \\ X & \xrightarrow{\Delta_{X/S}} & X \times_S X \end{array}$$

in which  $a(g, x) := (gx, x)$  and  $\pi$  satisfies  $\pi \circ e = \mathrm{id}_X$ . Thus,  $\det \Omega_{X/S}^1 \otimes \det(\mathrm{Lie}(\mathcal{H}))^* \simeq f^*(\det_S \mathrm{Lie}(G))^*$ , yielding the asserted description of  $\det(\mathrm{Lie}(\mathcal{H}))^*$ .

*Proof of Theorem 2.3.6.* Since  $N_G(H) = H$  and  $H$  is smooth (so  $G \rightarrow G/H$  admits sections étale-locally on  $G/H$ ), it follows by effective descent for closed subschemes that the quotient sheaf  $G/H$  coincides with the functor of smooth closed subgroups of  $G$  that are conjugate to  $H$  étale-locally on the base. To prove the representability of this functor and its properties as asserted in (1) and (2), we may and do assume that  $S$  is noetherian.

For  $n \geq 0$ , let  $H_n$  denote the  $n$ th infinitesimal neighborhood of the identity in  $H$  (i.e., the closed subscheme defined by the  $(n+1)$ th power of the ideal of the identity section). The construction of  $N_G(H)$  in the proof of Proposition 2.1.6 gives a description of  $N_G(H)$  as an infinite descending intersection, namely the intersection of the normalizers of the closed subschemes  $H_n$  over all  $n$  (where we define the “normalizer” of  $H_n$  in the evident manner; this makes sense even though  $H_n$  is usually not a subgroup scheme of  $H$ , and it exists by Proposition 2.1.2). The noetherian property of  $G$  implies that the intersection stabilizes for large  $n$ . In other words, for sufficiently large  $n$  we have an equality

$$N_G(H) = N_G(H_n)$$

inside  $G$ . Fix such an  $n > 0$ . By hypothesis  $H = N_G(H)$ , so  $N_G(H_n) = H$ .

Since  $G$  and  $H$  are smooth, with  $H$  closed in  $G$ , we can write the finite flat  $S$ -schemes  $G_n$  and  $H_n$  in the form  $G_n = \mathrm{Spec}_S(A_n)$  and  $H_n = \mathrm{Spec}_S(B_n)$  for finite flat  $\mathcal{O}_S$ -algebras  $A_n$  and  $B_n$  with  $B_n$  a quotient of  $A_n$ . The degrees of  $H_n$  and  $G_n$  over  $S$  are given by some universal formulas in terms of  $n$  and the relative dimensions of  $H$  and  $G$ ; let  $N$  denote the degree of  $H_n$  over  $S$ . Consider the Grassmannian  $\mathrm{Gr}_N(G_n)$  that classifies quotient vector bundles of  $A_n$  with rank  $N$ . The conjugation action of  $G$  on itself induces an action of  $G$  on  $G_n$ , and hence an action of  $G$  on  $\mathrm{Gr}_N(G_n)$ . Under this action, the  $S$ -point  $\xi$  of  $\mathrm{Gr}_N(G_n)$  corresponding to  $H_n \subset G_n$  has functorial stabilizer  $N_G(H_n) = N_G(H) = H$ . We conclude that the orbit map  $G \rightarrow \mathrm{Gr}_N(G_n)$  through  $\xi$  identifies the quotient sheaf  $G/H$  for the étale (or equivalently, fppf) topology with a subfunctor of the projective  $S$ -scheme  $\mathrm{Gr}_N(G_n)$ . Now we can run

through the same argument with algebraic spaces (and Zariski's Main Theorem) as in the proof of Theorem 2.3.1 to conclude that  $G/H$  is a smooth  $S$ -scheme that is quasi-affine over  $\mathrm{Gr}_N(G_n)$ . This completes the proof of (1).

Finally, suppose that every geometric fiber  $G_{\bar{s}}/H_{\bar{s}} = (G/H)_{\bar{s}}$  is projective. The properness of  $G/H \rightarrow S$  in such cases is a consequence of the following general fact: if  $f : X \rightarrow S$  is a separated flat map of finite type (with  $S$  noetherian) and if the fibers  $X_s$  are proper and geometrically connected then  $f$  is proper. To prove this fact (which is a special case of [EGA, IV<sub>3</sub>, 15.7.10]), first use direct limit considerations to reduce to working over the local rings on  $S$ ; i.e., we can assume  $S$  is local. Then by fpqc descent for the properness property of morphisms (which we only need in the quasi-projective case, for which it reduces to the topological property of closedness for a locally closed immersion), we may assume  $S = \mathrm{Spec}(A)$  for a *complete* local noetherian ring  $A$ .

By a deep result of Grothendieck on algebraization for formal schemes (see [EGA, III<sub>1</sub>, 5.5.1]), properness of the special fiber  $X_0$  provides an open and closed subscheme  $Z \subset X$  with  $Z_0 = X_0$ . (The existence of  $Z$  is a simple consequence of the theorem on formal functions if we assume  $X$  is open in a proper  $A$ -scheme  $\bar{X}$ , as is automatic when  $X$  is quasi-projective. To see this, note that the open subscheme  $X_0 \subset \bar{X}_0$  is closed by properness of  $X_0$ , so it is the zero scheme of an idempotent  $e_0$  on  $\bar{X}_0$ . Idempotents uniquely lift through infinitesimal thickenings, so by the theorem on formal functions we can lift  $e_0$  uniquely to an idempotent  $e$  on  $\bar{X}$ . Then the open and closed subscheme  $\bar{Z}$  of  $\bar{X}$  defined by the vanishing of  $e$  meets  $X$  in an open and closed subscheme  $Z$  that does the job. Such an  $A$ -proper  $\bar{X}$  actually exists even when  $X$  is not quasi-projective, due to the Nagata compactification theorem, but that lies much deeper than Grothendieck's construction of  $Z$  in general. Anyway, we only need the case of quasi-projective  $X$ , namely  $G/H$  above.) Since  $f$  is an open and closed map (as it is flat and proper), so  $f(Z) = S$  because the local scheme  $S$  is connected, we conclude that  $Z_s$  is non-empty for all  $s \in S$ . But each  $X_s$  is connected by hypothesis, and  $Z_s$  is open and closed in  $X_s$ , so  $Z_s = X_s$  for all  $s$ . Hence,  $Z = X$ , so  $X$  is  $S$ -proper.

We conclude from the properness of  $G/H$  that the monomorphism  $i : G/H \rightarrow \mathrm{Gr}_N(G_n)$  constructed above over  $S$  is proper, and hence  $i$  is a *closed immersion* (compare with Remark 2.3.5). Let  $\mathcal{O}_{G_n}$  denote the structure sheaf on  $G_n$ , viewed as a vector bundle on  $S$ . The canonical  $S$ -ample line bundle  $\mathcal{N}$  on the Grassmannian equips  $G/H$  with a line bundle  $\mathcal{L} = i^*(\mathcal{N})$  that is  $S$ -ample (since  $i$  is a closed immersion). Although  $\mathcal{L}$  depends on the choice of Grassmannian  $\mathrm{Gr}_N(G_n)$  (i.e., depends on the choice of  $n$ ), so it is not canonically attached to  $(G, H)$ , nonetheless  $\mathcal{L}$  serves a useful purpose: we will prove that it is a positive power of  $\det(\mathrm{Lie}(\mathcal{H}))^*$ , so this dual determinant bundle is also  $S$ -ample, as desired.

By construction of the Plücker embedding of the Grassmannian,  $\mathcal{N}$  is the determinant of the universal rank- $N$  quotient bundle  $\mathcal{Q}$  of  $\mathcal{O}_{G_n}$  over  $\mathrm{Gr}_N(G_n)$ . The  $n$ th infinitesimal neighborhood  $\mathcal{H}_n$  of  $\mathcal{H}$  along its identity section is finite locally free over  $G/H$  of rank  $N$ , so the structure sheaf  $\mathcal{O}_{\mathcal{H}_n}$  may be viewed as a rank- $N$  vector bundle quotient of  $\mathcal{O}_{G_n \times (G/H)}$  over  $G/H$ . By the definition of  $i$ , we have  $i^*(\mathcal{Q}) = \mathcal{O}_{\mathcal{H}_n}$  as quotients of  $\mathcal{O}_{G_n \times (G/H)}$ , so  $\mathcal{L} = \det_{G/H}(\mathcal{O}_{\mathcal{H}_n})$ .

There is an evident filtration of  $\mathcal{O}_{H_n}$  by powers of the augmentation ideal of  $\mathcal{H}$ , and this filtration has successive quotients  $\mathrm{Sym}_{G/H}^i(\mathrm{Lie}(\mathcal{H})^*)$  for  $0 \leq i \leq n$ . For any vector bundle  $\mathcal{E}$  of rank  $r$ , naturally  $\det(\mathrm{Sym}^i(\mathcal{E})) \simeq (\det \mathcal{E})^{m(i,r)}$  for an exponent  $m(i,r) > 0$  depending only on  $i$  and  $r$ . Hence, the line bundle  $\mathcal{L} = \det \mathcal{O}_{\mathcal{H}_n}$  is a positive power of  $\det(\mathrm{Lie}(\mathcal{H})^*)$ , as desired. (The power depends on  $n$  and the relative dimensions of  $H$  and  $G$  over  $S$ .)  $\square$

## 2.4. Exercises.

**Exercise 2.4.1.** Let  $M$  be a finitely generated abelian group, and  $k$  a field. Prove that every closed  $k$ -subgroup scheme  $H$  of  $D_k(M)$  has the form  $D_k(M/N)$  for a subgroup  $N \subset M$ . (Hint: reduce to the case  $k = \bar{k}$ , so  $H_{\text{red}}$  is a *smooth* subgroup. By considering  $H_{\text{red}}^0$  and a decomposition of  $M$  into a product of a finite free  $\mathbf{Z}$ -module and a finite abelian group, first treat the case when  $M$  is free and  $H$  is smooth and connected. Then pass to  $D_k(M)/H_{\text{red}}^0$  in general to reduce to the case of finite  $H$ , for which Cartier duality can be used.)

**Exercise 2.4.2.** Let  $H \rightarrow S$  be a group of multiplicative type, and  $j : K \hookrightarrow H$  a quasi-finite closed subgroup. This exercise proves  $K$  is *finite* over  $S$  (a special case of [SGA3, IX, 6.4]).

(i) Reduce to the case of noetherian  $S$ . Using that proper monomorphisms are closed immersions reduce to  $S = \text{Spec } R$  for a discrete valuation ring  $R$  (hint: valuative criterion). Further reduce to the case when  $H = D_R(M)$  for a finitely generated abelian group  $M$ .

(ii) With  $S = \text{Spec } R$  as in (i), use Exercise 2.4.1 to show that the schematic closure in  $H$  of the generic fiber of  $K$  is  $D_R(M/M')$  for  $M' \subset M$  of finite index.

(iii) Pass to quotients by  $D_R(M/M')$  to reduce to the case  $M' = 0$ , and conclude.

**Exercise 2.4.3.** Let  $f : H \rightarrow G$  be a homomorphism from a multiplicative type group  $H$  to an affine group  $G$  of finite presentation over a scheme  $S$ . Prove as follows that  $K := \ker f$  is multiplicative type (so  $f$  factors through the multiplicative type fppf quotient  $H/K$  that is an  $S$ -subgroup of  $G$ ); this is part of [SGA3, IX, 6.8].

(i) Reduce to noetherian  $S$ , split  $H = D_S(M)$ , and  $S = \text{Spec } A$  for local noetherian  $A$ .

(ii) Prove that  $K$  is closed in  $H$ , so by Exercise 2.4.1 the special fiber  $K_0$  equals  $D_k(M/M')$  for a subgroup  $M' \subset M$  (with  $k$  the residue field of  $A$ ). Prove that the map  $D_A(M/M') \rightarrow G$  vanishes (hint: use Corollary B.3.5), so  $D_A(M/M') \subset K$ .

(iii) Replace  $H$  with  $H/D_A(M/M') = D_A(M')$  so that  $K_0 = 1$ . By considering the special fiber of each *finite* (perhaps non-flat?)  $S$ -group  $K[n]$  with  $n \geq 1$ , prove  $K_s = 1$  for all  $s \in S$ . Use Lemma B.3.1 to show that  $e : S \rightarrow K$  is an isomorphism.

(iv) Relax the affineness hypothesis on  $G$  to separatedness.

**Exercise 2.4.4.** Let  $X, Y$ , and  $Z$  be schemes over a ring  $k$ , and  $\alpha : X \times Y \rightarrow Z$  a  $k$ -morphism. For a closed subscheme  $\iota : Z' \hookrightarrow Z$ , let  $\text{Transp}_X(Y, Z')$  be the functor of points  $x \in X(R)$  such that  $\alpha(x, \cdot)$  carries  $Y_R$  into  $Z'_R$  for  $k$ -algebras  $R$ .

(i) If  $Y$  is affine and  $k[Y]$  is  $k$ -free, prove  $\text{Transp}_X(Y, Z')$  is represented by a closed subscheme  $\text{Transp}_X(Y, Z')$  of  $X$ . (Hint: Reduce to affine  $X$ . If  $\{e_i\}$  is a  $k$ -basis of  $k[Y]$  and  $\{f_j\}$  generates the ideal in  $k[X \times Y] = k[X] \otimes_k k[Y]$  of the pullback of  $\iota$ , with  $f_j = \sum h_{ij} \otimes e_i$ , consider the zero scheme of the  $h_{ij}$  in  $X$ .) Remove the affineness hypothesis on  $Y$  if  $k$  is artinian and  $Y$  is  $k$ -flat (and see [SGA3, VIII, §6] for further generalizations).

(ii) Let  $Y$  be a closed subscheme of a  $k$ -group  $G$ . Using (i) with  $\Delta_{G/k}$  as  $\iota$ , construct a closed subscheme  $Z_G(Y) \subset G$  if  $k$  is a field or if  $Y$  is affine and  $k[Y]$  is  $k$ -free. Discuss the case  $Y = G$ . For a closed subscheme  $Y' \subset G$ , construct  $\text{Transp}_G(Y, Y')$ . How about  $N_G(Y)$ ?

(iii) Use (ii) to compute equations for  $N_G(G') \subset G$  over  $k = \mathbf{Z}[t]$  in Example 2.1.8.

(iv) Consider a finite type group  $G$  over a field  $k = k_s$  and a closed subscheme  $Y \subset G$ . If  $Y$  is smooth then prove  $Z_G(Y) = \bigcap_{y \in Y(k)} Z_G(y)$ , and if  $G$  is smooth then prove  $Y$  is normalized (resp. centralized) by  $G$  if it is normalized (resp. centralized) by  $G(k)$ .

**Exercise 2.4.5.** Let  $\pi : G \rightarrow S$  a finite group scheme, with  $\pi_*(\mathcal{O}_G)$  locally free over  $\mathcal{O}_S$ . Let  $G'$  be an affine  $S$ -group of finite presentation. Recall that the functor  $\underline{\text{Hom}}(X, Y)$  is represented by an affine  $S$ -scheme of finite presentation under the hypotheses of the first part of Lemma 2.1.3. Use several fiber products to represent  $\underline{\text{Hom}}_{S\text{-gp}}(G, G')$  by an affine  $S$ -scheme of finite presentation, thereby proving the second part of Lemma 2.1.3.

**Exercise 2.4.6.** (i) Let  $G$  be  $\text{SL}_n$  or  $\text{PGL}_n$  over a ring  $k$  and  $T$  the diagonal torus, or let  $G = \text{Sp}_{2n}$  and  $T$  the torus of points  $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$  for diagonal  $t \in \text{GL}_n$ . In all cases prove  $Z_G(T) = T$  (so  $T$  is a maximal torus on all geometric fibers) by using the Lie algebra.

(ii) Using (i), prove  $Z_{\text{SL}_n} = \mu_n$ ,  $Z_{\text{PGL}_n} = 1$ , and  $Z_{\text{Sp}_{2n}} = \mu_2$  as schemes.

**Exercise 2.4.7.** Consider a field  $k$  and a  $k$ -group  $H$  acting on a separated  $k$ -scheme  $Y$ . For a  $k$ -scheme  $S$ , let  $Y^H(S)$  be the set of  $y \in Y(S)$  invariant by the  $H_S$ -action on  $Y_S$ .

(i) Adapt Exercise 2.4.4(ii) to prove that  $Y^H$  is represented by a closed subscheme of  $H$ . In case  $H$  is smooth and  $k = k_s$ , prove  $Y^H = \bigcap_{h \in H(k)} Y^h$  where  $Y^h := \alpha_h^{-1}(\Delta_{Y/k})$  for  $\alpha_h : Y \rightarrow Y \times Y$  the map  $y \mapsto (y, h.y)$ . Relate this to Exercise 1.6.6.

(ii) For  $Y$  of finite type over  $k$  and  $y \in Y^H(k)$ , prove  $\text{Tan}_y(Y^H) = \text{Tan}_y(Y)^H$  for a suitable  $H$ -action on  $\text{Tan}_y(Y)$ .

(iii) Assume  $H$  is a closed subgroup of a  $k$ -group  $G$  of finite type. Let  $\mathfrak{g} := \text{Lie}(G)$  and  $\mathfrak{h} := \text{Lie}(H)$ . Prove  $\text{Tan}_e(Z_G(H)) = \mathfrak{g}^H$  (invariants under the adjoint action). Also prove  $\text{Tan}_e(N_G(H)) = \bigcap_{h \in H(k)} (\text{Ad}_G(h) - 1)^{-1}(\mathfrak{h})$  when  $k = k_s$  and  $H$  is smooth.

**Exercise 2.4.8.** Let  $G$  be a smooth connected affine group over a field  $k$ . For a smooth connected  $k$ -subgroup  $H \subset G$ , the proof of Theorem 2.3.6 constructs  $n > 0$  so that the  $n$ th-order infinitesimal neighborhood  $H_n$  of 1 in  $H$  satisfies  $N_G(H_n) = N_G(H)$ .

(i) If  $G$  is reductive and  $P \subset G$  is parabolic then the first-order neighborhood suffices: prove  $\text{Lie}(N_G(\text{Lie}(P))) = \text{Lie}(P)$  (hint: consider weight spaces in  $\text{Lie}(G)$  for a maximal torus  $T$  contained in  $P$  with  $k = \bar{k}$ , and don't confuse  $\text{Lie}(N_G(\text{Lie}(P)))$  with  $N_{\text{Lie}(G)}(\text{Lie}(P))$ ), and deduce that  $P = N_G(\text{Lie}(P))$  as schemes. (Recall that  $P$  is connected.)

(ii) Prove that all maximal tori in  $\text{SL}_2$  over a field of characteristic 2 have the *same* Lie algebra in  $\mathfrak{sl}_2$ . Find an explicit  $n$  so that  $N_{\text{SL}_2}(D_n) = N_{\text{SL}_2}(D)$  for the diagonal torus  $D$ .

**Exercise 2.4.9.** Let  $\Gamma = \text{Gal}(k_s/k)$  for a field  $k$ . For a  $k$ -group  $T$  of multiplicative type, the *character group*  $X(T) = \text{Hom}_{k_s}(T_{k_s}, \mathbf{G}_m)$  is a discrete  $\Gamma$ -module in an evident manner.

(i) If  $k'/k$  is a finite separable subextension of  $k_s$ , prove that the Weil restriction  $R_{k'/k}(T')$  is of multiplicative type over  $k$  when  $T'$  is over  $k'$ . (For  $T' = \mathbf{G}_m$  this is “ $k'^{\times}$  viewed as a  $k$ -group”.) By functorial considerations, prove  $X(R_{k'/k}(T')) = \text{Ind}_{\Gamma'}^{\Gamma}(X(T'))$  with  $\Gamma' = \text{Gal}(k_s/k')$ . For every  $k$ -torus  $T$ , construct a surjective  $k$ -homomorphism  $\prod_i R_{k'_i/k}(\mathbf{G}_m) \twoheadrightarrow T$  for finite separable extensions  $k'_i/k$ . Conclude that  $k$ -tori are *unirational* over  $k$ .

(ii) For a local field and  $k$ -torus  $T$ , prove  $T$  is  $k$ -anisotropic if and only if  $T(k)$  is compact.

(iii) For a finite extension field  $k'/k$ , define a *norm* map  $N_{k'/k} : R_{k'/k}(\mathbf{G}_m) \rightarrow \mathbf{G}_m$ . Prove its kernel is a torus when  $k'/k$  is separable. What if  $k'/k$  is not separable?

**Exercise 2.4.10.** Let  $X$  be a smooth separated scheme locally of finite type over a field  $k$ , and  $T$  a  $k$ -group of multiplicative type with a left action on  $X$ . This exercise is devoted to proving that  $X^T$  (as in Exercise 2.4.7) is *smooth*, generalizing Lemma 2.2.4 over fields.

(i) Reduce to the case  $k = \bar{k}$ . Fix a finite local  $k$ -algebra  $R$  with residue field  $k$ , and an ideal  $J$  in  $R$  with  $J\mathfrak{m}_R = 0$ . Choose  $\bar{x} \in X^T(R/J)$ , and for  $R$ -algebras  $A$  let  $E(A)$  be the fiber of  $X(A) \rightarrow X(A/JA)$  over  $\bar{x}_{A/JA}$ . Let  $x_0 = \bar{x} \bmod \mathfrak{m}_R \in X^T(k)$  and  $A_0 = A/\mathfrak{m}_R A$ . Prove  $E(A) \neq \emptyset$  and make it a torsor over the  $A_0$ -module  $F(A) := JA \otimes_k \mathrm{Tan}_{x_0}(X) = JA \otimes_{A_0} (A_0 \otimes_k \mathrm{Tan}_{x_0}(X))$  naturally in  $A$  (action  $v.x$  denoted as  $v + x$ ).

(ii) Define an  $A_0$ -linear  $T(A_0)$ -action on  $F(A)$  (hence a  $T_R$ -action on  $F$ ), and prove that  $E(A)$  is  $T(A)$ -stable in  $X(A)$  with  $t.(v + x) = t_0.v + t.x$  for  $x \in E(A)$ ,  $t \in T(A)$ ,  $v \in F(A)$ , and  $t_0 = t \bmod \mathfrak{m}_R A$ .

(iii) Choose  $\xi \in E(R)$  and define a map of functors  $h : T_R \rightarrow F$  by  $t.\xi = h(t) + \xi$  for points  $t$  of  $T_R$ ; check it is a 1-cocycle, and is a 1-coboundary if and only if  $E^{T_R}(R) \neq \emptyset$ . For  $V_0 = J \otimes_k \mathrm{Tan}_{x_0}(X)$  use  $h$  to define a 1-cocycle  $h_0 : T \rightarrow \underline{V}_0$ , and prove  $t.(v, c) := (t.v + ch_0(t), c)$  is a  $k$ -linear representation of  $T$  on  $V_0 \oplus k$ . Use a  $T$ -equivariant splitting (!) to prove  $h_0$  (and then  $h$ ) is a 1-coboundary; deduce  $X^T$  is smooth.

**Exercise 2.4.11.** Let  $S$  be a scheme. For any fppf affine  $S$ -group  $\mathcal{G}$ , let  $H^1(S, \mathcal{G})$  be the set of isomorphism classes of right  $\mathcal{G}$ -torsors over  $S$  for the fppf topology. For any homomorphism  $\mathcal{G} \rightarrow \mathcal{G}'$  between such groups, define  $H^1(S, \mathcal{G}) \rightarrow H^1(S, \mathcal{G}')$  via pushout for torsors:  $E \mapsto E \times^{\mathcal{G}} \mathcal{G}'$  (the quotient of  $E \times \mathcal{G}'$  by the anti-diagonal  $\mathcal{G}$ -action  $(e, g').g = (e.g, g^{-1}g')$ ).

(i) Prove that any right  $\mathcal{G}$ -torsor  $E$  is necessarily affine (and fppf), and that the quotient  $E \times^{\mathcal{G}} \mathcal{G}'$  exists as a scheme. Also use the affineness to prove that  $H^1(S, \mathcal{G})$  can be computed (functorially in  $\mathcal{G}$ !) by a non-commutative version of the usual Čech-type procedure generalizing non-abelian degree-1 Galois cohomology.

(ii) For any fppf  $S$ -affine  $S$ -group  $G$  and fppf closed  $S$ -subgroup  $H$  such that the fppf quotient sheaf  $G/H$  is represented by a scheme (see Theorem 2.3.6 for a sufficient criterion), identify  $G(S) \backslash (G/H)(S)$  with the kernel of the map  $H^1(S, H) \rightarrow H^1(S, G)$ .

**Exercise 2.4.12.** For  $n \geq 1$ , let  $X_n = \mathrm{Spec} A_n$  be the  $\mathbf{C}$ -scheme obtained by gluing  $2^n$  affine lines in a loop, with 0 on the  $i$ th line glued to 1 on the  $(i + 1)$ th line ( $i \in \mathbf{Z}/2^n\mathbf{Z}$ ).

(i) Prove that  $\pi_1(X_n) = \widehat{\mathbf{Z}}$  and the finite étale covers of  $X_n$  split Zariski-locally on  $X_n$ .

(ii) Define  $X_{n+1} \rightarrow X_n$  by collapsing odd-indexed lines to points and sending the  $2j$ th line in  $X_{n+1}$  to the  $j$ th line in  $X_n$ . Prove  $X_\infty := \varprojlim X_n$  is reducible and its local rings are discrete valuation rings or fields. (This is a slight variant of an example in the Stacks Project.)

(iii) Construct a nontrivial  $\mathbf{Z}$ -torsor  $E_1 \rightarrow X_1$  that is split Zariski-locally on  $X_1$  but not by any finite étale cover of  $X_1$ . Prove  $E_\infty := E_1 \times_{X_1} X_\infty \rightarrow X_\infty$  is a non-split  $\mathbf{Z}$ -torsor that splits Zariski-locally on  $X_\infty$ , and construct a rank-2 torus  $T \rightarrow X_\infty$  that splits Zariski-locally on  $X_\infty$  but is not isotrivial. Thus, “irreducible” cannot be relaxed to “connected” in Corollary B.3.6. (The preceding construction was suggested by Gabber.)

## 3. BASIC GENERALITIES ON REDUCTIVE GROUP SCHEMES

3.1. **Reductivity and semisimplicity.** In [SGA3, XIX, 2.7] a connectedness condition is imposed at the start of the relative theory of reductive groups:

**Definition 3.1.1.** Let  $S$  be a scheme. A *reductive  $S$ -group* is a smooth affine group scheme  $G \rightarrow S$  such that the geometric fibers  $G_{\bar{s}}$  are connected reductive groups. A *semisimple  $S$ -group* is a reductive  $S$ -group whose geometric fibers are semisimple.

In the theory of linear algebraic groups  $G$  over an algebraically closed field  $k$ , reductive groups are often permitted to be disconnected. One reason is that if  $g \in G(k)$  is semisimple then  $Z_G(g)$  may be disconnected (as happens already for  $\mathrm{PGL}_2$ ) but  $Z_G(g)^0$  is always reductive. Also, the Galois cohomological classification of connected semisimple groups  $G$  over a field  $k$  leads to the consideration of the automorphism scheme  $\mathrm{Aut}_{G/k}$ , and this is a smooth affine  $k$ -group whose identity component is semisimple but is usually disconnected when the Dynkin diagram has nontrivial automorphisms (e.g., if  $n > 2$  then  $\mathrm{Aut}_{\mathrm{SL}_n/k} = \mathrm{PGL}_n \rtimes \mathbf{Z}/2\mathbf{Z}$  with component group generated by transpose-inverse). In §7.1 we will discuss the existence and structure of automorphism schemes of reductive group schemes.

In the relative theory over a scheme that is not a single point, the disconnectedness of fibers presents new phenomena not seen in the classical case. For example, if  $G \rightarrow S$  is a smooth affine group scheme then the orders of the geometric fibral component groups  $\pi_0(G_{\bar{s}})$  can vary with  $\bar{s}$ , so these component groups can fail to arise as the fibers of a *finite étale  $S$ -group* (see Example 3.1.4).

Requiring connectedness of fibers is not unreasonable. A connected group scheme of finite type over a field is always geometrically connected (as for any connected finite type scheme  $X$  over a field  $k$  when  $X(k)$  is non-empty), so for a group scheme of finite type the property of having connected fibers is preserved by any extension of the ground field. Also, identity components vary well in smooth families of groups: for any smooth group scheme  $G \rightarrow S$  of finite presentation there exists a unique open subgroup scheme  $G^0 \subset G$  such that  $(G^0)_s$  is the identity component of  $G_s$  for all  $s \in S$  [EGA, IV<sub>3</sub>, 15.6.5]. The formation of  $G^0$  commutes with any base change on  $S$  since each  $G_s^0$  is geometrically connected, and by reduction to the case of noetherian  $S$  we see that  $G^0$  is finitely presented over  $S$ . Imposing connectedness of fibers amounts to passing to  $G^0$  in place of  $G$ . Beware that passage to  $G^0$  can exhibit some peculiar behavior relative to the theory over a field:

**Example 3.1.2.** For smooth affine  $S$ -groups  $G$ , the open subgroup  $G^0$  may not be closed. An interesting example is given in [SGA3, XIX, §5] over  $\mathrm{Spec} k[t]$  for any field  $k$  of characteristic 0. In that example  $G^0$  is affine, but there are smooth  $S$ -affine  $G$  for which  $G^0$  is not  $S$ -affine (so it is not closed in  $G$ ). See [Ra, VII, §3, (iii)] for such an example over  $S = \mathbf{A}_k^2$  with  $k$  any field of characteristic 0.

It turns out that with a reductive hypothesis on fibral identity components, the problems in Example 3.1.2 do not arise. This is made precise by the following result that we will never use and which rests on many later developments in the theory:

**Proposition 3.1.3.** *Let  $G \rightarrow S$  be a smooth separated group scheme of finite presentation such that  $G_{\bar{s}}^0$  is reductive for all  $s \in S$ . Then  $G^0$  is a reductive  $S$ -group that is open and closed in  $G$ , and  $G/G^0$  exists as a separated étale  $S$ -group of finite presentation.*

*Proof.* The open subgroup  $G^0 \rightarrow S$  is smooth and finitely presented with connected reductive fibers. Incredibly, in the definition of a reductive group scheme we can replace “affine” with “finitely presented” [SGA3, XVI, 5.2(i)], so  $G^0$  is a reductive  $S$ -group (in particular,  $G^0$  is  $S$ -affine). Moreover, in Theorem 5.3.5 we will show that any monic homomorphism from a reductive group scheme to a finitely presented and separated group scheme is always a closed immersion. Hence, for such  $G$  we see that  $G^0$  is both open *and* closed in  $G$ , so the quotient  $E := G/G^0$  that is initially a finitely presented and étale algebraic space over  $S$  is also *separated* over  $S$ . Thus, after a reduction to the case of noetherian  $S$ , we may apply Knutson’s criterion (as in the proof of Theorem 2.3.1) to conclude that  $E$  is a scheme.  $\square$

**Example 3.1.4.** The conclusion in Proposition 3.1.3 is “best possible”, in the sense that the relative component group  $G/G^0$ , which is always quasi-finite, separated, and étale over  $S$ , may not be finite over  $S$ . For example, let  $S$  be a connected  $\mathbf{Z}[1/2]$ -scheme and  $O(q)$  the orthogonal group of a nondegenerate quadratic space  $(V, q)$  over  $S$ , so  $O(q)^0 = \mathrm{SO}(q)$  and  $O(q)/\mathrm{SO}(q) = (\mathbf{Z}/2\mathbf{Z})_S$ . Let  $U \subset S$  be a finitely presented non-empty open subscheme with  $U \neq S$ , so  $U$  is not closed in  $S$ . The open subgroup  $E \subset (\mathbf{Z}/2\mathbf{Z})_S$  obtained by removing the closed non-identity locus over  $S - U$  is not  $S$ -finite, so its open preimage  $G \subset O(q)$  satisfies the hypotheses in Proposition 3.1.3 but  $G/G^0 = E$  is not  $S$ -finite.

In the classical theory of connected reductive groups, it is a fundamental fact that torus centralizers are again connected reductive [Bo91, 13.17, Cor.2]. In the relative case this remains valid: if  $G$  is a reductive  $S$ -group scheme and  $T$  is an  $S$ -torus in  $G$  then the closed subgroup  $Z_G(T)$  is  $S$ -smooth (see Lemma 2.2.4) and its geometric fibers are connected reductive by the classical theory, so  $Z_G(T)$  is reductive over  $S$ . This can be pushed a bit further, as explained in the following Remark that we will never use.

**Remark 3.1.5.** Let  $H$  be any multiplicative type subgroup of a reductive  $S$ -group  $G$ . The closed  $S$ -subgroup  $Z_G(H)$  in  $G$  is smooth (by Lemma 2.2.4) and its geometric fibers  $Z_G(H)_{\bar{s}}$  have reductive identity component by [CGP, Prop. A.8.12] (whose proof lies deeper than the case of torus centralizers, resting on an affineness result for coset spaces modulo connected reductive groups; see [Bo85] or [Ri]). However, the fibers of  $Z_G(H)$  can be disconnected. By Proposition 3.1.3, the open subgroup  $Z_G(H)^0$  is reductive (in particular, affine) over  $S$  and closed in  $Z_G(H)$ . The reader is referred to [CGP, Prop. A.8.10] for a wider class of  $H$  for which these conclusions remain valid (e.g., finite étale  $H$ , possibly non-commutative, whose order is a unit on the base).

To affirm that the notion of reductive group scheme is reasonable, we want to prove that reductivity of a fiber is inherited by nearby fibers for any smooth affine group scheme with connected fibers. This requires an improvement on the lifting of tori over adic noetherian rings in Corollary B.3.5, replacing completions with étale neighborhoods:

**Proposition 3.1.6.** *Let  $G \rightarrow S$  be a smooth affine group scheme, and  $H_0$  a multiplicative type subgroup of the fiber  $G_s$  over some  $s \in S$ . There exists an étale neighborhood  $(S', s')$  of  $(S, s)$  with  $k(s') = k(s)$  and a multiplicative type subgroup  $H' \subset G_{S'}$  such that  $H'_{s'} = H_0$  inside  $(G_{S'})_{s'} = G_s$ .*

*Proof.* We may assume  $S$  is noetherian and affine, and even finite type over  $\mathbf{Z}$  (by expressing a noetherian ring as a direct limit of its  $\mathbf{Z}$ -subalgebras of finite type; see [EGA, IV<sub>4</sub>, 17.8.7] for the descent of smoothness through such direct limits). Let  $A$  denote the completion  $\mathcal{O}_{S,s}^\wedge$ . By Corollary B.3.5, there exists a multiplicative type  $A$ -subgroup  $\widehat{H}$  in  $G_A$  lifting  $H_0$  in the special fiber  $G_s$ .

Let  $\{A_\alpha\}$  be the directed system of finite type  $\mathcal{O}_{S,s}$ -subalgebras of  $A$ , so  $A = \varinjlim A_\alpha$ . For sufficiently large  $\alpha$ , the closed  $A$ -subgroup  $\widehat{H} \subset G_A$  of multiplicative type descends to a closed  $A_\alpha$ -subgroup  $H_\alpha \subset G_\alpha := G_{A_\alpha}$ . By increasing  $\alpha$  we can arrange that  $H_\alpha$  is multiplicative type. Indeed, there exists an fppf cover  $\text{Spec } A' \rightarrow \text{Spec } A$  such that  $\widehat{H}_{A'} \simeq D_{A'}(M)$  for some finitely generated abelian group  $M$ , and by increasing  $\alpha$  we can arrange that  $\text{Spec } A' \rightarrow \text{Spec } A$  descends to an fppf cover  $\text{Spec } A'_\alpha \rightarrow \text{Spec } A_\alpha$ . Then the  $A'_\alpha$ -groups  $H_\alpha \otimes_{A_\alpha} A'_\alpha$  and  $D_{H'_\alpha}(M)$  become isomorphic over  $A'$ . However,  $A'$  is the direct limit of the rings  $A'_\beta := A'_\alpha \otimes_{A_\alpha} A_\beta$  over  $\beta \geq \alpha$ , so by increasing  $\alpha$  some more we can arrange that  $H_\alpha \otimes_{A_\alpha} A'_\alpha \simeq D_{A'_\alpha}(M)$ . Since  $\text{Spec } A'_\alpha$  is an fppf cover of  $\text{Spec } A_\alpha$ , it follows that  $H_\alpha$  is multiplicative type. Fix such an  $\alpha$ . Since  $\mathcal{O}_{S,s}$  is essentially of finite type over  $\mathbf{Z}$ , and  $A_\alpha$  is finite type over  $\mathcal{O}_{S,s}$ , we can apply the powerful Artin approximation theorem:

**Theorem 3.1.7** (Artin). *Let  $R$  be a local ring that is essentially of finite type over  $\mathbf{Z}$ , and  $B$  a finite type  $R$ -algebra equipped with a map  $f : B \rightarrow \widehat{R}$  over  $R$ . Pick  $N \geq 0$ .*

*The map  $f$  admits an  $N$ th-order “étale” approximation over  $R$  in the sense that there exists a residually trivial local-étale extension  $R \rightarrow R'$  and an  $R$ -algebra map  $\varphi : B \rightarrow R'$  such that the induced map to the completion*

$$\widehat{\varphi} : B \rightarrow \widehat{R}' = \widehat{R}$$

*agrees with  $f$  modulo  $\mathfrak{m}_{\widehat{R}}^{N+1}$ .*

Loosely speaking, this theorem says that any solution in  $\widehat{R}$  to a finite system of polynomial equations over  $R$  is well-approximated by a solution in the henselization of  $R$  (equivalently, a solution in some residually trivial local-étale extension of  $R$ ). Theorem 3.1.7 is a special case of the Artin approximation theorem; see [BLR, 3.6/16] for the proof of that theorem.

We apply Theorem 3.1.7 to  $R = \mathcal{O}_{S,s}$ ,  $B = A_\alpha$ ,  $N = 0$ , and the inclusion  $B \rightarrow \widehat{R}$  to obtain a residually trivial local-étale extension  $R \rightarrow R'$  and an  $R$ -algebra map  $A_\alpha \rightarrow R'$  that agrees residually with the reduction of the given inclusion  $A_\alpha \hookrightarrow \widehat{R}$ . Thus, the  $R'$ -group  $H_\alpha \otimes_{A_\alpha} R'$  of multiplicative type in  $G_{R'}$  has special fiber  $H_0$  in  $G_s$ ! In other words, we have found a multiplicative type subgroup lifting  $H_0$  over a local-étale neighborhood of  $(S, s)$ . Spreading this out over an étale neighborhood of  $(S, s)$  then does the job.  $\square$

**Corollary 3.1.8.** *Let  $G \rightarrow S$  be an fppf affine group scheme with connected fibers, and assume that  $G_s$  is a torus for some  $s \in S$ . Then  $G_U$  is a torus for some open neighborhood  $U$  of  $s$  in  $S$ .*

This result is [SGA3, X, 4.9].

*Proof.* First we assume that  $G$  is smooth, and then we reduce the general case to the smooth case. Since the property of being a torus is étale-local on the base, we may work in an étale neighborhood of  $(S, s)$ . Hence, by the smoothness of  $G$  we can use Proposition 3.1.6 to arrange that  $G$  contains a multiplicative type subgroup  $H$  such that  $H_s = G_s$ , and that  $G$  has constant fiber dimension. Passing to a further étale neighborhood makes  $H$  split, say  $H = D_S(M)$  for a finitely generated abelian group  $M$ . Since  $H_s$  is a torus,  $M$  is free. Hence,  $H$  is a torus. But the inclusion  $H \hookrightarrow G$  between smooth affine groups with connected fibers is an equality on  $s$ -fibers, so by smoothness and constancy of fiber dimensions over  $S$  it follows that  $H_{s'} = G_{s'}$  for all  $s' \in S'$ . That is,  $H \rightarrow G$  induces an isomorphism on fibers, so it is an isomorphism by the fibral isomorphism criterion (Lemma B.3.1).

In general (with  $G$  only assumed to be fppf rather than smooth over  $S$ ), we just need to prove that  $G$  is automatically smooth over an open neighborhood of  $S$ . We may reduce to the case when  $S$  is local noetherian, and the smoothness of the  $s$ -fiber and the fppf hypothesis on  $G$  implies that  $G \rightarrow S$  is smooth at all points of  $G_s$ , and so on an open neighborhood of  $G_s$  in  $S$ . This open neighborhood has open image in  $S$ , so by shrinking we can arrange that this open image is equal to  $S$ . In particular, each fiber group scheme  $G_{s'}$  has a non-empty smooth locus, so the fibers are smooth (due to homogeneity considerations on geometric fibers).  $\square$

**Proposition 3.1.9.** *Let  $G \rightarrow S$  be a smooth affine group scheme and suppose  $G_s^0$  is reductive for some  $s \in S$ .*

- (1) *There is an open  $U$  around  $s$  in  $S$  such that  $G_u^0$  is reductive for all  $u \in U$ . The same holds for semisimplicity.*
- (2) *If  $T \subset G$  is a torus such that  $T_s$  is maximal in  $G_s^0$  for some  $s \in S$  then there exists an open  $V$  around  $s$  in  $S$  such that  $Z_G(T)_v^0 = T_v$  and  $T_v$  is a maximal torus in  $G_v^0$  for all  $v \in V$ .*

This result is essentially [SGA3, XIX, 2.6] (where it is assumed that all fibers  $G_s$  are connected). Note that in part (2) we are *assuming* the existence of  $T$ . It is not evident that  $T$  exists even étale-locally around  $s$  (though we will prove this to be the case for reductive  $G$  in Corollary 3.2.7; see [SGA3, XIV, 3.20] for a generalization using the Zariski topology, building on Theorem A.1.1).

*Proof.* First we prove (2). The centralizer  $Z_G(T)$  in  $G$  is a smooth closed  $S$ -subgroup by Lemma 2.2.4, and obviously  $T \subset Z_G(T)^0$ . By working Zariski-locally around  $s$ , we may assume that the smooth  $S$ -groups  $T \rightarrow S$  and  $Z_G(T) \rightarrow S$  have constant fiber dimension. These fiber dimensions agree at  $s$ , so they agree on all fibers. For any  $\xi \in S$ , the closed subgroup  $Z_{G_\xi}(T_\xi) = Z_G(T)_\xi$  in  $G_\xi$  is smooth and contains  $T_\xi$ . But the dimensions agree,

so  $T_\xi = (Z_G(T)_\xi)^0 = (Z_G(T)^0)_\xi$  for all  $\xi \in S$ . We conclude that the  $S$ -map  $T \hookrightarrow Z_G(T)^0$  between smooth  $S$ -schemes is an isomorphism on fibers over  $S$ , so it is an isomorphism (Lemma B.3.1).

Although  $G^0$  might not be affine, for any torus  $T' \subset G$  it is clear that  $G^0 \cap Z_G(T')$  represents  $\underline{Z}_{G^0}(T')$ , so we denote it as  $Z_{G^0}(T')$ . In the classical theory it is shown that the centralizer of *any* torus in a smooth connected affine group over an algebraically closed field is connected, so  $Z_{G^0}(T) = Z_G(T)^0 = T$ . Also, for any geometric point  $\bar{\xi}$  over  $\xi \in S$  we have  $T_{\bar{\xi}} = Z_{G^0}(T)_{\bar{\xi}} = Z_{G^0_{\bar{\xi}}}(T_{\bar{\xi}})$ , so  $T_{\bar{\xi}}$  must be maximal as a torus in  $G^0_{\bar{\xi}}$ . This proves (2).

We next turn to (1), and we may assume  $G^0_s \neq 1$ . Fix an algebraic geometric point  $\bar{s} = \text{Spec}(\bar{k}(s))$  over  $s$ . Any maximal torus in  $G^0_{\bar{s}}$  descends to a split torus in  $G^0_K$  for some finite extension  $K/k(s)$  contained in  $\bar{k}(s)$ , and it is harmless to work fppf-locally around  $s$ . Thus, we can pass to an fppf neighborhood of  $(S, s)$  to increase  $k(s)$  to contain such a  $K$ . Now  $G^0_s$  contains a split torus  $T_s$  such that  $T_{\bar{s}}$  is maximal in  $G^0_{\bar{s}}$ . By Proposition 3.1.6, we may make a further étale base change on  $S$  around  $s$  to get to the case that  $T_s$  lifts to a torus  $T$  in  $G$  (hence in  $G^0$ ), and that  $T$  is even split. By (2), after some further Zariski-localization around  $s$  we may assume  $Z_{G^0}(T) = T$  and that

$$T \simeq D_S(M) := \underline{\text{Hom}}_{S\text{-gp}}(M_S, \mathbf{G}_m)$$

for a finite free  $\mathbf{Z}$ -module  $M$ .

The  $T$ -action on the vector bundle  $\mathfrak{g} = \text{Lie}(G) = \text{Lie}(G^0)$  over  $S$  decomposes it into a direct sum of quasi-coherent weight spaces indexed by elements of  $M$ , and the formation of these weight spaces commutes with base change on  $S$ . These weight spaces are vector bundles (being direct summands of  $\mathfrak{g}$ ), so the weight space decomposition on the  $s$ -fiber encodes the weight spaces on the nearby fibers (by Nakayama's Lemma at  $s$ ). Thus, by shrinking around  $s$  we can arrange that all weight spaces have constant rank, so the only characters  $m \in M$  for which the weight space  $\mathfrak{g}_m$  is nonzero are  $m = 0$  and the elements of the root system  $\Phi = \Phi(G^0_s, T_s)$  for the connected reductive geometric fiber  $G^0_{\bar{s}}$ .

For each  $\xi \in S$ , the weight space  $\mathfrak{g}_0$  for the trivial weight has  $\xi$ -fiber

$$\text{Lie}(Z_{G^0_s}(T_\xi)) = \text{Lie}(Z_{G^0}(T)_\xi) = \text{Lie}(T_\xi) = \text{Lie}(T)_\xi,$$

but the subbundle  $\mathfrak{t} := \text{Lie}(T)$  in  $\mathfrak{g}$  is clearly contained in  $\mathfrak{g}_0$ , so the inclusion  $\mathfrak{t} \subset \mathfrak{g}_0$  between subbundles must be an equality over  $S$  for rank reasons. In other words, the weight space decomposition over  $S$  is

$$\mathfrak{g} = \mathfrak{t} \bigoplus \left( \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right)$$

where each  $\mathfrak{g}_\alpha$  is a line bundle on  $S$ .

For each  $\alpha \in \Phi$ , let  $T_\alpha \subset T = D_S(M)$  be the unique relative codimension-1 subtorus contained in  $\ker \alpha$ ; explicitly,  $T_\alpha = D_S(M/L)$  where  $L \subset M$  is the saturation of  $\mathbf{Z}\alpha$  in  $M$ . Since  $\Phi$  is a reduced root system (by the classical theory, applied to  $(G^0_{\bar{s}}, T_{\bar{s}})$ ), elements of  $\Phi$  apart from  $\pm\alpha$  are linearly independent from  $\alpha$  and so cannot vanish on any fiber of  $T_\alpha$ .

By the following lemma, which is a variant of Exercise 1.6.15(ii), a geometric fiber  $G_{\bar{\xi}}^0$  is reductive provided that each fiber  $Z_{G^0}(T_\alpha)_{\bar{\xi}} = Z_{G^0}((T_{\bar{\xi}})_\alpha)$  is reductive.

**Lemma 3.1.10.** *Let  $G$  be a smooth connected affine group over an algebraically closed field  $k$ , and  $T$  a maximal torus in  $G$  such that  $Z_G(T) = T$ . For each nonzero  $T$ -weight  $\alpha$  on  $\mathfrak{g}$ , let  $T_\alpha$  be the codimension-1 subtorus  $(\ker \alpha)_{\text{red}}^0$ .*

*The group  $G$  is reductive if and only if the smooth connected subgroup  $Z_G(T_\alpha)$  is reductive for each  $\alpha$ .*

*Proof.* The implication “ $\Rightarrow$ ” is part of the classical theory of reductive groups. For the converse, let  $U = \mathcal{R}_u(G)$ . Thus,  $\mathfrak{u} := \text{Lie}(U)$  is a  $T$ -stable subspace of  $\mathfrak{g}$ , so it has a weight space decomposition. Each intersection  $Z_G(T_\alpha) \cap U$  is smooth and connected by Example 2.2.6, yet is also visibly unipotent and normal in  $Z_G(T_\alpha)$  since  $U$  is unipotent and normal in  $G$ . The reductivity of  $Z_G(T_\alpha)$  then forces  $Z_G(T_\alpha) \cap U = 1$ . The formation of Lie algebras of closed subgroup schemes is compatible with the formation of intersections (as one sees by consideration of dual numbers), so we conclude that  $\mathfrak{u}$  has trivial intersection with  $\text{Lie}(Z_G(T_\alpha))$ . But the functorial definition of  $Z_G(T_\alpha)$  implies (by consideration of dual numbers) that  $\text{Lie}(Z_G(T_\alpha)) = \mathfrak{g}^{T_\alpha}$ , so this contains the entire weight space for  $\alpha$  on  $\mathfrak{g}$ . In particular,  $\mathfrak{u}$  has vanishing intersection with each such weight space.

We conclude that  $\mathfrak{u}$  supports no nontrivial  $T$ -weights, so  $\mathfrak{u} \subseteq \mathfrak{g}^T = \text{Lie}(Z_G(T))$ . Thus,

$$\text{Lie}(U \cap Z_G(T)) = \mathfrak{u} \cap \text{Lie}(Z_G(T)) = \mathfrak{u}.$$

But  $U \cap Z_G(T)$  is smooth and connected (by Example 2.2.6), yet we have just seen that the containment  $U \cap Z_G(T) \subseteq U$  induces an equality on Lie algebras, so it must be an equality. In other words, necessarily  $U \subseteq Z_G(T)$ . We assumed  $Z_G(T) = T$ , so the smooth connected unipotent  $U$  must be trivial.  $\square$

Returning to the relative setting, the smooth centralizer  $Z_{G^0}(T_\alpha)$  has Lie algebra  $\mathfrak{g}^{T_\alpha} = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ . To verify that  $G^0$  has reductive fibers at all points of  $S$  near  $s$ , it suffices to treat each  $Z_G(T_\alpha)$  separately in place of  $G$  (by Lemma 3.1.10), so we have reduced to the case  $\Phi = \{\alpha, -\alpha\}$ .

Clearly  $G^0 \cap N_G(T)$  represents  $\underline{N}_{G^0}(T)$ , so we denote this as  $N_{G^0}(T)$ . Likewise,  $W_{G^0}(T) := N_{G^0}(T)/T$  is an open and finitely presented  $S$ -subgroup of the  $S$ -group  $W_G(T) = N_G(T)/T$  that is separated, étale, and quasi-finite (Theorem 2.3.1), so  $W_{G^0}(T)$  inherits these properties. The fiber  $W_{G^0}(T)_{\bar{s}}$  has order 2 and nontrivial element that acts by inversion on  $(T/T_\alpha)_{\bar{s}}$  (and so swaps  $\alpha$  and  $-\alpha$ ). Étaleness of  $W_{G^0}(T)$  and smoothness of  $T$  imply that by passing to an étale neighborhood of  $(S, s)$  we can arrange that  $W_{G^0}(T) \rightarrow S$  admits a section  $w$  that is the nontrivial point in the  $s$ -fiber. Thus, the  $w$ -action on the rank-1 torus  $T/T_\alpha$  is inversion over a neighborhood of  $s$  since it is inversion on the  $s$ -fiber.

By localizing more around  $s$ , we obtain that for *all* geometric points  $\bar{\xi}$  of  $S$  the fiber  $N_{G^0}(T)_{\bar{\xi}}$  contains an element that does not centralize  $T_{\bar{\xi}}$ . But in any smooth connected *solvable* group over an algebraically closed field, the normalizer of a maximal torus is equal

to its centralizer (as is immediately verified by considering the description of any such group as a semi-direct product of a maximal torus against the unipotent radical). It follows that all fibers  $G_\xi^0$  are *non-solvable*.

We have arranged that each fiber  $G_\xi^0$  contains the central torus  $(T_\alpha)_\xi$  with codimension 3 and is not solvable. Thus, the quotient  $(G/T_\alpha)_\xi^0$  is a non-solvable 3-dimensional smooth connected affine group. This leaves no room for a nontrivial unipotent radical (as the quotient by such a radical would be a smooth connected group of dimension at most 2, forcing solvability). Hence, every  $G_\xi^0$  is reductive.

It remains to check that when  $G^0 \rightarrow S$  has reductive geometric fibers, semisimplicity of a geometric fiber  $G_s^0$  implies semisimplicity of geometric fibers at points near  $s$ . In the presence of reductivity, semisimplicity of a connected linear algebraic group  $H$  over an algebraically closed field can be read off from the root system  $\Psi$ : it is equivalent to the condition that the  $\dim H = \#\Psi + \text{rank}(\Psi)$ . The preceding arguments (using a weight space decomposition of  $\mathfrak{g}$  fppf-locally near  $s$ ) show that these invariants are inherited by geometric fibers at points near  $s$ , so we are done.  $\square$

In later arguments involving reduction to the noetherian case, we need to ensure that reductivity (and semisimplicity) hypotheses can be descended:

**Corollary 3.1.11.** *Let  $\{A_i\}$  be a directed system of rings with direct limit  $A$ , and  $G_{i_0}$  a smooth affine  $A_{i_0}$ -group for some  $i_0$ . For all  $i \geq i_0$  define  $G_i = G_{i_0} \otimes_{A_{i_0}} A_i$  and  $G = G_{i_0} \otimes_{A_{i_0}} A$ . The fibers of  $G^0 \rightarrow \text{Spec } A$  are reductive if and only if the fibers of  $G_i^0 \rightarrow \text{Spec } A_i$  are reductive for all sufficiently large  $i \geq i_0$ , and  $G$  is a reductive  $A$ -group if and only if  $G_i$  is a reductive  $A_i$ -group for all sufficiently large  $i \geq i_0$ . The same holds for semisimplicity.*

*Proof.* For  $i \geq i_0$ , let  $U_i \subset \text{Spec } A_i$  be the locus of points at which the geometric fiber of  $G_i^0$  is reductive (resp. semisimple), and define  $U \subset \text{Spec } A$  similarly for  $G^0$ . By Proposition 3.1.9(1), these loci are open subsets. Our first problem is to prove that  $U = \text{Spec } A$  if and only if  $U_i = \text{Spec } A_i$  for all sufficiently large  $i$ .

Under the transition maps  $\text{Spec } A_{i'} \rightarrow \text{Spec } A_i$  (resp. the maps  $\text{Spec } A \rightarrow \text{Spec } A_i$ ), the preimage of  $U_i$  is  $U_{i'}$  (resp.  $U$ ), so the same holds for the respective closed complements  $Z_i$  of  $U_i$  in  $\text{Spec } A_i$  and  $Z$  of  $U$  in  $\text{Spec } A$ . We wish to show that  $Z$  is empty if and only if  $Z_i$  is empty for all sufficiently large  $i$ . Letting  $J_i \subset A_i$  and  $J \subset A$  be the respective radical ideals of these closed sets, we have  $A/J = \varinjlim A_i/J_i$ . Hence,  $A/J = 0$  if and only if  $A_i/J_i = 0$  for all sufficiently large  $i$  (by considering the equation  $1 = 0$ ).

It remains to show that if  $G^0 = G$  then  $G_i^0 = G$  for all sufficiently large  $i$ . By [EGA, IV<sub>3</sub>, 9.7.7(ii)], the subset  $Y_i \subset \text{Spec } A_i$  of points at which  $G_i$  has a geometrically connected fiber is a constructible set, and if  $i' \geq i$  then the preimage of  $Y_i$  in  $\text{Spec } A_{i'}$  is  $Y_{i'}$ . The common preimage of all  $Y_i$  in  $\text{Spec } A$  is the entire space, so by [EGA, IV<sub>3</sub>, 8.3.4] we have  $Y_i = \text{Spec } A_i$  for large  $i$ .  $\square$

In the proof of Proposition 3.1.9 we showed that when each  $G_s^0$  is reductive and there is a split torus  $T \subset G$  that is maximal on geometric fibers then for each  $s_0 \in S$  and varying

$s \in S$  near  $s_0$  the root systems  $\Phi(G_s^0, T_s)$  may be identified with  $\Phi(G_{s_0}^0, T_{s_0})$ . This has the following interesting consequence:

**Proposition 3.1.12.** *Let  $G \rightarrow S$  be a smooth affine group such that each  $G_s^0$  is reductive. The locus of  $s \in S$  such that  $G_s$  is connected is closed in  $S$ . In particular, if  $S$  is irreducible and the generic fiber is connected then all fibers are connected.*

Proposition 3.1.12 is immediate from the assertion in Proposition 3.1.3 that  $G/G^0$  is a separated étale  $S$ -group of finite presentation (as then  $\{s \in S \mid \#(G/G^0)_s \geq n\}$  is open). The proof of such separatedness rests on Theorem 5.3.5, which is proved much later, so we give a proof that avoids Proposition 3.1.3.

*Proof.* We may assume  $S$  is affine, and then by Corollary 3.1.11 we can assume  $S$  is noetherian. Since  $G_s$  is connected if and only if it is geometrically connected over  $k(s)$ , the set  $Y$  of points  $s \in S$  such that  $G_s$  is connected is a constructible [EGA, IV<sub>3</sub>, 9.7.7(ii)]. For any map  $S' \rightarrow S$ , the preimage of  $Y$  in  $S'$  is the locus of connected fibers for  $G_{S'} \rightarrow S'$ , so by the specialization criterion for closedness of a constructible set we may assume  $S = \text{Spec } R$  for a discrete valuation ring  $R$  and that the generic fiber is connected. We seek to prove that the special fiber is connected.

Without loss of generality,  $R$  is complete with an algebraically closed residue field  $k$ , so  $G_k^0$  contains a maximal torus  $T_0$  and by Proposition 3.1.6 there is an  $R$ -torus  $T \subset G$  lifting  $T_0 \subset G_0$ . The completeness of  $R$  ensures that  $T$  is split. Clearly  $T \subset Z_G(T)$ , so  $T$  is closed in  $Z_G(T)$ . Consider the special fibers  $T_0$  and  $Z_G(T)_k = Z_{G_k}(T_0)$ . The reductive identity component  $G_k^0$  meets  $Z_{G_k}(T_0)$  in  $Z_{G_k^0}(T_0) = T_0$  (equality due to the maximality of  $T_0$ ).

Let  $K$  be the fraction field of  $R$ . The closed subgroups  $T$  and  $Z_G(T)$  in  $G$  are both  $R$ -smooth, and we have shown that their special fibers have the same identity component, so their relative dimensions agree. Hence, on  $K$ -fibers the inclusion  $T_K \subset Z_G(T)_K = Z_{G_K}(T_K)$  must be an equality, as  $Z_{G_K}(T_K)$  is smooth and connected. In other words, the complement  $Z_G(T) - T$  is the union of the non-identity components of the special fiber, so it is closed in  $Z_G(T)$ . Hence,  $T$  is also open in  $Z_G(T)$ . But  $Z_G(T)$  is  $R$ -flat with irreducible generic fiber, so the total space of  $Z_G(T)$  is connected (even irreducible). Thus, the open and closed subset  $T$  in  $Z_G(T)$  must be the entire space, so the closed immersion  $T \hookrightarrow Z_G(T)$  must be an isomorphism due to  $R$ -smoothness.

By the observation immediately preceding the present proposition, the split reductive pairs  $(G_K, T_K)$  and  $(G_k^0, T_k)$  have isomorphic root systems. Let  $\Phi$  denote the common isomorphism class of  $\Phi(G_K, T_K)$  and  $\Phi(G_k^0, T_k)$ . The normalizer  $N_G(T)$  is a smooth closed subgroup of  $G$ , so the quotient

$$W_G(T) := N_G(T)/Z_G(T) = N_G(T)/T$$

is a separated étale  $R$ -group of finite type (Theorem 2.3.1). The generic fiber  $W_G(T)_K = W_{G_K}(T_K)$  is the finite constant group over  $K$  associated to the ordinary finite group  $W(\Phi)$ . Likewise,  $W_G(T)_k = W_{G_k}(T_k)$  is finite étale over  $k$  and contains as an open subgroup  $W_{G_k^0}(T_k) \simeq W(\Phi)$ . But the fiber degree for a separated étale map of finite type can only

“drop” under specialization, so we conclude that  $W_G(T)_k = W_{G_k^0}(T_k)$ . In other words,  $N_G(T) \subset G^0$ .

For any  $g \in G_k$ , the conjugate  $gT_k g^{-1}$  is a maximal torus of  $G_k^0$ . Hence, there exists  $g' \in G_k^0(k)$  such that  $gT_k g^{-1} = g'T_k g'^{-1}$ , so  $g^{-1}g' \in N_G(T)_k \subset G_k^0$ . This forces  $g \in G_k^0$ .  $\square$

See [PY06, Thm. 1.2] for a generalization of Proposition 3.1.12, and note that the locus of connected fibers can fail to be open (Example 3.1.4).

**3.2. Maximal tori.** There are several ways to define the notion of “maximal torus” in a smooth affine group scheme. Different approaches do lead to the same concept for reductive groups, at least Zariski-locally on the base. For our purposes, the following definition (taken from [SGA3, XII, 1.3]) is suitable for developing the general theory.

**Definition 3.2.1.** A *maximal torus* in a smooth affine group scheme  $G \rightarrow S$  is a torus  $T \subset G$  such that for each geometric point  $\bar{s}$  of  $S$  the fiber  $T_{\bar{s}}$  is not contained in any strictly larger torus in  $G_{\bar{s}}$ .

There are two sources of ambiguity in this definition. One is rather minor, namely to check that for any  $s \in S$  it suffices to consider a single geometric point over  $s$ . This is a special case of the general principle that properties of finitely presented structures in algebraic geometry are insensitive to scalar extension from one algebraically closed ground field to a bigger such field; this principle pervades the classical approach to linear algebraic groups (“independence of the universal domain”). In the case of interest, it comes out as follows.

**Proposition 3.2.2.** *If  $K/k$  is an extension of algebraically closed fields and  $T \subset G$  is a torus in a smooth affine  $k$ -group  $G$  then  $T$  is not contained in a strictly larger torus of  $G$  if and only if  $T_K$  is not contained in a strictly larger torus of  $G_K$ .*

An immediate consequence of this proposition is that if  $T \subset G$  is a torus in a smooth affine group scheme over a scheme  $S$  and if  $S' \rightarrow S$  is surjective then  $T$  satisfies Definition 3.2.1 in  $G$  over  $S$  if and only if  $T_{S'}$  does in  $G_{S'}$  over  $S'$ .

*Proof.* The implication “ $\Leftarrow$ ” is obvious. To verify the converse, we shall use the general technique of “spreading out and specialization”. Express  $K$  as the direct limit of its  $k$ -subalgebras  $A_i$  of finite type over  $k$ , so by limit considerations, any strict containment  $T_K \subsetneq T'$  between  $K$ -tori in  $G_K$  descends to a strict containment  $T_{A_i} \subsetneq T'_i$  between  $A_i$ -tori in  $G_{A_i}$  for some  $i$ . (This is the “spreading out” step, as  $\text{Spec}(A_i)$  is a finite type  $k$ -scheme, unlike  $\text{Spec}(K)$  in general. Note that an argument is required to verify that  $T'_i$  is a torus for large  $i$ ; e.g., it follows from the limit argument used at the start of the proof of Proposition 3.1.6.)

Viewing  $G_{A_i} = G \times \text{Spec}(A_i)$  as a finite-type reduced  $k$ -scheme in which  $T_{A_i}$  and  $T'_i$  are reduced closed subschemes (due to smoothness over the reduced  $\text{Spec}(A_i)$ ), since  $k = \bar{k}$  and  $T_{A_i}$  is a proper closed subset of  $T'_i$  it follows from the Nullstellensatz that some  $k$ -point of  $T'_i$  is not contained in  $T_{A_i}$ . This point lies over some  $\xi \in \text{Spec}(A_i)(k)$ , so we specialize there: passing to  $\xi$ -fibers yields a strict containment  $T \subsetneq (T'_i)_\xi$  between  $k$ -tori in  $G$ , a contradiction.  $\square$

A more serious ambiguity in the terminology of Definition 3.2.1 is to determine its relation with maximality relative to inclusions among tori in  $G$  over  $S$ . Any  $T$  that is maximal in the sense of Definition 3.2.1 is maximal among tori of  $G$ ; i.e., any containment  $T \subset T'$  among tori of  $G$  must be an equality. Indeed, the equality on fibers is clear, so by Lemma B.3.1 equality holds in  $G$  (and likewise  $T_U$  is maximal in  $G_U$  for all open subschemes  $U$  of  $S$ ).

Consider the converse: if a torus  $T \subset G$  is not contained in a strictly larger torus over  $S$  then is  $T$  maximal in the sense of Definition 3.2.1 (i.e., is  $T_{\bar{s}}$  maximal in  $G_{\bar{s}}$  for all geometric points  $\bar{s}$  of  $S$ )? There are Zariski-local obstructions:  $T_U$  may lie in a strictly larger  $G_U$ -torus for some open  $U \subset S$ . This is due to the fact that the dimension of the maximal tori in each geometric fiber  $G_{\bar{s}}$  may not be locally constant in  $s$ . For instance, in the Example at the end of [SGA3, XVI, §3] there is given an explicit smooth affine group over a discrete valuation ring such that the generic fiber is  $\mathbf{G}_m$  and the special fiber is  $\mathbf{G}_a$  (so  $T = 1$  is maximal relative to containment over the entire base, but not over the open generic point). Here is a more natural example.

**Example 3.2.3.** Let  $S = \text{Spec}(V)$  for a complete discrete valuation ring  $V$  with  $K := \text{Frac}(V)$ , and consider a separable quadratic extension  $K'/K$  that is not unramified. Let  $V'$  be the valuation ring of  $K'$ . The Weil restriction  $G = \mathbf{R}_{V'/V}(\mathbf{G}_m)$  (which represents the functor  $B \rightsquigarrow \mathbf{G}_m(B \otimes_V V') = (B \otimes_V V')^\times$  on  $V$ -algebras) is a smooth affine  $V$ -group of relative dimension 2; it is even an open subscheme of the  $V$ -scheme  $\mathbf{R}_{V'/V}(\mathbf{A}_{V'}^1) = \mathbf{A}_V^2$  (see [CGP, Prop. A.5.2]). In particular, the fibers of  $G$  are connected (as is also a consequence of general connectedness results for Weil restriction of *smooth* schemes [CGP, Prop. A.5.9].) The fibral connectedness for  $G$  can also be seen by inspection:  $G_K = \mathbf{R}_{K'/K}(\mathbf{G}_m)$  is a torus since  $K'/K$  is separable, and since  $K'/K$  is not unramified the geometric special fiber is  $\mathbf{R}_{\bar{k}[e]/\bar{k}}(\mathbf{G}_m) = \mathbf{G}_m \times \mathbf{G}_a$  (via  $(t, x) \mapsto t(1 + x\epsilon)$ ).

Thus, the evident  $\mathbf{G}_m$  as an  $R$ -subgroup of  $G$  is a torus not contained in any strictly larger torus of  $G$  (due to consideration of the special fiber), but its geometric generic fiber is not maximal in that of  $G$ .

For our purposes, “maximal torus” will always be taken in the sense of Definition 3.2.1. See Remark A.1.2 for the equivalence with another possible definition. The case of most interest to us is maximal tori in reductive group schemes. In such groups, the maximality property is robust with respect to the Zariski topology:

**Example 3.2.4.** Proposition 3.1.9(2) implies that if  $T$  is a torus in a reductive group scheme  $G \rightarrow S$  and it is maximal on the  $\bar{s}$ -fiber for some  $s \in S$  then it is maximal in the sense of Definition 3.2.1 over a Zariski-open neighborhood of  $s$ .

To use “reduction to the noetherian case” in the study of maximal tori, we require:

**Lemma 3.2.5.** *Let  $\{A_i\}$  be a directed system of rings with direct limit  $A$ , and let  $G$  be a smooth affine  $A$ -group equipped with an  $A$ -torus  $T \subset G$ . Pick  $i_0$  so that  $G$  descends to a smooth affine  $A_{i_0}$ -group  $G_{i_0}$  and  $T$  descends to an  $A_{i_0}$ -torus  $T_{i_0}$  in  $G_{i_0}$ . For  $i \geq i_0$ , define the pair  $(G_i, T_i)$  over  $A_i$  by scalar extension of the pair  $(G_{i_0}, T_{i_0})$  over  $A_{i_0}$ .*

If  $T$  is maximal in  $G$  then  $T_i$  is maximal in  $G_i$  for sufficiently large  $i$ .

The proof of this lemma uses “standard” direct limit arguments as well as constructions that are specific to group schemes.

*Proof.* We will use centralizers for tori (as in Lemma 2.2.4) and elementary affineness results for quotients by central tori (see [SGA3, IX, 2.3]). It is harmless to replace  $G_{i_0}$  with the finitely presented affine centralizer  $Z_{G_{i_0}}(T_{i_0})$  (and replace  $G_i$  with  $Z_{G_{i_0}}(T_{i_0})_{A_i} = Z_{G_i}(T_i)$  for all  $i \geq i_0$ , and replace  $G$  with their common base change  $Z_G(T)$  over  $A$ ), so we may assume that  $T_{i_0}$  is central in  $G_{i_0}$  over  $A_{i_0}$  (and similarly over  $A_i$  for  $i \geq i_0$ , as well as over  $A$ ). Then by passing to the affine quotient  $G_{i_0}/T_{i_0}$  over  $A_{i_0}$ , and similarly over every  $A_i$  and over  $A$ , we reduce to the case that all  $T_i$  and  $T$  are trivial.

Letting  $S = \text{Spec}(A)$  and  $S_i = \text{Spec}(A_i)$  for all  $i$ , we are brought to the case that the smooth affine geometric fibers  $G_{\bar{s}}$  all have no nontrivial tori (equivalently,  $G_{\bar{s}}^0$  is unipotent for all  $s \in S$ ). We seek to prove the same property holds for all geometric fibers of  $G_i \rightarrow S_i$  when  $i$  is sufficiently large. We may and do assume that  $S$  is non-empty, and hence likewise all  $S_i$  are non-empty. Since  $G_{i_0} \rightarrow S_{i_0}$  is finitely presented, the number of connected components for its geometric fibers is bounded [EGA, IV<sub>3</sub>, 9.7.9]. The same bound is valid for the geometric fibers of every  $G_i \rightarrow S_i$  for  $i \geq i_0$ , as well as for the geometric fibers of  $G \rightarrow S$ . Thus, if we choose a prime number  $p$  larger than such a bound then the unipotency of all  $G_{\bar{s}}^0$  implies that for all  $s \in S$  there are no nontrivial group homomorphisms  $\mu_p \rightarrow G_{\bar{s}}$ . In other words, for the affine and finitely presented  $S$ -scheme  $Y := \underline{\text{Hom}}_{S\text{-gp}}(\mu_p, G)$  and the canonical section  $\sigma : S \rightarrow Y$  over  $S$  corresponding to the *trivial*  $S$ -map  $\mu_p \rightarrow G$ , the map  $\sigma$  is surjective.

Let  $Y_i = \underline{\text{Hom}}_{S_i\text{-gp}}(\mu_p, G_i)$ , so  $Y_i = Y_{i_0} \otimes_{A_{i_0}} A_i$  compatibly with change in  $i \geq i_0$  and the limit of these affine schemes is  $Y$ . Since the canonical sections  $\sigma_i : S_i \rightarrow Y_i$  defined by the trivial  $S_i$ -maps  $\mu_p \rightarrow G_i$  yield the surjective map  $\sigma$  in the limit, for sufficiently large  $i \geq i_0$  the map  $\sigma_i$  is surjective [EGA, IV<sub>3</sub>, 8.10.5(vi)]. Fix such a large  $i$ , so the geometric fibers of the smooth  $S_i$ -group  $G_i \rightarrow S_i$  do not contain  $\mu_p$  as a subgroup scheme over  $S_i$  (since  $S_i$  is non-empty). It follows that the identity components of these fibers cannot contain a nontrivial torus. Hence, these identity components are unipotent, as desired.  $\square$

The following fundamental result concerning the “scheme of maximal tori” is the engine that makes the relative theory of reductive groups work (and it is proved in [SGA3, XII, 1.10] under weaker hypotheses than we impose):

**Theorem 3.2.6.** *Let  $G \rightarrow S$  be a smooth affine group scheme such that in the identity component  $G_{\bar{s}}^0$  of each geometric fiber the maximal tori are their own centralizers. Then the functor on  $S$ -schemes*

$$\text{Tor}_{G/S} : S' \rightsquigarrow \{\text{maximal tori in } G_{S'}\}$$

*is represented by a smooth quasi-affine  $S$ -scheme  $\text{Tor}_{G/S}$ , and  $\text{Tor}_{G/S} \rightarrow S$  is surjective.*

The main case of interest for which we can verify the hypothesis “Cartan subgroups of (identity components of) geometric fibers are tori” is the case of reductive group schemes, but

another interesting case that will arise later is parabolic subgroups of reductive group schemes (to be defined and studied in §5.2). In [SGA3, XII, 5.4] it is shown that  $\mathrm{Tor}_{G/S}$  is actually  $S$ -affine in Theorem 3.2.6. The proof of this finer property uses a hard representability theorem [SGA3, XI, 4.1] that we are avoiding. We do not need such affineness, so we say nothing further about it. See Exercise 3.4.7 for a generalization of Theorem 3.2.6 over fields.

*Proof.* By effectivity of fppf descent for schemes that are quasi-affine over the base, it suffices to work fppf-locally on  $S$  (since the functor in question is clearly an fppf sheaf of sets). Choose  $s \in S$ . As in the proof of Proposition 3.1.9(1), we may work fppf-locally around  $s$  to arrange that there exists a torus  $T \subset G$  such that  $T_{\bar{s}}$  is maximal in  $G_{\bar{s}}^0$ . The inclusion  $T \subset Z_G(T)^0$  between smooth  $S$ -groups is an isomorphism on geometric fibers at  $\bar{s}$ , so by connectedness and fiber dimension considerations it follows that the same holds on geometric fibers over points near  $s$ . Hence, by passing to a Zariski-open neighborhood of  $s$  in  $S$  we may arrange that  $T = Z_G(T)^0$ , so  $T$  is maximal in  $G$  (in the sense of Definition 3.2.1). Now we will just use that  $G \rightarrow S$  is a smooth affine group.

By Theorem 2.3.1, the quotient  $G/N_G(T)$  exists as a smooth quasi-affine  $S$ -scheme, and clearly  $G/N_G(T) \rightarrow S$  is surjective. There is an evident  $G$ -action on  $\mathrm{Tor}_{G/S}$  via conjugation, and the point  $T \in \mathrm{Tor}_{G/S}(S)$  has stabilizer subfunctor in  $G$  represented by the transporter scheme  $\mathrm{Transp}_G(T, T) = N_G(T)$ . Thus,  $G/N_G(T)$  is a subfunctor of  $\mathrm{Tor}_{G/S}$ , and we claim that it equals  $\mathrm{Tor}_{G/S}$ . It suffices to prove that the map  $G \rightarrow \mathrm{Tor}_{G/S}$  between fppf sheaves of sets is surjective, which is to say that for any  $S$ -scheme  $S'$  and  $T' \in \mathrm{Tor}_{G/S}(S')$  there exists an fppf covering  $S'' \rightarrow S'$  and a  $g \in G(S'')$  such that  $gT'_{S''}g^{-1} = T_{S''}$ . Put another way, we are claiming that the maximal tori  $T_{S'}$  and  $T'$  in  $G_{S'}$  are conjugate fppf-locally on  $S'$ . Upon renaming  $S'$  as  $S$ , our task is to prove that any two maximal tori  $T$  and  $T'$  in  $G$  are conjugate fppf-locally on  $S$ .

Direct limit arguments (along with Lemma 3.2.5) allow us to arrange that  $S$  is noetherian and affine, and even finite type over  $\mathbf{Z}$  (which will be relevant when we apply Artin approximation later in the argument). Pick  $s \in S$ , so  $T_{\bar{s}}$  and  $T'_{\bar{s}}$  are  $G(\bar{s})$ -conjugate for an algebraic geometric point  $\bar{s}$  over  $s$ . Express  $k(\bar{s})$  as a direct limit of subextensions of finite degree over  $k(s)$ , so there exists a finite extension  $K/k(s)$  such that  $T_K$  and  $T'_K$  are  $G(K)$ -conjugate and split. Passing to an fppf neighborhood of  $(S, s)$  then brings us to the case that  $T_s$  and  $T'_s$  are  $G(s)$ -conjugate and split. Let  $A$  denote the completion  $\mathcal{O}_{S,s}^\wedge$ .

By the same style of Artin approximation argument as used in the proof of Proposition 3.1.6 (this is where we need that  $S$  is finite type over  $\mathbf{Z}$ ), if  $T_A$  is  $G(A)$ -conjugate to  $T'_A$  then we can build an étale neighborhood  $U$  of  $(S, s)$  such that  $T_U$  is  $G(U)$ -conjugate to  $T'_U$ , so we would be done. Thus, we may and do replace  $S$  with  $\mathrm{Spec}(A)$  to arrange that (i)  $S$  is the spectrum of a complete local noetherian ring  $(A, \mathfrak{m})$  with residue field denoted  $k$ , and (ii) the special fibers  $T_0$  and  $T'_0$  are  $G(k)$ -conjugate and split. By lifting a conjugating element from  $G(k)$  into  $G(A)$  (as we may do since the local noetherian  $A$  is complete and  $G$  is smooth and affine), we can arrange that  $T_0 = T'_0$  inside the special fiber  $G_k$ .

It follows from the completeness of  $A$  that  $T$  and  $T'$  are split, hence abstractly isomorphic. The splitting isomorphisms  $T \simeq \mathbf{G}_m^r$  and  $T' \simeq \mathbf{G}_m^r$  can be chosen to coincide on the special fibers (i.e., they induced the same splitting of the torus  $T_0 = T'_0$  in  $G_0$ ), so the inclusions of  $T$  and  $T'$  into  $G$  may be viewed as a pair of inclusions from a common  $A$ -torus  $\mathbf{G}_m^r$  into  $G$  that agree residually. Thus, by Corollary B.3.5, these inclusions are  $G(A)$ -conjugate.  $\square$

In earlier arguments we have built maximal tori in reductive group schemes by working fppf-locally on the base. Now we show that it always suffices to work étale-locally:

**Corollary 3.2.7.** *Any reductive group scheme  $G \rightarrow S$  admits a maximal torus étale-locally on  $S$ . In particular, any connected reductive group over a field  $k$  admits a geometrically maximal torus defined over a finite separable extension of  $k$ .*

*Proof.* The structural morphism  $\mathrm{Tor}_{G/S} \rightarrow S$  from the scheme of maximal tori is a smooth surjection, so it admits sections étale-locally on  $S$ . But to give a section over an  $S$ -scheme  $U$  is to give an element of the set  $\mathrm{Tor}_{G/S}(U)$ , which is to say a maximal torus of  $G_U$ .  $\square$

The following result is a special case of part of [SGA3, XII, 5.4].

**Proposition 3.2.8.** *Let  $T$  be a maximal torus in a reductive group scheme  $G \rightarrow S$ . The Weyl group  $W_G(T) = N_G(T)/Z_G(T)$  is finite étale over  $S$ .*

Recall from Theorem 2.3.1 that a priori (without using reductivity) the quotient sheaf  $W_G(T)$  is represented by an  $S$ -scheme that is separated, étale, and finitely presented.

*Proof.* By Proposition 3.1.9 (and Corollary 3.1.11) we use direct limit arguments to reduce to the case that  $S$  is noetherian, so we can apply to  $W_G(T) \rightarrow S$  a general finiteness criterion for quasi-compact separated étale maps: it suffices that the number of points in the geometric fibers is Zariski-locally constant on the base. This finiteness criterion follows from étale localization on the base and the local structure theorem for such morphisms over a henselian local base in [EGA, IV<sub>4</sub>, 18.5.11(c)]. (See Exercise 3.4.1 for a generalization.) In other words, we claim that the order of the fibral Weyl group  $W_G(T)_{\bar{s}} = W_{G_{\bar{s}}}(T_{\bar{s}})$  is locally constant in  $s$ .

Since the Weyl group of a connected reductive group over an algebraically closed field coincides with the Weyl group of the associated root system, it suffices to check that the isomorphism class of the root system  $\Phi(G_{\bar{s}}, T_{\bar{s}})$  is locally constant in  $s$ . By passing to an étale neighborhood of a chosen point of  $S$  we can arrange that

$$T = D_S(M) = \underline{\mathrm{Hom}}_{S\text{-gp}}(M_S, \mathbf{G}_m)$$

for a finite free  $\mathbf{Z}$ -module  $M$  (thereby identifying the abelian sheaf  $\underline{\mathrm{Hom}}_{S\text{-gp}}(T, \mathbf{G}_m)$  with  $M_S$ ). In the proof of Proposition 3.1.9(1) we saw that the vector bundle  $\mathfrak{g}$  with its  $T$ -action decomposes into a direct sum of “root subbundles” and that this decomposition identifies the root system  $\Phi(G_{\bar{\xi}}, T_{\bar{\xi}}) \subset X(T_{\bar{\xi}}) = M = X(T_{\bar{s}})$  with  $\Phi(G_{\bar{s}}, T_{\bar{s}})$  for all  $\xi$  near  $s$ .  $\square$

**3.3. Scheme-theoretic and reductive center.** The final topic in this section is the functorial center of a reductive group scheme. This material is developed in [SGA3, XII, §4], and we will navigate our way towards a single result for reductive group schemes (which in turn is a special case of [SGA3, XII, 4.11]). We begin with two lemmas.

**Lemma 3.3.1.** *Let  $G$  be a smooth affine  $S$ -group with connected fibers, and  $H$  a subgroup of multiplicative type.*

- (1) *If  $H$  is normal in  $G$  then it is central in  $G$ .*
- (2) *Assume  $H$  is normal in  $G$ , and let  $\overline{G} = G/H$  be the associated smooth  $S$ -affine quotient [SGA3, IX, 2.3]. For every central multiplicative type subgroup  $\overline{H}'$  in  $\overline{G}$ , the preimage  $H'$  of  $\overline{H}'$  in  $G$  is a central multiplicative type subgroup of  $G$ .*

Part (2) of this lemma serves a role in our treatment akin to the role of [SGA3, XII, 4.7] in the general development of [SGA3, XII].

*Proof.* The normality of  $H$  in  $G$  implies that of each  $H[n]$  in  $G$ , and the relative schematic density of  $\{H[n]\}$  in  $H$  implies that  $H$  is central in  $G$  if all  $H[n]$  are central in  $G$ . Hence, to prove (1) we may and do replace  $H$  with  $H[n]$  for arbitrary  $n > 0$ , so we can assume that  $H$  is  $S$ -finite. Then the automorphism functor of  $H$  is identified (up to inversion) with that of its finite étale Cartier dual, so this functor is represented by a finite étale  $S$ -scheme (via effective descent for *finite* étale schemes over the base, applied to the automorphism functor of the constant Cartier dual over an étale cover of  $S$ ).

The conjugation action of  $G$  on  $H$  is classified by an  $S$ -group map from  $G$  to the finite étale automorphism scheme  $E$  of  $H$ . The identity section of  $E \rightarrow S$  is an open and closed immersion (as for any section to a finite étale map), so the kernel of the action map  $G \rightarrow \underline{\mathrm{Aut}}_{S\text{-gp}}(H) = E$  is an  $S$ -subgroup scheme of  $G$  that is both open and closed. Thus, (1) is reduced to the obvious fact that for any map of schemes  $X \rightarrow S$  with connected fibers, the only open and closed subscheme of  $X$  that maps onto  $S$  is  $X$ .

To prove (2), first note that  $H'$  is clearly finitely presented and flat (and affine) over  $S$ , and it is normal in  $G$  since  $\overline{H}'$  is central in  $\overline{G}$ . In particular, by (1) it suffices to prove that  $H'$  is multiplicative type. First we check directly that  $H'$  is central in  $G$  (and hence is commutative). Since  $H$  is central in  $G$  and the quotient  $H'/H = \overline{H}'$  is central in  $G/H = \overline{G}$ , the  $G$ -action on  $H'$  is classified by a homomorphism from  $G$  to the Hom-functor

$$\underline{\mathrm{Hom}}_{S\text{-gp}}(\overline{H}', H).$$

(Loosely speaking, this amounts to considering the upper-right entry in an upper-triangular  $2 \times 2$  matrix.) This classifying map vanishes if and only if  $H'$  is central in  $G$ , so by the relative schematic density of  $\{\overline{H}'[n]\}$  in  $\overline{H}'$  it suffices (for the proof of centrality) to replace  $\overline{H}'$  with  $\overline{H}'[n]$  (for an arbitrary  $n > 0$ ) and to replace  $H'$  with the preimage  $H'_n$  of  $\overline{H}'[n]$  in  $H'$ . But then we have the equality of functors

$$\underline{\mathrm{Hom}}_{S\text{-gp}}(\overline{H}', H) = \underline{\mathrm{Hom}}_{S\text{-gp}}(\overline{H}', H[n]),$$

and via Cartier duality the right side is identified with a Hom-functor between finite étale  $S$ -groups. In particular, this is represented by a finite étale  $S$ -group, so as in the proof of (1) any  $S$ -homomorphism from  $G$  to this Hom-scheme must vanish. Thus,  $H'$  is central in  $G$ .

It now follows that the fibers of  $H'$  are of multiplicative type. Indeed, we may check on geometric fibers, and over a field any commutative extension of a multiplicative type group by another is again of multiplicative type (Exercise 3.4.6(ii)). Our goal is to prove that  $H'$  is of multiplicative type.

To bootstrap from the fibers, we use deformation theory, as follows. The initial hypotheses are compatible with standard limit arguments on the base, so we may and do reduce to the case that  $S$  is noetherian at the outset, and then local, say  $S = \text{Spec}(A)$ . Let  $\widehat{S} = \text{Spec}(\widehat{A})$ . Suppose we can prove that  $H'_{\widehat{S}}$  is of multiplicative type. By Proposition B.3.4, this group is split by an étale cover of  $\widehat{S}$ . Any such étale cover is refined by a local finite étale cover  $\widehat{S}' \rightarrow \widehat{S}$  (since  $\widehat{A}$  is henselian local). But the categories of finite étale covers of  $A$  and  $\widehat{A}$  are equivalent (since  $A$  is henselian local), so  $\widehat{S}'$  uniquely descends to a local finite étale cover  $S' \rightarrow S$ . We may rename  $S'$  as  $S$  to reduce to the case that there is an isomorphism  $f : D_{\widehat{S}}(M) \simeq H'_{\widehat{S}}$  for a finitely generated abelian group  $M$ . Since  $H'$  is commutative, we may argue exactly as in the proof of Proposition B.3.4 to descend  $f$  to an isomorphism  $D_S(M) \simeq H'$ .

We may now rename  $\widehat{S}$  as  $S$  to reduce to the case that  $A$  is complete. Thus, by the final assertion in Theorem B.3.2, there exists an open and closed  $S$ -subgroup scheme  $\mathcal{H}'$  in  $H'$  of multiplicative type such that the special fibers of  $H'$  and  $\mathcal{H}'$  coincide inside the special fiber of  $G$ . We will prove that  $\mathcal{H}' = H'$ .

The multiplicative type  $S$ -subgroups  $H$  and  $\mathcal{H}'$  in the commutative  $H'$  satisfy the containment relation  $H_0 \subset \mathcal{H}'_0$  inside  $H'_0$  for their special fibers. We claim that  $H \subset \mathcal{H}'$ . The product map  $H \times \mathcal{H}' \rightarrow H'$  makes sense as an  $S$ -homomorphism since  $H'$  is commutative, and by Exercise 2.4.3 this map factors through an fppf quotient onto a multiplicative type  $S$ -subgroup  $K$ . By construction  $K$  contains  $\mathcal{H}'$ , but its special fiber is an fppf quotient of  $H_0 \times \mathcal{H}'_0$  via addition in  $H'_0$ , so since  $H_0 \subset \mathcal{H}'_0$  inside  $H'_0$  we conclude that the containment  $\mathcal{H}' \subset K$  of  $S$ -subgroups of  $H'$  induces an equality between special fibers. But  $K$  and  $\mathcal{H}'$  are both of multiplicative type and  $A$  is complete local noetherian, so  $K$  and  $\mathcal{H}'$  split over a finite étale extension of  $S$  (due to Proposition B.3.4 and the cofinality of such covers among all étale covers of  $S$ ). Hence, it is clear that  $\mathcal{H}' = K$ , so  $H \subset K = \mathcal{H}'$  as desired.

Consider the map  $\mathcal{H}' \rightarrow \overline{H}' := H'/H$  between  $S$ -groups of multiplicative type. On special fibers, this is the quotient map  $\mathcal{H}'_0 = H'_0 \rightarrow \overline{H}'_0$ , so the induced map between constant character groups is injective. Hence, the  $S$ -homomorphism  $\mathcal{H}' \rightarrow \overline{H}'$  is dual to an injection between character groups, so it is faithfully flat. Thus,  $\mathcal{H}' \rightarrow \overline{H}'$  is an fppf quotient homomorphism whose kernel is  $H$ , yet  $\mathcal{H}'$  is an  $S$ -subgroup of  $H'$  that is *also* an extension of  $\overline{H}'$  by  $H$  for the fppf topology (compatibly with the analogous such extension structure on  $\mathcal{H}'$ ). A snake lemma argument with fppf group sheaves then implies that the inclusion  $\mathcal{H}' \subset H'$  is an equality, so  $H'$  is of multiplicative type.  $\square$

The following lemma identifies an interesting property for multiplicative type subgroups that are central on geometric fibers.

**Lemma 3.3.2.** *Let  $G$  be a smooth affine  $S$ -group with connected fibers, and  $H \subset G$  an  $S$ -subgroup of multiplicative type. Assume that  $H_{\bar{s}}$  is central in  $G_{\bar{s}}$  and contains all central multiplicative type subgroups of  $G_{\bar{s}}$  for all geometric points  $\bar{s}$  of  $S$ . Then for any  $S$ -scheme  $S'$ , every central multiplicative type subgroup of  $G_{S'}$  is contained in  $H_{S'}$ .*

By a “spreading out and specialization” argument (as in the proof of Proposition 3.2.2), the fibral hypothesis in Lemma 3.3.2 is open on the base.

*Proof.* By Lemma 2.2.4, the functorial centralizer of  $H$  in  $G$  is represented by a finitely presented closed  $S$ -subgroup  $Z_G(H)$ . Any central subgroup scheme of  $G$  lies in  $Z_G(H)$ , so we may replace  $G$  with  $Z_G(H)$  to reduce to the case that  $H$  is central in  $G$ . The fppf quotient sheaf  $G/H$  then has a natural  $S$ -group structure, and by [SGA3, IX, 2.3] it is represented by an  $S$ -affine  $S$ -scheme of finite presentation.

By Exercise 2.4.3 and Lemma 3.3.1(2) we may rename  $G/H$  as  $G$  so that each  $G_{\bar{s}}$  contains no nontrivial central closed subgroup of multiplicative type. This property persists after any base change  $S' \rightarrow S$ , so upon renaming  $S'$  as  $S$  it remains to show that a central multiplicative type subgroup  $H' \subset G$  must be trivial. Every geometric fiber  $H'_{\bar{s}}$  is trivial, so consideration of the character group of the multiplicative type  $H'$  forces  $H'$  to be the trivial  $S$ -group.  $\square$

In view of the preceding lemma, the following definition (taken from [SGA3, XII, 4.1]) is reasonable as well as checkable in practice.

**Definition 3.3.3.** Let  $G \rightarrow S$  be a smooth affine group scheme with connected fibers. A *reductive center* of  $G$  is a central multiplicative type subgroup  $H \subset G$  that satisfies the conditions in Lemma 3.3.2.

It is clear that if a reductive center exists then it is unique, in which case we call it *the* reductive center. Finally, we arrive at a special case of [SGA3, XII, 4.11]:

**Theorem 3.3.4.** *Let  $G \rightarrow S$  be a reductive group scheme. Then  $G$  admits a reductive center  $Z$ , and  $Z$  coincides with the scheme-theoretic center  $Z_G$  of  $G$ . In particular,  $Z_G$  is  $S$ -flat. Moreover,  $Z$  represents the kernel of the action map*

$$u : G \rightarrow \underline{\mathrm{Aut}}_S(\mathrm{Tor}_{G/S}).$$

*Proof.* Since the formation of the reductive center commutes with base change if it exists, by effective fppf descent for schemes affine over the base we may work fppf-locally on  $S$ . Hence, we may and do assume that  $G$  contains a split maximal torus  $T$ . This identifies  $G/N_G(T)$  with  $\mathrm{Tor}_{G/S}$  via  $g \mapsto gTg^{-1}$ , so the natural  $G$ -action on  $\mathcal{T} := \mathrm{Tor}_{G/S}$  via conjugation goes over to the left translation action of  $G$  on  $G/N_G(T)$ . We will first construct a reductive center  $Z$ , and then prove that it is the center and represents the functorial kernel  $\ker u$ .

We may and do assume  $S$  is affine and  $T = D_S(M) \simeq \mathbf{G}_m^r$  for  $M \simeq \mathbf{Z}^r$  with some  $r \geq 0$ . Consider the  $M$ -graded decomposition  $\bigoplus_{m \in M} \mathfrak{g}_m$  of  $\mathfrak{g} = \mathrm{Lie}(G)$  into weight spaces under

the action by  $T$ . The formation of this decomposition commutes with base change on  $S$ , and these weight spaces are direct summands of  $\mathfrak{g}$  (hence vector bundles over  $S$ ). By working Zariski-locally on  $S$ , we may arrange that the weight spaces  $\mathfrak{g}_{m_0}$  that are nonzero have constant rank (necessarily rank 1 for  $m_0 \neq 0$ , by the reductivity hypothesis). Let  $\Phi \subset M$  be the finite set of nontrivial weights that arise. Define

$$H = \bigcap_{\alpha \in \Phi} \ker(\alpha : T \rightarrow \mathbf{G}_m).$$

This intersection of finitely many multiplicative type  $S$ -subgroups of  $T$  is a multiplicative type  $S$ -subgroup of  $T$  (corresponding to the quotient of  $M$  by the  $\mathbf{Z}$ -submodule spanned by  $\Phi$ ), and by construction the  $H$ -action on  $G$  via conjugation induces the trivial action on  $\mathfrak{g}$ .

The centralizer  $Z_G(H)$  is a smooth closed  $S$ -subgroup of  $G$  (Lemma 2.2.4) and its Lie algebra is  $\mathfrak{g}^H = \mathfrak{g}$  (due to the functorial characterization of  $Z_G(H)$ ). Thus, the map  $Z_G(H) \hookrightarrow G$  is an isomorphism between  $s$ -fibers for all  $s \in S$ , so it is an equality (as both  $Z_G(H)$  and  $G$  are fppf, even smooth, over  $S$ ). This shows that  $H$  is a central multiplicative type  $S$ -subgroup of  $G$ . But for any  $S$ -scheme  $S'$  and  $g' \in G(S')$  that centralizes  $G_{S'}$ , clearly  $g' \in Z_{G(S')}(T_{S'}) = (Z_G(T))(S') = T(S')$  (Lemma 3.1.10) and the adjoint action of  $T_{S'}$  on  $\mathfrak{g}_{S'} = \text{Lie}(G_{S'})$  makes the point  $g' \in T(S')$  act trivially, so  $g'$  is an  $S'$ -point of  $\bigcap_{\alpha \in \Phi} \ker(\alpha_{S'}) = H_{S'}$ . This shows that  $H$  is a reductive center of  $G$ , and that it represents the functorial center of  $G$ . The centrality also forces  $H \subset \ker u$ , so it remains to prove (after making a base change by any  $S' \rightarrow S$  and renaming  $S'$  as  $S$ ) that any  $g \in G(S)$  with trivial conjugation action on  $\mathcal{T}$  necessarily lies in  $H(S)$ .

By the usual limit arguments (including Corollary 3.1.11) we may reduce to the case when  $S$  is noetherian, and then artin local (by the Krull intersection theorem relative to the equations that defined  $H$  in  $G$ ), and finally artin local with algebraically closed residue field (via faithfully flat base change). Writing  $S = \text{Spec}(A)$  for an artin local ring  $A$ , the set of  $A$ -points of the  $A$ -smooth  $\mathcal{T}$  is relatively schematically dense [EGA, IV<sub>3</sub>, 11.10.9]; i.e., for any  $A$ -algebra  $A'$ , an  $A'$ -morphism from  $\mathcal{T}_{A'}$  to a separated  $A'$ -scheme is uniquely determined by its effect on the points of  $\mathcal{T}(A)$  (viewed as  $A'$ -points of  $\mathcal{T}_{A'}$ ). In other words,  $\ker u$  is *represented* by the intersection of the closed subschemes  $N_G(T_\sigma)$  where  $T_\sigma$  is the maximal  $A$ -torus in  $G$  corresponding to the varying point  $\sigma \in \mathcal{T}(A)$ .

We have a containment  $H \subset \ker u$  as closed subschemes of  $G$ , and seek to prove this is an equality. By Lemma B.3.1, this reduces to the consideration of the special fiber (as the artin local  $S$  has only one point). Now we are considering a connected reductive group  $G$  over an algebraically closed field  $k$ . The above description of  $\ker u$  becomes  $\ker u = \bigcap_{T'} T'$ , so to prove  $\ker u = H$  it suffices to show that  $\bigcap_{T'} T'$  is central in  $G$ .

Consider the smooth closed subgroup  $N$  of  $G$  generated by the maximal tori. This is normal, so  $G/N$  is a reductive group that contains no nontrivial tori (since quotient maps between linear algebraic  $k$ -groups carry maximal tori onto maximal tori). Hence,  $G/N$  is unipotent and therefore trivial. In other words,  $G$  is generated by its maximal tori, so there exists a finite set  $\{T'_i\}$  of maximal tori of  $G$  such that the multiplication map of  $k$ -schemes

$q : \prod T'_i \rightarrow G$  is dominant. A dominant map between  $k$ -varieties is generically flat, so there exists a dense open  $V \subset G$  and  $V' \subset q^{-1}(V)$  such that  $V' \rightarrow V$  is faithfully flat. Any (functorial) point of  $\bigcap_{T'} T' = \ker u$  centralizes all  $T'_i$  and hence centralizes  $V$ , so  $\ker u$  centralizes  $G$  (as  $V$  is a dense open subscheme of  $G$  over  $k$ ).  $\square$

**Corollary 3.3.5.** *Let  $G$  be a reductive  $S$ -group scheme, and  $Z$  a multiplicative type subgroup scheme of the center  $Z_G$ . The reductive quotient  $G/Z$  has center  $Z_G/Z$ ; in particular,  $G/Z_G$  has trivial center.*

*Moreover,  $T \mapsto T/Z$  defines a bijective correspondence between the set of maximal tori of  $G$  and the set of maximal tori of  $G/Z$ , with inverse given by scheme-theoretic preimage under the quotient map  $G \rightarrow G/Z$ .*

Note that any such  $T$  automatically contains the central  $Z$  in  $G$  since  $T = Z_G(T)$ , so  $T/Z$  makes sense. This corollary is a special case of [SGA3, XII, 4.7(b),(c)].

*Proof.* By Lemma 3.3.1, the central subgroups of multiplicative type in  $G/Z$  correspond bijectively (via the formation of preimages in  $G$ ) to central subgroups of multiplicative type in  $G$  that contain  $Z$ , which is to say the multiplicative type subgroups of  $Z_G$  that contain  $Z$  (as  $Z_G$  is the reductive center of  $G$ ). These in turn correspond to the multiplicative type subgroups of  $G/Z$  that are contained in  $Z_G/Z$ . Thus,  $Z_G/Z$  is the reductive center of  $G/Z$ , so it is also the center of  $G/Z$ .

To establish the bijective correspondence for maximal tori, we first note that the classical theory ensures that  $T/Z$  is maximal in  $G/Z$  when  $T$  is maximal in  $G$ . Hence, the proposed correspondence makes sense. It is also clear that  $T$  is the preimage of  $T/Z$  under the quotient map  $G \rightarrow G/Z$ , so the only problem is to prove that every maximal torus  $\bar{T}$  in  $G/Z$  has the form  $T/Z$  for some (necessarily unique) maximal torus  $T$  in  $G$ . Consider the preimage  $\tilde{T}$  of  $\bar{T}$  in  $G$ . We seek to prove that  $\tilde{T}$  is a maximal torus in  $G$ .

Since  $\bar{G}$  is the quotient of  $G$  modulo a central subgroup scheme, the conjugation action of  $G$  on itself factors through a left action of  $\bar{G}$  on  $G$ . As such, we get a left action of the torus  $\bar{T}$  on  $G$ , so the functorial centralizer  $G^{\bar{T}}$  for this action is represented by a smooth closed subgroup of  $G$  with connected geometric fibers (by arguing as in Example 2.2.6: we form the  $\bar{T}$ -centralizer in the reductive semi-direct product  $G \rtimes \bar{T}$ ). Consideration of geometric fibers shows that  $G^{\bar{T}}$  is reductive over  $S$ . But  $Z \subset G^{\bar{T}}$ , so we get a natural monomorphism

$$j : G^{\bar{T}}/Z \rightarrow Z_{\bar{G}}(\bar{T}) = \bar{T}$$

between smooth affine  $S$ -groups. This map is an isomorphism. Indeed, we can pass to geometric fibers (by the fibral isomorphism criterion as in Lemma B.3.1), and then surjectivity of  $j$  follows from the fact that surjections between smooth connected affine groups over a field induce surjections between centralizers for any compatible torus action [Bo91, 11.14, Cor. 2]. Thus,  $j$  is faithfully flat onto its image (by smoothness of its source and target, together with homogeneity and generic flatness), so it is a quotient map and therefore an isomorphism (as  $\ker j = 1$ , by monicity).

The isomorphism  $G^{\bar{T}}/Z = \bar{T}$  implies that  $G^{\bar{T}}$  coincides with the preimage  $\tilde{T}$  of  $\bar{T}$  in  $G$ . We can now apply Lemma 3.3.1(2) to the quotient map  $G^{\bar{T}} \rightarrow G^{\bar{T}}/Z = \bar{T}$  to conclude that the reductive  $G^{\bar{T}}$  is central in itself (i.e., commutative), and so  $G^{\bar{T}}$  is a torus. It suffices to prove that it is a maximal torus in  $G$  (in the sense of Definition 3.2.1). We first may pass to geometric fibers over  $S$ , so now  $S = \text{Spec}(k)$  for an algebraically closed field  $k$ . Consider a torus  $T'$  containing  $G^{\bar{T}}$ . The image  $T'/Z$  in  $\bar{G}$  is a torus (as it is multiplicative type and smooth), yet this image contains the torus  $G^{\bar{T}}/Z = \bar{T}$  that is maximal by hypothesis, so  $T'/Z = G^{\bar{T}}/Z$  and hence  $T' = G^{\bar{T}}$ . Thus,  $G^{\bar{T}}$  is a maximal torus in  $G$ .  $\square$

**Corollary 3.3.6.** *Let  $G \rightarrow S$  be a reductive group scheme, and  $T$  an  $S$ -torus in  $G$  that is maximal (in the sense of Definition 3.2.1).*

- (1) *The center  $Z_G$  is the kernel of the adjoint action  $T \rightarrow \text{GL}(\mathfrak{g})$ .*
- (2) *Assume  $S = \text{Spec } k$  for an algebraically closed field  $k$ . The center  $Z_G$  coincides with the scheme-theoretic intersection of all maximal tori  $T'$  in  $G$ .*

This corollary is a special case of [SGA3, XII, 4.7(d), 4.10]. The description in (1) provides a way to compute  $Z_G$ , since it is often easy to find a torus  $T$  such that  $Z_G(T) = T$  (equivalently  $\mathfrak{g}^T = \mathfrak{t}$ , due to smoothness and fibral connectedness of torus centralizers in  $G$ ).

*Proof.* To prove (1) we may work étale-locally on  $S$  so that  $T$  is split. In that case the equality in (1) was shown in the course of proving Theorem 3.3.4. For (2), since  $Z_G(T') = T'$  for all  $T'$  (as subschemes of  $G$ ), clearly every  $T'$  contains  $Z_G$ . It remains to show that the intersection  $H$  of all  $T'$  is in fact central in  $G$  (as a subgroup scheme). We saw in the proof of Theorem 3.3.4 that  $Z_G$  is the intersection of the normalizers  $N_G(T')$  as we vary through all  $T'$ , so indeed  $H \subset Z_G$ .  $\square$

**Example 3.3.7.** Corollary 3.3.6(1) provides a simple method to compute the center of connected reductive groups. We illustrate with  $G = \text{SL}_n$  over any scheme  $S$ . The diagonal torus  $T$  is maximal, since the case of geometric fibers is well-known. The action of  $T$  on the Lie algebra  $\mathfrak{sl}_n$  over any base  $S$  is the given by the habitual formulas, from which we see that the kernel of the action is  $\mu_n \subset T$ , so  $\mu_n = Z_{\text{SL}_n}$ .

Let  $G \rightarrow S$  be a reductive group scheme. By Corollary 3.3.6 and the existence of a maximal torus étale-locally on the base (Corollary 3.2.7),  $Z_G$  is the kernel of the adjoint representation  $\text{Ad}_G : G \rightarrow \text{GL}(\mathfrak{g})$ . In particular, passing to the Lie algebra, we see that

$$\text{Lie}(Z_G) = \ker(\text{Lie}(\text{Ad}_G)) = \ker(\text{ad}_{\mathfrak{g}})$$

(see [CGP, Prop. A.7.5] for the relation between  $\text{Lie}(\text{Ad}_G)$  and  $\text{ad}_{\mathfrak{g}}$ ). We say  $G$  is *adjoint* when  $Z_G = 1$ ; this property can be checked on geometric fibers since  $Z_G$  is multiplicative type, and is equivalent to the geometric fibers being semisimple of adjoint type (so  $G$  is semisimple).

For semisimple  $G$  of adjoint type, the  $S$ -homomorphism  $\text{Ad}_G : G \rightarrow \text{GL}(\mathfrak{g})$  has trivial kernel. In view of Remark B.1.4 it is not obvious if  $\text{Ad}_G$  is a closed immersion in the adjoint

semisimple case over a general scheme. In fact, it is always a closed immersion for such  $G$ , because any monomorphism from a *reductive* group scheme to a separated group of finite presentation is a closed immersion (Theorem 5.3.5). We will not use this fact.

**Definition 3.3.8.** A homomorphism  $f : G' \rightarrow G$  between smooth affine  $S$ -groups is an *isogeny* if it is a finite flat surjection, and a *central isogeny* if moreover  $\ker f$  is central in  $G'$ .

Over any field of characteristic  $> 0$ , the Frobenius isogeny of a nontrivial connected semisimple group is non-central. In the classical setting there exist examples of isogenies between connected semisimple groups such that the kernel is *commutative* and non-central, though these only exist in characteristic 2. See Remark C.3.6 and [PY06, Lemma 2.2].

It is natural to ask if the isogeny (or central isogeny) property can be checked on fibers over the base. That is, if  $f : G' \rightarrow G$  is a homomorphism between smooth affine  $S$ -groups and if  $f_s$  is an isogeny for all  $s \in S$  then is  $f$  finite flat? Such an  $f$  is certainly surjective, and also flat due to the fibral flatness criterion [EGA, IV<sub>3</sub>, 11.3.10]. Hence,  $\ker f$  is a quasi-finite flat closed normal  $S$ -subgroup of  $G'$ , and  $G = G'/\ker f$  in the sense of fppf sheaves. By fppf descent,  $f$  is finite if and only if  $\ker f$  is  $S$ -finite (cf. Exercise 3.4.5(iii)). But is  $\ker f$  actually  $S$ -finite? And if  $\ker f_s$  is central in  $G_s$  for all  $s \in S$  then is  $\ker f$  central in  $G'$ ? We will give affirmative answers in the reductive case. The case of central isogenies will be treated now; the general case lies a bit deeper (see Proposition 6.1.10).

**Proposition 3.3.9.** *A surjective homomorphism  $f : G' \rightarrow G$  between reductive groups over a scheme  $S$  is a central isogeny if and only if  $f_{\bar{s}}$  has finite central kernel for all  $s \in S$ .*

*Proof.* The implication “ $\Rightarrow$ ” is obvious. For the converse, the flatness of  $f$  follows from the fibral flatness criterion [EGA, IV<sub>3</sub>, 11.3.10], so it remains to show that  $K := \ker f$  is  $S$ -finite and central in  $G'$ . We may assume that  $S$  is noetherian, then local (by direct limit considerations), and finally complete (by faithfully flat descent).

First we show that  $K$  is central. This asserts that the  $G'$ -action on  $K$  by conjugation is trivial, an identity that is sufficient to check on artin local points over  $S$ . Thus, we may assume that  $S = \text{Spec } A$  for an artin local ring  $A$ . Hence, the quasi-finite flat  $K$  is now finite flat. The special fiber  $K_0 \subset Z_{G'_0}$  uniquely lifts to a finite multiplicative type subgroup  $\tilde{K} \subset Z_{G'}$  since  $A$  is artin local and  $Z_{G'}$  is of multiplicative type, and the map  $\tilde{K} \rightarrow G$  induced by  $f$  is trivial on the special fiber, so it is trivial (by rigidity of homomorphisms from groups of multiplicative type; see [SGA3, IX, 3.5]). This implies that  $\tilde{K} \subset \ker f = K$  inside  $G'$ . But this inclusion between finite flat  $A$ -groups induces an equality on special fibers (by construction of  $\tilde{K}$ ), so it is an equality. Hence,  $K$  is central in  $G'$ .

Over a general  $S$ ,  $K$  is closed in the multiplicative type  $Z_{G'}$ . By Exercise 2.4.2, all quasi-finite closed subgroups of a multiplicative type group are finite. Thus,  $K$  is  $S$ -finite.  $\square$

### 3.4. Exercises.

**Exercise 3.4.1.** This exercise proves a very useful lemma of Deligne and Rapoport [DR, II, 1.19] that is a generalization of the finiteness criterion used in the proof of Proposition 3.2.8. Let  $f : X \rightarrow Y$  be a quasi-finite flat and separated map between noetherian schemes, and assume its fiber degree is constant. We seek to prove that  $f$  is finite.

(i) Using that a proper quasi-finite map is finite, reduce to the case  $Y = \text{Spec } R$  for a discrete valuation ring  $R$  (hint: use the valuative criterion for properness).

(ii) By Zariski's Main Theorem, the quasi-finite separated  $X$  over  $Y$  admits an open immersion  $j : X \hookrightarrow \overline{X}$  into a finite  $Y$ -scheme  $\overline{X}$ . With  $Y = \text{Spec } R$  as in (i), arrange that  $\overline{X}$  is also  $R$ -flat and has the same generic fiber as  $X$ .

(iii) Using constancy of fiber degree, deduce that  $j$  is an isomorphism and conclude.

(iv) In case  $f$  is étale, express the result in the language of specialization for constructible étale sheaves.

(v) Remove the noetherian hypotheses without requiring  $f$  to be of finite presentation.

**Exercise 3.4.2.** This exercise proves the fibral isomorphism criterion (Lemma B.3.1) over a noetherian base  $S$ . (The case of general  $S$  reduces to this case by standard limit arguments.)

(i) Reduce to the separated case by using that  $\Delta_h : Y \rightarrow Y \times_{Y'} Y$  satisfies the given hypotheses and is separated.

(ii) Now taking  $h$  to be separated, use the result from Exercise 3.4.1 to reduce to proving  $h$  is flat, and then reduce to the case when  $S$  is artin local.

(iii) For artin local  $S = \text{Spec } A$ , prove  $h$  is a closed immersion, and use  $S$ -flatness of  $Y$  to deduce that the ideal defining it in  $Y'$  vanishes modulo  $\mathfrak{m}_A$ , so  $Y = Y'$ .

**Exercise 3.4.3.** (i) Prove that Corollary 3.3.5 is valid when  $G$  is replaced with a parabolic subgroup of a connected reductive group.

(ii) Over any field  $k$ , show that the Heisenberg group  $U \subset \text{GL}_3$  (the standard upper triangular unipotent subgroup) is a central extension of  $U'' \simeq \mathbf{G}_a \times \mathbf{G}_a$  by  $U' \simeq \mathbf{G}_a$ , with  $U'$  the scheme-theoretic center of  $U$ . Assuming  $\text{char}(k) = p > 0$ , show that the Frobenius kernel  $\ker F_{U/k}$  (see Exercise 1.6.7) is likewise a central extension of  $\ker F_{U''/k} \simeq \alpha_p \times \alpha_p$  by  $\ker F_{U'/k} \simeq \alpha_p$ , and deduce that  $U \rightarrow \overline{U} := U/(\ker F_{U'/k})$  and  $\overline{U} \rightarrow U/(\ker F_{U/k}) = U^{(p)}$  are central isogenies between smooth connected affine  $k$ -groups such that the composite isogeny is not a central isogeny.

(iii) Show that any nontrivial smooth connected unipotent group over a field is a counterexample to both parts of Corollary 3.3.6 when “reductive” is relaxed to “smooth connected affine”.

**Exercise 3.4.4.** Let  $A$  be a finite-dimensional associative algebra over a field  $k$ . Consider the ring functor  $\underline{A} : R \rightsquigarrow A \otimes_k R$  and the group functor  $\underline{A}^\times : R \rightsquigarrow (A \otimes_k R)^\times$  on  $k$ -algebras.

(i) Prove that  $\underline{A}$  is represented by an affine space over  $k$ . Using the  $k$ -scheme map  $N_{A/k} : \underline{A} \rightarrow \mathbf{A}_k^1$  defined functorially by  $u \mapsto \det(m_u)$ , where  $m_u : A \otimes_k R \rightarrow A \otimes_k R$  is left multiplication by  $u \in \underline{A}(R)$ , prove that  $\underline{A}^\times$  is represented by the open *affine* subscheme

$N_{A/k}^{-1}(\mathbf{G}_m)$ . (This is often called “ $A^\times$  viewed as a  $k$ -group”, a phrase that is, strictly speaking, meaningless, since  $A^\times$  does not encode the  $k$ -algebra  $A$ .)

(ii) For  $A = \text{Mat}_n(k)$  show that  $\underline{A}^\times = \text{GL}_n$ , and for  $k = \mathbf{Q}$  and  $A = \mathbf{Q}(\sqrt{d})$  identify it with an explicit  $\mathbf{Q}$ -subgroup of  $\text{GL}_2$  (depending on  $d$ ). Prove  $\underline{A}^\times$  is connected reductive in general.

(iii) For  $A = \text{Mat}_n(k)$ , show that  $N_{A/k} : \underline{A}^\times \rightarrow \mathbf{G}_m$  is  $\det^n$ .

**Exercise 3.4.5.** A diagram  $1 \rightarrow G' \xrightarrow{j} G \xrightarrow{\pi} G'' \rightarrow 1$  of fppf groups over a ring  $k$  is *exact* if  $\pi$  is faithfully flat and  $G' = \ker \pi$ .

(i) For any such diagram, prove  $G'' = G/G'$  via  $\pi$ . Prove a diagram of  $k$ -groups of multiplicative type  $1 \rightarrow H' \rightarrow H \rightarrow H'' \rightarrow 1$  is exact if and only if the associated diagram of  $\mathbf{G}_m$ -dual étale sheaves is exact.

(ii) Prove that  $G''$  is smooth when  $G$  is smooth, even if  $G'$  is not smooth.

(iii) If  $G'$  is finite then prove that  $\pi$  is *finite flat* with fibral degree locally constant on  $\text{Spec } k$ , and that  $\pi_n : \text{SL}_n \rightarrow \text{PGL}_n$  has degree  $n$ . Compute  $\text{Lie}(\pi_n)$ ; when is it surjective?

(iv) Assume  $k$  is a field. Prove that the left exact sequence of Lie algebras arising from a short exact sequence of finite type  $k$ -groups is short exact if  $G'$  is smooth. Give a counterexample to short exactness on Lie algebras with smooth  $G$  and non-smooth  $G'$ .

**Exercise 3.4.6.** Let  $k$  be a ring, and let  $1 \rightarrow H' \rightarrow E \rightarrow H'' \rightarrow 1$  be a short exact sequence of fppf group sheaves with  $H''$  and  $H'$  group schemes of multiplicative type.

(i) Using descent theory and the affineness of  $H'$ , prove that  $E$  is affine and fppf over  $k$ .

(ii) Assume  $E$  is commutative. Prove that if  $k$  is a field then  $E$  is of multiplicative type. What if  $k$  is an arbitrary ring? (See Corollary B.4.2(3).)

(iii) By considering the  $E$ -conjugation action on  $H'$  and étaleness of the automorphism functor of  $H'$ , prove that if  $H''$  has connected fibers (e.g., a torus) then  $H'$  is central in  $E$  and in fact  $E$  is commutative (hint: once centrality is proved, show the commutator of  $E$  factors through a bi-additive pairing  $H'' \times H'' \rightarrow H'$ ). What can we then conclude via (ii)?

**Exercise 3.4.7.** Let  $G$  be a smooth affine group over a field  $k$ . Using fppf descent and the existence of a (geometrically!) maximal torus over some finite extension, generalize Theorem 3.2.6 to apply to  $G$  without restriction on the Cartan subgroups of  $G_{\bar{k}}$ . Deduce that  $G$  admits such a torus over a finite *separable* extension  $k'/k$ .

## 4. ROOTS, COROOTS, AND SEMISIMPLE-RANK 1

Let  $G \rightarrow S$  be a reductive group scheme. By Corollary 3.2.7, at the cost of passing to an étale cover on  $S$  we may arrange that  $G$  contains a maximal torus  $T$ . By deeper work with Cartan subalgebras of  $\mathfrak{g}$  one can even make a maximal torus *Zariski-locally* on the original  $S$  [SGA3, XIV, 3.20]. We do not use this fact (but see Exercise 7.3.7(i)).

Suppose for a moment that  $T$  splits, so there is an isomorphism

$$T \simeq D_S(M) = \underline{\mathrm{Hom}}_{S\text{-gp}}(M_S, \mathbf{G}_m)$$

for a finite free  $\mathbf{Z}$ -module  $M$ . In the evident manner, we get a map of groups  $M \rightarrow \mathrm{Hom}_{S\text{-gp}}(T, \mathbf{G}_m)$ . The  $T$ -action on  $\mathfrak{g} = \mathrm{Lie}(G)$  then corresponds to an  $M$ -grading  $\bigoplus_{m \in M} \mathfrak{g}_m$  of the vector bundle  $\mathfrak{g}$ , where  $t \in T$  acts on the subbundle  $\mathfrak{g}_m$  via multiplication by the unit  $m(t)$  [CGP, Lemma A.8.8]. The formation of the subbundles  $\mathfrak{g}_m$  in  $\mathfrak{g}$  commutes with any base change on  $S$ .

By passing to geometric fibers and using the classical theory of root spaces for connected reductive groups, we see that the locally constant rank of each vector bundle  $\mathfrak{g}_m$  takes values in  $\{0, 1\}$  when  $m \neq 0$  and that  $\mathfrak{g}_0 = \mathrm{Lie}(T)$  (since  $\mathrm{Lie}(T) \subset \mathfrak{g}_0$  as subbundles of  $\mathfrak{g}$ , with equality on geometric fibers over  $S$ ). We will use these observations to develop a general theory of root spaces and root groups, leading (in §5) to both a complete classification in the split case when the geometric fibers have semisimple-rank 1 as well as a deeper understanding of the Weyl group scheme  $W_G(T)$  in the split case.

**4.1. Roots and the dynamic method.** When  $T$  is split as above, any character  $T \rightarrow \mathbf{G}_m$  arises from an element of  $M$  Zariski-locally on  $S$  since (i)  $T \simeq \mathbf{G}_m^r$  for some  $r \geq 0$ , and (ii) any endomorphism of  $\mathbf{G}_m$  over  $S$  is given, Zariski-locally on  $S$ , by  $t \mapsto t^n$  for some  $n \in \mathbf{Z}$ . This leads to the construction of some subbundles of  $\mathfrak{g}$ :

**Definition 4.1.1.** Let  $S$  be a non-empty scheme. Assume that  $G$  admits a split maximal torus  $T$  over  $S$  and fix an isomorphism  $T \simeq D_S(M)$  for a finite free  $\mathbf{Z}$ -module  $M$ . A *root* for  $(G, T)$  is a nonzero element  $a \in M$  such that  $\mathfrak{g}_a$  is a line bundle.

We may and do also view roots as fiberwise nontrivial characters  $T \rightarrow \mathbf{G}_m$  corresponding to *constant* sections of the étale sheaf  $\underline{\mathrm{Hom}}_{S\text{-gp}}(T, \mathbf{G}_m) = M_S$  that are fiberwise nonzero and induce roots in the classical sense on geometric fibers. In Exercise 4.4.1 it is shown that any root for  $(G_{\bar{s}}, T_{\bar{s}})$  arises from a root for  $(G_U, T_U)$  in the above sense for some Zariski-open neighborhood  $U$  of  $s$  in  $S$ .

We will later study root systems arising from reductive group schemes equipped with a split maximal torus, but for now we study a single root  $a : T \rightarrow \mathbf{G}_m$ . Since  $a$  is fiberwise nontrivial, its kernel  $\ker a$  is readily seen to be an  $S$ -group of multiplicative type (in particular, it is  $S$ -flat).

**Example 4.1.2.** In the relative theory, we encounter a new phenomenon that is never seen in the theory over a field: the root spaces  $\mathfrak{g}_a$  are line bundles on  $S$  but they may be *nontrivial* as such, even in the presence of a split maximal torus. The most concrete version is seen

over a Dedekind domain  $A$  with nontrivial class group: if  $J$  is a non-principal integral ideal of  $A$  then for the rank-2 vector bundle  $M = A \oplus J$  the  $A$ -group  $\mathrm{GL}(M)$  has generic fiber  $\mathrm{GL}_2$  and its points valued in any  $A$ -algebra  $R$  consist of matrices in  $\begin{pmatrix} R & J^{-1} \otimes_A R \\ J \otimes_A R & R \end{pmatrix}$  with unit determinant. This is a form of  $\mathrm{GL}_2$  over  $A$  whose root spaces relative to the split diagonal torus are  $J^{\pm 1}$  inside of the standard root spaces for  $\mathfrak{gl}_2$  over the fraction field of  $A$ .

More generally, consider a nontrivial line bundle  $\mathcal{L}$  over a scheme  $S$  (such as  $\mathcal{L} = \mathcal{O}(1)$  on  $S = \mathbf{P}_k^1$  for a field  $k$ ). Let  $G = \mathrm{GL}(\mathcal{E})$  for  $\mathcal{E} = \mathcal{O} \oplus \mathcal{L}$ . This is a Zariski-twisted form of  $\mathrm{GL}_2$  over  $S$ , and it contains the split maximal torus  $T = \mathbf{G}_m^2 = D_S(\mathbf{Z}^2)$  acting as ordinary unit scaling on both  $\mathcal{O}$  and  $\mathcal{L}$ . In this case the Lie algebra of  $G$  is

$$\mathfrak{g} = \underline{\mathrm{End}}(\mathcal{E}) = \mathcal{E} \otimes \mathcal{E}^* = \mathcal{O}^{\oplus 2} \oplus \mathcal{L}^{-1} \oplus \mathcal{L},$$

where  $\mathcal{O}^{\oplus 2} = \mathrm{Lie}(T)$  and the root spaces  $\mathfrak{g}_{\pm a}$  are the subbundles  $\mathcal{L}^{\mp 1}$  (with roots  $\pm a : T \rightarrow \mathbf{G}_m$  corresponding to  $(c_1, c_2) \mapsto (c_1/c_2)^{\pm 1}$ ). Since  $\mathcal{L}$  a nontrivial line bundle, the root spaces that are nontrivial line bundles.

As in the classical theory, we describe characters of  $T$  using additive notation, so we write  $-a$  rather than  $1/a$ , and  $a + b$  rather than  $ab$  (and 0 denotes the trivial element of  $\mathrm{Hom}_{S\text{-gp}}(T, \mathbf{G}_m)$ ).

**Lemma 4.1.3.** *Let  $T = D_S(M)$  be a split maximal torus of a reductive group scheme  $G$  over a non-empty scheme  $S$ . A fiberwise nontrivial  $a : T \rightarrow \mathbf{G}_m$  is a root of  $(G, T)$  if and only if  $-a$  is, in which case the common kernel  $\ker a = \ker(-a)$  contains a unique subtorus  $T_a = T_{-a}$  of relative codimension 1 in  $T$ .*

*In such cases, the reductive centralizer  $G_a := Z_G(T_a)$  has geometric fibers with semisimple-rank 1 and Lie algebra  $\mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$  inside  $\mathfrak{g}$ .*

This is essentially [SGA3, XIX, 3.5].

*Proof.* By the classical theory on geometric fibers, an element  $a \in M$  is a root if and only if its negative  $-a \in M$  is a root. Thus, the same holds over  $S$ . Suppose these are both roots (over  $S$ ). To prove the rest, consider the relative codimension-1 torus  $T_a$  in  $T$  corresponding to the maximal torsion-free quotient of  $M/(\mathbf{Z}a)$  is contained in  $\ker a$ . Its uniqueness as a relative codimension-1 torus killed by  $a$  is clear on (geometric) fibers by the classical theory, and so uniqueness holds over  $S$  by the duality between tori and étale sheaves of finite free  $\mathbf{Z}$ -modules on  $S$ .

The centralizer  $Z_G(T_a)$  has Lie algebra  $\mathfrak{g}^{T_a}$  that is a *subbundle* of  $\mathfrak{g}$  (as for the Lie algebra of any smooth closed subgroup of  $G$ ) and contains the subbundle  $\mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$  of  $\mathfrak{g}$ , so to prove that this containment is an equality we may pass to geometric fibers and use the classical theory. The semisimple-rank 1 property of the geometric fibers of  $G_a$  over  $S$  is likewise classical (and is also evident from the description of the Lie algebra).  $\square$

Now that we have built root spaces  $\mathfrak{g}_a$ , the next step is to build root groups. Let us first recall how root groups are characterized and constructed in the classical theory, so we can see why the classical viewpoint cannot be used when working over a base scheme  $S$ .

Over an algebraically closed field  $k$ , if  $a \in \Phi(G, T)$  and  $T_a = (\ker a)_{\text{red}}^0$  is the unique codimension-1 torus in  $T$  killed by  $a$ , then the centralizer scheme  $Z_G(T_a)$  is a connected reductive subgroup of  $G$  with Lie algebra  $\mathfrak{g}^{T_a}$  containing the nonzero  $a$ -weight space  $\mathfrak{g}_a$ . Thus, the codimension-1 torus  $T_a$  in  $T$  is the maximal central torus in  $Z_G(T_a)$ , so  $\text{Lie}(Z_G(T_a)) = \mathfrak{g}^{T_a}$  has as its  $T$ -roots precisely the nonzero  $\mathbf{Q}$ -multiples of  $a$  in  $\Phi(G, T) \subset X(T)_{\mathbf{Q}}$  since  $a$  is a nontrivial character of  $T/T_a \simeq \mathbf{G}_m$ . In particular,  $\mathcal{D}(Z_G(T_a))$  is a semisimple group having as a maximal torus the 1-dimensional isogeny complement  $T' := (T \cap \mathcal{D}(Z_G(T_a)))_{\text{red}}^0$  to  $T_a$  in  $T$ . By the semisimple rank-1 classification, the group  $\mathcal{D}(Z_G(T_a))$  is isomorphic to either  $\text{SL}_2$  or  $\text{PGL}_2$ , and the isomorphism can be chosen to carry  $T'$  over to the diagonal torus. In particular, the roots for  $(G, T)$  that are  $\mathbf{Q}$ -multiples of  $a$  are precisely  $\pm a$ , and (by composing with conjugation on  $\text{SL}_2$  or  $\text{PGL}_2$  if necessary) we obtain a central isogeny  $q_a : \text{SL}_2 \rightarrow \mathcal{D}(Z_G(T_a))$  carrying the diagonal torus  $D$  onto  $T'$  and carrying the standard upper triangular unipotent subgroup  $U^+$  isomorphically onto a subgroup  $U_a \subset \mathcal{D}(Z_G(T_a))$  that is isomorphic to  $\mathbf{G}_a$  and normalized by  $T$  with  $\text{Lie}(U_a) = \mathfrak{g}_a$ .

A direct inspection of  $\text{SL}_2$  and  $\text{PGL}_2$  then shows that  $U_a$  is uniquely determined by these properties relative to  $(G, T, a)$ , and it is called the *root group* for  $a$ . The composition of  $q_a$  with the standard parameterization  $t \mapsto \text{diag}(t, 1/t)$  yields a unique cocharacter  $a^\vee : \mathbf{G}_m \rightarrow T'$  satisfying  $\langle a, a^\vee \rangle = 2$ , and this is called the *coroot* for  $(G, T, a)$ . We emphasize that the definitions of root groups and coroots in the classical theory rest on the notions of “derived group” and “unipotent subgroup” that are not (yet) available when the base scheme is more general than a field. Also, in the classical theory the definition of a coroot rests on the semisimple-rank 1 classification.

To introduce root groups and coroots attached to roots of a reductive group scheme over a general scheme  $S$ , we require an entirely different approach to build them in the relative theory over  $S$ . Moreover, this will *not* give a new approach in the classical case because in our proofs over  $S$  we will appeal to the known theory of root groups and coroots on geometric fibers over  $S$ . The absence of a “derived group” for reductive groups over  $S$  (which will only become available rather far into the development of the general theory; see Theorem 5.3.1ff.) also makes the statement of the split semisimple-rank 1 classification over  $S$  somewhat more complicated than in the classical case, as there is no elementary way to “split off” the maximal central torus in the relative setting.

In the relative theory, root groups will be instances of a general group construction that we now explain. First, we need some notation. For any finite-rank vector bundle  $\mathcal{E}$  on  $S$ , let  $\mathbf{W}(\mathcal{E}) \rightarrow S$  denote the associated additive  $S$ -group whose set of  $S'$ -points is the additive group of global sections of  $\mathcal{E}_{S'}$  for any  $S$ -scheme  $S'$ . Explicitly,

$$\mathbf{W}(\mathcal{E}) = \text{Spec}_S(\text{Sym}(\mathcal{E}^*))$$

as  $S$ -groups, compatibly with base change on  $S$ . Canonically  $\text{Lie}(\mathbf{W}(\mathcal{E})) \simeq \mathcal{E}$  as vector bundles on  $S$  (with trivial Lie bracket); this isomorphism respects base change on  $S$ . Here is the relative approach to root groups:

**Theorem 4.1.4.** *Let  $G \rightarrow S$  be a reductive group scheme over a non-empty scheme  $S$ ,  $T \simeq D_S(M)$  a split maximal torus, and  $a \in M$  a root. Let  $T$  act on  $\mathbf{W}(\mathfrak{g}_a)$  via  $t.v = a(t)v$  using the vector bundle structure on  $\mathfrak{g}_a$ . There is a unique  $S$ -group homomorphism*

$$\exp_a : \mathbf{W}(\mathfrak{g}_a) \rightarrow G$$

*inducing the canonical inclusion  $\mathfrak{g}_a \hookrightarrow \mathfrak{g}$  on Lie algebras and intertwining the  $T$ -action on  $G$  via conjugation and the  $T$ -action on  $\mathbf{W}(\mathfrak{g}_a)$  via  $a$ -scaling.*

*The map  $\exp_a$  is also a closed immersion factoring through  $Z_G(T_a)$ , its formation commutes with base change on  $S$ , and the multiplication map*

$$(4.1.1) \quad \mathbf{W}(\mathfrak{g}_{-a}) \times T \times \mathbf{W}(\mathfrak{g}_a) \rightarrow Z_G(T_a)$$

*is an isomorphism onto an open subscheme  $\Omega_a \subset Z_G(T_a)$ . Moreover, the semi-direct product of  $T$  against either  $\mathbf{W}(\mathfrak{g}_{\pm a})$  is a closed  $S$ -subgroup of  $G$ .*

The closed subgroup  $\exp_a(\mathbf{W}(\mathfrak{g}_a)) \subset G$  will be called the  $a$ -root group for  $(G, T, M)$ . The proof of Theorem 4.1.4 (to be given in §4.2) uses a “dynamic method” entirely different from the approach given in [SGA3, XX, 1.5–1.14] (which rests on Hochschild cohomology, deformation theory, and fpqc descent). To be precise, the method in [SGA3, XX, §1] gives slightly less; the properties that the maps  $\exp_{\pm a}$  are closed immersions (rather than mere monomorphisms) and that the semi-direct products  $T \ltimes \mathbf{W}(\mathfrak{g}_{\pm a})$  are closed  $S$ -subgroups of  $G$  are not obtained until [SGA3, XX, 5.9]. The dynamic approach will yield these properties immediately, and will provide a more streamlined route through the semisimple-rank 1 classification over  $S$ . Thus, we now digress and introduce the dynamic method.

As motivation for what will follow, we now explain how to extract the standard upper triangular Borel subgroup and its unipotent radical in  $\mathrm{SL}_2$  over an algebraically closed field by means of a  $\mathbf{G}_m$ -action on  $\mathrm{SL}_2$  without ever needing to say “solvable subgroup” or “unipotent subgroup” as in the classical theory. Rather generally, if  $\mathbf{G}_m \times G \rightarrow G$  is an action of  $\mathbf{G}_m$  on a group scheme  $G$  over a base scheme  $S$ , for any  $g \in G(S)$  we say  $\lim_{t \rightarrow 0} t.g$  exists if the orbit map  $\mathbf{G}_m \rightarrow G$  defined by  $t \mapsto t.g$  extends to an  $S$ -scheme morphism  $\mathbf{A}_S^1 \rightarrow G$ ; such an extension is unique if it exists since  $k[x] \subset k[x, 1/x]$  for any ring  $k$ . In such cases, the image of 0 in  $G(S)$  is called  $\lim_{t \rightarrow 0} t.g$ . This limit concept has the following interesting application for  $\mathrm{SL}_2$ :

**Example 4.1.5.** Let  $\lambda : \mathbf{G}_m \rightarrow G := \mathrm{SL}_2$  be the homomorphism  $\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}$ . Define a  $\mathbf{G}_m$ -action on  $G$  as follows: for any  $k$ -algebra  $k'$ ,  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(k')$ , and  $t \in \mathbf{G}_m(k') = k'^{\times}$ , let

$$t.g = \lambda(t) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \lambda(t)^{-1} = \begin{pmatrix} a & t^2 b \\ t^{-2} c & d \end{pmatrix}.$$

Thus,  $\lim_{t \rightarrow 0} t.g$  exists if and only if  $c = 0$  in  $k'$ , and this limit exists and equals 1 if and only if  $c = 0$  and  $a = d = 1$  in  $k'$ .

In other words, the upper triangular subgroup  $B \subset G$  represents the functor of points  $g$  of  $G$  for which  $\lim_{t \rightarrow 0} t.g$  exists, and the strictly upper triangular subgroup  $U$  represents the functor of points  $g$  of  $G$  for which  $\lim_{t \rightarrow 0} t.g$  exists and equals 1. This gives a mechanism for

constructing  $B$  and  $U$  entirely in terms of  $G$  and the  $\mathbf{G}_m$ -action on it, without reference to notions such as solvability or unipotence that are well-suited to working over a field but are not available (at least not in a useful manner) when the base is anything more complicated than a field (such as a discrete valuation ring or non-reduced artin local ring).

Note that if we replace  $\lambda$  with its reciprocal 1-parameter subgroup  $-\lambda : t \mapsto \lambda(1/t)$  then the analogous limiting process above recovers the lower triangular subgroup  $B'$  and its strictly lower triangular subgroup  $U'$ .

The subgroup constructions in the preceding example can be carried out more generally. In [CGP, Ex. 2.1.1] the case  $G = \mathrm{GL}_n$  is worked out over any ring  $k$  when using a diagonal 1-parameter subgroup

$$\lambda(t) = \mathrm{diag}(t^{e_1}, \dots, t^{e_n})$$

for integers  $e_1 \geq \dots \geq e_n$ . In such cases one gets various “parabolic” subgroups and their “unipotent radicals” (keep in mind that the base ring  $k$  may not be a field, hence the quotation marks), all depending on the  $e_j$ 's.

The general context for these limit considerations involves a separated  $S$ -scheme  $X$  equipped with a left action by the  $S$ -group  $\mathbf{G}_m$  (only the case of  $S$ -affine  $X$  will be used below). For an  $S$ -scheme  $S'$  and  $x \in X(S')$ , it makes sense to ask if the orbit map  $\mathbf{G}_m \rightarrow X_{S'}$  over  $S'$  defined by  $t \mapsto t.x$  extends to an  $S'$ -map  $\mathbf{A}_{S'}^1 \rightarrow X_{S'}$ . If such an extension exists then it is unique (because  $X$  is separated and a closed subscheme of  $\mathbf{A}_{S'}^1$  that contains  $\mathbf{G}_m$  must be the entire affine line); we then say that “ $\lim_{t \rightarrow 0} t.x$  exists” and denote the image of  $0 \in \mathbf{A}^1(S')$  in  $X(S')$  as this limit. A fundamental source of such examples arises as follows.

**Definition 4.1.6.** For a ring  $k$  and an affine  $k$ -group  $G$ , a 1-parameter subgroup of  $G$  is a  $k$ -homomorphism  $\lambda : \mathbf{G}_m \rightarrow G$ . (We allow that  $\ker \lambda \neq 1$ , and even that  $\lambda = 1$ , though the latter option is not very useful.)

Any 1-parameter subgroup defines a  $\mathbf{G}_m$ -action on  $G$  via  $t.g = \lambda(t)g\lambda(t)^{-1}$ . By working directly with a weight space decomposition of the coordinate ring  $k[G]$ , a variant of the above procedures in  $\mathrm{SL}_2$  and  $\mathrm{GL}_n$  can be carried out for 1-parameter subgroups  $\lambda : \mathbf{G}_m \rightarrow G$  of rather general affine groups  $G$  without requiring the crutch of a  $\mathrm{GL}_n$ -embedding of  $G$  (which is *not* known to exist locally on the base in general, even for smooth affine groups over the dual numbers over a field; cf. [SGA3, XI, 4.3, 4.6]). It will also be extremely useful to consider an abstract action of  $\mathbf{G}_m$  on  $G$ , not only actions arising from conjugation against a 1-parameter subgroup, so in our formulation below of a vast generalization of Example 4.1.5 we treat abstract  $\mathbf{G}_m$ -actions. The following “dynamic” result summarizes the main conclusions in [CGP, 2.1]. It is the key ingredient in our approach to root groups in reductive group schemes.

**Theorem 4.1.7.** *Let  $G$  be a finitely presented affine group over a ring  $k$ , and consider an action  $\lambda : \mathbf{G}_m \times G \rightarrow G$  by  $\mathbf{G}_m$  on the  $k$ -group  $G$ . Consider the following subfunctors of  $G$  on the category of  $k$ -algebras:*

$$\underline{P}_G(\lambda)(k') = \{g \in G(k') \mid \lim_{t \rightarrow 0} \lambda(t, g) \text{ exists}\}, \quad \underline{U}_G(\lambda)(k') = \{g \in \underline{P}_G(\lambda)(k') \mid \lim_{t \rightarrow 0} \lambda(t, g) = 1\}.$$

Likewise, let  $\underline{Z}_G(\lambda)$  be the subfunctor of points of  $G$  that commute with the  $\mathbf{G}_m$ -action  $\lambda$ .

- (1) These functors are unaffected by replacing  $\lambda$  with  $\lambda^n$  for  $n > 0$ , and they are represented by respective finitely presented closed subgroups  $P_G(\lambda)$ ,  $U_G(\lambda)$ , and  $Z_G(\lambda)$  of  $G$ , with  $U_G(\lambda)$  normalized by  $Z_G(\lambda)$ .
- (2) The fibers of  $U_G(\lambda) \rightarrow S$  are connected. and so are the fibers of  $P_G(\lambda)$  and  $Z_G(\lambda)$  if  $G \rightarrow S$  has connected fibers.
- (3) The multiplication map  $Z_G(\lambda) \times U_G(\lambda) \rightarrow P_G(\lambda)$  is an isomorphism.
- (4) Assume  $G$  is smooth. The subgroups  $P_G(\lambda)$ ,  $U_G(\lambda)$ , and  $Z_G(\lambda)$  are smooth and the multiplication map

$$U_G(-\lambda) \times P_G(\lambda) \rightarrow G$$

is an open immersion. Writing  $\bigoplus_{n \in \mathbf{Z}} \mathfrak{g}_n$  for the weight space decomposition of  $\mathfrak{g}$  under the  $\mathbf{G}_m$ -action (so  $\mathfrak{g}_n = \{v \in \mathfrak{g} \mid t.v = t^n v\}$ ), the Lie algebras of these subgroups are

$$\mathfrak{z}_G(\lambda) = \mathfrak{g}_0 = \mathfrak{g}^{\mathbf{G}_m}, \quad \mathfrak{u}_G(\lambda) = \mathfrak{g}_+ := \bigoplus_{n>0} \mathfrak{g}_n, \quad \mathfrak{p}_G(\lambda) = \mathfrak{z}_G(\lambda) \oplus \mathfrak{u}_G(\lambda).$$

Also, the fibers of  $U_G(\lambda) \rightarrow S$  are unipotent.

In terms of the theory of connected reductive groups over an algebraically closed field, part (3) is analogous to a Levi decomposition of a parabolic subgroup and part (4) is analogous to an open Bruhat cell. When we apply Theorem 4.1.7 to examples in which the action arises from conjugation against a 1-parameter subgroup  $\lambda : \mathbf{G}_m \rightarrow G$ , we shall denote the resulting closed subgroups as  $P_G(\lambda)$ ,  $U_G(\lambda)$ , and  $Z_G(\lambda)$ .

*Proof.* By considering the semi-direct product group  $G' = \mathbf{G}_m \ltimes G$  defined by the given action, and using its evident 1-parameter subgroup  $t \mapsto (t, 1)$ , the general case can be reduced to the special case that  $\lambda$  arises from the conjugation action against a 1-parameter subgroup. This reduction step is explained in [CGP, Rem. 2.1.11], so we now may and do assume the action is conjugation against a 1-parameter subgroup, also denoted  $\lambda : \mathbf{G}_m \rightarrow G$ .

It is clear from the definitions that the three subfunctors of  $G$  under consideration are subgroup functors. The invariance of the functor under passage to  $\lambda^n$  with  $n > 0$  is rather elementary; see Exercise 4.4.2(i) (or [CGP, Rem. 2.1.7]). The existence of  $P_G(\lambda)$  as a closed subscheme of  $G$  is a special case of [CGP, Lemma 2.1.4] (defining the closed subscheme by the ideal of  $k[G]$  generated by the negative weight spaces  $k[G]_n$  for  $n < 0$  relative to the  $\mathbf{G}_m$ -action on  $k[G]$  through  $\lambda$ ; this is typically larger than the  $k$ -linear span of these negative weight spaces). The existence of  $Z_G(\lambda)$  and  $U_G(\lambda)$  are given in [CGP, Lemma 2.1.5], where the finite presentation property is also established (by reduction to the case of noetherian  $k$ ). Explicitly,  $Z_G(\lambda) = P_G(\lambda) \cap P_G(-\lambda)$  and  $U_G(\lambda)$  is the fiber over 1 for the limit morphism  $P_G(\lambda) \rightarrow G$  defined by  $g \mapsto \lim_{t \rightarrow 0} t.g$ . This settles (1). (The proof of the existence of  $P_G(\lambda)$ ,  $Z_G(\lambda)$ , and  $U_G(\lambda)$  as closed subgroup schemes does not require  $G$  to be finitely presented over  $k$ .)

Part (3) is [CGP, Prop. 2.1.8(2)], and part (4) apart from the Lie algebra and unipotence assertions is [CGP, Prop. 2.1.8(3)] (which is the hardest part of the proof). The description

of the Lie algebras in (4) is [CGP, Prop. 2.1.8(1)] (proved by a functorial calculation with dual numbers). The fibral unipotence for  $U_G(\lambda)$  in (4) is part of [CGP, Lemma 2.1.5] (which does not require smoothness of  $G$ , once one has developed a good theory of unipotent group schemes over a field; see [SGA3, XVII, 1.3, 2.1]). Part (2) is [CGP, Prop. 2.1.8(4)]; the idea for  $U_G(\lambda)$  being fiberwise connected is that the limiting process provides paths  $t \mapsto t.g$  linking all points  $g$  of  $U_G(\lambda)$  to 1, and for  $P_G(\lambda)$  and  $Z_G(\lambda)$  the fibral connectedness in the case of smooth  $G$  follows from part (4) whereas in the general case for (2) it requires further work that is specific to groups over fields (e.g., the existence of a  $\mathrm{GL}_n$ -embedding).  $\square$

**Example 4.1.8.** For  $G = \mathrm{SL}_n$  over any ring  $k$  and  $\lambda(t) = \mathrm{diag}(t^{e_1}, \dots, t^{e_n})$  for a strictly decreasing sequence of integers  $e_1 > \dots > e_n$ ,  $P_G(\lambda)$  is the standard upper triangular  $k$ -subgroup,  $U_G(\lambda)$  is its  $k$ -subgroup of strictly upper triangular matrices, and  $Z_G(\lambda)$  is the  $k$ -subgroup of diagonal elements. Passing to  $-\lambda$  yields the lower triangular analogues, exactly as in Example 4.1.5. If the  $e_j$ 's are pairwise distinct but not strictly monotone then  $P_G(\lambda)$  is the conjugate of the upper triangular subgroup by a suitable permutation matrix (corresponding to rearranging the  $e_j$ 's to be strictly decreasing).

**Example 4.1.9.** For any connected reductive group  $G$  over an algebraically closed field  $k$  and any maximal torus  $T$  in  $G$ , as  $\lambda$  varies through the cocharacters of  $T$  the resulting smooth connected subgroups  $P_G(\lambda)$  of  $G$  containing  $T$  are precisely the parabolic subgroups of  $G$  that contain  $T$ . We refer the reader to [CGP, Prop. 2.2.9] for a proof.

Note that  $P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda)$  with  $U_G(\lambda)$  a smooth connected unipotent normal subgroup and  $Z_G(\lambda)$  the centralizer of the torus  $\lambda(\mathbf{G}_m)$  in  $G$  (so  $Z_G(\lambda)$  is connected reductive). Thus,  $U_G(\lambda)$  is the unipotent radical of  $P_G(\lambda)$  and  $Z_G(\lambda)$  is a Levi subgroup of  $P_G(\lambda)$ .

It is immediate from the definitions that the formation of  $P_G(\lambda)$ ,  $U_G(\lambda)$ , and  $Z_G(\lambda)$  commutes with any base change on  $k$ , and that the theory of these groups works over any base scheme  $S$  (not just affine schemes). Here are some easy but very useful “functorial” properties of these subgroups.

**Proposition 4.1.10.** *Let  $(G, \lambda)$  be as in Theorem 4.1.7.*

- (1) *For any finitely presented closed subgroup  $H$  of  $G$  that is stable under the  $\mathbf{G}_m$ -action, with the restricted action on  $H$  also denoted as  $\lambda$ ,*

$$H \cap P_G(\lambda) = P_H(\lambda), \quad H \cap U_G(\lambda) = U_H(\lambda), \quad H \cap Z_G(\lambda) = Z_H(\lambda).$$

*In particular, if  $H$  and  $G$  are smooth then  $H \cap P_G(\lambda)$  is smooth and likewise with  $U_G(\lambda)$  and  $Z_G(\lambda)$ .*

- (2) *Let  $(G', \lambda')$  be another such pair over  $k$ , and  $f : G \rightarrow G'$  a  $\mathbf{G}_m$ -equivariant map. Then  $f$  carries  $P_G(\lambda)$  into  $P_{G'}(\lambda')$ ,  $U_G(\lambda)$  into  $U_{G'}(\lambda')$ , and  $Z_G(\lambda)$  into  $Z_{G'}(\lambda')$ . When  $f$  is flat and surjective and  $G \rightarrow \mathrm{Spec} k$  has connected fibers then the maps*

$$(4.1.2) \quad P_G(\lambda) \rightarrow P_{G'}(\lambda'), \quad U_G(\lambda) \rightarrow U_{G'}(\lambda'), \quad Z_G(\lambda) \rightarrow Z_{G'}(\lambda')$$

are surjections that are flat on fibers over  $\text{Spec } k$ . In particular, when  $f$  is a flat surjection and  $G$  is  $k$ -smooth with connected fibers then the surjections in (4.1.2) are flat.

*Proof.* The assertion in (1) is immediate from the functorial definitions of these subgroups. The functoriality in (2) is likewise obvious, and it suffices to check the surjectivity assertion on fibers over  $\text{Spec } k$ . The surjectivity problem when  $k$  is a field is [CGP, Cor. 2.1.9], which also gives the flatness of the surjections in such cases. (Strictly speaking, [CGP, Cor. 2.1.9] considers the case of conjugation actions against 1-parameter subgroups. The general case reduces to this; see [CGP, Rem. 2.1.11].) When  $f$  is a flat surjection and  $G$  is smooth then  $G'$  in (2) is smooth as well, so all of the subgroups of interest in (4.1.2) are smooth, by Theorem 4.1.7(4). Hence, in such cases the flatness of the induced surjective maps arising from  $f$  can be checked on fibers over  $\text{Spec } k$ , for which we have already noted that the flatness holds.  $\square$

**4.2. Root groups and coroots.** Returning to our study of a split maximal torus  $T = D_S(M)$  in a reductive group scheme  $G \rightarrow S$  over a non-empty scheme  $S$ , now we use the preceding dynamic constructions to build “root groups”  $U_{\pm a}$  in  $G_a = Z_G(T_a) = G_{-a}$  for any root  $a$  of  $(G, T)$  (with  $T_a = T_{-a}$  as in Lemma 4.1.3). Before we prove Theorem 4.1.4, we work out what it is saying for  $\text{SL}_2$  over any ring.

**Example 4.2.1.** We saw in Example 4.1.5 that for the standard 1-parameter subgroup  $\lambda(t) = \text{diag}(t, 1/t)$  in  $G = \text{SL}_2$ , the resulting subgroups  $U(\lambda)$  and  $U(-\lambda)$  are respectively the strictly upper and strictly lower triangular subgroups, corresponding to the roots  $\pm a : T \rightrightarrows \mathbf{G}_m$  satisfying  $a(\lambda(t)) = t^2$  and  $(-a)(\lambda(t)) = t^{-2}$ . The proof of Theorem 4.1.4 will show that these subgroups are respectively the images of  $\exp_a$  and  $\exp_{-a}$ . Via the standard trivializations of  $\mathfrak{g}_{\pm a}$  given by  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  respectively (to identify  $\mathbf{W}(\mathfrak{g}_{\pm a})$  with  $\mathbf{W}(\mathcal{O}_S) = \mathbf{G}_a$ ), we have  $\exp_a(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$  and  $\exp_{-a}(z) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ . Note that the “exponential” terminology is reasonable: the nilpotent matrices  $n_+(z) = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}$  and  $n_-(z) = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}$  satisfy  $n_{\pm}(z)^2 = 0$ , so we imagine that  $e^{n_{\pm}(z)}$  should mean  $1 + n_{\pm}(z) = \exp_{\pm a}(z)$ .

Here is the dynamic proof of Theorem 4.1.4.

*Proof.* By  $T$ -equivariance, since  $T_a$  acts trivially on  $\mathbf{W}(\mathfrak{g}_a)$  it follows that if  $\exp_a$  is to exist then it must factor through the reductive subgroup  $Z_G(T_a)$ . We may therefore replace  $G$  with  $Z_G(T_a)$ , so by Lemma 4.1.3 we are reduced to the case that  $G$  has geometric fibers of semisimple-rank 1 with  $T_a$  central in  $G$  and  $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$ .

In view of the asserted uniqueness, the compatibility with base change will be automatic and we may work étale-locally on  $S$  to prove the theorem. Choose a cocharacter  $\lambda : \mathbf{G}_m \rightarrow T$  corresponding to an element of the dual lattice  $M^\vee$  such that the pairing  $\langle a, \lambda \rangle \in \mathbf{Z}$  (corresponding to the element  $a \circ \lambda \in \text{End}_{S\text{-gp}}(\mathbf{G}_m) = \mathbf{Z}_S(S)$  that is a constant section) lies in  $\mathbf{Z}_{>0}$ . Note that it may be impossible to arrange that this pairing is 1, since  $a$  may be divisible by 2 in the character group of  $T$  (as happens for the long roots of  $\text{Sp}_{2n}$ ).

The inclusion  $T \subset Z_G(\lambda)$  of smooth closed subgroups of  $G$  is an equality. Indeed, it suffices to prove equality on geometric fibers over  $S$ , both of which are connected, so it is enough to

compare their Lie algebras inside  $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$ . Since  $\langle \pm a, \lambda \rangle \neq 0$ , the description of  $\text{Lie}(Z_G(\lambda))$  in Theorem 4.1.7(4) implies that this Lie algebra must coincide with  $\mathfrak{t}$ , as desired.

Using the indicated  $T$ -action on  $\mathbf{W}(\mathfrak{g}_a)$ , composing with  $\lambda$  defines a  $\mathbf{G}_m$ -action on  $\mathbf{W}(\mathfrak{g}_a)$ , namely a point  $c$  of  $\mathbf{G}_m$  acts via multiplication by  $a(\lambda(c)) = c^{\langle a, \lambda \rangle}$ . Since  $\langle a, \lambda \rangle > 0$ , it follows that for  $H := \mathbf{W}(\mathfrak{g}_a)$  we have  $H = U_H(\lambda)$ . The  $T$ -equivariance requirement on  $\exp_a$  implies that (if it exists) it must be  $\mathbf{G}_m$ -equivariant, so it must carry  $H$  into  $U_G(\lambda)$ . By Theorem 4.1.7(4),  $\text{Lie}(U_G(\lambda))$  is the “positive” weight space for the  $\mathbf{G}_m$ -action on  $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$ . Since  $\mathbf{G}_m$  acts on  $\mathfrak{t}$  trivially and on  $\mathfrak{g}_{\pm a}$  via  $t.v = t^{\langle \pm a, \lambda \rangle} v$  with  $\langle a, \lambda \rangle > 0$ , we conclude that  $\text{Lie}(U_G(\lambda)) = \mathfrak{g}_a$ . Hence, if  $\exp_a$  is to exist at all then it must factor through an  $S$ -homomorphism  $\mathbf{W}(\mathfrak{g}_a) \rightarrow U_G(\lambda)$  that induces an *isomorphism* on Lie algebras. But  $U_G(\lambda)$  is  $S$ -smooth with connected fibers, so if  $\exp_a$  is to exist then it must be an étale homomorphism onto  $U_G(\lambda)$ .

We can do better: such an étale map  $\mathbf{W}(\mathfrak{g}_a) \rightarrow U_G(\lambda)$  must be an isomorphism. Indeed, it suffices to check the isomorphism property on geometric fibers over  $S$ , and both sides have geometric fiber  $\mathbf{G}_a$  equipped with a  $\mathbf{G}_m$ -action inducing the *same* scaling action  $t.v = t^{\langle a, \lambda \rangle} v$  on the Lie algebra. But the only  $\mathbf{G}_m$ -actions on  $\mathbf{G}_a$  over a field are  $t.x = t^n x$  for  $n \in \mathbf{Z}$ , and the effect on the Lie algebra detects  $n$  by the same formula. In other words, the induced map between geometric fibers (if  $\exp_a$  exists) must be an étale endomorphism of  $\mathbf{G}_a$  that is equivariant for the action  $t.x = t^{\langle a, \lambda \rangle} x$ , and it is easy to check that over a field  $F$  the only such endomorphisms are  $x \mapsto cx$  for  $c \in F^\times$ , which are visibly isomorphisms.

To summarize, if  $\exp_a$  is to exist then it must be a  $\mathbf{G}_m$ -equivariant isomorphism  $\mathbf{W}(\mathfrak{g}_a) \simeq U_G(\lambda)$  (so in particular it must be a closed immersion into  $G$ ). Note conversely that any such  $\mathbf{G}_m$ -equivariant isomorphism is  $T$ -equivariant as a map to  $G$  since  $T_a$  acts trivially on both sides and  $\mathbf{G}_m \times T_a \rightarrow T$  defined by  $(c, t) \mapsto \lambda(c)t$  is an isogeny of tori (ensuring that  $T$ -equivariance is equivalent to the combination of  $T_a$ -equivariance and  $\mathbf{G}_m$ -equivariance). Thus, for the existence and uniqueness of  $\exp_a$  it is necessary and sufficient to prove the existence and uniqueness of a  $\mathbf{G}_m$ -equivariant isomorphism of  $S$ -groups  $\mathbf{W}(\mathfrak{g}_a) \simeq U_G(\lambda)$  that induces the identity map on Lie algebras. Once we have  $\exp_{\pm a}$  in hand, the open immersion claim in the theorem is immediate from Theorem 4.1.7(4) since  $T = Z_G(\lambda)$  and necessarily  $\exp_{\pm a}$  carries  $\mathbf{W}(\mathfrak{g}_{\pm a})$  over to  $U_G(\pm\lambda)$ . This also gives that each  $T \times \mathbf{W}(\mathfrak{g}_{\pm a})$  is a closed  $S$ -subgroup of  $G$ , as each is identified with  $Z_G(\pm\lambda) \times U_G(\pm\lambda) = P_G(\pm\lambda)$ .

The uniqueness of  $\exp_a$  amounts to the assertion that  $\mathbf{W}(\mathfrak{g}_a)$  has no nontrivial automorphism that is  $\mathbf{G}_m$ -equivariant and induces the identity on the Lie algebra. Working Zariski-locally so that  $\mathfrak{g}_a$  admits a trivialization as a line bundle, this is the assertion that  $\mathbf{G}_a$  over  $S$  admits no nontrivial automorphism that is equivariant for the  $\mathbf{G}_m$ -action  $t.x = t^{\langle a, \lambda \rangle} x$  and induces the identity on  $\text{Lie}(\mathbf{G}_a)$ . An endomorphism of  $\mathbf{G}_a$  over a ring  $k$  is precisely an additive polynomial, and equivariance for  $t.x = t^n x$  with  $n \neq 0$  says precisely that the polynomial is  $x \mapsto cx$  for some  $c \in k$ , so the effect on the Lie algebra is multiplication by  $c$ . Thus, the identity condition on the Lie algebra forces  $c = 1$ , as desired.

It remains to prove the existence of  $\exp_a$ . The smooth  $S$ -group  $U_G(\lambda)$  has fibers that are connected and unipotent (see Theorem 4.1.7(2),(4)) and of dimension 1 (as the Lie algebra

is the line bundle  $\mathfrak{g}_a$ ), so the geometric fibers of  $U_G(\lambda)$  are  $\mathbf{G}_a$ . Beware that it is not obvious that even the actual fibers of  $U_G(\lambda)$  over  $S$  are isomorphic to  $\mathbf{G}_a$  (let alone that this holds Zariski-locally over  $S$ ); the classification of forms of  $\mathbf{G}_a$  is rather subtle, even over (imperfect) fields, because the automorphism functor of  $\mathbf{G}_a$  is quite bad (see [Ru]). The key to bypassing such difficulties is the  $\mathbf{G}_m$ -action on  $U_G(\lambda)$ , as we now explain.

The uniqueness of  $\exp_a$  allows us to work étale-locally to prove its existence, so we may assume that the smooth surjection  $U_G(\lambda) \rightarrow S$  admits a section  $\sigma$  disjoint from the identity section. In this case, we have:

**Lemma 4.2.2.** *The  $S$ -group  $U_G(\lambda)$  is isomorphic to  $\mathbf{G}_a$ , via an isomorphism carrying its  $\mathbf{G}_m$ -action over to  $t.x = t^{\langle a, \lambda \rangle} x$ .*

Granting this lemma, let us conclude the argument. Fix such an isomorphism of  $S$ -groups. This induces a canonical basis of  $\mathrm{Lie}(U_G(\lambda)) = \mathfrak{g}_a$ , which in turn identifies  $\mathbf{W}(\mathfrak{g}_a)$  with  $\mathbf{G}_a$  carrying the  $\mathbf{G}_m$ -action over to  $t.x = t^{\langle a, \lambda \rangle} x$ . Thus, visibly  $\mathbf{W}(\mathfrak{g}_a)$  and  $U_G(\lambda)$  are  $\mathbf{G}_m$ -equivariantly isomorphic as  $S$ -groups (namely, isomorphic to  $\mathbf{G}_a$  with the indicated  $\mathbf{G}_m$ -action). Pick one such isomorphism, so its effect on the Lie algebra is multiplication on  $\mathfrak{g}_a$  by some global unit. Scaling on  $\mathbf{W}(\mathfrak{g}_a)$  by the reciprocal of that unit then provides the desired  $\exp_a$ .

Now it remains to prove Lemma 4.2.2. By the functorial definition of  $U_G(\lambda)$ , the orbit map  $\mathbf{G}_m \rightarrow U_G(\lambda)$  defined by  $t \mapsto t.\sigma$  over  $S$  extends to an  $S$ -scheme map

$$q : \mathbf{A}_S^1 \rightarrow U_G(\lambda)$$

that carries 1 to  $\sigma$  and is  $\mathbf{G}_m$ -equivariant when using the *usual*  $\mathbf{G}_m$ -scaling action on  $\mathbf{A}_S^1$  (as it suffices to check such equivariance on the open  $\mathbf{G}_m$  inside the affine  $S$ -line). Note also that  $q(0) = 1$  by passing to the “limit” as  $t \rightarrow 0$  on the formula  $q(t) = t.\sigma$ . On fibers over a geometric point  $\bar{s}$  of  $S$ , we may identify the group  $U_G(\lambda)_{\bar{s}}$  with  $\mathbf{G}_a$  carrying the point  $\sigma(\bar{s})$  over to 1, and the  $\mathbf{G}_m$ -action on  $U_G(\lambda)_{\bar{s}}$  goes over to scaling by  $t^{\langle a, \lambda \rangle}$  on  $\mathbf{G}_a$ . Thus,  $q_{\bar{s}}$  is identified with an endomorphism of the affine line over  $k(\bar{s})$  that satisfies  $q_{\bar{s}}(t) = t^{\langle a, \lambda \rangle}$  for  $t \in \mathbf{G}_m$  and hence for  $t \in \mathbf{A}^1$ .

Letting  $n = \langle a, \lambda \rangle > 0$ , we claim that  $\mu_n$  acts trivially on  $U_G(\lambda)$ . Indeed, the centralizer subgroup scheme  $U_G(\lambda)^{\mu_n}$  is a *smooth* closed subgroup of  $U_G(\lambda)$  (as  $\mu_n$  is multiplicative type), so this subgroup equals  $U_G(\lambda)$  if and only if it does so on fibers over  $S$ . This reduces the  $\mu_n$ -triviality claim on  $U_G(\lambda)$  to the case of geometric fibers, where it is clear from our concrete description ( $U_G(\lambda)_{\bar{s}} = \mathbf{G}_a$  with  $\mathbf{G}_m$ -action  $t.x = t^n x$ ).

It follows from the  $\mathbf{G}_m$ -equivariance of the map  $q$  that it is invariant under the natural  $\mu_n$ -action on the affine line. The  $n$ th-power endomorphism of the affine  $S$ -line is a categorical quotient by the  $\mu_n$ -action in the category of  $S$ -affine schemes, so  $q$  factors through an  $S$ -scheme map

$$\bar{q} : \mathbf{A}_S^1 \rightarrow U_G(\lambda)$$

carrying 0 to 1 that is  $\mathbf{G}_m$ -equivariant when using the action  $t.x = t^n x$  on the affine line (and the conjugation action on  $U_G(\lambda)$  via  $\lambda$ ). On geometric fibers over  $S$ , our earlier calculations

with each  $q_{\bar{s}}$  imply that each  $\bar{q}_{\bar{s}}$  is identified with an *automorphism* of the affine line over  $k(\bar{s})$  (as a scheme), so  $\bar{q}$  is an isomorphism of  $S$ -schemes. Thus, it remains to prove that  $\bar{q}$  is an  $S$ -homomorphism.

We have seen above that the  $\mathbf{G}_m$ -action on  $U_G(\lambda)$  makes  $\mu_n$  act trivially, and the same holds for the  $\mathbf{G}_m$ -action on the affine line over  $S$  that is the source for  $q$ . Thus, the domain of  $\bar{q}$  inherits an action by the quotient  $\mathbf{G}_m/\mu_n \simeq \mathbf{G}_m$  that makes the  $S$ -scheme isomorphism  $\bar{q}$  identify  $U_G(\lambda)$  with an  $S$ -group structure on  $\mathbf{A}_S^1$  that has 0 as the identity and is equivariant for the ordinary  $\mathbf{G}_m$ -scaling. The  $S$ -homomorphism property for  $\bar{q}$  is reduced to checking that addition is the only such group law on the affine  $S$ -line.

We may assume  $S = \text{Spec } k$  for a ring  $k$ , so the abstract group law transferred from  $U_G(\lambda)$  via  $\bar{q}$  is an  $m \in k[x, y]$  satisfying  $m(x, 0) = x$ ,  $m(0, y) = y$ , and  $m(tx, ty) = tm(x, y)$  for  $t \in \mathbf{G}_m$ . It is clear by homogeneity considerations in  $t$  that the final condition forces  $m = cx + c'y$  for some  $c, c' \in k$ , so the first two conditions imply  $m = x + y$ .  $\square$

**Definition 4.2.3.** The image  $\exp_a(\mathbf{W}(\mathfrak{g}_a)) \subset G$  is denoted  $U_a$  and is called the *a-root group*.

Note that if the line bundle  $\mathfrak{g}_a$  is trivialized (as may be done Zariski-locally on  $S$ ) then  $U_a$  is identified with the additive group  $\mathbf{G}_a$  over  $S$ .

**Remark 4.2.4.** The existence of the  $T$ -equivariant  $\exp_a$  implies that to give an  $S$ -group isomorphism  $p_a : \mathbf{G}_a \simeq U_a$  intertwining the  $T$ -action on  $U_a$  and the action  $t.x = a(t)x$  on  $\mathbf{G}_a$  is precisely the same as to choose a global trivializing section  $X$  of the line bundle  $\mathfrak{g}_a$  (via  $p_a(z) = \exp_a(zX)$ ). This follows from the faithful flatness of  $a : T \rightarrow \mathbf{G}_m$  and the easy fact that the only automorphisms of the  $S$ -group  $\mathbf{G}_a$  that are equivariant for the standard  $\mathbf{G}_m$ -action are scaling by global units of  $S$ . Such isomorphisms  $p_a$  are called *parameterizations* of the root group  $U_a$ .

With root spaces  $\mathfrak{g}_a$  and root groups  $U_a$  now constructed for roots  $a$  arising from triples  $(G, T, M)$  over any non-empty scheme  $S$ , the remaining ingredient before we can discuss the split semisimple-rank 1 classification over  $S$  is the definition of coroots in the relative setting. As we have already noted, this requires a viewpoint rather different from the classical case over a field. Indeed, in the classical case coroots are defined using the classification of semisimple groups of rank 1 (over an algebraically closed field), whereas in the relative setting over a base scheme everything gets turned upside down: we need coroots even to state the split semisimple-rank 1 classification over a base scheme.

In the classical theory over an algebraically closed field, it makes sense to consider the closed subgroup  $\langle U_a, U_{-a} \rangle$  of  $G$  generated by a pair of “opposite” root groups, and one shows that this is either  $\text{SL}_2$  or  $\text{PGL}_2$ . But in the theory over rings it does not make sense to form a (smooth closed) subgroup “generated” by a pair of smooth closed subgroups of a general smooth affine group. Likewise, one cannot use the alternative description  $\mathcal{D}(Z_G(T_a))$  of  $\langle U_a, U_{-a} \rangle$  since the notion of “derived group” does not generally make sense for smooth closed subgroups of a general smooth affine group over a ring. The key to our success over  $S$  is to simultaneously characterize the coroot  $a^\vee : \mathbf{G}_m \rightarrow Z_G(T_a)$  and compatible trivializations of  $\mathfrak{g}_a$  and  $\mathfrak{g}_{-a}$  in intrinsic terms. For inspiration, once again we turn to the case of  $\text{SL}_2$ :

**Example 4.2.5.** Let  $G = \mathrm{SL}_2$  over a non-empty scheme  $S$  and let  $D$  be the diagonal split maximal torus  $D_S(\mathbf{Z})$  equipped with the standard positive root  $a : D \simeq \mathbf{G}_m$  inverse to  $\lambda : t \mapsto \mathrm{diag}(t, 1/t)$ . This yields the standard open cell

$$\Omega_a = U_{-a} \times D \times U_a = U_G(-\lambda) \times Z_G(\lambda) \times U_G(\lambda),$$

and likewise there is the other open cell

$$\Omega_{-a} = U_a \times D \times U_{-a} \subset G$$

(with multiplication involving the root groups  $U_{\pm a}$  appearing in the opposite order).

Consider the product  $U_a \times U_{-a}$  in  $\Omega_{-a}$ . Its points are

$$\exp_a(z) \exp_{-a}(z') = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z' & 1 \end{pmatrix} = \begin{pmatrix} 1 + zz' & z \\ z' & 1 \end{pmatrix}.$$

When does such a point lie in other other open cell  $\Omega_a = U_{-a} \times D \times U_a$ ? The points of  $\Omega_a$  are those of the form

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix} \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c & cu' \\ cu & 1/c + cuu' \end{pmatrix},$$

which are precisely the points of  $\mathrm{SL}_2$  whose upper left entry is a unit. Hence,  $\exp_a(z) \exp_{-a}(z')$  lies in  $\Omega_a$  if and only if  $1 + zz'$  is a unit (with  $z, z'$  arbitrary points of  $\mathbf{G}_a$ ), in which case  $c = 1 + zz'$ ,  $u = z'/(1 + zz')$ , and  $u' = z/(1 + zz')$ .

Varying over points of  $\mathrm{SL}_2$  valued in all  $S$ -schemes, we see that the duality between  $\mathfrak{g}_a$  and  $\mathfrak{g}_{-a}$  given by multiplication in  $\mathcal{O}_S$  characterizes when  $\exp_a(z) \exp_{-a}(z')$  lies in  $\Omega_a$  via the condition “ $1 + zz' \in \mathbf{G}_m$ ”. Likewise, since

$$\exp_a(z) \exp_{-a}(z') = \begin{pmatrix} 1 & 0 \\ z'/(1 + zz') & 1 \end{pmatrix} \begin{pmatrix} 1 + zz' & 0 \\ 0 & (1 + zz')^{-1} \end{pmatrix} \begin{pmatrix} 1 & z/(1 + zz') \\ 0 & 1 \end{pmatrix},$$

we see that the coroot  $a^\vee(t) := \mathrm{diag}(t, 1/t)$  is recovered by noting that the  $D$ -component of  $\exp_a(z) \exp_{-a}(z')$  in  $\Omega_a$  is exactly  $a^\vee(1 + zz')$ .

The calculations with  $\mathrm{SL}_2$  in Example 4.2.5 motivate:

**Theorem 4.2.6.** *Let  $G$  be a reductive group over a scheme  $S$ , with fibers of semisimple-rank 1. Assume there exists a split maximal torus  $T = D_S(M)$  in  $G$  and a root  $a : T \rightarrow \mathbf{G}_m$  arising from  $M$ . Let  $U_{\pm a}$  and  $\Omega_a := U_{-a} \times T \times U_a$  and  $\Omega_{-a} := U_a \times T \times U_{-a}$  be the associated root groups and “open cells” in  $G$  as in Theorem 4.1.4.*

*There is a unique  $\mathcal{O}_S$ -bilinear (hence  $\mathbf{G}_m$ -equivariant) pairing of line bundles  $\mathfrak{g}_a \times \mathfrak{g}_{-a} \rightarrow \mathcal{O}_S$  (denoted  $(X, Y) \mapsto XY$ ) and  $S$ -homomorphism  $a^\vee : \mathbf{G}_m \rightarrow T$  such that the following two conditions hold:*

- (1) *for any  $S$ -scheme  $S'$  and points  $\exp_a(X) \in U_a(S')$  and  $\exp_{-a}(Y) \in U_{-a}(S')$ , the  $S'$ -valued point*

$$\exp_a(X) \exp_{-a}(Y) \in \Omega_{-a} \subset G$$

*lies in the “open cell”  $\Omega_a$  if and only if  $1 + XY$  is a unit on  $S'$ .*

(2) when this unit condition is satisfied,

$$(4.2.1) \quad \exp_a(X) \exp_{-a}(Y) = \exp_{-a} \left( \frac{Y}{1+XY} \right) a^\vee(1+XY) \exp_a \left( \frac{X}{1+XY} \right) \in \Omega_a.$$

In particular, the formation of this bilinear pairing and  $a^\vee$  commute with base change on  $S$ .

Moreover, the pairing  $(X, Y) \mapsto XY$  is a perfect duality, and  $a \circ a^\vee = 2$  (i.e.,  $a(a^\vee(c)) = c^2$  for  $c \in \mathbf{G}_m$ ).

This result is [SGA3, XX, 2.1], whose proof there involves heroic calculations. Our proof of Theorem 4.2.6 will be long, but it involves very few calculations and will yield some auxiliary results that rapidly lead to a Zariski-local version of the classification of “split” reductive groups of semisimple-rank 1 (in Theorem 5.1.8). First, we make some observations.

An interesting consequence of the duality in Theorem 4.2.6 is that the line bundle  $\mathfrak{g}_a$  is globally trivial if and only if  $\mathfrak{g}_{-a}$  is, so likewise  $U_a$  admits a parameterization in the sense of Remark 4.2.4 if and only if  $U_{-a}$  does. When such parameterizations  $p_{\pm a} : \mathbf{G}_a \simeq U_{\pm a}$  exist, we say (following [SGA3, XX, 2.6.1]) that they are *linked* if they correspond to dual bases for  $\mathfrak{g}_a$  and  $\mathfrak{g}_{-a}$ . (Another convention, advocated by Demazure in more recent times, is to declare bases of  $\mathfrak{g}_a$  and  $\mathfrak{g}_{-a}$  to be linked when they are negative dual to each other. This has the advantage that the open subscheme  $\Omega_a \cap \Omega_{-a}$  in  $\Omega_a$  is defined by  $XY \neq 1$  rather than  $XY \neq -1$ . It thereby eliminates signs in certain equations.)

Clearly for a given parameterization of  $U_a$ , there exists a unique one of  $U_{-a}$  to which it is linked. Note also that necessarily  $(-a)^\vee = -a^\vee$ . Indeed, this is a known fact in the classical theory, and in general it can be deduced from geometric fibers over  $S$ .

*Proof of Theorem 4.2.6.* We may and do assume  $S$  is non-empty. To prove uniqueness, we first note that  $Z_G = \ker a$  by Corollary 3.3.6(1). Concretely, inside  $T = D_S(M)$  we have  $Z_G = \ker a = D_S(M/\mathbf{Z}a)$ . (Note that  $M/\mathbf{Z}a$  may not be torsion-free.)

The quotient  $G/Z_G$  makes sense as a reductive group scheme in which  $T/Z_G = D_S(\mathbf{Z}a)$  is a maximal torus such that the induced character  $\bar{a} : T/Z_G \rightarrow \mathbf{G}_m$  is an isomorphism. In particular, since  $\ker \bar{a} = 1$ , it follows that  $G/Z_G$  has trivial schematic center (Corollary 3.3.5). Moreover, the behavior of the open immersion (4.1.1) under  $Z_G$ -scaling shows that the natural maps  $\mathbf{W}(\mathfrak{g}_{\pm a}) \rightrightarrows G/Z_G$  are isomorphisms onto closed subgroups  $V_{\pm \bar{a}}$  normalized by  $T/Z_G$ , and moreover that the adjoint action of  $G/Z_G$  on its Lie algebra makes  $T/Z_G$  have  $\text{Lie}(V_{\pm \bar{a}})$  as a weight space for the character  $\pm \bar{a}$ . In other words, the quotient map  $G \rightarrow G/Z_G$  induces an isomorphism  $U_{\pm a} \simeq U_{\pm \bar{a}}$  and identifies the “open cell”  $\Omega_{\pm \bar{a}}$  as  $\Omega_{\pm a}/Z_G$ .

We conclude that for the purpose of constructing a bilinear pairing between  $\mathfrak{g}_a$  and  $\mathfrak{g}_{-a}$  that characterizes the points of  $U_a \times U_{-a}$  whose product in  $G$  lies in the open subscheme  $\Omega_a = U_{-a} \times T \times U_a$ , it is harmless to pass to  $G/Z_G$ . Likewise, for the proof of uniqueness it is harmless to pass to  $G/Z_G$  provided that we can settle uniqueness in general over an algebraically closed field, since any  $S$ -homomorphism  $\mathbf{G}_m \rightarrow T$  is uniquely determined by its effect on geometric fibers over  $S$ . Thus, for the proof of uniqueness it suffices to treat two basic cases:  $S = \text{Spec}(k)$  for an algebraically closed field  $k$ , and the case  $Z_G = 1$  over a general  $S$ .

Consider the situation over an algebraically closed field  $k$ . In this case we know that  $G/Z_G = \mathrm{PGL}_2$  by the classical theory, and by conjugacy of maximal tori we can choose this identification to carry the maximal torus  $T/Z_G$  over to the diagonal torus  $\overline{D}$  parameterized by  $\overline{\lambda}(t) := \mathrm{diag}(t, 1)$ . Applying conjugation by a representative of the nontrivial element of  $W_{\mathrm{PGL}_2}(\overline{D})$  (such as the standard Weyl element) if necessary, we can also arrange that  $\overline{a}$  goes over to the unique root for  $\overline{D}$  that satisfies  $\overline{a}(\overline{\lambda}(t)) = t$ . In this case, existence is settled by using the calculations in Example 4.2.5 and composing with the degree-2 central isogeny  $\mathrm{SL}_2 \rightarrow \mathrm{PGL}_2$  (e.g., we take the coroot to be the composition of  $t \mapsto \mathrm{diag}(t, 1/t) \in \mathrm{SL}_2$  with the central isogeny to  $\mathrm{PGL}_2$ , which is to say  $\overline{a}^\vee(t) = \mathrm{diag}(t^2, 1)$ ). These calculations also imply uniqueness of the pairing of root spaces, since we can pass to  $G/Z_G = \mathrm{PGL}_2$  and observe that any possibility for the bilinear pairing must be a multiple of the standard one by some  $c \in k$ , yet the unit conditions on  $1 + xy$  and  $1 + cxy$  for varying  $x, y \in k$  do not coincide unless  $c = 1$  since  $k$  is an algebraically closed field.

Since bilinear pairing between root spaces is uniquely determined (over the algebraically closed field  $k$ ) by composing the formula (4.2.1) over  $k$  with projection to  $G/Z_G$  we see that any possibility for the coroot  $a^\vee : \mathbf{G}_m \rightarrow T$  over  $k$  must have composition with  $T \rightarrow T/Z_G = \overline{D} \subset \mathrm{PGL}_2$  given by  $c \mapsto \mathrm{diag}(c, 1/c) = \mathrm{diag}(c^2, 1) \bmod \mathbf{G}_m$ . Thus,  $a^\vee$  is unique up to multiplication against a cocharacter  $\mu : \mathbf{G}_m \rightarrow Z_G$ . But any such cocharacter  $\mu$  factors through the torus  $(Z_G)_{\mathrm{red}}^0$  that is the maximal central torus in  $G$ , and this has finite intersection with the connected semisimple  $\mathcal{D}(G)$ . Since (4.2.1) over  $k$  forces  $a^\vee$  to be valued in  $\langle U_a, U_{-a} \rangle = \mathcal{D}(G)$ , it follows that  $\mu$  is trivial, so  $a^\vee$  is also unique. This completes the proof of existence and uniqueness over an algebraically closed field, and the additional properties (perfectness of the bilinear pairing, and the identity  $a \circ a^\vee = 2$ ) are immediate from these calculations (since  $a$  factors through  $T/Z_G$ ).

Returning to the situation over a general (non-empty) base  $S$ , the results over an algebraically closed field imply uniqueness of the coroot in general, as well as perfectness of the bilinear pairing (if it exists) and the identity  $a \circ a^\vee = 2$  (as the latter concerns an endomorphism of  $\mathbf{G}_m$  and so can be checked on geometric fibers). We will next prove uniqueness in general, and then it will remain to address existence. As we have already noted, for the proof of uniqueness (of the bilinear pairing, as the case of the coroot is settled) we may and do assume  $Z_G = 1$ . In this case  $G$  has all geometric fibers isomorphic to  $\mathrm{PGL}_2$  by the classical theory, so the roots  $\pm a : T \rightarrow \mathbf{G}_m$  are isomorphisms on geometric fibers over  $S$  and hence are isomorphisms over  $S$ . Thus, we can apply:

**Proposition 4.2.7.** *Let  $G \rightarrow S$  be a reductive group with trivial center and geometric fibers of semisimple-rank 1. If there exists a split maximal torus  $T \subset G$  then Zariski-locally on  $S$  there exists a group isomorphism  $G \simeq \mathrm{PGL}_2$ . This isomorphism may be chosen to carry  $T$  over to the diagonal torus.*

The Zariski-local nature of this result could be improved to a unique global isomorphism at the cost of using a relative notion of pinning and carrying out some preliminary arguments with the relative notion of Borel subgroup and to prove that  $\mathrm{PGL}_2$  is its own automorphism

functor. We prefer to postpone such considerations until we treat the general Existence and Isomorphism Theorems, as Zariski-local results will be entirely sufficient for our present purposes.

*Proof.* The split property of  $T$  provides a weight space decomposition of the rank-3 vector bundle  $\mathfrak{g}$ , so by working Zariski-locally on  $S$  we may arrange that there exists a root  $a : T \simeq \mathbf{G}_m$ . Let  $\lambda : \mathbf{G}_m \simeq T$  be the inverse of  $a$ , so  $U_G(\pm\lambda) = U_{\pm a}$ . Let  $B = P_G(\lambda)$ . By Exercise 4.4.2(i) and the  $\mathrm{PGL}_2$ -variant of Example 4.1.5, on geometric fibers this is a Borel subgroup. Proposition 2.1.6 provides a closed normalizer subscheme  $N_G(B) \subset G$ . We have not shown this normalizer to be flat, but we claim more: the closed immersion  $B \hookrightarrow N_G(B)$  inside  $G$  is an equality. Since  $B$  is flat, by Lemma B.3.1 it suffices to prove the equality on geometric fibers.

Now consider the situation over an algebraically closed field  $k$ . An elementary calculation with  $\mathrm{PGL}_2$  over  $k$  shows that  $B$  and  $N_G(B)$  have the same  $k$ -points, so it suffices to show that  $N_G(B)$  has the same Lie algebra as  $B$  inside of  $\mathfrak{pgl}_2$ . By dimension considerations, this is just a matter of ruling out the possibility that  $\mathrm{Lie}(N_G(B)) = \mathfrak{pgl}_2$ . But  $\mathfrak{b}$  is an ideal in  $\mathrm{Lie}(N_G(B))$  and it is clearly not an ideal in  $\mathfrak{pgl}_2$ . Thus, indeed  $B = N_G(B)$  as  $S$ -subgroups of  $G$ . We can therefore apply Theorem 2.3.6 to conclude that the quotient sheaf  $G/B$  is a smooth proper  $S$ -scheme admitting a canonical  $S$ -ample line bundle.

The formation of  $G/B$  commutes with any base change, such as passage to geometric fibers over  $S$ , so these fibers are identified with the quotient scheme of  $\mathrm{PGL}_2$  modulo a Borel subgroup over an algebraically closed field, which is to say that the geometric fibers  $(G/B)_{\bar{s}}$  are isomorphic to  $\mathbf{P}^1$ . That is,  $G/B \rightarrow S$  is a smooth proper curve with connected geometric fibers of genus 0. Moreover, the identity section of  $G \rightarrow S$  provides a section  $\sigma$  to  $G/B \rightarrow S$ .

By standard arguments with cohomology and base change (applied to the direct image on  $S$  of the inverse of the ideal sheaf of  $\sigma$  on  $G/B$ , after reducing to the case of noetherian  $S$ ) we see that Zariski-locally on  $S$  there exists an isomorphism  $G/B \simeq \mathbf{P}^1$  carrying the section  $\sigma$  over to  $\infty$ . Hence, we may assume  $G/B \simeq \mathbf{P}_S^1$  carrying  $1 \bmod B$  to  $\infty$ . In particular, the automorphism functor  $\underline{\mathrm{Aut}}(G/B)$  on  $S$ -schemes is represented by  $\mathrm{PGL}_2$  (Exercise 1.6.3(iv)) with the stabilizer of  $1 \bmod B$  going over to the stabilizer of  $\infty \in \mathbf{P}^1$ , which is to say the standard upper triangular subgroup  $B_\infty$  of  $\mathrm{PGL}_2$ .

The left translation action of  $G$  on  $G/B$  defines an  $S$ -homomorphism

$$G \rightarrow \underline{\mathrm{Aut}}(G/B) = \mathrm{PGL}_2.$$

On geometric fibers this is an isomorphism by the classical theory, so it is an isomorphism of  $S$ -groups. Points of  $B$  are carried into  $B_\infty$ , and the resulting map  $B \rightarrow B_\infty$  is an isomorphism since it is so on geometric fibers over  $S$ . Thus, the torus  $T \subset B$  is carried over to a maximal torus of  $\mathrm{PGL}_2$  contained in  $B_\infty$ .

It remains to prove that any maximal torus  $T$  of  $\mathrm{PGL}_2$  over  $S$  that is contained in  $B_\infty$  can be conjugated to the diagonal torus  $D = \mathbf{G}_m$ , at least Zariski-locally on  $S$ . Consider the smooth transporter scheme  $Y = \mathrm{Transp}_{B_\infty}(T, D)$  over  $S$ . This is stable under left multiplication by  $D$  in  $\mathrm{PGL}_2$ , making  $Y$  a left  $D$ -torsor for the étale topology. Indeed, in

view of the smoothness of  $Y$ , the torsor assertion is equivalent to the condition that the map

$$D \times_S Y \rightarrow Y \times_S Y$$

defined by  $(d, y) \mapsto (d \cdot y, y)$  is an isomorphism. By the smoothness of both sides it is sufficient to check the isomorphism property on fibers over geometric points  $\bar{s}$  of  $S$ . But  $T_{\bar{s}}$  is  $B_\infty(\bar{s})$ -conjugate to  $D_{\bar{s}}$  by the classical theory, so  $Y_{\bar{s}}$  is a torsor for the *smooth* normalizer scheme  $N_{B_\infty}(D)_{\bar{s}}$  that is equal to  $D_{\bar{s}}$  (via computation on geometric points). Since  $D = \mathbf{G}_m$ , and every  $\mathbf{G}_m$ -torsor for the étale topology is also a torsor for the Zariski topology (by descent theory for line bundles), it follows that  $Y \rightarrow S$  admits sections Zariski-locally over  $S$ , so the desired  $B_\infty$ -conjugation of  $T$  into  $D$  can be found Zariski-locally on  $S$ .  $\square$

By Proposition 4.2.7, for the proof of uniqueness in Theorem 4.2.6 we may assume  $G = \mathrm{PGL}_2$  with  $T$  the diagonal torus  $\bar{D}$  parameterized by  $\lambda : \mathbf{G}_m \simeq \bar{D}$  via  $\lambda(t) = \mathrm{diag}(t, 1)$ . Any root  $\bar{a} : \bar{D} \rightarrow \mathbf{G}_m$  must be inverse to one of  $\pm\lambda$  Zariski-locally on  $S$ , as it suffices to check this on geometric fibers (where it follows from the classical theory). Thus, by working Zariski-locally on  $S$  and composing with conjugation by the standard Weyl element  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  if necessary, we can arrange that  $\bar{a}$  is inverse to  $\lambda$ . It follows that  $\langle \bar{a}, \lambda \rangle = 1 > 0$ , so the root group  $U_{\bar{a}} = U_{\mathrm{PGL}_2}(\lambda)$  is the strictly upper triangular subgroup of  $\mathrm{PGL}_2$  and the root group  $U_{-\bar{a}}$  is the strictly lower triangular subgroup. If we use the standard bases of the Lie algebras of these subgroups of  $\mathrm{PGL}_2$  then  $\exp_{\pm\bar{a}}$  are the standard parameterizations of  $U_{\pm\bar{a}}$  (see Example 4.2.1, using the root  $\mathrm{diag}(c, 1) \mapsto c$  and composition with the central isogeny  $\mathrm{SL}_2 \rightarrow \mathrm{PGL}_2$ ). Thus, the calculations in  $\mathrm{SL}_2$  in Example 4.2.5 show that the standard duality between the root spaces (using their standard bases) and the 1-parameter subgroup  $\bar{a}^\vee(c) = \mathrm{diag}(c^2, 1) = \mathrm{diag}(c, 1/c) \bmod \mathbf{G}_m$  satisfy the requirements.

We have just proved existence for Theorem 4.2.6 over a general base  $S$  when  $Z_G = 1$ , and the argument gives uniqueness in such cases too. Indeed, any possibility for the bilinear pairing must be  $(X, Y) \mapsto cXY$  for some global unit  $c$  on  $S$ , and the equivalence of the unit conditions on  $1 + XY$  and  $1 + cXY$  on all  $S$ -schemes forces  $c = 1 + \xi$  for some nilpotent  $\xi$  on  $S$ . Then (4.2.1) and analogous established formula using the standard pairing and standard coroot forces  $\xi = 0$  and forces any possibility for the coroot to agree with the standard coroot on any unit of the form  $1 + xy$  with functions  $x$  and  $y$  on varying  $S$ -schemes. Any unit can be expressed in this form (take  $y = 1$ ), so uniqueness is established in general when  $Z_G = 1$ . But we already noted above that uniqueness over a general base when  $Z_G = 1$  implies uniqueness in general (without restriction on  $Z_G$ ), since uniqueness is already known for the coroot (due to the case of an algebraically closed ground field, which has been completely settled).

Finally, it remains prove existence without assuming  $Z_G = 1$ . Thanks to the settled uniqueness in general, we may work Zariski-locally on the base so that  $S = \mathrm{Spec} k$  for a ring  $k$ . As has already been explained, the bilinear pairing whose existence has been established for  $G/Z_G$  does provide the expected characterization for where  $\Omega_a$  meets  $U_a \times U_{-a} \subset \Omega_{-a}$ , so the direct product decomposition of  $\Omega_a$  does give an  $S$ -scheme map

$$f : \{(X, Y) \in \mathbf{A}_S^2 \mid 1 + XY \in \mathbf{G}_m\} = \mathrm{Spec} k[X, Y]_{1+XY} \rightarrow T$$

making (4.2.1) hold when using  $f(X, Y)$  in the  $T$ -factor on the right side. Applying  $T$ -conjugation to both sides of (4.2.1) implies  $f(a(t)X, a(t)^{-1}Y) = f(X, Y)$  for  $t \in \mathbf{G}_m$  whenever  $1 + XY \in \mathbf{G}_m$ . Since  $f = h/(1 + XY)^r$  for some  $r \geq 0$  and some  $h \in k[X, Y]$ , necessarily  $h$  is a polynomial in  $XY$ , or equivalently in  $1 + XY$ . Thus, we can define an  $S$ -scheme map  $a^\vee : \mathbf{G}_m \rightarrow T$  by the requirement  $a^\vee(1 + XY) = f(X, Y)$ .

In other words, (4.2.1) is proved except for the property that  $a^\vee$  is an  $S$ -homomorphism. Even without this homomorphism property, such a map  $a^\vee$  is uniquely determined, so to prove the homomorphism property we may assume  $S$  is noetherian, and then artin local (by the Krull intersection theorem), even with algebraically closed residue field (by faithfully flat base change). We wish to reduce this problem to a situation for which Proposition 4.3.1 can be applied to make everything explicit. To that end, we now carry out some considerations with central extensions.

The quotient  $G/Z_G$  and its split maximal torus  $T/Z_G$  admit an isomorphism

$$(G/Z_G, T/Z_G) \simeq (\mathrm{PGL}_2, \bar{D})$$

by Proposition 4.2.7. Consider the pullback diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & Z_G & \longrightarrow & \tilde{G} & \longrightarrow & \mathrm{SL}_2 \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & Z_G & \longrightarrow & G & \longrightarrow & \mathrm{PGL}_2 \longrightarrow 1 \end{array}$$

in which the top row is a central extension. By Proposition 4.3.1 below, the top row uniquely splits, so we get an isomorphism  $\tilde{G} \simeq \mathrm{SL}_2 \times Z_G$  in which the preimage  $\tilde{T}$  of  $T$  goes over to  $D \times Z_G$  (since the diagonal torus  $D \subset \mathrm{SL}_2$  is the full preimage of the diagonal torus  $\bar{D} \subset \mathrm{PGL}_2$ ). Thus, we get a cocharacter

$$b : \mathbf{G}_m = D \rightarrow T$$

via the identification  $\mu : \mathbf{G}_m \simeq D$  defined by  $\mu : t \mapsto \mathrm{diag}(t, 1/t)$ . We shall prove that  $b = a^\vee$ , so  $a^\vee$  is indeed a homomorphism.

The composition of  $\mu$  with the quotient map  $D \rightarrow T$  is a cocharacter  $\bar{\mu} : \mathbf{G}_m \rightarrow T$  whose composition with  $T \rightarrow T/Z_G = \bar{D}$  is the cocharacter  $\bar{\lambda} : t \mapsto \mathrm{diag}(t, 1/t) \bmod \mathbf{G}_m = \mathrm{diag}(t^2, 1) \in \mathrm{PGL}_2$ . Thus,

$$\langle a, \bar{\mu} \rangle = \langle \bar{a}, \bar{\mu} \bmod Z_G \rangle = \langle \bar{a}, \bar{\lambda} \rangle > 0,$$

so the standard root groups  $U^\pm = U_{\mathrm{SL}_2}(\pm\mu)$  in  $\mathrm{SL}_2$  map into  $U_G(\pm\bar{\mu}) = U_{\pm a}$  and these maps  $U^\pm \rightarrow U_{\pm a}$  are isomorphisms on geometric fibers for smoothness, dimension, and unipotence reasons because  $\ker(\tilde{G} \rightarrow G) = \mu_2$ . Hence, the maps  $U^\pm \rightarrow U_{\pm a}$  are isomorphisms. In this way, the induced map

$$U^- \times (Z_G \times D) \times U^+ \rightarrow U_{-a} \times T \times U_a = \Omega_a$$

is the quotient by the  $\mu_2$ -action, so our explicit knowledge of the standard coroot and bilinear pairing of root spaces for  $(\mathrm{SL}_2, D)$  (as in Example 4.1.5) imply that  $b : \mathbf{G}_m \rightarrow T$  and

the bilinear pairing constructed between  $\mathfrak{g}_a$  and  $\mathfrak{g}_{-a}$  satisfy the desired requirements that uniquely characterize the coroot and bilinear pairing of root spaces. This completes the proof that indeed  $b = a^\vee$ , so Theorem 4.2.6 is proved (conditional on the splitting of central extensions of  $\mathrm{SL}_2$ , provided by Proposition 4.3.1 below).  $\square$

**4.3. Central extensions of  $\mathrm{SL}_2$ .** This section is devoted to proving a general splitting result for central extensions of  $\mathrm{SL}_2$  (needed to complete the proof of Theorem 4.2.6) and recording an  $\mathrm{SL}_2$ -variant on Proposition 4.2.7. We begin with the result on central extensions:

**Proposition 4.3.1** (Gabber). *Let  $S$  be a scheme, and  $Z$  a commutative separated  $S$ -group. Any fppf central extension of group sheaves*

$$1 \rightarrow Z \rightarrow G' \rightarrow \mathrm{SL}_2 \rightarrow 1$$

*is uniquely split as a central extension.*

We only need the case that  $G'$  is an affine  $S$ -group of finite presentation (possibly not smooth!) with  $Z$  of multiplicative type, so the reader may safely restrict attention to that case. I am grateful to Gabber for explaining the result to me in the generality above; my initial proof via deformation theory was only for  $Z$  of multiplicative type.

*Proof.* In  $G := \mathrm{SL}_2$ , define the usual parameterizations  $h(t) = \mathrm{diag}(t, 1/t)$  of the diagonal torus  $D$  and  $x(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$  and  $y(v) = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}$  for the strictly upper triangular subgroup  $U$  and strictly lower triangular subgroup  $V$  respectively. We first address the uniqueness of the splitting, which is to say the vanishing of any  $S$ -homomorphism  $G = \mathrm{SL}_2 \rightarrow Z$ . Since  $h(t)x(u)h(t)^{-1}x(u)^{-1} = x((t^2 - 1)u)$  and  $h(t)y(v)h(t)^{-1}y(v)^{-1} = y((t^{-2} - 1)v)$ , relative schematic density considerations in  $U$  and  $V$  imply that any homomorphism  $f : G \rightarrow Z$  to an  $S$ -separated commutative target must kill the subgroups  $U$  and  $V$ . But for  $g(t) := y(-1/t)x(t)y(-1/t)$  (with  $t \in \mathbf{G}_m$ ) the standard formula  $g(t)g(1)^{-1} = h(t)$  implies that  $f$  also kills  $D$ , so  $\ker f$  contains the open cell  $\Omega := U \times D \times V \subset G$ . Thus, again using relative schematic density considerations,  $f = 1$ . This proves the uniqueness.

To build a splitting of the given central extension, let  $D', U', V' \subset G'$  respectively denote the preimages of  $D, U, V \subset G$ , so these are each central extensions by  $Z$  of their images in  $G$ . A key point is to verify that  $D'$  is necessarily commutative. As for any central extension of one commutative group object by another, the commutator of  $D'$  factors through a bi-additive pairing

$$D \times D \rightarrow Z$$

and the vanishing of this pairing is equivalent to the commutativity of  $D'$ ; see Exercise 4.4.3. Hence, it suffices to show that there is no nontrivial bi-additive pairing  $\mathbf{G}_m \times \mathbf{G}_m \rightarrow Z$  into a separated commutative  $S$ -group. The collection of closed subgroups  $\{\mu_n\}$  in  $\mathbf{G}_m$  is relatively schematically dense over  $S$ , so by the separatedness of  $Z$  over  $S$  it suffices to prove the vanishing of any bi-additive pairing  $\mu_n \times \mathbf{G}_m \rightarrow Z$ . But  $[n] : \mathbf{G}_m \rightarrow \mathbf{G}_m$  is an epimorphism of sheaves, so such vanishing is clear.

Next, we use the commutativity of  $D'$  to prove the commutativity of  $U'$  and  $V'$ . By symmetry, it suffices to treat  $U'$ . Note that  $D'$  normalizes  $U'$ , and the  $G'$ -action on  $G'$  by

conjugation factors through an action by the central quotient  $G = G'/Z$ , so we get a natural action by  $D$  on  $U'$ . The bi-additive pairing

$$c : U \times U \rightarrow Z$$

induced by the commutator on  $U'$  is therefore  $D$ -invariant in the sense that  $c(h.u_1, h.u_2) = c(u_1, u_2)$  for all  $h \in D = \mathbf{G}_m$  and  $u_1, u_2 \in U = \mathbf{G}_a$ . That is, for all  $t \in \mathbf{G}_m$  and  $u_1, u_2 \in \mathbf{G}_a$  we have  $c(tu_1, tu_2) = c(u_1, u_2)$ . Equivalently,  $c(tu_1, u_2) = c(u_1, t^{-1}u_2)$ . Consider fppf-local units  $t$  such that  $t+1$  is a unit and  $t' := (t+1)^{-1} - t^{-1} - 1$  is a unit. Bi-additivity of  $c$  gives

$$\begin{aligned} c(u_1, t'u_2) &= c(u_1, (t+1)^{-1}u_2)c(u_1, t^{-1}u_2)^{-1}c(u_1, u_2)^{-1} \\ &= c((t+1)u_1, u_2)c(tu_1, u_2)^{-1}c(u_1, u_2)^{-1} \\ &= c((t+1)u_1 - tu_1 - u_1, u_2) \\ &= 1 \end{aligned}$$

in  $Z$ . As an algebraic identity it is clear that such  $t'$  cover a relatively schematically dense open locus in  $\mathbf{G}_m$ , so  $S$ -separatedness of  $Z$  then forces  $c = 1$  as desired.

**Lemma 4.3.2.** *The  $D$ -equivariant quotient maps  $U' \rightarrow U$  and  $V' \rightarrow V$  admit unique  $D$ -equivariant splittings.*

*Proof.* By symmetry it suffices to treat  $U'$ . First we address the uniqueness, so then we may work fppf-locally on  $S$  to make the construction. Uniqueness amounts to the assertion that there are no nontrivial  $D$ -equivariant  $S$ -homomorphisms  $f : U \rightarrow Z$ . Since  $D = \mathbf{G}_m$  acts trivially on  $Z$  but acts on  $U = \mathbf{G}_a$  via  $t.x(u) = x(t^2u)$ ,  $U - \{0\}$  is a single “ $D$ -orbit”, so relative schematic density of  $U - \{0\}$  in  $U$  and the  $S$ -separatedness of  $Z$  then give the vanishing of any such  $f$ .

Now working fppf-locally on  $S$ , we may arrange that there exists a unit  $t$  on  $S$  such that  $t^2 - 1$  is also a unit, and also that the element  $h(t) \in D(S)$  admits a lift  $h' \in D'(S)$ . The  $h(t)$ -action on  $U'$  is induced by  $h'$ -conjugation, and as an endomorphism of the commutative  $U'$  it induces the identity on the subgroup  $Z$ . Thus, the endomorphism  $\varphi : u' \mapsto h'u'h'^{-1} - u' = h(t).u' - u'$  of the abstract commutative  $S$ -group  $U'$  kills  $Z$  and lies over the endomorphism of  $U = \mathbf{G}_a$  given by  $x(u) \mapsto x(t^2u)x(u)^{-1} = x((t^2 - 1)u)$ . But this endomorphism of  $U$  is an automorphism. Thus,  $\varphi$  factors through an  $S$ -homomorphism  $U = U'/Z \rightarrow U'$  lifting an automorphism  $\bar{\varphi}$  of  $U$ , and by construction it is  $D$ -equivariant (due to the commutativity of  $D'$ ). Precomposing  $\varphi$  with the inverse of  $\bar{\varphi}$  then provides the desired splitting.  $\square$

Using the unique  $D$ -equivariant  $S$ -group isomorphisms  $U' \simeq U \times Z$  and  $V' \simeq V \times Z$ , we obtain  $D$ -equivariant  $S$ -homomorphisms  $x' : \mathbf{G}_a \rightarrow U'$  and  $y' : \mathbf{G}_a \rightarrow V'$  lifting the respective standard parameterizations  $x$  of  $U$  and  $y$  of  $V$ . The next step is to use these to build an  $S$ -homomorphism  $h' : \mathbf{G}_m \rightarrow D'$  lifting the parameterization  $h : \mathbf{G}_m \simeq D$ . Define  $g'(t) = y'(-1/t)x'(t)y'(-1/t)$  for  $t \in \mathbf{G}_m$ , so this lifts the point  $g(t) := y(-1/t)x(t)y(-1/t) = \begin{pmatrix} 0 & t \\ -1/t & 0 \end{pmatrix} \in \mathrm{SL}_2 = G$ . Hence,  $h'(t) = g'(t)g'(1)^{-1} \in G'$  lifts  $g(t)g(1)^{-1} = h(t)$ , so  $h'$  is valued in  $D'$ .

**Lemma 4.3.3.** *The map  $h' : \mathbf{G}_m \rightarrow D'$  is a homomorphism lifting  $h : \mathbf{G}_m \simeq D$ , and  $h'(s)g'(t) = g'(st)$  for all  $s, t \in \mathbf{G}_m$ .*

*Proof.* Conjugation by  $h'(s)$  on  $g'(t) = y'(-1/t)x'(t)y'(-1/t)$  is the same as the action by  $h(s)$ , and so by the  $D$ -equivariance of the construction of  $x'$  and  $y'$  as respective lifts of  $x$  and  $y$  we have  $h(s).y'(v) = y'(v/s^2)$  and  $h(s).x'(u) = x'(s^2u)$  for any  $u, v \in \mathbf{G}_a$ . Thus,  $h(s).g'(t) = y'(-1/s^2t)x'(s^2t)y'(-1/s^2t) = g'(s^2t)$ , or in other words

$$h'(s)g'(t)h'(s)^{-1} = g'(s^2t).$$

Multiplying this against the inverse of the special case  $t = 1$  then gives  $h'(s)h'(t)h'(s)^{-1} = g'(s^2t)g'(s^2)^{-1} = h'(s^2t)h'(s^2)^{-1}$ . But  $D'$  is commutative, so we obtain  $h'(t) = h'(s^2t)/h'(s^2)$  for any points  $s, t \in \mathbf{G}_m$ . This establishes that  $h'$  is a homomorphism (visibly lifting  $h$ ). By the definition of  $h'$ , the identity  $h'(st) = h'(s)h'(t)$  says

$$g'(st)g'(1)^{-1} = h'(s)g'(t)g'(1)^{-1},$$

so  $g'(st) = h'(s)g'(t)$ . □

By going back to the definition of  $g'$ , the identity  $h'(s)g'(t) = g'(st)$  says

$$h'(s)y'(-1/t)x'(t)y'(-1/t) = y'(-1/st)x'(st)y'(-1/st).$$

The  $D$ -equivariance underlying  $y'$  gives that  $h'(s)y'(-1/t) = y'(-1/s^2t)h'(s)$ , so

$$y' \left( -\frac{1}{s^2t} \right) h'(s)x'(t) = y' \left( -\frac{1}{st} \right) x'(st)y' \left( -\frac{1}{st} + \frac{1}{t} \right).$$

Bringing  $y'(-1/st)$  to the left side, we arrive at the relation

$$y' \left( \frac{1}{s} \cdot \left( -\frac{1}{st} + \frac{1}{t} \right) \right) h'(s)x'(t) = x'(st)y' \left( -\frac{1}{st} + \frac{1}{t} \right)$$

in  $G'$  for any units  $s$  and  $t$ , or equivalently

$$(4.3.1) \quad x'(u)y'(v) = y' \left( \frac{v}{1+uv} \right) h'(1+uv)x' \left( \frac{u}{1+uv} \right)$$

for  $u, v \in \mathbf{G}_a$  such that  $1+uv$  and  $u$  are units.

We can establish (4.3.1) without the unit condition on  $u$  by working fppf-locally, as follows. For any points  $u, v$  of  $\mathbf{G}_a$  such that  $1+uv$  is a unit, fppf-locally we may write  $u = u' + u''$  where  $u', u'', 1+u''v \in \mathbf{G}_m$  (as we see by treating separately the cases when  $v$  vanishes or does not vanish at a geometric point of interest), so by using that the commutation relation for  $h'$  against  $x'$  and  $y'$  coincides with that of  $h$  against  $x$  and  $y$  (due to the  $D$ -equivariance

underlying the construction of  $x'$  and  $y'$  we get

$$\begin{aligned}
 x'(u)y'(v) &= x'(u')x'(u'')y'(v) \\
 &= x'(u')y'\left(\frac{v}{1+u''v}\right)h'(1+u''v)x'\left(\frac{u''}{1+u''v}\right) \\
 &= y'\left(\frac{v}{1+uv}\right)h'\left(\frac{1+uv}{1+u''v}\right)x'\left(\frac{u'(1+u''v)}{1+uv}\right)h'(1+u''v)x'\left(\frac{u''}{1+u''v}\right) \\
 &= y'\left(\frac{v}{1+uv}\right)h'(1+uv)x'\left(\frac{u'}{(1+u''v)(1+uv)}\right)x'\left(\frac{u''}{1+u''v}\right) \\
 &= y'\left(\frac{v}{1+uv}\right)h'(1+uv)x'\left(\frac{u}{1+uv}\right)
 \end{aligned}$$

when  $1+uv$  is a unit. Passing to inverses and negating  $u$  and  $v$ , we likewise get

$$y'(v)x'(u) = x'\left(\frac{u}{1+uv}\right)h'(1+uv)y'\left(\frac{v}{1+uv}\right)$$

when  $1+uv$  is a unit.

Recall that in  $\mathrm{SL}_2$ ,  $x(u)y(v)$  lies in the open cell  $U^- \times D \times U^+$  precisely when  $1+uv$  is a unit, and the preceding calculations show that  $x', h', y'$  satisfy the *same* commutation relations that govern the  $S$ -birational group law on the open cell  $\Omega = U \times D \times V$  of  $G = \mathrm{SL}_2$  (but beware that  $U' \times D' \times V' \rightarrow G'$  is not an open immersion when  $Z \neq 1$ , since  $Z$  lies in all three groups  $V'$ ,  $D'$ , and  $U'$ ). It follows that the  $S$ -morphism section  $\sigma : \Omega \rightarrow G'$  defined by  $x(u)h(s)y(v) \mapsto x'(u)h'(s)y'(v)$  for  $(u, s, v) \in \mathbf{G}_a \times \mathbf{G}_m \times \mathbf{G}_a$  is multiplicative in the “ $S$ -birational” sense that on the fiberwise-dense (hence relatively schematically dense) open locus  $m_G^{-1}(\Omega) \cap (\Omega \times \Omega)$  in  $\Omega \times \Omega$  consisting of points  $(\omega_1, \omega_2)$  whose product in  $G = \mathrm{SL}_2$  lies in  $\Omega$ , we have  $\sigma(\omega_1)\sigma(\omega_2) = \sigma(\omega_1\omega_2)$ .

If  $G'$  is a separated  $S$ -scheme then we can appeal to facts about  $S$ -birational group laws to conclude that  $\sigma$  extends (uniquely) to an  $S$ -homomorphism  $G \rightarrow G'$  that is the sought-after splitting of the given central extension of  $G$  by  $Z$ . Here is an alternative concrete procedure that works even when  $G'$  is just a group sheaf. Zariski-locally, every point of  $G$  either lies in the open cell  $\Omega$  or its translate by the point  $x(1) \in U$  lies in  $\Omega$ , so as a group sheaf  $G$  is generated by finite products of points of  $\Omega$ . Hence, to construct the desired homomorphic section  $G \rightarrow G'$  extending  $\sigma$  we just have to check that if  $a_1, \dots, a_n$  and  $b_1, \dots, b_m$  are points of  $\Omega$  (valued in some  $S$ -scheme) such that  $a_1 \cdots a_n = b_1 \cdots b_m$  in  $G$  then

$$(4.3.2) \quad \sigma(a_1) \cdots \sigma(a_n) \stackrel{?}{=} \sigma(b_1) \cdots \sigma(b_m).$$

To deduce this equality from the weaker  $S$ -birational multiplicativity already established for  $\sigma$ , observe that for such given  $a_i$  and  $b_j$  valued in some  $S$ -scheme  $S'$ , fppf-locally on  $S'$  there exists a point  $\omega$  of  $\Omega$  such that  $\omega a_1 \cdots a_i$  and  $\omega b_1 \cdots b_j$  lie in  $\Omega$  for all  $i, j \geq 0$ . Letting  $g = a_1 \cdots a_n = b_1 \cdots b_m$ , we have

$$\omega g = (\omega a_1 \cdots a_{n-1})a_n, \quad \omega g = (\omega b_1 \cdots b_{m-1})b_m.$$

Thus, the  $S$ -birational multiplicativity gives that

$$\sigma(\omega g) = \sigma(\omega a_1 \cdots a_{n-1})\sigma(a_n), \quad \sigma(\omega g) = \sigma(\omega b_1 \cdots b_{m-1})\sigma(b_m).$$

Continuing in this way (using the conditions on  $\omega$ ) we get

$$\sigma(\omega a_1 \cdots a_{n-1}) = \sigma(\omega)\sigma(a_1) \cdots \sigma(a_{n-1}), \quad \sigma(\omega b_1 \cdots b_{m-1}) = \sigma(\omega)\sigma(b_1) \cdots \sigma(b_{m-1}),$$

so (4.3.2) holds up to a harmless left multiplication by  $\sigma(\omega)$ .  $\square$

For later purposes, we now establish the  $\mathrm{SL}_2$ -analogue of Proposition 4.2.7.

**Proposition 4.3.4.** *Let  $G \rightarrow S$  be a reductive group scheme whose geometric fibers have finite center of order 2 and semisimple-rank 1, and let  $T$  a split maximal torus of  $G$ . Zariski-locally on  $S$  there exists an  $S$ -group isomorphism  $G \simeq \mathrm{SL}_2$ , and it can be chosen to carry  $T$  over to the diagonal torus.*

*Proof.* Since the multiplicative type center  $Z_G$  has fibers of order 2, its Cartier dual is finite étale of order 2. But  $(\mathbf{Z}/2\mathbf{Z})_S$  has no nontrivial étale  $S$ -forms since  $(\mathbf{Z}/2\mathbf{Z})^\times = 1$ . Hence,  $Z_G \simeq \mu_2$ . Likewise, the geometric fibers  $G_{\bar{s}}$  must be  $\mathrm{SL}_2$  by the classical theory. The quotient  $G/Z_G$  has trivial center (Corollary 3.3.5), and it has a split maximal torus  $T/Z_G$ , so by Proposition 4.2.7 there exists an isomorphism  $G/Z_G \simeq \mathrm{PGL}_2$  Zariski-locally on  $S$  that moreover carries  $T/Z_G$  over to the diagonal torus. In view of the bijective correspondence between maximal tori in  $G$  and  $G/Z_G$ , as well as in  $\mathrm{SL}_2$  and  $\mathrm{SL}_2/\mu_2 = \mathrm{PGL}_2$  (by applying Corollary 3.3.5), if we can lift the isomorphism  $G/Z_G \simeq \mathrm{PGL}_2$  to an isomorphism  $G \simeq \mathrm{SL}_2$  (at least Zariski-locally on  $S$ ) then it will carry  $T$  to the diagonal torus.

It remains to show that Zariski-locally on  $S$ , any smooth central extension  $G$  of  $\mathrm{PGL}_2$  by  $\mu_2$  over  $S$  with connected geometric fibers is isomorphic to  $\mathrm{SL}_2$  (as a central extension). Using central pushout along  $\mu_2 \hookrightarrow \mathbf{G}_m$  embeds  $G$  as a closed subgroup of a central extension  $G'$  of  $\mathrm{PGL}_2$  by  $\mathbf{G}_m$ . Pulling back this latter extension by the  $\mu_2$ -quotient map  $\mathrm{SL}_2 \rightarrow \mathrm{PGL}_2$  yields a central extension

$$1 \rightarrow \mathbf{G}_m \rightarrow \tilde{G}' \rightarrow \mathrm{SL}_2 \rightarrow 1$$

that is a central  $\mu_2$ -extension of  $G'$ . By Proposition 4.3.1 there exists a splitting  $\tilde{G}' = \mathrm{SL}_2 \times \mathbf{G}_m$ , and Zariski-locally on  $S$  this can be arranged to carry any given maximal torus  $\tilde{T}'$  of  $\tilde{G}'$  over to  $D \times \mathbf{G}_m$  (since we just have to arrange that the corresponding isomorphism  $\tilde{G}'/Z_{\tilde{G}'} \simeq \mathrm{PGL}_2$  carries  $\tilde{T}'/Z_{\tilde{G}'}$  over to the diagonal torus, as we know we can do Zariski-locally on  $S$  by Proposition 4.2.7).

On geometric fibers over  $S$ , the central subgroup  $\mu = \ker(\tilde{G}' \twoheadrightarrow G') \simeq \mu_2$  in  $\tilde{G}' = \mathrm{SL}_2 \times \mathbf{G}_m$  must either be the center of the  $\mathrm{SL}_2$ -factor or the diagonally embedded  $\mu_2$  in the product  $\mathrm{SL}_2 \times \mathbf{G}_m$ . The first option cannot occur over any geometric point  $\bar{s}$  of  $S$ : it would imply that  $G'_{\bar{s}}$  is a split extension  $\mathrm{PGL}_2 \times \mathbf{G}_m$ , yet the smooth connected subgroup  $G_{\bar{s}} \subset G'_{\bar{s}}$  is semisimple, so necessarily  $G_{\bar{s}} = \mathcal{D}(G'_{\bar{s}}) = \mathrm{PGL}_2$ , contradicting the hypothesis that the given central quotient map  $G_{\bar{s}} \rightarrow \mathrm{PGL}_2$  has a nontrivial kernel. Thus, the central  $\mu \subset Z_{\tilde{G}'} = \mu_2 \times \mu_2$  must be the diagonal  $S$ -subgroup since this holds on geometric fibers over  $S$ .

We conclude that  $G'$  is the pushout  $\mathrm{SL}_2 \times^{\mu_2} \mathbf{G}_m = \mathrm{GL}_2$  over  $S$ . Moreover, via the bijection between the sets of maximal tori in  $\tilde{G}'$  and its central quotient  $G' = \tilde{G}'/\mu$  (Corollary 3.3.5), we see that the isomorphism  $G' \simeq \mathrm{SL}_2 \times^{\mu_2} \mathbf{G}_m = \mathrm{GL}_2$  can be arranged to carry any given maximal torus of  $G'$  over to  $D \times^{\mu_2} \mathbf{G}_m$ , which is the diagonal torus  $\tilde{D}$  in  $\mathrm{GL}_2$ . Apply this to the maximal torus  $T' := T \times^{\mu_2} \mathbf{G}_m$  inside  $G \times^{\mu_2} \mathbf{G}_m = G'$ .

It now suffices to prove that the subgroup  $G \subset G'$  (which meets  $T'$  in  $T$ ) coincides with the subgroup  $\mathrm{SL}_2 \subset \mathrm{SL}_2 \times^{\mu_2} \mathbf{G}_m = \mathrm{GL}_2$  (which meets  $T = D \times^{\mu_2} \mathbf{G}_m = \tilde{D}$  in  $D$ ). Indeed, since the natural quotient map  $G \rightarrow G/\mu_2 = G'/\mathbf{G}_m = \mathrm{SL}_2/\mu_2 = \mathrm{PGL}_2$  is the central quotient map  $G \rightarrow G/Z_G \simeq \mathrm{PGL}_2$  arranged at the start, an equality  $G = \mathrm{SL}_2$  inside  $G'$  must respect the structures of both sides as central extensions of  $\mathrm{PGL}_2$  by  $\mu_2$ , so we would be done.

To relate  $G$  and  $\mathrm{SL}_2$  inside  $G'$ , we shall use root groups. More specifically, we may work Zariski-locally on  $S$  so as to acquire a pair of opposite roots  $\pm a$  for  $(G, T)$  whose root spaces  $\mathfrak{g}_{\pm a}$  are trivial line bundles over  $S$ . The torus  $T'$  is generated by  $T$  and the central  $\mathbf{G}_m$  in  $G'$ , so the  $T$ -weight spaces on  $\mathfrak{g}$  are also  $T'$ -weight spaces in  $\mathfrak{g}'$ . Letting  $\pm a' : T' \rightarrow \mathbf{G}_m$  be the corresponding fiberwise nontrivial  $T'$ -weights, it is clear that  $U_{\pm a}$  equipped with  $\exp_{\pm a}$  satisfies the properties uniquely characterizing  $(U_{\pm a'}, \mathfrak{g}'_{\pm a'})$ . The identification

$$(G', T') = (\mathrm{GL}_2, \tilde{D})$$

must carry the root groups  $U_{\pm a'}$  for  $(G', T')$  over to the standard root groups  $U^{\pm}$  on the right side Zariski-locally on  $S$  (as we see via comparison over the local rings on  $S$ ).

We have shown that  $U_{\pm a'} = U_{\pm a} \subset G$  inside  $G'$ , and obviously  $U^{\pm} \subset \mathrm{SL}_2$  inside  $G' = \mathrm{GL}_2$ . For  $x(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in U^+$  and  $y(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \in U^-$  we have

$$y(-1/t)x(t)y(-1/t)y(-1)x(1)y(-1) = \mathrm{diag}(t, 1/t)$$

for any  $t \in \mathbf{G}_m$ , so the standard open cell  $\Omega \subset \mathrm{SL}_2$  inside  $G' = \mathrm{GL}_2$  lies in the closed  $S$ -subgroup  $G \subset G'$ . By relative schematic density of  $\Omega$  in  $\mathrm{SL}_2$  over  $S$ , it follows that  $\mathrm{SL}_2 \subset G$  as closed subgroups of  $G'$ . But  $G$  and  $\mathrm{SL}_2$  are each  $S$ -smooth with connected geometric fibers of dimension 3, so the inclusion  $\mathrm{SL}_2 \subset G$  is an equality on geometric fibers over  $S$  and hence is an equality as closed subgroups of  $G'$ .  $\square$

#### 4.4. Exercises.

**Exercise 4.4.1.** Let  $G$  be a reductive group scheme over a scheme  $S$  and let  $T$  be a split maximal torus of  $G$ , with a fixed isomorphism  $T \simeq D_S(M)$  for a finite free  $\mathbf{Z}$ -module  $M$ . For each  $s \in S$  and root  $\alpha \in \Phi(G_{\bar{s}}, T_{\bar{s}})$  construct a Zariski-open neighborhood  $U$  of  $s$  in  $S$  and a root  $a$  of  $(G_U, T_U)$  in the sense of Definition 4.1.1 such that  $a(\bar{s}) = \alpha$ . Prove moreover that any two such  $a$  (for the same  $\alpha$ ) coincide on a Zariski-open neighborhood of  $s$  in  $S$ .

**Exercise 4.4.2.** Let  $G$  be a finitely presented affine group over a ring  $k$ , and choose a 1-parameter subgroup  $\lambda : \mathbf{G}_m \rightarrow G$  over  $k$ .

(i) Prove that  $P_G(\lambda^n) = P_G(\lambda)$  for any  $n > 0$ , and likewise for  $U_G(\lambda^n)$  and  $Z_G(\lambda^n)$ .

(ii) Let  $k/k'$  be a finite flat extension of noetherian rings,  $G'$  the Weil restriction  $\mathbf{R}_{k/k'}(G)$  (an affine  $k'$ -group of finite type), and  $\lambda' : \mathbf{G}_m \rightarrow G'$  the  $k'$ -morphism corresponding to the  $k$ -homomorphism  $\lambda : \mathbf{G}_m \rightarrow G$  via the universal property of  $\mathbf{R}_{k/k'}$ . Prove that  $\lambda'$  is a  $k'$ -homomorphism and  $P_{G'}(\lambda') = \mathbf{R}_{k/k'}(P_G(\lambda))$ , and similarly for  $U_{G'}(\lambda')$  and  $Z_{G'}(\lambda')$ .

**Exercise 4.4.3.** Prove the following fact that was used in the proof of Proposition 4.3.1: for any central extension  $1 \rightarrow Z \rightarrow H' \rightarrow H \rightarrow 1$  of one commutative group sheaf by another (on any site), the commutator of  $H'$  factors through a bi-additive pairing

$$H \times H \rightarrow Z$$

and the vanishing of this pairing is equivalent to the commutativity of  $H'$ . (This generalizes part of Exercise 3.4.6(iii).)

**Exercise 4.4.4.** Let  $A$  be a finite-dimensional associative algebra over a field  $k$ , and  $\underline{A}^\times$  the associated  $k$ -group of units as in Exercise 3.4.4. Prove  $\mathrm{Tan}_e(\underline{A}^\times) = A$  naturally, and that the Lie algebra structure is  $[a, a'] = aa' - a'a$ . Using  $A = \mathrm{End}(V)$ , recover  $\mathfrak{gl}(V)$  without coordinates. Use this to compute the Lie algebras  $\mathfrak{sl}(V)$ ,  $\mathfrak{pgl}(V)$ ,  $\mathfrak{sp}(\psi)$  (for a symplectic form  $\psi$ ),  $\mathfrak{gsp}(\psi)$ , and  $\mathfrak{so}(q)$  without coordinates.

**Exercise 4.4.5.** Let  $K$  be a degree-2 finite étale algebra over a field  $k$  (i.e., a separable quadratic field extension or  $k \times k$ , the latter called the *split* case), and let  $\sigma$  be the unique nontrivial  $k$ -automorphism of  $K$ ; note that  $K^\sigma = k$ . A  $\sigma$ -hermitian space is a pair  $(V, h)$  consisting of a finite free  $K$ -module equipped with a perfect  $\sigma$ -semilinear form  $h : V \times V \rightarrow K$  (i.e.,  $h(cv, v') = ch(v, v')$ ,  $h(v, cv') = \sigma(c)h(v, v')$ , and  $h(v', v) = \sigma(h(v, v'))$ ). Note that  $v \mapsto h(v, v)$  is a quadratic form  $q_h : V \rightarrow k$  over  $k$  satisfying  $q_h(cv) = N_{K/k}(c)q_h(v)$  for  $c \in K$  and  $v \in V$ , and  $\dim_k V$  is even ( $\mathrm{char}(k) = 2$  is allowed!). (One similarly defines the notion of a  $\sigma$ -anti-hermitian space by requiring  $h(v', v) = -\sigma(h(v, v'))$ .)

The *unitary group*  $U(h)$  over  $k$  is the subgroup of  $\mathbf{R}_{K/k}(\mathrm{GL}(V))$  preserving  $h$ . Using  $\mathbf{R}_{K/k}(\mathrm{SL}(V))$  gives the *special unitary group*  $\mathrm{SU}(h)$ . Example:  $V = F$  finite étale over  $K$  with an involution  $\sigma'$  lifting  $\sigma$ , and  $h(v, v') := \mathrm{Tr}_{F/K}(v\sigma'(v'))$ ; e.g., CM fields  $F$  and  $K$ , totally real  $k$ , and complex conjugations  $\sigma'$  and  $\sigma$ .

(i) If  $K = k \times k$ , prove  $V \simeq V_0 \times V_0^*$  with  $h((v, \ell), (v', \ell')) = (\ell'(v), \ell(v'))$  for a  $k$ -vector space  $V_0$ . Identify  $U(h)$  with  $\mathrm{GL}(V_0)$  carrying  $\mathrm{SU}(h)$  to  $\mathrm{SL}(V_0)$ . Compute  $q_h$  and prove that  $q_h$  is non-degenerate.

(ii) In the non-split case prove that  $U(h)_K \simeq GL_n$  carrying  $SU(h)$  to  $SL_n$  ( $n = \dim_K V$ ). Prove  $U(h)$  is smooth and connected with derived group  $SU(h)$  and center  $\mathbf{G}_m$ , and that  $q_h$  is non-degenerate. Compute  $\mathfrak{su}(h)$ .

(iii) Identify  $U(h)$  with a  $k$ -subgroup of  $SO(q_h)$ . Discuss the split case, and the case  $k = \mathbf{R}$ .

**Exercise 4.4.6.** Consider a  $k$ -torus  $T \subset GL(V)$  containing  $Z_{GL(V)} = \mathbf{G}_m$ , with  $k$  infinite. Let  $A_T \subset \text{End}(V)$  be the commutative  $k$ -subalgebra generated by  $T(k)$ .

(i) When  $k = k_s$ , prove  $A_T$  is a product of copies of  $k$  and that  $T(k) \hookrightarrow A_T^\times$  is an equality.

(ii) Using Galois descent and the end of Exercise 2.4.9(i), prove  $(A_T)_{k_s} = A_{T_{k_s}}$ , and deduce  $T(k) = A_T^\times$ . Construct a natural isomorphism  $T \simeq R_{A/k}(\mathbf{G}_m)$ , and a bijective correspondence between  $k$ -subtori in  $GL(V)$  containing  $Z_{GL(V)}$  and étale finite commutative  $k$ -subalgebras of  $\text{End}(V)$ . Generalize to *finite*  $k$  by using Galois descent to reduce to the case just handled.

**Exercise 4.4.7.** Let  $(V, q)$  be a non-degenerate quadratic space over a field  $k$  with  $\dim V \geq 2$ .

(i) If  $q = 0$  has a nonzero solution  $v$  in  $V$ , prove that  $v$  lies in a hyperbolic plane  $H$  with  $H \oplus H^\perp = V$ . (If  $\text{char}(k) = 2$  and  $\dim V$  is even, work over  $\bar{k}$  to show  $v \notin V^\perp$ .) Use this to construct a  $\mathbf{G}_m$  inside  $SO(q)$ .

(ii) If  $SO(q)$  contains a  $k$ -subgroup  $S \simeq \mathbf{G}_m$ , prove conversely that  $q = 0$  has a nonzero solution in  $V$ . (Hint: prove that  $V^S \neq V$  and compute  $q(tv)$  in two ways for  $t \in S$  and a nonzero  $v$  in a weight space for a nontrivial  $S$ -weight.)

**Exercise 4.4.8.** Let  $G$  be a connected semisimple group over an algebraically closed field  $k$ , and let  $T$  be a maximal torus and  $B$  a Borel subgroup of  $G$  containing  $T$ . Let  $\Delta$  be the set of simple positive roots relative to the positive system of roots  $\Phi^+ = \Phi(B, T)$ .

(i) Using Corollary 3.3.6, prove that  $Z_G = 1$  (equivalently,  $\text{Ad}_G$  is a closed immersion) if and only if  $\Delta$  is a basis of  $X(T)$ . (Do *not* assume the Existence and Isomorphism Theorems, as was done in Exercise 1.6.12(ii).)

(ii) Assume  $G$  is adjoint, and let  $\{\omega_i^\vee\}$  denote the basis of  $X_*(T)$  dual to the basis  $\{a_i\} = \Delta$  of  $X(T)$ . For each subset  $I \subset \Delta$ , let  $\lambda_I \in X_*(T)$  be the cocharacter  $\sum_{a_i \in I} \omega_i^\vee$ . Prove that the parabolic subgroup  $P_G(\lambda_I)$  coincides with the “standard” parabolic subgroup  $P_{\Delta-I}$  containing  $B$  that arose in the proof of Proposition 1.4.6 (so  $B = P_\emptyset = P_G(\lambda_\Delta)$ ). This gives a “dynamic” description of the parabolic subgroups of  $G$  containing  $B$ . (Hint: By Proposition 1.4.6, it suffices to compare Lie algebras inside  $\mathfrak{g}$ .)

(iii) Prove that  $\rho^\vee := \lambda_\Delta$  coincides with  $(1/2) \sum_{a \in \Phi^+} a^\vee$ . Equivalently (by consideration of the dual root datum and Exercise 1.6.16), for each  $a_i \in \Delta$  prove that  $\langle \sum_{a \in \Phi^+} a, a_i^\vee \rangle = 2$ . (Hint: Using that the reflection  $s_{a_i} : v \mapsto v - \langle v, a_i^\vee \rangle a_i$  preserves  $\Phi = \Phi^+ \amalg -\Phi^+$ , show that  $s_{a_i}$  permutes  $\Phi^+ - \{a_i\}$ . Apply the resulting “change of variables”  $a \mapsto s_{a_i}(a)$  to show that  $\langle \sum_{a \in \Phi^+ - \{a_i\}} a, a_i^\vee \rangle$  vanishes.)

## 5. SPLIT REDUCTIVE GROUPS AND PARABOLIC SUBGROUPS

**5.1. Split groups and the open cell.** In the theory of connected reductive groups  $G$  over a field  $k$  (not assumed to be algebraically closed), one says that  $G$  is *split* if it admits a maximal  $k$ -torus  $T \subset G$  that is  $k$ -split. (Keep in mind that for us, “maximal” means “geometrically maximal”, as in Definition 3.2.1. The equivalence with other possible notions of maximality over a field, which we never use, rests on Remark A.1.2 and Grothendieck’s existence theorem for geometrically maximal tori over any field.) For such  $(G, T)$ , the weight spaces  $\mathfrak{g}_a$  in  $\mathfrak{g}$  for the nontrivial weights  $a$  of  $T$  that occur on  $\mathfrak{g}$  are all 1-dimensional (as may be inferred from the theory over  $\bar{k}$ ). More specifically, each  $\mathfrak{g}_a$  is free of rank 1 over  $k$  since  $k$  is a field. In the relative theory, such module-freeness for the root spaces  $\mathfrak{g}_a$  must be imposed as a condition (following [SGA3, XXII, 1.13]):

**Definition 5.1.1.** Let  $G$  be a reductive group over a non-empty scheme  $S$ . It is *split* if there exists a maximal torus  $T$  equipped with an isomorphism  $T \simeq D_S(M)$  for a finite free  $\mathbf{Z}$ -module  $M$  such that:

- (1) the nontrivial weights  $a : T \rightarrow \mathbf{G}_m$  that occur on  $\mathfrak{g}$  arise from elements of  $M$  (so in particular, such  $a$  are roots for  $(G, T)$  and are “constant sections” of  $M_S$ ),
- (2) each root space  $\mathfrak{g}_a$  is free of rank 1 over  $\mathcal{O}_S$ ,
- (3) each coroot  $a^\vee : \mathbf{G}_m \rightarrow T$  arises from an element of the dual lattice  $M^\vee$  (i.e.,  $a^\vee$  as a global section of  $M_S^\vee$  over  $S$  is a constant section).

Note that the definition of  $a^\vee$  uses the reductive subgroup  $Z_G(T_a)$ , to which Theorem 4.2.6 is applicable. Also, although conditions (1) and (3) are automatic when  $S$  is connected (as the global sections of  $M_S$  in general are the locally constant  $M$ -valued functions on  $S$ , and similarly for  $M_S^\vee$ ), we do *not* assume  $S$  is connected. The reason is that when developing the theory of split reductive group schemes we want to work locally on  $S$  and use descent theory in some proofs, but localization on the base and (especially) descent theory often lead to disconnected base schemes. For this reason, we avoid the notation “ $\Phi(G, T)$ ” except when  $S$  is connected (e.g.,  $S = \text{Spec } k$  for a field or domain  $k$ ).

**Example 5.1.2.** Let  $S = \text{Spec}(k_1 \times k_2) = \text{Spec } k_1 \amalg \text{Spec } k_2$  for fields  $k_1$  and  $k_2$ , so an  $S$ -group  $G$  is precisely  $G = G_1 \amalg G_2$  where  $G_i$  is a  $k_i$ -group. In the case  $G_1 = \text{PGL}_2 \times \mathbf{G}_m$  and  $G_2 = \text{GL}_2$  (with their split diagonal tori), conditions (1) and (2) in Definition 5.1.1 are satisfied but condition (3) fails.

**Lemma 5.1.3.** *Any reductive group scheme over a non-empty scheme becomes split étale-locally on the base.*

A basic example of this lemma is that a connected reductive group over a field  $k$  becomes split over a finite separable extension of  $k$ .

*Proof.* There exists a maximal torus  $T$  étale-locally on the base (Corollary 3.2.7), and further étale localization provides an isomorphism  $T \simeq D_S(M)$ . Working Zariski-locally then makes

the weight space decomposition for  $\mathfrak{g}$  under the  $T$ -action have all nontrivial  $T$ -weights  $a$  arising from  $M$  with  $\mathfrak{g}_a$  having constant rank 1. A final Zariski-localization (which is necessary, by Example 5.1.2) makes each coroot arise from  $M^\vee$ .  $\square$

There is a case when the split property is automatic in the presence of a maximal torus:

**Example 5.1.4.** Let  $S$  be a (non-empty) connected normal noetherian scheme with trivial Picard group and trivial étale fundamental group. For example,  $S$  may be  $\text{Spec } \mathbf{Z}$  or  $\mathbf{A}_k^n$  for an algebraically closed field  $k$  of characteristic 0. Then we claim that every reductive group scheme  $G$  over  $S$  admitting a maximal torus  $T$  over  $S$  is automatically split. Over  $\mathbf{Z}$  this corresponds to the fact that the action of  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$  on  $X(T_{\overline{\mathbf{Q}}})$  is unramified at all primes and hence is trivial (Hermite–Minkowski). The general case proceeds along similar lines, as follows.

First note that by the normality of  $S$ , every non-empty open subset  $V$  of a connected finite étale  $S$ -scheme is irreducible. Hence, the étale fundamental group of  $V$  is a quotient of the Galois group at the generic point  $\eta$  of  $S$ . Thus, tori over  $S$  correspond to representations  $\pi_1(S, \bar{\eta})$  on  $\mathbf{Z}$ -lattices. But  $\pi_1(S, \bar{\eta}) = 1$  by hypothesis, so  $T$  is split. Hence, we may choose an isomorphism  $T \simeq D_S(M)$ . The global sections of  $M_S$  are the elements of  $M$  since  $S$  is connected, and likewise the global sections of  $M_S^\vee$  are the elements of  $M^\vee$ . Finally, each  $\mathfrak{g}_a$  for  $a \neq 0$  has constant rank (0 or 1) since  $S$  is connected, and when nontrivial it is free of rank 1 since  $\text{Pic}(S) = 1$ .

**Remark 5.1.5.** The split semisimple groups over  $\mathbf{Z}$  are called (semisimple) *Chevalley groups*. By Example 5.1.4, among all semisimple  $\mathbf{Z}$ -groups these are precisely the ones that admit a maximal  $\mathbf{Z}$ -torus (in the sense of Definition 3.2.1). A connected semisimple  $\mathbf{Q}$ -group admits a semisimple  $\mathbf{Z}$ -model if and only if it is split over  $\mathbf{Q}_p$  for all  $p$ . (Indeed, for the implication “ $\Leftarrow$ ” we can first spread out to a semisimple group over some  $\mathbf{Z}[1/N]$ , and then we can “glue” with split models over  $\mathbf{Z}_p$  for each  $p|N$  to make a semisimple  $\mathbf{Z}$ -model; see the proof of [CG, Lemma 1.3] for references on this gluing technique over a Dedekind base. There are semisimple  $\mathbf{Z}$ -groups that are not split, or equivalently do not admit a maximal torus, such as special orthogonal groups of even unimodular lattices (e.g., the  $E_8$  and Leech lattices).

The set of elements of  $M$  that occur in condition (1) of Definition 5.1.1 is denoted  $\Phi$  (with the choice of isomorphism  $T \simeq D_S(M)$  understood from context), and the set of corresponding coroots in  $M^\vee$  is denoted  $\Phi^\vee$ . We have

$$\mathfrak{g} = \mathfrak{t} \bigoplus \left( \bigoplus_{a \in \Phi} \mathfrak{g}_a \right).$$

The subsets  $\Phi \subset M - \{0\}$  and  $\Phi^\vee \subset M^\vee - \{0\}$  inherit some combinatorial properties from the classical theory on geometric fibers [SGA3, XXII, 1.14, 3.4]:

**Proposition 5.1.6.** *The 4-tuple  $(M, \Phi, M^\vee, \Phi^\vee)$  is a reduced root datum. The Weyl group  $W_G(T) = N_G(T)/T \subset \underline{\text{Aut}}(M_S^\vee) = \text{Aut}(M^\vee)_S$  is the constant subgroup  $W(\Phi)_S$ .*

*Proof.* The required combinatorial conditions to be a root datum can be checked on a single geometric fiber (recall that  $S$  is non-empty), where it follows from the classical theory. Likewise, since  $W_G(T)$  is a finite étale  $S$ -subgroup of the (opposite group of the) automorphism scheme of  $M_S$  (as  $W_G(T)$  acts faithfully on  $T = D_S(M)$ ), to compare it with the constant subgroup arising from the Weyl group of  $\Phi$  we may again pass to geometric fibers and appeal to the classical theory.  $\square$

We sometimes call  $(G, T, M)$  a “split” group, with the isomorphism  $T \simeq D_S(M)$  and subset  $\Phi \subset M$  tacitly understood to be specified. This really comes in three parts: the pair  $(G, T)$ , the root datum  $(M, \Phi, M^\vee, \Phi^\vee)$ , and the isomorphism  $T \simeq D_S(M)$  that carries  $\Phi$  over to roots for  $(G, T)$ . Keep in mind that the axioms for a root datum uniquely determine the bijection  $a \mapsto a^\vee$  between roots and coroots (Remark 1.3.4).

**Example 5.1.7.** Consider a split triple  $(G, T, M)$ . The center  $Z_G$  is  $D_S(M/Q)$ , where  $Q \subset M$  is the  $\mathbf{Z}$ -span of the roots (the *root lattice*). Indeed, by Corollary 3.3.6,  $Z_G$  is the kernel of the adjoint action of  $T = D_S(M)$  on  $\mathfrak{g}$ , and by definition the nontrivial weights for this action are the elements of  $\Phi \subset M$  viewed as characters on  $T$ . Hence,  $D_S(M/Q) \subset Z_G$ , and to prove equality we may pass to geometric fibers, where it is clear (since all multiplicative type groups over an algebraically closed field are split).

Since semisimplicity is equivalent to finiteness of the center, it follows that  $G$  is semisimple if and only if the elements of  $\Phi$  span  $M_{\mathbf{Q}}$  over  $\mathbf{Q}$ . Now suppose that  $G$  is semisimple. In such cases  $G$  has trivial center (i.e., it is *adjoint*) precisely when  $\Phi$  spans  $M$  over  $\mathbf{Z}$ . Since  $M$  lies inside the *weight lattice*  $P$  in  $M_{\mathbf{Q}}$  that is (by definition) dual to the coroot lattice (i.e., the  $\mathbf{Z}$ -span of  $\Phi^\vee$ ) in  $M_{\mathbf{Q}}^\vee$ , the center  $Z_G = D_S(M/Q)$  is a quotient of  $D_S(P/Q)$ . In particular, if  $M = P$  (i.e., if  $M$  is as big as possible) then the geometric fibers of  $G$  admit no nontrivial central isogenous cover: in the cases with  $M = P$  we say  $G$  is *simply connected*. The Existence Theorem implies that a split semisimple group scheme is simply connected if and only if it has no nontrivial central isogenous cover extension by a finite flat group scheme (necessarily of multiplicative type, as we can check on geometric fibers over  $S$ ); see Exercise 6.5.3.

To make the constancy of  $W_G(T)$  over  $S$  in Proposition 5.1.6 more concrete, note that for each root  $a$  there exists a natural map  $W_{Z_G(T_a)}(T) \rightarrow W_G(T)$  that on geometric fibers computes the order-2 subgroup generated by the involution  $s_a : t \mapsto t/a^\vee(a(t))$  of  $T = D_S(M)$  (dual to the involution  $m \mapsto m - a^\vee(m)a$  of  $M$ ). Since endomorphisms of multiplicative type  $S$ -groups are uniquely determined by their effect on geometric fibers over  $S$ , we conclude that each subgroup  $W_{Z_G(T_a)}(T)$  is identified with  $(\mathbf{Z}/2\mathbf{Z})_S$  having the unique everywhere-nontrivial section correspond to  $m \mapsto m - a^\vee(m)a$ . Hence, to construct elements  $n_a \in N_G(T)(S)$  representing the reflections  $s_a$  in the Weyl group of the root datum (as in the classical theory) the problem is reduced to the case of split reductive groups with semisimple-rank 1. In such cases we wish to show that  $N_G(T)(S) \rightarrow W_G(T)(S)$  is surjective by exhibiting an explicit element  $n_a \in N_G(T)(S)$  representing  $s_a$ . This will be deduced (in

Corollary 5.1.11) from the following classification of split semisimple-rank 1 groups Zariski-locally on the base.

**Theorem 5.1.8.** *Let  $(G, T, M)$  be a split reductive group with geometric fibers of semisimple-rank 1 over a non-empty scheme  $S$ . Up to forming a direct product against a split central torus, Zariski-locally on  $S$  the pair  $(G, T)$  is isomorphic to exactly one of the following:*

- $(\mathrm{SL}_2, D)$  with  $D$  the diagonal torus,
- $(\mathrm{PGL}_2, \overline{D})$  with  $\overline{D}$  the diagonal torus,
- $(\mathrm{GL}_2, \tilde{D})$  with  $\tilde{D}$  the diagonal torus.

We will later refine this result by constructing a unique such isomorphism globally, subject to some additional conditions that can always be imposed in the split case (such as compatibility with linked trivializations of the root spaces  $\mathfrak{g}_{\pm a}$ ). This provides an explicit isomorphism  $W_G(T)(S) \simeq W(\Phi)$  as in [SGA3, XXII, 3.4].

*Proof.* The three proposed cases are fiberwise non-isomorphic, so there are no repetitions in the list. The roots  $\pm a$  provide a central torus  $T_a$  of relative codimension 1 in  $T$ , and the classical theory on geometric fibers implies that the center  $Z_G/T_a$  of  $G/T_a$  is finite. That is, each geometric fiber of  $G/T_a$  is either  $\mathrm{SL}_2$  or  $\mathrm{PGL}_2$ . Note that  $T_a$  is the split torus corresponding to the quotient of  $M$  by the saturation of  $\mathbf{Z}a \subset M$ . Since the root datum determines the structure of the center  $Z_G \subset T$  on geometric fibers, it follows that  $Z_G/T_a$  must have constant fiber degree, either 1 or 2. We will treat the two possibilities separately.

First suppose that  $G/T_a$  has center of order 2. By Proposition 4.3.4, working Zariski-locally on  $S$  provides an isomorphism  $G/T_a \simeq \mathrm{SL}_2$  carrying  $T/T_a$  over to  $D$ . Thus,  $G$  is a central extension of  $\mathrm{SL}_2$  by the split torus  $T_a$ . Applying Proposition 4.3.1, there exists a unique splitting  $G = \mathrm{SL}_2 \times T_a$ , and clearly  $T$  must then go over to  $D \times T_a$ .

Next, suppose that  $G/T_a$  has trivial center. By Proposition 4.2.7, working Zariski-locally on  $S$  provides an isomorphism  $G/T_a \simeq \mathrm{PGL}_2$  carrying  $T/T_a$  over to  $\overline{D}$ . Pulling back along the central isogeny  $\mathrm{SL}_2 \rightarrow \mathrm{PGL}_2$  yields a central extension  $\tilde{G}$  of  $\mathrm{SL}_2$  by  $T_a$  that is also a central extension of  $G/T_a$  by  $\mu_2$ . In particular,  $\tilde{G}$  is a reductive group scheme having  $G$  as a central quotient by  $\mu_2$ , so there is a unique maximal torus  $\tilde{T}$  of  $\tilde{G}$  satisfying  $\tilde{T}/\mu_2 = T$  inside  $\tilde{G}/\mu_2 = G$ . The torus  $\tilde{T}$  is split Zariski-locally on  $S$  since  $T$  is split, so  $\tilde{G} = \mathrm{SL}_2 \times T_a$  with  $\tilde{T}$  going over to  $D \times T_a$ . The central subgroup  $\mu_2 \subset \tilde{G} = \mathrm{SL}_2 \times T_a$  has two possibilities on fibers: it is  $\mu_2$  in the  $\mathrm{SL}_2$ -factor or it is a diagonally embedded  $\mu_2$  in  $\mathrm{SL}_2 \times T_a$  via some inclusion  $\mu_2 \hookrightarrow T_a$ . These cases (on fibers) are distinguished by whether or not the projection to  $T_a$  kills this central subgroup.

Since a homomorphism between multiplicative type  $S$ -groups is determined over a Zariski-open neighborhood of a point  $s \in S$  by its effect on  $\bar{s}$ -fibers, we conclude that Zariski-locally on  $S$  either (i)  $G = \mathrm{PGL}_2 \times T_a$  with  $T = \overline{D} \times T_a$ , or (ii)  $G = \mathrm{SL}_2 \times^{\mu_2} T_a$  with  $T = D \times^{\mu_2} T_a$  for some inclusion  $\mu_2 \hookrightarrow T_a$ . Case (i) corresponds to  $Z_G = T_a$  being a torus, and the second case corresponds to  $Z_G \simeq \mu_2 \times T_a$  not being a torus since the structure of  $Z_G$  is determined across all fibers by the root datum (so its fibral isomorphism class is “constant”). Thus, it

remains to address the situation when  $(G, T)$  falls into case (ii) Zariski-locally on  $S$ . In such cases the inclusion  $\mu_2 \hookrightarrow T_a$  corresponds Zariski-locally on  $S$  to an index-2 subgroup of the constant group dual to  $T_a$ , so we can Zariski-locally split off this  $\mu_2$  inside a  $\mathbf{G}_m$ -factor of  $T_a$ . This provides a description of  $G$  (Zariski-locally on  $S$ ) as the direct product of a split torus against  $\mathrm{SL}_2 \times^{\mu_2} \mathbf{G}_m = \mathrm{GL}_2$  equipped with the maximal torus  $D \times^{\mu_2} \mathbf{G}_m = \tilde{D}$ .  $\square$

**Corollary 5.1.9.** *Let  $G$  be a reductive group over a non-empty scheme  $S$ , and  $T$  a maximal torus for which there exists a root  $a$  (and hence a root  $-a$ ). Let  $\mathbf{W}(\mathfrak{g}_a)^\times$  denote the open complement of the identity section in  $\mathbf{W}(\mathfrak{g}_a)$ . For every section  $X$  of  $\mathbf{W}(\mathfrak{g}_a)^\times$ , let  $X^{-1}$  denote the dual section of  $\mathbf{W}(\mathfrak{g}_{-a})^\times$ . Define  $w_a : \mathbf{W}(\mathfrak{g}_a)^\times \rightarrow G$  by*

$$w_a(X) := \exp_a(X) \exp_{-a}(-X^{-1}) \exp_a(X).$$

- (1) *The values of  $w_a$  lie in  $N_{Z_G(T_a)}(T)$  and map onto the unique everywhere nontrivial section of  $W_{Z_G(T_a)}(T) = (\mathbf{Z}/2\mathbf{Z})_S$ ; i.e.,*

$$t \mapsto w_a(X) t w_a(X)^{-1}$$

*is the reflection  $t \mapsto t/a^\vee(a(t))$  in  $W(\Phi(G_{\bar{s}}, T_{\bar{s}}))$  associated to  $a_{\bar{s}}$  for all geometric points  $\bar{s}$  of  $S$ .*

- (2) *For any unit  $c$  on  $S$  and sections  $X, X'$  of  $\mathbf{W}(\mathfrak{g}_a)^\times$ ,*

$$w_a(cX) = a^\vee(c)w_a(X) = w_a(X)a^\vee(c)^{-1}, \quad w_a(X)w_a(X') = a^\vee(-XX'^{-1}).$$

- (3) *The conjugation action of  $w_a(X)$  on  $U_a$  is valued in  $U_{-a}$  and is given by*

$$w_a(X) \exp_a(X') w_a(X)^{-1} = \exp_{-a}(-(X^{-1}X')X^{-1}).$$

*In particular,  $w_a(X) \exp_a(X) w_a(X)^{-1} = \exp_{-a}(-X^{-1})$  and the adjoint action of  $w_a(X)$  on  $\mathfrak{g}$  satisfies  $\mathrm{Ad}_G(w_a(X))(X') = -(X^{-1}X')X^{-1}$ .*

- (4) *The following relations hold:*

$$w_{-a}(X^{-1}) = w_a(X)^{-1} = w_a(-X), \quad w_a(X)w_{-a}(Y) = a^\vee(XY), \quad w_a(X)^2 = a^\vee(-1) \in T.$$

*Proof.* Since  $T_a$  centralizes  $U_{\pm a}$ , it is clear that  $w_a$  takes its values in  $Z_G(T_a)$ . We may therefore replace  $G$  with its reductive closed subgroup  $Z_G(T_a)$  (that contains  $T$  and  $U_{\pm a}$ ) to reduce to the case that  $G$  has all geometric fibers of semisimple-rank 1. The asserted identities are all fppf-local on the base, so by working étale-locally (or fppf-locally) we can assume that  $T$  is split. Thus, by Theorem 5.1.8 we get an explicit description of  $(G, T)$  up to forming a direct product against a split torus. Such an additional central torus factor has no effect on the root groups or the proposed relations, so we are reduced to the three special cases in Theorem 5.1.8. The third case in Theorem 5.1.8 reduces to the first case because in that central pushout the subtorus  $\mathbf{G}_m$  is central and  $\mathrm{SL}_2$  contains the root groups for  $D$ .

Summarizing, we are reduced to checking the special cases  $\mathrm{SL}_2$  and  $\mathrm{PGL}_2$  equipped with their diagonal maximal torus. By composing with the conjugation by the standard Weyl element if necessary (which induces inversion on the diagonal torus), we may arrange that  $a$  is the root whose root group  $U_a$  is the strictly upper triangular subgroup. For the  $\mathrm{SL}_2$ -case, Example 4.2.1 makes everything explicit. The formulas in Example 4.2.1 are inherited by the

central quotient  $\mathrm{PGL}_2$  when using the diagonal torus and its associated root  $\bar{a}(\mathrm{diag}(t, 1)) = t$  and coroot  $a^\vee(t) = \mathrm{diag}(t^2, 1) = \mathrm{diag}(t, 1/t) \bmod \mathbf{G}_m$ , so now everything is reduced to straightforward calculations with the standard root groups, roots, and coroots for  $\mathrm{SL}_2$  and  $\mathrm{PGL}_2$  equipped with their diagonal tori and standard linked trivializations of their root spaces. For example, if  $G = \mathrm{SL}_2$  and we take  $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  then  $w_a(cX) = \begin{pmatrix} 0 & c \\ -1/c & 0 \end{pmatrix}$ , so  $w_a(X) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  is the standard Weyl element and the first identity in part (2) says  $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/c & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ .  $\square$

**Remark 5.1.10.** A standard calculation using the proof of Corollary 5.1.9 yields

$$(w_a(X) \exp_a(X))^3 = 1$$

for any section  $X$  of  $\mathbf{W}(\mathfrak{g}_a)^\times$ ; this encodes the identity  $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}^3 = 1$  in  $\mathrm{SL}_2$ . This relation is needed when constructing homomorphisms from reductive group schemes to other group schemes (e.g., isogenies or isomorphisms between reductive group schemes); see Theorem 6.2.4. Our approach to getting the group-theoretic relations among  $w_{\pm a}$  and  $\exp_{\pm a}$  involves reduction to calculations with  $\mathrm{SL}_2$  and  $\mathrm{PGL}_2$  (because we have already obtained a Zariski-local classification result). The approach in [SGA3, XX] involves calculations in a slightly more indirect manner.

Here is the promised surjectivity of  $N_G(T)(S) \rightarrow W_G(T)(S)$  in the split case:

**Corollary 5.1.11.** *Let  $(G, T, M)$  be a split reductive group over a non-empty scheme  $S$ . Fix linked trivializations  $X_a \in \mathbf{W}(\mathfrak{g}_a)^\times(S)$  for all  $a \in \Phi$  (so  $X_{-a}$  is dual to  $X_a$ ). The natural map  $N_G(T)(S) \rightarrow W_G(T)(S)$  is surjective, with  $n_a := w_a(X_a)$  mapping to the reflection  $s_a$  and satisfying*

$$n_{-a} = a^\vee(-1)n_a, \quad n_a^2 = a^\vee(-1), \quad n_a \exp_a(X_a)n_a^{-1} = \exp_{-a}(-X_{-a}).$$

*Proof.* Since  $W_G(T)$  is the constant group associated to  $W(\Phi)$ , an element of  $W_G(T)(S)$  is a locally constant function valued in  $W(\Phi)$ . Thus, by passing to the constituents of a covering of  $S$  by pairwise disjoint open sets, for the proof of surjectivity on  $S$ -points we can focus on constant functions. But  $W(\Phi)$  is generated by the reflections  $s_a$ , so it remains to prove the assertions concerning  $n_a$ , which are all special cases of (3) and (4) in Corollary 5.1.9.  $\square$

The infinitesimal version of the relations among roots and root spaces [SGA3, XX, 2.10] will not be used in what follows, but we record it for completeness:

**Corollary 5.1.12.** *Let  $(G, T, a)$  be as in Corollary 5.1.9. Let  $\overline{\pm a} = \mathrm{Lie}(\pm a) : \mathfrak{t} \rightarrow \mathcal{O}_S$  and define  $H_a = \mathrm{Lie}(a^\vee)(1) \in \mathfrak{t}$  using the canonical basis of  $\mathrm{Lie}(\mathbf{G}_m)$ . Then*

$$\overline{-a} = -\bar{a}, \quad H_{-a} = -H_a, \quad \bar{a}(H_a) = 2,$$

and for all points  $t$  of  $T$  and local sections  $X, X' \in \mathfrak{g}_a$ ,  $Y \in \mathfrak{g}_{-a}$ , and  $H \in \mathfrak{t}$ , we have:

$$(5.1.1) \quad \mathrm{Ad}_G(t)(H) = H, \quad \mathrm{Ad}_G(t)(X) = a(t)X, \quad \mathrm{Ad}_G(t)(Y) = a(t)^{-1}Y,$$

$$(5.1.2) \quad \mathrm{Ad}_G(\exp_a(X))(H) = H - \bar{a}(H)X, \quad \mathrm{Ad}_G(\exp_a(X))(X') = X',$$

$$(5.1.3) \quad \mathrm{Ad}_G(\exp_a(X))(Y) = Y + (XY)H_a - (XY)X,$$

$$(5.1.4) \quad [H, X] = \bar{a}(H)X, \quad [H, Y] = -\bar{a}(H)Y, \quad [X, Y] = (XY)H_a.$$

*Proof.* As in the proof of Corollary 5.1.9, we reduce to the groups  $\mathrm{SL}_2$  and  $\mathrm{PGL}_2$  equipped with their diagonal tori and standard linked root space trivializations, with  $a$  as the standard positive root (whose root group consists of the strictly upper triangular matrices). In these cases the assertions are straightforward (and classical) calculations.  $\square$

The following relative version of the “open cell” generalizes (4.1.1) by replacing  $Z_G(T_a)$  with  $G$  (subject to the hypothesis that  $(G, T)$  is split).

**Theorem 5.1.13.** *Let  $(G, T, M)$  be a split reductive group over a non-empty scheme  $S$ . Fix a positive system of roots  $\Phi^+ \subset \Phi$ .*

*For any enumeration  $\{a_i\}$  of  $\Phi^+$ , the multiplication map  $\prod_i U_{a_i} \rightarrow G$  is an isomorphism onto a smooth closed subgroup  $U_{\Phi^+}$  that is normalized by  $T$ , independent of the choice of enumeration, and has connected unipotent fibers. The multiplication map*

$$U_{-\Phi^+} \times T \times U_{\Phi^+} \rightarrow G$$

*is an isomorphism onto an open subscheme  $\Omega_{\Phi^+}$ . Moreover, the semi-direct product  $T \times U_{\Phi^+} \rightarrow G$  is a closed  $S$ -subgroup.*

The existence of  $\Omega_{\Phi^+}$  is given in [SGA3, XXII, 4.1.2], but the construction of  $U_{\Phi^+}$  as a closed subgroup in [SGA3, XXII, §5] is completely different, resting on a detailed study of Lie algebras and smoothness properties of normalizers in  $G$  for certain subalgebras of  $\mathfrak{g}$  (see [SGA3, XXII, 5.3.4, 5.6.5]). Our proof gives closedness results as a consequence of the construction of  $U_{\Phi^+}$  via the dynamic method, which builds the desired “Borel subgroup” and its “unipotent radical” without any considerations with root groups.

*Proof.* Pick  $\lambda \in M^\vee$  so that the open half-space  $\{\lambda > 0\}$  in  $M_{\mathbf{R}}$  meets  $\Phi$  in  $\Phi^+$ . Interpreting  $\lambda$  as a cocharacter  $\mathbf{G}_m \rightarrow T$ , it makes sense to form the smooth closed  $S$ -subgroups  $U_G(\lambda)$ ,  $P_G(\lambda)$ , and  $Z_G(\lambda)$  with connected fibers as in Theorem 4.1.7. The tangent space to  $Z_G(\lambda)$  coincides with  $\mathfrak{t} = \mathfrak{g}_0$  since the only  $T$ -weight on  $\mathfrak{g}$  whose pairing with  $\lambda$  vanishes is the trivial weight (as  $\Phi = \Phi^+ \amalg -\Phi^+$ ). Thus, the inclusion  $T \subset Z_G(\lambda)$  between smooth closed  $S$ -subgroups with connected fibers induces an equality on Lie algebras (inside  $\mathfrak{g}$ ) and hence is an equality inside  $G$ . That is,  $Z_G(\lambda) = T$ , so  $P_G(\lambda) = Z_G(\lambda) \times U_G(\lambda) = T \times U_G(\lambda)$ . The  $S$ -group  $U_G(\lambda)$  has (connected) unipotent fibers, by Theorem 4.1.7(4), so  $P_G(\lambda)$  has (connected) solvable fibers.

By Theorem 4.1.7, the multiplication map

$$U_G(-\lambda) \times T \times U_G(\lambda) \rightarrow G$$

is an open immersion. Note that  $U_G(\lambda)$  has nothing to do with a choice of enumeration of  $\Phi^+$ . Also, for  $a \in \Phi$  the root group  $U_a$  is normalized by  $T$  and hence is normalized by the  $\mathbf{G}_m$ -action through conjugation by  $\lambda$ , with  $\lambda(t)$  acting on  $\mathrm{Lie}(U_a) = \mathfrak{g}_a$  via scaling by  $t^{\langle a, \lambda \rangle}$ .

Thus,  $U_{U_a}(\lambda)$  has Lie algebra  $\mathfrak{g}_a$  if  $\langle a, \lambda \rangle > 0$  (i.e., if  $a \in \Phi^+$ ) and has vanishing Lie algebra otherwise (i.e., if  $a \in -\Phi^+$ ). Since  $U_{U_a}(\lambda)$  must be  $S$ -smooth with connected fibers (as  $U_a$  is!), this  $S$ -subgroup of  $U_a$  vanishes when  $a \in -\Phi^+$  and coincides with  $U_a$  when  $a \in \Phi^+$ . In particular,  $U_a \subset U_G(\lambda)$  for all  $a \in \Phi^+$ .

It now suffices to show that for any choice of enumeration  $\{a_i\}$  of  $\Phi^+$ , the multiplication mapping

$$\prod_i U_{a_i} \rightarrow U_G(\lambda)$$

is an isomorphism of  $S$ -schemes. By smoothness of both sides, it suffices to check the isomorphism property on geometric fibers, so we may and do assume  $S = \text{Spec } k$  for an algebraically closed field  $k$ . Since  $P_G(\lambda) = T \times U_G(\lambda)$  is connected and solvable, dimension considerations for its Lie algebra imply that it is a Borel subgroup with  $U_G(\lambda)$  as its unipotent radical, and the subgroups  $U_{a_i}$  must be its root groups. In the classical theory it is proved that the unipotent radical of a Borel subgroup is directly spanned (in any order) by its root groups, though this also follows from general considerations using just the reducedness of the root system; see [CGP, Thm. 3.3.11] for such an alternative proof of direct spanning in the classical case (applying [CGP, Thm. 3.3.11] to the smooth connected unipotent  $U_G(\lambda)$ ).  $\square$

As in the classical case, we say that  $U_{\Phi^+}$  is *directly spanned* (in any order) by the  $U_{a_i}$ 's for  $a_i \in \Phi^+$ , and we call  $\Omega_{\Phi^+}$  the *open cell* (or *big cell*) associated to  $\Phi^+$ . The link between the root system and the commutation relations among positive root groups carries over as in the classical theory:

**Proposition 5.1.14.** *Let  $(G, T, M)$  be a split reductive group over a non-empty scheme  $S$ . Pick roots  $a, b \in \Phi$  such that  $b \neq \pm a$ . Choose trivializations of the root spaces  $\mathfrak{g}_c$  for all roots  $c$  of the form  $ia + jb$  with integers  $i, j \geq 1$ . Consider the associated parameterizations  $p_c : \mathbf{G}_a \simeq U_c$ , and fix an enumeration of this set of roots  $c$ .*

*The root groups  $U_a$  and  $U_b$  commute if there are no roots of the form  $ia + jb$  with  $i, j \geq 0$ , and in general the commutation relation is given by*

$$(5.1.5) \quad (p_a(x), p_b(y)) := p_a(x)p_b(y)p_a(-x)p_b(-y) = \prod_{i,j>0} p_{ia+jb}(C_{i,j,a,b}x^i y^j)$$

*where the product is taken over all roots  $ia + jb$  with  $i, j > 0$  and the coefficients  $C_{i,j,a,b}$  are global functions on  $S$ .*

As in the classical case, the “structure constants”  $C_{i,j,a,b}$  are rather mysterious at this stage of the theory, and a detailed study of rank-2 cases will be required to clean them up. Note that these structure constants depend on the choice of ordering among the terms in the product on the right side of (5.1.5); also, this product involves at most 6 terms (as we see by inspecting the classification of reduced rank-2 root systems). Finally, observe that the commutation relations in (5.1.5) describe the group law on  $U_{\Phi^+}$ ; this is one reason that we need to clean up the structure constants.

*Proof.* We argue exactly as in the classical case by considering  $T$ -equivariance, exercising more care due to the base scheme being rather general. Pick a positive system of roots  $\Phi^+$  containing  $a$  and  $b$  (as we can do since  $b \neq \pm a$ ), and choose an enumeration  $\{c_m\}$  of  $\Phi^+$  extending the choice of enumeration of the roots of the form  $ia + jb$  with  $i, j > 0$ .

A priori the commutator  $(p_a(x), p_b(y))$  lies in  $U_{\Phi^+} = \prod_m U_{c_m}$ , and we have to show that the only factors  $U_c$  which can have a nontrivial component are for the roots  $c = ia + jb$  for  $i, j > 0$ , and that the factor in such a component has the form  $p_{ia+jb}(C_{i,j,a,b}x^i y^j)$ .

Consider the expression

$$(p_a(x), p_b(y)) = \prod_{c \in \Phi^+} p_c(h_c(x, y))$$

where the product on the right side is taken in the order according to the chosen enumeration of  $\Phi^+$ , and  $h_c : U_a \times U_b \rightarrow U_c$  is a  $T$ -equivariant map of  $S$ -schemes. That is,  $h_c(a(t)x, b(t)y) = c(t)h_c(x, y)$  for the  $S$ -map  $h_c : \mathbf{G}_a \times \mathbf{G}_a \rightarrow \mathbf{G}_a$  that is given by some 2-variable polynomial Zariski-locally over  $S$ . Writing  $h_c = \sum_{i,j \geq 0} f_{i,j} x^i y^j$  for some Zariski-local functions  $f_{i,j}$  on  $S$ , we have  $f_{i,j} a(t)^i b(t)^j = c(t) f_{i,j}$  for all  $i, j$ . Setting  $y = 0$  gives  $f_{i,0} = 0$  for all  $i$  since  $p_b(0) = 1$ , and likewise  $f_{0,j} = 0$  for all  $j$ . If  $i, j > 0$  then  $f_{i,j}$  is killed by the character  $c - (ia + jb)$  on  $T = D_S(M)$  that arises from an element of  $M$ . Such a character is either trivial or fiberwise nontrivial (and hence faithful flat onto  $\mathbf{G}_m$ ), so  $f_{i,j} = 0$  except possibly when  $c = ia + jb$ , in which case such  $i$  and  $j$  are uniquely determined by  $c$  (since the distinct positive roots  $a$  and  $b$  are linearly independent). In other words, each  $h_c$  that is not identically zero is a single monomial of some constant bi-degree  $(i, j)$  such that  $ia + jb \in \Phi^+$  and  $i, j > 0$ . In particular, if there are no such roots  $ia + jb$  with  $i, j > 0$  then  $U_a$  commutes with  $U_b$ .  $\square$

It is useful to generalize the construction of  $U_{\Phi^+}$  (even going beyond Remark 5.2.6 in the classical case) by constructing fiberwise unipotent smooth closed subgroups  $U_\Psi \subset G$  directly spanned in any order by certain subsets  $\Psi \subset \Phi$ . To characterize the  $\Psi$  that we shall consider, we make a brief digression concerning general root systems.

Let  $(V, \Phi)$  be a (possibly non-reduced) root system, with  $V$  a  $\mathbf{Q}$ -vector space. Recall that a subset  $\Psi \subset \Phi$  is called *closed* if  $a + b \in \Psi$  for any  $a, b \in \Psi$  such that  $a + b \in \Phi$ . Interesting examples of such  $\Psi$  are  $\Phi_{\lambda > q} = \{a \in \Phi \mid \lambda(a) > q\}$  and  $\Phi_{\lambda \geq q} = \{a \in \Phi \mid \lambda(a) \geq q\}$  for  $\lambda \in V^*$  and  $q \in \mathbf{Q}$ , as well as the sets of roots

$$[a, b] = \{ia + jb \in \Phi \mid i, j \geq 0\}, \quad (a, b) = \{ia + jb \in \Phi \mid i, j \geq 1\}, \quad [a, b) = \{ia + jb \mid i \geq 1, j \geq 0\}$$

for linearly independent  $a, b \in \Phi$ .

By [CGP, 2.2.7], the closed sets in  $\Phi$  are precisely the subsets of the form  $\Phi \cap A$  for a subset  $A \subseteq V$  that is a subsemigroup (i.e.,  $a + a' \in A$  for all  $a, a' \in A$ ; we allow  $A$  to be empty). When  $\Psi = \Phi_{\lambda > q}$  we can use  $A = \{v \in V \mid \lambda(v) > q\}$ , but when  $\Psi = [a, b)$  for linearly independent  $a, b \in \Phi$  there is no ‘‘obvious’’ choice for  $A$ . For any closed  $\Psi \subseteq \Phi$  there is a unique minimal choice for  $A$ , namely the subsemigroup  $\langle \Psi \rangle$  generated by  $\Psi$  (which is empty when  $\Psi$  is empty). We are interested in closed  $\Psi$  that lie in a positive system of

roots. Such a positive system of roots is not uniquely determined by  $\Psi$ , but there is a simple characterization for when one exists:

**Lemma 5.1.15.** *Let  $(V, \Phi)$  be a root system. For  $\Psi \subseteq \Phi$ , the following are equivalent:*

- (1)  $\Psi$  is closed and is contained in a positive system of roots;
- (2)  $\Psi = \Phi \cap A$  for a subsemigroup  $A \subseteq V$  such that  $0 \notin A$ ;
- (3)  $\Psi$  is closed and  $\Psi \cap -\Psi$  is empty.

*Proof.* Consider a closed set  $\Psi$ , so  $\Psi = \Phi \cap A$  for  $A = \langle \Psi \rangle$ . The positive systems of roots in  $\Phi$  are precisely the subsets  $\Phi^+ = \Phi_{\lambda > 0}$  with  $\lambda \in V^*$  that is nonzero on all roots. If  $\Psi$  is contained in some  $\Phi^+ = \Phi_{\lambda > 0}$  then  $\langle \Psi \rangle$  lies in  $\{v \in V \mid \lambda(v) > 0\}$ , so  $0 \notin \langle \Psi \rangle$ . Thus, (1) implies (2). The implication “(2)  $\Rightarrow$  (3)” is trivial, and “(3)  $\Rightarrow$  (1)” is precisely [Bou2, VI, §1.7, Prop. 22] (due to the characterization of positive systems of roots in  $\Phi$  in terms of Weyl chambers for  $\Phi$  in  $V_{\mathbf{R}}$ , given by [Bou2, VI, §1.7, Cor. 1, Cor. 2 to Prop. 20]).  $\square$

Here is a generalization of  $U_{\Phi^+}$  (inspired by [SGA3, XXII, 5.9.5]).

**Proposition 5.1.16.** *Let  $(G, T, M)$  be a split reductive group over a non-empty scheme  $S$ , and let  $\Psi$  be a closed set in  $\Phi$  such that  $\Psi \cap -\Psi = \emptyset$ .*

- (1) *For any enumeration  $\{a_i\}$  of  $\Psi$ , the multiplication map  $\prod U_{a_i} \rightarrow G$  is an isomorphism onto a smooth closed subgroup  $U_{\Psi}$  that is normalized by  $T$ , independent of the choice of enumeration, and has connected unipotent fibers.*
- (2) *Choose  $\lambda \in M^V$  that is non-vanishing on  $\Phi$  such that the positive system of roots  $\Phi^+ := \Phi_{\lambda > 0}$  contains  $\Psi$ . For every integer  $n > 0$ , let  $\Psi_{\geq n} = \Psi \cap \Phi_{\lambda \geq n}$ . The subgroups  $U_{\geq n} := U_{\Psi_{\geq n}}$  are normal in  $U_{\Psi}$  and the multiplication map*

$$\prod_{a \in \Psi, \lambda(a) = n} U_a \rightarrow U_{\geq n} / U_{\geq n+1}$$

*(with the product taken in any order) is an  $S$ -group isomorphism. In particular,  $U_{\geq n} / U_{\geq n+1}$  is a power of  $\mathbf{G}_a$  as an  $S$ -group.*

By Lemma 5.1.15, there always exists  $\lambda$  as in (2).

*Proof.* Choose  $\lambda$  as in (2) and let  $\Phi^+ = \Phi_{\lambda > 0}$ . Since  $U_{\Phi^+}$  is directly spanned in any order by the root groups  $U_a$  for  $a \in \Phi^+$ , for any enumeration  $\{a_i\}$  of  $\Psi$  the multiplication map  $\prod U_{a_i} \rightarrow G$  is a monomorphism onto a direct factor scheme of  $U_{\Phi^+}$ . Hence, if we can prove that this is an  $S$ -subgroup for one choice of enumeration then the same holds for any enumeration. Thus, to prove (1) it suffices to consider a single enumeration. Also, once the existence of  $U_{\Psi_{\geq n}}$  is proved for all  $n \geq 1$ , it is immediate from (5.1.5) that  $U_{\Psi_{\geq n}}$  is normal in  $U_{\Psi}$  for all  $n \geq 1$ . It is also obvious that such subgroups are normalized by  $T$ .

For large  $m$ ,  $\Psi_{\geq m}$  is empty. By descending induction on  $m$  we shall prove (1) for  $\Psi_{\geq m}$  and then (2) for  $\Psi_{\geq m}$  when using our initial choice of  $\lambda$ . Since  $\Psi = \Psi_{\geq 1}$  and  $\lambda$  was arbitrary, this induction will prove (1) and (2) in general. The base of the induction (large  $m$ ) is obvious, with  $U_{\Psi_{\geq m}} = 1$  for large  $m$ .

Now suppose the cases  $m' \geq m + 1$  are settled, and consider (1) and (2) for  $\Psi_{\geq m}$ . We know that to prove (1) for  $\Psi_{\geq m}$  it suffices to consider one enumeration. We will use an enumeration that is adapted to  $\lambda$ . It is immediate from (5.1.5) that  $U_{\geq m+1}$  is normalized by  $U_a$  for all  $a \in \Psi_{\geq m}$ . Likewise, if  $a, b \in \Psi_{\geq m}$  then for any  $S$ -scheme  $S'$  and  $u_a \in U_a(S')$  and  $u_b \in U_b(S')$  we see that  $u_a u_b u_a^{-1} u_b^{-1} \in U_{\geq m+1}(S')$  since  $\lambda(ia + jb) \geq m + 1$  for any  $i, j \geq 1$ . Letting  $\Psi_m = \{a \in \Psi \mid \lambda(a) = m\}$ , it follows that for any  $S$ -scheme  $S'$  and  $a, b \in \Psi_m$  the subgroups  $U_a(S')$  and  $U_b(S')$  in  $G(S')$  commute modulo the subgroup  $U_{\geq m+1}(S')$  that they normalize. Thus, for any enumeration  $\{c_i\}$  of  $\Psi_m$  the monic multiplication map  $\prod U_{c_i}(S') \times U_{\geq m+1}(S') \rightarrow G(S')$  has image that is a subgroup. This proves (1) for  $\Psi_{\geq m}$ , and (2) for  $n = m$  is now obvious (by consideration of  $S'$ -valued points for any  $S$ -scheme  $S'$ ).  $\square$

We end this section with applications of the open cell over a field. (See [Bo91, 14.10] for an alternative approach via the structure of automorphisms of connected semisimple groups).

**Proposition 5.1.17.** *Let  $G$  be a split nontrivial connected semisimple group over a field  $k$ . The set  $\{G_i\}_{i \in I}$  of minimal nontrivial normal smooth connected  $k$ -subgroups of  $G$  is finite, the  $G_i$ 's pairwise commute with each other, and the multiplication homomorphism  $\prod G_i \rightarrow G$  is a central isogeny.*

*Proof.* Let  $T$  be a split maximal  $k$ -torus in  $G$ , and  $\Phi = \Phi(G, T) \neq \emptyset$ . For each irreducible component  $\Phi_i$  of  $\Phi$ , let  $G_i$  be the smooth connected  $k$ -subgroup of  $G$  generated by the root groups  $U_a$  for  $a \in \Phi_i$ . For any  $i' \neq i$  and roots  $a' \in \Phi_{i'}$  and  $a \in \Phi_i$ ,  $a + a' \notin \Phi$  and we can put  $a$  and  $a'$  into a common positive system of roots. Hence,  $U_a$  and  $U_{a'}$  commute (Proposition 5.1.14), so  $G_i$  and  $G_{i'}$  commute. Since the root groups  $U_a$  and  $U_{-a}$  generate a subgroup containing  $a^\vee(\mathbf{G}_m)$ , and the coroots generate a finite-index subgroup of  $X_*(T)$  (as  $G$  is semisimple), the collection of all root groups generates a smooth closed subgroup containing all factors of the open cell in Theorem 5.1.13. Hence, the  $G_i$ 's generate  $G$ , so each  $G_i$  is normal in  $G$  and the product map

$$\pi : \prod G_i \rightarrow G$$

is a surjective homomorphism. Normality of  $G_i$  in  $G$  implies that  $G_i$  inherits semisimplicity from  $G$ . For  $i' \neq i$ , the subgroups  $G_i$  and  $G_{i'}$  commute and are nontrivial and semisimple, so  $G_i \neq G_{i'}$ .

A straightforward induction shows that if  $\{N_j\}$  is a finite collection of pairwise commuting normal smooth closed  $k$ -subgroups of a smooth  $k$ -group  $H$  of finite type then the multiplication homomorphism  $\prod N_i \rightarrow H$  has central kernel. (This can be generalized to the setting of group sheaves, as the interested reader can check.) Hence,  $\ker \pi$  is central. But  $\prod G_i$  is semisimple, so  $\pi$  is a central isogeny.

It remains to show that these  $G_i$  are precisely the minimal nontrivial normal smooth connected  $k$ -subgroups of  $G$ . It suffices to do this over  $\bar{k}$  since the formation of the  $G_i$  commutes with extension of the ground field. That is, we may assume that  $k = \bar{k}$ .

Let  $N$  be a nontrivial normal smooth connected  $k$ -subgroup of  $G$ , so  $N$  inherits semisimplicity from  $G$ . Hence,  $N$  is non-commutative. Since  $G$  is generated by the pairwise commuting subgroups  $G_i$ , there must be some  $i$  such that the commutator subgroup  $(N, G_i)$  is nontrivial. But  $(N, G_i)$  is normal in  $G$  and is contained in  $G_i$ , so it suffices to show that each  $G_i$  is minimal as a nontrivial normal smooth connected  $k$ -subgroup of  $G$ . Now we can assume  $N$  is contained in some  $G_{i_0}$  and we seek to show that  $N = G_{i_0}$ . By normality in  $G$ ,  $S := T \cap N$  is a maximal torus in  $N$  (so  $S \neq 1$ ).

Since  $T$  is a split maximal torus of  $G$ , each  $T_i := T \cap G_i$  is a split maximal torus of  $G_i$  and  $\pi$  carries  $\prod T_i$  isogenously onto  $T$ . Clearly  $S = N \cap T_{i_0}$ , and the connected reductive subgroup  $N \cdot T_{i_0}$  in  $G$  has maximal torus  $T_{i_0}$  and derived group  $N$ , so  $T_{i_0}$  is the almost direct product of  $S$  and the maximal central torus  $Z$  of  $N \cdot T_{i_0}$ . Hence, the isomorphism  $X(T_{i_0})_{\mathbf{Q}} \simeq X(S)_{\mathbf{Q}} \oplus X(Z)_{\mathbf{Q}}$  induces a bijection  $\Phi(N \cdot T_{i_0}, T_{i_0}) \simeq \Phi(N, S) \times \{0\}$ . In this way  $\Phi(N, S)$  spans a nonzero subspace of  $X(T_{i_0})_{\mathbf{Q}}$  that is stable under the action of  $W_{G_{i_0}}(T_{i_0}) = W(\Phi_{i_0})$ . The Weyl group of an irreducible root system  $(V, \Psi)$  acts irreducibly on  $V$  [Bou2, VI, §1.2, Cor.], so if the root system  $\Phi(G_{i_0}, T_{i_0})$  is irreducible then  $\Phi(N, S)$  spans  $X(T_{i_0})_{\mathbf{Q}} = X(S)_{\mathbf{Q}} \oplus X(Z)_{\mathbf{Q}}$ , so  $S = T_{i_0}$  for dimension reasons. This would force  $T_{i_0} \subset N$ , so the connected semisimple  $G_{i_0}/N$  would have trivial maximal torus and thus  $N = G_{i_0}$  as desired.

Finally, we show that each  $(X(T_i)_{\mathbf{Q}}, \Phi(G_i, T_i))$  is irreducible by relating  $\Phi(G_i, T_i)$  to the irreducible component  $\Phi_i$  of  $\Phi$ . The center of a connected reductive group lies in any maximal torus, so the direct product structure of open cells in Theorem 5.1.13 implies that a central isogeny  $H' \rightarrow H$  between connected reductive  $k$ -groups induces (for compatible maximal tori of  $H'$  and  $H$ ) a natural bijection between the collections of root groups as well as an isomorphism between the root systems and root groups for corresponding roots (see Exercise 1.6.8(i)). More specifically, the isomorphism  $X(T)_{\mathbf{Q}} \simeq \prod X(T_i)_{\mathbf{Q}}$  identifies  $\Phi(G, T)$  with  $\prod \Phi(G_i, T_i)$ . But if  $i' \neq i$  then  $T_{i'}$  centralizes  $G_i$  and hence centralizes all root groups of  $(G_i, T_i)$ , so each  $a \in \Phi_i$  kills the image of  $T_{i'}$  in  $T$ . Thus,  $\Phi_i \subset \Phi(G_i, T_i)$  inside  $X(T)_{\mathbf{Q}}$ , so the definition of the  $\Phi_i$  as the irreducible components of  $\Phi$  forces  $\Phi_i = \Phi(G_i, T_i)$  for all  $i$ .  $\square$

A connected semisimple group  $H$  over a field  $k$  is *k-simple* if  $H \neq 1$  and  $H$  has no nontrivial normal smooth connected proper  $k$ -subgroups, and *absolutely simple* if  $H_K$  is  $K$ -simple for some (equivalently, any) algebraically closed extension  $K/k$ . For the groups  $G_i$  in Proposition 5.1.17, if  $T$  is a split maximal torus of  $G$  then  $T_i := T \cap G_i$  is a split maximal torus of  $G_i$  and the proof of Proposition 5.1.17 shows that the isomorphism  $X(T)_{\mathbf{Q}} \simeq \prod X(T_i)_{\mathbf{Q}}$  identifies the  $\Phi(G_i, T_i)$  with the irreducible components of  $\Phi$ . In particular:

**Corollary 5.1.18.** *A nontrivial connected semisimple group  $G$  over a field is absolutely simple if and only if the root system for  $G_{k_s}$  is irreducible.*

It follows that the  $G_i$  in Proposition 5.1.17 are absolutely simple. Here is a generalization of Proposition 5.1.17 beyond the split case.

**Theorem 5.1.19** (Decomposition theorem for semisimple groups). *Let  $G$  be a nontrivial connected semisimple group over a field  $k$ . The set  $\{G_i\}_{i \in I}$  of minimal nontrivial normal*

smooth connected  $k$ -subgroups of  $G$  is finite, each  $G_i$  is  $k$ -simple, the  $G_i$ 's pairwise commute, and the multiplication homomorphism

$$\prod G_i \rightarrow G$$

is a central isogeny.

For each  $J \subset I$  the normal connected semisimple  $k$ -subgroup  $G_J \subset G$  generated by  $\{G_i\}_{i \in J}$  has as its minimal nontrivial normal smooth connected  $k$ -subgroups precisely the  $G_i$  for  $i \in J$ , and every normal smooth connected  $k$ -subgroup  $N \subset G$  equals  $G_J$  for a unique  $J$ . In particular, for each  $N$  there exists a unique  $N'$  that commutes with  $N$  and makes the multiplication homomorphism  $N \times N' \rightarrow G$  a central isogeny.

See Exercise 5.4.2 for the generalization to connected reductive  $k$ -groups  $G$ .

*Proof.* We first treat the case  $k = k_s$ , and then we will deduce the general case by Galois descent. Assuming  $k = k_s$ ,  $G$  is split and we can apply Proposition 5.1.17. Letting  $G_i$  be as in that result, we have proved their simplicity and that they pairwise commute and define a central isogeny  $\prod G_i \rightarrow G$ . Note also that each  $G_J$  is semisimple, due to normality in  $G$ .

For each non-empty  $J$  the natural map  $\prod_{i \in J} G_i \rightarrow G_J$  is a central isogeny, so by root system considerations (applying Proposition 5.1.17 to the split  $G_J$ ), the set of minimal nontrivial normal smooth connected subgroups of  $G_J$  is exhausted by the  $G_i$ 's for  $i \in J$ . To show that every  $N$  has the form  $G_J$  for some  $J$ , we can assume  $N \neq 1$ . Thus,  $N$  contains some  $G_{i_0}$  and (by consideration of root systems) the minimal nontrivial normal smooth connected subgroups of  $\overline{G} = G/G_{i_0}$  are the images  $\overline{G}_i$  of the  $G_i$  for  $i \neq i_0$ . By induction on dimension,  $\overline{N} := N/G_{i_0}$  is equal to  $\overline{G}_{J_0}$  for a subset  $J_0 \subset I - \{i_0\}$ , so  $N = G_J$  for  $J = J_0 \cup \{i_0\}$ .

Finally, we consider general  $k$ . Let  $\Gamma = \text{Gal}(k_s/k)$ . Let  $\{G'_i\}_{i \in I}$  be the set of minimal nontrivial normal smooth connected  $k_s$ -subgroups of  $G' = G_{k_s}$ , so  $\Gamma$  naturally permutes these subgroups and hence acts on the index set  $I$ . For each  $\Gamma$ -stable subset  $J \subset I$ ,  $G'_J$  descends to a normal smooth connected  $k$ -subgroup  $G_J \subset G$ , and by Galois descent these  $G_J$  are precisely the normal smooth connected  $k$ -subgroups of  $G$ . Hence, the minimal nontrivial ones are the groups  $G_J$  for  $J$  a  $\Gamma$ -orbit in  $I$ . Since the  $\Gamma$ -stable subsets of  $I$  are precisely the unions of  $\Gamma$ -orbits, we are done.  $\square$

The  $G_i$  in Theorem 5.1.19 are called the  $k$ -simple factors of  $G$ . The formation of this set is sensitive to extension of the ground field:

**Example 5.1.20.** Consider the Weil restriction  $G = R_{k'/k}(G')$  for a finite separable extension  $k'/k$  and an absolutely simple and semisimple  $k'$ -group  $G'$ . Since  $k' \otimes_k k_s$  is a product of copies of  $k_s$  indexed by the set of  $k$ -embeddings  $\sigma : k' \rightarrow k_s$ ,  $G_{k_s} = \prod_{\sigma} G'_{\sigma}$  where  $G'_{\sigma} = k_s \otimes_{\sigma, k'} G'$ . In particular,  $G$  is connected semisimple and its simple factors over  $k_s$  are the  $G'_{\sigma}$ . But these are permuted transitively by  $\text{Gal}(k_s/k)$ , so  $G$  is  $k$ -simple. Also, if  $G'_{k'_s}$  has a root datum that is semisimple and simply connected (resp. adjoint) then clearly so does  $G_{k_s}$ .

Theorem 5.1.19 shows that, up to central isogeny, to classify connected semisimple groups over a field  $k$ , the main case to consider is the  $k$ -simple case. Remarkably, the construction in Example 5.1.20 gives all examples, subject to a simply connected hypothesis. We will address this more fully in Example 6.4.6, as an application of the classification theorems in terms of root data.

**5.2. Parabolic subgroups and conjugacy.** In the classical theory one defines parabolic subgroups  $P \subset G$  in terms of the structure of  $G/P$  and uses this to infer properties such as  $P = N_G(P)$  (at least on geometric points) and the connectedness of such subgroups. In the version over a scheme we will directly prove that parabolic subgroups are self-normalizing and use that fact to construct  $G/P$  as a scheme projective over the base.

**Definition 5.2.1.** A *parabolic subgroup* of a reductive group scheme  $G$  is a smooth affine  $S$ -group  $P$  equipped with a monic homomorphism  $P \rightarrow G$  such that  $P_{\bar{s}}$  is a parabolic subgroup of  $G_{\bar{s}}$  (i.e.,  $G_{\bar{s}}/P_{\bar{s}}$  is proper) for all  $s \in S$ .

Note that all fibers  $P_{\bar{s}}$  are connected, by the classical theory. We do not require  $P \rightarrow G$  to be a closed immersion, but it will soon be proved that this condition does hold. Here is a natural class of examples arising from the dynamic method in §4.1.

**Example 5.2.2.** Suppose that  $G$  contains a maximal torus  $T$ , and let  $\lambda : \mathbf{G}_m \rightarrow T$  be a cocharacter. The smooth closed  $S$ -subgroup  $P_G(\lambda)$  is parabolic. Indeed, its fiber at a geometric point  $\bar{s}$  of  $S$  is  $P_{G_{\bar{s}}}(\lambda_{\bar{s}})$ , and the classical theory implies that such subgroups are always parabolic (see Example 4.1.9).

The dynamic description parabolic subgroups in the theory over an algebraically closed field (Example 4.1.9) admits the following relative formulation over any scheme:

**Proposition 5.2.3.** *Let  $G \rightarrow S$  be a reductive group scheme, and  $P$  a parabolic subgroup of  $G$ . Then  $P \rightarrow G$  is a closed immersion, and étale-locally on  $S$  there exists a maximal torus  $T$  of  $G$  such that  $T \subset P$ . If  $G$  admits a maximal torus  $T \subset P$  and  $(G, T)$  is split then Zariski-locally on  $S$  there exists  $\lambda : \mathbf{G}_m \rightarrow T$  such that  $P = P_G(\lambda)$ .*

Our proof of the closedness of  $P$  in  $G$  uses the dynamic method; the proof in [SGA3, XXII, 5.8.5] is rather different. A reader who wonders about removing the split hypothesis on  $(G, T)$  should look at Remark 5.2.4.

*Proof.* On geometric fibers over  $S$ , since  $P_{\bar{s}}$  is parabolic in  $G_{\bar{s}}$  we see that a maximal torus in  $P_{\bar{s}}$  is its own scheme-theoretic centralizer in  $P_{\bar{s}}$ . Hence, we may apply Theorem 3.2.6 to  $P$ , so by working étale-locally on  $S$  (as we may do for verifying that  $P \rightarrow G$  is a closed immersion) we obtain a maximal torus  $T$  of  $P$ . Obviously  $T$  is also a maximal torus of  $G$  (as this is an assertion on geometric fibers that is trivial to verify).

By further étale localization on  $S$ , we may suppose that  $(G, T)$  arises from a split triple  $(G, T, M)$  and  $S \neq \emptyset$ . For each  $s \in S$ , the cocharacters of  $T_{\bar{s}}$  coincide with the cocharacters of  $T_s$  over  $k(s)$  since  $T_s$  is split, so by Example 4.1.9 there exists a cocharacter  $\lambda_s : \mathbf{G}_m \rightarrow T_s$  over  $k(s)$  such that  $P_{\bar{s}} = P_{G_{\bar{s}}}(\lambda_{\bar{s}}) = P_{G_s}(\lambda_s)_{\bar{s}}$ . Hence,  $P_s = P_{G_s}(\lambda_s)$ . The split condition on  $T$

provides a Zariski-open neighborhood of  $s$  over which  $\lambda_s$  lifts to a cocharacter  $\lambda : \mathbf{G}_m \rightarrow T$ . Then we may work Zariski-locally around  $s$  in  $S$  to arrange that  $\lambda \in M^\vee$ . Clearly  $P_G(\lambda)$  is a *closed* parabolic subgroup of  $G$  that contains  $T$  and has  $s$ -fiber  $P_s$ . We will prove that  $P = P_G(\lambda)$  over a Zariski-open neighborhood of  $s$  in  $S$ .

Consider the Lie algebra  $\mathfrak{p}$  of  $P$  inside  $\mathfrak{g}$ . Although we do not yet know that  $P$  is closed in  $G$ , nonetheless the inclusion  $\mathfrak{p} \hookrightarrow \mathfrak{g}$  of  $\mathcal{O}_S$ -modules is a subbundle because on geometric fibers over  $S$  the inclusion  $P_{\bar{s}} \hookrightarrow G_{\bar{s}}$  is a closed immersion. Since  $\mathfrak{p}$  is stable under the adjoint action on  $\mathfrak{g}$  by the split torus  $T \subset P$ , by working Zariski-locally on  $S$  around  $s$  we can arrange that  $\mathfrak{p}$  is a direct sum of  $\mathfrak{t}$  and weight spaces  $\mathfrak{g}_a$  for some roots  $a \in \Phi \subset M$ . Since  $P_{\bar{s}} = P_G(\lambda)_{\bar{s}}$ , the  $a$  which arise in this way are precisely those that satisfy  $\lambda(a) \geq 0$ . Hence, the smooth closed subgroup  $P_P(\lambda) \subset P$  has full Lie algebra, forcing  $P_P(\lambda) = P$ .

By the functoriality in Proposition 4.1.10(2) (applied to the  $\mathbf{G}_m$ -equivariant homomorphism  $P \rightarrow G$ ), it follows that  $P \rightarrow G$  factors through  $P_G(\lambda)$ . The resulting map of smooth affine  $S$ -groups  $j : P \rightarrow P_G(\lambda)$  is an isomorphism on Lie algebras inside  $\mathfrak{g}$ . But on geometric fibers over  $S$  the map induced by  $j$  is a closed immersion between smooth connected affine groups, so the isomorphism property on Lie algebras forces  $j$  to be an isomorphism between geometric fibers over  $S$ . Hence,  $j$  is an isomorphism. In particular,  $P$  is closed in  $G$ .  $\square$

**Remark 5.2.4.** In the Borel–Tits structure theory for connected reductive groups over fields, the dynamic description  $P_G(\lambda)$  of  $P$  in terms of a cocharacter  $\lambda : \mathbf{G}_m \rightarrow T$  over the base field is valid without a split hypothesis on  $(G, T)$  (see [CGP, Prop. 2.2.9]). Thus, if  $S = \text{Spec } R$  for a henselian local ring  $R$  then we can remove the split hypothesis on  $T$  in Proposition 5.2.3. Indeed, it suffices to show that any cocharacter  $\lambda_0 : \mathbf{G}_m \rightarrow T_0$  over the residue field  $k$  lifts to a cocharacter  $\mathbf{G}_m \rightarrow T$  over  $R$ . In terms of the étale sheaf  $E$  dual to  $T$ , this is precisely the surjectivity of  $E(R) \rightarrow E(k)$ , which in turn is an immediate consequence of the henselian property of  $R$ . Removing the split hypothesis over more general local rings (and hence removing it from Proposition 5.2.3 over more general schemes  $S$ ) is tied up with delicate questions of comparing Zariski-local and étale-local properties of maximal tori in reductive groups, and we will not address the matter further here.

We now get many nice consequences, which we give in a series of corollaries.

**Corollary 5.2.5.** *Let  $P$  be a parabolic subgroup of a reductive group scheme  $G \rightarrow S$ . There is a unique smooth closed normal  $S$ -subgroup  $\mathcal{R}_u(P) \subset P$  whose geometric fiber  $\mathcal{R}_u(P)_{\bar{s}}$  coincides with the unipotent radical  $\mathcal{R}_u(P_{\bar{s}})$  for all  $s \in S$ .*

*The quotient  $P/\mathcal{R}_u(P)$  is represented by a reductive group scheme, and any surjective homomorphism from  $P$  onto a reductive  $S$ -group uniquely factors through  $P/\mathcal{R}_u(P)$ .*

We call  $\mathcal{R}_u(P)$  the *unipotent radical* of  $P$ . The proof of Corollary 5.2.5 uses the dynamic method; see [SGA3, XXII, 5.11.3, 5.11.4(ii)] for an alternative approach.

*Proof.* In view of the uniqueness we may work étale-locally on  $S$ , so by Proposition 5.2.3 we may arrange that  $G$  contains a split maximal torus  $T$  and that  $P = P_G(\lambda) = Z_G(\lambda) \rtimes U_G(\lambda)$  for some cocharacter  $\lambda : \mathbf{G}_m \rightarrow T$  (so  $T \subset Z_G(\lambda) \subset P_G(\lambda) = P$ ). It is clear that  $U_G(\lambda)$

satisfies the requirements to be  $\mathcal{R}_u(P)$  except possibly for the uniqueness and the universal mapping property relative to homomorphisms from  $P$  onto reductive  $S$ -groups.

Suppose that  $U \subset P$  is a smooth closed normal  $S$ -subgroup such that  $U_{\bar{s}} = \mathcal{R}_u(P_{\bar{s}})$  for all  $s \in S$ . Normality of  $U$  in  $P$  implies that  $U$  is normalized by  $T$  (as  $T \subset P$ ), so it makes sense to form the smooth closed  $S$ -subgroup  $U_U(\lambda)$  in  $U$ . The  $\bar{s}$ -fiber of  $U_U(\lambda)$  has the same Lie algebra as  $U$  (since  $U_{\bar{s}} = \mathcal{R}_u(P_{\bar{s}}) = U_{G_{\bar{s}}}(\lambda_{\bar{s}})$ ), so the closed immersion  $U_U(\lambda) \hookrightarrow U$  between smooth  $S$ -groups is an isomorphism on Lie algebras and hence an isomorphism on fibers (due to the *connectedness* of each  $U_{\bar{s}}$ ). It follows that  $U = U_U(\lambda) \subset U_G(\lambda)$ . But by hypothesis this inclusion between smooth closed  $S$ -subgroups of  $G$  induces an equality on geometric fibers over  $S$ , so  $U = U_G(\lambda)$ . This establishes the uniqueness of  $\mathcal{R}_u(P)$ .

Next, consider a surjective homomorphism  $f : P \rightarrow \bar{G}$  onto a reductive  $S$ -group. We need to prove that this map kills  $\mathcal{R}_u(P)$ . Working étale-locally on  $S$ , by Proposition 5.2.3 we may arrange that  $P = P_G(\lambda)$ , so  $\mathcal{R}_u(P) = U_G(\lambda)$ . By Proposition 4.1.10(2),  $f$  carries  $U_G(\lambda)$  onto  $U_{\bar{G}}(f \circ \lambda)$ . But  $U_G(\lambda)$  is normal in  $P$  and  $f$  is an fppf covering, so it follows that  $U_{\bar{G}}(f \circ \lambda)$  is normal in  $\bar{G}$ . But  $U_{\bar{G}}(f \circ \lambda)$  is  $S$ -smooth with connected unipotent fibers, so normality in the reductive  $S$ -group  $\bar{G}$  forces  $U_{\bar{G}}(f \circ \lambda)$  to have relative dimension 0 and therefore be trivial. This says exactly that  $f$  kills  $\mathcal{R}_u(P)$ .  $\square$

**Remark 5.2.6.** As an application of Corollary 5.2.5, we can construct many smooth closed subgroups of  $G$  directly spanned in any order by certain collections of root groups.

For any root system  $\Phi$ , there is a notion of *parabolicity* for subsets of  $\Phi$ ; see [Bou2, VI, §1.7, Def. 4]. These are precisely the subsets

$$\Phi_{\lambda \geq 0} := \{a \in \Phi \mid \lambda(a) \geq 0\}$$

for linear forms  $\lambda$  on the  $\mathbf{Q}$ -vector space spanned by  $\Phi$  (see [CGP, Prop. 2.2.8] for a proof), and they always contain a positive system of roots. (See Definition 1.4.5ff.)

Consider subsets  $\Psi \subset \Phi$  whose complement is parabolic, which is to say  $\Psi = \Phi_{\lambda < 0}$  for some  $\lambda$ , or equivalently  $\Psi = \Phi_{\lambda > 0}$  for some  $\lambda$ . An interesting example of such a subset is  $\Psi = \Phi^+ - \{a\}$  for a positive system of roots  $\Phi^+$  and a simple positive root  $a$  when  $\Phi$  is *reduced*.

For a split reductive group  $(G, T, M)$ , let  $\Psi$  be the complement of a parabolic subset in  $\Phi$ , so  $\Psi = \Phi_{\lambda > 0}$  for some  $\lambda$ . The smooth closed  $S$ -subgroup  $U_G(\lambda)$  depends only on  $\Psi$  and not the choice of  $\lambda$ . In fact, we claim that  $U_G(\lambda)$  coincides with the  $S$ -group  $U_{\Psi}$  from Proposition 5.1.16. Indeed,  $\text{Lie}(U_G(\lambda))$  is spanned by the weight spaces  $\mathfrak{g}_a$  for such  $a$ , as is  $\text{Lie}(U_{\Psi})$ , so if there is an inclusion  $U_{\Psi} \subset U_G(\lambda)$  as smooth closed  $S$ -subgroups of  $G$  then it must be an equality (as we may check on geometric fibers, where smoothness and connectedness reduces the problem to the known equality of Lie algebras). Hence, it suffices to show that  $U_G(\lambda)$  contains  $U_a$  for all  $a \in \Phi_{\lambda > 0}$ . The explicit description of the  $T$ -conjugation action on  $U_a$  and the functorial definition of  $U_G(\lambda)$  imply that  $U_a \subset U_G(\lambda)$  since  $\langle \lambda, a \rangle > 0$ .

The dependence of  $U_G(\lambda)$  on  $\Psi$  rather than on  $\lambda$  can be proved in another way: this  $S$ -group is the unipotent radical of the parabolic subgroup  $P_G(\lambda)$  containing  $T$  (in the sense of Corollary 5.2.5), and  $P_G(\lambda)$  only depends on  $\Psi$  due to Corollary 5.2.7(2) below.

**Corollary 5.2.7.** *Let  $G \rightarrow S$  be a reductive group,  $s \in S$  a point, and  $P, Q \subset G$  parabolic subgroups.*

- (1) *If  $P_{\bar{s}}$  is conjugate to  $Q_{\bar{s}}$  for some  $s \in S$  then there exists an étale neighborhood  $U$  of  $(S, s)$  such that  $P_U$  is  $G(U)$ -conjugate to  $Q(U)$ . In particular, if  $P$  and  $Q$  are conjugate over all geometric points of  $S$  then they are conjugate étale-locally on  $S$ .*
- (2) *Assume  $P$  and  $Q$  contain a common maximal torus  $T$  of  $G$ . If  $\mathrm{Lie}(P_{\bar{s}}) \subset \mathrm{Lie}(Q_{\bar{s}})$  inside  $\mathfrak{g}_{\bar{s}}$  then  $P_V \subset Q_V$  for some Zariski-open neighborhood  $V$  of  $s$  in  $S$ . In particular, if  $\mathrm{Lie}(P_{\bar{s}}) = \mathrm{Lie}(Q_{\bar{s}})$  inside  $\mathfrak{g}_{\bar{s}}$  then  $P$  and  $Q$  coincide over a Zariski-open neighborhood of  $s$  in  $S$ .*

This result is a special case of [SGA3, XXII, 5.3.7, 5.3.11].

*Proof.* We first treat (2). Since étale maps are open, it is harmless to pass to an étale neighborhood of  $(S, s)$  to split  $T$ . By working Zariski-locally around  $s$  we can arrange that  $(G, T)$  arises from a split triple  $(G, T, M)$ . Further Zariski localization brings us to the case  $P = P_G(\lambda)$  and  $Q = P_G(\mu)$  for cocharacters  $\lambda, \mu : \mathbf{G}_m \rightrightarrows T$  arising from  $M^\vee$  (Proposition 5.2.3). The containment  $\mathrm{Lie}(P_{\bar{s}}) \subset \mathrm{Lie}(Q_{\bar{s}})$  implies that  $\Phi_{\lambda \geq 0} \subseteq \Phi_{\mu \geq 0}$  inside  $\Phi$ , as these are precisely the roots that appear in the respective Lie algebras of  $P_{\bar{s}}$  and  $Q_{\bar{s}}$  inside  $\mathfrak{g}_{\bar{s}}$ . But the containment  $\Phi_{\lambda \geq 0} \subset \Phi_{\mu \geq 0}$  implies that the smooth closed subgroup  $P_P(\mu) \subset P$  has full Lie algebra, so  $P = P_P(\mu) \subset P_G(\mu) = Q$  inside  $G$ .

Now we consider (1). By working étale-locally around  $(S, s)$  we may arrange that  $P$  contains a split maximal torus  $T$  of  $G$  and that  $Q$  contains a split maximal torus  $T'$  of  $G$ . By Corollary 2.1.5 we may pass to a further étale neighborhood so that  $T$  and  $T'$  are  $G(S)$ -conjugate, so applying such conjugacy brings us to the case that  $T' = T$ ; i.e.,  $P$  and  $Q$  contain a common split maximal torus  $T$ . Further Zariski localization provides a split triple  $(G, T, M)$ . By hypothesis there exists  $g \in G(\bar{s})$  such that  $gP_{\bar{s}}g^{-1} = Q_{\bar{s}}$ . Since  $T_{\bar{s}}$  and  $gT_{\bar{s}}g^{-1}$  are maximal tori in  $Q_{\bar{s}}$ , there exists  $q \in Q(\bar{s})$  such that  $qgT_{\bar{s}}g^{-1}q^{-1} = T_{\bar{s}}$ , so in other words  $qg \in N_G(T)(\bar{s})$  and its class  $w_0 \in W_G(T)(\bar{s})$  carries  $P_{\bar{s}}$  to  $Q_{\bar{s}}$ .

The  $S$ -group  $W_G(T) \rightarrow S$  is the finite constant group  $W(\Phi)_S$  (Proposition 5.1.6), so  $w_0$  spreads over a Zariski-open neighborhood of  $(S, s)$ . Further Zariski-localization then lifts the resulting point of  $W_G(T)(S) = (N_G(T)/T)(S)$  to  $N_G(T)(S)$  since  $T$  is  $S$ -split. Passing to such a neighborhood yields some  $n \in N_G(T)(S)$  such that  $nPn^{-1}$  and  $Q$  have the same  $s$ -fiber inside  $G_s$ . But these contain  $T$ , so by (2) there exists a Zariski-open neighborhood of  $s$  in  $S$  over which  $nPn^{-1}$  and  $Q$  coincide.  $\square$

**Corollary 5.2.8.** *For any parabolic subgroup  $P$  in a reductive group scheme  $G$ , the normalizer functor  $\underline{N}_G(P)$  is represented by  $P$ .*

*The quotient sheaf  $G/P$  for the étale topology on the category of  $S$ -schemes coincides with the functor of subgroups of  $G$  étale-locally conjugate to  $P$ , and it is represented by a smooth proper  $S$ -scheme equipped with a canonical  $S$ -ample line bundle. Explicitly, if  $\mathcal{P} \subset G \times (G/P)$  is the “universal parabolic subgroup locally conjugate to  $P$ ” then  $\det(\mathrm{Lie}(\mathcal{P}))^*$  is  $S$ -ample on  $G/P$ .*

The self-normalizer property in the special case  $G = \mathrm{PGL}_2$  was handled in the proof of Proposition 4.2.7, using the smallness of  $\dim \mathrm{PGL}_2$  to give a simple argument. The general case is proved in [SGA3, XXII, 5.8.5] by a method different from the one below.

*Proof.* Since the  $S$ -smooth  $P$  is closed in  $G$  and has connected fibers, Proposition 2.1.6 ensures that  $\underline{N}_G(P)$  is represented by a finitely presented closed subscheme  $N_G(P)$  of  $G$ . Beware that whereas normalizers of multiplicative type subgroups in smooth affine groups are always smooth, in the setting of Proposition 2.1.6 not even flatness of the normalizer is assured. Nonetheless, we do have an inclusion  $P \subset N_G(P)$  as finitely presented closed subschemes of  $G$ , with  $P$  flat (even smooth) over  $S$ . Hence, by Lemma B.3.1, this inclusion is an equality if it is so on geometric fibers over  $S$ . That is, we are reduced to the classical case  $S = \mathrm{Spec} k$  for an algebraically closed field  $k$ .

The classical theory gives that  $P(k) = N_G(P)(k)$  inside  $G(k)$ , so the closed  $k$ -subgroup scheme  $N_G(P)$  in  $G$  has the same dimension as  $P$ . We need to establish the equality  $P = N_G(P)$  as schemes. It suffices to verify that  $N_G(P)$  is smooth, or equivalently (since its dimension is  $\dim P$ ) that  $\mathrm{Lie}(N_G(P)) = \mathrm{Lie}(P)$  inside  $\mathfrak{g}$ . By using the explicit description  $P = P_G(\lambda)$  for some cocharacter  $\lambda : \mathbf{G}_m \rightarrow G$  (which is valued in some maximal torus  $T$  of  $G$  that necessarily lies in  $P$  since  $P_G(\lambda) \supset Z_G(\lambda) \supset T$ ), the equality of Lie algebras becomes an exercise in weights for the  $T$ -action on  $\mathfrak{g}$ . This calculation is given in the proof of [CGP, Prop. 3.5.7] (which is written in the greater generality of pseudo-reductive groups but can be applied as written in the reductive case, and even simplifies a bit in the reductive case since we may replace  $Z_G(T)$  with  $T$  everywhere).

The established equality  $P = N_G(P)$  and Theorem 2.3.6 imply the assertions concerning the functorial meaning of  $G/P$  and its existence as a smooth proper  $S$ -scheme equipped with a canonical  $S$ -ample line bundle.  $\square$

**Corollary 5.2.9.** *Let  $G \rightarrow S$  be a reductive group scheme. The functor on  $S$ -schemes*

$$\underline{\mathrm{Par}}_{G/S} : S' \rightsquigarrow \{\text{parabolic subgroups of } G_{S'}\}$$

*is represented by a smooth proper  $S$ -scheme  $\mathrm{Par}_{G/S}$  equipped with the canonical  $S$ -ample line bundle  $\det(\mathrm{Lie}(\mathcal{P}))^*$ , where  $\mathcal{P} \subset G \times \mathrm{Par}_{G/S}$  is the universal parabolic subgroup of  $G$ .*

The existence aspect of this corollary is part of [SGA3, XXVI, 3.3(ii)]. Over an algebraically closed field  $k$  the proof gives that  $\mathrm{Par}_{G/k} = \coprod G/P_i$  where  $P_i$  varies through representatives of the finite set of conjugacy classes of parabolic subgroups of  $G$  (parameterized by the parabolic subsets of the root system  $\Phi$  for  $G$  containing a fixed positive system of roots  $\Phi^+$ , or equivalently by the subsets of the base  $\Delta$  of  $\Phi^+$ ). In particular, in the classical case  $\mathrm{Par}_{G/k}$  is generally disconnected.

*Proof.* Since we aim to construct  $\mathrm{Par}_{G/S}$  as a proper  $S$ -scheme equipped with a canonical  $S$ -ample line bundle arising from the universal parabolic subgroup, by effective descent in the presence of a relatively ample line bundle it suffices to work étale-locally on  $S$ . Thus, by Theorem 3.2.6 we can assume that  $G$  admits a split maximal torus  $T$ .

The isomorphism class of the fibral root system for  $(G, T)$  is Zariski-locally constant on  $S$ , so we can arrange that  $S \neq \emptyset$  and there exists a split 4-tuple  $(G, T, M, \Phi)$ . Choose a positive system of roots  $\Phi^+ \subset \Phi$  and let  $\{\lambda_j\}_{j \in J} \subset M^\vee$  be a finite set of cocharacters such that  $\Phi_{\lambda_j \geq 0}$  varies (without repetition) through the finitely many parabolic subsets of  $\Phi$  containing  $\Phi^+$ .

For any  $S$ -scheme  $S'$  and parabolic subgroup  $Q \subset G_{S'}$ , each geometric fiber  $Q_{\bar{s}'}$  is conjugate to  $P_{G_{\bar{s}'}}(\lambda_j)$  for a unique  $j = j(s')$  (by the classical theory). Hence, by Corollary 5.2.7,  $Q$  is becomes conjugate to  $P_G(\lambda_j)$  on an étale neighborhood of  $(S, \bar{s}')$ , so  $j(s')$  is locally constant in  $s' \in S'$ . This defines a covering of  $S'$  by pairwise disjoint open sets  $S'_j$  in  $S'$  such that  $Q|_{S'_j}$  becomes conjugate to  $P_G(\lambda_j)|_{S'_j}$  étale-locally on  $S'_j$ . The conjugating point of  $G$  is well-defined modulo  $P_G(\lambda_j)$  due to the self-normalizing property of parabolic subgroup schemes, so the quotient  $\coprod_j G/P_G(\lambda_j)$  represents  $\underline{\text{Par}}_{G/S}$ . (Keep in mind that the functor of points of a disjoint union  $\coprod_{i \in I} X_i$  of  $S$ -schemes  $X_i$  indexed by a set  $I$  assigns to every  $S$ -scheme  $S'$  a disjoint union decomposition  $\coprod S'_i$  of  $S'$  indexed by  $I$  and a point in  $X_i(S'_i)$  for each  $i \in I$ .)

Letting  $\mathcal{P}$  denote the universal parabolic subgroup over  $\text{Par}_{G/S}$ , Theorem 2.3.6 ensures that the line bundle  $\det(\text{Lie}(\mathcal{P}))^*$  on  $\text{Par}_{G/S}$  is  $S$ -ample.  $\square$

**Definition 5.2.10.** A *Borel subgroup* of a reductive group scheme  $G \rightarrow S$  is a parabolic subgroup  $P \subset G$  such that  $P_{\bar{s}}$  is a Borel subgroup of  $G_{\bar{s}}$  for all  $s \in S$ . A reductive group  $G \rightarrow S$  is *quasi-split* over  $S$  if it admits a Borel subgroup scheme over  $S$ . (The notion of “quasi-split” is defined with an additional requirement in [SGA3, XXIV, 3.9] involving the scheme of Dynkin diagrams, but for semi-local  $S$  the two notions coincide [SGA3, XXIV, 3.9.1]. For our purposes, the definition we have given will be sufficient.)

**Theorem 5.2.11.** *Let  $G \rightarrow S$  be a reductive group scheme.*

- (1) *Let  $P \subset G$  be a parabolic subgroup. If  $P_{\bar{s}}$  is a Borel subgroup of  $G_{\bar{s}}$  then  $P_U$  is a Borel subgroup of  $G_U$  for some open  $U \subset S$  around  $s$ , and the open locus of  $s \in S$  such that  $P_{\bar{s}}$  is a Borel subgroup is also closed.*
- (2) *Any two Borel subgroups of  $G$  are conjugate étale-locally on  $S$ .*
- (3) *The functor on  $S$ -schemes*

$$\underline{\text{Bor}}_{G/S} : S' \rightsquigarrow \{\text{Borel subgroups of } G_{S'}\}$$

*is represented by a smooth proper  $S$ -scheme  $\text{Bor}_{G/S}$  equipped with the canonical  $S$ -ample line bundle  $\det(\text{Lie}(\mathcal{B}))^*$ , where  $\mathcal{B} \subset G \times \text{Bor}_{G/S}$  is the universal Borel subgroup of  $G$ .*

The existence and properties of  $\text{Bor}_{G/S}$  are part of [SGA3, XXII, 5.8.3(i)].

*Proof.* Fiber dimension considerations for the smooth map  $P \rightarrow S$  settle (1), since the isomorphism class of the fibral root datum for  $G \rightarrow S$  is locally constant over  $S$ . Part (2) follows from Corollary 5.2.7. Finally,  $\text{Bor}_{G/S}$  is the open and closed subscheme of  $\text{Par}_{G/S}$  over which the fibers of the universal parabolic subgroup are Borel subgroups (using part (1)).  $\square$

**Proposition 5.2.12.** *Let  $G \rightarrow S$  be a reductive group scheme, and  $T \subset G$  a maximal torus. If  $B \subset G$  is a Borel subgroup containing  $T$  then there exists a unique Borel subgroup  $B' \subset G$  satisfying  $B' \cap B = T$ .*

*Proof.* The uniqueness assertion allows us to work étale-locally on  $S$ , so we may assume that  $S \neq \emptyset$  and  $(G, T)$  arises from a split triple  $(G, T, M)$  such that  $B = P_G(\lambda)$  for some  $\lambda \in M^\vee$ . The inclusion  $T \subset Z_G(\lambda)$  is an equality and  $B' := P_G(-\lambda) = Z_G(-\lambda) \rtimes U_G(-\lambda) = T \rtimes U_G(-\lambda)$  is a Borel subgroup of  $G$  containing  $T$ . For any  $S$ -scheme  $S'$ , the only  $S'$ -maps  $\mathbf{P}_{S'}^1 \rightarrow G_{S'}$  are the “constant” maps to  $S'$ -points of  $G$ . Thus,  $B' \cap B = Z_G(\lambda) = T$ .

To establish the uniqueness of  $B'$ , note that if  $B''$  is another such Borel subgroup then necessarily  $B''$  contains  $T$ , so Corollary 5.2.7(2) reduces the uniqueness problem to the case of geometric fibers over  $S$ , which is well-known.  $\square$

In the split case, we get a Zariski-local conjugacy result for Borel subgroups:

**Corollary 5.2.13.** *Let  $(G, T, M)$  be a split reductive group over a non-empty scheme  $S$ , and let  $B$  be a Borel subgroup of  $G$ .*

- (1) *Every point  $s \in S$  admits an open neighborhood  $U$  such that some  $G(U)$ -conjugate of  $B_U$  contains  $T_U$ .*
- (2) *Any two Borel subgroups of  $G$  that contain  $T$  are  $N_G(T)$ -conjugate Zariski-locally over  $S$ .*

*Proof.* Pick a positive system of roots  $\Phi^+$  in  $\Phi$ , so there exists a cocharacter  $\lambda \in M^\vee$  such that  $\Phi_{\lambda \geq 0} = \Phi^+$ . Consideration of Lie algebras shows that the parabolic subgroup  $B' := P_G(\lambda)$  containing  $T$  is a Borel subgroup. Since  $B$  and  $B'$  are  $G$ -conjugate étale-locally on  $S$ , the transporter scheme  $\text{Transp}_G(B, B')$  provided by Proposition 2.1.6 is a torsor for  $N_G(B') = B'$  in the étale topology. Thus, to prove (1) it is enough to construct Zariski-local sections for any  $B'$ -torsor in the étale topology on  $S$ .

We will construct a composition series for the  $S$ -group  $B'$  consisting of smooth closed  $S$ -subgroups such that the successive quotients are  $\mathbf{G}_a$  or  $\mathbf{G}_m$ . Since torsors for  $\mathbf{G}_a$  or  $\mathbf{G}_m$  in the étale topology are always Zariski-locally trivial, we will then get the desired Zariski-local sections. To build the composition series for  $B'$ , we use its description as  $P_G(\lambda) = Z_G(\lambda) \rtimes U_G(\lambda)$ . The choice of  $\lambda$  implies that the inclusion  $T \subset Z_G(\lambda)$  is an equality, so since  $T$  is  $S$ -split we are reduced to considering  $U_G(\lambda)$ . The desired composition series for  $U_G(\lambda) = \mathcal{R}_u(P_G(\lambda))$  (see Corollary 5.2.5) is given by the subgroups  $U_{\Phi_{\lambda \geq n}}$  as in Proposition 5.1.16(2).

Now we turn to the proof of (2). By Corollary 5.2.7, it suffices to show that for any  $s \in S$  there exists a section  $n$  of  $N_G(T)$  over an open neighborhood  $U$  of  $s$  such that  $nB_U n^{-1}$  and  $B'_U$  have the same  $s$ -fiber. But  $N_G(T) \rightarrow W_G(T)$  is surjective for the Zariski topology since  $T$  is split, and  $W_G(T)$  is a finite constant  $S$ -group (so it has Zariski-local sections through any point of a fiber over  $S$ ). Hence, we just need to recall the fact from the classical theory that the geometric fiber  $W_G(T)_{\bar{s}} = W_{G_{\bar{s}}}(T_{\bar{s}})$  acts transitively on the set of Borel subgroups of  $G_{\bar{s}}$  that contain  $T_{\bar{s}}$ .  $\square$

**Corollary 5.2.14.** *Let  $G$  be a reductive group over a henselian local ring  $R$  with finite residue field. Then  $G$  is quasi-split and it becomes split over a finite étale extension of  $R$ .*

This corollary is very useful when  $R$  is the valuation ring of a non-archimedean local field. See Definition 5.2.10 for the notion of quasi-split for reductive group schemes.

*Proof.* By [Bo91, 16.6] (or Exercise 6.5.7), the special fiber  $G_0$  over the finite residue field  $k$  admits a Borel subgroup  $B_0$ . By Corollary 3.2.7, there exists a finite (separable) extension  $k'/k$  such that  $(G_0)_{k'}$  admits a split maximal torus  $T'_0$ . Since  $\mathrm{Bor}_{G/R}$  is  $R$ -smooth and  $R$  is henselian local, any  $k$ -point in the special fiber lifts to an  $R$ -point. Hence,  $B_0$  lifts to a Borel subgroup of  $G$ . The scheme  $\mathrm{Tor}_{G_0/k}$  has a  $k'$ -point corresponding to  $T'_0$ , and this may be viewed as a  $k'$ -point of  $\mathrm{Tor}_{G/R}$ . But  $\mathrm{Tor}_{G/R}$  is a smooth scheme over the henselian local ring  $R$ , so any  $k'$ -point must lift to an  $R'$ -point, where  $R \rightarrow R'$  is the local finite étale extension inducing the residual extension  $k'/k$ . Hence,  $\mathrm{Tor}_{G/R}(R') \neq \emptyset$ , so  $G_{R'}$  contains an  $R'$ -torus that lifts the  $k'$ -split  $T'_0$  and hence is  $R'$ -split too (due to the henselian property of  $R'$ ).  $\square$

**5.3. Applications to derived groups and closed immersions.** In the classical theory, there is a good notion of derived group for any smooth affine group. In the relative theory over a scheme, a new idea is needed to construct a satisfactory theory of the derived group (at least in the reductive case). The structure of the open cell (Theorem 5.1.13) and the construction of parabolic subgroups via cocharacters in the split case (Proposition 5.2.3) will enable us to build the “derived group” of any reductive group scheme (as in [SGA3, XXII, 6.1–6.2]):

**Theorem 5.3.1.** *Let  $G \rightarrow S$  be a reductive group scheme. There is a unique semisimple closed normal  $S$ -subgroup  $\mathcal{D}(G) \subset G$  such that  $G/\mathcal{D}(G)$  is a torus, and  $\mathcal{D}(G)$  represents the fppf-sheafification of the “commutator subfunctor”  $S' \rightsquigarrow [G(S'), G(S')]$  on the category of  $S$ -schemes.*

*In particular, the quotient map  $G \rightarrow G/\mathcal{D}(G)$  is initial among all homomorphisms from  $G$  to an abelian sheaf, and the formation of  $\mathcal{D}(G)$  commutes with any base change on  $S$ .*

*Proof.* By the asserted uniqueness, we may use étale descent to arrange that  $S$  is non-empty and  $T = D_S(M)$  for a finite free  $\mathbf{Z}$ -module  $M$  such that  $(G, T, M)$  is a split triple encoding a root datum  $(M, \Phi, M^\vee, \Phi^\vee)$ .

Pick a positive system of roots  $\Phi^+$  in  $\Phi$ , and consider the resulting open cell

$$\Omega := U_- \times T \times U_+ \subset G$$

where  $U_+ = U_{\Phi^+}$  and  $U_- = U_{-\Phi^+}$ . Let  $T' \subset T$  be the split subtorus “generated” by the coroots; i.e.,  $T' = D_S(M/L)$  where  $L \subset M$  is the saturated sublattice that is the annihilator of the coroot lattice  $\mathbf{Z}\Phi^\vee \subset M^\vee$ . This is the minimal subtorus of  $T$  through which all coroots factor, and for a geometric point  $\bar{s}$  of  $S$  the fiber  $T'_{\bar{s}}$  is the subtorus of  $T_{\bar{s}}$  generated by the coroots for  $(G_{\bar{s}}, T_{\bar{s}})$ . The idea is to show that  $\Omega' := U_- \times T' \times U_+$  is the open cell for a split closed semisimple  $S$ -subgroup of  $G$  that will be  $\mathcal{D}(G)$ .

Consider the  $S$ -morphism

$$(5.3.1) \quad f : \Omega \rightarrow T/T'$$

defined by  $f(u_-tu_+) = t \bmod T'$ . There is clearly at most one  $S$ -homomorphism  $\bar{f} : G \rightarrow T/T'$  that extends  $f$ , and we will prove that  $\bar{f}$  exists and is smooth with  $\mathcal{D}(G) := \ker \bar{f}$  satisfying the desired properties. The key result to be shown is that the condition  $u_+u_- \in \Omega'$  holds over on a fiberwise-dense open subscheme  $V \subset U_+ \times U_-$ . Indeed, we can take such a  $V$  to be the preimage under multiplication  $U_+ \times U_- \rightarrow G$  of the open  $V_0 \subset G$  provided by:

**Lemma 5.3.2.** *There exists an open subscheme  $V_0 \subset G$  containing the identity section such that for all  $u_\pm \in U_\pm$ , if  $u_+u_- \in V_0$  then  $u_+u_- \in \Omega'$ .*

This result is [SGA3, XXII, 6.16], whose proof rests on “abstract” rank-1 calculations in [SGA3, XX, 3.12]. We provide an exposition of the argument that instead rests on the explicit classification in Theorem 5.1.8.

*Proof.* Let  $w_0 \in W(\Phi)$  denote the long Weyl element relative to  $\Phi^+$  (i.e., the product in any order of the reflections in the positive roots), so the  $w_0$ -action swaps  $\Phi^+$  and  $-\Phi^+$ . By Corollary 5.1.11 we may choose  $n_0 \in N_G(T)(S)$  that is a representative for  $w_0$ , so  $n_0$ -conjugation swaps  $U_+$  and  $U_-$  and therefore  $n_0\Omega'n_0^{-1} = U_+ \times T' \times U_-$ . Hence, it suffices to find an open  $V_{n_0} \subset G$  containing the identity section such that  $n_0\Omega'n_0^{-1} \cap V_{n_0} \subset \Omega'$  inside  $G$  (as we may then take  $V_0 = V_{n_0}$ ). We shall prove the analogous result for any  $n \in N_G(T)(S)$ . Note that this problem is Zariski-local on  $S$ .

Using the map  $N_G(T)(S) \rightarrow W_G(T)(S)$  and the identification of  $W_G(T)$  with  $W(\Phi)_S$ , by working Zariski-locally on  $S$  we may assume that the image of  $n$  in  $W_G(T)(S)$  is a constant section arising from some  $w \in W(\Phi)$ . For each  $a$  in the set  $\Delta$  of simple positive roots we pick  $n_a \in N_G(T)(S)$  as in Corollary 5.1.11 representing the simple reflection  $s_a$  (viewed as a constant section in  $W_G(T)(S)$ ). Thus,  $n_a \in Z_G(T_a)$  for all  $a \in \Delta$  and  $n = n_{a_1} \cdots n_{a_m} t$  for some finite sequence  $\{a_j\}$  in  $\Delta$  and some  $t \in T(S)$ . To construct an open  $V_n \subset G$  around the identity section such that  $n\Omega'n^{-1} \cap V_n \subset \Omega'$ , we will argue by induction on the number  $m$  of simple roots that appear in the expression for  $n$ . The case  $m = 0$  is trivial by taking  $V_n = G$  ( $T$  normalizes  $U_\pm$ ), so in general we can arrange  $t = 1$ . For  $m = 1$  we will soon show that we can take  $V_{n_a} = \Omega$  for all  $a \in \Delta$ . Granting this for a moment, when  $m > 1$  we write  $n = n_{a_1} n'$  and by induction may assume that  $V_{n'}$  has been found. Then for the open subscheme  $V_n := n_{a_1} V_{n'} n_{a_1}^{-1} \cap \Omega \subset G$  we have

$$\begin{aligned} n\Omega'n^{-1} \cap V_n &= n_{a_1} (n'\Omega'n'^{-1} \cap V_{n'}) n_{a_1}^{-1} \cap \Omega \\ &\subset n_{a_1} \Omega' n_{a_1}^{-1} \cap \Omega \\ &\subset \Omega', \end{aligned}$$

where the final containment follows from our temporary hypothesis that we may take  $V_{n_a} = \Omega$  for all  $a \in \Delta$ .

It remains to prove that  $n_a \Omega' n_a^{-1} \cap \Omega \subset \Omega'$  for all  $a \in \Delta$ . By applying Remark 5.2.6 with  $\Psi = \Phi^+ - \{a\}$ , the root groups  $U_b$  for  $b \in \Phi^+ - \{a\}$  directly span *in any order* a smooth

closed  $S$ -subgroup  $U_{\widehat{a}} \subset U_+ = U_+$  (clearly normalized by  $T$ ). In particular,  $U_{\widehat{a}} \times U_a = U_+$  via multiplication. We similarly get  $U_{\widehat{-a}} \subset U_-$  such that  $U_{-a} \times U_{\widehat{-a}} = U_-$  via multiplication, so every point  $\omega'$  of  $\Omega'$  valued in an  $S$ -scheme  $S'$  has the form

$$\omega' = g_- \exp_{-a}(X_-) t' \exp_a(X_+) g_+$$

where  $g_{\pm} \in U_{\widehat{\pm a}}(S')$ ,  $t' \in T'(S')$ , and  $X_{\pm} \in \mathfrak{g}_{\pm a}(S')$ . The action on  $\Phi$  by the reflection  $s_a$  preserves  $\Phi^+ - \{a\}$  (as all elements of  $\Phi^+ - \{a\}$  have a positive coefficient away from  $a$  somewhere in their  $\Delta$ -expansion, and this property is not affected by applying  $s_a$ ), so  $n_a$ -conjugation preserves each  $U_{\widehat{\pm a}}$ . Thus, the property  $n_a \omega' n_a^{-1} \in \Omega'$  is insensitive to replacing  $\omega'$  with  $\exp_{-a}(X_-) t' \exp_a(X_+)$  (as  $\Omega'$  is stable under left multiplication by  $U_-$  and right multiplication by  $U_+$ ). Likewise, the condition  $n_a \omega' n_a^{-1} \in \Omega$  is insensitive to replacing  $\omega'$  with  $\exp_{-a}(X_-) t' \exp_a(X_+)$ . In other words, we may pass to the case  $g_{\pm} = 1$ .

With  $g_{\pm} = 1$ , our problem now takes place within the split reductive group  $Z_G(T_a)$  with semisimple-rank 1 and root system  $\{\pm a\}$  relative to its split maximal torus  $T$ , so we can assume  $G = Z_G(T_a)$ . If  $G = G_1 \times T_1$  for a split torus  $T_1$  then our problem takes place inside  $G_1$ , so we may pass to  $G_1$ . Thus, after Zariski-localization on the base we may assume that  $(G, T)$  is one of the three explicit pairs in the split semisimple-rank 1 classification in Theorem 5.1.8 (up to direct product against a split torus), and conjugating by the standard Weyl element  $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  if necessary allows us to arrange that  $a$  is the standard positive root (i.e.,  $U_a$  is the subgroup of strictly upper triangular matrices in  $\mathrm{SL}_2$  or  $\mathrm{PGL}_2$ ). In cases (1) and (2) of Theorem 5.1.8 we have  $T' = D = T$ , so  $\Omega = \Omega'$  and there is nothing to do. Thus, we may assume  $(G, T)$  is as in case (3):  $G = \mathrm{SL}_2 \times^{\mu_2} \mathbf{G}_m = \mathrm{GL}_2$  and  $T$  is the diagonal torus. The torus  $T'$  generated by  $a^\vee$  is the diagonal torus  $D$  in the subgroup  $\mathrm{SL}_2$ , and replacing  $n_a$  with a suitable left  $T(S')$ -multiple (as we may do) allows us to take  $n_a = w$ .

Now  $\Omega'$  is the open subgroup of  $\mathrm{SL}_2$  given by the unit condition on the upper left matrix entry, and  $\Omega = \Omega' \times^{\mu_2} \mathbf{G}_m$ . Thus,  $n \Omega' n^{-1} \cap \Omega \subset \mathrm{SL}_2 \cap \Omega = \Omega'$  inside  $G$ .  $\square$

By the choice of  $V$  using Lemma 5.3.2, the relatively schematically dense open

$$U_- \times T \times V \times T \times U_+ \subset \Omega \times \Omega$$

lies in  $m_G^{-1}(\Omega)$  and the map  $f$  in (5.3.1) satisfies  $f((u_1^- t_1 u_1^+)(u_2^- t_2 u_2^+)) = t_1 t_2 \bmod T'$ . Hence, by consideration of  $S$ -birational group laws (see [SGA3, XVIII, 2.3]),  $f$  extends uniquely to an  $S$ -homomorphism  $\bar{f}$ . On fibers over a geometric point  $\bar{s}$  of  $S$  we have a good theory of the semisimple derived group  $\mathcal{D}(G_{\bar{s}})$ . In particular,  $T'_{\bar{s}}$  is a maximal torus of  $\mathcal{D}(G_{\bar{s}})$  and the identification of root systems  $\Phi(\mathcal{D}(G_{\bar{s}}), T'_{\bar{s}}) = \Phi(G_{\bar{s}}, T_{\bar{s}})$  via the isogeny  $T'_{\bar{s}} \times (Z_{G_{\bar{s}}})_{\mathrm{red}}^0 \rightarrow T_{\bar{s}}$  identifies  $\Phi^+$  with a positive system of roots for  $\Phi(\mathcal{D}(G_{\bar{s}}), T'_{\bar{s}})$ . The corresponding open cell in  $\mathcal{D}(G_{\bar{s}})$  is clearly  $\Omega'_{\bar{s}}$ . Hence,  $(\ker \bar{f}_{\bar{s}})^0 = \mathcal{D}(G_{\bar{s}})$ , so  $\bar{f}_{\bar{s}}$  factors as

$$G_{\bar{s}} \rightarrow G_{\bar{s}}/\mathcal{D}(G_{\bar{s}}) \rightarrow T'_{\bar{s}}/T'_{\bar{s}},$$

where the second map is an étale isogeny of tori. This isogeny is an isomorphism since the maximal torus  $T'_{\bar{s}}$  maps onto the torus quotient  $G_{\bar{s}}/\mathcal{D}(G_{\bar{s}})$  and  $T'_{\bar{s}} \subset \mathcal{D}(G_{\bar{s}})$ . Thus,  $\bar{f}_{\bar{s}}$  is the quotient by the derived group of  $G_{\bar{s}}$ , so  $\bar{f}$  is a smooth surjection whose kernel  $\ker \bar{f}$  has

$\bar{s}$ -fiber  $\mathcal{D}(G_{\bar{s}})$ . Thus,  $\ker \bar{f}$  is a semisimple  $S$ -group that is closed and normal in  $G$ ; we define  $\mathcal{D}(G)$  to be this group.

By construction,  $\mathcal{D}(G)$  contains  $U_{\pm}$  and  $T'$  as subgroups, with  $T'$  a split maximal torus of  $\mathcal{D}(G)$ . There is a split triple  $(\mathcal{D}(G), T', M')$ , where  $M'$  is the maximal torsion-free quotient of  $M$  that kills  $(\mathbf{Z}\Phi^{\vee})^{\perp} \subset M$  and we let  $\Phi' \subset M'$  be the image of  $\Phi$ . Let  $\Phi'^+$  be the positive system of roots corresponding to  $\Phi^+$  under the bijection  $\Phi \rightarrow \Phi'$ . The direct product subfunctor  $\Omega'$  in  $\Omega$  lies in  $\mathcal{D}(G)$ . It is obvious that  $\Omega'$  must be the open cell of  $\mathcal{D}(G)$  associated to  $\Phi'^+$ . The commutativity of the quotient  $G/\mathcal{D}(G) = T/T'$  implies that  $\mathcal{D}(G)$  contains the commutator subsheaf of  $G$ . Provided that every semisimple  $S$ -group (such as  $\mathcal{D}(G)$ ) coincides with its own commutator subsheaf (for the fppf topology), the asserted uniqueness of  $\mathcal{D}(G)$  will be clear and so we will be done.

Finally, consider a semisimple  $S$ -group  $G$ . We seek to show that  $G$  is its own commutator subsheaf. We may work étale-locally on the base, so we can assume  $S \neq \emptyset$  and that  $G$  is part of a split triple  $(G, T, M)$  over  $S$ . Pick a positive system of roots in  $\Phi$ . The associated open cell generates  $G$  for the fppf topology (as for any open neighborhood of the identity section of a smooth group scheme with connected fibers), and the map  $\mathbf{G}_m^r \rightarrow T$  defined by the simple positive coroots is an isogeny. Thus, to prove that  $G$  is its own commutator subsheaf (for the fppf topology), the structure of the open cell as a direct product scheme reduces the problem to the case of semisimple groups with semisimple-rank 1 case, and more specifically to the case of  $\mathrm{SL}_2$  equipped with its diagonal torus (due to Theorem 5.1.8).

Once again passing to the standard open cell reduces us to some classical identities, as follows. For  $x(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ ,  $y(v) = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}$ , and  $h(t) = \mathrm{diag}(t, 1/t)$  we have  $h(t)x(u)h(t)^{-1}x(u)^{-1} = x((t^2 - 1)u)$  and  $h(t)y(v)h(t)^{-1}y(v)^{-1} = y((t^{-2} - 1)v)$ , so the commutator subsheaf contains both standard root groups. These in turn generate the diagonal torus via the identity

$$(5.3.2) \quad h(t) = y(-1/t)x(t)y(-1/t)(y(-1)x(1)y(-1))^{-1},$$

so  $\mathrm{SL}_2$  is indeed its own commutator subsheaf.  $\square$

The group  $\mathcal{D}(G)$  is called the *derived group* of  $G$ . Note that by uniqueness,  $\mathcal{D}(G) = G$  if and only if  $G$  is semisimple, and by the universal property it is clear that the formation of  $\mathcal{D}(G)$  is functorial in  $G$ . It follows formally that any  $S$ -homomorphism from a semisimple  $S$ -group to  $G$  must factor through  $\mathcal{D}(G)$ .

**Corollary 5.3.3.** *Let  $G \rightarrow S$  be a reductive group scheme,  $\mathcal{D}(G)$  its derived group. Let  $Z \subset Z_G$  be the maximal central torus, and  $T' = G/\mathcal{D}(G)$  the maximal torus quotient. The natural  $S$ -homomorphisms  $f : Z \times \mathcal{D}(G) \rightarrow G$  and  $h : G \rightarrow T' \times (G/Z)$  are central isogenies.*

This result is [SGA3, XXII, 6.2.4]. See Definition 3.3.8 for the notion of a central isogeny.

*Proof.* The classical theory implies that  $f_s$  and  $h_s$  are central isogenies for all  $s \in S$ , we may apply Proposition 3.3.9 to conclude.  $\square$

**Proposition 5.3.4.** *Let  $G \rightarrow S$  be a reductive group scheme, and  $Z$  the maximal torus of  $Z_G$ . The map  $T \mapsto T \cap \mathcal{D}(G)$  is a bijective correspondence between the set of maximal tori*

in  $G$  and the set of maximal tori in  $\mathcal{D}(G)$ . Conversely, if  $T'$  is a maximal torus of  $\mathcal{D}(G)$  then  $T' \times Z \rightarrow G$  is an isogeny onto a maximal torus  $T$  of  $G$ , and this reverses the bijective correspondence.

The same holds for parabolic subgroups, with the analogous procedures using intersection and product against  $Z$ .

*Proof.* Consider the correspondence for maximal tori. By Example 2.2.6, since  $\mathcal{D}(G)$  is normalized by  $T$  it follows that  $T \cap \mathcal{D}(G)$  is smooth with connected fibers, so it is a torus. The classical theory implies that this intersection is a maximal torus of  $\mathcal{D}(G)$ . Likewise, the classical theory shows that if we define  $T' := T \cap \mathcal{D}(G)$  then the multiplication map  $Z \times T' \rightarrow T$  between tori is an isogeny. Conversely, for a maximal torus  $T'$  of  $\mathcal{D}(G)$ , we have to show that the multiplication map  $T' \times Z \rightarrow G$  is an isogeny onto a maximal torus of  $G$ . This map factors through  $Z_G(T')$ , so it suffices to show that the reductive group  $Z_G(T')$  is a torus and  $T' \times Z \rightarrow Z_G(T')$  is an isogeny of tori. These assertions can be checked on geometric fibers over  $S$ , where they are well-known in the classical theory.

Now we turn to the consideration of parabolic subgroups. For any parabolic subgroup  $P \subset G$ , we have  $Z_G \subset P$ . Indeed, it suffices to check this étale-locally on  $S$ , and by Proposition 5.2.3 we can perform such localization on  $S$  so that  $P = P_G(\lambda)$  for some  $\lambda : \mathbf{G}_m \rightarrow G$ . Thus,  $Z_G \subset Z_G(\lambda) \subset P_G(\lambda) = P$ . It follows that  $Z \subset P$ , so the isogeny  $Z \times \mathcal{D}(G) \rightarrow G$  implies that  $P$  is uniquely determined by  $P' = P \cap \mathcal{D}(G)$ , and  $P'$  is smooth by Proposition 4.1.10(1). For each  $s \in S$  we have  $\mathcal{D}(G)_{\bar{s}}/P'_{\bar{s}} \simeq G_{\bar{s}}/P_{\bar{s}}$ , so  $P'_{\bar{s}}$  is a parabolic subgroup of  $\mathcal{D}(G)_{\bar{s}}$ . Hence,  $P'$  is parabolic in  $\mathcal{D}(G)$ . The multiplication map  $m : Z \times P' \rightarrow P$  is an isogeny on fibers, so  $m$  is a quasi-finite flat surjection, and  $\ker m$  is visibly central. But  $\ker m$  is closed in the  $S$ -finite kernel of  $Z \times G' \rightarrow G$ , so it is  $S$ -finite and hence  $m$  is a central isogeny.

Finally, it remains to show that every parabolic subgroup  $Q$  of  $\mathcal{D}(G)$  arises as  $P \cap \mathcal{D}(G)$  for a parabolic subgroup  $P$  of  $G$ . In view of the uniqueness of such a  $P$  we may work étale-locally on  $S$ , so by Proposition 5.2.3 we may arrange that  $Q = P_{\mathcal{D}(G)}(\mu)$  for some  $\mu : \mathbf{G}_m \rightarrow \mathcal{D}(G)$ . But then  $P := P_G(\mu)$  is a parabolic subgroup of  $G$  (Example 5.2.2) and  $P \cap \mathcal{D}(G) = P_{\mathcal{D}(G)}(\mu) = Q$ .  $\square$

For a split reductive group scheme  $(G, T, M)$  and parabolic subgroup  $P = P_G(\lambda)$  with  $\lambda \in M^\vee$ , Proposition 5.1.16(2) provides a composition series  $\{U_{\geq n}\}_{n \geq 1}$  for the unipotent radical  $U = \mathcal{R}_u(P) = U_G(\lambda)$  with  $U_{\geq n} = U_{\Phi_{\lambda \geq n}}$  a smooth closed subgroup directly spanned in any order by the root groups  $U_c$  for  $c$  satisfying  $\lambda(c) \geq n$ . Moreover,  $U_{\geq n+1}$  is normal in  $U_{\geq n}$  and the quotient  $U_{\geq n}/U_{\geq n+1}$  is commutative and identified (as an  $S$ -group) with the direct product of the root groups  $U_c \simeq \mathbf{G}_a$  with  $\lambda(c) = n$ . This general filtration of unipotent radicals of parabolic subgroups has an interesting application (that we will never need) when it is combined with the dynamic method of §4.1:

**Theorem 5.3.5.** *Let  $G \rightarrow S$  be a reductive group. Any monic homomorphism  $f : G \rightarrow G'$  to a separated  $S$ -group of finite presentation is a closed immersion. In particular, if  $G$  is an adjoint semisimple  $S$ -group then  $\mathrm{Ad}_G : G \rightarrow \mathrm{GL}(\mathfrak{g})$  is a closed immersion.*

This result is proved in another way (without the dynamic method) in [SGA3, XVI, 1.5(a)]. When  $G$  is not  $S$ -affine, our proof uses a difficult theorem due to Raynaud. We will not use Theorem 5.3.5 in the development of the general theory below.

*Proof.* The monicity condition is expressed by the fact that the diagonal  $\Delta_f : G \rightarrow G \times_{G'} G$  is an isomorphism, so by direct limit arguments we can reduce to the case when  $S$  is noetherian. Since  $f$  is a monomorphism, it is a closed immersion if and only if it is proper [EGA, IV<sub>3</sub>, 8.11.5]. Thus, by the valuative criterion for properness, we are reduced to checking that if  $R$  is a discrete valuation ring with fraction field  $K$  and  $\text{Spec } R \rightarrow S$  is a morphism of schemes then  $G(R) = G(K) \cap G'(R)$  inside  $G'(K)$ . Applying base change along  $\text{Spec } R \rightarrow S$  then reduces us to the case  $S = \text{Spec } R$ .

Next, we reduce to the case when  $G'$  is affine and  $R$ -flat. (The reader who is only interested in the case of affine  $G'$  can ignore this step.) The map on generic fibers  $G_K \rightarrow G'_K$  is a closed immersion, so the schematic closure  $\overline{G}$  of  $G$  in  $G'$  is an  $R$ -flat separated subgroup of  $G'$  through which  $G$  factors (since  $G$  is  $R$ -flat). But  $\overline{G}$  is a separated flat  $R$ -group of finite type with affine generic fiber, so it must be affine by a result of Raynaud (see [SGA3, VI<sub>B</sub>, 12.10(iii), 12.10.1] or [PY06, Prop. 3.1]). Hence, we may replace  $G'$  with  $\overline{G}$  to reduce to the case that  $G'$  is affine and  $R$ -flat.

Over a regular base of dimension  $\leq 1$ , every flat affine group scheme of finite type is a closed subgroup of some  $\text{GL}_n$ . This is easy to prove by adapting arguments from the case when the base is a field (see Exercise 5.4.7); in fact, the result is true over any regular affine base of dimension  $\leq 2$  [SGA3, VI<sub>B</sub>, 13.2]. Thus, we may identify  $G'$  as a closed subgroup of some  $\text{GL}_n$ , so we can replace  $G'$  with  $\text{GL}_n$ . This reduces the problem to the case that  $G'$  is a reductive  $S$ -group, but we allow  $S$  to be an arbitrary scheme (to clarify the generality of the steps that follow).

By working étale-locally on  $S$ , we may assume  $G$  arises from a split triple  $(G, T, M)$  (and that  $S$  is non-empty). Choose  $\lambda \in M^\vee$  such that  $T = Z_G(\lambda)$ , so  $B := P_G(\lambda)$  is a Borel subgroup of  $G$ . Let  $U_+ = U_G(\lambda) = \mathcal{R}_u(B)$  and  $U_- = U_G(-\lambda) = \mathcal{R}_u(B_-)$ , where  $B_- = P_G(-\lambda)$  is the opposite Borel subgroup of  $G$  containing  $T$  (see Proposition 5.2.12). Let  $\Omega \subset G$  be the open cell  $U_- \times B$ .

For  $\lambda' = f \circ \lambda$  we likewise get smooth closed subgroups  $Z_{G'}(\lambda'), U_{G'}(\pm\lambda') \subset G'$  such that the multiplication map

$$U_{G'}(-\lambda) \times Z_{G'}(\lambda') \times U_{G'}(\lambda') \rightarrow G'$$

is an open immersion; we let  $\Omega' \subset G'$  denote this open subscheme. Since  $Z_{G'}(\lambda')$  is reductive, by working étale-locally on  $S$  we can arrange that  $Z_{G'}(\lambda')$  contains a split maximal torus  $T'$ , so  $T'$  is maximal in  $G'$  (by the classical theory) and  $\lambda'$  factors through  $T'$  (since  $T' = Z_{G'}(T')$ ). Further localization on  $S$  brings us to the case that  $(G', T')$  arises from a split triple  $(G', T', M')$ . Hence, for  $\Psi' = \Phi'_{\lambda' \geq 0}$ ,  $U_{G'}(\pm\lambda')$  coincides with the subgroup  $U_{\pm\Psi'}$  from Remark 5.2.6 that is directly spanned by the root groups  $U_{c'}$  for  $c' \in \pm\Psi' = \Phi'_{\pm\lambda' > 0}$ .

Proposition 4.1.10(2) gives that  $\Omega \subset f^{-1}(\Omega')$ , and the key point is that this containment is an *equality*. To verify this equality between open subschemes of  $G$  we may pass to geometric

fibers over  $S$ , in which case the equality is [CGP, Prop. 2.1.8(3)] (which has nothing to do with smoothness or reductivity). Thus, the restriction of  $f$  over the open subscheme  $\Omega' \subset G'$  is the map  $\Omega \rightarrow \Omega'$  that is the direct product of the maps

$$U_G(-\lambda) \rightarrow U_{G'}(-\lambda'), \quad T = Z_G(\lambda) \rightarrow Z_{G'}(\lambda'), \quad U_G(\lambda) \rightarrow U_{G'}(\lambda').$$

We will prove that each of these three maps is a closed immersion, so  $f$  is a closed immersion when restricted over  $\Omega'$ .

Since  $T \rightarrow Z_{G'}(\lambda')$  is a monic homomorphism from a multiplicative type group to a smooth affine  $S$ -group, it is a closed immersion (Lemma B.1.3). To prove that  $U := U_G(\lambda) \rightarrow U_{G'}(\lambda') =: U'$  is a closed immersion, consider the filtrations  $\{U_{\geq n}\}_{n \geq 1}$  and  $\{U'_{\geq n}\}_{n \geq 1}$  on these as described immediately before Theorem 5.3.5. We claim that  $f$  carries  $U_{\geq n}$  into  $U'_{\geq n}$  for all  $n$ . More specifically:

**Lemma 5.3.6.** *For any  $c \in \Phi^+$ ,  $f|_{U_c}$  factors through the closed subgroup  $U_{Z_{G'}(\ker c)}(c^\vee) \subset G'$ , and this closed subgroup is directly spanned in any order by the root groups  $U'_{c'}$  for  $c' \in \Phi'$  such that  $c'|_T$  is a positive integral multiple of  $c$ .*

The group  $Z_{G'}(\ker c)$  is smooth since  $\ker c$  is multiplicative type, but beware that its fibers over  $S$  may not be connected (since  $\ker c$  may not be a torus).

*Proof.* Since  $U_c$  is normalized by  $T$  with trivial action by  $\ker c$ , it is carried into the smooth closed subgroup  $Z_{G'}(\ker c)$ . It follows that  $U_c$  is carried into  $U_{Z_{G'}(\ker c)}(c^\vee)$  since  $\langle c, c^\vee \rangle = 2 > 0$ . Likewise, if  $c' \in \Phi'$  satisfies  $c'|_T = nc$  with  $n \geq 1$  then  $c'$  kills  $\ker c$  and  $\langle c', c^\vee \rangle = \langle nc, c^\vee \rangle = 2n > 0$ , so  $U'_{c'}$  is contained in  $U_{Z_{G'}(\ker c)}(c^\vee)$ . It remains to show that for any choice of enumeration  $\{c'_i\}$  of the set of such  $c'$ , the multiplication map of  $S$ -schemes

$$(5.3.3) \quad \prod U'_{c'_i} \rightarrow U_{Z_{G'}(\ker c)}(c^\vee)$$

is an isomorphism. Since this is a map between smooth  $S$ -schemes, we may and do pass to geometric fibers, so  $S = \text{Spec } k$  for an algebraically closed field  $k$ .

Connectedness of  $U_{Z_{G'}(\ker c)}(c^\vee)$  implies that it coincides with  $U_{Z_{G'}(\ker c)^0}(c^\vee)$ . The group  $Z_{G'}(\ker c)^0$  is smooth since  $\ker c$  is of multiplicative type, and its Lie algebra is the trivial weight space  $\text{Lie}(G')^{\ker c}$  for the linear action of the split multiplicative type group  $\ker c$  on  $\text{Lie}(G')$ . Hence, the  $T'$ -weights on  $Z_{G'}(\ker c)^0$  are the elements  $c' \in \Phi'$  that are trivial on  $\ker c$ , which is to say  $c'|_T$  is an integral multiple of  $c$ . For such  $c'$ , the condition  $\langle c', c^\vee \rangle > 0$  says exactly that  $c'|_T$  is a positive integral multiple of  $c$ . It follows that (5.3.3) is an isomorphism on tangent spaces at the identity, so  $U_{Z_{G'}(\ker c)}(c^\vee)$  is generated by the  $U'_{c'_i}$ . Hence, we just have to check that these root groups directly span (in any order) a unipotent smooth connected subgroup of  $G'$ . The subset  $\{c'_i\} \subset \Phi'$  is closed and disjoint from its negative (since  $\langle c'_i, c^\vee \rangle > 0$  for all  $i$ ), so Proposition 5.1.16 provides this direct spanning result. (See [Bo91, 14.5(2)] and [CGP, 3.3.11, 3.3.13(1)] for related direct spanning results in the theory over a field.)  $\square$

By Lemma 5.3.6, for every  $n \geq 1$  we get homomorphisms  $U_{\geq n} \rightarrow U'_{\geq n}$ . Consider the resulting homomorphisms between vector groups

$$f_n : U_{\geq n}/U_{\geq n+1} \rightarrow U'_{\geq n}/U'_{\geq n+1}.$$

We claim that each  $f_n$  is a closed immersion. By construction,  $f_n$  is  $\mathbf{G}_m$ -equivariant with source and target identified with a power of  $\mathbf{G}_a$  on which  $\mathbf{G}_m$  acts through the  $n$ th-power map. Thus,  $f_n$  is a linear map of vector bundles, so to check if it is a closed immersion it suffices to pass to geometric fibers and verify injectivity on Lie algebras. For this purpose we may now assume that  $S = \text{Spec } k$  for an algebraically closed field  $k$ . The Lie algebra of  $U_{\geq n}/U_{\geq n+1}$  is the direct product of the root groups  $U_c$  for  $c \in \Phi$  such that  $\lambda(c) = n$ , and similarly for  $U'_{\geq n}/U'_{\geq n+1}$  using  $\lambda' = f \circ \lambda$ . If  $\text{Lie}(f_n)$  is not injective then by the equivariance of  $f_n$  with respect to the closed immersion of tori  $T \hookrightarrow T'$  it follows that  $\ker(\text{Lie}(f_n))$  would have to contain some root space  $\mathfrak{g}_c$  for  $c \in \Phi$  satisfying  $\lambda(c) = n$ . The vanishing of  $\text{Lie}(f_n)$  on  $\mathfrak{g}_c$  implies that  $\text{Lie}(f)$  carries  $\mathfrak{g}_c$  into the span of the root spaces  $\mathfrak{g}_{c'}$  for  $c' \in \Phi'$  such that  $\lambda'(c') \geq n+1$ . But the  $T$ -action on that span has as its weights precisely the  $T$ -restrictions of these roots  $c'$ , so the containment of  $\mathfrak{g}_c$  in the space forces  $c'|_T = c$  for at least one such  $c'$ . For that  $c'$  we have  $n+1 \leq \lambda'(c') = \lambda(c'|_T) = \lambda(c) = n$ , a contradiction.

Returning to the relative setting over  $S$ , since the maps  $U_{\geq n}/U_{\geq n+1} \rightarrow U'_{\geq n}/U'_{\geq n+1}$  are all closed immersions, the map  $U \rightarrow U'$  is a closed immersion by repeated applications of:

**Lemma 5.3.7.** *In a commutative diagram of short exact sequences of flat, separated, finitely presented  $S$ -group schemes*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H'_1 & \longrightarrow & H_1 & \longrightarrow & H''_1 & \longrightarrow & 1 \\ & & j' \downarrow & & \downarrow j & & \downarrow j'' & & \\ 1 & \longrightarrow & H'_2 & \longrightarrow & H_2 & \longrightarrow & H''_2 & \longrightarrow & 1 \end{array}$$

*if the outer vertical maps are closed immersions then so is the middle one.*

*Proof.* As usual, we may reduce to the case that  $S$  is noetherian. Clearly  $j$  is a monomorphism, so it suffices to prove that it is proper. Using the valuative criterion for properness, it suffices to show that if  $R$  is a discrete valuation ring with fraction field  $K$  then the containment  $H_1(R) \subset H_1(K) \cap H_2(R)$  is an equality. Choose  $h_1 \in H_1(K) \cap H_2(R)$ , so the image  $h''_1$  of  $h_1$  in  $H''_1(K)$  lies in  $H''_1(K) \cap H''_2(R) = H''_1(R)$ . To extend  $h_1$  to an  $R$ -point of  $H_1$  it is harmless to replace  $R$  with a flat local extension by another discrete valuation ring. Since  $H_1 \rightarrow H''_1$  is fppf, we can choose such an extension  $\tilde{R}$  of  $R$  so that  $h''_1$  viewed as an  $\tilde{R}$ -point of  $H''_1$  lifts to  $H_1(\tilde{R})$ . By renaming  $\tilde{R}$  as  $R$  and multiplying  $h_1$  by the inverse of an  $R$ -lift of  $h''_1$ , we reduce to the case that  $h''_1$  is trivial, so  $h_1$  arises from some  $h'_1 \in H'_1(K)$ . The image  $h_2 \in H_2(K)$  of  $h_1$  lies in  $H_2(R) \cap H'_2(K) = H'_2(R)$ , so  $h'_1 \in H'_1(K) \cap H'_2(R) = H'_1(R)$ . Hence,  $h_1 \in H_1(R)$ .  $\square$

We have completed the proof that  $U_G(\lambda) \rightarrow U_{G'}(\lambda')$  is a closed immersion. Likewise,  $U_G(-\lambda) \rightarrow U_{G'}(-\lambda')$  is a closed immersion, so  $\Omega = f^{-1}(\Omega') \rightarrow \Omega'$  is a closed immersion.

That is,  $f : G \rightarrow G'$  is a closed immersion when restricted over the open subscheme  $\Omega'$  in the reductive group  $G'$ . Since  $(G, T)$  is split,  $N_G(T)(S) \rightarrow W_G(T)(S)$  is surjective (Corollary 5.1.11) and  $W_G(T) = W(\Phi)_S$  is a finite constant  $S$ -group (Proposition 5.1.6). Thus, the Bruhat decomposition on geometric fibers implies that  $G$  is generated by  $N_G(T)(S)$ -translates of  $\Omega$ . Hence,  $f$  is a closed immersion into the open union of translates of  $\Omega'$  by the image in  $G'(S)$  of representatives in  $N_G(T)(S)$  for the finitely many elements of  $W(\Phi)$ . We conclude that  $f$  is a (finitely presented) closed immersion into a (finitely presented) open subscheme, or in other words it is a quasi-compact immersion.

By the valuative criterion for properness, to prove that  $f$  is a closed immersion we may reduce to the case  $S = \text{Spec } R$  for a discrete valuation ring  $R$  with fraction field  $K$ . Over such a Dedekind base, the schematic closure  $\overline{G}$  of  $G$  in  $G'$  is a closed  $S$ -subgroup with generic fiber  $G_K$  and it contains  $G$  as an open subgroup. In particular, the special fiber  $\overline{G}_0$  has identity component  $G_0$  that is smooth (and reductive), so  $\overline{G}$  is smooth. Replacing  $G'$  with  $\overline{G}$  reduces us to the case that  $G'$  contains  $G$  as a dense open subgroup. It then suffices to prove that  $G'_0$  is connected (so  $G' = G$ ), and this is the content of Proposition 3.1.12.  $\square$

**Example 5.3.8.** As an application of the open cell and the closed immersion property for the adjoint representation of a semisimple group scheme of adjoint type in Theorem 5.3.5, consider such group schemes  $G$  over  $\mathbf{Z}$  that are split and have an irreducible root system; these are the (simple semisimple) Chevalley groups of adjoint type. Let  $T$  be a split maximal  $\mathbf{Z}$ -torus, and  $\Phi = \Phi(G, T)$ . Fix a positive system of roots  $\Phi^+$  in  $\Phi$ , and let  $\Delta \subset \Phi^+$  be the base of simple positive roots. For any field  $k$ , we claim that  $G(k)$  is the subgroup of  $\text{Aut}(\mathfrak{g}_k)$  generated by the elements  $\text{Ad}_G(\exp_{\pm a}(X))$  for  $X \in \mathfrak{g}_{\pm a}$  and  $\text{Ad}_G(t)$  where  $t \in \prod_{a \in \Delta} k^\times$  via the isomorphism  $T \simeq \prod_{a \in \Delta} \mathbf{G}_m$  defined by  $t \mapsto (a(t))$  (isomorphism due to the adjoint property; see Exercise 5.4.4).

The groups  $U_{\pm a}(k)$  generate the representative  $w_a(X)$  of  $s_a \in W(\Phi)$  using any  $X \in \mathfrak{g}_a - \{0\}$ , and these simple reflections  $s_a$  generate  $W(\Phi)$ . But  $N_G(T)(k) \rightarrow W_G(T)(k) = W(\Phi)$  is surjective since  $T$  is  $k$ -split (and  $W_G(T) = N_G(T)/T$ ), so conjugation by the elements  $w_a(X)$  carries the groups  $U_a(k)$  for simple positive  $a$  to the groups  $U_b(k)$  for all roots  $b$ . This provides the factors  $U_{\pm\Phi^+}(k)$  in the  $k$ -points of the open cell  $\Omega$ . The standard locally closed Bruhat cells over  $\overline{k}$  are clearly defined over  $k$  (using representatives for  $W(\Phi)$  in  $N_G(T)(k)$ , such as the elements  $w_a(X)$ ), so the Bruhat decomposition over  $\overline{k}$  implies that  $\Omega(k)$  generates  $G(k)$ . This gives the desired list of generators by applying the inclusion  $\text{Ad}_G : G(k) \hookrightarrow \text{Aut}(\mathfrak{g}_k)$ .

This can be pushed further via the split semisimple-rank 1 classification and the Existence Theorem over  $\mathbf{Z}$ , as follows. By the Existence Theorem, there exists a simply connected central cover  $\tilde{G} \rightarrow G$  over  $\mathbf{Z}$  (see Exercise 6.5.3), so  $G = \tilde{G}/Z_{\tilde{G}}$ . We claim that inside  $\text{Aut}(\mathfrak{g}_k)$ , the elements  $\text{Ad}_G(\exp_{\pm a}(X))$  for  $X \in \mathfrak{g}_{\pm a}(k)$  generate the image of  $\tilde{G}(k)/Z_{\tilde{G}}(k)$  in  $G(k) = (\tilde{G}/Z_{\tilde{G}})(k)$ .

To prove this, first note that (as for any semisimple central extension of  $G$  by a finite group scheme of multiplicative type) the preimage  $\tilde{T}$  of  $T$  in  $\tilde{G}$  is a split maximal torus in  $\tilde{G}$ , and there exists a natural identification of root systems and isomorphisms between corresponding

root groups for  $(G, T)$  and  $(\tilde{G}, \tilde{T})$  (Exercise 1.6.12(i)). Moreover, the simply connected property for  $\tilde{G}$  implies that the simple positive coroots are a basis of the cocharacter group of  $\tilde{T}$ . Thus, no coroot is divisible in the cocharacter lattice, so each pair of opposite root groups of  $\tilde{G}_k$  generates an  $\mathrm{SL}_2$  and not a  $\mathrm{PGL}_2$ . But in  $\mathrm{SL}_2(k)$  the diagonal points are generated by the  $k$ -points of the standard root groups (see (5.3.2) for a classical formula), so the elements  $U_{\pm a}(k)$  in  $G(k)$  for  $a \in \Delta$  generate the image of  $\tilde{G}(k)$  in  $G(k)$ . This establishes our claim.

By using a well-chosen choice of  $\mathbf{Z}$ -basis  $X_b$  of each  $\mathfrak{g}_b$  for every  $b \in \Phi$  (a ‘‘Chevalley system’’, as in the proof of the Existence Theorem; see Definition 6.3.2), the elements  $\exp_a(X_a) \in \mathrm{Aut}(\mathfrak{g})$  can be described explicitly over  $\mathbf{Z}$  in terms of the combinatorics of  $\Phi$ . This yields explicit ‘‘universal’’ formulas for  $\exp_a(cX_a)$  for all  $c \in k^\times$ . For finite  $k$ , this explicit list of generators inside  $\mathrm{Aut}(\mathfrak{g}_k)$  is used in some references to define the finite Chevalley groups (rather than defining them to be  $\tilde{G}(k)/Z_{\tilde{G}}(k)$ ). The basic structural properties of the finite Chevalley groups (such as the simplicity of  $\tilde{G}(k)/Z_{\tilde{G}}(k)$  for finite  $k$  away from the three exceptional cases  $\mathrm{SL}_2(\mathbf{F}_2)$ ,  $\mathrm{SL}_2(\mathbf{F}_3)/\mathbf{F}_3^\times$ , and  $\mathrm{Sp}_4(\mathbf{F}_2) \simeq S_6$ ) can be established by using the structure theory of split semisimple groups over general fields (via Tits’ theory of  $(B, N)$ -pairs).

## 5.4. Exercises.

**Exercise 5.4.1.** Let  $G$  be a smooth connected affine group over a field  $k$ .

(i) For a maximal  $k$ -torus  $T$  in  $G$  (see Remark A.1.2) and a smooth connected  $k$ -subgroup  $N$  in  $G$  that is normalized by  $T$ , prove that  $T \cap N$  is a maximal  $k$ -torus in  $N$  (e.g., smooth and connected!). Show by example that  $S \cap N$  can be disconnected for a non-maximal  $k$ -torus  $S$ . Hint: first analyze  $Z_G(T) \cap N$  using  $T \times N$  to reduce to the case when  $T$  is central in  $G$ , and then pass to  $G/T$ .

(ii) Let  $H$  be a smooth connected normal  $k$ -subgroup of  $G$ , and  $P$  a parabolic  $k$ -subgroup. Prove  $(P_{\bar{k}} \cap H_{\bar{k}})_{\text{red}}^0$  is a parabolic subgroup of  $H_{\bar{k}}$ , and use Theorem 1.1.8 (applied to  $H$ ) to prove  $P \cap H$  is connected (hint: work over  $\bar{k}$ ).

(iii) For  $H$  as in (ii), by using that  $Q = N_H(Q)$  scheme-theoretically for parabolic  $Q$  in  $H$  (Corollary 5.2.8), prove  $P \cap H$  in (ii) is smooth and therefore parabolic in  $H$ . (Hint: when  $k = \bar{k}$ , prove  $(P \cap H)_{\text{red}}^0$  is normal in  $P$ , hence in  $P \cap H$ .) In particular, prove that the scheme-theoretic intersection  $B \cap H$  is a Borel  $k$ -subgroup of  $H$  for all Borels  $B$  of  $G$ .

**Exercise 5.4.2.** This exercise generalizes Theorem 5.1.19 to the reductive case. Let  $G$  be a connected reductive group over a field  $k$ ,  $Z$  its maximal central  $k$ -torus, and  $G' = \mathcal{D}(G)$  its semisimple derived group. Let  $\{G'_i\}$  be the  $k$ -simple factors of  $G'$ . Prove that they are precisely the minimal nontrivial normal smooth connected non-central  $k$ -subgroups of  $G$ , and that the multiplication homomorphism

$$Z \times \prod G_i \rightarrow G$$

is a central isogeny. (Keep in mind that if  $k$  is finite then  $G(k)$  is not Zariski-dense in  $G$ , so in general an argument is needed to prove that the  $G'_i$  are normal in  $G$ .) Also prove that the normal connected semisimple  $k$ -subgroups of  $G'$  are necessarily normal in  $G$  (the converse being obvious).

**Exercise 5.4.3.** Let  $R$  be Dedekind with fraction field  $K$ , and  $G$  a connected reductive  $K$ -group. A reductive  $R$ -group scheme is *quasi-split* if it has a Borel subgroup over the base.

(i) Show that  $G = \mathcal{G}_K$  for a reductive group scheme  $\mathcal{G}$  over a dense open  $U \subset \text{Spec } R$ .

(ii) Assume  $R$  is a henselian (e.g., complete) discrete valuation ring and that  $G = \mathcal{G}_K$  for a reductive  $R$ -group  $\mathcal{G}$ . Using  $\text{Bor}_{\mathcal{G}/R}$ , prove that if the special fiber  $\mathcal{G}_0$  is split (resp. quasi-split) then so is  $G$  over  $K$ . What if  $R$  is not assumed to be henselian?

(iii) Using (i) and (ii), show that if  $G$  is a connected reductive group over a global field  $F$  then  $G_{F_v}$  is quasi-split for all but finitely many places  $v$  of  $F$ . Likewise show that any  $G$ -torsor over  $F$  admits an  $F_v$ -point for all but finitely many  $v$ . See Exercise 7.3.5 for the property of being split.

**Exercise 5.4.4.** For a split adjoint semisimple group  $(G, T, M)$  over a non-empty scheme  $S$ , Example 5.3.8 used that any base  $\Delta$  for  $\Phi$  is a basis for  $M$ . Using Corollary 3.3.6, explain why  $\Delta$  being a basis for  $M$  characterizes the adjoint property for  $G$ .

**Exercise 5.4.5.** This exercise develops a very important special case of Exercise 3.4.4, the group of “norm-1 units” in a central simple algebra.

(i) The linear derivations of a matrix algebra over a field are precisely the inner derivations (i.e.,  $x \mapsto yx - xy$  for some  $y$ ); see [DF, Ch. 3, Exer. 30] for a proof based on a clever application of the Skolem–Noether theorem. Combining this with length-induction on artin local rings, prove the Skolem–Noether theorem for  $\text{Mat}_n(R)$  for any artin local ring  $R$  (i.e., all  $R$ -algebra automorphisms of  $\text{Mat}_n(R)$  are conjugation by a unit).

(ii) Let  $A$  be a finite-dimensional central simple algebra over a field  $k$ . Construct an affine  $k$ -scheme  $I$  of finite type such that naturally in  $k$ -algebras  $R$ ,

$$I(R) = \text{Isom}_{R\text{-alg}}(A_R, \text{Mat}_n(R)).$$

Note that  $I(\bar{k})$  is non-empty. Prove  $I$  is smooth by checking the infinitesimal criterion for  $I_{\bar{k}}$  with the help of (i). Deduce that  $A_K \simeq \text{Mat}_n(K)$  for a finite *separable* extension  $K/k$ .

(iii) By (ii), we can choose a finite Galois extension  $K/k$  and a  $K$ -algebra isomorphism  $\theta : A_K \simeq \text{Mat}_n(K)$ , and by Skolem–Noether this is unique up to conjugation by a unit. Prove that for any choice of  $\theta$ , the determinant map transfers to a multiplicative map  $\underline{A}_K \rightarrow \mathbf{A}_K^1$  which is independent of  $\theta$ . Deduce that it is  $\text{Gal}(K/k)$ -equivariant, and so descends to a multiplicative map  $\text{Nrd}_{A/k} : \underline{A} \rightarrow \mathbf{A}_k^1$  which “becomes” the determinant over *any* extension  $F/k$  for which  $A_F \simeq \text{Mat}_n(F)$ . Prove that  $\text{Nrd}_{A/k}^n = \text{N}_{A/k}$  (explaining the name *reduced norm* for  $\text{Nrd}_{A/k}$ ), and conclude that  $\underline{A}^\times = \text{Nrd}_{A/k}^{-1}(\mathbf{G}_m)$ .

(iv) Let  $\text{SL}(A) = \ker(\text{Nrd}_{A/k} : \underline{A}^\times \rightarrow \mathbf{G}_m)$  (denoted  $\text{SL}_{n,D}$  if  $A = \text{Mat}_n(D)$  for a central division algebra  $D$  over  $k$ ). Prove that its formation commutes with any extension of the ground field, and that it becomes isomorphic to  $\text{SL}_n$  over  $\bar{k}$ . In particular,  $\text{SL}(A)$  is a connected semisimple  $k$ -group that is the derived group of the connected reductive  $\underline{A}^\times$ . (Note that  $\ker \text{N}_{A/k}$  is non-smooth whenever  $\text{char}(k) | n$ .)

(v) Using the preceding constructions and Galois descent, generalize the bijective correspondence in Exercise 4.4.6(ii) to central simple algebras over any field (possibly finite).

**Exercise 5.4.6.** This exercise builds on Exercise 5.4.5 to prove a special case of a general conjugacy result of Borel and Tits for maximal split tori in connected reductive groups over fields. Let  $A$  be a central simple algebra over a field  $k$ ,  $T$  a  $k$ -torus in  $\underline{A}^\times$  containing  $Z_{\underline{A}^\times} = \mathbf{G}_m$ , and  $A_T$  the corresponding étale commutative  $k$ -subalgebra of  $A$  (with  $\dim_k A_T = \dim T$ ) as in Exercise 4.4.6.

(i) Prove that  $\text{SL}(A)$  is  $k$ -anisotropic if and only if  $A$  is a division algebra.

(ii) Prove that the centralizer  $B_T = Z_A(A_T)$  is a semisimple  $k$ -algebra with center  $A_T$ .

(iii) If  $T$  is  $k$ -split, prove  $A_T \simeq k^r$  and that the simple factors  $B_i$  of  $B_T$  are *central* simple  $k$ -algebras.

(iv) Assume  $T$  is  $k$ -split. Using (iii), prove  $T$  is maximal as a  $k$ -split torus in  $\underline{A}^\times$  if and only if the (central!) simple factors  $B_i$  of  $B_T$  are *division algebras*.

(v) Fix an isomorphism  $A \simeq \text{End}_D(V)$  for a right module  $V$  over a central division algebra  $D$ , so  $V$  is a left  $A$ -module and  $V = \coprod V_i$  with *nonzero* left  $B_i$ -modules  $V_i$ . If  $T$  is maximal

as a  $k$ -split torus in  $\underline{A}^\times$ , prove  $V_i$  has rank 1 over  $B_i$  and  $D$ , so  $B_i \simeq D$ . Using  $D$ -bases, deduce that *all maximal  $k$ -split tori in  $\underline{A}^\times$  are  $\underline{A}^\times(k)$ -conjugate*.

**Exercise 5.4.7.** In the proof of Theorem 5.3.5, we used the fact that any flat affine group scheme  $G$  of finite type over a Dedekind domain  $R$  occurs as a closed subgroup of some  $\mathrm{GL}_n$  over  $R$ .

(i) Prove the analogous result over fields by adapting whatever is your favorite proof for *smooth* affine groups over fields.

(ii) Make your argument in (i) work over  $R$  (for flat affine groups of finite type) by using that any finitely generated torsion-free  $R$ -module is projective (and hence a direct summand of a finite free  $R$ -module).

**Exercise 5.4.8.** Let  $G$  be a reductive group over a scheme  $S$ . Show that if  $P$  is a parabolic subgroup of  $G$  then  $Z_G \subset P$  and that  $P \mapsto P/Z_G$  is a bijective correspondence between the set of parabolic subgroups of  $G$  and of  $G^{\mathrm{ad}} = G/Z_G$ , with inverse given by the formation of inverse images under the quotient map  $G \rightarrow G^{\mathrm{ad}}$ .

Construct natural isomorphisms  $\mathrm{Par}_{G/S} \simeq \mathrm{Par}_{G^{\mathrm{ad}}/S}$  and  $\mathrm{Bor}_{G/S} \simeq \mathrm{Bor}_{G^{\mathrm{ad}}/S}$ .

**Exercise 5.4.9.** Let  $G$  be a reductive group over a non-empty scheme  $S$ , and  $Z \subset Z_G$  a flat central closed subgroup scheme (so  $Z$  is of multiplicative type). This exercise addresses splitting properties for  $G/Z$  given splitting hypotheses on  $G$  and  $Z$ .

(i) Prove that the smooth affine quotient  $G' = G/Z$  is a reductive  $S$ -group, and that if  $T \subset G$  is a maximal torus (so  $Z \subset Z_G \subset Z_G(T) = T$ ) then so is  $T' := T/Z \subset G'$ . Give an example over a field in which  $T'$  is split,  $T$  is non-split, and  $Z_G$  is not a direct factor of  $T$ .

(ii) Consider a split triple  $(G, T, M)$ , and assume  $Z$  is split, so  $X(Z) = \overline{M}_S$  for a quotient  $\overline{M} = M/M'$  of  $M$ . For each  $a \in \Phi \subset M$  (so  $a|_{Z_G} = 1$ ), let  $a'$  denote the induced character of  $T' = T/Z$ . Prove that  $X(T') = M'_S$  inside  $X(T) = M_S$ , and that  $a' \in M'$  for all  $a$ .

(iii) For  $(G, T, M)$  as in (ii), prove that the natural map  $U_a \rightarrow U_{a'}$  between root groups is an isomorphism (hint: fibral isomorphism criterion). Deduce that the line bundle  $\mathfrak{g}'_{a'}$  on  $S$  is *globally trivial*, and that  $(G', T', M')$  is split in the sense of Definition 5.1.1 (note Example 5.1.2).

## 6. EXISTENCE, ISOMORPHISM, AND ISOGENY THEOREMS

**6.1. Pinnings and main results.** In §1.5 we introduced the notion of a *pinning* on a triple  $(G, T, B)$  over an algebraically closed field  $k$ . The purpose of that concept was to “rigidify” the triple (eliminating the action of the adjoint torus) so that passage to the root datum loses no information concerning isomorphisms. The Isomorphism Theorem for split reductive group schemes over an arbitrary non-empty scheme  $S$  requires a relative version of pinnings, and there is a generalization (the Isogeny Theorem) that incorporates isogenies. The purpose of this preliminary section is to develop several concepts related to pinnings and morphisms of root data. At the end of this section we state the Existence, Isomorphism, and Isogeny Theorems over any non-empty scheme  $S$ .

**Definition 6.1.1.** Let  $(G, T, M)$  be a split reductive group over a non-empty scheme  $S$ , and let  $R(G, T, M) = (M, \Phi, M^\vee, \Phi^\vee)$  be its associated root datum. A *pinning* on  $(G, T, M)$  is a pair  $(\Phi^+, \{X_a\}_{a \in \Delta})$  consisting of a positive system of roots  $\Phi^+ \subset \Phi$  (or equivalently, a base  $\Delta$  of  $\Phi$ ) and trivializing sections  $X_a \in \mathfrak{g}_a(S)$  for each simple positive root  $a \in \Delta$ .

The 5-tuple  $(G, T, M, \Phi^+, \{X_a\}_{a \in \Delta})$  is a *pinned* split reductive  $S$ -group.

Since  $\Delta$  determines  $\Phi^+$ , we will usually write  $(G, T, M, \{X_a\}_{a \in \Delta})$  and suppress the explicit mention of  $\Phi^+$ . In Exercise 6.5.1 a more “group-theoretic” definition of pinnings over  $S$  is given, replacing the trivializations of simple positive root spaces  $\mathfrak{g}_a$  with suitable homomorphisms from  $\mathrm{SL}_2$  into  $\mathcal{D}(Z_G(T_a))$  for each  $a \in \Delta$ . Keep in mind that the definition of the “split” property for  $(G, T, M)$  in Definition 5.1.1 includes the condition that the line bundles  $\mathfrak{g}_a$  are free of rank 1, so a pinning  $(\Phi^+, \{X_a\}_{a \in \Delta})$  can be chosen for any  $\Phi^+ \subset \Phi$ .

**Remark 6.1.2.** It may be tempting to expect that the choice of  $\Phi^+$  is “the same” as a choice of Borel subgroup of  $G$  containing  $T$  as in the classical case, but that it not true when  $S$  is disconnected (and we must allow the base scheme to be disconnected for descent theory arguments). More precisely, by Proposition 5.2.3 and Corollary 5.2.7(2),  $\Phi^+ = \Phi(B, T)$  for a unique Borel subgroup  $B \subset G$  containing  $T$ , and the Borel subgroups of  $G$  containing  $T$  that we obtain by varying  $\Phi^+$  are precisely those  $B$  for which the subgroup

$$\Phi(B, T) \subset \mathrm{Hom}_{S\text{-gp}}(T, \mathbf{G}_m) = \Gamma(S, X_*(T)) = \Gamma(S, \underline{M})$$

lies inside the subgroup  $M$  of “constant sections”. In particular, when  $S$  is disconnected there are Borel subgroups  $B'$  of  $G$  containing  $T$  that do not arise from any choice of  $\Phi^+ \subset \Phi$ . For this reason, in the relative theory we work throughout with a choice of  $\Phi^+$  rather than with a choice of  $B$  (although the two viewpoints coincide when  $S$  is connected, such as in the theory over a field, domain, or local ring).

There is an evident notion of *isomorphism* between pinned split reductive  $S$ -groups. The right notion of *isogeny* (refining the ordinary notion for smooth affine  $S$ -groups as in Definition 3.3.8) requires some care in order to account for Frobenius isogenies between root groups on geometric fibers in positive characteristic. To motivate the definition, consider

two pinned split reductive  $S$ -groups

$$(G', T', M', \{X'_{a'}\}_{a' \in \Delta'}), \quad (G, T, M, \{X_a\}_{a \in \Delta})$$

over  $S$  and a quasi-finite surjective  $S$ -homomorphism  $f : G' \rightarrow G$  such that  $f(T') \subset T$ . (In Proposition 6.1.10 we will show that  $f$  is necessarily finite and flat, hence an isogeny.) Note that  $f : T' \rightarrow T$  is an isogeny, since the map

$$X(f) : \underline{M} = X_*(T) \rightarrow X_*(T') = \underline{M}'$$

induces a finite-index inclusion of lattices on geometric fibers over  $S$ .

There is an open cover  $\{U_i\}$  of  $S$  such that the map induced by  $X(f)$  on  $U_i$ -sections carries  $M$  into  $M'$ . We now suppose (as may be achieved by working Zariski-locally on  $S$ ) that the map induced by  $X(f)$  on global sections over  $S$  carries  $M$  into  $M'$ . In particular, we get an isomorphism  $M_{\mathbf{Q}} \simeq M'_{\mathbf{Q}}$  between  $\mathbf{Q}$ -vector spaces.

By the classical theory on geometric fibers, for each  $a' \in \Phi'$  the root group  $(U_{a'})_{\bar{s}}$  for  $(G'_{\bar{s}}, T'_{\bar{s}})$  is carried isogenously by  $f_{\bar{s}}$  onto the root group of  $(G_{\bar{s}}, T_{\bar{s}})$  for a unique  $a(s) \in \Phi$ . Since  $f_{\bar{s}}$  is a (possibly non-central) isogeny, every root in  $\Phi(G_{\bar{s}}, T_{\bar{s}})$  arises in this way from a unique  $a' \in \Phi'$ . Each resulting map between root groups of the  $\bar{s}$ -fibers is identified with an endomorphism of  $\mathbf{G}_a$  having the form  $x \mapsto cx^{q(s)}$  for some  $c \in k(\bar{s})^\times$  and some integral power  $q(s) \geq 1$  of the characteristic exponent of  $k(s)$ , due to the equivariance of  $f_{\bar{s}}$  with respect to the isogeny  $T'_{\bar{s}} \rightarrow T_{\bar{s}}$ . It follows that  $X(f_{\bar{s}})(a') = q(s)a(s)$ . Likewise,  $X_*(f_{\bar{s}})(a(s)^\vee) = q(s)a'^\vee$  by the construction of coroots in the classical theory. The map  $\Phi' \rightarrow \Phi$  defined by  $a' \mapsto a(s)$  has very weak dependence on  $s$ :

**Lemma 6.1.3.** *For  $a' \in \Phi'$ , the associated function  $s \mapsto a(s) \in \Phi(G_{\bar{s}}, T_{\bar{s}}) = \Phi$  is Zariski-locally constant in  $s$ .*

*Proof.* We may assume that  $S$  is noetherian, so every pair of distinct points  $\{s, \eta\}$  in  $S$  with  $s$  in the closure of  $\eta$  can be dominated by the spectrum of a discrete valuation ring. Since our problem is to prove a constancy result on the connected components of  $S$ , and every point  $s$  of  $S$  is in the closure of the generic point of each irreducible component of  $S$  through  $s$ , by pullback to the spectra of discrete valuation rings we may and do assume  $S = \text{Spec } R$  for a discrete valuation ring  $R$ . In this case we have to prove that  $a(s) = a(\eta)$  in  $\Phi$ , where  $s$  is the closed point of  $S$  and  $\eta$  is the generic point of  $S$ .

Viewing  $a(s)$  as an  $S$ -homomorphism  $T \rightarrow \mathbf{G}_m$  global section of  $X_*(T)$  via the inclusion  $\Phi \subset M$ , the saturation of  $\mathbf{Z}a(s)$  in  $M$  defines a split  $S$ -subtorus  $T_{a(s)}$  of relative codimension 1 in  $T$ , and the isogeny  $T' \rightarrow T$  between split  $S$ -tori must carry  $T'_{a'}$  into  $T_{a(s)}$  since we can check this on the fibers over a single geometric closed point of  $S$  (such as  $s$ ). Likewise using a geometric generic point of  $S$  gives that  $T'_{a'}$  is carried into  $T_{a(\eta)}$ . Hence,  $a(\eta) = \pm a(s)$  (since  $a(s)$  and  $a(\eta)$  lie in the reduced root system  $\Phi$ ). We can therefore pass to the induced map  $\mathcal{D}(Z_{G'}(T'_{a'})) \rightarrow \mathcal{D}(Z_G(T_{a(s)}))$  to reduce to the case when the fibers are semisimple of rank 1.

Consider the cocharacters  $\lambda' = a'^\vee \in M'^\vee$  and  $\lambda = f \circ \lambda' = a(s)^\vee \in M^\vee$ . The map  $f$  carries  $U_{a'} := U_{G'}(a'^\vee) = U_{G'}(\lambda')$  into  $U_G(\lambda) := U_G(a(s)^\vee) = U_G(q_a a(s)^\vee) = U_{a(s)}$  (see

Theorem 4.1.7(1) and Proposition 4.1.10(2)), so passing to  $\bar{\eta}$ -fibers gives that the elements  $a(\eta)$  and  $a(s)$  in  $\Phi$  have the same root groups for  $(G_{\bar{\eta}}, T_{\bar{\eta}})$ . Hence,  $a(\eta) = a(s)$ .  $\square$

By Lemma 6.1.3, it is reasonable to impose the additional requirement on  $f$  that there exists a (necessarily unique) bijection  $d : \Phi' \rightarrow \Phi$  and a prime power  $q_{a'} \geq 1$  for each  $a' \in \Phi'$  so that the map  $M \rightarrow M'$  induced by  $X(f)$  carries  $d(a')$  to  $q_{a'}a'$  and the map induced by its dual  $X_*(f)$  carries  $a'^{\vee}$  to  $q_{a'}d(a')^{\vee}$ , where the integer  $q_{a'}$  is an integral power of the characteristic exponent of  $k(s)$  for each  $s \in S$ . Indeed, the preceding discussion shows that Zariski-locally on  $S$ , every quasi-finite surjection  $(G', T') \rightarrow (G, T)$  satisfies these conditions.

Since  $X(f)_{\mathbf{Q}}$  is an isomorphism and  $d$  is injective, the conditions  $d(-a') \mapsto q_{-a'}(-a') = -q_{-a'}a'$  and  $d(a') \mapsto q_{a'}a'$  force  $d(-a')$  and  $d(a')$  to be distinct linearly dependent elements of the reduced root system  $\Phi$ , so  $d(-a') = -d(a')$ . Likewise,  $q_{-a'} = q_{a'}$ . Moreover:

**Lemma 6.1.4.** *The quasi-finite surjection  $f$  carries the  $S$ -subgroup scheme  $U'_{a'}$  of  $G'$  into the  $S$ -subgroup  $U_{d(a')}$  of  $G$  via a homomorphism that is  $\mathbf{G}_m$ -equivariant for respective scaling through  $a'^{\vee} = q_{a'}d(a')^{\vee}$  and  $d(a')^{\vee}$ . Moreover,  $(-1)^{q_{a'}} = -1$  on  $S$ , if  $q_{a'} = p^n$  with a prime  $p$  and  $n > 0$  then  $p = 0$  on  $S$ , and for all  $a' \in \Phi'$  there exists a unique isomorphism of line bundles*

$$f_{a'} : (\mathfrak{g}'_{a'})^{\otimes q_{a'}} \simeq \mathfrak{g}_{d(a')}$$

such that  $U'_{a'} \rightarrow U_{d(a')}$  is given by the isogeny  $\exp_{a'}(X') \mapsto \exp_{d(a')}(f_{a'}(X'^{\otimes q_{a'}}))$  for all  $X' \in \mathfrak{g}'_{a'}$ .

*Proof.* Since  $d(a') \circ f = q_{a'}a'$ ,  $f$  carries  $T'_{a'}$  into (hence onto)  $T_{d(a')}$ . Thus,  $f$  carries  $Z_{G'}(T'_{a'})$  into  $Z_G(T_{d(a')})$  via a quasi-finite surjection. But  $f \circ a'^{\vee} = q_{a'}d(a')^{\vee}$ , so we can pass to the induced map between semisimple derived groups by working with the rank-1 split tori  $a'^{\vee}(\mathbf{G}_m)$  and  $d(a')^{\vee}(\mathbf{G}_m)$ . This brings us to the case that  $G'$  and  $G$  are semisimple with fibers of rank 1. For  $\lambda' = a'^{\vee}$  and  $\lambda = f \circ \lambda' = q_{a'}d(a')^{\vee}$  we have  $U'_{a'} = U_{G'}(a'^{\vee}) = U_{G'}(\lambda')$  and  $U_{d(a')} = U_G(d(a')^{\vee}) = U_G(\lambda)$  (see Theorem 4.1.7(1)), so  $f$  carries  $U'_{a'}$  into  $U_{d(a')}$  by Proposition 4.1.10(2).

By the definition of a split reductive  $S$ -group, the root spaces admit global trivializations (as line bundles on  $S$ ). Choose such trivializations for  $(G, T)$  and  $(G', T')$ , so we get  $S$ -group isomorphisms  $U'_{a'} \simeq \mathbf{G}_a$  and  $U_{d(a')} \simeq \mathbf{G}_a$  that are respectively  $T'$ -equivariant and  $T$ -equivariant via the respective scaling actions on  $\mathbf{G}_a$  by  $a'$  and  $d(a')$ . In other words, the map induced by  $f$  between the root groups becomes a surjective endomorphism  $f : \mathbf{G}_a \rightarrow \mathbf{G}_a$  that satisfies

$$f(a'(t')x) = d(a')(f(t'))f(x) = a'(t')^{q_{a'}}f(x)$$

for points  $t'$  of  $T'$  and  $x$  of  $\mathbf{G}_a$  over  $S$ . Since  $a' : T' \rightarrow \mathbf{G}_m$  is an fppf covering, it follows that  $f(ux) = u^{q_{a'}}f(x)$  for all points  $u$  of  $\mathbf{G}_m$  and  $x$  of  $\mathbf{G}_a$ . By the relative schematic density of  $\mathbf{G}_m$  in  $\mathbf{G}_a$  over  $S$ , the same identity holds with  $u$  permitted to be any point of  $\mathbf{G}_a$ .

Letting  $q = q_{a'}$ , we claim that  $f(x) = cx^q$  for a unique unit  $c$  on the base (so the existence and uniqueness of  $f_{a'}$  will follow). The uniqueness is clear, so we may work Zariski-locally on  $S$  for existence. Hence, we can assume that  $f$  is given by a polynomial map  $x \mapsto$

$c_0 + c_1x + \cdots + c_nx^n$  for some integer  $n \geq 0$  and some global functions  $c_0, \dots, c_n$  on the base. Since  $f(0) = 0$  we have  $c_0 = 0$ , and the identity  $f(ux) = u^q f(x)$  implies that  $c_j u^j = u^q c_j$  for all  $j$ . If  $j \neq q$  then fppf-locally on  $S$  there exists a unit  $u$  such that  $u^{q-j} - 1$  is a unit, so  $c_j = 0$  on  $S$ . Hence,  $f(x) = cx^q$  for some  $c$  on  $S$ . The maps induced by  $f$  between root groups on geometric fibers over  $S$  are isogenies, so  $c$  is nowhere-vanishing, which is to say that  $c$  is a unit on  $S$ .

Finally, we have to show that  $(-1)^q = -1$  on  $S$  and also that  $p = 0$  on  $S$  if  $q = p^n$  for a prime  $p$  and  $n > 0$ . The homomorphism property for  $f$  and unit property for  $c$  imply that  $x \mapsto x^q$  is additive on  $S$  (so  $(-1)^q = -1$  on  $S$ ). Assume  $q = p^n$  for a prime  $p$  and  $n > 0$ , so  $(x + y)^q - x^q - y^q$  involves the monomial  $x^{p^{n-1}}y^{q-p^{n-1}}$  with a binomial coefficient whose  $p$ -adic ordinal is 1 and the monomial  $xy^{q-1}$  with a coefficient of  $q$ , so the greatest common divisor of all monomial coefficients (in  $\mathbf{Z}$ ) is  $p$ . Hence,  $p = 0$  on  $S$  in such cases.  $\square$

**Proposition 6.1.5.** *In the setting of Lemma 6.1.4, the  $\mathcal{O}_S$ -linear isomorphisms  $f_{a'}$  and  $f_{-a'}$  are dual relative to the canonical duality pairings between  $\mathfrak{g}'_{a'}$  and  $\mathfrak{g}'_{-a'}$  and between  $\mathfrak{g}_{d(a')}$  and  $\mathfrak{g}_{-d(a')} = \mathfrak{g}_{d(-a')}$ . Moreover, if  $X'$  is a trivializing section of  $\mathfrak{g}'_{a'}$  and  $X := f_{a'}(X'^{\otimes q(a')})$  is the associated trivialization of  $\mathfrak{g}_{d(a')}$  then  $f(w_{a'}(X')) = w_a(X)$ .*

See Corollary 5.1.9 for the definition of  $w_a(X)$  for any  $a \in \Phi$  and any trivializing section  $X$  of  $\mathfrak{g}_a$ .

*Proof.* By passing to derived groups of torus centralizers, we can reduce to the case of groups that are fiberwise semisimple of rank 1. Thus, now assume we are considering such groups, so we can let  $q = q(a') = q(-a')$  and  $a = d(a')$  (so  $-a = d(-a')$ ). Since  $f_{\pm a'}$  are  $\mathcal{O}_S$ -linear, consideration of the induced map between open cells and the unique characterization of the duality pairing between root spaces in (4.2.1) easily gives (after a straightforward calculation) that if local sections  $X'$  of  $\mathfrak{g}'_{a'}$  and  $Y'$  of  $\mathfrak{g}'_{-a'}$  satisfy  $1 + X'Y' \in \mathbf{G}_m$  then  $1 + f_{a'}(X'^{\otimes q})f_{-a'}(Y'^{\otimes q}) \in \mathbf{G}_m$  and  $1 + f_{a'}(X'^{\otimes q})f_{-a'}(Y'^{\otimes q}) = 1 + (X'Y')^q$ . By taking  $Y' := uX'^{-1}$  for a unit  $u$  such that  $1 + u$  is a unit (as may be done fppf locally on  $S$ ), it follows that the respective trivializing sections  $f_{a'}(X'^{\otimes q})$  and

$$f_{-a'}(Y'^{\otimes q}) = f_{-a'}(u^q(X'^{-1})^{\otimes q}) = u^q f_{-a'}((X'^{-1})^{\otimes q})$$

of  $\mathfrak{g}_a$  and  $\mathfrak{g}_{-a}$  have pairing equal to  $u^q$ . Thus, the asserted duality compatibility between  $f_{a'}$  and  $f_{-a'}$  holds.

By definition,  $w_{a'}(X') = \exp_{a'}(X') \exp_{-a'}(-Y') \exp_{a'}(X')$  where  $Y'$  is the trivialization of  $\mathfrak{g}'_{-a'}$  linked to  $X'$ . The preceding discussion shows that  $Y := f_{-a'}(Y'^{\otimes q})$  is the trivialization of  $\mathfrak{g}_{-d(a')}$  linked to  $X$ , so (using that  $(-1)^{q_{a'}} = -1$  on  $S$ )

$$f(w_{a'}(X')) = \exp_{d(a')}(X) \exp_{d(-a')}(-Y) \exp_{d(a')}(X) = w_{d(a')}(X)$$

since  $d(-a') = -d(a')$ .  $\square$

**Definition 6.1.6.** Let  $(G, T, M)$  and  $(G', T', M')$  be split reductive groups over a scheme  $S \neq \emptyset$ . A quasi-finite surjection  $f : (G', T') \rightarrow (G, T)$  is *compatible with the splittings* if there

exists a homomorphism  $h : M \rightarrow M'$ , a bijection  $d : \Phi' \rightarrow \Phi$ , and a function  $q : \Phi' \rightarrow \mathbf{Z}_{\geq 1}$  valued in prime powers such that:

- (1) the induced map  $X(f) : \underline{M} = X(T) \rightarrow X(T') = \underline{M}'$  arises from  $h$ ,
- (2) for all  $a' \in \Phi'$ ,  $h(d(a')) = q(a')a'$  and  $h^\vee(a'^\vee) = q(a')d(a')^\vee$ ,
- (3) if  $q(a') > 1$  is a power of a prime  $p(a')$  then  $S$  is a  $\mathbf{Z}/p(a')\mathbf{Z}$ -scheme.

In the setting of the preceding definition,  $h_{\mathbf{Q}}$  is an isomorphism (since each  $f_s$  is an isogeny) and  $f$  uniquely determines  $h$ ,  $d$ , and  $q$ . We will sometimes write  $f : (G', T', M') \rightarrow (G, T, M)$  to denote that  $f$  is compatible with the splittings. In [SGA3, XXII, 4.2.1] it is only required that  $f$  is quasi-finite on the derived subgroups (or rather, that property on derived groups is a consequence of other conditions imposed there), in which case  $h_{\mathbf{Q}}$  may be neither injective nor surjective when there are nontrivial central tori; we only consider quasi-finite surjective  $f$ . Our interest in such maps that are compatible with splittings is due to the following immediate consequence of the preceding considerations.

**Proposition 6.1.7.** *Let  $(G, T, M)$  and  $(G', T', M')$  be split reductive groups over a scheme  $S \neq \emptyset$ . Any quasi-finite surjection  $f : (G', T') \rightarrow (G, T)$  is compatible with the splittings Zariski-locally on  $S$ . If  $S$  is connected then  $f$  is compatible with the splittings over  $S$ .*

We are now led to:

**Definition 6.1.8.** Let  $R' = (X', \Phi', X'^\vee, \Phi'^\vee)$  and  $R = (X, \Phi, X^\vee, \Phi^\vee)$  be reduced root data, and  $p$  a prime or 1. A  $p$ -morphism  $R' \rightarrow R$  is a triple  $(h, d, q)$  consisting of a homomorphism  $h : X \rightarrow X'$ , a bijection  $d : \Phi' \rightarrow \Phi$ , and a function  $q : \Phi' \rightarrow \{p^n\}_{n \geq 0}$  such that

- (1) the induced map  $h_{\mathbf{Q}}$  is an isomorphism (i.e.,  $h$  is a finite-index injection),
- (2) for all  $a' \in \Phi'$ ,  $h(d(a')) = q(a')a'$  and  $h^\vee(a'^\vee) = q(a')d(a')^\vee$ ,

For  $p$ -morphisms  $(h, d, q) : R' \rightarrow R$  and  $(h', d', q') : R'' \rightarrow R'$ , the composition  $R'' \rightarrow R$  is  $(h' \circ h, d \circ d', (q \circ d') \cdot q')$ . If  $S$  is a non-empty scheme, a  $p$ -morphism  $(h, d, q)$  is a  $p(S)$ -morphism if  $S$  is a  $\mathbf{Z}/p(a')\mathbf{Z}$ -scheme whenever  $q(a') > 1$  is a power of a prime  $p(a')$ .

Since the root data are reduced and  $q$  takes values in  $\mathbf{Z}_{\geq 1}$  (and a root datum uniquely determines the bijection  $a \mapsto a^\vee$  between sets of roots and coroots), the condition  $h^\vee(a'^\vee) = q(a')d(a')^\vee$  implies that  $h$  determines  $d$  and  $q$ . The notion of *isomorphism* between root data is the evident one, and clearly a  $p$ -morphism that is an isomorphism must be a 1-morphism. Note also that for a  $p$ -morphism  $R' \rightarrow R$ , the map  $h : X \rightarrow X'$  goes in the “other” direction, whereas the map  $d : \Phi' \rightarrow \Phi$  goes in the “same” direction. This is motivated by the examples arising from split reductive group schemes:

**Example 6.1.9.** For a non-empty scheme  $S$  and quasi-finite surjection  $f : (G', T', M') \rightarrow (G, T, M)$  compatible with the splittings, the associated triple  $R(f) := (h, d, q)$  is a  $p(S)$ -morphism between the root data where either  $p = 1$  or  $\text{char}(k(s)) = p > 1$  for all  $s \in S$ , and this is compatible with composition and base change.

By Proposition 3.3.9, a quasi-finite surjection  $f : (G', T') \rightarrow (G, T)$  over  $S$  is a central isogeny if and only if  $\ker f_{\bar{s}} \subset T'_{\bar{s}}$  for all  $s \in S$  (since  $T'$  contains  $Z_{G'}$  and any normal subgroup

scheme of multiplicative type in a connected group scheme over a field is necessarily central). Thus, consideration of the open cell on geometric fibers implies that  $f$  is a central isogeny if and only if the  $p(S)$ -morphism  $R(f)$  of root data satisfies  $q(a') = 1$  for all  $a' \in \Phi'$  (see Exercise 6.5.2).

In view of the characterization of central isogenies between split reductive  $S$ -group schemes in Example 6.1.9, a 1-morphism  $(h, d, q) : R' \rightarrow R$  between reduced root data is also called a *central isogeny*. The condition that  $q(a') = 1$  for all  $a' \in \Phi'$  says precisely that  $h(d(a')) = a'$  and  $h^\vee(a'^\vee) = d(a')^\vee$  for all  $a' \in \Phi'$ , or equivalently  $h$  induces a bijection between the sets of roots (with inverse  $d$ ) and  $h^\vee$  induces a bijection between the sets of coroots (with inverse equal to the “dual” of  $d$ ). In non-central cases,  $h$  does not carry  $\Phi$  into  $\Phi'$ .

**Proposition 6.1.10.** *Let  $f : G' \rightarrow G$  be a homomorphism between reductive group schemes. If  $f_s$  is an isogeny for all  $s \in S$  then  $f$  is an isogeny.*

*Proof.* As we saw in the discussion preceding Proposition 3.3.9,  $f$  is necessarily a quasi-finite flat surjection and it is equivalent to show that  $\ker f$  is  $S$ -finite. By limit arguments, we may assume that  $S$  is noetherian. In view of the quasi-finiteness of  $f$ , finiteness is equivalent to properness. Thus, by the valuative criterion we can assume that  $S = \text{Spec } R$  for a discrete valuation ring  $R$ , with fraction field  $K$  and residue field  $k$ . We may and do assume that  $R$  is strictly henselian (so  $G'$  is  $S$ -split).

Since  $R$  is henselian, we can apply the structure theorem for quasi-finite morphisms [EGA, IV<sub>4</sub>, 18.5.11(a),(c)]: for any quasi-finite separated  $S$ -scheme  $X$ , there exists a unique open and closed subscheme  $X' \subset X$  that is  $S$ -finite and satisfies  $X'_k = X_k$ . In particular, the formation of  $X'$  is functorial in  $X$  and compatible with products over  $S$ , so if  $X$  is an  $S$ -group then  $X'$  is an  $S$ -subgroup. Consider the unique open and closed finite  $S$ -subgroup  $H' \subset \ker f$  with special fiber  $\ker f_k$  (so  $H'$  is also flat). Clearly  $\ker f$  is finite if and only if  $H' = \ker f$ .

We claim that  $H'$  is normal in  $G'$ ; i.e., the closed subgroup  $N_{G'}(H') \subset G'$  from Proposition 2.1.2 is equal to  $G'$ . Since  $G'$  is  $S$ -split, by consideration of an open cell we see that  $G'(S)$  is fiberwise dense in  $G'$ , so  $G'(S)$  is relatively schematically dense in  $G'$  over  $S$ . Thus, to prove  $N_{G'}(H') = G'$  it suffices to check equality on  $R$ -points, which is to say that  $G'(S)$  normalizes  $H'$ . By the uniqueness of  $H'$ , this follows from the normality of  $\ker f$  in  $G'$ .

The group  $G'$  can be replaced with the reductive quotient  $G'/H'$ , so now  $f_k$  is an isomorphism. Our problem is to show that  $f$  is an isomorphism, and it is equivalent to check this on the generic fiber. Let  $T'$  be a maximal torus in  $G'$ , so it is split (as  $R$  is strictly henselian); fix an isomorphism  $T' \simeq D_S(M)$  for a finite free  $\mathbf{Z}$ -module  $M$ . The kernel of  $f|_{T'} : T' \rightarrow G$  is a quasi-finite closed subgroup of  $T'$  with trivial special fiber, and by Exercise 2.4.2 any quasi-finite closed subgroup of the torus  $T'$  is finite. Thus, the  $S$ -group  $\ker(f|_{T'})$  is  $S$ -finite with trivial special fiber, so it is trivial. It follows that  $f|_{T'}$  is a monomorphism, so it is a closed immersion since  $T'$  is of multiplicative type (Lemma B.1.3). Hence, we may and do also view  $T'$  as a maximal torus of  $G$  and  $f$  as a map  $(G', T', M) \rightarrow (G, T', M)$  between reductive  $S$ -groups that is compatible with the splittings (using the identity map on  $M$ ).

As in Example 6.1.9 (and the discussion preceding it), we get an induced  $p(S)$ -morphism  $R(f)$  between the root data. But this map of root data can be computed using any fiber, so working with the special fibers implies that  $R(f)$  is an isomorphism. In particular,  $R(f)$  is central. Passing to the (split) generic fibers, it follows that the isogeny  $f_K$  is a central (Example 6.1.9), so the degree of  $f_K$  can be read off from the restriction  $T'_K \rightarrow T_K$ . But  $R(f_K) = R(f)$  is an isomorphism, so  $f_K$  is an isomorphism.  $\square$

We now deduce some elementary properties of  $p$ -morphisms  $(h, d, q)$  between reduced root data that were established immediately above Lemma 6.1.4 for triples  $(h, d, q)$  that arise from quasi-finite surjections between split reductive group schemes. Namely, we claim that always  $d(-a') = -d(a')$  and  $q(-a') = q(a')$ . To prove this, first note that by the isomorphism property for  $h_{\mathbf{Q}}$ ,  $d(a')$  and  $d(-a')$  are linearly dependent in  $M$ . By injectivity of  $d$ , we have  $d(a') \neq d(-a')$ . Hence,  $d(-a') = -d(a')$  for all  $a' \in \Phi'$  (due to reducedness of the root data). The identity  $h(d(-a')) = q(-a') \cdot (-a')$  then implies that  $q(-a') = q(a')$  for all  $a' \in \Phi'$ .

**Remark 6.1.11.** In [SGA3, XXI, 6.1.1, 6.8.1], the notions of *morphism* and *p-morphism* between reduced root data are defined (with an integer  $p \geq 1$ ). In the definition of a *morphism* there,  $q$  is identically 1 and  $h$  is only required to be  $\mathbf{Z}$ -linear (so  $\ker h$  may be nontrivial and  $h_{\mathbf{Q}}$  may not be surjective); this is intended to encode homomorphisms between split reductive group schemes with an isogeny condition between the derived groups but no such condition between the maximal central tori. The  $p$ -morphisms in [SGA3, XXI, 6.8.1] are a variant on our notion of  $p$ -morphism in which  $p$  is any integer  $\geq 1$  and  $h$  is only required to be  $\mathbf{Z}$ -linear.

**Lemma 6.1.12.** *Let  $(h, d, q) : R' \rightarrow R$  be a  $p$ -morphism between reduced root data, and let  $\Phi'^+$  be a positive system of roots in  $\Phi'$ , with  $\Delta'$  its base of simple roots. Then  $\Phi^+ := d(\Phi'^+)$  is a positive system of roots in  $\Phi$  and  $\Delta := d(\Delta')$  is its base of simple roots.*

*Proof.* Pick a linear form  $\lambda'$  on  $X'_{\mathbf{Q}}$  such that  $\Phi'^+ = \Phi'_{\lambda' > 0}$ , and let  $\lambda = \lambda' \circ h_{\mathbf{Q}}$ . The relations  $h(d(a')) = q(a')a'$  for  $a' \in \Phi$  with  $q(a') \in \mathbf{Q}_{>0}$  and the isomorphism property for  $h_{\mathbf{Q}}$  imply that  $\lambda$  is non-vanishing on  $d(\Phi')$  and that  $\Phi_{\lambda > 0} = d(\Phi'^+) =: \Phi^+$ , so indeed  $\Phi^+$  is a positive system of roots. It is likewise clear from the isomorphism property of  $h_{\mathbf{Q}}$  that  $d(\Delta')$  is a linearly independent set whose span  $\mathbf{Q}_{\geq 0} \cdot d(\Delta')$  over  $\mathbf{Q}_{\geq 0}$  satisfies

$$\Phi \subset \mathbf{Q}_{\geq 0} \cdot d(\Delta') \cup -\mathbf{Q}_{\geq 0} \cdot d(\Delta').$$

It then follows from elementary inductive arguments with reduced root systems (see [SGA3, XXI, 3.1.5], which avoids a reducedness hypothesis) that this forces  $d(\Delta')$  to be the base of a positive system of roots in  $\Phi$ . But clearly  $d(\Delta') \subseteq \Phi^+$ , so  $d(\Delta')$  must be the set of simple roots of  $\Phi^+$ .  $\square$

**Proposition 6.1.13.** *Let  $(G', T', M')$  and  $(G, T, M)$  be split reductive groups over a non-empty scheme  $S$ ,  $f : (G', T', M') \rightarrow (G, T, M)$  an isogeny compatible with the splittings, and  $(h, d, q) := R(f)$  the associated  $p(S)$ -morphism between the root data. Let  $\Phi'^+$  be a positive*

system of roots in  $\Phi'$ ,  $\Delta'$  its base of simple roots,  $\Phi^+ := d(\Phi'^+)$  the associated positive system of roots in  $\Phi$ , and  $\Delta = d(\Delta')$  its base of simple roots.

- (1) If  $B' \subset G'$  is the Borel subgroup containing  $T'$  that corresponds to  $\Phi'^+$  and  $B \subset G$  is the Borel subgroup containing  $T$  that corresponds to  $\Phi^+$  then  $f$  carries  $B'$  into  $B$ .
- (2) For any pinning  $\{X'_{a'}\}_{a' \in \Delta'}$  of  $(G', T', M', \Delta')$  and the pinning of  $(G, T, M, \Delta)$  given by the sections  $X_{d(a')} = f_{a'}(X'_{a'}{}^{\otimes q(a')})$ ,  $f$  is uniquely determined by  $R(f)$  and the pinnings  $\{X'_{a'}\}_{a' \in \Delta'}$  and  $\{X_a\}_{a \in \Delta}$ . In particular, if  $G$  is semisimple then an automorphism of  $(G, T, M, \{X_a\}_{a \in \Delta})$  whose effect on  $\mathfrak{g}$  is the identity on each  $X_a$  ( $a \in \Delta$ ) must be the identity.

*Proof.* Since  $f$  carries  $T'$  into  $T$  and  $U'_{a'}$  into  $U_{d(a')}$ , part (1) is immediate from the equalities

$$B' = T' \times \prod_{a' \in \Phi'^+} U'_{a'}, \quad B = T \times \prod_{a \in \Phi^+} U_a$$

respectively defined in  $G'$  and  $G$  via multiplication (using any enumeration of the positive root groups). To prove part (2), first note that for any  $a' \in \Delta'$  the restriction  $f : U'_{a'} \rightarrow U_{d(a')}$  is uniquely determined because

$$f(\exp_{a'}(cX_{d(a')})) = \exp_{d(a')}(f_{a'}((cX'_{a'})^{\otimes q(a')})) = \exp_{d(a')}(c^{q(a')}X_{d(a')})$$

for any  $c \in \mathbf{G}_a$ . By Proposition 6.1.5 we have  $f(w_{a'}(X'_{a'})) = w_a(X_a)$ , so  $f$  is also uniquely determined on the global sections  $w_{a'}(X'_{a'})$  that represent simple reflections generating the Weyl group  $W(\Phi')$ .

Since the Weyl group carries any element of  $\Phi'$  into  $\Delta'$ , it follows that  $f$  is uniquely determined on  $U'_{a'}$  for every  $a' \in \Phi'$ . The constituent  $h$  in  $R(f)$  determines  $f : T' \rightarrow T$ , so we conclude that  $f$  is uniquely determined on the open cell  $\Omega'$  of  $(G', T', M', \Phi'^+)$ . The relative schematic density of  $\Omega'$  in  $G'$  then implies that  $f$  is uniquely determined.  $\square$

**Definition 6.1.14.** Let  $(G', T', M', \{X'_{a'}\}_{a' \in \Delta'})$  and  $(G, T, M, \{X_a\}_{a \in \Delta})$  be pinned split reductive groups over a non-empty scheme  $S$ . An isogeny  $f : (G', T', M') \rightarrow (G, T, M)$  compatible with the splittings (in the sense of Definition 6.1.6) is *compatible with the pinnings* if the bijection  $d : \Phi' \rightarrow \Phi$  arising from  $R(f)$  carries  $\Delta'$  into  $\Delta$  and  $f(\exp_{a'}(X'_{a'})) = \exp_{d(a')}(X_{d(a')}^{\otimes q(a')})$  for all  $a' \in \Delta'$ .

For any isogeny  $f : (G', T', M') \rightarrow (G, T, M)$  compatible with the splittings and any pinning  $\{X'_{a'}\}_{a' \in \Delta'}$  of  $(G', T', M')$ , we have shown in Proposition 6.1.13(1) that there exists a unique pinning  $\{X_a\}_{a \in \Delta}$  of  $(G, T, M)$  such that  $f$  is compatible with these pinnings. The crucial fact is that in such situations, the pinning-compatible  $f$  is *uniquely determined* by  $R(f)$ , due to Proposition 6.1.13(2). In particular:

**Corollary 6.1.15.** *Let  $S$  be a non-empty scheme, and consider the category of pinned split reductive groups  $(G, T, M, \{X_a\}_{a \in \Delta})$  over  $S$ , using as morphisms the isogenies that are compatible with the splittings and pinnings. The functor*

$$(6.1.1) \quad (G, T, M, \{X_a\}_{a \in \Delta}) \rightsquigarrow R(G, T, M) = (M, \Phi, M^\vee, \Phi^\vee)$$

into the category of root data equipped with  $p(S)$ -morphisms is faithful.

Moreover, if  $(G', T')$  and  $(G, T)$  are reductive  $S$ -groups equipped with (possibly non-split) maximal tori and  $f, F : (G', T') \rightrightarrows (G, T)$  are isogenies then  $f$  and  $F$  induce the same isogeny  $T' \rightarrow T$  if and only if  $f = c_{\bar{t}} \circ F$  for some  $\bar{t} \in (T/Z_G)(S)$ , where  $c_{\bar{t}}$  denotes the natural action of  $\bar{t} \in (G/Z_G)(S)$  on  $G$  induced by conjugation. In such cases,  $\bar{t}$  is unique.

This result is essentially the content of [SGA3, XXIII, 1.9.1, 1.9.2] (except that we record the role of the  $(T/Z_G)(S)$ -action when we do not require splittings or pinnings).

*Proof.* The faithfulness is immediate from Proposition 6.1.13(2), so it remains to address the assertion concerning the equality of  $f, F : T' \rightrightarrows T$  in the absence of splittings and pinnings. It is clear that  $c_{\bar{t}} \circ F$  and  $F$  induce the same isogeny from  $T'$  to  $T$  for any  $\bar{t}$  (since the  $T/Z_G$ -action on  $G$  is the identity on  $T$ ), and also that  $c_{\bar{t}} \circ F$  uniquely determines  $\bar{t}$  (since  $F$  is faithfully flat). To prove the existence of  $\bar{t}$  when  $f|_{T'} = F|_{T'}$  in  $\text{Hom}_{S\text{-gp}}(T', T)$ , we may work étale-locally on  $S$  due to the uniqueness. Hence, we can assume that  $T' = D_S(M')$  and  $T = D_S(M)$  making  $(G', T', M')$  and  $(G, T, M)$  split as well as making the common isogeny  $f, F : T' \rightrightarrows T$  arise from a homomorphism  $h : M \rightarrow M'$  (so  $f$  and  $F$  are compatible with the splittings).

Choose a pinning  $\{X_{a'}\}_{a' \in \Delta'}$  of  $(G', T', M')$ . The induced  $p(S)$ -morphisms  $R(f), R(F) : R(G', T', M') \rightrightarrows R(G, T, M)$  have the same  $h$ , so they coincide; let  $(h, d, q)$  denote this common  $p(S)$ -morphism. We get two pinnings of  $(G, T, M)$  relative to  $\Delta = d(\Delta')$ , namely

$$X_{d(a')} = f_{a'}(X_{a'}^{\otimes q(a')}), \quad Y_{d(a')} = F_{a'}(X_{a'}^{\otimes q(a')})$$

for  $a' \in \Delta'$ .

The action of  $G$  on  $G$  via conjugation factors through an action of  $G/Z_G$  on  $G$ . Upon restricting this to an action of  $T/Z_G$  on a root group  $U_a$  for  $a \in \Phi$ , we recover the fact that the character  $a : T \rightarrow \mathbf{G}_m$  factors through  $T/Z_G$ . Let  $\bar{a} : T/Z_G \rightarrow \mathbf{G}_m$  denote the character obtained from  $a$ , so these are the elements of the root system  $\bar{\Phi}$  in the root datum for the split group  $(G/Z_G, T/Z_G, \bar{M})$ , where  $\bar{M} = \sum_{a \in \Phi} \mathbf{Z}a \subset M$  is the possibly non-saturated subgroup corresponding to the quotient  $T/Z_G$  of the split torus  $T$  modulo the split multiplicative type subgroup  $Z_G$  (see Corollary 3.3.6(1)).

For each  $a \in \Delta$  there exists a unique unit  $u_a \in \mathbf{G}_m(S)$  such that  $Y_a = u_a X_a$ , so the necessary and sufficient condition for  $c_{\bar{t}} \circ F$  and  $f$  to agree as isogenies compatible with splittings and pinnings (and hence to be equal) is that  $\bar{a}(\bar{t}) = u_a$  for all  $a \in \Delta$ . Hence, it is necessary and sufficient to show that the subset  $\bar{\Delta} \subset \bar{\Phi}$  corresponding to  $\Delta$  is a  $\mathbf{Z}$ -basis of  $\bar{M}$ . But this basis property is obvious because

$$\bar{M} = \sum_{a \in \Phi} \mathbf{Z}a = \sum_{a \in \Delta} \mathbf{Z}a = \bigoplus_{a \in \Delta} \mathbf{Z}a$$

(due to the fact that  $\Delta$  is a base for a positive system of roots for  $\Phi$ ). □

The following theorem records the main results to be proved in the remainder of §6.

**Theorem 6.1.16.** *Let  $S$  be a non-empty scheme.*

- (1) (Isogeny Theorem) *For split reductive  $S$ -groups  $(G', T', M')$  and  $(G, T, M)$ , any  $p(S)$ -morphism  $R(G', T', M') \rightarrow R(G, T, M)$  is induced by an isogeny  $f : (G', T', M') \rightarrow (G, T, M)$  compatible with the splittings, and  $f$  is unique up to the faithful action of  $(T/Z_G)(S)$  on  $G$ .*
- (2) (Existence Theorem) *Every root datum is isomorphic to the root datum of a split reductive  $S$ -group.*

The Isogeny Theorem is a slight weakening of [SGA3, XXV, 1.1] (which allows  $S$ -group maps that are not isogenies between maximal central tori, and similarly on root data). The Existence Theorem is [SGA3, XXV, 1.2]. An immediate consequence of the Isogeny Theorem and the arguments with the  $\mathbf{Z}$ -basis  $\Delta$  of  $X(T/Z_G)$  in the proof of Corollary 6.1.15 is:

**Theorem 6.1.17** (Isomorphism Theorem). *Let  $S$  be a non-empty scheme. The functor*

$$(G, T, M, \{X_a\}_{a \in \Delta}) \rightsquigarrow (R(G, T, M), \Delta)$$

*from the category of pinned split reductive  $S$ -groups (equipped with isomorphisms) to the category of based root data (equipped with isomorphisms) is an equivalence.*

*In particular, every split reductive  $S$ -group  $(G, T, M)$  is uniquely determined up to isomorphism by its root datum, and every isomorphism of root data  $R(G', T', M') \simeq R(G, T, M)$  arises from an isomorphism  $f : (G', T', M') \simeq (G, T, M)$  compatible with the splittings, with  $f$  uniquely determined up to the faithful action of  $(T/Z_G)(S)$  on  $G$ .*

In the classical theory, the Existence, Isomorphism, and Isogeny Theorems are proved over a general algebraically closed field (see [Spr, 9.6.2, 9.6.5, 10.1.1]). The traditional proof of the Existence Theorem in the classical theory builds a group from its open cell via delicate procedures guided by the Bruhat decomposition and the structure of the Dynkin diagram. The approach over a general base scheme is different, because “points” do not have the same geometric meaning in the relative theory as in the classical case. (For example, if  $R$  is a nonzero ring then  $\mathbf{A}_R^1$  is stratified by  $Z = \{0\}$  and  $U = \mathbf{G}_m$  but  $Z(R) \cup U(R) = \{0\} \cup R^\times \neq R = \mathbf{A}_R^1(R)$  whenever  $R$  is not a field.) In place of arguments inspired by the Bruhat decomposition, Weil’s theory of birational group laws will be used to prove the Existence Theorem over  $\mathbf{Z}$  (from which the Existence Theorem is deduced in general via base change).

To prove the Existence Theorem over  $\mathbf{Z}$ , we need to know the Existence Theorem over *some* algebraically closed field of characteristic 0, such as  $\mathbf{C}$ . Based on such input, we will use the Isomorphism Theorem and descent to prove the Existence Theorem over  $\mathbf{Q}$  via pinnings (to rigidify structures). Thus, the Isomorphism Theorem (and hence the Isogeny Theorem) will be proved before the Existence Theorem. Split reductive groups over  $\mathbf{Q}$  will be “spread out” over  $\mathbf{Z}$  via arguments with open cells, structure constants, and birational group laws.

We emphasize that the proofs of the Isogeny and Isomorphism Theorems over a general (non-empty) scheme  $S$  will not use the classical case as input, and the proof of the Existence Theorem over  $S$  will not require the classical case of the Existence Theorem in positive characteristic. Apart from a few simplifications via the dynamic method, our treatment of the proofs is just an exposition of the proof presented in [SGA3, XXIII, XXV].

**6.2. The Isogeny and Isomorphism Theorems.** In this section we prove the Isogeny Theorem (i.e., Theorem 6.1.16(1)) and record some consequences of the Isomorphism Theorem (i.e., Theorem 6.1.17). Fix a non-empty scheme  $S$  and split triples  $(G', T', M')$  and  $(G, T, M)$  over  $S$ . Since the root spaces are trivial as line bundles, these admit pinnings. For any  $p(S)$ -morphism  $\phi : R' \rightarrow R$  between the corresponding root data, we seek an isogeny  $f : (G', T', M') \rightarrow (G, T, M)$  compatible with the splittings such that  $R(f) = \phi$ . (The uniqueness of  $f$  up to the action of  $(T/Z_G)(S)$  on  $G$  is provided by Corollary 6.1.15.)

To construct  $f$ , we require criteria for the existence of an  $S$ -homomorphism  $f$  from a split reductive  $S$ -group  $(G, T, M)$  to an  $S$ -group  $H$  when the restrictions of  $f$  to  $T$  and its associated root groups  $U_a$  are all specified. More precisely, suppose that  $(G, T, M, \{X_a\}_{a \in \Delta})$  is a pinned split reductive  $S$ -group, and for  $a \in \Delta$  let

$$n_a = w_a(X_a) = \exp_a(X_a) \exp_{-a}(-X_a^{-1}) \exp_a(X_a),$$

where  $X_a^{-1}$  is the trivialization of  $\mathfrak{g}_{-a}$  linked to  $X_a$ . (In [SGA3], the element  $n_a \in N_G(T)(S)$  is denoted as  $w_a$ .) Any  $S$ -homomorphism  $f : G \rightarrow H$  to an  $S$ -group scheme  $H$  is uniquely determined by its restriction to the open cell  $\Omega$  arising from  $T$  and the positive system of roots  $\Phi^+$  with base  $\Delta$ , so  $f$  is uniquely determined by its restrictions

$$f_T : T \rightarrow H, \quad f_a : U_a \rightarrow H$$

for  $a \in \Phi$ . Since  $\Phi$  is covered by the  $W(\Phi)$ -orbits of  $\Delta$ , and the elements  $n_a$  represent the simple positive reflections that generate  $W(\Phi)$ , instead of keeping track of the maps  $f_a$  for all  $a \in \Phi^+$  it is enough to record the maps  $f_a$  for  $a \in \Delta$  provided that we also record the images  $h_a = f(n_a) \in H(S)$  for  $a \in \Delta$ .

Note that  $N_G(T)$  is the disjoint union of translates  $nT$  for a set of elements  $n \in N_G(T)(S)$  representing  $W(\Phi)$ , such as products of the elements  $n_a \in N_G(T)(S)$  for  $a \in \Delta$  (upon writing each  $w \in W(\Phi)$  as a word in the simple positive reflections). Thus, a preliminary step towards an existence criterion for a homomorphism  $f : G \rightarrow H$  gluing maps on  $T$  and the  $U_a$  ( $a \in \Delta$ ) as well as values  $h_a = f(n_a) \in H(S)$  is to settle the case when there is given a homomorphism  $f_N : N_G(T) \rightarrow H$  (rather than just  $f_T$ ) and homomorphisms  $U_a \rightarrow H$  for all roots  $a \in \Phi$  (not just for  $a \in \Delta$ ). Such a preliminary gluing criterion is provided by the following result [SGA3, XXIII, 2.1]:

**Theorem 6.2.1.** *For  $S$ -homomorphisms  $f_N : N_G(T) \rightarrow H$  and  $f_a : U_a \rightarrow H$  for  $a \in \Phi$ , there exists an  $S$ -homomorphism  $f : G \rightarrow H$  extending  $f_N$  and the maps  $f_a$  if and only the following three conditions hold:*

- (1) *For all  $a \in \Delta$  and  $b \in \Phi$ ,*

$$f_N(n_a) f_b(u_b) f_N(n_a)^{-1} = f_{s_a(b)}(n_a u_b n_a^{-1})$$

*for all  $u_b \in U_b$ .*

- (2) *The triple  $(f_a, f_{-a}, f_N|_{N_{Z_G(T_a)}(T)})$  extends to an  $S$ -homomorphism  $Z_G(T_a) \rightarrow H$  for all  $a \in \Delta$ .*

- (3) For all distinct  $a, b \in \Delta$  and the subgroup  $U_{[a,b]}$  directly spanned in any order by the groups  $U_c$  for

$$c \in [a, b] := \{ia + jb \in \Phi \mid i, j \geq 0\},$$

there exists an  $S$ -homomorphism  $U_{[a,b]} \rightarrow H$  restricting to  $f_c$  on  $U_c$  for all  $c \in [a, b]$ .

**Remark 6.2.2.** The existence of  $U_{[a,b]}$  is a special case of Proposition 5.1.16.

The idea of the proof of Theorem 6.2.1 is that since  $f_N|_T$  and the  $f_a$  determine what  $f$  must be on the open cell  $\Omega = U_{-\Phi^+} \times T \times U_{\Phi^+}$ , and the translates  $n\Omega$  by products  $n$  among the  $\{n_a\}_{a \in \Delta}$  cover  $G$  (Corollary 1.4.14), we just have to keep track of the homomorphism property when extending  $f$  across translates of  $\Omega$ .

By using induction on word length in the Weyl group (relative to the simple positive reflections  $s_a$  for  $a \in \Delta$ ), the base of the induction amounts to checking that for *some* enumeration of  $\Phi^+$ , the  $S$ -morphism  $f_U : U = U_{\Phi^+} := \prod_{a \in \Phi^+} U_a \rightarrow H$  defined by  $(u_a) \mapsto \prod_{a \in \Phi^+} f_a(u_a)$  is a homomorphism. Of course, once this is proved for some choice of enumeration, it follows that the enumeration of  $\Phi^+$  does not matter, as  $U$  is directly spanned in any order by the positive root groups. The case of  $U_{-\Phi^+}$  is also needed, but this will follow formally from the case of  $U_{\Phi^+}$  by using a representative for the long Weyl element to swap  $\Phi^+$  and  $-\Phi^+$ .

For the convenience of the reader, we now sketch the proof of the base case for the induction (i.e., the homomorphism property for  $f_U$ ). Fix a structure of ordered vector space on  $M_{\mathbf{Q}}$  so that  $\Phi^+$  is the associated positive system of roots (see the discussion following Definition 1.4.1, and note that  $W(\Phi)$  acts transitively on the set of positive systems of roots in  $\Phi$ ). Consider the resulting enumeration  $c_0 < \cdots < c_m$  of  $\Phi^+$ .

**Lemma 6.2.3.** *The map  $f_U$  is an  $S$ -homomorphism when it is defined using  $\{c_j\}$ .*

This is [SGA3, XXIII, 2.1.4].

*Proof.* For  $i \geq 1$ , consider the direct product scheme  $U_{\geq i} := \prod_{c \geq c_i} U_c$  and the map  $U_{\geq i} \rightarrow U$  of  $S$ -schemes defined by multiplication in strictly increasing order of the roots. By Proposition 5.1.16, this identifies  $U_{\geq i}$  with a closed  $S$ -subgroup of  $U$  that is moreover normalized by  $U_{c_{i-1}}$  when  $i > 0$ . The homomorphism property for  $f_U$  on  $U = U_{\geq 0}$  will be proved by descending induction: for all  $i$  we claim that  $f_U$  restricts to a homomorphism on  $U_{\geq i}$ . The case  $i = m$  is obvious (as the restriction to  $U_{\geq m} = U_{c_m}$  is  $f_{c_m}$ ).

In general, if the result holds for some  $i > 0$  then by writing  $U_{\geq i-1} = U_{c_{i-1}} \times U_{\geq i}$  and using the homomorphism property for  $f_U$  on  $U_{\geq i}$  it is straightforward to reduce to verifying the identities

$$(6.2.1) \quad f_b(u_b)^{-1} f_a(u_a) f_b(u_b) \stackrel{?}{=} f_U(u_b^{-1} u_a u_b)$$

for  $b = c_{i-1}$ ,  $a > c_{i-1} = b$ , and all points  $u_a$  of  $U_a$  and  $u_b$  of  $U_b$ . Note that these identities make sense because  $u_b^{-1} u_a u_b \in U_{\geq i}$  since  $a \geq c_i$  and  $b = c_{i-1}$ . To summarize, we have reduced ourselves to proving that for any  $a, b \in \Phi^+$  with  $a > b$ , the identities (6.2.1) are satisfied.

For any  $u_a \in U_a$  and  $u_b \in U_b$ , clearly  $u_b^{-1} u_a u_b \in U_{[a,b]}$ . Thus, if  $a, b \in \Delta$  then the desired identities are a consequence of hypothesis (3) of Theorem 6.2.1. To reduce the case of a

general pair to these special cases, we use a result in the theory of root systems: there exists  $w \in W(\Phi)$  such that  $w(a) \in \Delta$  and  $w(b) \in [w(a), b_0]$  for some  $b_0 \in \Delta$  necessarily distinct from  $w(a)$ . This result is proved in [SGA3, XXI, 3.5.4] (and its main content is [Bou2, VI, §1.7, Cor. 2]), applied to the reverse lexicographical ordering on  $\Phi^+$  relative to an enumeration of the base  $\Delta$ ). It enables us to reduce to the settled special case  $a, b \in \Delta$  provided that if  $n \in N_G(T)(S)$  represents  $w \in W(\Phi)$  then

$$f_N(n)f_a(u_a)f_N(n)^{-1} = f_{w(a)}(nu_an^{-1})$$

for all  $a \in \Delta$  and points  $u_a \in U_a$ . The case  $n = n_b$  for  $b \in \Delta$  is exactly hypothesis (1) in Theorem 6.2.1. The general case is deduced from this via induction on word length in  $W(\Phi)$  relative to the simple positive reflections  $\{s_c\}_{c \in \Delta}$  and applications of hypotheses (1) and (2) in Theorem 6.2.1. See [SGA3, XXIII, 2.1.3] for further details.  $\square$

Now we are ready to formulate the main criterion for constructing homomorphisms, building on Theorem 6.2.1. We will eventually obtain a criterion that reduces all difficulties to the case of groups with semisimple-rank at most 2. That is, the serious computational effort will only be required with low-rank groups. Keep in mind that we have explicitly described the split cases with semisimple-rank 1 in Theorem 5.1.8, at least Zariski-locally on the base. (The intervention of Zariski-localization can be removed from Theorem 5.1.8 by using a pinning and Proposition 6.1.13(2), but we do not need that minor improvement here.)

Let  $(G, T, M, \{X_a\}_{a \in \Delta})$  be a pinned split reductive  $S$ -group, and let  $H$  be an  $S$ -group. For each  $a \in \Delta$ , let  $n_a = w_a(X_a)$ . For  $S$ -homomorphisms  $f_T : T \rightarrow H$  and  $f_a : U_a \rightarrow H$  and elements  $h_a \in H(S)$  for all  $a \in \Delta$ , we have seen that there exists at most one  $S$ -homomorphism  $f : G \rightarrow H$  such that  $f|_T = f_T$  and  $f|_{U_a} = f_a$  and  $f(n_a) = h_a$  for all  $a \in \Delta$ . But when does  $f$  exist? There are some necessary conditions. For example, for all  $a \in \Delta$  we must have

$$(6.2.2) \quad f_T(t)f_a(u_a)f_T(t)^{-1} = f_a(tu_at^{-1}), \quad h_af_T(t)h_a^{-1} = f_T(s_a(t))$$

for all points  $t \in T$  and  $u \in U_a$  (valued in a common  $S$ -scheme).

There are additional conditions imposed by relations in the Weyl group. More specifically, note that if  $a, b \in \Delta$  then  $(s_a s_b)^{m_{ab}} = 1$  in  $W(\Phi)$  where  $m_{ab} = m_{ba}$  is the  $ab$ -entry in the symmetric Cartan matrix (it is the order of  $s_a s_b$  in  $W(\Phi)$ , so  $m_{aa} = 1$ ). In particular,  $t_{ab} := (n_a n_b)^{m_{ab}} \in T(S)$ , so  $t_{aa} = n_a^2 = a^\vee(-1)$ . This yields the further necessary conditions

$$(6.2.3) \quad h_a^2 = f_T(a^\vee(-1)), \quad (h_a h_b)^{m_{ab}} = f_T(t_{ab})$$

for  $a, b \in \Delta$  with  $b \neq a$ . The relation  $(n_a \exp_a(X_a))^3 = 1$  (Remark 5.1.10) yields the necessary condition

$$(6.2.4) \quad (h_a f_a(\exp_a(X_a)))^3 = 1$$

for all  $a \in \Delta$ . Finally, if  $a, b \in \Delta$  are distinct then in order that a homomorphism  $f_{ab} : U_{[a,b]} \rightarrow H$  extending  $f_a$  on  $U_a$  and  $f_b$  on  $U_b$  actually arises from an  $f : G \rightarrow H$  of the desired

type, the following conjugation relations must hold on the root groups  $U_c$  for  $c \in [a, b]$ . If  $c \neq a$  then

$$(6.2.5) \quad h_a f_{ab}(u_c) h_a^{-1} = f_{ab}(n_a u_c n_a^{-1}).$$

(The right side makes sense because (i)  $n_a$  conjugates  $U_c$  to  $U_{s_a(c)}$ , and (ii)  $s_a(c) = c - \langle c, a^\vee \rangle a$  lies in  $[a, b]$ , due to  $s_a(c)$  having a positive  $b$ -coefficient with  $a$  and  $b$  distinct elements of the base  $\Delta$  of  $\Phi^+$ .) Likewise, if  $c \neq b$  then

$$(6.2.6) \quad h_b f_{ab}(u_c) h_b^{-1} = f_{ab}(n_b u_c n_b^{-1}).$$

The remarkable fact is that the preceding necessary conditions for the existence of  $f$ , each of which only involves subgroups  $Z_G(T_a)$  of semisimple-rank 1 and subgroups  $Z_G(T_{ab})$  of semisimple-rank 2 (with  $T_{ab}$  denoting the unique torus of relative codimension-2 in  $T$  contained in  $\ker a \cap \ker b$  for distinct  $a, b \in \Phi^+$ ), are also sufficient. This is [SGA3, XXIII, 2.3]:

**Theorem 6.2.4.** *If the conditions (6.2.2), (6.2.3), (6.2.4), (6.2.5), and (6.2.6) all hold then  $f : G \rightarrow H$  exists satisfying  $f|_T = f_T$ ,  $f|_{U_a} = f_a$  for all  $a \in \Delta$ , and  $f(n_a) = h_a$  for all  $a \in \Delta$ .*

*Proof.* The proof is largely a systematic (and intricate) argument with word length in Weyl groups, bootstrapping from the given conditions to eventually establish the hypotheses in Theorem 6.2.1. This entails constructing  $f_N$  and the homomorphisms  $f_a$  for all roots  $a \in \Phi$  (recovering the given homomorphisms for  $a \in \Delta$ ).

An elementary argument (see [SGA3, XXIII, 2.3.1]) constructs the  $S$ -homomorphism  $f_N : N_G(T) \rightarrow H$  that extends  $f_T$  and carries  $n_a$  to  $h_a$  for all  $a \in \Delta$ ; such an  $f_N$  is unique since products among the  $n_a$ 's represent all elements of  $W(\Phi)$  (and  $N_G(T)$  is covered by the left-translates of its open and closed subgroup  $T$  by any set of representatives of  $W(\Phi)$  in  $N_G(T)(S)$ ). The construction of well-defined maps  $f_a$  for all  $a \in \Phi$  is harder, and rests on a lemma of Tits in the theory of root systems (stated as Exercise 21 in [Bou2, Ch. VI], and proved in [SGA3, XXI, 5.6]).

We refer to [SGA3, XXIII, 2.3.2–2.3.6] for the details, and sketch the main group-theoretic argument that establishes the requirement in Theorem 6.2.1(2). This requirement says that for each  $a \in \Delta$  there exists an  $S$ -homomorphism  $F_a : Z_G(T_a) \rightarrow H$  such that  $F_a|_T = f_T$ ,  $F_a|_{U_a} = f_a$ , and  $F_a(n_a) = h_a$ . (Such an  $F_a$  is visibly unique if it exists, since  $n_a$ -conjugation swaps the opposite root groups that appear in the open cell of  $Z_G(T_a)$  relative to its split maximal torus  $T$  and roots  $\pm a$ , and it satisfies  $F_a|_{U_{-a}} = f_{-a}$  by defining  $f_{-a}(u) = h_a^{-1} f_a(n_a u n_a^{-1}) h_a$  for  $u \in U_{-a}$ .) The construction of  $F_a$  in [SGA3, XXIII., 2.3.2] rests on calculations in [SGA3, XX, 6.2] with an “abstract” split reductive group of semisimple-rank 1. The explicit classification of such split groups in Theorem 5.1.8 will now be used to simplify those calculations.

We may replace  $G$  with  $Z_G(T_a)$ , so our problem becomes exactly the special case of  $G$  with semisimple-rank 1. In particular,  $\Delta = \{a\}$ , so the conditions (6.2.5) and (6.2.6) become vacuous and (6.2.3) only involves the first relation there. The uniqueness allows us to work étale-locally on  $S$  for existence, so the central torus direct factor as in Theorem 5.1.8 can

be dropped and we are reduced to three special cases:  $(G, T)$  is either  $(\mathrm{SL}_2, D)$ ,  $(\mathrm{PGL}_2, \overline{D})$ , or  $(\mathrm{GL}_2, \tilde{D}) = (\mathrm{SL}_2 \times^{\mu_2} \mathbf{G}_m, D \times^{\mu_2} \mathbf{G}_m)$ . The third case trivially reduces to the first case (since  $a^\vee(-1) \in \mathcal{D}(G)$  in general), and so does the second case since the natural degree-2 central isogeny  $\mathrm{SL}_2 \rightarrow \mathrm{PGL}_2$  has kernel  $\mu_2 = D[2] \subset D$ . Hence, we may and do assume that  $(G, T, M) = (\mathrm{SL}_2, D, \mathbf{Z})$  where the element  $1 \in \mathbf{Z}$  goes over to the isomorphism  $D \simeq \mathbf{G}_m$  inverse to  $c \mapsto \mathrm{diag}(c, 1/c)$ , and likewise we can arrange that  $a$  is the standard positive root (i.e.,  $U_a$  is the strictly upper triangular subgroup of  $G = \mathrm{SL}_2$ ) and  $X_a$  corresponds to  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ .

In this special case, the challenge is not to define  $f$  on the open cell  $\Omega$  (it is clear what the unique possibility for that must be), nor how to define  $f$  on another translate of  $\Omega$  that (together with  $\Omega$ ) covers  $\mathrm{SL}_2$ . The hard part is to verify that one gets a globally well-defined  $S$ -morphism that is moreover a homomorphism. The calculations to verify this in [SGA3, XX, 6.2] are done in the absence of an explicit classification in semisimple-rank 1, and they become simpler for the explicit case we need, namely  $(\mathrm{SL}_2, D)$  (e.g., the auxiliary parameters  $u \in U_a = \mathbf{G}_a$  and  $\tilde{u} \in U_{-a} = \mathbf{G}_a$  there become 1).  $\square$

**Remark 6.2.5.** One may wonder about a finer result beyond Theorem 6.2.4 that replaces (6.2.5) and (6.2.6) with a more explicit presentation of  $U_{[a,b]}$  in terms of “generators and relations”. This viewpoint is systematically developed in [SGA3, XXIII, 2.6, 3.1.3, 3.2.8, 3.3.7, 3.4.10, 3.5], but it is not logically relevant to the proofs of the Isogeny, Isomorphism, or Existence Theorems, so we will say nothing more about it.

**Example 6.2.6.** We can make explicit what Theorem 6.2.4 says concerning the construction of isogenies  $f : (G', T', M', \{X'_{a'}\}) \rightarrow (G, T, M, \{X_a\})$  between pinned split reductive  $S$ -group with semisimple-rank 1 (so  $\Phi(G', T') = \{\pm a'\}$  and  $\Phi(G, T) = \{\pm a\}$ ) such that  $f$  is compatible with the pinnings and the splittings. That is, we want  $f$  to restrict to an isogeny  $T' \rightarrow T$  dual to a given finite-index lattice inclusion  $h : M \rightarrow M'$  satisfying  $h(a) = qa'$  for a prime power  $q = p^n \geq 1$  such that  $p = 0$  on  $S$  if  $q > 1$  (duality forces  $h^\vee(a'^\vee) = qa^\vee$  due to being in the case of semisimple-rank 1), and we want  $f$  to restrict to an isogeny  $f_{a'} : U'_{a'} \rightarrow U_a$  given by  $\exp_{a'}(cX'_{a'}) \mapsto \exp_a(c^q X_a)$  (which is a homomorphism when  $q > 1$  because we assume that  $S$  is a  $\mathbf{Z}/p\mathbf{Z}$ -scheme in such cases).

We claim that for any such  $h$  and  $q$  there exists a unique such  $f$ . Note that necessarily  $R(f)$  coincides with the  $p(S)$ -morphism  $(h, d, q)$  where  $d(a') = a$  and  $d(-a') = -a$ , so our claim is exactly the Isogeny Theorem for all cases with semisimple-rank 1. (This case of the Isogeny Theorem is [SGA3, XXIII, 4.1.2], whose proof via [SGA3, XX, 4.1] rests on extensive calculations. The reason that we will be able to avoid those calculations is because we have Theorem 5.1.8.)

The key feature of semisimple-rank 1 is that the conditions in Theorem 6.2.4 become very concrete in such cases. First of all, (6.2.3) says  $n_a^2 \stackrel{?}{=} f_T(a'^\vee(-1))$ , and this is automatic since

$$n_a^2 = a^\vee(-1) = a^\vee((-1)^q) = (a^\vee)^q(-1) = h^\vee(a'^\vee)(-1) = f_T(a'^\vee(-1))$$

(we have used that  $(-1)^q = -1$  on  $S$ , since  $p$  vanishes on  $S$  when  $q = p^n > 1$ ). Likewise, the second relation in (6.2.2) merely says that  $f_T$  intertwines inversion on  $T'$  and  $T$ , and the first

relation in (6.2.2) is automatic since  $h(a) = qa'$  (and  $f_T = D_S(h)$ ). The relations in (6.2.5) and (6.2.6) are vacuous in cases with semisimple-rank 1. Finally, (6.2.4) is automatic since it asserts  $(w_a(X_a) \exp_a(X_a))^3 = 1$ , which always holds (see Remark 5.1.10).

The necessary and sufficient conditions in Theorem 6.2.4 involve only reductive closed subgroups with semisimple-rank  $\leq 2$ , so we immediately deduce the following crucial existence criterion for homomorphisms [SGA3, XXIII, 2.4] that is expressed entirely in terms of closed reductive subgroups with such low semisimple-rank.

**Corollary 6.2.7.** *Let  $(G, T, M)$  be a split reductive  $S$ -group with semisimple-rank  $\geq 2$ . Let  $\Phi^+$  be a positive system of roots in  $\Phi$ , and let  $\Delta$  be the corresponding base of simple roots. For each  $a, b \in \Delta$ , let  $T_{ab} \subset T$  be the unique subtorus of relative codimension-2 contained in  $\ker a \cap \ker b$  when  $a \neq b$  and let  $T_{aa} = T_a$ .*

*For an  $S$ -group  $H$ , assume there are given  $S$ -homomorphisms  $f_{ab} : Z_G(T_{ab}) \rightarrow H$  for all  $a, b \in \Delta$  such that  $f_{ab} = f_{ba}$  and  $f_{ab}|_{Z_G(T_a)} = f_{aa}$  for all  $a, b \in \Delta$ . Then there exists a unique  $S$ -homomorphism  $f : G \rightarrow H$  such that  $f|_{Z_G(T_{ab})} = f_{ab}$  for all  $a, b \in \Delta$ .*

Now we are ready to carry out the proof of the Isogeny Theorem (i.e., Theorem 6.1.16(1)). The main work will be in the case of groups with semisimple-rank 2.

*Proof of Isogeny Theorem.* Let  $\Delta'$  be the base of a positive system of roots  $\Phi'^+$  in  $\Phi'$ , so  $\Delta := d(\Delta')$  is a base for a positive system of roots  $\Phi^+ = d(\Phi'^+)$  in  $\Phi$ . The triviality hypothesis on the root spaces allows us to choose pinning  $\{X'_{a'}\}_{a' \in \Delta'}$  and  $\{X_a\}_{a \in \Delta}$ . Corollary 6.1.15 shows that  $f$  is unique up to the action of  $(T/Z_G)(S)$ , so it remains to prove the existence of  $f$ .

The root systems of the split reductive  $S$ -groups  $G$  and  $G'$  are isomorphic via the given isogeny between their root data, so the constant fibral semisimple-ranks for these groups are the same. The case of semisimple-rank 0 is trivial, and the case of semisimple-rank 1 is exactly Example 6.2.6, so we now assume the common semisimple-rank of  $(G', T')$  and  $(G, T)$  is  $\geq 2$ .

**Step 1.** We reduce to the case when  $G$  and  $G'$  have semisimple-rank 2 as follows. For each pair of (possibly equal) elements  $a', b' \in \Delta'$ , let  $a = d(a')$  and  $b = d(b')$  in  $\Delta$  and consider the closed reductive  $S$ -subgroups  $Z_{G'}(T'_{a'b'}) \subset G'$  and  $Z_G(T_{ab}) \subset G$  with respective maximal tori  $T' = D_S(M')$  and  $T = D_S(M)$ . In these  $S$ -subgroups the respective sets of roots  $\Phi_{a'b'}$  and  $\Phi_{ab}$  lie in the respective subsets  $\Phi' \subset M' - \{0\}$  and  $\Phi \subset M - \{0\}$ , and the root spaces are free of rank 1 as line bundles since they coincide with root spaces for  $(G', T')$  and  $(G, T)$  respectively. Hence, we get root data

$$R(Z_{G'}(T'_{a'b'}), T') = (M', \Phi'_{a'b'}, M'^{\vee}, \Phi'^{\vee}_{a'b'})$$

and

$$R(Z_G(T_{ab}), T) = (M, \Phi_{ab}, M^{\vee}, \Phi^{\vee}_{ab})$$

equipped with positive systems of roots  $\Phi'_{a'b'} \cap \Phi'^+$  and  $\Phi_{ab} \cap \Phi^+$ .

Since  $a', b' \in \Delta'$  and  $a, b \in \Delta$ , the positive systems of roots  $\Phi'_{a'b'} \cap \Phi'^+$  and  $\Phi_{ab} \cap \Phi^+$  (with rank 1 when  $a' = b'$  and rank 2 when  $a' \neq b'$ ) have as their respective bases  $\{a', b'\}$  and  $\{a, b\}$

when  $a' \neq b'$  and  $\{a'\}$  and  $\{a\}$  when  $a' = b'$ . Thus, the split reductive  $S$ -groups  $Z_{G'}(T'_{a'b'})$  and  $Z_G(T_{ab})$  with semisimple-rank  $\leq 2$  are also pinned (using our pinnings for  $(G', T', M')$  and  $(G, T, M)$ ). Concretely,  $\Phi'_{a'b'}$  consists of the roots that are trivial on the torus  $T'_{a'b'}$ , and likewise for  $\Phi_{ab}$  using  $T_{ab}$  (as we may check on geometric fibers, using the classical theory). But  $X(T'_{a'b'})$  is the quotient of  $X(T')$  modulo the saturation of  $\mathbf{Z}a' + \mathbf{Z}b'$ , and likewise for  $X(T_{ab})$  as a quotient of  $X(T)$  using  $\mathbf{Z}a + \mathbf{Z}b$ , so it follows from the definition of an isogeny of reduced root data that for the given  $p(S)$ -morphism of root data  $\phi = (h, d, q)$  the map  $h : X(T) \rightarrow X(T')$  induces a compatible map  $X(T_{ab}) \rightarrow X(T'_{a'b'})$ . Hence,  $\phi$  restricts to a  $p(S)$ -morphism of root data

$$\phi_{a'b'} : R(Z_{G'}(T'_{a'b'}), T') \rightarrow R(Z_G(T_{ab}), T)$$

for all (possibly equal)  $a', b' \in \Delta'$ . Note that  $\phi_{a'b'} = \phi_{b'a'}$ .

Now assume that all cases of semisimple-rank 2 are settled (as is true for all cases with semisimple-rank 1), so we obtain isogenies

$$f_{a'b'} : (Z_{G'}(T'_{a'b'}), T', M') \rightarrow (Z_G(T_{ab}), T, M)$$

that are compatible with the splittings and satisfy  $R(f_{a'b'}) = \phi_{a'b'}$ . By Corollary 6.1.15, replacing such an  $f_{a'b'}$  with its composition against the action of a *unique*  $\bar{t} \in (T/Z_G)(S)$  makes  $f_{a'b'}$  also compatible with the pinnings  $\{X'_{a'}, X'_{b'}\}$  and  $\{X_a, X_b\}$  (by which we mean  $\{X'_{a'}\}$  and  $\{X_a\}$  when  $a' = b'$ ). Hence, the equality  $R(f_{a'b'}) = \phi_{a'b'} = \phi_{b'a'} = R(f_{b'a'})$  forces  $f_{a'b'} = f_{b'a'}$ . Likewise, if  $a' \neq b'$  then  $\phi_{a'b'}$  restricts to  $\phi_{a'a'}$  on the root datum  $R(Z_{G'}(T'_{a'}), T')$ , so  $f_{a'b'}|_{Z_{G'}(T'_{a'})} = f_{a'a'}$ . By Corollary 6.2.7, there is a unique isogeny  $f : (G', T') \rightarrow (G, T)$  inducing all  $f_{a'b'}$ , so  $f$  respects the splittings and the pinnings. This completes the reduction of the Isogeny Theorem to the case of semisimple-rank 2.

**Step 2.** Assume that  $G$  and  $G'$  have semisimple-rank 2. By the classification of rank-2 reduced root systems, there are four possibilities for each root system:  $A_1 \times A_1$  (e.g.,  $\mathrm{SL}_2 \times \mathrm{SL}_2$ ),  $A_2$  (e.g.,  $\mathrm{SL}_3$ ),  $B_2$  (e.g.,  $\mathrm{Sp}_4$ ), and  $G_2$ . In each of these four cases we will define a “universal” choice of trivialization of all positive root spaces in a manner that only depends on the based root system  $(\Phi, \Delta)$  and an enumeration  $\xi : \{1, 2\} \simeq \Delta$ .

[Since we have not yet proved the Existence Theorem, we may not yet know that there exists a pinned split reductive  $S$ -group whose root system has type  $G_2$  (if we haven’t constructed  $G_2$  already by some other means; e.g., octonion algebras). This is not logically relevant, since at present we are only aiming to prove that *if* we are given a pair of pinned split reductive  $S$ -groups then we can relate  $p(S)$ -morphisms between the root data to isogenies between the  $S$ -groups. To prove the Isogeny Theorem we only need to consider each of the root systems of rank 2 that *might* occur, without constructing specific  $S$ -groups.]

Consider a pinned split reductive  $S$ -group  $(G, T, M, \{X_a\}_{a \in \Delta})$  whose root system has rank 2. We use the pinning to define a specific representative  $n_a = w_a(X_a) \in N_G(T)(S)$  for the simple reflection  $s_a \in W(\Phi)$  for each  $a \in \Delta$ . If  $c \in \Phi^+$  is a positive root (relative to  $\Delta$ ) then there exists  $w \in W(\Phi)$  such that  $w^{-1}(c) \in \Delta$ , so for any sign  $\varepsilon_c \in \{\pm 1\}$  and any product

$n_w \in N_G(T)(S)$  among  $\{n_a\}_{a \in \Delta}$  such that  $n_w$  represents  $w$ ,

$$X_{c,n_w} := \varepsilon_c \text{Ad}(n_w)(X_{w^{-1}(c)})$$

is a trivialization of  $\mathfrak{g}_c$ . This trivialization depends on both  $w$  and  $n_w$ , neither of which is determined by  $c$ , so  $X_{c,n_w}$  is not intrinsic even if we set  $\varepsilon_c = 1$  (e.g., if  $S = \text{Spec } \mathbf{Z}$  then there is an ambiguity from scaling by  $\mathbf{Z}^\times = \{1, -1\}$ ).

In [SGA3, XXIII, 3.1–3.4] each of the four reduced rank-2 root systems  $\Phi$  is considered separately, along with a choice of base  $\Delta$  and an enumeration  $\xi : \{1, 2\} \simeq \Delta$  (in order of increasing root length for types  $B_2$  and  $G_2$ ). In each case, for every  $c \in \Phi^+$  an explicit choice is made for  $\varepsilon_c \in \{\pm 1\}$ ,  $w(c) \in \Phi^+$ , and  $n_{w(c)}$  as a product among  $\{n_a\}_{a \in \Delta}$  to define a trivialization  $X_c$  of  $\mathfrak{g}_c$ . There is nothing canonical about the choices of  $\varepsilon_c$ ,  $w(c)$ , or  $n_{w(c)}$ , but these choices are made only depending on the based root system  $(\Phi, \Delta)$  and enumeration  $\xi : \{1, 2\} \simeq \Delta$ . In this way, we obtain a trivialization  $\{X_c\}_{c \in \Phi^+}$  for the positive root spaces in each pinned split reductive group scheme  $(G, T, M, \{X_a\}_{a \in \Delta})$  with semisimple-rank 2 when  $\Delta$  is equipped with an enumeration (say in order of increasing root length for types  $B_2$  and  $G_2$ ). In [SGA3, XXIII, 3.4.1(ii)], the signs  $\{\varepsilon_c\}_{c \in \Phi^+}$  are taken to be 1 except for type  $G_2$ .

[The enumeration  $\xi : \{1, 2\} \simeq \Delta$  is most important for type  $A_2$ , since for types  $B_2$  and  $G_2$  the root lengths give an intrinsic distinction between the two elements of  $\Delta$ , whereas for type  $A_1 \times A_1$  the only positive roots are the simple ones and their root groups commute. The effect of the choice of enumeration for type  $A_2$  is seen via the signs that break the symmetry in the formulas for  $\{X_c\}_{c \in \Phi^+}$  in [SGA3, XXIII, 3.2.1(ii)] if one swaps the order of enumeration of the two simple positive roots.]

Any two isomorphisms between pinned split reductive groups of semisimple-rank 2 that induce the same bijection between the  $\Delta$ 's must coincide on derived groups (Proposition 6.1.13(2)), and so coincide on the “universal” trivializations of all positive root spaces. The induced bijection between the  $\Delta$ 's can be controlled by demanding compatibility with the chosen enumeration of  $\Delta$ . Thus, the  $X_c$ 's are functorial with respect to isomorphisms between pinned split reductive  $S$ -groups with semisimple-rank 2 when we demand that the isomorphism be compatible with a fixed choice of enumeration of  $\Delta$ . In this sense, our choice of  $\{X_c\}_{c \in \Phi^+}$  is “universal” for any based reduced root system  $(\Phi, \Delta)$  of rank 2 equipped with an enumeration of  $\Delta$ .

**Step 3.** The trivializations  $X_c$  will now be used to unambiguously define “structure constants” (global functions on  $S$ , to be precise) that encode the  $S$ -group law. Choose a pinned split reductive  $S$ -group  $(G, T, M, \{X_a\}_{a \in \Delta})$ . For all  $c \in \Phi^+$ , define  $p_c : \mathbf{G}_a \simeq U_c$  via  $p_c(x) = \exp_c(xX_c)$ . For any  $a \in \Delta$  and  $c \in \Phi^+ - \{a\}$  there exists a unique unit  $u(a, c) \in \mathbf{G}_m(S)$  defined by

$$(6.2.7) \quad \text{Ad}_G(n_a)(X_c) = u(a, c)X_{s_a(c)}.$$

(Note that  $s_a(c) \in \Phi^+$  because the  $\Delta$ -expansion of each positive  $c \neq a$  has some positive coefficient away from  $a$  and hence  $s_a(c) = c - \langle c, a^\vee \rangle a$  does as well.) Likewise, by introducing some universal signs in the definitions of the coefficients in (5.1.5) we see that for  $b, b' \in \Phi^+$

and roots  $ib + jb' \in \Phi$  with  $i, j \geq 1$  there are unique  $C_{i,j,b,b'} \in \mathbf{G}_a(S)$  such that

$$p_{b'}(y)p_b(x) = p_b(x)p_{b'}(y) \prod_{i,j} p_{ib+jb'}(C_{i,j,b,b'}x^i y^j),$$

where the product on the right side is taken relative to the ordering on  $\Phi^+$  defined by lexicographical order relative to the chosen enumeration of  $\Delta$ . (Recall that for types  $B_2$  and  $G_2$ , we made the convention to enumerate  $\Delta$  by putting the short root first. This choice is implicit in the formulas in [SGA3, XXIII, 3.3, 3.4].)

A priori,  $u(a, c)$  and  $C_{i,j,b,b'}$  may depend on  $(G, T, M, \{X_a\}_{a \in \Delta})$  over  $S$  and the enumeration  $\xi : \{1, 2\} \simeq \Delta$ . We call these the *structure constants* for  $(G, T, M, \{X_a\}_{a \in \Delta})$ . (They are global functions on  $S$ ). If we already knew the Isomorphism and Existence Theorems then the following lemma would be immediate, and the real miracle at the heart of the Isogeny, Isomorphism, and Existence Theorems is that this lemma can be proved directly:

**Lemma 6.2.8.** *For each based reduced root system  $(\Phi, \Delta)$  of rank 2 equipped with an enumeration of  $\Delta$ , there are signs  $u(a, c) \in \mathbf{Z}^\times$  and integers  $C_{i,j,b,b'} \in \mathbf{Z}$  that induce the structure constants for every pinned split reductive group  $(G, T, M, \{X_a\}_{a \in \Delta})$  with based root system  $(\Phi, \Delta)$  over any non-empty scheme  $S$ .*

*Proof.* The idea of the proof is to exploit additional relations arising from the  $N_G(T)(S)$ -action on the root spaces and the  $W(\Phi)$ -action on  $\Phi$ . For example, if  $a, b \in \Phi$  with  $b \neq -a$  and if  $n \in N_G(T)(S)$  represents  $w \in W(\Phi)$  satisfying  $w(a) = b$  then for any trivializations  $X$  of  $\mathfrak{g}_a$  and  $Y$  of  $\mathfrak{g}_b$  the unit  $u \in \mathbf{G}_m(S)$  defined by  $\text{Ad}_G(n)(X) = uY$  satisfies

$$(6.2.8) \quad nw_a(X)n^{-1} = b^\vee(u)w_b(Y)$$

(where  $w_a(X) = \exp_a(X) \exp_{-a}(-X^{-1}) \exp_a(X)$ , and similarly for  $b$  using  $Y$ ). An equivalent formulation is to say that  $w_b(uY) = b^\vee(u)w_b(Y)$ , which is the first identity in Corollary 5.1.9(2).

The relations (6.2.8) lead to nontrivial conditions on the units  $u(a, c)$  in (6.2.7), and for each  $\Phi$  these extra relations yield unique universal solutions  $u(a, c) \in \mathbf{Z}^\times = \{1, -1\}$  that are the same for *all*  $(G, T, M, \{X_a\}_{a \in \Delta})$  with root system  $\Phi$  over *any* non-empty  $S$ . The details are in [SGA3, XXIII, 3.1–3.4], working case-by-case depending on the rank-2 root system  $\Phi$ .

Relations between root groups and the  $W(\Phi)$ -action on  $\Phi$  are given in [SGA3, XXIII, 3.1.1] by applying conjugation against each  $n_a = w_a(X_a)$  ( $a \in \Delta$ ) to the commutation relations among root groups. This allows one to uniquely solve for some of the  $C_{i,j,b,b'}$ 's. Miraculously, these solutions arise from constants in  $\mathbf{Z}$  that depend only on  $(\Phi, \Delta, \xi)$  and do *not* depend on  $S$  or  $(G, T, M, \{X_a\}_{a \in \Delta})$ . There are more commutation relations than remaining unknown coefficients  $C_{i,j,b,b'}$ , and case-by-case arguments in [SGA3, XXIII, 3.1–3.4] depending only on  $(\Phi, \Delta, \xi)$  yield linear equations on the  $C_{i,j,b,b'}$ 's over  $\mathbf{Z}$  that depend only on  $(\Phi, \Delta, \xi)$  and do *not* depend on  $(G, T, M, \{X_a\}_{a \in \Delta})$  (or on  $S$ ). Remarkably, these equations over  $\mathbf{Z}$  admit a unique solution over any non-empty scheme  $S$ . The solution over  $S$  arises from the unique solution over  $\mathbf{Z}$ .  $\square$

**Example 6.2.9.** Before we use Lemma 6.2.8 to complete (our sketch of) the proof of the Isogeny Theorem, we illustrate the lemma for type  $G_2$  (cf. [SGA3, XXIII, 3.4.1]). We have  $\Delta = \{a_0, b_0\}$  with  $a_0$  short and  $b_0$  long, so

$$\Phi^+ = \{a_0, b_0, a_0 + b_0, 2a_0 + b_0, 3a_0 + b_0, 3a_0 + 2b_0\}.$$

For any  $(G, T, M, \{X_a\}_{a \in \Delta})$  with root system of type  $G_2$  over any non-empty scheme  $S$ , define a “universal” trivialization  $X_c$  of  $\mathfrak{g}_c$  over  $S$  for every  $c \in \Phi^+$  via

$$X_{a_0+b_0} = \text{Ad}_G(n_{b_0})(X_{a_0}), \quad X_{2a_0+b_0} = \text{Ad}_G(n_{a_0})(X_{a_0+b_0}) = \text{Ad}_G(n_{a_0}n_{b_0})(X_{a_0})$$

and

$$X_{3a_0+b_0} = -\text{Ad}_G(n_{a_0})(X_{b_0}), \quad X_{3a_0+2b_0} = \text{Ad}_G(n_{b_0})(X_{3a_0+b_0}) = -\text{Ad}_G(n_{b_0}n_{a_0})(X_{b_0})$$

(note the signs).

The universality of the units  $u(a, c) \in \mathbf{G}_m(S)$  ( $a \in \Delta$ ,  $c \in \Phi^+$ ) is illustrated by the fact that necessarily  $u(a_0, 2a_0 + b_0) = -1$ ,  $u(a_0, 3a_0 + b_0) = 1$ , and  $u(b_0, 3a_0 + 2b_0) = -1$ ; that is,

$$\text{Ad}_G(n_{a_0})(X_{2a_0+b_0}) = -X_{a_0+b_0}, \quad \text{Ad}_G(n_{a_0})(X_{3a_0+b_0}) = X_{b_0}, \quad \text{Ad}_G(n_{b_0})(X_{3a_0+2b_0}) = -X_{3a_0+b_0}.$$

Likewise, the commutation relations among the parameterizations  $p_c(x) = \exp_c(xX_c)$  for the positive root groups involve coefficients that are universal constants in  $\mathbf{Z}$ ; e.g.,

$$p_{a_0+b_0}(y)p_{a_0}(x) = p_{a_0}(x)p_{a_0+b_0}(y)p_{2a_0+b_0}(2xy)p_{3a_0+b_0}(3x^2y)p_{3a_0+2b_0}(3xy^2),$$

$$(6.2.9) \quad p_{2a_0+b_0}(y)p_{a_0}(x) = p_{a_0}(x)p_{2a_0+b_0}(y)p_{3a_0+b_0}(3xy),$$

and

$$p_{3a_0+b_0}(y)p_{b_0}(x) = p_{b_0}(x)p_{3a_0+b_0}(y)p_{3a_0+2b_0}(-xy).$$

The universal coefficients in the commutation relations are  $\pm 1$  for types  $A_1 \times A_1$  and  $A_2$ , but coefficients in  $\{\pm 2\}$  arise for type  $B_2$  and coefficients in  $\{\pm 2, \pm 3\}$  arise for type  $G_2$ . An interesting consequence is that for type  $B_2$  (resp.  $G_2$ ) there are some root groups that commute in characteristic 2 (resp. characteristic 3) but not in any other characteristic. For example, (6.2.9) gives an “extra” commutation among root groups of  $G_2$  in characteristic 3.

**Step 4.** Returning to the proof of the Isogeny Theorem, recall that we have reduced the problem to groups of semisimple-rank 2. Lemma 6.2.8 implies that the “structure constants” describing both the adjoint action of the  $n_a$ ’s on the positive root spaces and the commutation relations among the positive root groups are absolute constants in  $\mathbf{Z}$  that depend only on  $(\Phi, \Delta)$  (and our enumeration of  $\Delta$ ); they do not depend on the base scheme  $S$  or the pinned split reductive  $S$ -group with root system  $\Phi$ . This immediately yields a weak form of the Isomorphism Theorem: a split reductive  $S$ -group is determined up to isomorphism by its root datum. Indeed, the  $S$ -birational group law on the open cell is determined by these “structure constants” and the root datum  $R = (M, \Phi, M^\vee, \Phi^\vee)$  (e.g., the split maximal torus is  $D_S(M)$ ), so by the universality of these constants we see that the open cell equipped with its  $S$ -birational group law only depends on  $(R, \Delta)$  and the enumeration of  $\Delta$ .

To go further and prove the Isogeny Theorem, one first determines all  $p$ -morphisms among reduced semisimple root data of rank 2 (especially with  $p$  a prime rather than  $p = 1$ ); this is an elementary combinatorial problem since we are only considering rank 2. We have noted above that the  $S$ -birational group law is described on the open cell in terms of the root datum and universal structure constants in  $\mathbf{Z}$ . Also, there is a variant on Corollary 6.2.7 given in [SGA3, XXIII, 2.5] (as an immediate consequence of Theorem 6.2.4) that provides an existence criterion for homomorphisms out of a pinned split reductive group of semisimple-rank 2. Combining this criterion with case-by-case arguments (depending on  $\Phi$ ), one builds isogenies between pinned split reductive  $S$ -groups realizing any  $p(S)$ -morphism between the reduced root data. The details are elegantly explained in [SGA3, XXIII, 4.1.3–4.1.8]. (The hardest cases are type  $B_2$  over  $\mathbf{Z}/2\mathbf{Z}$ -schemes with  $q \in \{2^n\}_{n \geq 1}$  and type  $G_2$  over  $\mathbf{Z}/3\mathbf{Z}$ -schemes with  $q \in \{3^n\}_{n \geq 1}$ .)  $\square$

**Remark 6.2.10.** The Isogeny Theorem underlies the classification of “exceptional” isogenies between connected semisimple groups over fields. More specifically, in characteristic 0 all isogenies are central, so let us focus on connected semisimple groups over a field  $k$  with characteristic  $p > 0$ . There are two evident classes of isogenies over  $k$ : central isogenies and Frobenius isogenies  $F_{G/k} : G \rightarrow G^{(p)}$ . It is natural to wonder if every isogeny is a composition among these. A map factors through a Frobenius isogeny on the source if and only if it induces the zero map on Lie algebras (as the infinitesimal  $k$ -subgroups of  $G$  that have full Lie algebra are those which contain  $\ker F_{G/k}$ ), so it is equivalent to determine if there are non-central isogenies that are nonzero on Lie algebras. This problem can be reduced to the case  $k = k_s$ , so we can restrict attention to the split case, for which the Isogeny Theorem is applicable.

By passing to simply connected central covers (Exercise 6.5.3) and direct factors thereof, the problem becomes: when do there exist non-central  $p$ -morphisms between irreducible and reduced root data that do not factor through “multiplication by  $p$ ”? (Here we are using the Isogeny Theorem and the fact that Frobenius isogenies in characteristic  $p > 0$  correspond to “multiplication by  $p$ ” on the root datum.) This is a purely combinatorial problem for each prime  $p$ , and by considering the classification of root systems the answer is affirmative if and only if  $p \in \{2, 3\}$ . Some explicit examples in characteristic 2 are classical in the theory of quadratic forms, namely the isogenies  $SO_{2n+1} \rightarrow Sp_{2n}$  with infinitesimal non-central commutative kernel  $\alpha_2^{2n}$  for  $n \geq 1$ ; see [PY06, Lemma 2.2] for further discussion of these isogenies. For  $n = 2$  this isogeny  $SO_5 \rightarrow Sp_4$  gives rise to an exotic *endomorphism* of  $Sp_4$  since  $Spin_5 = Sp_4$  (as  $B_2 = C_2$ ; see Example C.6.5), and this is a “square root” of the Frobenius isogeny over  $\mathbf{F}_2$ . Similarly, one gets an exotic endomorphism of  $F_4$  in characteristic 2 and of  $G_2$  in characteristic 3. These endomorphisms underlie the existence of the Suzuki and Ree groups in the classification of finite simple groups.

We end this section with some interesting applications of the Isomorphism Theorem. (See [SGA3, XXIII, §5] for a more extensive discussion.)

**Proposition 6.2.11.** *Let  $G$  and  $G'$  be reductive groups over a non-empty scheme  $S$ .*

- (1) If  $G$  and  $G'$  are isomorphic fpqc-locally on  $S$  then they are so étale-locally on  $S$ .
- (2) Assume that  $G$  and  $G'$  are isomorphic étale-locally on  $S$ , that  $S$  is connected with  $\text{Pic}(S) = 1$ , and that  $G$  and  $G'$  have respective split maximal tori  $T$  and  $T'$ . The pairs  $(G, T)$  and  $(G', T')$  are isomorphic, as are the triples  $(G, B, T)$  and  $(G', B', T')$  for Borel subgroups  $B \supset T$  and  $B' \supset T'$ . If  $G' = G$  and  $S$  is also affine then these isomorphisms can be chosen to arise from  $G(S)$ -conjugation.

*Proof.* For (1) we may work étale-locally on  $S$  to reach the split case with the same root datum. Then we can apply the Isomorphism Theorem. For (2), fix isomorphisms  $T \simeq D_S(M)$  and  $T' \simeq D_S(M')$  for finite free  $\mathbf{Z}$ -modules  $M$  and  $M'$ . Since  $S$  is connected, so constant sheaves on  $S$  have only constant global sections,  $M = \text{Hom}_{S\text{-gp}}(T, \mathbf{G}_m)$  and similarly for  $M'$  and  $T'$ . Likewise, the root spaces for  $(G, T)$  and  $(G', T')$  are free of rank 1 since  $\text{Pic}(S) = 1$ . Thus,  $(G, T, M)$  and  $(G', T', M')$  are split. In particular, Borel subgroups  $B \supset T$  and  $B' \supset T'$  do exist; choose such  $S$ -subgroups.

The connectedness of  $S$  ensures that the choices for  $B$  correspond bijectively to the positive systems of roots in  $\Phi \subset M$ , and similarly for  $B'$  (see Remark 6.1.2). Hence, by choosing suitable pinnings, the Isomorphism Theorem in the precise form of Theorem 6.1.17 provides an isomorphism  $(G, B, T) \simeq (G', B', T')$ . In the special case  $G' = G$  with  $S$  also affine, it remains to show that  $G(S)$  acts transitively on the set of pairs  $(B, T)$ .

Let  $B$  be a Borel subgroup of  $G$ , so the orbit map  $G \rightarrow \text{Bor}_{G/S}$  through  $B$  identifies  $\text{Bor}_{G/S}$  with  $G/B$  by Corollary 5.2.7(1) (since  $N_G(B) = B$ , by Corollary 5.2.8). The  $S$ -group  $B$  has a composition series whose successive quotients are  $\mathbf{G}_a$  and  $\mathbf{G}_m$ , so the vanishing of both  $H^1(S_{\text{ét}}, \mathbf{G}_a) = H^1(S_{\text{Zar}}, \mathcal{O}_S)$  (as  $S$  is affine) and  $H^1(S_{\text{ét}}, \mathbf{G}_m) = H^1(S_{\text{Zar}}, \mathcal{O}_S^\times)$  (as  $\text{Pic}(S) = 1$ ) implies that  $H^1(S_{\text{ét}}, B) = 1$  (concretely, every étale  $B$ -torsor over  $S$  is split). Hence, the  $B$ -torsor  $G \rightarrow G/B$  induces a surjection  $G(S) \rightarrow (G/B)(S)$ , so  $G(S)$  acts transitively on the set of Borel  $S$ -subgroups of  $G$ .

As we saw above, any split maximal torus  $T$  in  $G$  lies in some Borel subgroup  $B$ . In view of the  $G(S)$ -conjugacy of Borel  $S$ -subgroups of  $G$ , it remains to show that any two split maximal tori  $T, T' \subset G$  contained in  $B$  are  $B(S)$ -conjugate. Any such tori are  $B$ -conjugate étale-locally on  $S$  (Proposition 2.1.2), so  $\text{Transp}_B(T, T')$  is a torsor over  $S_{\text{ét}}$  for  $N_B(T) = B \cap N_G(T) = T$  (the final equality due to  $N_G(T)/T$  being the finite constant group for  $W(\Phi)$ , with  $W(\Phi)$  acting simply transitively on the set of  $B \supset T$  on geometric fibers over  $S$ ). But  $T = \mathbf{G}_m^r$  and  $\text{Pic}(S) = 1$ , so we are done.  $\square$

The Existence Theorem over  $\mathbf{C}$  is well-known in the classical theory (see Appendix D). As an application of the Isomorphism Theorem, we now improve on the Existence Theorem over  $\mathbf{C}$  by pushing it down to  $\mathbf{Q}$ . This will be an ingredient in the proof of the Existence Theorem in general.

**Proposition 6.2.12.** *For each reduced root datum  $R$ , there exists a split connected reductive  $\mathbf{Q}$ -group  $(G, T)$  such that  $R(G, T) \simeq R$ .*

*Proof.* Choose a split connected reductive  $\mathbf{C}$ -group  $(G, T, M)$  having  $R$  as its root datum. Writing  $\mathbf{C} = \varinjlim A_i$  for finite type  $\mathbf{Q}$ -subalgebras  $A_i$ , the triple  $(G, T, M)$  descends to a split

reductive group scheme  $(\mathcal{G}, \mathcal{T}, M)$  over  $\text{Spec } A$  for some finite type  $\mathbf{Q}$ -subalgebra  $A = A_{i_0} \subset \mathbf{C}$ . Its root datum is  $R$ , so by passing to the fiber at a closed point we find a split triple  $(G', T', M)$  over a number field  $F$  with root datum isomorphic to  $R$ ; fix this isomorphism.

By replacing  $F$  with a finite extension we may assume that  $F$  is Galois over  $\mathbf{Q}$ . We will now carry out Galois descent down to  $\mathbf{Q}$  via the crutch of a pinning. Choose a positive system of roots  $\Phi^+$  in  $\Phi = \Phi(G', T')$ , and let  $\Delta$  be the corresponding base. For each  $a \in \Delta$ , pick a basis  $X_a$  of the  $F$ -line  $\mathfrak{g}_a$ , so we get a pinned split reductive group  $(G', T', M, \{X_a\}_{a \in \Delta})$ . We have a chosen isomorphism  $\phi : R(G', T', M) \simeq R$ , and for all  $\gamma \in \Gamma := \text{Gal}(F/\mathbf{Q})$  we get another pinned split reductive group

$$(\gamma^*(G'), \gamma^*(T'), M, \{\gamma^*(X_a)\}_{a \in \Delta})$$

via the evident identifications  $\Phi(\gamma^*(G'), \gamma^*(T')) \simeq \Phi(G', T') = \Phi \subset M - \{0\}$  and  $X(\gamma^*(T')) \simeq X(T') = M$  (defined by functoriality of scalar extension along the  $\Gamma$ -action on  $F$ ).

It is easy to check that the resulting isomorphisms of root data

$$(6.2.10) \quad R(G', T', M) \simeq R(\gamma^*(G'), \gamma^*(T'), M, \{\gamma^*(X_a)\}_{a \in \Delta})$$

satisfy the cocycle condition. By the Isomorphism Theorem, (6.2.10) arises from a unique isomorphism between *pinned* split reductive  $F$ -groups

$$(G', T', M, \{X_a\}_{a \in \Delta}) \simeq (\gamma^*(G'), \gamma^*(T'), M, \{\gamma^*(X_a)\}_{a \in \Delta}),$$

and the uniqueness implies that these isomorphisms inherit the cocycle condition from that aspect of the isomorphisms of root data. Note that these isomorphisms between pinned groups use the identity automorphism on  $M$ , so they use the identity bijection on  $\Delta$ . (That is,  $X_a$  is carried to  $\gamma^*(X_a)$ .) Hence, by Galois descent we obtain a pinned split reductive  $\mathbf{Q}$ -group descending  $(G', T', M, \{X_a\}_{a \in \Delta})$ , and its root datum is clearly  $R$ .  $\square$

**6.3. Existence Theorem.** Let  $R = (M, \Phi, M^\vee, \Phi^\vee)$  be a reduced root datum. By base change, to prove the Existence Theorem for  $R$  (i.e., Theorem 6.1.16(2)) over an arbitrary non-empty scheme  $S$  it suffices to treat the case  $S = \text{Spec } \mathbf{Z}$ . By the following lemma, whose proof is a formal argument with root data, it suffices to consider only  $R$  that is semisimple and simply connected (i.e.,  $\Phi$  spans  $M_{\mathbf{Q}}$  over  $\mathbf{Q}$ , and  $\Phi^\vee$  spans  $M^\vee$  over  $\mathbf{Z}$ ) and such that the root system associated to  $R$  is irreducible.

**Lemma 6.3.1.** *To prove the Existence Theorem over a non-empty scheme  $S$ , it suffices to treat semisimple root data  $(X, \Phi, X^\vee, \Phi^\vee)$  that are simply connected and have associated root system  $(X_{\mathbf{Q}}, \Phi)$  that is irreducible.*

*Proof.* The idea is to use a preliminary central isogeny of root data to separate the maximal central torus from the derived group, and then to treat tori and semisimple groups separately. Let  $R = (X, \Phi, X^\vee, \Phi^\vee)$  be a root datum, so  $X$  contains  $\mathbf{Z}\Phi \oplus (\mathbf{Z}\Phi^\vee)^\perp$  with finite index, where the annihilator  $(\mathbf{Z}\Phi^\vee)^\perp$  in  $X$  is saturated but  $\mathbf{Z}\Phi$  may not be saturated. In general, if  $L \rightarrow L'$  is an injective map between finite free  $\mathbf{Z}$ -modules, we write  $L_{\text{sat}}$  to denote the

saturation of  $L$  in  $L'$  (i.e., the kernel of  $L' \rightarrow (L'/L)_{\mathbf{Q}}$ ). The natural map

$$X \rightarrow (X/(\mathbf{Z}\Phi)_{\text{sat}}) \oplus (X/(\mathbf{Z}\Phi^{\vee})^{\perp}) =: X'$$

is a finite-index inclusion that carries  $\Phi$  onto a subset  $\Phi'$  that lies in the second summand of  $X'$ . The  $\mathbf{Z}$ -dual of  $X'$  is naturally identified with the direct sum

$$X'^{\vee} = (\mathbf{Z}\Phi)^{\perp} \oplus (\mathbf{Z}\Phi^{\vee})_{\text{sat}}$$

and  $\Phi'^{\vee}$  is defined to be the image of  $\Phi^{\vee}$  under inclusion into the second factor.

Clearly  $R' := (X', \Phi', X'^{\vee}, \Phi'^{\vee})$  is a root datum, and the natural isogeny  $(h, d, q) : R' \rightarrow R$  is central in the sense that  $q(a') = 1$  for all  $a' \in \Phi'$ . If  $R'$  arises from a split connected reductive  $k$ -group  $(G', T')$  then the cokernel of  $h : X \rightarrow X' = X(T')$  corresponds to a split finite multiplicative type  $S$ -subgroup  $\mu \subset T'$  such that  $X(T'/\mu) = X$  inside  $X(T') = X'$ . In particular, the centrality of  $(h, d, q)$  implies that all roots of  $(G', T')$  lie in  $X(T'/\mu)$ , which is to say that  $\mu$  is a central  $S$ -subgroup scheme of  $G'$  (Corollary 3.3.6(1)). Hence, the central quotient  $G := G'/\mu$  makes sense as a semisimple  $S$ -group in which  $T := T'/\mu$  is a split maximal torus. Since the inclusion  $h : X \hookrightarrow X'$  carries  $\Phi$  onto  $\Phi'$  (as  $q$  is identically 1), and the dual map  $X'^{\vee} \rightarrow X^{\vee}$  carries  $\Phi'^{\vee}$  onto  $\Phi^{\vee}$  (due to the unique characterization of coroots for a root system), the root datum  $R(G, T)$  is visibly identified with  $(X, \Phi, X^{\vee}, \Phi^{\vee})$ . Now it suffices to treat  $R'$  instead of  $R$ .

Let  $L = X/(\mathbf{Z}\Phi^{\vee})^{\perp}$ ,  $L^{\vee} = (\mathbf{Z}\Phi^{\vee})_{\text{sat}}$ ,  $\Psi = \Phi \bmod (\mathbf{Z}\Phi^{\vee})^{\perp} \subset L$ , and  $\Psi^{\vee} = \Phi^{\vee} \subset L^{\vee}$ . Clearly  $R'' := (L, \Psi, L^{\vee}, \Psi^{\vee})$  is a root datum that is semisimple, and

$$R' = R'' \oplus (X/(\mathbf{Z}\Phi)_{\text{sat}}, \emptyset, (\mathbf{Z}\Phi)^{\perp}, \emptyset).$$

It suffices to treat the two summands separately (as we can then form the direct product of the corresponding split connected reductive groups). The second summand is trivially handled by using the split torus with character group  $X/(\mathbf{Z}\Phi)_{\text{sat}}$ , so we may now focus our attention on  $R''$ . That is, we may assume that our root datum  $R$  is semisimple.

As in (1.3.2), we have  $\mathbf{Z}\Phi \subset X \subset (\mathbf{Z}\Phi^{\vee})^*$ . Let  $X' = (\mathbf{Z}\Phi^{\vee})^*$ ,  $\Phi' = \Phi$ ,  $X'^{\vee} = \mathbf{Z}\Phi^{\vee}$ , and  $\Phi'^{\vee} = \Phi^{\vee}$ , so  $R' = (X', \Phi', X'^{\vee}, \Phi'^{\vee})$  is a root datum that is semisimple and simply connected. There is an evident central isogeny of root data  $R' \rightarrow R$ , so by repeating the central quotient construction above we see that the Existence Theorem for  $R$  over  $S$  is reduced to the Existence Theorem for  $R'$  over  $S$ . Thus, it remains to treat the case of semisimple root data that are simply connected. The equality  $X^{\vee} = \mathbf{Z}\Phi^{\vee}$  ensures that the decomposition of the root system into its irreducible components is also valid at the level of the root datum. Hence, it remains to settle the case of semisimple root data that are simply connected and have an irreducible associated root system. This is precisely the case that we are assuming is established.  $\square$

Fix a semisimple reduced root datum  $R$  that is simply connected. (We will not require irreducibility for  $R$ .) Proposition 6.2.12 provides a split reductive group  $(G, T, M)$  over  $\mathbf{Q}$  with root datum  $R$ . This yields the Existence Theorem for  $R$  over some  $\mathbf{Z}[1/N]$ , but  $N$  might depend on  $R$ . The problem is to get the result over the entirety of  $\text{Spec } \mathbf{Z}$ , not ignoring any

small primes. The rest of §6.3 is devoted to the construction of such a split  $\mathbf{Z}$ -group by a method that works uniformly across all (simply connected and semisimple)  $R$ . In view of the classification of irreducible and reduced root systems, it would suffice to exhibit an explicit example for each Killing–Cartan type.

For the classical types  $A_n$  ( $n \geq 1$ ),  $B_n$  ( $n \geq 3$ ),  $C_n$  ( $n \geq 2$ ), and  $D_n$  ( $n \geq 4$ ) we can use the  $\mathbf{Z}$ -groups  $\mathrm{SL}_{n+1}$  ( $n \geq 1$ ),  $\mathrm{Spin}_{2n+1}$  ( $n \geq 3$ ),  $\mathrm{Sp}_{2n}$  ( $n \geq 2$ ), and  $\mathrm{Spin}_{2n}$  ( $n \geq 4$ ) respectively. (To make sense of spin groups over  $\mathbf{Z}$  and not just over  $\mathbf{Z}[1/2]$ , we need a characteristic-free viewpoint on non-degeneracy for quadratic spaces over rings. This is discussed in Appendix C.) Thus, the arguments that follow are only needed to handle the exceptional types  $E_6$ ,  $E_7$ ,  $E_8$ ,  $F_4$ , and  $G_2$ . Even some of these types can be settled by direct construction (e.g.,  $G_2$  and  $F_4$  can be handled by using octonion and Jordan algebras over  $\mathbf{Z}$ ). Explicit constructions can require special care at small primes (e.g., residue characteristic 2 for spin groups and type  $F_4$ , and residue characteristics 2 and 3 for type  $G_2$ ). The uniform approach below is insensitive to the peculiar demands of small primes or of specific irreducible root systems.

Before we take up the proof of the Existence Theorem, we need to digress and discuss the following concept:

**Definition 6.3.2.** Let  $(G, T, M)$  be a split reductive group over a non-empty scheme  $S$ . A *Chevalley system* for  $(G, T, M)$  is a collection of trivializing sections  $X_a \in \mathfrak{g}_a(S)$  for all  $a \in \Phi$  so that

$$\mathrm{Ad}_G(w_a(X_a))(X_b) = \pm X_{s_a(b)}$$

for all  $a, b \in \Phi$ , where the sign ambiguity is global over  $S$  (possibly depending on  $a$  and  $b$ ) and

$$w_c(X) := \exp_c(X) \exp_{-c}(-X^{-1}) \exp_c(X) \in N_G(T)(S)$$

for every trivializing section  $X$  of  $\mathfrak{g}_c$  and every  $c \in \Phi$ .

The existence of a Chevalley system is vacuous for semisimple-rank 0, and for semisimple-rank 1 we can build one by using any  $X_a$  whatsoever and defining  $X_{-a} := X_a^{-1}$  to be the linked trivialization of  $\mathfrak{g}_{-a}$  (this works, since  $\mathrm{Ad}_G(w_a(X))(X) = -X^{-1}$  for any trivializing section  $X$  of  $\mathfrak{g}_a$ ; see Corollary 5.1.9(3)). By setting  $b = a$  and using that  $s_a(a) = -a$ , it likewise follows that for any Chevalley system  $\{X_a\}_{a \in \Phi}$  we must have  $X_{-a} = \pm X_a^{-1}$  (i.e.,  $X_a$  and  $X_{-a}$  are linked, up to a global sign).

**Example 6.3.3.** Chevalley systems are closely related to the notion of a “Chevalley basis” for a complex semisimple Lie algebra (cf. [Hum72, 25.1–25.2]). To explain this, consider a connected semisimple  $\mathbf{C}$ -group  $G$  equipped with maximal torus  $T$ , so  $\mathfrak{g} := \mathrm{Lie}(G)$  is a semisimple Lie algebra and  $\mathfrak{t} := \mathrm{Lie}(T)$  is a Cartan subalgebra. Fix a positive system of roots in  $\Phi(G, T) = \Phi(\mathfrak{g}, \mathfrak{t})$ , and let  $\Delta$  be the corresponding set of simple roots, so the vectors  $v_a = \mathrm{Lie}(a^\vee)(1)$  are a basis of  $\mathfrak{t}$ . Let  $X_c$  be a basis of  $\mathfrak{g}_c$  for each  $c \in \Phi$ , so the collection of  $v_a$ ’s and  $X_c$ ’s is a basis of  $\mathfrak{g}$ . Let’s consider the resulting “structure constants”.

For  $a \in \Delta$  and  $c \in \Phi$ , we have  $[v_a, X_c] = \mathrm{Ad}_G(v_a)(X_c) = \langle c, a^\vee \rangle X_c$  since conjugation by  $a^\vee(t)$  on  $U_c = \mathbf{G}_a$  is via scaling by  $c(a^\vee(t)) = t^{\langle c, a^\vee \rangle}$ . Consider  $c, c' \in \Phi$  with  $c' \neq \pm c$ . There

is a positive system of roots containing  $c$  and  $c'$ , so by Proposition 5.1.14 the groups  $U_c$  and  $U_{c'}$  commute if  $c + c' \notin \Phi$  (forcing  $[X_c, X_{c'}] = 0$ ) and otherwise  $[X_c, X_{c'}] = r(c, c')X_{c+c'}$  for some  $r(c, c') \in \mathbf{C}$ . The special feature of  $\{X_c\}_{c \in \Phi}$  being a Chevalley system is that the numbers  $r(c, c')$  are (nonzero) *integers* that are moreover determined up to sign by the root system; see Remark 6.3.5. This provides an explicit  $\mathbf{Z}$ -form for every complex semisimple Lie algebra, and Chevalley used this viewpoint to construct adjoint split semisimple  $\mathbf{Z}$ -groups (see Theorem 5.3.5, as well as [Hum72, 25.5, §26]).

**Proposition 6.3.4.** *Let  $(G, T, M, \{X_a\}_{a \in \Delta})$  be a pinned split reductive group over a non-empty scheme  $S$ . There is a Chevalley system  $\{X_c\}_{c \in \Phi}$  extending the pinning, and each  $X_c$  is unique up to a sign.*

For semisimple-rank 2, the main computations for the construction of a Chevalley system were carried out in the proof of the Isomorphism Theorem, but additional arguments are required even in that case (since the definition of a Chevalley system involves the adjoint action for  $w_a(X_a)$  for all  $a \in \Phi$ ).

*Proof.* Let  $n_a = w_a(X_a)$  for all  $a \in \Delta$ . Since every element of  $W(\Phi)$  is represented by a product among the elements of  $\{n_a\}_{a \in \Delta}$ , for any  $c \in \Phi$  we can find such a product  $n$  representing an element  $w \in W(\Phi)$  so that  $w(a) = c$  for some  $a \in \Delta$ . Thus,  $X_c = \pm \text{Ad}_G(n)(X_a)$ . This shows the uniqueness up to a sign.

To prove existence, we begin by running the uniqueness proof in reverse. For each  $c \in \Phi$  not in  $\Delta$ , choose some  $w \in W(\Phi)$  such that  $w^{-1}(c) \in \Delta$ . Pick a word  $a_1 \cdots a_m$  in elements  $a_i$  of  $\Delta$  so that  $s_{a_1} \cdots s_{a_m} = w$  in  $W(\Phi)$ . For the element  $n := n_{a_1} \cdots n_{a_m} \in N_G(T)(S)$ , use the isomorphism  $\text{Ad}_G(n) : \mathfrak{g}_{w^{-1}(c)} \simeq \mathfrak{g}_c$  to define

$$X_c := \text{Ad}_G(n)(X_{w^{-1}(c)}).$$

It suffices to show that  $\{X_c\}_{c \in \Phi}$  is a Chevalley system. (The definition of the  $X_c$ 's depends on the choice of  $w$  and the word  $a_1 \cdots a_m$ . However, once the proof is done, it will follow that changing these choices only affects  $X_c$  by at most a global sign change.)

For  $c \in \Phi$  and  $b \in \Delta$ , we need to prove that

$$(6.3.1) \quad \text{Ad}_G(n_b)(X_c) = \pm X_{s_b(c)}.$$

By definition,  $X_c = \text{Ad}_G(n_0)(X_{a_0})$  for some  $a_0 \in \Delta$  and product  $n_0$  among the elements of  $\{n_a\}_{a \in \Delta}$  so that  $n_0$  represents an element  $w_0 \in W(\Phi)$  satisfying  $w_0(a_0) = c$ . Likewise,  $X_{s_b(c)} = \text{Ad}_G(n_1)(X_{a_1})$  for some  $a_1 \in \Delta$  and product  $n_1$  among the elements of  $\{n_a\}_{a \in \Delta}$  so that  $n_1$  represents an element  $w_1 \in W(\Phi)$  satisfying  $w_1(a_1) = s_b(c) = (s_b w_0)(a_0)$ . Thus,  $n := n_0^{-1} n_b^{-1} n_1$  represents some  $w \in W(\Phi)$  satisfying  $w(a_1) = a_0$ , and our problem is to show that  $\text{Ad}_G(n)(X_{a_1}) = \pm X_{a_0}$  (as then applying  $\text{Ad}_G(n_b n_0)$  to both sides will give

$$X_{s_b(c)} = \text{Ad}_G(n_1)(X_{a_1}) = \pm \text{Ad}_G(n_b)(\text{Ad}_G(n_0)(X_{a_0})) = \pm \text{Ad}_G(n_b)(X_c)$$

as desired).

Although  $n = n_0^{-1} n_b^{-1} n_1$  is not written as a product among the  $\{n_a\}_{a \in \Delta}$ , due to the intervention of some inversions, these inversions can be absorbed into the sign ambiguity in

(6.3.1). The reason is as follows. We have  $n_a^{-1} = a^\vee(-1)n_a$  with  $a^\vee(-1) \in T[2]$ , and for any  $t \in T$  and  $a' \in \Delta$  the identity

$$tn_{a'}t^{-1} = tw_{a'}(X_{a'})t^{-1} = w_{a'}(a'(t)X_{a'}) = a'^\vee(a'(t))w_{a'}(X_{a'}) = t^2n_{a'}$$

(using Corollary 5.1.9(2)) implies that  $T[2]$  centralizes  $n_{a'}$ . In particular, for  $n = n_0^{-1}n_b^{-1}n_1$ , the product  $n' := n_0n_b n_1$  among the elements of  $\{n_a\}_{a \in \Delta}$  lifts the *same* word in the involutions  $s_a$  and we have  $n = tn'$  with  $t := \lambda(-1)$  for some cocharacter  $\lambda \in M^\vee \in \text{Hom}_{S\text{-gp}}(\mathbf{G}_m, T)$ . Thus,  $\text{Ad}_G(n) = \text{Ad}_G(t) \circ \text{Ad}_G(n')$ , and the effect of  $\text{Ad}_G(t)$  on each  $\mathfrak{g}_c$  is scaling by  $c(t) = (-1)^{\langle c, \lambda \rangle} = \pm 1$ .

To summarize, we are reduced to proving a general fact about words in the elements  $n_a$ : if  $a, a' \in \Delta$  and  $\{a_1, \dots, a_m\}$  is a sequence in  $\Delta$  such that  $(s_{a_m} \circ \dots \circ s_{a_1})(a) = a'$  then

$$\text{Ad}_G(n_{a_m} \circ \dots \circ n_{a_1})(X_a) \stackrel{?}{=} \pm X_{a'}$$

in  $\mathfrak{g}(S)$ , for some global sign  $\pm 1$ . Note that this equality is obvious when the semisimple-rank is at most 1, and it is also obvious when the pinning extends to a Chevalley system. Thus, to settle it for all cases with semisimple-rank 2 we just need to construct *some* Chevalley system extending the pinning in every case. But the rank-2 calculations in the proof of Lemma 6.2.8 (for the aspect concerning the units  $u(a, c)$ ) achieve exactly this! To be precise, those calculations construct a “positive” Chevalley system: a collection of trivializations  $\{X_c\}_{c \in \Phi^+}$  of the positive root spaces that extends the pinning and satisfies  $\text{Ad}_G(n_a)(X_c) = \pm X_{s_a(c)}$  for any  $a \in \Delta$  and  $c \in \Phi - \{a\}$ . Hence, to settle the case of semisimple-rank 2 we just need to extend any “positive” Chevalley system  $\{X_c\}_{c \in \Phi^+}$  to an actual Chevalley system.

Define  $X_{-c} = X_c^{-1}$  for all  $c \in \Phi^+$ . We claim that  $\{X_c\}_{c \in \Phi}$  is a Chevalley system. This amounts to checking that  $\text{Ad}_G(n_a)(X_c) = \pm X_{s_a(c)}$  for all  $c \in \Phi$ . The cases  $(a, c)$  and  $(a, -c)$  are equivalent, by the functoriality of duality of opposite root spaces with respect to  $n_a$ -conjugation on  $G$ . Hence, we may assume  $c \in \Phi^+$ . The case  $c \in \Phi^+ - \{a\}$  is known by hypothesis (as  $s_a(c) \in \Phi^+$  for all  $c \in \Phi^+ - \{a\}$ ), and  $\text{Ad}_G(n_a)(X_a) = -X_a^{-1}$  by Corollary 5.1.9(2). This completes the argument for semisimple-rank 2.

For the case of semisimple-rank  $> 2$ , one needs to apply several results in the theory of root systems and make artful use of presentations of Weyl groups as reflection groups to ultimately reduce to the settled case of semisimple-rank 2. We refer to [SGA3, XXIII, 6.3] for the details, which rest on two ingredients: many of the case-by-case “universal” formulas established for pinned split reductive groups with a root system of rank 2 in [SGA3, XXIII, 3.1–3.4], and root system arguments from [SGA3, XXIII, 2.3] that were used in the proof of Theorem 6.2.4.  $\square$

**Remark 6.3.5.** A very useful application of the existence of Chevalley systems is “Chevalley’s rule” [SGA3, XXIII, 6.5] that computes – up to a sign – universal formulas for the structure constants in the Lie algebra of a split semisimple group scheme  $(G, T, M)$  relative to a Chevalley system in the Lie algebra. Explicitly, if  $\{X_c\}_{c \in \Phi}$  is a Chevalley system for  $(G, T, M)$  then  $[X_a, X_b] = \pm(p(a, b) + 1)X_{a+b}$  whenever  $a, b, a + b \in \Phi$ , where  $p(a, b)$  is the

greatest integer  $z \geq 0$  such that  $a - zb \in \Phi$ . This result is proved by inspecting the universal constants in Lemma 6.2.8 for the commutation relations among positive root groups in pinned split reductive groups with semisimple-rank 2.

Returning to the proof of the Existence Theorem, we have reduced to the task of extending a split reductive  $\mathbf{Q}$ -group  $(G, T, M)$  to a split reductive  $\mathbf{Z}$ -group. We have also seen that it suffices to treat cases in which the root datum  $R = (M, \Phi, M^\vee, \Phi^\vee)$  for  $(G, T, M)$  is semisimple and simply connected (i.e.,  $\Phi$  spans  $M_{\mathbf{Q}}$  over  $\mathbf{Q}$  and  $\Phi^\vee$  spans  $M^\vee$  over  $\mathbf{Z}$ ), and we can assume moreover that the associated root system is irreducible.

To construct the required split semisimple  $\mathbf{Z}$ -group extending  $(G, T, M)$ , choose a base  $\Delta$  of  $\Phi$  and a pinning  $\{X_a\}_{a \in \Delta}$  of  $(G, T, M)$ . Using Proposition 6.3.4, extend this to a Chevalley system  $\{X_c\}_{c \in \Phi}$ . Since  $X_{-c} = \pm X_c^{-1}$  for all  $c$ , by using sign changes if necessary we may and do arrange that  $X_{-c} = X_c^{-1}$  for all  $c \in \Phi$ .

Fix an enumeration  $\{a_1, \dots, a_r\}$  of  $\Delta$ . Use this to define the lexicographical ordering on  $M_{\mathbf{Q}}$ , so we get an ordering  $\{c_0, \dots, c_m\}$  of  $\Phi^+$ . Identify the unipotent radical  $U_+ := U_{\Phi^+}$  of the Borel subgroup  $B = T \ltimes U_+$  corresponding to  $\Phi^+$  with a direct product (as  $\mathbf{Q}$ -schemes) of the positive root groups  $U_c$  via the chosen ordering  $\{c_j\}$  on  $\Phi^+$ . Let  $U_- = U_{-\Phi^+}$ .

The coroots in  $\Delta^\vee$  are a  $\mathbf{Z}$ -basis for the cocharacter group  $M^\vee$  of  $R$ , due to  $R$  being simply connected, so we get an isomorphism  $\mathbf{G}_m^\Delta \simeq T$  over  $\mathbf{Q}$  via  $(t_a)_{a \in \Delta} \mapsto \prod a^\vee(t_a)$ . For each  $c \in \Phi$ , use  $X_c$  to identify  $U_c$  with  $\mathbf{G}_a$  via  $p_c : x \mapsto \exp_c(xX_c)$ , so the open cell  $\Omega = U_- \times T \times U_+$  in the pinned split group  $(G, T, M, \{X_a\}_{a \in \Delta})$  over  $\mathbf{Q}$  is identified with a product of  $\mathbf{G}_m$ 's and  $\mathbf{G}_a$ 's as a  $\mathbf{Q}$ -scheme:

$$(6.3.2) \quad \prod_{j=-m}^0 \mathbf{G}_a \times \prod_{a \in \Delta} \mathbf{G}_m \times \prod_{i=1}^m \mathbf{G}_a \simeq U_- \times T \times U_+ = \Omega$$

via

$$((x'_j), (t_a)_{a \in \Delta}, (x_i)) \mapsto \prod_{j=-m}^0 p_{-c_{-j}}(-x'_j) \cdot \prod_{a \in \Delta} a^\vee(t_a) \cdot \prod_{i=0}^m p_{c_i}(x'_i)$$

in which the product description for  $U_+$  uses the ordering on  $\Phi^+$  and the one for  $U_-$  uses the ordering on  $-\Phi^+$ . (The specific choice of ordering for the product description of  $U_\pm$  will eventually turn out not to matter, but we need to make some definite choice at the outset.)

**Lemma 6.3.6.** *The  $\mathbf{Q}$ -scheme isomorphisms  $\prod_{j=-m}^0 \mathbf{G}_a \simeq U_-$  and  $\prod_{i=0}^m \mathbf{G}_a \simeq U_+$  defined above carry the  $\mathbf{Q}$ -group structures on  $U_\pm$  over to  $\mathbf{Q}$ -group structures on  $\mathbf{G}_a^{m+1}$  that are defined over  $\mathbf{Z}$ .*

*Proof.* Once the case of  $U_+$  is settled, the same conclusion holds for  $U_+$  using *any* ordering on  $\Phi^+$  to define the identification of  $U_+$  with the  $\mathbf{Q}$ -scheme  $\mathbf{G}_a^{m+1}$  (since the permutation of factors on the  $\mathbf{Q}$ -scheme  $\mathbf{G}_a^{m+1}$  is an automorphism defined over  $\mathbf{Z}$ ). The element  $n = n_{a_1} \cdots n_{a_r} \in N_G(T)(\mathbf{Q})$  represents the long element  $w = s_{a_1} \cdots s_{a_r} \in W(\Phi)$  relative to  $\Delta$ , so  $n$ -conjugation carries  $U_+$  to  $U_-$  but the bijection  $\Phi^+ \simeq -\Phi^+$  defined by the  $w$ -action need not carry the ordering on  $\Phi^+$  to an easily described ordering on  $-\Phi^+$  (since the bijection

$\Delta \simeq -\Delta$  may be hard to understand). Regardless, since  $\{X_c\}$  is a Chevalley system, so  $np_c(x)n^{-1} = \exp_{w(c)}(\text{Ad}_G(n)(xX_c)) = p_{w(c)}(\pm x)$  for some universal sign (depending only on  $n$ ,  $c$ , and the choice of Chevalley system), it follows that the result for  $U_-$  is a formal consequence of the result for  $U_+$ . Hence, we may and do now focus our attention on the case of  $U_+$ .

Consider the  $\mathbf{Q}$ -scheme isomorphism  $U_+ \simeq \mathbf{G}_a^{m+1}$  as defined in (6.3.2); we use this to equip  $U_+$  with a  $\mathbf{Z}$ -group structure. Clearly the identity section of  $U_+$  is defined over  $\mathbf{Z}$ , and inversion on  $U_+$  corresponds to reversing the order of multiplication and replacing each  $p_c(x)$  with  $p_c(-x)$ , so it suffices to check that the multiplication law on  $U_+$  is defined over  $\mathbf{Z}$  (as then the inversion on  $U_+$  is defined over  $\mathbf{Z}$ , and all of the group scheme diagrams commute over  $\mathbf{Z}$  since they commute over  $\mathbf{Q}$ ). In other words, we can focus on the multiplication law and not dwell on inversion.

The multiplication description of  $U_+$  as a product of root groups over  $\mathbf{Q}$  is defined relative to the lexicographical order on  $\Phi^+$  using some ordering of  $\Delta$ . Thus, for each  $1 \leq i \leq m$ , the product  $U_{\geq c_i} := U_{c_i} \cdots U_{c_m}$  is a closed  $\mathbf{Q}$ -subgroup of  $U$  normalized by  $U_{c_{i-1}}$  (as we saw in the proof of Lemma 6.2.3) and this computes the direct product structure that we have built into the  $\mathbf{Z}$ -structure. Hence, it suffices to prove by descending induction on  $i$  that the  $\mathbf{Q}$ -group law on each  $U_{\geq c_i}$  is defined over  $\mathbf{Z}$  (on the corresponding direct product of copies of  $\mathbf{G}_a$ ). The base case  $i = m$  is obvious, and likewise the  $\mathbf{Q}$ -group structure on each  $U_c = \mathbf{G}_a$  is visibly defined over  $\mathbf{Z}$ , so to carry out the induction it is enough to check that the normalization action by  $U_{c_{i-1}}$  on  $U_{\geq c_i}$  is defined over  $\mathbf{Z}$ . We can also assume that the semisimple-rank is at least 2 (or else there is nothing to do).

More generally, for  $c, c' \in \Phi^+$  with  $c < c'$ , consider the conjugation action by  $p_c(x)$  on  $p_{c'}(x')$ , assuming that the group law on  $U_{\geq c'}$  is already known to be defined over  $\mathbf{Z}$ . By (5.1.5) we have

$$(6.3.3) \quad p_c(x)p_{c'}(x')p_c(x)^{-1} = \prod_{i \geq 0, j > 0} p_{ic+jc'}(C_{i,j,c,c'}x^i x'^j)$$

where the product (in  $U_+$ ) is taken relative to the ordering on  $\Phi^+$  and the coefficients  $C_{i,j,c,c'}$  lie in  $\mathbf{Q}$ . This product lies in  $U_{\geq c'}$ , and it suffices to prove that  $C_{i,j,c,c'} \in \mathbf{Z}$  for all  $(i, j)$ . The first term on the right in (6.3.3) is  $p_{c'}(C_{0,1,c,c'}x')$  because  $c' < ic + jc'$  in  $\Phi^+$  for all  $i, j \geq 1$ , and (5.1.5) implies that  $C_{0,1,c,c'} = 1$ . Hence, our problem really concerns the commutator  $p_{c'}(-x')p_c(x)p_{c'}(-x')^{-1}p_c(x)^{-1}$ . We may now replace  $G$  with  $Z_G(T_{c,c'})$  to reduce to the case of semisimple-rank 2 (keeping in mind that the ordering on  $\Phi^+$  is immaterial once the full result is proved).

Now consider the case of pinned split groups of semisimple-rank 2. The Chevalley system extending a pinning is unique up to signs, so we can use whatever Chevalley system we like that extends an initial choice of pinning. By Lemma 6.2.8, when using the lexicographical ordering on  $\Phi^+$  relative to *some* choice of enumeration of  $\Delta$ , the structure constants in the commutation relations for the positive root groups are in  $\mathbf{Z}$ . This establishes the result in

the semisimple-rank 2 case for some choice of enumeration on  $\Delta$ , and it is sufficient to prove the result for one such choice.  $\square$

The evident  $\mathbf{Z}$ -scheme  $\Omega_{\mathbf{Z}}$  extending the left side of (6.3.2) has generic fiber identified with  $\Omega$ , and it is a direct product scheme  $\mathcal{U}_- \times \mathcal{T} \times \mathcal{U}_+$  with  $\mathcal{T} = \mathbf{G}_m^{\Delta}$  a split  $\mathbf{Z}$ -torus having cocharacter group  $\mathbf{Z}\Phi^{\vee} = M^{\vee}$  and  $\mathcal{U}_{\pm}$  a  $\mathbf{Z}$ -group extending  $U_{\pm}$  (by Lemma 6.3.2). For  $c \in \Phi^+$  let  $\tilde{p}_c : \mathbf{G}_a \rightarrow \mathcal{U}_+$  be the evident inclusion extending  $p_c$  over  $\mathbf{Q}$ , and likewise using  $\mathcal{U}_-$  for  $c \in -\Phi^+$ , so each  $\tilde{p}_c$  is a closed immersion of  $\mathbf{Z}$ -groups (as we can check the homomorphism property over  $\mathbf{Q}$ ); let  $\mathcal{U}_c$  denote its image. By construction the  $\mathbf{Z}$ -group  $\mathcal{U}_+$  is directly spanned in *some* order by the  $\mathcal{U}_c$ 's for  $c \in \Phi^+$ , and similarly for  $\mathcal{U}_-$  using  $c \in -\Phi^+$ . Likewise, the  $T$ -action on  $\Omega$  extends to a  $\mathcal{T}$ -action on  $\Omega_{\mathbf{Z}}$  that normalizes  $\mathcal{U}_{\pm}$ , as this amounts to some factorization assertions for flat closed subschemes that can be checked over  $\mathbf{Q}$ ; more explicitly,  $a^{\vee}(t)\tilde{p}_c(x)a^{\vee}(t)^{-1} = \tilde{p}_c(t^{(c, a^{\vee})}x)$  over  $\mathbf{Z}$  since this holds over  $\mathbf{Q}$ .

**Lemma 6.3.7.** *The  $\mathbf{Z}$ -group  $\mathcal{U}_+$  is directly spanned in any order by the  $\mathcal{U}_c$ 's for  $c \in \Phi^+$ . Likewise,  $\mathcal{U}_-$  is directly spanned in any order by the  $\mathcal{U}_c$ 's for  $c \in -\Phi^+$ .*

This result is [SGA3, XXV, 2.5], and the proof there uses a theory of Lazard from [BIBLE, 13.1] for torus actions on unipotent groups. We replace that with a related result proved in [CGP, 3.3] by the dynamic method.

*Proof.* The cases of  $\mathcal{U}_+$  and  $\mathcal{U}_-$  are identical, so we just discuss the case of  $\mathcal{U}_+$ . We have to show that for any enumeration of  $\Phi^+$ , the resulting multiplication map  $\prod_{c \in \Phi^+} \mathcal{U}_c \rightarrow \mathcal{U}$  between  $\mathbf{Z}$ -schemes is an isomorphism. It suffices to check the isomorphism result on geometric fibers over  $\text{Spec } \mathbf{Z}$ , so we are reduced to the following general claim over an algebraically closed field  $k$ . Let  $U$  be a unipotent smooth connected affine  $k$ -group directly spanned in some order by unipotent smooth closed subgroups  $U_i$ . Assume there is given an action on  $U$  by a torus  $T$  normalizing each  $U_i$  such that the  $T$ -action on each  $\text{Lie}(U_i)$  is through a nontrivial character  $a_i$  with  $a_i \neq a_{i'}$  for all  $i \neq i'$ . Then we claim that  $U$  is directly spanned in any order by the  $U_i$ 's. This is a special case of a general criterion in [CGP, 3.3.11] (proved by the dynamic method) for a unipotent smooth affine  $k$ -group to be directly spanned by appropriate smooth closed subgroups in the presence of a sufficiently nontrivial torus action.  $\square$

We conclude from Lemma 6.3.7 that the smooth affine  $\mathbf{Z}$ -groups  $\mathcal{U}_{\pm}$  extending  $U_{\pm}$  can be defined using arbitrary orderings on  $\Phi^+$  and  $-\Phi^+$ . Define the ‘‘identity section’’  $\tilde{e} \in \Omega_{\mathbf{Z}}(\mathbf{Z})$  to correspond to the direct product of the identity sections of  $\mathcal{U}_{\pm}$  and  $\mathcal{T}$  (so this extends the identity section  $e \in \Omega(\mathbf{Q}) \subset G(\mathbf{Q})$ ).

The  $\mathbf{Q}$ -group structure on  $G$  defines a birational group law on  $\Omega$ , and we seek to extend it to a  $\mathbf{Z}$ -birational group law on  $\Omega_{\mathbf{Z}}$  in a manner that interacts well with the  $\mathbf{Z}$ -groups  $\mathcal{T}$ ,  $\mathcal{U}_+$ , and  $\mathcal{U}_-$  (so that we can control the  $\mathbf{Z}$ -group that will be constructed from  $\Omega_{\mathbf{Z}}$  via Weil's theory of birational group laws). This proceeds in several steps. Guided by the Bruhat decomposition of  $G(\overline{\mathbf{Q}})$ , the first step is to consider the effect on  $\Omega$  by  $n_a$ -conjugation for all  $a \in \Delta$ . For any  $n \in N_G(T)(\mathbf{Q})$ , we have  $nU_cn^{-1} = U_{w(c)}$  for  $w \in W(\Phi)$  represented by  $n$ . In

the special case  $n = n_a$  for  $a \in \Delta$ , we have  $w(c) = s_a(c) \in \Phi^+$  for all  $c \in \Phi^+ - \{a\}$ , whereas  $w(a) = -a$ . Also,  $n_a$  normalizes  $T$ .

**Lemma 6.3.8.** *For each  $a \in \Delta$ , there exists an open subscheme  $\mathcal{V}_a \subset \Omega_{\mathbf{Z}}$  containing  $\mathcal{T}$  and every  $\mathcal{U}_c$  ( $c \in \Phi$ ) such that the automorphism  $g \mapsto n_a g n_a^{-1}$  of  $G$  carries  $(\mathcal{V}_a)_{\mathbf{Q}}$  into  $\Omega$ . Moreover,  $\mathcal{V}_a$  can be chosen so that the resulting map  $(\mathcal{V}_a)_{\mathbf{Q}} \rightarrow \Omega$  extends to a  $\mathbf{Z}$ -morphism  $f_a : \mathcal{V}_a \rightarrow \Omega_{\mathbf{Z}}$  restricting to an automorphism of the  $\mathbf{Z}$ -group  $\mathcal{T}$  and carrying the  $\mathbf{Z}$ -group  $\mathcal{U}_c$  isomorphically onto  $\mathcal{U}_{s_a(c)}$  for all  $c \in \Phi$ . In particular,  $\tilde{e}$  factors through  $\mathcal{V}_a$  and  $f_a(\tilde{e}) = \tilde{e}$ .*

This result in [SGA3, XXV, 2.7].

*Proof.* For  $c \in \Phi$ , the isomorphism  $U_c \simeq U_{s_a(c)}$  defined by  $n_a$ -conjugation is given by

$$p_c(x) \mapsto p_{s_a(c)}(\text{Ad}_G(n_a)(xX_c)) = p_{s_a(c)}(\pm xX_{s_a(c)})$$

since  $\{X_c\}_{c \in \Phi}$  is a Chevalley system. Hence, this visibly extends to a  $\mathbf{Z}$ -group isomorphism  $\mathcal{U}_c \simeq \mathcal{U}_{s_a(c)}$ . Likewise, on the  $\mathbf{Q}$ -group  $T = \mathbf{G}_m^{\Delta} = D_{\mathbf{Q}}(M)$ , the effect of  $n_a$ -conjugation is given by the  $\mathbf{Z}$ -group automorphism of  $\mathcal{T} = \mathbf{G}_m^{\Delta} = D_{\mathbf{Z}}(M)$  induced by  $s_a$ .

By Lemma 6.3.7, the  $\mathbf{Z}$ -group  $\mathcal{U}_+$  is directly spanned in any order by the  $\mathcal{U}_c$  for  $c \in \Phi^+$  and the  $\mathbf{Z}$ -group  $\mathcal{U}_-$  is directly spanned in any order by the  $\mathcal{U}_c$  for  $c \in -\Phi^+$ . Thus,

$$(6.3.4) \quad \Omega_{\mathbf{Z}} = \prod_{c \in \Phi^+ - \{a\}} \mathcal{U}_{-c} \times \mathcal{U}_{-a} \times \mathcal{T} \times \mathcal{U}_a \times \prod_{c \in \Phi^+ - \{a\}} \mathcal{U}_c$$

using some fixed choice of enumeration of  $\Phi^+ - \{a\}$  in both products. Finally, since we arranged that  $X_{-c} = X_c^{-1}$  for all  $c \in \Phi$ ,  $n_a$ -conjugation swaps  $U_a$  and  $U_{-a}$  via negation on the standard coordinate of  $\mathbf{G}_a$  relative to the parameterizations  $p_a$  and  $p_{-a}$  (Corollary 5.1.9(3)). Thus, using the chosen enumeration of  $\Phi^+ - \{a\}$  to define the order of multiplication,  $n_a$ -conjugation on  $\Omega$  carries

$$\prod_{c \in \Phi^+ - \{a\}} p_{-c}(x_{-c}) \cdot p_{-a}(x') \cdot t \cdot p_a(x) \cdot \prod_{c \in \Phi^+ - \{a\}} p_c(x_c)$$

to

$$\prod_{c \in \Phi^+ - \{a\}} p_{-s_a(c)}(\pm x_{-c}) \cdot p_a(-x) \cdot D_{\mathbf{Z}}(s_a)(t) \cdot p_{-a}(-x') \cdot \prod_{c \in \Phi^+ - \{a\}} p_{s_a(c)}(\pm x_c)$$

for some universal signs. The terms  $p_a(-x)$  and  $p_{-a}(-x')$  appear in the “wrong” places; we want to swap their positions (at the cost of changing the  $\mathcal{T}$ -component) so that we can make things work over  $\mathbf{Z}$ .

For  $t' := D_{\mathbf{Z}}(s_a)(t)$  we have  $p_a(-x)t'p_{-a}(-x') = p_a(-x)p_{-a}(-x'/a(t))t'$ . Since  $X_{-a} = X_a^{-1}$ , Theorem 4.2.6(1) gives that  $p_a(-x)p_{-a}(-x'/a(t))$  lies in  $U_{-a} \times T \times U_a$  if and only if  $1 + xx'a(t)^{-1}$  is a unit, or equivalently  $a(t) + xx'$  is a unit. Under this unit hypothesis, since  $a^{\vee}(1 + xx'/a(t)) = t^{-2}a^{\vee}(a(t) + xx')$  we deduce from (4.2.1) that

$$p_a(-x)p_{-a}(-x'/a(t)) = p_{-a}\left(\frac{-x'}{a(t) + xx'}\right) (t^{-2}a^{\vee}(a(t) + xx')) p_a\left(\frac{-x}{a(t) + xx'}\right).$$

Using the  $\mathbf{Z}$ -group law on  $\mathcal{U}_\pm$ , we conclude that for the open subscheme  $\mathcal{V}_a \subset \Omega_{\mathbf{Z}}$  defined by the unit condition  $a(t) + x_a x_{-a} \in \mathbf{G}_m$  (using the coordinatization on  $\Omega_{\mathbf{Z}}$  relative to the product decomposition (6.3.4)),  $n_a$ -conjugation on  $\Omega$  carries  $(\mathcal{V}_a)_{\mathbf{Q}}$  into  $\Omega$  and the resulting map  $(\mathcal{V}_a)_{\mathbf{Q}} \rightarrow \Omega$  extends to a  $\mathbf{Z}$ -morphism  $f_a : \mathcal{V}_a \rightarrow \Omega_{\mathbf{Z}}$ .

By definition it is clear that  $\mathcal{V}_a$  contains  $\mathcal{T}$  and every  $\mathcal{U}_c$ . The map  $f_a$  carries  $\mathcal{T}$  into itself and  $\mathcal{U}_c$  into  $\mathcal{U}_{s_a(c)}$  for all  $c \in \Phi$  because such factorization through flat closed subschemes can be checked over  $\mathbf{Q}$  (where it is obvious). By the same argument, the induced map  $\mathcal{T} \rightarrow \mathcal{T}$  is a  $\mathbf{Z}$ -group involution and the induced map  $\mathcal{U}_c \rightarrow \mathcal{U}_{s_a(c)}$  is a  $\mathbf{Z}$ -homomorphism with inverse given by the induced map  $\mathcal{U}_{s_a(c)} \rightarrow \mathcal{U}_{s_a^2(c)} = \mathcal{U}_c$ .  $\square$

In addition to conjugation by the representatives  $n_a$  for the simple reflections  $s_a \in W(\Phi)$ , we need to address conjugation by some representative  $n$  for the long Weyl element  $w \in W(\Phi)$  relative to  $\Delta$ . Explicitly,  $w = \prod_{a \in \Delta} s_a$  in  $W(\Phi)$  using multiplication taken in the order of *any* enumeration of  $\Delta$ , but the product  $\prod_{a \in \Delta} n_a \in N_G(T)(\mathbf{Q})$  generally depends on the choice of enumeration of  $\Delta$  (i.e., if we change the order of multiplication then the product in  $N_G(T)(\mathbf{Q})$  changes by a possibly nontrivial  $T(\mathbf{Q})$ -multiplication). For our purposes it is only necessary to work with *some* enumeration of  $\Delta$ , so we shall use the enumeration chosen earlier to define the lexicographical ordering on  $M_{\mathbf{Q}}$  (which defined our ordering on  $\Phi^+$  and  $-\Phi^+$ ).

Let  $n$  denote the resulting product  $\prod_{a \in \Delta} n_a \in N_G(T)(\mathbf{Q})$ , so  $n$ -conjugation on  $G$  restricts to an automorphism of  $T$  and swaps  $U_+$  and  $U_-$ .

**Lemma 6.3.9.** *For  $n$  as defined above, there are open subschemes  $\mathcal{V}, \mathcal{V}' \subset \Omega_{\mathbf{Z}}$  containing  $\mathcal{U}_\pm$  and  $\mathcal{T}$  such that the automorphism  $g \mapsto ngn^{-1}$  of  $G$  carries  $\mathcal{V}_{\mathbf{Q}}$  and  $\mathcal{V}'_{\mathbf{Q}}$  into  $\Omega$  and the resulting maps  $\mathcal{V}_{\mathbf{Q}}, \mathcal{V}'_{\mathbf{Q}} \rightrightarrows \Omega$  extend to  $\mathbf{Z}$ -morphisms*

$$f : \mathcal{V} \rightarrow \Omega_{\mathbf{Z}}, \quad f' : \mathcal{V}' \rightarrow \Omega_{\mathbf{Z}}$$

*satisfying the following properties:*

- (1)  $f|_{\mathcal{U}_\pm}$  factors through a  $\mathbf{Z}$ -group morphism onto  $\mathcal{U}_\mp$ , and similarly for  $f'$ ;
- (2)  $f$  and  $f'$  restrict to  $\mathbf{Z}$ -group endomorphisms of  $\mathcal{T}$ ;
- (3)  $f' \circ f|_{f^{-1}(\mathcal{V}')} : \mathcal{V}' \rightarrow \Omega_{\mathbf{Z}}$  is the canonical open immersion.

*In particular,  $f^{-1}(\mathcal{V}')$  is fiberwise dense in  $\mathcal{V}$  and  $\tilde{e}$  factors through  $\mathcal{V}$  and  $\mathcal{V}'$ , with  $f(\tilde{e}) = \tilde{e}$  and  $f'(\tilde{e}) = \tilde{e}$ .*

This result is [SGA3, XXV, 2.8] (except that we include some  $\mathbf{Z}$ -group compatibilities in the statement).

*Proof.* By  $\mathbf{Z}$ -flatness considerations and the evident properties on the  $\mathbf{Q}$ -fiber, once we find  $\mathcal{V}$  and  $\mathcal{V}'$  containing  $\mathcal{T}$  and  $\mathcal{U}_\pm$  so that  $n$ -conjugation carries their  $\mathbf{Q}$ -fibers into  $\Omega$  and the resulting maps  $\mathcal{V}_{\mathbf{Q}} \rightarrow \Omega$  and  $\mathcal{V}'_{\mathbf{Q}} \rightarrow \Omega$  extend to  $\mathbf{Z}$ -morphisms, the additional properties in (1), (2), and (3) are immediate. The actual construction of  $\mathcal{V}$  and  $\mathcal{V}'$  involves an inductive argument on word length in  $W(\Phi)$ , with Lemma 6.3.8 used to carry out the induction. The

identity that makes it work is  $n_a^4 = 1$  in  $N_G(T)(\mathbf{Q})$  for all  $a \in \Delta$  (since  $n_a^2 = a^\vee(-1)$ ). We refer the reader to [SGA3, XXV, 2.8] for the details.  $\square$

Now we bring in birational group laws. (See [SGA3, XVIII] or [BLR, Ch. 5] for general discussions of this concept.) By artful use of  $f$  and  $f'$  from Lemma 6.3.9, one can put a  $\mathbf{Z}$ -birational group law on  $\Omega_{\mathbf{Z}}$ :

**Proposition 6.3.10.** *There are open subschemes  $\mathcal{V}_1 \subset \Omega_{\mathbf{Z}} \times \Omega_{\mathbf{Z}}$  and  $\mathcal{V}_2 \subset \Omega_{\mathbf{Z}}$  such that:*

- (1)  $\mathcal{U}_{\pm}, \mathcal{T} \subset \mathcal{V}_2$  and  $\mathcal{U}_+ \times \mathcal{U}_+, \mathcal{U}_- \times \mathcal{U}_-, \mathcal{T} \times \mathcal{T}, \Omega_{\mathbf{Z}} \times \{\tilde{e}\}, \{\tilde{e}\} \times \Omega_{\mathbf{Z}} \subset \mathcal{V}_1$ ,
- (2) for the generic fibers  $V_j := (\mathcal{V}_j)_{\mathbf{Q}}$ , the multiplication morphism  $m : G \times G \rightarrow G$  carries  $V_1$  into  $\Omega$  and inversion  $\iota : G \simeq G$  carries  $V_2$  into  $\Omega$ ,
- (3) the induced maps  $m : V_1 \rightarrow \Omega$  and  $\iota : V_2 \rightarrow \Omega$  extend to  $\mathbf{Z}$ -morphisms  $m_{\mathbf{Z}} : \mathcal{V}_2 \rightarrow \Omega_{\mathbf{Z}}$  and  $\iota_{\mathbf{Z}} : \mathcal{V}_1 \rightarrow \Omega_{\mathbf{Z}}$ .

Moreover, the  $\mathbf{Q}$ -birational group law  $(\Omega, m)$  extends to a  $\mathbf{Z}$ -birational group law  $(\Omega_{\mathbf{Z}}, m_{\mathbf{Z}})$  with inverse  $\iota_{\mathbf{Z}}$  and identity  $\tilde{e}$  that restricts to the  $\mathbf{Z}$ -group laws on  $\mathcal{U}_+, \mathcal{U}_-$ , and  $\mathcal{T}$ .

This is [SGA3, XXV, 2.9], except that the assertions concerning containments of  $\mathcal{U}_{\pm}$  and  $\mathcal{T}$  as well as compatibility with their  $\mathbf{Z}$ -group structures are not mentioned there (but are immediate from inspecting the construction there and using known identities for maps between the  $\mathbf{Q}$ -fibers).

*Proof.* The motivation for the construction of  $\mathcal{V}_1$  and  $\mathcal{V}_2$  can be seen by considering the special case  $G = \mathrm{SL}_2$ . In that special case,  $\Omega_{\mathbf{Z}} = \mathbf{G}_a \times \mathbf{G}_m \times \mathbf{G}_a$  with the action of  $\mathcal{T} = \mathbf{G}_m$  on  $\mathcal{U}_{\pm} = \mathbf{G}_a$  given by  $t.x = t^{\pm 2}x$ . The Zariski-open condition  $t^2 + xx' \in \mathbf{G}_m$  defines a suitable  $\mathcal{V}_2 \subset \Omega_{\mathbf{Z}}$  for exactly the same reason that the condition  $a(t) + xx' \in \mathbf{G}_m$  arose in the proof of Lemma 6.3.8. To find a Zariski-open condition on points

$$((x'_2, t_2, x_2), (x'_1, t_1, x_1)) \in \Omega_{\mathbf{Z}} \times \Omega_{\mathbf{Z}}$$

to define  $\mathcal{V}_1$ , we use Theorem 4.2.6(1) to see that the condition  $1 + x_2x'_1 \in \mathbf{G}_m$  does the job.

The construction of  $\mathcal{V}_1$  and  $\mathcal{V}_2$  in general is given in [SGA3, XXV, 2.9], modeled on the case of  $\mathrm{SL}_2$ . One uses Lemma 6.3.9 to overcome the absence in the general case of formulas as explicit as in the case of  $\mathrm{SL}_2$ . The  $\mathbf{Z}$ -birational group law property amounts to an associativity identity that can be checked on the  $\mathbf{Q}$ -fiber, and likewise for the inversion and identity assertions for this birational group law.  $\square$

Now we are in position to apply results that promote birational group laws to group schemes. For a scheme  $S$ , if  $X \rightarrow S$  is a smooth surjective separated map of finite presentation equipped with an  $S$ -birational group law  $m$ , a *solution* is a smooth separated  $S$ -group  $(X', m')$  of finite presentation equipped with an  $S$ -birational isomorphism between  $X$  and  $X'$  that is compatible with  $m$  and  $m'$  (i.e., an  $S$ -isomorphism  $f : \Omega \simeq \Omega'$  between fiberwise dense open subschemes  $\Omega \subset X$  and  $\Omega' \subset X'$  such that  $m' \circ (f \times f) = (f \times f) \circ m$  as  $S$ -rational maps from  $X \times_S X$  to  $X'$ ). A preliminary result in the theory of  $S$ -birational group laws is that a solution is unique up to unique  $S$ -isomorphism (not just  $S$ -birationally) if it exists [BLR, 5.1/3]. This is proved by translation arguments, using that a smooth surjective map

has many étale-local sections. We emphasize that it is *not* required that  $X$  is open in  $X'$ ; i.e., we allow for the possibility that only some fiberwise dense open subscheme of  $X$  appears as an open subscheme of  $X'$  (compatibly with the  $S$ -birational group laws).

For practical applications, it is useful to have a criterion which ensures that a given  $S$ -birational group law  $(X, m)$  occurs as an open subscheme of an  $S$ -group, with no shrinking of  $X$  required. To motivate the criterion, we now consider  $S$ -birational group laws that arise from fiberwise dense open subschemes of  $S$ -groups and we identify some properties that such  $S$ -birational group laws must satisfy.

**Example 6.3.11.** Let  $G \rightarrow S$  be a smooth separated  $S$ -group of finite presentation, and  $X \subset G$  a fiberwise dense open subscheme. Then  $U = m_G^{-1}(X) \cap (X \times_S X)$  is the open domain of definition of the associated  $S$ -birational group law on  $X$ , and it is  $X$ -dense in  $X \times_S X$  in the sense that  $U$  is fiberwise dense relative to both projections  $X \times_S X \rightrightarrows X$ . Indeed, for any geometric point  $\bar{s}$  of  $S$  and  $\bar{x} \in X(\bar{s})$ , the  $\bar{x}$ -fibers of  $U_{\bar{s}}$  under the projections  $X_{\bar{s}} \times X_{\bar{s}} \rightrightarrows X_{\bar{s}}$  are the open overlaps  $X_{\bar{s}} \cap (\bar{x}^{-1} \cdot X_{\bar{s}})$  and  $X_{\bar{s}} \cap (X_{\bar{s}} \cdot \bar{x}^{-1})$  in  $G_{\bar{s}}$  that are dense in  $X_{\bar{s}}$ .

Moreover, the universal left and right “translation” maps  $U \rightrightarrows X \times_S X$  defined by  $u = (x_1, x_2) \mapsto (x_1, m(x_1, x_2)), (m(x_1, x_2), x_2)$  are open immersions with  $X$ -dense image because they are obtained by restriction to  $U \subset G \times_S G$  of the universal translation maps  $(g, g') \mapsto (gg', g'), (g, gg')$  that are automorphisms of the  $S$ -scheme  $G \times_S G$ .

Motivated by Example 6.3.11, an  $S$ -birational group law  $(X, m)$  is called *strict* if there exists an open subscheme  $\Omega$  of the domain of definition of  $m$  in  $X \times_S X$  such that  $\Omega$  is  $X$ -dense and the maps  $\Omega \rightrightarrows X \times_S X$  defined by  $(x, x') \mapsto (x, m(x, x'))$  and  $(x, x') \mapsto (m(x, x'), x')$  are open immersions whose respective images in  $X \times_S X$  are  $X$ -dense. (This is an equivalent formulation of the definition of a “group germ” in [SGA3, XVIII, 3.1].) We have just seen in Example 6.3.11 that strictness is a necessary condition for a “solution”  $(X', m')$  to an  $S$ -birational group law  $(X, m)$  not to require any shrinking of  $X$ ; i.e., it is necessary in order that  $X$  be open in a solution  $X'$  (as  $S$ -birational groups).

Remarkably, strictness is also sufficient for the existence of a solution  $(X', m')$  containing  $(X, m)$  as a fiberwise dense open subscheme. This deep result is essentially [SGA3, XVIII, 3.7, 3.13(iii)], except that  $X'$  is built there only as an fppf (and hence smooth) algebraic space group, resting on the special case of strictly henselian local  $S$  for which  $X'$  is built as a scheme. (The theory of algebraic spaces did not exist in 1963, so at that time the language of fppf sheaves was used; [Ar74, Cor. 6.3] provides the link with algebraic spaces.) The construction of this solution  $X'$  as a scheme is given in [BLR, 5.2/3] for the cases that  $S$  is the spectrum of a separably closed field or strictly henselian discrete valuation ring.

By [BLR, 6.6/1] the algebraic space  $X'$  is always a scheme, but we are only interested in the special case that  $S$  is Dedekind (namely,  $S = \text{Spec } \mathbf{Z}$ ). One can establish the result over such  $S$  without any (implicit) use of algebraic spaces as follows. First consider the “local” version:  $S = \text{Spec } R$  for  $R$  a field or discrete valuation ring. By using the known solution over a strict henselization of any such  $R$ , a solution can be built over  $R$  by descent arguments;

see [BLR, 6.5/2]. In particular, for a general Dedekind scheme  $S$  we get a solution over the generic points (though in the cases of interest over  $\mathbf{Z}$  we are even given a solution  $G$  over  $\mathbf{Q}$ ). This “spreads out” to a solution over a dense open subscheme  $V$  of  $S$ . By limit considerations, the general Dedekind case reduces to local versions of the problem at the finitely many closed points of  $S - V$ , which are instances of the settled local case that  $R$  is a discrete valuation ring.

To summarize, the  $\mathbf{Z}$ -birational group  $\Omega_{\mathbf{Z}}$  is fiberwise dense in a smooth finite type and separated  $\mathbf{Z}$ -group  $G_{\mathbf{Z}}$  once we verify:

**Proposition 6.3.12.** *The  $\mathbf{Z}$ -birational group law on  $\Omega_{\mathbf{Z}}$  is strict.*

*Proof.* The universal left and right “translation” morphisms  $\mathcal{V}_1 \rightrightarrows \Omega_{\mathbf{Z}} \times \Omega_{\mathbf{Z}}$  are maps between smooth separated  $\mathbf{Z}$ -schemes of finite type, and on the  $\mathbf{Q}$ -fibers they are open immersions since  $V_1$  is a non-empty open in the smooth connected  $\mathbf{Q}$ -group  $G$ . Thus, these maps are birational. We claim that these  $\mathbf{Z}$ -maps are étale on an open subscheme  $\mathcal{W}$  of  $\mathcal{V}_1$  that contains  $\{\tilde{e}\} \times \Omega_{\mathbf{Z}}$  and  $\Omega_{\mathbf{Z}} \times \{\tilde{e}\}$ . The case of the universal left “translation”  $(\omega, \omega') \mapsto (\omega, m_{\mathbf{Z}}(\omega, \omega'))$  will be treated, and right “translation” goes similarly.

In view of  $\mathbf{Z}$ -smoothness for the source and target, it suffices to check that the tangent maps at  $(x, \tilde{e}(s))$  and  $(\tilde{e}(s), x)$  are isomorphisms for every geometric point  $s$  of  $\text{Spec } \mathbf{Z}$  and closed point  $x \in \Omega_s := (\Omega_{\mathbf{Z}})_s$ . Restriction to the respective open neighborhoods

$$\mathcal{V}'_{1,s} = (\mathcal{V}_1)_s \cap (\Omega_s \times \{\tilde{e}(s)\}) \subset \Omega_s, \quad \mathcal{V}''_{1,s} = (\mathcal{V}_1)_s \cap (\{x\} \times \Omega_s) \subset \Omega_s$$

of  $x$  and  $\tilde{e}(s)$  in  $\Omega_s$  gives maps  $\mathcal{V}'_{1,s}, \mathcal{V}''_{1,s} \rightrightarrows \Omega_s \times \Omega_s$  that are respectively the identity map and “left translation” by  $x$  into the slice  $\{x\} \times \Omega_s$  (on an open domain containing  $\tilde{e}(s)$ ). Since  $\tilde{e}$  is an identity for the  $\mathbf{Z}$ -birational group law, this latter map at  $(x, \tilde{e}(s))$  is the canonical inclusion on the tangent space. Hence, by using the canonical decomposition

$$\text{Tan}_{(x, \tilde{e}(s))}((\Omega_{\mathbf{Z}} \times \Omega_{\mathbf{Z}})_s) = \text{Tan}_x(\Omega_s) \bigoplus \text{Tan}_{\tilde{e}(s)}(\Omega_s)$$

we deduce the isomorphism property for the tangent map at  $(x, \tilde{e}(s))$  since  $V \bigoplus V$  is the direct sum of the diagonal and  $\{0\} \bigoplus V$  for any vector space  $V$ . The same argument works at the points  $(\tilde{e}(s), x)$ .

Pick an open subscheme  $\mathcal{W} \subset \mathcal{V}_1$  containing  $\Omega_{\mathbf{Z}} \times \{\tilde{e}\}$  and  $\{\tilde{e}\} \times \Omega_{\mathbf{Z}}$  on which the universal left and right “translations” are étale maps to  $\Omega_{\mathbf{Z}} \times \Omega_{\mathbf{Z}}$ , so these define a pair of maps  $\mathcal{W} \rightrightarrows \Omega_{\mathbf{Z}} \times \Omega_{\mathbf{Z}}$  that are birational, separated, and quasi-finite. By Zariski’s Main Theorem [EGA, III<sub>1</sub>, 4.4.9], any birational, separated, and quasi-finite map between connected normal noetherian schemes is an open immersion.  $\square$

Since  $\Omega_{\mathbf{Z}}$  is now a fiberwise dense open subscheme of a unique smooth and separated  $\mathbf{Z}$ -group  $G_{\mathbf{Z}}$  of finite type (compatibly with  $\mathbf{Z}$ -birational group laws),  $G_{\mathbf{Z}} \rightarrow \text{Spec } \mathbf{Z}$  has connected fibers. By the uniqueness of solutions to birational group laws, the inclusion  $\Omega \hookrightarrow G_{\mathbf{Q}}$  extends to an isomorphism  $G \simeq G_{\mathbf{Q}}$ , so we can view  $G_{\mathbf{Z}}$  as a  $\mathbf{Z}$ -model for  $G$ . The section  $\tilde{e} \in \Omega_{\mathbf{Z}}(\mathbf{Z}) \subset G_{\mathbf{Z}}(\mathbf{Z})$  is the identity section, as this holds over  $\mathbf{Q}$  and equalities of maps between separated flat  $\mathbf{Z}$ -schemes can be checked over  $\mathbf{Q}$ . Likewise, the immersions

$\mathcal{T} \rightarrow G_{\mathbf{Z}}$  and  $\mathcal{U}_{\pm} \rightarrow G_{\mathbf{Z}}$  are  $\mathbf{Z}$ -homomorphisms, since we can check over  $\mathbf{Q}$ . We do not yet know if these immersions are closed immersions, nor if  $G_{\mathbf{Z}}$  is affine.

To prove that the smooth separated finite type group  $G_{\mathbf{Z}}$  is semisimple, we will first show that its geometric fibers over  $\text{Spec } \mathbf{Z}$  (which we know are connected) are semisimple, and then deduce from this that  $G_{\mathbf{Z}}$  is affine. The key to fibral semisimplicity is:

**Lemma 6.3.13.** *For every geometric point  $s$  of  $\text{Spec } \mathbf{Z}$  and every  $c \in \Phi$ , the fibral subgroups  $(\mathcal{U}_c)_s$  and  $(\mathcal{U}_{-c})_s$  in  $(G_{\mathbf{Z}})_s$  generate a subgroup that contains the nontrivial torus  $c_s^{\vee}(\mathbf{G}_m)$ . In particular, the subgroup of  $(G_{\mathbf{Z}})_s$  generated by the two unipotent subgroups  $(\mathcal{U}_{\pm c})_s = \mathbf{G}_a$  is not unipotent.*

*Proof.* Identify  $\mathcal{U}_{\pm c}$  with  $\mathbf{G}_a$  via  $\tilde{p}_{\pm c}$ . The calculations in the proof of Lemma 6.3.8 show that if  $x, x'$  are points of  $\mathbf{G}_a$  such that  $1 + xx' \in \mathbf{G}_m$  and  $(\tilde{p}_c(x), \tilde{p}_{-c}(x')) \in \mathcal{V}_1$  then

$$(6.3.5) \quad \tilde{p}_c(x)\tilde{p}_{-c}(x') = \tilde{p}_{-c}\left(\frac{x'}{1+xx'}\right)c^{\vee}(1+xx')\tilde{p}_c\left(\frac{x}{1+xx'}\right)$$

in the group law of  $G_{\mathbf{Z}}$ . The direct product scheme  $\mathcal{U}_c \times \mathcal{U}_{-c}$  inside  $\Omega_{\mathbf{Z}} \times \Omega_{\mathbf{Z}}$  meets the open neighborhood  $\mathcal{V}_1$  of  $(\tilde{e}, \tilde{e})$ , and imposing the additional condition “ $1 + xx' \in \mathbf{G}_m$ ” defines a fiberwise dense open subscheme on which (6.3.5) holds. Hence, by separatedness and smoothness considerations, it follows that (6.3.5) holds on the open subscheme of  $\mathcal{U}_c \times \mathcal{U}_{-c} = \mathbf{A}_{\mathbf{Z}}^2$  where  $1 + xx' \in \mathbf{G}_m$ . Now pass to  $s$ -fibers.  $\square$

The proof that  $G_{\mathbf{Z}}$  is affine will use some structural input on the fibers, so we first address the fibral structure:

**Proposition 6.3.14.** *For each geometric point  $s \in \text{Spec } \mathbf{Z}$ ,  $(G_{\mathbf{Z}})_s$  is semisimple and  $\mathcal{T}_s$  is a maximal torus. In particular, the fibers of  $G_{\mathbf{Z}}$  are affine.*

*Proof.* By construction,  $G_{\mathbf{Z}}$  contains the torus  $\mathcal{T} = D_{\mathbf{Z}}(M)$  as an  $S$ -subgroup (which we have not yet shown to be closed, as  $G_{\mathbf{Z}}$  is not yet shown to be affine, so we cannot apply Lemma B.1.3 and we wish to avoid using the deep Theorem 5.3.5). Consider the action by this torus on

$$\mathfrak{g} := \text{Lie}(G_{\mathbf{Z}}) = \text{Tan}_{\tilde{e}}(\Omega_{\mathbf{Z}}) = \text{Lie}(\mathcal{T}) \oplus \bigoplus_{c \in \Phi} \text{Lie}(\mathcal{U}_c).$$

These direct summands are stable under the adjoint action of  $\mathcal{T}$ , with  $\text{Lie}(\mathcal{T})$  centralized by the action and the line subbundle  $\text{Lie}(\mathcal{U}_c)$  acted upon through through character  $c \in \Phi \subset M - \{0\}$ . It follows that under the  $\mathcal{T}$ -action on  $\mathfrak{g}$ , the trivial weight space is the subbundle  $\text{Lie}(\mathcal{T})$  and the set of nontrivial weights consists of the “constant sections”  $c \in \Phi \subset M - \{0\}$ , with the  $c$ -weight space equal to  $\text{Lie}(\mathcal{U}_c)$  for  $c \in \Phi$ .

Now pass to fibers at a geometric point  $s$  of  $\text{Spec } \mathbf{Z}$ . The  $s$ -group  $G_s := (G_{\mathbf{Z}})_s$  contains the  $s$ -subgroups  $U_{c,s} := (\mathcal{U}_c)_s \simeq \mathbf{G}_a$  and  $T_s := \mathcal{T}_s = D_s(M)$  such that  $T_s$  normalizes each  $U_{c,s} = \mathbf{G}_a$  via the action  $t.x = c(t)x$ , so in the commutative quotient  $G_s^{\text{ab}} = G_s/\mathcal{D}(G_s)$  the image of  $U_{c,s}$  is trivial. Then (6.3.5) implies that for each  $c \in \Phi$  the image of  $c^{\vee}(\mathbf{G}_m)_s$  in this quotient is also trivial. But the subgroups  $U_{c,s}$  and  $c^{\vee}(\mathbf{G}_m)$  for  $c \in \Phi$  generate  $G_s$  (as  $\Omega_{\mathbf{Z}}$  is

a fiberwise dense open subscheme of  $G_{\mathbf{Z}}$  whose direct product decomposition as a  $\mathbf{Z}$ -scheme is defined by multiplication in the  $\mathbf{Z}$ -group law on  $G_{\mathbf{Z}}$ , so  $G_s/\mathcal{D}(G_s) = 1$ .

We have proved that  $G_s$  has no nontrivial commutative quotient modulo a normal smooth connected closed subgroup, and that  $G_s = \mathcal{D}(G_s)$ . It is a theorem of Chevalley [Chev60] (and Barsotti [Bar] and Rosenlicht [Ro, Thm. 16]) that every smooth connected group  $H$  over an algebraically closed field is an extension of an abelian variety by a connected linear algebraic group. (See [Con02] for a contemporary exposition of the proof.) In particular, if such a group  $H$  has no nontrivial commutative quotient modulo a normal smooth connected closed subgroup then  $H$  is a linear algebraic group! Hence,  $G_s$  is a connected linear algebraic group that is equal to its own derived subgroup. In particular, if it is reductive then it must be semisimple.

It remains to show that  $G_s$  is reductive and  $T_s$  is a maximal torus in  $G_s$ . The inclusion  $T_s \hookrightarrow Z_{G_s}(T_s)$  between *connected* smooth linear algebraic groups is an equality on Lie algebras, so it is an isomorphism. Hence,  $T_s$  is a maximal torus in  $G_s$ . To establish the reductivity, we first require a dynamic characterization of  $U_{c,s}$  in terms of the  $T_s$ -action on  $G_s$  for each  $c \in \Phi$ .

Fix  $c \in \Phi$ . For the codimension-1 subtorus  $T'_s = ((\ker c)_s)_{\text{red}}$  in  $T_s$ , the smooth connected centralizer  $Z_{G_s}(T'_s)$  has Lie algebra that is the  $T'_s$ -centralizer in  $\mathfrak{g}_s$ , which is the span of the weight spaces for the  $T_s$ -weights that are trivial on  $T'_s$ . Hence,  $\text{Lie}(Z_{G_s}(T'_s))$  is the span of  $\text{Lie}(T_s)$  and  $\text{Lie}(U_{\pm c,s})$  since  $\Phi$  is a reduced root system in  $M = X(T_s)$ . Consider the smooth connected unipotent subgroup  $U_{Z_{G_s}(T'_s)}(c^\vee)$ . This clearly has Lie algebra  $\text{Lie}(U_{c,s})$ , so the inclusion  $U_{c,s} \subset U_{Z_{G_s}(T'_s)}(c^\vee)$  between smooth connected linear algebraic groups is an equality on Lie algebras. In other words, we have established the “dynamic” characterization

$$(6.3.6) \quad U_{c,s} = U_{Z_{G_s}(T'_s)}(c^\vee)$$

in terms of the  $T_s$ -action on  $G_s$ .

Now suppose that  $U := \mathcal{R}_u(G_s)$  is nontrivial. By normality in  $G_s$ , the nonzero subspace  $\text{Lie}(U)$  in  $\mathfrak{g}_s$  has a weight space decomposition relative to the  $T_s$ -action. Since  $U \cap T_s = 1$  (due to the unipotence of  $U$ ), passing to Lie algebras gives  $\text{Lie}(U) \cap \text{Lie}(T_s) = 0$ . Hence, all  $T_s$ -weights that occur on  $\text{Lie}(U)$  are nontrivial, so for some  $c_0 \in \Phi$  the 1-dimensional  $c_0$ -weight space  $\text{Lie}(U_{c_0,s})$  is contained in  $\text{Lie}(U)$ .

The intersection  $U \cap Z_{G_s}(T'_s)$  is smooth and connected since

$$T'_s \times (U \cap Z_{G_s}(T'_s)) = Z_{T'_s \times U}(T'_s),$$

so by applying Proposition 4.1.10(1) to  $H = U \cap Z_{G_s}(T'_s)$  equipped with its action by  $c_0^\vee(\mathbf{G}_m)$  we conclude via the dynamic description (6.3.6) of  $U_{c_0,s}$  that  $U \cap U_{c_0,s}$  is smooth and connected. But  $\text{Lie}(U \cap U_{c_0,s}) = \text{Lie}(U) \cap \text{Lie}(U_{c_0,s}) = \text{Lie}(U_{c_0,s})$ , so the inclusion  $U \cap U_{c_0,s} \hookrightarrow U_{c_0,s}$  between smooth connected groups is an equality on Lie algebras. This forces  $U \cap U_{c_0,s} = U_{c_0,s}$ , so  $U_{c_0,s} \subset U$  inside  $G_s$ . Thus, the subgroup  $U_{-c_0,s} = n_{c_0} U_{c_0,s} n_{c_0}^{-1}$  is contained in  $U$  by normality of  $U$  in  $G_s$ . But the subgroups  $U_{c_0,s}$  and  $U_{-c_0,s}$  of  $U$  generate

a non-unipotent subgroup of  $G_s$  (by Lemma 6.3.13), which is absurd since  $U$  is unipotent. Hence,  $U = 1$ .  $\square$

**Proposition 6.3.15.** *The  $\mathbf{Z}$ -group  $G_{\mathbf{Z}}$  is affine with the split torus  $\mathcal{T}$  as a maximal torus.*

*Proof.* Consider the adjoint action  $\mathrm{Ad}_{G_{\mathbf{Z}}} : G_{\mathbf{Z}} \rightarrow \mathrm{GL}(\mathfrak{g})$ . The fibral semisimplicity in Proposition 6.3.14 implies that  $\ker \mathrm{Ad}_{G_{\mathbf{Z}}}$  has finite geometric fibers (by the classical theory), so  $\mathrm{Ad}_{G_{\mathbf{Z}}}$  is quasi-finite. But it is a general fact that any quasi-finite homomorphism between separated flat groups of finite type over a Dedekind base is necessarily an affine morphism; see [SGA3, XXV, §4] for the proof (which is based on a clever translation argument). Hence,  $\mathrm{Ad}_{G_{\mathbf{Z}}}$  is an affine morphism, so  $G_{\mathbf{Z}}$  inherits affineness from  $\mathrm{GL}(\mathfrak{g})$ .

Now that  $G_{\mathbf{Z}}$  is affine, it follows that  $\mathcal{T}$  is a closed subgroup of  $G_{\mathbf{Z}}$  (as for any multiplicative type subgroup of a smooth affine group (Lemma B.1.3)). The maximality of this torus in geometric fibers was proved in Proposition 6.3.14.  $\square$

Since the  $\mathbf{Q}$ -fiber of  $(G_{\mathbf{Z}}, \mathcal{T}, M)$  is the triple  $(G, T, M)$  whose root datum is the original  $R$  of interest, Proposition 6.3.15 completes the proof of the Existence Theorem (since the root spaces for  $(G_{\mathbf{Z}}, \mathcal{T}, M)$  are free of rank 1 by construction, or because  $\mathrm{Pic}(\mathbf{Z}) = 1$ ).

**6.4. Applications of Existence and Isomorphism Theorems.** Chevalley groups were originally defined to be the output of a certain explicit construction over  $\mathbf{Z}$  given in [Chev61] for any split connected semisimple  $\mathbf{Q}$ -group descending a given connected semisimple  $\mathbf{C}$ -group. Turning the history around, we define a *Chevalley group* to be a reductive group scheme  $G$  over  $\mathbf{Z}$  that admits a maximal torus  $T$  over  $\mathbf{Z}$ . Since all tori over  $\mathbf{Z}$  are split (as  $\mathrm{Spec} \mathbf{Z}$  is normal and connected with no nontrivial connected finite étale cover) and all line bundles over  $\mathbf{Z}$  are trivial, Chevalley groups are necessarily split. These are precisely the  $\mathbf{Z}$ -groups constructed by Chevalley, at least in the semisimple case, due to:

**Proposition 6.4.1.** *Chevalley groups up to isomorphism are in bijective correspondence with root data, and two Chevalley groups are isomorphic over  $\mathbf{Z}$  if and only if they are isomorphic over  $\mathbf{C}$ .*

*Proof.* The bijectivity assertion is the combination of the Isomorphism and Existence Theorems over  $\mathbf{Z}$ . By the Isomorphism Theorem, the equivalence between  $\mathbf{Z}$ -isomorphism and  $\mathbf{C}$ -isomorphism is immediate (as root data do not “know” the base scheme).  $\square$

**Remark 6.4.2.** Let  $R$  be a semisimple root datum,  $\Phi$  its underlying root system, and  $\mathfrak{g}$  a split Lie algebra over  $\mathbf{Q}$  with root system  $\Phi$ . Chevalley initially proved the Existence Theorem over  $\mathbf{Z}$  for adjoint  $R$  by making an explicit construction inside the automorphism algebra of a Lie algebra over  $\mathbf{Z}$  generated by a Chevalley basis of  $\mathfrak{g}$ . In his 1961 Bourbaki report [Chev61], he removed the adjoint condition on  $R$  by working with a split semisimple  $\mathbf{Q}$ -group  $(G, T, M)$  having root datum  $R$  rather than with the Lie algebra  $\mathfrak{g}$  over  $\mathbf{Q}$  having root system  $\Phi$ .

Chevalley’s idea was to pick a faithful representation  $(V, \rho)$  and use a Chevalley system  $\{X_a\}_{a \in \Phi}$  to construct a lattice  $\Lambda$  in  $V$  so that the schematic closure  $\mathcal{G}$  of  $G$  in  $\mathrm{GL}(\Lambda)$  has an

“open cell” structure over  $\mathbf{Z}$  extending one on  $G$ . The fibral connectedness and semisimplicity properties of  $\mathcal{G}$  were analyzed via the open cell structure.

**Example 6.4.3.** Here is a useful application of Chevalley groups. Let  $S$  be a non-empty connected scheme, and  $(G, T, M)$  a split reductive group over  $S$ . The short exact sequence of  $S$ -groups

$$(6.4.1) \quad 1 \rightarrow T \rightarrow N_G(T) \rightarrow W_G(T) \rightarrow 1$$

induces an exact sequence of groups

$$1 \rightarrow \mathrm{Hom}(M, \mathbf{G}_m(S)) \rightarrow N_G(T)(S) \rightarrow W(\Phi) \rightarrow H^1(S_{\text{ét}}, T) = M^\vee \otimes_{\mathbf{Z}} \mathrm{Pic}(S).$$

We claim that (6.4.1) is the central pushout of an exact sequence

$$(6.4.2) \quad 1 \rightarrow T[2] \rightarrow \widetilde{W}_G(T) \rightarrow W_G(T) \rightarrow 1$$

for a finite flat  $S$ -subgroup  $\widetilde{W}_G(T) \subset N_G(T)$  such that  $\widetilde{W}_G(T)(S)$  is carried onto the finite group  $W(\Phi)$  of constant sections. In particular, if  $S = \mathrm{Spec} R$  for a domain  $R$  (or more generally, if  $\mu_2(S)$  is finite) then  $N_G(T)(S)$  contains a *finite* subgroup mapping into  $W(\Phi)$ .

The construction of (6.4.2) rests on a choice of pinning, or more specifically on a choice of  $\mathbf{Z}$ -descent of  $(G, T, M)$ . In other words, it suffices to make the construction when  $S = \mathrm{Spec} \mathbf{Z}$  (i.e., for Chevalley groups), and in such cases we claim that the finite flat  $\mathbf{Z}$ -subgroup  $\widetilde{W}_G(T) \subset N_G(T)$  is *canonical*; it does not depend on a pinning (and is called the *Tits group* for  $(G, T)$ ). The special features of  $\mathbf{Z}$  are that  $\mathrm{Pic}(\mathbf{Z}) = 1$  and  $\mathbf{G}_m(\mathbf{Z}) = \mu_2(\mathbf{Z})$  (so  $T(\mathbf{Z}) = T[2](\mathbf{Z})$ ); these imply that on  $\mathbf{Z}$ -points (6.4.1) yields a short exact sequence

$$1 \rightarrow T[2](\mathbf{Z}) \rightarrow N_G(T)(\mathbf{Z}) \rightarrow W(\Phi) \rightarrow 1,$$

so  $N_G(T)(\mathbf{Z})$  is *finite*. Hence, we can define  $\widetilde{W}_G(T)$  to be the finite flat  $\mathbf{Z}$ -subgroup of  $N_G(T)$  generated by the  $N_G(T)(\mathbf{Z})$ -translates of  $T[2]$ . Explicitly,  $\widetilde{W}_G(T)(\mathbf{Z})$  contains the elements  $n_a = w_a(X_a)$  for any  $a \in \Phi$  and trivializing section  $X_a$  of  $\mathfrak{g}_a$  (well-defined up to a sign). This same description gives the pinning-dependent construction over a general non-empty base  $S$  (over which unit scaling on the pinning may go beyond sign changes, thereby making the Tits group depend on the pinning). See [Ti66b] and [Ti66c] for further discussion (where  $\widetilde{W}_G(T)(\mathbf{Z})$  is called the *extended Weyl group*), and [Bou3, IX, §4, Exer. 12(d)] for an interpretation via compact Lie groups.

Now we turn our attention to a relative version (and refinement) of the decomposition of a connected semisimple group over a field  $k$  into an “almost direct product” of its  $k$ -simple factors. In Theorem 5.1.19 we canonically described every nontrivial connected semisimple group  $G$  over a field  $k$  as a central isogenous quotient of a product of  $k$ -simple semisimple groups  $G_i$ . For the simply connected central covers  $\widetilde{G}_i \rightarrow G_i$ , each  $\widetilde{G}_i$  is  $k$ -simple (since  $G_i$  is) and the composite map

$$\prod \widetilde{G}_i \rightarrow \prod G_i \rightarrow G$$

is a central isogeny (due to Corollary 3.3.5). Thus, this map identifies  $\prod \tilde{G}_i$  with the simply connected central cover of  $G$ .

The problem of classifying all possible  $G$  over  $k$  is thereby largely reduced to the case of  $k$ -simple  $G$  that are simply connected. We wish to explain why the absolutely simple case (over finite separable extensions of  $k$ ) is the most important case. This rests on:

**Proposition 6.4.4.** *Let  $G \rightarrow S$  be a fiberwise nontrivial simply connected semisimple group over a non-empty scheme  $S$ . There is a finite étale cover  $S' \rightarrow S$  and a simply connected semisimple group  $G' \rightarrow S'$  with absolutely simple fibers such that  $G \simeq R_{S'/S}(G')$ .*

*The pair  $(S'/S, G')$  is uniquely determined up to unique  $S$ -isomorphism in the following sense: if  $(S''/S, G'')$  is another such pair then every  $S$ -group isomorphism  $R_{S'/S}(G') \simeq R_{S''/S}(G'')$  arises from a unique pair  $(\alpha, f)$  consisting of an  $S$ -isomorphism  $\alpha : S' \simeq S''$  and a group isomorphism  $f : G' \simeq G''$  over  $\alpha$ .*

*Proof.* In view of the uniqueness assertions, by étale descent we may work étale-locally on  $S$ . Thus, we can assume that  $G$  is split, say with a split maximal torus  $T = D_S(M)$  whose root spaces  $\mathfrak{g}_\alpha$  are free of rank 1. The semisimple root datum  $R(G, T, M)$  is simply connected, so it decomposes as a direct product  $\prod_{i \in I} R_i$  of simply connected root data  $R_i$  whose underlying root systems are irreducible (and  $I \neq \emptyset$ ). By the Existence Theorem there exists a split semisimple  $S$ -group  $(G_i, T_i, M_i)$  whose root datum is  $R_i$ . The geometric fibers of  $G_i \rightarrow S$  are simply connected and simple (Corollary 5.1.18). By the Isomorphism Theorem,  $G \simeq \prod G_i$ . For  $S' = \prod_{i \in I} S$  and the  $S'$ -group  $G' = \prod G_i$  we have  $R_{S'/S}(G') = \prod G_i = G$ .

It remains to prove the asserted unique description of isomorphisms (so then the preceding construction in the split case does indeed settle the general case, via descent theory). Consider two pairs  $(S'/S, G')$  and  $(S''/S, G'')$  and an  $S$ -group isomorphism  $\varphi : R_{S'/S}(G') \simeq R_{S''/S}(G'')$ . We seek to show that  $\varphi$  arises from a unique pair  $(\alpha, f)$ . The uniqueness allows us to work étale-locally on  $S$  for existence, so we can assume that  $S = \text{Spec } A$  for a strictly henselian local ring  $A$ . Then  $S'$  and  $S''$  are each a non-empty finite disjoint union of copies of  $S$ , so the assertion can be reformulated as follows: if  $\{G'_i\}$  and  $\{G''_j\}$  are non-empty finite collections of simply connected semisimple  $A$ -groups with absolutely simple fibers then any  $A$ -group isomorphism

$$\varphi : \prod G'_i \simeq \prod G''_j$$

arises from a unique bijection  $\alpha : I \simeq J$  and  $A$ -group isomorphisms  $f_i : G'_i \simeq G''_{\alpha(i)}$ .

The uniqueness of  $\alpha$  is immediate from passage to the special fiber, and then the uniqueness of  $\{f_i\}$  is clear. To prove the existence of  $(\alpha, f)$ , we shall use the crutch of maximal tori over the strictly henselian local ring  $A$ . Let  $T'_i \subset G'_i$  be a maximal torus, so  $T' := \prod T'_i$  is a maximal torus in  $\prod G'_i$ . Over the strictly henselian local ring  $A$ , all maximal tori in a reductive  $A$ -group  $H$  are  $H(A)$ -conjugate to each other (Corollary 2.1.5). Hence, the product construction of maximal tori in  $\prod G'_i$  gives all maximal tori. The same applies to  $\prod G''_j$ , so  $\varphi(T') = \prod T''_j$  for maximal tori  $T''_j \subset G''_j$ .

We claim that the isomorphism  $\varphi : \prod T'_i \simeq \prod T''_j$  arises from a bijection  $\alpha : I \simeq J$  and collection of isomorphisms  $h_i : T'_i \simeq T''_{\alpha(i)}$ . Since these tori are split over the local  $A$ , it is

equivalent to verify the assertion on the special fiber. But over the residue field we can appeal to Theorem 5.1.19 to identify the absolutely simple special fibers  $\{(G'_i)_0\}$  and  $\{(G''_j)_0\}$  with the simple factors of the respective product groups  $\prod (G'_i)_0$  and  $\prod (G''_j)_0$ . Hence,  $\varphi_0$  must permute these factors according to some bijection  $\alpha : I \simeq J$  and carry  $(G'_i)_0$  isomorphically onto  $(G''_{\alpha(i)})_0$ . This latter isomorphism must carry  $(T'_i)_0$  isomorphically onto  $(T''_{\alpha(i)})_0$ , and these torus isomorphism (together with  $\alpha$ ) do the job.

Having built  $\alpha : I \simeq J$  and  $h_i : T'_i \simeq T''_{\alpha(i)}$  compatible with  $\varphi$ , we can relabel the indices so that  $\varphi$  is an isomorphism of  $A$ -groups  $\prod G'_i \simeq \prod G''_i$  carrying the  $A$ -subgroup  $T'_i$  isomorphically to the  $A$ -subgroup  $T''_i$  for each  $i$ . It remains to show that the  $A$ -subgroup  $G'_{i_0}$  is carried isomorphically to the  $A$ -subgroup  $G''_{i_0}$  for each  $i_0$ . We will do this via an intrinsic characterization of  $G'_{i_0}$  in terms of  $\{T'_i\}$  and  $G' = \prod G'_i$ . The centralizer of  $\prod_{i \neq i_0} T'_i$  in  $G'$  is  $G'_{i_0} \times \prod_{i \neq i_0} T'_i$ , so the derived group of this centralizer is  $G'_{i_0}$ . A similar description applies to  $G''_{i_0}$  in terms of  $\{T''_i\}$  and  $G'' = \prod G''_i$ , so we are done.  $\square$

**Remark 6.4.5.** The assertions in Proposition 6.4.4 remain true, with the same proof, when “simply connected” is replaced by “adjoint” everywhere.

**Example 6.4.6.** Let  $k$  be a field and  $G$  a nontrivial connected semisimple  $k$ -group that is simply connected. Proposition 6.4.4 provides a canonical isomorphism  $G \simeq R_{k'/k}(G')$  for a unique pair  $(k'/k, G')$  consisting of a nonzero finite étale  $k$ -algebra  $k'$  and a semisimple  $k'$ -group  $G'$  such that all fibers of  $G' \rightarrow \text{Spec } k'$  are connected, simply connected, and absolutely simple. Using the decomposition into factor fields  $k' = \prod k'_i$  and letting  $G'_i$  denote the  $k'_i$ -fiber of  $G'$ , we have  $G \simeq \prod_i R_{k'_i/k}(G'_i)$ . By Example 5.1.20, these factors are  $k$ -simple. In particular,  $G$  is  $k$ -simple if and only if it has the form  $R_{k'/k}(G')$  for a finite separable extension  $k'/k$  and a connected semisimple  $k'$ -group  $G'$  that is *absolutely simple* and simply connected. This argument shows that the pair  $(k'/k, G')$  is canonically attached to  $G$  (not merely up to  $k$ -isomorphism). An extrinsic characterization of  $k'/k$  is given in Exercise 6.5.9.

**Example 6.4.7.** Let  $G$  be a simply connected semisimple  $\mathbf{Z}$ -group. By Proposition 6.4.4,  $G \simeq R_{A/\mathbf{Z}}(G')$  for a nonzero finite étale  $\mathbf{Z}$ -algebra  $A$  and a simply connected semisimple  $A$ -group  $G'$  whose fibers are absolutely simple. By Hermite–Minkowski,  $A = \prod A_i$  with  $A_i = \mathbf{Z}$ . Thus, if  $G_i$  denotes the restriction of  $G'$  over  $\text{Spec } A_i = \text{Spec } \mathbf{Z}$  then  $R_{A/\mathbf{Z}}(G') = \prod G_i$ . Hence, to classify all such  $G$  one loses no generality by restricting attention to the case when  $G$  has absolutely simple fibers (i.e., an irreducible root system for the geometric fibers).

By the same reasoning, over any connected normal noetherian scheme  $S$  whatsoever, to classify simply connected semisimple  $S$ -groups  $G$  one can pass to the case of  $G$  with absolutely simple fibers at the cost of replacing  $S$  with some connected finite étale covers (namely, the connected components of  $S'$  as in Proposition 6.4.4).

## 6.5. Exercises.

**Exercise 6.5.1.** Let  $(G, T, M)$  be a split semisimple group over a non-empty scheme  $S$ .

(i) Choose  $a \in \Phi$ . Show that any  $\phi : \mathrm{SL}_2 \rightarrow G$  satisfying  $\phi(\mathrm{diag}(c, 1/c)) = a^\vee(c)$  with  $\ker \phi$  central factors uniquely through a central isogeny  $\bar{\phi} : (\mathrm{SL}_2, D) \rightarrow (\mathcal{D}(Z_G(T_a)), a^\vee(\mathbf{G}_m))$  between split semisimple groups of rank 1, carrying the strictly upper-triangular subgroup  $U^+$  isomorphically onto  $U_a$ . Prove that such  $\phi$  exist, and that  $\phi \mapsto \mathrm{Lie}(\bar{\phi})(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})$  is a bijection between the set of such  $\phi$  and the set of global bases  $X_a$  of  $\mathfrak{g}_a = \mathrm{Lie}(U_a)$ .

(ii) Under the dictionary in (i), we can pass from a choice of  $X_a$  to a choice of  $\phi$ , and then to an element  $X_{-a} := \mathrm{Lie}(\phi)(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}) \in \mathfrak{g}_{-a}(S)$ . Prove that  $X_{-a}$  is a nowhere-vanishing section of  $\mathfrak{g}_{-a}$ , and that it is the trivialization of  $\mathfrak{g}_{-a}$  dual to  $X_a$  via the canonical perfect pairing between  $\mathfrak{g}_a$  and  $\mathfrak{g}_{-a}$  in Theorem 4.2.6.

(iii) Use (i) to give a group-theoretic definition of pinings that does not mention bases for root spaces; cf. [CGP, A.4.12].

**Exercise 6.5.2.** Fill in the missing details at the end of Example 6.1.9: prove that if an isogeny  $f : (G', T') \rightarrow (G, T)$  between connected reductive groups equipped with maximal tori over an algebraically closed field induces an isomorphism between the corresponding root groups then  $\ker f \subset Z_{G'}$ . Then state and prove an analogous result for an isogeny between split reductive group schemes over a non-empty scheme.

**Exercise 6.5.3.** Using the Existence and Isomorphism Theorems, the equivalence in Exercise 1.6.12(ii) among definitions for “simply connected” in the classical case is now proved. A semisimple group over a non-empty scheme  $S$  is *simply connected* when its geometric fibers are simply connected (recovering the definition in Example 5.1.7 in the split case).

(i) For any semisimple  $S$ -group  $G$ , prove the existence and uniqueness of a central isogeny  $\pi : \tilde{G} \rightarrow G$  from a semisimple  $S$ -group  $\tilde{G}$  that is simply connected. By “uniqueness” we mean that for any two such pairs  $(\tilde{G}, \pi)$  and  $(\tilde{G}', \pi')$  there exists a *unique* isomorphism  $\tilde{G} \simeq \tilde{G}'$  carrying  $\pi$  to  $\pi'$ . We call  $(\tilde{G}, \pi)$  the *simply connected central cover* of  $G$ .

(ii) For a semisimple  $S$ -group  $G$ , prove that  $\tilde{G}/Z_{\tilde{G}} \rightarrow G/Z_G$  is an isomorphism and  $G \rightarrow G/Z_G$  is uniquely covered by an isomorphism between the simply connected central covers of  $G$  and  $G/Z_G$ . Deduce that the isomorphism classes of  $\tilde{G}$  and  $G/Z_G$  determine each other.

(iii) Prove that for a simply connected semisimple  $S$ -group  $G$ , any central extension of  $G$  by a group  $H$  of multiplicative type is uniquely split. (Hint: work étale-locally on  $S$  to split  $H$ , and note that the central pushout along an inclusion of  $H$  into a torus is reductive and hence admits a “derived group” that is semisimple.) Using this, prove that any homomorphism between semisimple  $S$ -groups uniquely lifts to a homomorphism between their simply connected central covers; in this sense  $(\tilde{G}, \pi)$  is uniquely functorial in  $G$ .

(iv) Let  $T'$  be a torus in a simply connected semisimple  $S$ -group  $G$ . Prove that  $\mathcal{D}(Z_G(T'))$  is simply connected. (Hint: pass to  $S = \mathrm{Spec} k$  for  $k = \bar{k}$  and find  $\lambda \in X_*(T')$  so that  $Z_G(T') = Z_G(\lambda)$ . Pick a maximal torus  $T \supseteq T'$  and closed Weyl chamber in  $X_*(T)_{\mathbf{Q}}$  containing  $\lambda$ , with associated bases  $\Delta$  and  $\Delta^\vee$ , so  $\langle a, \lambda \rangle \geq 0$  for all  $a \in \Delta$ . Prove that  $Z_G(\lambda)$

supports precisely the  $T$ -roots spanned over  $\mathbf{Q}$  by  $\Delta' = \{a' \in \Delta \mid \langle a', \lambda \rangle = 0\}$ . Deduce that  $X_*(\mathcal{Z}(Z_G(T')) \cap T)$  is spanned over  $\mathbf{Z}$  by  $\Delta'^{\vee}$ . Using  $T' = (\ker a)_{\text{red}}^0$  for  $a \in \Phi(G, T)$ , show by example that the analogue with “adjoint” replacing “simply connected” is false.

**Exercise 6.5.4.** Does Proposition 4.3.1 remain valid with  $\text{SL}_2$  replaced by any semisimple group scheme that is simply connected in the sense of Exercise 6.5.3?

**Exercise 6.5.5.** If  $G$  is a connected semisimple group over a field  $k$  and if  $\mathcal{L} \rightarrow G$  is a  $\mathbf{G}_m$ -torsor with a chosen basis  $e' \in \mathcal{L}(e)$ , prove that  $\mathcal{L}$  admits a unique structure of central extension of  $G$  by  $\mathbf{G}_m$  with identity  $e'$ . (The proof imitates the case of abelian varieties after passing to the split case and using the triviality of line bundles on the open cell, together with a classic result of Chevalley that any pointed map from a smooth connected  $k$ -group to  $(\mathbf{G}_m, 1)$  is a homomorphism.) Does this result generalize to semisimple group schemes  $G$  over a non-empty scheme  $S$  when a trivialization of  $\mathcal{L}$  along  $e \in G(S)$  is given?

**Exercise 6.5.6.** Let  $G$  be a connected linear algebraic group over an algebraically closed field  $k$  of characteristic 0. Prove without recourse to the classification theorems that  $G$  is semisimple if and only if  $\text{Lie}(G)$  is semisimple. (Hint for “ $\Rightarrow$ ”: Let  $\mathfrak{t}$  be a solvable Lie ideal. By the Levi–Malcev Theorem [Bou1, I, 6.8, Thm. 5], semisimplicity is equivalent to the vanishing of all such  $\mathfrak{t}$ . Prove that  $\mathfrak{t}$  is stable under the adjoint action of  $G$ , and consider its possible weights under the action of a maximal torus of  $G$ .)

**Exercise 6.5.7.** Recall Lang’s Theorem [Bo91, 16.5(i)]: if a connected linear algebraic group  $G$  over a finite field  $k$  acts on a finite type  $k$ -scheme  $V$  such that  $G(\bar{k})$  is transitive on  $V(\bar{k})$  then  $V(k) \neq \emptyset$ . Apply it to  $\text{Tor}_{G'/k}$  and  $\text{Bor}_{G'/k}$  for  $G' = G/\mathcal{R}_u(G)$  to prove that  $G$  contains a (geometrically) maximal  $k$ -torus and a Borel  $k$ -subgroup.

**Exercise 6.5.8.** Let  $S' \rightarrow S$  be a finite étale cover of schemes,  $G'$  a reductive  $S'$ -group, and  $G := \text{R}_{S'/S}(G')$ . Prove that  $G$  is a reductive  $S$ -group, and adapt the method of proof of Proposition 6.4.4 to show that  $T' \mapsto \text{R}_{S'/S}(T')$  is a bijection between the sets of maximal tori of  $G'$  and of  $G$ . Do likewise for Borel subgroups, and use this to construct natural isomorphisms  $\text{Tor}_{G/S} \simeq \text{R}_{S'/S}(\text{Tor}_{G'/S'})$  and  $\text{Bor}_{G/S} \simeq \text{R}_{S'/S}(\text{Bor}_{G'/S'})$ .

**Exercise 6.5.9.** Let  $k$  be a field and  $G$  a connected semisimple  $k$ -group that is  $k$ -simple and simply connected. According to Example 6.4.6,  $G \simeq \text{R}_{k'/k}(G')$  for a canonically associated pair  $(k'/k, G')$  where  $k'/k$  is a finite separable extension field and the simply connected semisimple  $k'$ -group  $G'$  is absolutely simple.

(i) For an extension field  $L/k$ , prove that  $G_L$  is  $L$ -split if and only if  $k' \otimes_k L$  is a split  $L$ -algebra (i.e., all of its factor fields coincide with  $L$ ) and  $G' \otimes_{k', j'} L$  is  $L$ -split for all  $k$ -embeddings  $j' : k' \rightarrow L$ . (Hint: use Exercise 6.5.8.)

(ii) Prove that if  $k$  is a global field and  $v$  is a place of  $k$  such that  $G_{k_v}$  is  $k_v$ -split then  $v$  is totally split in  $k'/k$ . Deduce that a  $k$ -simple connected semisimple  $k$ -group that is split at all finite places must be absolutely simple.

(ii) Define a natural action by  $\text{Gal}(k_s/k)$  on the set of  $k_s$ -simple factors of  $G_{k_s}$ , and relate  $k'$  to the open subgroup that preserves one of these factors.

## 7. AUTOMORPHISM SCHEME

**7.1. Structure of automorphisms.** Many arithmetic properties of non-degenerate quadratic forms  $q$  of rank  $\geq 3$  over number fields are encoded in terms of the corresponding connected semisimple group  $\mathrm{SO}(q)$  (or the disconnected orthogonal group  $\mathrm{O}(q)$ ). All such groups become isomorphic over  $\overline{\mathbf{Q}}$ , so the problem of classifying quadratic forms can be related to the problem of classifying connected semisimple groups with a fixed geometric isomorphism type (equivalently, with a fixed root datum for its geometric fiber). The classification of connected semisimple  $\mathbf{Q}$ -groups sharing a common isomorphism type over  $\overline{\mathbf{Q}}$  is tied up with understanding the structure of the automorphism groups of split connected semisimple groups over number fields.

To see the link between automorphism groups and classification problems, consider the situation over a general field  $k$ . Let  $G$  be a connected reductive  $k$ -group, so  $G_{k_s}$  admits a splitting and pinning. This gives rise to a reduced root datum  $R$  whose isomorphism class is intrinsic to  $G$  (and can be computed over any separably closed extension of  $k$ ). Let  $G_0$  be the split connected reductive  $k$ -group with the same root datum (so  $G_0$  is unique up to  $k$ -isomorphism). Since  $G$  and  $G_0$  become isomorphic over a finite Galois extension of  $k$  (as both are split over  $k_s$  with the same root datum, and hence are isomorphic over  $k_s$ ), we may view  $G$  as a “ $k_s/k$ -form” of  $G_0$ . To classify the possibilities for  $G$ , we use:

**Lemma 7.1.1.** *Let  $R$  be a reduced root datum, and  $G_0$  a split connected reductive  $k$ -group with root datum  $R$ . The set of  $k$ -isomorphism classes of connected reductive  $k$ -groups  $G$  whose associated root datum over  $k_s$  is isomorphic to  $R$  is in natural bijection with the Galois cohomology set  $H^1(k_s/k, \mathrm{Aut}((G_0)_{k_s}))$ .*

*Proof.* Choose  $G$  and pick a finite Galois extension  $K/k$  such that there exists a  $K$ -group isomorphism  $\varphi : G_K \simeq (G_0)_K$ . Typically this isomorphism is not defined over  $k$  (i.e., it does not descend to a  $k$ -group isomorphism  $G \simeq G_0$ ), and to measure this possible failure we examine how  $\varphi$  interacts with the canonical  $K$ -isomorphisms  $\gamma^*(G_K) \simeq G_K$  for  $\gamma \in \mathrm{Gal}(K/k)$  that encode the  $k$ -descent  $G$  of  $G_K$ , as well as the analogous  $K$ -isomorphisms for  $G_0$  (where  $\gamma^*(X)$  denotes the base change of a  $K$ -scheme  $X$  through scalar extension by  $\gamma : K \simeq K$ ). That is, for each  $\gamma$  we get a  $K$ -isomorphism

$$c_\gamma : (G_0)_K \simeq \gamma^*((G_0)_K) \xrightarrow{\gamma^*(\varphi)^{-1}} \gamma^*(G_K) \simeq G_K \xrightarrow{\varphi} (G_0)_K,$$

and it is straightforward to check that the cocycle condition

$$c_{\gamma'\gamma} = c_{\gamma'} \circ (\gamma'.c_\gamma)$$

holds in  $\mathrm{Aut}((G_0)_K)$ , where  $c \mapsto \gamma'.c$  denotes the natural action of  $\gamma' \in \mathrm{Gal}(K/k)$  on  $\mathrm{Aut}((G_0)_K)$  through scalar extension along  $\gamma' : K \simeq K$  (combined with the descent isomorphism  $\gamma'^*((G_0)_K) \simeq (G_0)_K$  defined by the  $k$ -structure  $G_0$  on the  $K$ -group  $(G_0)_K$ ).

The choice of  $(K, \varphi)$  defines the function  $c : \mathrm{Gal}(K/k) \rightarrow \mathrm{Aut}((G_0)_K)$  via  $\gamma \mapsto c_\gamma$ , and in terms of the non-abelian cohomology conventions in [Ser97, I, §5.1] this lies in the set  $Z^1(K/k, \mathrm{Aut}((G_0)_K))$  of 1-cocycles on  $\mathrm{Gal}(K/k)$  with coefficients in the  $\mathrm{Gal}(K/k)$ -group

$\text{Aut}((G_0)_K)$ . This 1-cocycle depends on  $\varphi$ , but the other choices are precisely  $a \circ \varphi$  for  $a \in \text{Aut}((G_0)_K)$ , which leads to the 1-cocycle  $\gamma \mapsto a \circ c_\gamma \circ (\gamma.a)^{-1}$ . As we vary through all such  $a$ , these cocycles vary through precisely the ones that are cohomologous to  $c$ . In particular, the cohomology class  $[c] \in H^1(K/k, \text{Aut}((G_0)_K))$  is independent of  $\varphi$ . Note that  $c = 1$  as a function precisely when  $\varphi$  is defined over  $k$ , and more generally  $c$  is a coboundary  $\gamma \mapsto a^{-1} \circ \gamma.a$  for  $a \in \text{Aut}((G_0)_K)$  precisely when  $a \circ \varphi$  is defined over  $k$ .

Since Galois descent is effective for affine schemes, it follows that  $H^1(K/k, \text{Aut}((G_0)_K))$  is identified with the pointed set of isomorphism classes of  $k$ -groups  $G$  such that there exists a  $K$ -group isomorphism  $G_K \simeq (G_0)_K$ . This is a special case of the general formalism of “twisted forms” as in [Ser97, III] (i.e., it has nothing to do with  $G_0$  being a split connected reductive  $k$ -group), and for a  $k$ -embedding  $K \rightarrow K'$  into another finite Galois extension of  $k$  the resulting inflation map of pointed sets

$$H^1(K/k, \text{Aut}((G_0)_K)) \rightarrow H^1(K'/k, \text{Aut}((G_0)_{K'}))$$

“relaxing” the condition of existence of a  $K$ -isomorphism (to  $(G_0)_K$ ) to the condition of existence of a  $K'$ -isomorphism (to  $(G_0)_{K'}$ ). Passing to the direct limit over all  $K/k$  inside  $k_s/k$  gives the result.  $\square$

**Remark 7.1.2.** If we fix a separable closure  $k_s/k$  then we can define a *canonical* based root datum attached to any (possibly non-split) connected reductive  $k$ -group  $G$  as follows. For each pair  $(T, B)$  in  $G_{k_s}$  consisting of a (geometrically) maximal  $k_s$ -torus  $T$  and a Borel  $k_s$ -subgroup  $B$  that contains it, we get an associated based root datum  $R(G, T, B)$ . If  $(T', B')$  is another choice then by Proposition 6.2.11(2) there exists  $g \in G(k_s)$  such that  $g$ -conjugation carries  $(T, B)$  to  $(T', B')$ . The choice of  $g$  is unique up to  $N_G(B)(k_s) \cap N_G(T)(k_s) = B(k_s) \cap N_G(T)(k_s) = T(k_s)$ , so the induced isomorphism  $R(G, T, B) \simeq R(G, T', B')$  of based root data is independent of the choice of  $g$ . More specifically, we get *canonical* isomorphisms among all of the based root data  $R(G, T, B)$  as we vary  $(T, B)$ , and these isomorphisms are compatible with respect to composition.

By forming the (inverse or direct) limit along this system of isomorphisms, we get “the” based root datum of  $G$ , to be denoted  $(R(G), \Delta)$ . This generally depends on  $k_s/k$  when  $G$  is not  $k$ -split. That is, if  $\gamma \in \text{Gal}(k_s/k)$  and  $(T, B)$  is a pair in  $G_{k_s}$ , then the isomorphism  $R(G_{k_s}, T, B) \simeq R(G_{k_s}, \gamma^*(T), \gamma^*(B))$  defined via scalar extension along  $\gamma$  and the  $k$ -structure  $G$  on  $G_{k_s}$  may not coincide with the effect of conjugation by an element of  $G(k_s)$  that carries  $(T, B)$  to  $(\gamma^*(T), \gamma^*(B))$ . Put in other terms, we have just defined a natural action of  $\gamma \in \text{Gal}(k_s/k)$  on  $(R(G), \Delta)$  via

$$(R(G), \Delta) = R(G_{k_s}, T, B) \simeq R(G_{k_s}, \gamma^*(T), \gamma^*(B)) = (R(G), \Delta);$$

this is independent of the choice of  $(T, B)$  and may be nontrivial. It is the “\*-action” of  $\text{Gal}(k_s/k)$  on  $\text{Dyn}(\Phi)$  that appears in the Borel–Tits approach to classifying isotropic connected semisimple groups over fields (modulo the “anisotropic kernel”).

When  $k_s/k$  is understood from context,  $(R(G), \Delta)$  is called the *based root datum* for  $G$ . By the Existence and Isomorphism Theorems, up to unique  $k$ -isomorphism there exists

a unique pinned split connected reductive  $k$ -group  $(G_0, T_0, M_0, \{X_a\}_{a \in \Delta})$  with based root datum  $(R(G), \Delta)$ . As we vary through the different choices for the base, the resulting pinned split  $k$ -groups are canonically isomorphic to each other. This all depends on  $k_s/k$  except when  $G$  is  $k$ -split. The dependence on  $k_s/k$  can be eliminated by working with suitable étale  $k$ -schemes, such as the finite étale  $k$ -scheme associated to  $\Delta$  equipped with the above action of  $\text{Gal}(k_s/k)$ . (This is the “scheme of Dynkin diagrams”  $\text{Dyn}(G)$ ; see Example 7.1.11.)

The preceding considerations can be adapted to a general base scheme. To explain this, it is convenient to introduce some notation.

**Definition 7.1.3.** If  $G \rightarrow S$  is a group scheme, its *automorphism functor*  $\underline{\text{Aut}}_{G/S}$  on the category of  $S$ -schemes is  $\underline{\text{Aut}}_{G/S} : S' \rightsquigarrow \text{Aut}_{S'\text{-gp}}(G_{S'})$ . A representing object (if one exists) is denoted  $\text{Aut}_{G/S}$  and is called the *automorphism scheme* of  $G$ .

The automorphism functor is a sheaf for the fppf (and even fpqc) topology, and we will see that representability and structural properties of the automorphism functor in the case of split reductive  $G$  are extremely useful in the classification of all reductive  $G$ . As an example, we will show that  $\text{Aut}_{\text{SL}_2/S}$  exists and is identified with  $\text{PGL}_2$  (see Theorem 7.1.9(3)). Note that in general  $\underline{\text{Aut}}_{G/S}(S)$  is the automorphism group of  $G$  in the usual sense, which we also denote as  $\text{Aut}(G)$  (as for any category).

Consider a reductive group  $G$  over a *connected* non-empty scheme  $S$ . Connectedness of the base ensures that all geometric fibers have the same reduced root datum  $R$  (see Lemma 6.1.3), so it makes sense to define  $G_0$  to be the split reductive  $S$ -group with root datum  $R$ . (This defines  $G_0$  uniquely up to  $S$ -isomorphism, by the Isomorphism Theorem.) The  $S$ -groups  $G$  and  $G_0$  become isomorphic étale-locally on  $S$ , due to the Isomorphism Theorem and the existence étale-locally on  $S$  of splittings and pinnings for  $G \rightarrow S$ . Since étale descent is effective for schemes affine over the base, we can use the same formalism as over fields to identify the set of isomorphism classes of  $G$  having fibral root datum isomorphic to  $R$  with the étale cohomology set

$$(7.1.1) \quad H^1(S_{\text{ét}}, \underline{\text{Aut}}_{G_0/S}) := \varinjlim_{S'/S} \check{H}^1(S'/S, \underline{\text{Aut}}_{G_0/S}),$$

where we vary through (a cofinal set of) “all” étale covers  $S' \rightarrow S$  (see Exercise 7.3.3(i) and Exercise 2.4.11(i)). Explicitly, the Čech-type cohomology set  $\check{H}^1(S'/S, \underline{\text{Aut}}_{G_0/S})$  is identified with the set of  $S$ -isomorphism classes of those  $G$  that become split over  $S'$ . This interpretation explains why the transition maps in the direct limit (7.1.1), defined by pullback along  $S$ -maps  $S'' \rightarrow S'$ , form a directed system relative to the partial order of refinement among covers  $S' \rightarrow S$  (i.e., these transition maps do not depend on the specific  $S$ -maps between such covers). This is analogous to the canonicity of transition maps defining Čech cohomology on ordinary topological spaces as a direct limit.

**Remark 7.1.4.** The classical theory over an algebraically closed field  $k$  does not address representability properties of the automorphism functor on  $k$ -algebras, but it does suggest

that the automorphism group over  $k$  should be viewed as having a “geometric” structure. More specifically (by Proposition 1.5.1, the identification (1.5.2), and Proposition 1.5.5), if  $G$  is a simply connected (and connected) semisimple group over an algebraically closed field  $k$  and  $\Phi$  is its root system then  $\text{Aut}(G)$  is naturally the group of  $k$ -points of

$$(7.1.2) \quad (G/Z_G) \rtimes \text{Aut}(\text{Dyn}(\Phi)),$$

where the semi-direct product structure rests on a choice of splitting and pinning. If we want to use (7.1.2) as anything deeper than a bookkeeping device (e.g., exploit that  $G/Z_G$  is connected), we should prove representability of  $\underline{\text{Aut}}_{G/k}$  and not just make a construction that has the “right” geometric points. This will be done in Theorem 7.1.9.

As a prelude to the case of a general base scheme, now consider the problem of describing  $\underline{\text{Aut}}_{G/k}(k) = \text{Aut}(G)$  for a split connected reductive group  $G$  over a general field  $k$ . This goes beyond the classical arguments related to (7.1.2) over algebraically closed fields because maps such as  $G(k) \rightarrow (G/Z_G)(k)$  can fail to be surjective when  $k \neq \bar{k}$  (e.g.,  $\text{SL}_n(k) \rightarrow \text{PGL}_n(k)$  for any field  $k$  such that  $k^\times$  is not  $n$ -divisible).

Let  $(G, T, M)$  be a split connected reductive group over a field  $k$ . The action of  $G$  on itself by conjugation factors through an action of  $G/Z_G$  on  $G$ . This identifies the adjoint quotient  $G/Z_G$  with a subfunctor of the automorphism functor  $\underline{\text{Aut}}_{G/k}$ , so  $(G/Z_G)(k)$  is a subgroup of  $\underline{\text{Aut}}_{G/k}(k) = \text{Aut}(G)$ . Note that this works even when  $G(k) \rightarrow (G/Z_G)(k)$  fails to be surjective, and that it defines a normal subgroup of  $\text{Aut}(G)$ . (See Exercise 7.3.2.)

Due to the canonicity of the associated based root datum  $(R(G), \Delta)$  as in Remark 7.1.2,  $\text{Aut}(G)$  naturally acts on  $(R(G), \Delta)$ . Here is how it works in concrete terms, by identifying  $\text{Aut}(G)/(G/Z_G)(k)$  with  $\text{Aut}(R(G), \Delta)$ . Pick a positive system of roots  $\Phi^+$  in  $\Phi = \Phi(G, T)$ , with base denoted  $\Delta$ . Equivalently, choose a Borel subgroup  $B$  in  $G$  containing  $T$  (with  $\Phi(B, T) = \Phi^+$ ). Let  $\phi$  be an arbitrary automorphism of  $G$ , so  $T' = \phi(T)$  is a split (geometrically) maximal  $k$ -torus of  $G$  and  $B' = \phi(B)$  is a Borel  $k$ -subgroup of  $G$  containing  $T'$ . By Proposition 6.2.11(2), there exists  $g \in G(k)$  such that  $T' = gTg^{-1}$  and  $B' = gBg^{-1}$ . Thus, by composing  $\phi$  with an automorphism arising from  $G(k)$  we can arrange that  $\phi(T) = T$  and  $\phi(\Phi^+) = \Phi^+$ . These additional requirements are preserved under composition of  $\phi$  with the action of an element  $g \in (G/Z_G)(k)$  if and only if  $g \in (T/Z_G)(k)$ , since  $B/Z_G = N_{G/Z_G}(B/Z_G)$  and  $N_{G/Z_G}(T/Z_G) \cap (B/Z_G) = T/Z_G$ .

The automorphism of the based root datum  $(R(G, T), \Delta)$  induced by  $\phi$  is unaffected by composing  $\phi$  with the action of  $(T/Z_G)(k)$  since the  $T/Z_G$ -action on  $G$  is trivial on  $T$ . The resulting action of  $\phi$  on the set  $\Delta$  has more structure: it is an automorphism of the Dynkin diagram  $\text{Dyn}(\Phi)$ . This can be nontrivial:

**Example 7.1.5.** Let  $G = \text{SL}_n$ , and let  $w \in G(k)$  be the anti-diagonal matrix whose entries alternate  $1, -1, 1, -1, \dots$  beginning in the upper right (so  $ww^\top = 1$ ). The automorphisms  $g \mapsto (g^\top)^{-1}$  and  $g \mapsto wgw^{-1}$  of  $G$  swap the upper triangular and lower triangular Borel subgroups while preserving the diagonal torus  $D$ . The composite automorphism  $\iota : g \mapsto w(g^\top)^{-1}w^{-1}$  is an involution of  $G$  that preserves  $D$  and the upper triangular Borel subgroup, inducing the involution of the Dynkin diagram when  $n > 2$ .

Returning to a general split  $(G, T)$  over  $k$ , let  $\{X_a\}_{a \in \Delta}$  be a pinning of  $(G, T, B)$ . For  $X'_a := \text{Lie}(\phi)(X_{\phi^{-1}(a)})$  we get another pinning  $\{X'_a\}_{a \in \Delta}$ . Clearly  $X'_a = c_a X_a$  for a unique  $(c_a) \in (k^\times)^\Delta$ . Since  $\Delta$  is a  $\mathbf{Z}$ -basis of the character group  $\mathbf{Z}\Phi$  of the the split (geometrically) maximal torus  $T/Z_G \subset G/Z_G$  (Corollary 3.3.6(1)), we have  $T/Z_G \simeq \prod_{a \in \Delta} \mathbf{G}_m$  via  $t \bmod Z_G \mapsto (a(t))_{a \in \Delta}$ . Hence,  $(T/Z_G)(k)$  acts simply transitively on the set of all pinnings of  $(G, T, B)$ , so by composing  $\phi$  with a *unique* automorphism arising from  $(T/Z_G)(k)$  we can arrange that  $\text{Lie}(\phi)(X_a) = X_{\phi(a)}$  for all  $a \in \Delta$  (i.e.,  $X'_a = X_a$  for all  $a \in \Delta$ ).

Let  $\Theta = \text{Aut}(R(G, T), \Delta)$  be the automorphism group of the based root datum, so by Corollary 6.1.15 the conditions we have imposed on  $\phi$  relative to  $(T, B, \{X_a\}_{a \in \Delta})$  make it determined *uniquely* by its image in  $\Theta$ . That is, we have identified  $\text{Aut}(G)/(G/Z_G)(k)$  with a subgroup of  $\Theta$ . In fact, this subgroup inclusion is an equality, or equivalently *every* element of  $\Theta$  arises from a unique  $\phi$  satisfying the conditions imposed relative to  $(T, B, \{X_a\}_{a \in \Delta})$ . This is exactly the content of the precise form of the Isomorphism Theorem given in Theorem 6.1.17. In other words, the choice of pinning defines an injective homomorphism  $\Theta \hookrightarrow \text{Aut}(G)$  with image equal to the automorphism group of  $(G, T, B, \{X_a\}_{a \in \Delta})$ , so  $\Theta$  is carried to a subgroup of  $\text{Aut}(G)$  that preserves  $T$  and  $B$  and maps isomorphically onto  $\text{Aut}(G)/(G/Z_G)(k)$ .

We have constructed an isomorphism of groups

$$(7.1.3) \quad (G/Z_G)(k) \rtimes \text{Aut}(R(G, T), \Delta) \simeq \text{Aut}(G).$$

This depends on the choice of pinning, as well as the split hypothesis on  $G$ . (Although the inclusion of  $(G/Z_G)(k)$  as a normal subgroup of  $\text{Aut}(G)$  is available without any split hypothesis on  $G$ , in the non-split case there may be no section to  $\text{Aut}(G) \rightarrow \text{Aut}(G)/(G/Z_G)(k)$ ; see Example 7.1.12.) Since the preceding discussion is compatible with extension of the ground field, we likewise have  $(G/Z_G)(K) \rtimes \text{Aut}(R(G, T), \Delta) \simeq \text{Aut}(G_K) = \underline{\text{Aut}}_{G/k}(K)$  for any extension field  $K/k$ . To summarize:

**Proposition 7.1.6.** *Let  $(G, T, M, \{X_a\}_{a \in \Delta})$  be a pinned split connected reductive group over a field  $k$ . The group  $\text{Aut}(G)$  is naturally an extension of  $\text{Aut}(R(G, T), \Delta)$  by  $(G/Z_G)(k)$ , and the pinning naturally splits this extension as a semi-direct product.*

**Remark 7.1.7.** By Proposition 1.5.1, if  $(R, \Delta)$  is a semisimple based root datum with root system  $\Phi$  then  $\text{Aut}(R, \Delta) \subset \text{Aut}(\text{Dyn}(\Phi))$ , with equality when  $R$  is simply connected or adjoint or when the “fundamental group”  $(\mathbf{Z}\Phi^\vee)^*/\mathbf{Z}\Phi$  is cyclic (which includes all irreducible reduced root systems except type  $D_{2n}$ ,  $n \geq 2$ ).

**Example 7.1.8.** The root system  $\Phi = A_{n-1}$  ( $n \geq 2$ ) has cyclic “fundamental group”  $(\mathbf{Z}\Phi^\vee)^*/(\mathbf{Z}\Phi) \simeq \mathbf{Z}/n\mathbf{Z}$  and  $\text{Aut}(\text{Dyn}(\Phi))$  is equal to  $\mathbf{Z}/2\mathbf{Z}$  when  $n > 2$  (and  $\{1\}$  when  $n = 2$ ). Thus,  $\text{Aut}(\text{SL}_n) = \text{PGL}_n(k) \rtimes (\mathbf{Z}/2\mathbf{Z})$  for  $n > 2$ , with the factor  $\mathbf{Z}/2\mathbf{Z}$  generated by the involution  $g \mapsto (g^T)^{-1}$  (due to Example 7.1.5). This involution is available for  $\text{SL}_2$ , but in that case it is inner (arising from conjugation by  $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ).

[The equality  $\text{Aut}_k(\text{SL}_2) = \text{PGL}_2(k)$  for  $n = 2$  can be proved in elementary terms, as follows. Using  $k$ -rational conjugacy results, it suffices to show that an automorphism of  $\text{SL}_2$  that is the identity on the upper triangular Borel subgroup  $B^+$  is the identity on the open

cell  $\Omega^+$  (and hence is the identity). The effect on  $U^-$  must be  $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ cx & 1 \end{pmatrix}$  for some  $c \in k^\times$ . But the two standard open cells  $\Omega^+$  and  $\Omega^-$  have intersection defined by  $1 + xy \neq 0$  inside  $\mathbf{G}_a \times \mathbf{G}_m \times \mathbf{G}_a = \Omega^+$  via

$$(y, t, x) \mapsto \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

so we must have that the open loci  $1 + xy \neq 0$  and  $1 + cxy \neq 0$  on  $\mathbf{G}_a \times \mathbf{G}_a$  coincide. This forces  $c = 1$ , as desired.]

In Example 7.1.10 we will compute the section  $\mathbf{Z}/2\mathbf{Z} = \text{Aut}(\text{Dyn}(\Phi)) \hookrightarrow \text{Aut}(\text{SL}_n)$  associated to a “standard” choice of splitting and pinning for  $\text{SL}_n$ . This section carries the diagram involution to  $\iota$  from Example 7.1.5. In particular, the “coordinate dependence” of the involution  $\iota$  reflects the fact that the semi-direct product structure in (7.1.3) is not intrinsic to  $G$  or even to  $(G, T)$ : it depends on the choice of the splitting and pinning.

The same conclusions apply to  $\text{Aut}(\text{PGL}_n)$ , as well as to  $\text{Aut}(\text{SL}_n/\mu)$  for any  $k$ -subgroup  $\mu \subset \mu_n$ . This illustrates that automorphisms of a connected semisimple  $k$ -group lift uniquely to automorphisms of the simply connected central cover (Exercise 6.5.3(iii)).

The preceding arguments suggest that in the split case we should represent  $\underline{\text{Aut}}_{G/k}$  by a smooth  $k$ -group with identity component  $G/Z_G$  and constant component group  $\text{Aut}(R, \Delta)$ , where  $R = R(G, T)$ . In particular, if  $G$  is semisimple then  $\underline{\text{Aut}}_{G/k}$  should be represented by a linear algebraic  $k$ -group having constant component group that is a subgroup of the finite group  $\text{Aut}(\text{Dyn}(\Phi))$ . This can be made rather concrete: define the  $G/Z_G$ -group isomorphism

$$\alpha^0 : (G/Z_G) \times G \rightarrow (G/Z_G) \times G$$

by passage to the  $Z_G$ -quotient on the first factor relative to the map of  $G$ -group schemes  $(g, g') \mapsto (g, gg'g^{-1})$ , and use translation against the injection  $\text{Aut}(R, \Delta) \hookrightarrow \text{Aut}(G)$  (defined by a choice of pinning) to obtain a “universal automorphism”

$$\alpha : ((G/Z_G) \rtimes \text{Aut}(R, \Delta)) \times G \simeq ((G/Z_G) \rtimes \text{Aut}(R, \Delta)) \times G$$

between group schemes over  $(G/Z_G) \rtimes \text{Aut}(R, \Delta)$ . The fiber map  $\alpha_\xi$  over any  $\xi \in ((G/Z_G) \rtimes \text{Aut}(R, \Delta))(K)$  (for any field extension  $K/k$ ) is the automorphism corresponding to  $\xi$  via the analogue of (7.1.3) for  $K$ -valued points.

This construction defines a morphism of  $k$ -group functors

$$(7.1.4) \quad (G/Z_G) \rtimes \text{Aut}(R, \Delta) \rightarrow \underline{\text{Aut}}_{G/k},$$

and the classical approach would end here since this map has been constructed to be bijective on points valued in any extension field  $K/k$  (which is sufficient for many applications). For problems related to deforming automorphisms or working over  $k$ -algebras that are not fields, it is useful to go beyond field-valued points and prove that (7.1.4) is an isomorphism of functors on  $k$ -algebras.

Now we turn to the general case. Let  $G$  be a reductive  $S$ -group, with  $S \neq \emptyset$ . For the maximal central torus  $Z$  and semisimple derived group  $G' = \mathcal{D}(G)$ , the multiplication map  $Z \times G' \rightarrow G$  is a central isogeny (Corollary 5.3.3). The kernel  $\mu$  of this isogeny is a

finite  $S$ -group of multiplicative type, so an automorphism of  $G$  is “the same” as a pair of automorphisms of  $G'$  and  $Z$  that coincide on  $\mu$ . By Exercise 7.3.1,  $\underline{\text{Aut}}_{\mu/S}$  is a finite étale  $S$ -group and  $\underline{\text{Aut}}_{Z/S}$  is represented by a separated étale  $S$ -group that is an étale form of  $\text{GL}_r(\mathbf{Z})$  where  $r$  is the rank of  $Z$  (locally constant on  $S$ ). Hence, to understand properties of  $\underline{\text{Aut}}_{G/S}$ , the real content is in the semisimple case. Here is the main result.

**Theorem 7.1.9.** *Let  $G$  be a reductive group over a non-empty scheme  $S$ .*

- (1) *The functor  $\underline{\text{Aut}}_{G/S}$  is represented by a separated and smooth  $S$ -group  $\text{Aut}_{G/S}$  that fits into a short exact sequence*

$$1 \rightarrow G/Z_G \rightarrow \text{Aut}_{G/S} \rightarrow \text{Out}_{G/S} \rightarrow 1$$

*where  $G/Z_G$  is closed in  $\text{Aut}_{G/S}$  and  $\text{Out}_{G/S}$  is a separated étale  $S$ -group that is locally constant for the étale topology on  $S$ .*

- (2) *The  $S$ -group  $\text{Out}_{G/S}$  has finite geometric fibers if and only if the maximal central torus of  $G$  has rank  $\leq 1$ , in which case  $\text{Out}_{G/S}$  is  $S$ -finite and  $\text{Aut}_{G/S}$  is  $S$ -affine.*
- (3) *For a pinned split reductive  $S$ -group  $(G, T, M, \{X_a\}_{a \in \Delta})$ ,  $\text{Out}_{G/S}$  is identified with the constant  $S$ -group associated to the group  $\text{Aut}(R(G, T, M), \Delta)$ . Moreover, the pinning defines a semi-direct product splitting of  $S$ -groups*

$$\text{Aut}_{G/S} \simeq (G/Z_G) \rtimes \text{Aut}(R(G, T, M), \Delta)_S.$$

*Proof.* Étale descent is effective for schemes that are separated and étale over the base. (Indeed, one can reduce to the finite type case, which is [SGA1, IX, 4.1]. Alternatively, the descent trivially exists as an algebraic space, and any algebraic space that is separated and locally quasi-finite over a scheme is a scheme [LMB, A.2].) Also, any fppf group-sheaf extension  $E$  of a group scheme  $H$  by an fpqc affine group scheme is necessarily representable since  $E$  is an algebraic space that is affine over  $H$  and hence is a scheme. (Algebraic spaces are not really needed to prove that  $E$  is a scheme; one just has to use the effectivity of fpqc descent for schemes affine over the base.) Thus, for the proof of (1) it suffices to work étale-locally on  $S$ . It is likewise enough to work étale-locally on  $S$  for the proof of (2).

For the proof of the entire theorem, it is now enough to work with a pinned split group  $(G, T, M, \{X_a\}_{a \in \Delta})$ . The arguments leading up to the proof of (7.1.3) were written to apply verbatim at the level of  $S'$ -points over any  $S$ -scheme  $S'$ , upon noting that the appeal to Proposition 6.2.11(2) is valid over the local rings on  $S'$  (and we can “spread out” isomorphisms over local rings on  $S'$  to isomorphisms over Zariski-open sets in  $S'$ ). To conclude, we just need to make three observations: (i) the semi-direct product structure in (3) implies that  $G/Z_G$  is closed in  $\text{Aut}_{G/S}$ , (ii) a constant  $S$ -group is  $S$ -finite if and only if it has finite geometric fibers, (iii) a based root datum  $(R, \Delta)$  has finite automorphism group if and only if the underlying root datum  $R = (M, \Phi, M^\vee, \Phi^\vee)$  satisfies  $\dim M_{\mathbf{Q}}/\mathbf{Q}\Phi \leq 1$  (since  $\text{GL}_1(\mathbf{Z})$  is finite).  $\square$

**Example 7.1.10.** Let  $(G, T, M, \{X_a\}_{a \in \Delta})$  be a pinned split semisimple  $S$ -group whose root system  $\Phi \subset M$  is irreducible and not of type  $D_{2n}$  ( $n \geq 2$ ). By Proposition 1.5.1 we have

$\text{Aut}(R(G, T), \Delta) = \text{Aut}(\text{Dyn}(\Phi))$ , so  $\text{Aut}_{G/S} = (G/Z_G) \rtimes \text{Aut}(\text{Dyn}(\Phi))_S$ . For example, consider  $G = \text{SL}_n$  with the diagonal torus  $T = D_S(M)$  for  $M = \mathbf{Z}^n/\text{diag}(\mathbf{Z})$ ,  $\Delta$  corresponding to the upper triangular Borel subgroup  $B$ , and the standard pinning  $\{X_a\}_{a \in \Delta}$ . We claim that the associated section  $\text{Aut}(\text{Dyn}(\Phi))_S \rightarrow \text{Aut}_{\text{SL}_n}$  carries the diagram involution  $\varphi$  to the involution  $\iota$  of  $\text{SL}_n$  from Example 7.1.5.

Since  $\iota$  is an involution that preserves  $(T, B)$  and induces the nontrivial involution on  $\text{Dyn}(\Phi)$ , we just have to check that its effect on root groups is a permutation of the  $X_a$ 's. The standard root group  $U_{ij}$  for  $i < j$  is carried by  $\iota$  to  $U_{n+1-j, n+1-i}$ , and relative to the standard parameterizations  $\mathbf{G}_a \simeq U_c$  the isomorphism  $U_{ij} \simeq U_{n+1-j, n+1-i}$  goes over to the automorphism  $x \mapsto (-1)^{1+j-i}x$ . Thus, for  $j = i + 1$  we get  $\iota(X_a) = X_{\varphi(a)}$  as desired.

**Example 7.1.11.** Let  $(G, T, M, \{X_a\}_{a \in \Delta})$  be a pinned split reductive  $S$ -group, with  $\Phi \subset M$  the set of roots. The composite map  $\text{Out}_{G/S} \simeq \text{Aut}(R(G, T, M), \Delta)_S \rightarrow \text{Aut}(\text{Dyn}(\Phi))_S$  into the constant group associated to the finite automorphism group of the Dynkin diagram of  $\Phi$  induces a map

$$\mathrm{H}^1(S_{\text{ét}}, \text{Aut}_{G/S}) \rightarrow \mathrm{H}^1(S_{\text{ét}}, \text{Aut}(\text{Dyn}(\Phi))_S)$$

into the pointed set of  $S$ -twisted forms of the Dynkin diagram. More specifically, if  $G'$  is a form of  $G$  over  $S$  then its class  $[G']$  in  $\mathrm{H}^1(S_{\text{ét}}, \text{Aut}_{G/S})$  gives rise to a finite étale  $S$ -scheme  $\text{Dyn}(G')$ , the “scheme of Dynkin diagrams” for  $G'$ . The edges and their multiplicities on geometric fibers are encoded in terms of a finite étale closed subscheme  $\text{Edge}(G') \subset \text{Dyn}(G') \times \text{Dyn}(G')$  disjoint from the diagonal (no loops) and an  $S$ -map  $\text{Dyn}(G') \rightarrow \{1, 2, 3\}_S$  that assigns “squared length”. These  $S$ -schemes  $\text{Dyn}(G')$  and  $\text{Edge}(G')$  are constant when  $[G']$  arises from  $\mathrm{H}^1(S_{\text{ét}}, G/Z_G)$  (i.e., inner twisting of the split form  $G$ ). See [SGA3, XXIV, §3] for further details.

If  $G$  is not  $k$ -split then there may not be a  $k$ -group section to  $\text{Aut}_{G/k} \rightarrow \text{Out}_{G/k}$ , in contrast with the split case in Theorem 7.1.9(3), as even on  $k$ -points this map can be non-surjective. This is related to the existence of Borel  $k$ -subgroups, as we will see in Proposition 7.2.13. Here are some basic examples.

**Example 7.1.12.** Let  $A$  be a central simple algebra of rank  $n^2$  over a field  $k$ , with an integer  $n > 2$ . (Class field theory provides many such  $A$  that are central division algebras when  $k$  is a global or non-archimedean local field.) The algebraic group  $G = \text{SL}(A)$  of units of reduced-norm 1 in  $A$  (Exercise 5.4.5(iv)) is an inner form of  $G_0 = \text{SL}_n$ , and  $\text{Out}_{G/k}$  is a  $k$ -form of the constant  $k$ -group  $\text{Out}_{G_0/k} = (\mathbf{Z}/2\mathbf{Z})_k$  that has no nontrivial  $k$ -forms (since  $\text{Aut}(\mathbf{Z}/2\mathbf{Z}) = 1$ ), so  $\text{Out}_{G/k} = (\mathbf{Z}/2\mathbf{Z})_k$ . We claim that the map  $\text{Aut}_{G/k} \rightarrow \text{Out}_{G/k} = (\mathbf{Z}/2\mathbf{Z})_k$  is not surjective on  $k$ -points (and so has no  $k$ -group section) if and only if  $A$  is not 2-torsion in  $\text{Br}(k)$ . Equivalently, we claim that  $G$  admits a  $k$ -automorphism that is not “geometrically inner” if and only if  $A \simeq A^{\text{opp}}$  as  $k$ -algebras.

To prove the implication “ $\Leftarrow$ ”, we first note that  $G^{\text{opp}} = \text{SL}(A^{\text{opp}})$ , so a  $k$ -algebra isomorphism  $A \simeq A^{\text{opp}}$  induces a  $k$ -isomorphism  $G \simeq G^{\text{opp}}$  that is the identity on the common center (whose finite étale Cartier dual is geometrically cyclic order  $n$ ). Thus, composing with inversion  $G^{\text{opp}} \simeq G$  yields a  $k$ -automorphism of  $G$  that is inversion on the center. But

since  $n > 2$ , inversion on the dual-to-cyclic center is nontrivial. Hence, we have obtained a  $k$ -automorphism of  $G$  that is not geometrically inner.

For the converse implication “ $\Rightarrow$ ”, assume that  $G$  admits a  $k$ -automorphism  $f : G \rightarrow G$  that is not geometrically inner. We claim that  $f$  acts as inversion on the finite center  $Z_G$ , and that it uniquely extends via gluing with inversion on the central  $\mathbf{G}_m$  in  $\underline{A}^\times$  to yield a  $k$ -automorphism of  $\underline{A}^\times$  whose composition with inversion uniquely extends to a  $k$ -algebra isomorphism  $A \simeq A^{\text{opp}}$  (as desired). In view of the uniqueness statements we may (by Galois descent) extend the ground field to  $k_s$ , so  $A = \text{Mat}_n$  and  $G = \text{SL}_n$ . Since  $G/Z_G = \underline{A}^\times/\mathbf{G}_m$ , so  $(G/Z)(k) = A^\times/k^\times$ , it is harmless to precompose  $f$  with the effect on  $G$  of an inner automorphism of  $A$ . Thus, we can focus on the case of a single  $k$ -automorphism of  $\text{SL}_n$  that is not geometrically inner (as any two are related through composition against an inner automorphism of  $A$  over  $k$ ). Hence, we may assume that  $f$  is the transpose-inverse automorphism of  $\text{SL}_n$  (here we use that  $n > 2$ ). This is inversion on the central  $\mu_n$ , which visibly uniquely glues with inversion on the central  $\mathbf{G}_m$  to define a  $k$ -automorphism of  $\text{GL}_n$  (namely, transpose-inverse). By the Zariski-density of  $\text{GL}_n$  in  $\text{Mat}_n$ , its composition with inversion uniquely extends to an algebra anti-automorphism of  $\text{Mat}_n$  (namely, transpose).

**Example 7.1.13.** The automorphism scheme has an interesting application in the non-split case over  $\mathbf{R}$ . Let  $G$  be an  $\mathbf{R}$ -anisotropic connected semisimple  $\mathbf{R}$ -group. (Such  $G$  correspond precisely to the connected compact Lie groups via  $G \rightsquigarrow G(\mathbf{R})$ , and  $G(\mathbf{R})$  is a maximal compact subgroup of  $G(\mathbf{C})$ ; see Theorem D.2.4 and Proposition D.3.2.) The  $\mathbf{R}$ -group  $\text{Aut}_{G/\mathbf{R}}$  has identity component  $G/Z_G$ , so it is also  $\mathbf{R}$ -anisotropic. In [Ser97, III, 4.5, Ex. (b)], the anisotropy of  $\text{Aut}_{G/\mathbf{R}}$  is used to recover E. Cartan’s classification of  $\mathbf{R}$ -forms of a connected semisimple  $\mathbf{C}$ -group: the  $\mathbf{R}$ -forms of  $G_{\mathbf{C}}$  (up to isomorphism) are in bijection with conjugacy classes of involutions  $\iota$  of the maximal compact subgroup  $G(\mathbf{R})$  of  $G(\mathbf{C})$ .

This can be pushed a step further: the map  $\iota \mapsto \iota_{\mathbf{C}}$  from conjugacy classes of involutions of  $G(\mathbf{R})$  to conjugacy classes of involutions of the “complexification”  $G_{\mathbf{C}}$  of  $G(\mathbf{R})$  is a bijection (apply [Ser97, III, 4.5, Thm. 6] to the anisotropic  $\text{Aut}_{G/\mathbf{R}}$ ), so the  $\mathbf{R}$ -forms  $H$  of a connected semisimple  $\mathbf{C}$ -group  $\mathcal{G}$  are classified up to isomorphism by conjugacy classes of involutions of  $\mathcal{G}$ . This can be made more concrete by using Mostow’s description [Mos, §6] of maximal compact subgroups of  $H(\mathbf{R})$  in terms of maximal compact subgroups of  $\mathcal{G}(\mathbf{C})$ : one can show that the conjugacy class of involutions of  $\mathcal{G}$  corresponding to  $H$  contains  $(\theta_K)_{\mathbf{C}}$ , where  $\theta_K$  is the Cartan involution of  $H$  associated to a maximal compact subgroup  $K$  of  $H(\mathbf{R})$ . Equivalently, if  $\mathcal{G}_0$  is the split  $\mathbf{R}$ -form of  $\mathcal{G}$  then the descent datum on  $\mathcal{G}$  relative to the  $\mathbf{R}$ -structure  $H$  corresponds to the involution  $g \mapsto (\theta_K)_{\mathbf{C}}(\bar{g})$  of the real Lie group  $\mathcal{G}(\mathbf{C}) = \mathcal{G}_0(\mathbf{C})$  for any maximal compact subgroup  $K \subset H(\mathbf{R})$ .

**7.2. Cohomological approach to forms.** As an application of the structure of the automorphism scheme, we can gain some insight into the cohomological description (in Lemma 7.1.1) of the set of isomorphism classes of forms of a given connected reductive group over a field. We begin with the situation over a field because the relevant cohomology is Galois

cohomology, which is more concrete than étale cohomology. Later we will generalize to the case of an arbitrary connected non-empty base scheme.

Let  $G$  be a connected reductive group over a field  $k$ , so for the split  $k$ -form  $G_0$  of  $G$  the  $k$ -group  $\mathrm{Aut}_{G/k}$  is a  $k$ -form of  $\mathrm{Aut}_{G_0/k}$ . We shall explicitly describe how to build this  $k$ -form as an instance of “inner cocycle-twisting” from [Ser97, I, §5.3]. Fix a  $k_s$ -isomorphism  $\varphi : G_{k_s} \simeq (G_0)_{k_s}$ , so we get a  $k_s$ -isomorphism

$$\xi : (\mathrm{Aut}_{G/k})_{k_s} = \mathrm{Aut}_{G_{k_s}/k_s} \simeq \mathrm{Aut}_{(G_0)_{k_s}/k_s} = (\mathrm{Aut}_{G_0/k_s})_{k_s}$$

via  $f \mapsto \varphi \circ f \circ \varphi^{-1}$ . The obstruction to  $\mathrm{Gal}(k_s/k)$ -equivariance for  $\xi$  is expressed by the following identity for  $\gamma \in \mathrm{Gal}(k_s/k)$ :

$$\xi(\gamma^*(f)) = (\varphi \circ \gamma^*(\varphi)^{-1}) \circ \gamma^*(\xi(f)) \circ (\varphi \circ \gamma^*(\varphi)^{-1})^{-1}.$$

In other words, for the 1-cocycle  $c : \gamma \mapsto \varphi \circ \gamma^*(\varphi)^{-1}$  in  $Z^1(k_s/k, \mathrm{Aut}((G_0)_{k_s}))$  whose cohomology class classifies  $G$  as a form of  $G_0$  in Lemma 7.1.1, the  $k$ -form  $\mathrm{Aut}_{G/k}$  of  $\mathrm{Aut}_{G_0/k}$  is obtained through inner twisting by the 1-cocycle  $c$ .

Using cocycle-twisting notation of [Ser97, I, §5.3, Ex. 2] defines an isomorphism

$$(7.2.1) \quad \mathrm{Aut}_{G/k} = {}_c \mathrm{Aut}_{G_0/k}.$$

This identification depends on  $c$ , not just the cohomology class of  $c$ . (Replacing  $\varphi$  with  $\psi \circ \varphi$  for  $\psi \in \mathrm{Aut}((G_0)_{k_s})$  has the effect of replacing  $c$  with a cohomologous 1-cocycle  $c' : \gamma \mapsto \psi^{-1} \circ c_\gamma \circ (\gamma^*(\psi))$ , and  $\psi$  defines an isomorphism  ${}_c \mathrm{Aut}_{G_0/k} \simeq {}_{c'} \mathrm{Aut}_{G_0/k}$  respecting the identifications with  $\mathrm{Aut}_{G/k}$ .) Since this twisting process is defined through cocycles valued in inner automorphisms of the group  $\mathrm{Aut}_{G_0/k}(k_s)$ , which in turn preserve *normal*  $k_s$ -subgroups of  $(\mathrm{Aut}_{G_0/k})_{k_s}$ , we have the compatible equalities of normal  $k$ -subgroups  $G/Z_G = {}_c(G_0/Z_{G_0})$  and quotients  $\mathrm{Out}_{G/k} = {}_c \mathrm{Out}_{G_0/k}$  (where we are now twisting by 1-cocycles valued in the automorphism functors of the  $k$ -groups  $G_0/Z_{G_0}$  and  $\mathrm{Out}_{G_0/k}$ ).

In general, if  $\mathcal{G}$  is a smooth affine  $k$ -group then a continuous 1-cocycle  $a : \mathrm{Gal}(k_s/k) \rightarrow \mathrm{Aut}(\mathcal{G}_{k_s})$  defines a  $k$ -form  ${}_a \mathcal{G}$  of  $\mathcal{G}$  whose isomorphism class as a  $k$ -group only depends on the class of  $a$  in  $H^1(k_s/k, \mathrm{Aut}(\mathcal{G}_{k_s}))$ . It is an important fact that if  $a$  lifts to a 1-cocycle  $\tilde{a} \in Z^1(k_s/k, \mathcal{G}(k_s))$  then a choice of such  $\tilde{a}$  defines a bijection of sets  $t_{\tilde{a}} : H^1(k_s/k, {}_a \mathcal{G}) \simeq H^1(k_s/k, \mathcal{G})$  functorially in the pair  $(\mathcal{G}, \tilde{a})$  and carrying the base point to the class of  $\tilde{a}$  [Ser97, I, §5.3, Prop. 35bis]. This can be described in terms of the more conceptual language of torsors (say for the étale topology, though it is equivalent to use the fppf topology since  $\mathcal{G}$  is smooth): if  $Y$  is a right  $\mathcal{G}$ -torsor whose isomorphism class in  $H^1(k, \mathcal{G})$  is represented by  $\tilde{a}$  then  ${}_a \mathcal{G}$  is the automorphism scheme  $\underline{\mathrm{Aut}}_{\mathcal{G}}(Y)$  of the right  $\mathcal{G}$ -torsor  $Y$  and  $t_{\tilde{a}}^{-1}$  carries the class of a right  $\mathcal{G}$ -torsor  $X$  to the class of the right  $\underline{\mathrm{Aut}}_{\mathcal{G}}(Y)$ -torsor  $\underline{\mathrm{Isom}}(X, Y)$ . See [Ser97, I, 5.3] or [Con11, App. B] for further details.

It is important to distinguish between the images in  $H^1(k_s/k, \mathrm{Aut}(\mathcal{G}_{k_s}))$  of  $H^1(k, \mathcal{G})$  and  $H^1(k, \mathcal{G}/Z_{\mathcal{G}})$ . The  $k$ -groups  ${}_a \mathcal{G}$  arising from  $a$  in the image of  $H^1(k, \mathcal{G}/Z_{\mathcal{G}})$  are called *inner forms* of  $\mathcal{G}$  (because  $\mathcal{G}/Z_{\mathcal{G}}$  is considered to be the scheme of “inner automorphisms” of  $\mathcal{G}$ ), and the inner forms arising from  $a$  in the image of  $H^1(k, \mathcal{G})$  are called *pure inner forms*. In general, many inner forms are not pure inner forms, and the map  $H^1(k, \mathcal{G}) \rightarrow H^1(k, \mathcal{G}/Z_{\mathcal{G}})$

is neither injective nor surjective. Pure inner forms play an important role in local harmonic analysis and the local Langlands correspondence; see [GGP, §2] for some examples with classical groups. We have seen above that passage to a pure inner form does not change the degree-1 Galois cohomology (as a set). The same is not true for passage to general inner forms:

**Example 7.2.1.** For  $n > 1$  we have  $H^1(k, \mathrm{SL}_n) = 1$ . Thus,  $\mathrm{SL}_n$  has no nontrivial pure inner forms. The quotient  $\mathrm{SL}_n/Z_{\mathrm{SL}_n}$  is  $\mathrm{PGL}_n$ , and the pointed set  $H^1(k, \mathrm{PGL}_n)$  classifies rank- $n^2$  central simple  $k$ -algebras  $A$ . The associated inner forms of  $\mathrm{SL}_n$  are the  $k$ -groups  $\mathrm{SL}(A)$ .

For a rank- $n^2$  central division algebra  $D$  over  $k$ , we claim that the  $k$ -form  $\mathrm{SL}_{1,D} := \mathrm{SL}(D)$  of  $\mathrm{SL}_n$  can have *nontrivial* degree-1 Galois cohomology (in which case  $H^1(k, \mathrm{SL}_{1,D})$  has no bijection with  $H^1(k_s/k, \mathrm{SL}_n)$ ). To see this, consider the exact sequence of  $k$ -groups

$$1 \rightarrow \mathrm{SL}_{1,D} \rightarrow \underline{D}^\times \xrightarrow{\mathrm{Nrd}} \mathbf{G}_m \rightarrow 1.$$

The pointed set  $H^1(k, \underline{D}^\times)$  is trivial: it classifies étale sheaves of rank-1 left modules over the quasi-coherent sheaf  $D$  on  $\mathrm{Spec} k$ , and the only such object up to isomorphism is  $D$  (due to effectivity of étale descent for quasi-coherent sheaves and the freeness of finitely generated  $D$ -modules). Thus,  $H^1(k, \mathrm{SL}_{1,D})$  is trivial if and only if the reduced norm map  $\nu_D : \underline{D}^\times \rightarrow k^\times$  is surjective. Already with  $n = 2$ , among quaternion division algebras over  $\mathbf{Q}$  one finds many examples where such surjectivity fails. For example, if  $D_{\mathbf{R}} \simeq \mathbf{H}$  (there are many such  $D$ , by class field theory) then  $\nu_D$  does not hit  $\mathbf{Q}_{<0}$ . (The surjectivity of the reduced norm map for all  $D$  over  $k$  is closely related to  $k$  being a  $C_r$ -field with  $r \leq 2$ ; see [Ser97, II, 4.5].)

Twisting against the  $\mathrm{Aut}_{G_0/k}(k_s)$ -valued 1-cocycle  $c$  and using (7.2.1) defines a bijection of sets

$$H^1(k, \mathrm{Aut}_{G/k}) \simeq H^1(k, \mathrm{Aut}_{G_0/k}).$$

By construction this carries the trivial point to the class of  $G$  in  $H^1(k, \mathrm{Aut}_{G_0/k})$ , and it induces a compatible bijection  $H^1(k, \mathrm{Out}_{G/k}) \simeq H^1(k, \mathrm{Out}_{G_0/k})$  (viewing  $c$  with values in the automorphism functor of  $\mathrm{Out}_{G_0/k} = \mathrm{Aut}(R, \Delta)_k$ ). That is, we have a commutative diagram of exact sequences of pointed sets

$$(7.2.2) \quad \begin{array}{ccccc} H^1(k, G/Z_G) & \longrightarrow & H^1(k, \mathrm{Aut}_{G/k}) & \longrightarrow & H^1(k, \mathrm{Out}_{G/k}) \\ & & \downarrow \simeq & & \downarrow \simeq \\ H^1(k, G_0/Z_{G_0}) & \longrightarrow & H^1(k, \mathrm{Aut}_{G_0/k}) & \longrightarrow & H^1(k, \mathrm{Out}_{G_0/k}) \end{array}$$

We do *not* claim to construct a map between the leftmost terms in (7.2.2) making a commutative square. But the commutativity and exactness imply that the fiber of  $H^1(k, \mathrm{Aut}_{G_0/k}) \rightarrow H^1(k, \mathrm{Out}_{G_0/k})$  through  $[G]$  is identified with the image of  $H^1(k, G/Z_G) \rightarrow H^1(k, \mathrm{Aut}_{G/k})$ ; cf. [Ser97, I, §5.5, Prop. 39, Cor. 2]. The image of  $H^1(k, G/Z_G)$  in  $H^1(k, \mathrm{Aut}_{G/k})$  consists of the  $k$ -forms of  $G$  obtained by twisting by 1-cocycles valued in  $(G/Z_G)(k_s)$ , which we called the *inner forms* of  $G$ . We have proved:

**Proposition 7.2.2.** *The fibers of  $H^1(k, \text{Aut}_{G_0/k}) \rightarrow H^1(k, \text{Out}_{G_0/k})$  are precisely the classes that are inner forms of each other.*

The commutativity of (7.2.2) also gives that  $H^1(k, \text{Aut}_{G/k}) \rightarrow H^1(k, \text{Out}_{G/k})$  is *surjective*, since the analogous surjectivity holds for  $G_0$  (due to a  $k$ -group section to  $\text{Aut}_{G_0/k} \rightarrow \text{Out}_{G_0/k}$  defined by a choice of pinning).

If  $k$  is a finite field then  $H^1(k, G/Z_G)$  is a single point, due to Lang's Theorem [Bo91, 16.5(i)]. In equivalent terms, the fibers of  $\text{Aut}_{G/k} \rightarrow \text{Out}_{G/k}$  over  $k$ -points are  $G/Z_G$ -torsors and hence all have a  $k$ -point (as  $G/Z_G$  is a smooth connected  $k$ -group). Let's push this case further:

**Example 7.2.3.** Let  $k$  be a finite field, and let  $(G_0, T_0, M)$  be a split connected reductive  $k$ -group. Let  $R$  be the root datum and  $\Delta$  a base of the associated root system, so  $\Theta := \text{Aut}(R, \Delta)$  is identified with the constant component group  $\text{Out}_{G_0/k}$  of  $\text{Aut}_{G_0/k}$ . The natural surjective map  $H^1(k, \text{Aut}_{G_0/k}) \rightarrow H^1(k, \Theta_k)$  is bijective because the fiber through each class  $[G]$  is an image of the set  $H^1(k, G/Z_G)$  that consists of a single point.

Since  $\Theta_k$  is a constant group and  $\text{Gal}(k_s/k) = \widehat{\mathbf{Z}}$  with generator given by  $\text{Frob}_k$ , the cohomology set  $H^1(k, \Theta_k)$  consists of conjugacy classes of elements  $\theta \in \Theta = \text{Aut}(R, \Delta)$ . Hence, the isomorphism classes of  $k$ -forms of  $G_0$  are classified by the conjugacy classes in  $\Theta = \text{Aut}(R, \Delta)$ . In more concrete terms, if we fix a pinning  $\{X_a\}_{a \in \Delta}$  and identify  $\Theta = \text{Aut}(R, \Delta)$  with  $\text{Aut}(G_0, T_0, M, \{X_a\}_{a \in \Delta})$  via Theorem 6.1.17 then to each  $\theta \in \Theta$  we associate the  $k$ -form  $G_\theta$  of  $G_0$  obtained by replacing the  $\text{Frob}_k$ -action on  $G_0(k_s)$  with the usual action followed by the action of  $\theta$  on  $G_0$  via the pinning.

For example, if  $G_0$  is semisimple and either simply connected or adjoint (or  $\Phi$  is irreducible with cyclic fundamental group) then the set of isomorphism classes of  $k$ -forms of  $G_0$  is in bijective correspondence with the set of conjugacy classes in  $\text{Aut}(\text{Dyn}(\Phi))$ . Since a connected semisimple  $k$ -group is split if and only if its simply connected central cover is split, we conclude that a connected semisimple  $k$ -group with irreducible root system  $\Phi$  over  $k_s$  is necessarily split whenever  $\text{Dyn}(\Phi)$  has no nontrivial automorphisms. This holds for types  $A_1, B, C, E_7, E_8, F_4$ , and  $G_2$ . (For type  $A_1$ , this expresses the fact that  $\mathbf{P}^1$  has no nontrivial forms over the finite field  $k$ .)

Among the simply connected semisimple  $k$ -groups of a fixed type  $A_n$  with  $n > 1$  or  $D_n$  with  $n \geq 5$ , there is exactly one nontrivial  $k$ -form since  $\text{Aut}(\text{Dyn}(\Phi)) = \mathbf{Z}/2\mathbf{Z}$  for such  $\Phi$ . For type  $D_4$  there are two nontrivial  $k$ -forms since  $\text{Aut}(\text{Dyn}(D_4)) = S_3$  has two nontrivial conjugacy classes. For example, in type  $A_n$  ( $n \geq 2$ ) a non-split simply connected form is  $\text{SU}_{n+1}(k'/k)$ , where  $k'/k$  is a quadratic extension.

**Remark 7.2.4.** Let  $S = \text{Spec } R$  for a henselian local ring  $R$  with finite residue field  $k$  (e.g., the valuation ring of a non-archimedean local field). The classification in Example 7.2.3 via conjugacy classes in  $\Theta$  works over  $S$  because  $\pi_1(S) = \pi_1(\text{Spec } k)$  and every smooth surjection  $X \rightarrow S$  acquires a section over the connected finite étale cover  $S' \rightarrow S$  corresponding to any finite extension  $k'/k$  such that  $X_k(k') \neq \emptyset$ .

Consider a reductive group  $G$  over a non-empty scheme  $S$ . The *inner forms* of  $G$  are the forms of  $G$  that correspond to the image of  $H^1(S_{\text{ét}}, G/Z_G)$  in  $H^1(S_{\text{ét}}, \text{Aut}_{G/S})$ . If all geometric fibers of  $G$  have the same root datum  $R$  (as occurs when  $S$  is connected) and if  $G_0$  denotes the split reductive  $S$ -group with root datum  $R$ , then the inner forms of  $G$  correspond to the fiber through the class  $[G]$  of  $G$  under the map

$$H^1(S_{\text{ét}}, \text{Aut}_{G_0/S}) \rightarrow H^1(S_{\text{ét}}, \text{Out}_{G/S}).$$

Indeed, when  $S = \text{Spec } k$  for a field  $k$  this is Proposition 7.2.2, and a variant of the method of proof works in general (Exercise 7.3.8).

**Example 7.2.5.** The inner forms of  $\text{SL}_n$  over a field  $k$  are the  $k$ -groups  $\text{SL}(A)$  for central simple  $k$ -algebras  $A$  of rank  $n^2$  (see Exercise 5.4.5(iv) for  $\text{SL}(A)$ ). Indeed,  $\text{SL}(A)$  is the image under  $f : H^1(k, \text{PGL}_n) \rightarrow H^1(k, \text{Aut}_{\text{SL}_n})$  of the class of  $A$  in  $H^1(k, \text{PGL}_n) \subset \text{Br}(k)[n]$ . Since  $\ker f = 1$ , so  $\text{SL}(A) \not\cong \text{SL}_n$  when  $A$  is not split, the uniqueness in Proposition 7.2.13 implies that  $\text{SL}(A)$  is not quasi-split over  $k$  when  $A$  is not a matrix algebra over  $k$ .

An automorphism  $\phi \in \text{Aut}_{G/S}(S')$  is *inner* if it arises from  $(G/Z_G)(S')$  (such an automorphism may not arise from  $G(S')!$ ). Since the  $S$ -group  $\text{Aut}_{G/S}/(G/Z_G) = \text{Out}_{G/S}$  is étale and separated over  $S$ , so an element of  $\text{Out}_{G/S}(S)$  is trivial if and only if it is so on geometric fibers over  $S$ , an automorphism of  $G$  is inner if and only if it is so on geometric fibers over  $S$ . Here is an interesting application:

**Proposition 7.2.6.** *Let  $G \rightarrow S$  be a reductive group scheme, and  $H \rightarrow S$  a group scheme with connected fibers. Any action of  $H$  on  $G$  must be through inner automorphisms; i.e., it is the composition of a unique  $S$ -homomorphism  $H \rightarrow G/Z_G$  followed by the natural action of  $G/Z_G$  on  $G$ .*

*Proof.* To give an action of  $H$  on  $G$  is to give an  $S$ -homomorphism  $H \rightarrow \text{Aut}_{G/S}$ , so we just have to show that the composite map  $H \rightarrow \text{Out}_{G/S}$  is trivial. Since  $\text{Out}_{G/S}$  is locally constant over  $S$ , it suffices to work on geometric fibers over  $S$ , where the result is clear.  $\square$

We now discuss the classification of forms in the semisimple case over a general connected non-empty scheme  $S$ . Fix a semisimple reduced root datum  $R$ , and let  $G_0$  be the split semisimple  $S$ -group with root datum  $R$ . For any semisimple  $S$ -group  $G$  whose geometric fibers have root datum  $R$ , the simply connected central cover  $\tilde{G}$  is a form of  $\tilde{G}_0$  with root datum  $R^{\text{sc}}$  equal to the simply connected “cover” of  $R$ . Likewise, the adjoint quotient  $G/Z_G$  is a form of  $G_0/Z_{G_0}$  with root datum  $R^{\text{ad}}$  equal to the adjoint “quotient” of  $R$ . The kernel of the covering map  $\tilde{G}_0 \rightarrow G_0/Z_{G_0} = \tilde{G}_0/Z_{\tilde{G}_0}$  is the finite multiplicative type group  $Z_{\tilde{G}_0} = D_S(\Pi_0)$  where  $\Pi_0$  is the “fundamental group”  $(\mathbf{Z}\Phi^\vee)^*/\mathbf{Z}\Phi$ . The kernel  $\mu = \ker(\tilde{G} \rightarrow G) \subset Z_{\tilde{G}}$  is a form of  $\mu_0 := \ker(\tilde{G}_0 \rightarrow G_0) \subset D_S(\Pi_0)$ .

The problem of classifying the possibilities for  $G$  falls into two parts: classify the forms of  $\tilde{G}_0$ , and then for each such form determine if the twisting process applied to  $D_S(\Pi_0)$  preserves the subgroup  $\mu_0$ . This second part is always affirmative when  $\Pi_0$  is cyclic, such as for irreducible  $\Phi$  not of type  $D_{2n}$ . Also, it can be studied on a single geometric fiber

and is always a purely combinatorial problem since the action of the  $S$ -subgroup  $G_0/Z_{G_0} = \widetilde{G}_0/Z_{\widetilde{G}_0} \subset \text{Aut}_{\widetilde{G}_0/S}$  has no effect on the center of  $\widetilde{G}_0$ .

It follows that we lose little of the real content of the classification problem in the semisimple case by focusing on simply connected  $G$ . Thus, the root datum decomposes as a direct product according to the irreducible components of the root system  $\Phi$ . Setting aside the combinatorial problem of permutations of irreducible components of  $\Phi$  in the twisting process, we likewise lose little generality by assuming that  $\Phi$  is irreducible. Then the automorphism group of the based root datum coincides with the automorphism group of the Dynkin diagram (i.e., no problems arise for type  $D_{2n}$  as in Example 1.5.2), so we obtain:

**Corollary 7.2.7.** *Let  $\Phi$  be an irreducible reduced root system, and  $S$  a connected non-empty scheme. The set of isomorphism classes of simply connected and semisimple  $S$ -groups with root system  $\Phi$  on geometric fibers is in natural bijection with*

$$(7.2.3) \quad H^1(S_{\text{ét}}, (G_0/Z_{G_0}) \rtimes \text{Aut}(\text{Dyn}(\Phi))_S),$$

where  $(G_0, T_0, M, \{X_a\}_{a \in \Delta})$  is the pinned split simply connected and semisimple  $S$ -group with root system  $\Phi$ .

The cohomological description of forms via (7.2.3) is useful in multiple ways. Firstly, if we make many constructions of forms then the cohomological viewpoint can be helpful for proving that all possibilities have been exhausted. Secondly, cohomology provides an efficient mechanism for understanding the conceptual meaning of invariants that enter into a classification theorem, such as auxiliary Galois extensions that occur in a construction (e.g., the local invariants that arise in the Hasse–Minkowski theorem for non-degenerate quadratic forms over global fields).

Here is an example that illustrates the usefulness of the fact that the automorphism scheme of a semisimple group scheme is smooth and affine.

**Example 7.2.8.** Let  $F$  be a global field, and  $\Sigma$  a non-empty finite set of places of  $F$  containing the archimedean places. The étale cohomology set  $H^1(\mathcal{O}_{F,\Sigma}, \mathcal{G})$  is finite for any smooth affine  $\mathcal{O}_{F,\Sigma}$ -group  $\mathcal{G}$  with reductive fibers such that the order of the component group of  $\mathcal{G}_{\overline{F}}$  is not divisible by  $\text{char}(F)$  [GM, 5.1, §7]. (The hypothesis of reductive fibers can easily be removed when  $\text{char}(F) = 0$ , but to do so when  $\text{char}(F) > 0$  requires a local-global finiteness result [Con11, 1.3.3(i)] which rests on the structure theory of pseudo-reductive groups.) In particular, if  $G_0$  is a split semisimple  $\mathcal{O}_{F,\Sigma}$ -group whose root system  $\Phi$  is irreducible then  $H^1(\mathcal{O}_{F,\Sigma}, \text{Aut}_{G_0/\mathcal{O}_{F,\Sigma}})$  is finite provided that  $\text{char}(F)$  does not divide the order of  $\text{Aut}(\text{Dyn}(\Phi))$ . By inspecting the list of Dynkin diagrams, the restriction on  $\text{char}(F)$  only arises when  $\text{char}(F) \in \{2, 3\}$ .

[This restriction on the characteristic is genuine: when  $G_0$  is simply connected and its connected Dynkin diagram has an automorphism of order  $p = \text{char}(F) \in \{2, 3\}$  then the infinitely many degree- $p$  Artin–Schreier extensions of  $F$  unramified outside  $\Sigma$  give rise to infinitely many pairwise non-isomorphic  $\mathcal{O}_{F,\Sigma}$ -forms of  $G_0$ . There is a similar infinitude

phenomenon over local function fields of characteristic 2 or 3. Examples include special unitary groups associated to quadratic Galois extensions  $F'/F$  in characteristic 2.]

We conclude that as long as  $\text{char}(F) \neq 2, 3$ , up to isomorphism there are only *finitely many* semisimple  $\mathcal{O}_{F,\Sigma}$ -groups with a given irreducible root datum over  $\overline{F}$ . Beware that this is *not* saying anything about the number of (isomorphism classes of) connected semisimple  $F$ -groups that admit a specific semisimple  $\mathcal{O}_{F,\Sigma}$ -model. Indeed, since  $\text{Aut}_{G_0/\mathcal{O}_{F,\Sigma}}$  is not  $\mathcal{O}_{F,\Sigma}$ -proper, if  $\mathcal{G}$  and  $\mathcal{G}'$  are  $\mathcal{O}_{F,\Sigma}$ -forms of  $G_0$  then the Isom-scheme  $\text{Isom}(\mathcal{G}, \mathcal{G}')$  is not  $\mathcal{O}_{F,\Sigma}$ -proper and hence may have an  $F$ -point that does not extend to an  $\mathcal{O}_{F,\Sigma}$ -point. This is illustrated in the next example.

**Example 7.2.9.** Let  $\Phi$  be an irreducible and reduced root system such that  $\text{Aut}(\text{Dyn}(\Phi)) = 1$  (i.e., types  $A_1, B, C, E_7, E_8, F_4, G_2$ ). Let  $G_0$  be the split simply connected semisimple  $S$ -group with root system  $\Phi$ . The cohomology set in Corollary 7.2.7 is  $H^1(S_{\text{ét}}, G_0/Z_{G_0})$ , so we get an exact sequence of pointed sets

$$H^1(S, Z_{G_0}) \rightarrow H^1(S, G_0) \rightarrow H^1(S, \text{Aut}_{G_0/S}) \rightarrow H^2(S, Z_{G_0})$$

(using the fppf topology if  $Z_{G_0}$  is not smooth; see Exercise 7.3.4 for the effect on  $H^1(S, G_0)$  and  $H^1(S, \text{Aut}_{G_0/S})$  when passing from the étale topology to the fppf topology). For example, if  $S = \text{Spec } \mathbf{Z}$  then  $Z_{G_0}$  is a product of various  $\mu_n$ 's and  $H^2(\mathbf{Z}, \mu_n) = H^2(\mathbf{Z}, \mathbf{G}_m)[n]$  (using fppf cohomology). But  $H^2(\mathbf{Z}, \mathbf{G}_m)$  is the same whether we use fppf or étale topologies [BrIII, 11.7], so since  $\mathbf{Z}$  has only one archimedean place, the group  $H^2(\mathbf{Z}, \mathbf{G}_m)$  vanishes by global class field theory (see [Mi, III, Ex. 2.22(f)]). Likewise, since  $\text{Pic}(\mathbf{Z}) = 1$  and  $\mathbf{Z}^\times = \{\pm 1\}$ , we have  $H^1(\mathbf{Z}, Z_{G_0}) = \Pi_0/(2)$  and

$$H^1(\mathbf{Z}, \text{Aut}_{G_0/S}) = H^1(\mathbf{Z}, G_0)/H^1(\mathbf{Z}, Z_{G_0}).$$

One can give an a Weil-style adelic description of  $H^1(\mathbf{Z}, G_0)$  when it is partitioned according to the genus (i.e., classes that have the same image in every  $H^1(\mathbf{Z}_p, G_0)$ ). See [CG, §4] for examples of non-isomorphic semisimple  $\mathbf{Z}$ -groups whose  $\mathbf{Q}$ -fibers are isomorphic.

The case  $S = \text{Spec } K$  for a global or local field  $K$  works out nicely: since  $G_0$  is simply connected,  $H^1(K, G_0)$  is rather small and entirely understood (by work of Kneser, Harder, Bruhat–Tits, and Chernousov), and the contribution from  $H^2(K, Z_{G_0}) \subset \text{Br}(K)$  is well-understood. This illustrates the general fact that the classification of connected semisimple groups over a field is intimately tied up with the Galois cohomological properties of the field (e.g., the structure of the Brauer group).

To prove more definitive classification results over a field, especially over arithmetically interesting fields, an entirely different approach is required. One has to use the finer structure theory of Borel–Tits that involves rational conjugacy for maximal  $k$ -split tori and minimal parabolic  $k$ -subgroups, relative root systems, and the “ $*$ -action” of  $\text{Gal}(k_s/k)$  on the Dynkin diagram (see Remark 7.1.2). This gives a classification of  $k$ -groups “modulo the  $k$ -anisotropic groups” (whose structure depends on special features of  $k$ ).

As we noted in Remark 7.1.2, the  $*$ -action of  $\text{Gal}(k_s/k)$  on the diagram can be refined to an action on the based root datum  $(R(G), \Delta)$ . If  $(G_0, T_0, M_0, \{X_a\}_{a \in \Delta_0})$  denotes the pinned

split connected reductive  $k$ -group with  $(R(G_0), \Delta_0) = (R(G), \Delta)$  then the homomorphism

$$\mathrm{Gal}(k_s/k) \rightarrow \mathrm{Aut}((R(G), \Delta)) = \mathrm{Out}_{G_0/k}(k)$$

defines a class in  $H^1(k, \mathrm{Out}_{G_0/k})$ . The image of this class under the natural section to  $H^1(k, \mathrm{Aut}_{G_0/k}) \rightarrow H^1(k, \mathrm{Out}_{G_0/k})$  is the quasi-split inner form of  $G$ , in view of the proof of uniqueness of this inner form in Proposition 7.2.13. See the tables at the end of [Ti66a] and [Spr] for the classification of  $k$ -forms for general as well as special fields  $k$ . (Examples without this extra technology are also discussed in [Ser97, III, 1.4, 2.2–2.3, 3.2] and [Ser97, III, App. 2, §3].) See [PS] for a relativization over arbitrary semilocal rings.

**Example 7.2.10.** The semi-direct product structure of  $\mathrm{Aut}_{G_0/S}$  in Theorem 7.1.9(3) is generally destroyed under passage to a form of  $G_0$  that is not quasi-split (Example 7.1.12), and this presents serious issues in any attempt to explicitly describe the degree-1 cohomology. For example, if  $G_0 = \mathrm{SL}_n$  over a field  $k$  with  $n > 2$ , the exact sequence

$$1 \rightarrow \mathrm{PGL}_n \rightarrow \mathrm{Aut}_{\mathrm{SL}_n/k} \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow 1$$

induces an exact sequence of pointed sets

$$(7.2.4) \quad H^1(k, \mathrm{PGL}_n) \rightarrow H^1(k, \mathrm{Aut}_{\mathrm{SL}_n/k}) \rightarrow H^1(k, \mathbf{Z}/2\mathbf{Z}).$$

The map on the right must be surjective, since the automorphism scheme splits as a semi-direct product over  $k$ , and likewise the map on the left has trivial kernel (as a map of pointed sets) though it is not generally injective (Example 7.1.12). In Example 7.2.5 we addressed the image of  $H^1(k, \mathrm{PGL}_n)$ .

For any form  $G$  of  $\mathrm{SL}_n$  there is an associated class in  $H^1(k, \mathbf{Z}/2\mathbf{Z})$  via (7.2.4). This class is trivial precisely when  $G$  comes from  $H^1(k, \mathrm{PGL}_n)$ , which is to say that it is one of the norm-1 unit groups as in Example 7.2.5. Let us focus on the case when the class in  $H^1(k, \mathbf{Z}/2\mathbf{Z})$  is nontrivial, so it corresponds to a quadratic separable extension field  $k'/k$ . The fiber in  $H^1(k, \mathrm{Aut}_{\mathrm{SL}_n/k})$  over the class  $[k']$  of  $k'/k$  in  $H^1(k, \mathbf{Z}/2\mathbf{Z})$  contains a unique quasi-split form  $G$ . We wish to describe it explicitly (and then its inner forms will exhaust the entire fiber). The group  $G_{k'}$  is identified with  $\mathrm{SL}_n$  so that for any commutative  $k$ -algebra  $A$  the subgroup

$$G(A) \subset G(k' \otimes_k A) = \mathrm{SL}_n(k' \otimes_k A) = \mathrm{R}_{k'/k}(\mathrm{SL}_n)(A)$$

consists of the points  $g' \in \mathrm{SL}_n(k' \otimes_k A)$  satisfying  $g' = \iota(\bar{g}')$ , where  $\iota$  is as in Example 7.1.10 and  $c' \mapsto \bar{c}'$  is the nontrivial  $k$ -automorphism of  $k'$ . Since  $\iota$  is an involution, for  $w$  as in Example 7.1.5 the condition  $g' = \iota(\bar{g}')$  says exactly

$$\bar{g}' = \iota(g') = w(g'^{\top})^{-1}w^{-1} = w^{-1}(g'^{\top})^{-1}w$$

(the final equality uses that  $w^{-1} = \pm w$ ). This means that  $g'$  preserves the non-degenerate  $(-1)^{n+1}$ -hermitian pairing  $h : k'^n \times k'^n \rightarrow k'$  defined by  $h(x', y') = x'^{\top} w \bar{y}'$ , so  $G = \mathrm{SU}(h)$  (see Exercise 4.4.5). Note that  $h$  has an isotropic subspace of the maximal possible dimension,  $\lfloor n/2 \rfloor$ , due to the matrix for  $w$ .

The  $k$ -group  $G' = \mathrm{SU}_n(k'/k)$  defined using the hermitian  $H(x', y') = x'^{\top} \bar{y}'$  on  $k'^n$  is an inner form of  $G$ . Detecting when it is distinct from  $G$  (equivalently: quasi-split) is an

arithmetic problem. For example, if  $k$  is finite then  $G' \simeq G$  (all connected semisimple groups over finite fields are quasi-split) whereas if  $k = \mathbf{R}$  then  $G' \not\simeq G$  (since  $G'(\mathbf{R})$  is compact).

It is a very useful fact (for the classification of forms and other purposes) that over any non-empty scheme  $S$  every reductive  $S$ -group  $G$  has an inner form that is quasi-split (see Definition 5.2.10). Put another way, every  $G$  can be obtained from a quasi-split reductive  $S$ -group via inner twisting. We develop this result in several ways.

**Proposition 7.2.11.** *Let  $S$  be a non-empty scheme,  $(G_0, T_0, M)$  a split reductive  $S$ -group with root datum  $R$ , and  $G$  a reductive  $S$ -group whose geometric fibers have root datum  $R$ . There exists a quasi-split inner form  $G'$  of  $G$  such that  $\mathrm{Aut}_{G'/S} \rightarrow \mathrm{Out}_{G'/S}$  has an  $S$ -group section and there exists a Borel subgroup  $B' \subset G'$  containing a maximal torus  $T'$  of  $G'$ .*

*Proof.* Let  $\Delta$  be the base of the positive system of roots  $\Phi^+ \subset \Phi$ , and let  $B_0$  be the Borel subgroup containing  $T_0$  that corresponds to  $\Phi^+$ . Define  $\Theta = \mathrm{Aut}(R, \Delta)$ . Choose a pinning  $\{X_a\}_{a \in \Delta}$ ; this defines an  $S$ -group section to  $\mathrm{Aut}_{G_0/S} \rightarrow \mathrm{Out}_{G_0/S} \simeq \Theta_S$  which (by construction) lands inside the subgroup scheme of automorphisms of  $G_0$  that preserve  $T_0$  and  $B_0$ .

Consider the cohomology class  $[G] \in H^1(S_{\text{ét}}, \mathrm{Aut}_{G_0/S})$  classifying  $G$ . If  $\xi$  denotes its image in  $H^1(S_{\text{ét}}, \Theta_S)$ , then the section  $\Theta \rightarrow \mathrm{Aut}(G_0)$  defined by the pinning carries  $\xi$  to a class  $[G'] \in H^1(S_{\text{ét}}, \mathrm{Aut}_{G_0/S})$  in the fiber over  $\xi$ . By Exercise 7.3.8, all classes in the same fiber of the map  $H^1(S_{\text{ét}}, \mathrm{Aut}_{G_0/S}) \rightarrow H^1(S_{\text{ét}}, \mathrm{Out}_{G_0/S})$  are inner forms of each other, so  $G'$  is an inner form of  $G$ . Since the class of  $G'$  in  $H^1(S_{\text{ét}}, \mathrm{Aut}_{G_0/S})$  is represented by a 1-cocycle with values in the subgroup of  $\mathrm{Aut}_{G_0/S}$  that preserves  $T_0$  and  $B_0$ ,  $G'$  admits a maximal torus  $T'$  and Borel subgroup  $B'$  containing  $T'$ . This cocycle is valued in  $\Theta_S$ , so the twisting process that constructs  $\mathrm{Aut}_{G'/S}$  as a form of  $\mathrm{Aut}_{G_0/S}$  also twists  $\Theta_S$  into an  $S$ -subgroup of  $\mathrm{Aut}_{G'/S}$  that maps isomorphically onto the twist  $\mathrm{Out}_{G'/S}$  of  $\mathrm{Out}_{G_0/S}$ .  $\square$

Here is another quasi-split inner form, with additional properties for a Borel subgroup:

**Proposition 7.2.12.** *Let  $S$  be a non-empty scheme and  $G$  a reductive  $S$ -group. There is a quasi-split inner form  $G'$  of  $G$  containing a Borel subgroup  $B'$  satisfying:*

- (1) *the unipotent radical  $U' = \mathcal{R}_u(B') \simeq \mathcal{R}_u(B'/Z_{G'})$  admits a composition series  $U' = U'_0 \supset U'_1 \supset \cdots \supset U'_N = 1'$  by smooth closed normal subgroups of  $B'$  such that each  $U'_n/U'_{n+1}$  is a vector group;*
- (2) *for the torus  $T' = B'/U'$ , the natural map  $Z_{G'} \rightarrow T'$  is a closed immersion and  $T'/Z_{G'} \simeq \mathbf{R}_{S'/S}(\mathbf{G}_m)$  for a finite étale cover  $S' \rightarrow S$ ;*
- (3)  *$T'$  lifts to a maximal torus in  $B'$ .*

This result is treated separately from Proposition 7.2.11 due to difficulties with the outer automorphism group when passing from reductive groups to semisimple groups.

*Proof.* In (2), the map  $Z_{G'} \rightarrow T'$  between multiplicative type groups is a closed immersion because it can be checked on geometric fibers (where  $Z_{G'} \cap U'$  must be trivial). To construct  $G'$ , we reduce to treating  $\mathcal{D}(G)$  in place of  $G$  as follows. Let  $Z$  be the maximal central torus in  $G$ , so  $\mu := Z \cap \mathcal{D}(G)$  is a finite  $S$ -group of multiplicative type and  $G = Z \times^\mu \mathcal{D}(G)$  (Corollary

5.3.3). In particular, the natural map  $\mathcal{D}(G) \rightarrow G^{\text{ad}} := G/Z_G$  is a central isogeny that identifies  $G/Z_G$  with the adjoint quotient of  $\mathcal{D}(G)$  (Proposition 3.3.9). This identification of adjoint quotients is compatible with the natural map  $\text{Aut}_{G/S} \rightarrow \text{Aut}_{\mathcal{D}(G)/S}$ , so the map

$$H^1(S_{\text{ét}}, \text{Aut}_{G/S}) \rightarrow H^1(S_{\text{ét}}, \text{Aut}_{\mathcal{D}(G)/S})$$

corresponding to  $G' \rightsquigarrow \mathcal{D}(G')$  (which may not be injective) carries the set of inner forms of  $G$  onto the set of inner forms of  $\mathcal{D}(G)$ .

Suppose there exists an inner form  $H$  of  $\mathcal{D}(G)$  admitting a Borel subgroup  $B$  satisfying the desired properties modulo  $Z_H$ . Pick a class  $\xi \in H^1(S_{\text{ét}}, G^{\text{ad}})$  mapping to the class of  $H$  in  $H^1(S_{\text{ét}}, \text{Aut}_{\mathcal{D}(G)})$ . The inner twisting by  $\xi$  on  $\mathcal{D}(G)$  has no effect on the central subgroup  $\mu$ , so  $\mu$  is identified with a central subgroup of  $H$ . The central pushout  $G' = Z \times^\mu H$  is the inner form of  $G$  obtained from  $\xi$ , and  $\mathcal{D}(G') = H$ . Thus,  $B' := Z \times^\mu B$  is a Borel subgroup of  $G'$ ,  $B'/Z_{G'} = B/Z_H$ , and  $U := \mathcal{R}_u(B)$  is identified with  $\mathcal{R}_u(B')$  (so  $T' := B'/U' = (Z \times^\mu B)/U =: T$ , and hence  $T'/Z_{G'} = T/Z_H$ ). Likewise, maximal tori in  $\mathcal{D}(G)$  correspond bijectively to maximal tori in  $G$ . It is therefore enough to consider  $\mathcal{D}(G)$  in place of  $G$ , so we may and do now assume that  $G$  is semisimple.

There is a bijective correspondence between the Borel subgroups of a semisimple group and Borel subgroups of its adjoint quotient (Exercise 5.4.8), and likewise for maximal tori. Any inner form of  $G^{\text{ad}}$  is the adjoint quotient of an inner form of  $G$ , so we may replace  $G$  with  $G^{\text{ad}}$  to make  $G$  adjoint. We may also assume that all fibers  $G_s$  are nontrivial (as the locus of trivial fibers is open and closed in  $S$ ). By the adjoint version of Proposition 6.4.4,  $G \simeq R_{\mathcal{S}/S}(\mathcal{G})$  for a finite étale cover  $\mathcal{S} \rightarrow S$  and an adjoint semisimple  $\mathcal{S}$ -group  $\mathcal{G}$  with absolutely simple fibers. Weil restriction through a finite étale cover carries inner forms to inner forms, Borel subgroups to Borel subgroups, vector groups to vector groups, and preserves short exactness. We may assume that the geometric fibers of  $G$  all have the same root datum (as the locus for each isomorphism type is open and closed in  $S$ ), so we may pass to  $\mathcal{G} \rightarrow \mathcal{S}$  in order that the geometric fibers of  $G \rightarrow S$  have a common root datum  $R$  with an irreducible root system. Let  $\Phi$  be this root system.

Let  $(G_0, T_0, M_0, \{X_a\}_{a \in \Delta})$  be a pinned split adjoint semisimple  $S$ -group with root datum  $R$ . Let  $B_0$  be the Borel subgroup corresponding to  $\Delta$ , and  $\Phi^+$  the positive system of roots with base  $\Delta$ . Applying the proof of Proposition 7.2.11 in this situation, we get a triple  $(G', B', T')$  with  $G'$  a quasi-split inner form of  $G$ . We claim that this satisfies all of the asserted properties. Note that  $T' \rightarrow B'/\mathcal{R}_u(B')$  is an isomorphism, as we may check on geometric fibers over  $S$ .

The twisting process is done against a cocycle representing some  $\xi \in H^1(S_{\text{ét}}, \Theta_S)$  for  $\Theta = \text{Aut}(\text{Dyn}(\Phi))$ . The group  $\Theta$  is a subgroup of  $\text{Aut}(\Delta)$  since  $G_0$  is semisimple. Since  $T_0 = \prod_{a \in \Delta} \mathbf{G}_m$  via  $t \mapsto (a(t))$  (as  $G_0$  is adjoint), we have  $T' = R_{S'/S}(\mathbf{G}_m)$  where  $S' \rightarrow S$  is the twist of  $S \times \Delta$  corresponding to the image of  $\xi$  in  $H^1(S_{\text{ét}}, \text{Aut}(\Delta))$ .

To describe  $\mathcal{R}_u(B')$ , let  $\lambda_0 = \sum_{c \in \Phi^+} c^\vee : T_0 \rightarrow \mathbf{G}_m$ . For all  $a \in \Delta$  we have  $\langle a, \lambda_0 \rangle = 2 > 0$  (Exercise 4.4.8(iii)), so  $\Phi_{\lambda_0 > 0} = \Phi^+$ . Hence,  $P_{G_0}(\lambda_0) = B_0$  (as we can check on geometric fibers, by Corollary 5.2.7(2)), so  $U_{G_0}(\lambda_0) = \mathcal{R}_u(B_0)$ . The cocharacter  $\lambda_0$  of  $T_0$  is invariant

under the action of  $\Theta$  on  $G_0$  defined via the pinning, so  $\xi$ -twisting carries  $\lambda_0$  over to a *fiberwise nontrivial* cocharacter  $\lambda : T' \rightarrow \mathbf{G}_m$  such that  $U_G(\lambda) = U' := \mathcal{R}_u(B')$ . For  $n \geq 1$ , let  $\Phi_{\geq n}$  denote the set of  $c \in \Phi^+$  such that  $\lambda_0(c) \geq n$ . By Proposition 5.1.16, the root groups  $U_c \subset U_0$  for  $c \in \Phi_{\geq n}$  directly span (in any order) a smooth closed subgroup  $U_{\geq n}$  in  $U_0$  with  $U_{\geq n}$  is normalized by all positive root groups (and hence by  $B_0$ ) and  $U_{\geq n}/U_{\geq n+1} \simeq \prod_{\lambda_0(c)=n} U_c$  thereby inheriting a trivial  $U_0$ -action. Under the resulting action of  $T_0 = B_0/U_0$ , the  $\mathbf{G}_m$ -action through  $\lambda_0$  is via  $n$ th-power scaling on each  $U_c \simeq \mathbf{G}_a$  (using any trivialization of  $\mathfrak{g}_c$ ). This  $\mathbf{G}_m$ -action encodes the vector group structure on  $U_{\geq n}/U_{\geq n+1}$  (via the action of  $\mathbf{G}_m/\mu_n$ ).

The twisting process carries  $\lambda_0$  to  $\lambda$  and so carries each  $U_{\geq n}$  over to a smooth closed subgroup  $U'_n \subset U'$  that is normal in  $B'$ . Each  $U'_n/U'_{n+1}$  is commutative with a trivial  $U'$ -action and an induced action by  $T' = B'/U'$  that provides a  $\mathbf{G}_m$ -action through  $\lambda : \mathbf{G}_m \rightarrow T'$ . By descent, this action makes  $\mu_n$  act trivially, and the resulting action of  $\mathbf{G}_m/\mu_n = \mathbf{G}_m$  on  $U'_n/U'_{n+1}$  is a  $\mathbf{G}_m$ -equivariant étale twist of  $U_{\geq n}/U_{\geq n+1}$  with its natural linear vector group structure. The linear structure on a vector group is encoded in the  $\mathbf{G}_m$ -action, so by étale descent for vector bundles each  $U_{\geq n}/U_{\geq n+1}$  is a vector group over  $S$ .  $\square$

When  $S$  is semi-local we can combine Proposition 7.2.11 and Proposition 7.2.12:

**Proposition 7.2.13.** *For semi-local  $S$ , up to isomorphism every reductive  $S$ -group  $G$  admits a unique quasi-split inner form  $G'$ . The quotient map  $\text{Aut}_{G'/S} \rightarrow \text{Out}_{G'/S}$  admits an  $S$ -group section, and there is a Borel subgroup  $B' \subset G'$  satisfying the properties in Proposition 7.2.12.*

*Proof.* Once uniqueness is proved, the asserted properties follow from Proposition 7.2.11 and Proposition 7.2.12. To prove uniqueness, we may assume  $G$  admits a Borel subgroup  $B$  satisfying the properties in Proposition 7.2.12. Letting  $G'$  be a quasi-split inner form of  $G$ , with  $B'$  a Borel subgroup, we seek to prove that  $G' \simeq G$ . Consider a class  $\xi \in H^1(S_{\text{ét}}, G/Z_G)$  which twists  $G$  into  $G'$ . Since Borel subgroups are conjugate étale-locally on the base (Corollary 5.2.13), we can arrange that  $\xi$  is valued in  $N_{G/Z_G}(B/Z_G) = B/Z_G$ . Hence, it suffices to prove that  $H^1(S_{\text{ét}}, B/Z_G) = 1$ .

The unipotent radical  $U = \mathcal{R}_u(B/Z_G) \simeq \mathcal{R}_u(B)$  has a composition series whose successive quotients are vector groups, and the torus  $(B/Z_G)/U$  has the form  $R_{S'/S}(\mathbf{G}_m)$  for a finite étale cover  $S' \rightarrow S$  (so  $S'$  is semi-local). The vanishing of  $H^1(S_{\text{ét}}, B/Z_G)$  is therefore reduced to the vanishing of  $H^1(S_{\text{ét}}, \mathcal{V})$  for vector groups  $\mathcal{V}$  and the vanishing of  $H^1(S_{\text{ét}}, R_{S'/S}(\mathbf{G}_m))$ . The higher vector group cohomology vanishes because it agrees with the Zariski cohomology and  $S$  is affine. The Weil restriction through  $S'/S$  is a finite pushforward for the étale sites, so by a degenerating Leray spectral sequence

$$H^1(S_{\text{ét}}, R_{S'/S}(\mathbf{G}_m)) \simeq H^1(S'_{\text{ét}}, \mathbf{G}_m) = \text{Pic}(S')$$

(the final equality by descent theory). Since  $S'$  is semi-local,  $\text{Pic}(S') = 1$ .  $\square$

The only role of semi-locality for the affine  $S$  in the proof of Proposition 7.2.13 is to ensure that every finite étale cover of  $S$  has trivial Picard group. Thus, the conclusion also holds (with the same proof) for  $S = \text{Spec } \mathbf{Z}$  and  $S = \mathbf{A}_k^n$  for a field  $k$  of characteristic 0.

### 7.3. Exercises.

**Exercise 7.3.1.** Let  $T$  be a group of multiplicative type over a scheme  $S$ . Prove that the automorphism functor  $\underline{\text{Aut}}_{T/S}$  on  $S$ -schemes is represented by a separated étale  $S$ -group, and that if  $T = D_S(M)$  for a finitely generated  $\mathbf{Z}$ -module  $M$  then this functor is represented by the constant  $S$ -group associated to  $\text{Aut}(M)^{\text{opp}}$ .

Deduce that if  $T$  is normal in an  $S$ -group  $G$  with connected fibers then  $T$  is central in  $G$ . (If  $G \rightarrow S$  is smooth and affine then Theorem 2.3.1 yields another proof of this fact.)

**Exercise 7.3.2.** Let  $G$  be a smooth group scheme over a field  $k$ .

(i) Prove that the image of  $G(k)$  in  $(G/Z_G)(k)$  is a normal subgroup and that the quotient by this image is a subgroup of the commutative fppf cohomology group  $H^1(k, Z_G)$  (so  $(G/Z_G)(k)/G(k)$  is *commutative*). Make this explicit when  $G = \text{SL}_n$ , and see Exercise 7.3.4.

(ii) The  $G$ -action on itself through conjugation factors through an action by  $G/Z_G$  on  $G$ . Explain how this identifies  $(G/Z_G)(k)$  with a subgroup of  $\text{Aut}(G)$  even when  $(G/Z_G)(k)$  contains points not lifting to  $G(k)$ . Make it explicit for  $G = \text{SL}_n$  via  $\text{SL}_n \hookrightarrow \text{GL}_n$ .

(iii) Show that the  $k$ -automorphisms of  $G$  arising from  $(G/Z_G)(k)$  are precisely the  $k$ -automorphisms which become inner on  $G(\bar{k})$ . Deduce that  $(G/Z_G)(k)$  is normal in  $\text{Aut}(G)$ , and over every separably closed field  $k \neq \bar{k}$  give an example in which the action on  $G(k)$  by some element of  $(G/Z_G)(k)$  is not inner in the sense of abstract group theory.

**Exercise 7.3.3.** Let  $G$  be a reductive group over a connected scheme  $S$ , with  $R$  the common root datum of the geometric fibers.

(i) Explain in terms of descent theory why the set of isomorphism classes of reductive  $S$ -groups with root datum  $R$  on geometric fibers is in natural bijection with the set  $H^1(S_{\text{ét}}, \text{Aut}_{G/S})$ . The classes arising from  $H^1(S_{\text{ét}}, G/Z_G)$  are called *inner forms* of  $G$ .

(ii) Let  $G_0$  be the unique split reductive  $S$ -group with root datum  $R$ . Prove that the set of (isomorphism classes of) inner forms of  $G$  is identified with the fiber through the class of  $G$  for  $h : H^1(S_{\text{ét}}, \text{Aut}_{G_0/S}) \rightarrow H^1(S_{\text{ét}}, \text{Out}_{G_0/S}) = H^1(S_{\text{ét}}, \Theta_S)$ , where  $\Theta = \text{Aut}(R(G_0, T_0), \Delta)$ .

(iii) For  $h$  as in (ii), prove  $\ker h = 1$  but use Example 7.1.12 to show that  $h$  is *not* injective for  $G_0 = \text{SL}_n$  over a field  $k$  admitting a rank- $n^2$  central simple algebra not in  $\text{Br}(k)[2]$ .

(iv) For  $n \geq 2$ , prove that the forms of  $\text{SO}_{2n+1}$  (type  $B_n$ ) over  $S$  are precisely the groups  $\text{SO}(q)$  for non-degenerate  $(V, q)$  of rank  $2n+1$  over  $S$ , with non-isomorphic quadratic spaces giving non-isomorphic groups. What happens for  $\text{SO}_{2n}$  with  $n \geq 4$  (type  $D_n$ )?

**Exercise 7.3.4.** Let  $S$  be scheme, and  $G$  a smooth affine  $S$ -group. Use descent theory to prove that the natural map  $H^1(S_{\text{ét}}, G) \rightarrow H^1(S_{\text{fppf}}, G)$  is bijective. If you are familiar with algebraic spaces, prove the same result without the affineness restriction on  $G$ . (For a remarkable generalization to higher cohomology in the commutative case, see [BrIII, 11.7].)

**Exercise 7.3.5.** Let  $G$  be a connected semisimple group over a global field  $k$ .

(i) If  $G$  is an inner form of a  $k$ -split group, prove  $G_{k_v}$  is split for all but finitely many  $v$ .

(ii) If  $G$  is *not* an inner form of a  $k$ -split group, prove  $G_{k_v}$  is non-split for a set of  $v$  with positive Dirichlet density. Make this explicit for  $\text{SU}_n(k'/k)$ .

**Exercise 7.3.6.** Let  $(G, T, M)$  be a split semisimple group over a connected non-empty scheme  $S$ . For each positive system of roots  $\Phi^+ \subset \Phi$  (with base  $\Delta$ ) and choice of pinning  $\{X_a\}_{a \in \Delta}$ , let  $\{X_{-a}\}_{a \in \Delta}$  be the “linked” pinning for  $(G, T, -\Phi^+)$  (i.e.,  $X_{-a} = X_a^{-1}$ ).

(i) Use the Isomorphism Theorem and the functoriality of duality between  $\mathfrak{g}_a$  and  $\mathfrak{g}_{-a}$  to construct an *involution* of  $G$  that restricts to inversion on  $T$  and swaps  $\Phi^+$  and  $-\Phi^+$ . Prove that up to a  $(G/Z_G)(S)$ -conjugation, this involution of  $G$  is independent of the choice of  $T$ ,  $\Phi^+$ , and pinning (hint: Proposition 6.2.11(2) and Corollary 3.3.6(1)). Deduce the existence of a canonical  $(G/Z_G)(S)$ -conjugacy class of involutions of  $G$  (called *Chevalley involutions*). What are the Chevalley involutions of  $\mathrm{SL}_n$ ? Of  $\mathrm{Sp}_{2n}$ ?

(ii) Prove that the Chevalley involutions are inner (i.e., arise from  $(G/Z_G)(S)$ ) precisely when the long element of  $W(\Phi)$  (relative to a choice of  $\Phi^+$ ) acts via negation on  $M$ .

(iii) Via the dictionary in Example 7.1.13, show that if  $S = \mathrm{Spec} \mathbf{R}$  then the Chevalley involutions of  $G$  are the Cartan involutions. See [AV] for more on Chevalley involutions.

**Exercise 7.3.7.** This exercise considers groups with no maximal torus or no Borel subgroup.

(i) Prove that if  $S$  is a connected normal noetherian scheme with  $\pi_1(S) = 1$  and if  $V$  is a vector bundle on  $S$  of rank  $r > 1$  that is not a direct sum of line bundles then  $\mathrm{SL}(V)$  is a semisimple  $S$ -group with no maximal  $S$ -torus. Show that an example of such  $(S, V)$  is  $S_0 = \mathbf{P}_k^2$  for any separably closed field  $k$  and  $V_0 = \mathrm{Tan}_{S_0/k}$ . (Hint:  $\dim V_0(S_0) = 8$ .)

(ii) Let  $K = \mathbf{Q}(\sqrt{5})$ , and let  $D$  be the quaternion division  $K$ -algebra ramified at precisely the two archimedean places. Use gluing methods as in Remark 5.1.5 to extend the anisotropic  $K$ -group  $G = \mathrm{SL}(D)$  to a semisimple  $\mathcal{O}_K$ -group  $\mathcal{G}$ . Prove that  $\mathcal{G}$  contains no maximal  $\mathcal{O}_K$ -torus. (Hint: show that  $K$  has no quadratic extension unramified at all finite places.)

(iii) Let  $(L, q)$  be a positive-definite even unimodular lattice with rank  $\geq 3$ . Prove that  $\mathrm{SO}(q)$  is a semisimple  $\mathbf{Z}$ -group with no maximal  $\mathbf{Z}$ -torus.

(iv) Over  $\mathbf{R}$  or any non-archimedean local field  $k$ , prove that a nontrivial connected semisimple  $k$ -group  $G$  has no Borel  $k$ -subgroup if  $G(k)$  is compact. Verify the compactness for  $G = \mathrm{SL}(D)$  with any central division algebra  $D$  over  $k$ .

**Exercise 7.3.8.** Prove that Proposition 7.2.2 remains valid for a split reductive group  $G_0$  over any non-empty scheme  $S$  in place of  $\mathrm{Spec} k$ .

**Exercise 7.3.9.** Let  $G$  be reductive over a scheme  $S \neq \emptyset$  with all  $G_{\bar{s}}$  of some type  $\Phi$ . Let  $(G_0, T_0, M)$  be split reductive over  $S$  of type  $\Phi$ , and  $B_0 \subset G_0$  a Borel subgroup with  $T_0 \subset B_0$ .

(i) Using Proposition 6.2.11 and Corollary 5.2.8, prove  $\mathrm{Aut}_{(G_0, B_0)} = (B_0/Z_{G_0}) \rtimes \mathrm{Aut}(\Phi)$ .

(ii) If  $B \subset G$  is a Borel subgroup, classify  $(G, B)$  up to isomorphism by a class in the cohomology set  $\mathrm{H}^1(S_{\text{ét}}, B_0/Z_{G_0} \rtimes \mathrm{Aut}(\Phi))$  and prove that  $G$  splits if it is an inner form of  $G_0$  and  $\mathrm{H}^1(S, \mathcal{O}_S) = 0$  and  $\mathrm{Pic}(S) = 1$ .

(iii) Using the properness of  $\mathrm{Bor}_{G/S}$ , prove that if  $S$  is connected and Dedekind then  $\mathrm{Bor}_{G/S}(S) \neq \emptyset$  when the generic fiber  $G_\eta$  is split (hint: valuative criterion).

(iv) If  $R$  is Dedekind with fraction field  $K$  and  $\mathrm{Pic}(R) = 1$ , show that a reductive  $R$ -group  $G$  is split if  $G_K$  is split. When  $\mathrm{Pic}(R) \neq 1$ , construct non-split  $R$ -forms of  $\mathrm{SL}_2$  that are split *Zariski-locally* on  $R$  (see Example 4.1.2).

## APPENDIX A. GROTHENDIECK'S THEOREM ON TORI

**A.1. Motivation and definitions.** In the early days of the theory of linear algebraic groups, the ground field was assumed to be algebraically closed (as in work of Chevalley). The needs of Lie theory, number theory, and finite group theory (“finite simple groups of Lie type”) led to the development (independently by Borel, Satake, and Tits) of a theory of connected reductive groups over any perfect field (using Galois-theoretic techniques to apply results in the algebraically closed case). Problems over local and global function fields provided motivation to eliminate the perfectness assumption, as was done by Borel and Tits [BoTi]. The initial breakthrough that made it possible to work over an arbitrary field is the following result [SGA3, XIV, 1.1]:

**Theorem A.1.1** (Grothendieck). *Let  $G$  be a smooth connected affine group over a field  $k$ . Then  $G$  contains a  $k$ -torus  $T$  such that  $T_{\bar{k}}$  is maximal in  $G_{\bar{k}}$ .*

The hardest case of the proof is when  $k$  is imperfect, and it was for this purpose that Grothendieck's scheme-theoretic ideas were essential, at first. (In [SGA3, XIV, 1.5(d)] he gave a second scheme-theoretic proof for infinite  $k$ , using the scheme of maximal tori for general smooth connected affine groups over a field (Exercise 3.4.7), which he proved to be unirational in [SGA3, XIV, 6.1], so he could invoke the elementary fact that unirational varieties over infinite fields have rational points.) Borel and Springer eliminated the use of schemes via Lie-theoretic methods (see [Bo91, 18.2(i)]); this amounts to working with certain infinitesimal group schemes in disguise, as we shall see. The aim of this appendix is to give a scheme-theoretic interpretation of their argument.

**Remark A.1.2.** As an application of Theorem A.1.1, we can show via torus-centralizer arguments that if  $K/k$  is any extension field and if  $T \subset G$  is a  $k$ -torus not contained in a strictly larger one then  $T_K \subset G_K$  is not contained in a strictly larger  $K$ -torus of  $G_K$ . In particular, taking  $K = \bar{k}$ , it follows that over a field  $k$ , “maximality” for  $k$ -tori in the geometric sense of Definition 3.2.1 is the same as maximality in the naive sense of containment of  $k$ -tori; i.e., there is no ambiguity about the meaning of the phrase “maximal  $k$ -torus”, and all such tori have the same dimension (due to conjugacy over  $\bar{k}$ ). (One can also consider the same problem over artin local  $S$ ; see [SGA3, XIV, 1.4]. The common dimension of the maximal  $k$ -tori is sometimes called the *reductive rank* of  $G$  because it coincides with the same invariant for the reductive quotient  $G_{\bar{k}}/\mathcal{R}_u(G_{\bar{k}})$ .)

The equivalence of maximality for  $T$  over  $k$  and for  $T_K$  over  $K$  was mentioned in the Introduction, and we now sketch how it works out. For any field extension  $K/k$ , a torus of  $G_K$  containing  $T_K$  must lie in the closed subgroup scheme  $Z_{G_K}(T_K) = Z_G(T)_K$ , but  $Z_G(T)$  is also smooth (see Lemma 2.2.4), so we may replace  $G$  with  $Z_G(T)$  to reduce to the special case that  $T$  is *central* in  $G$ . Then we can pass to the affine quotient  $k$ -group  $G/T$  to arrive at the case  $T = 1$ . We may also replace  $G$  with  $G^0$ .

A smooth connected affine group over an algebraically closed field contains no nontrivial torus if and only if it is unipotent, so we are reduced to the following: if a smooth connected affine group  $G$  over a field  $k$  contains no nontrivial  $k$ -torus then must it be unipotent? The

general problem, for an arbitrary smooth connected affine  $k$ -group, is the existence of *one*  $k$ -torus that is maximal over  $\bar{k}$ ; i.e., a  $k$ -torus that is maximal in the sense of Definition 3.2.1. This is exactly Theorem A.1.1. It therefore follows from this theorem that if the only such  $k$ -torus is the trivial one then the group must be unipotent, as desired.

**Remark A.1.3.** Beware that if  $k \neq k_s$  then typically there are *many*  $G(k)$ -conjugacy classes of maximal  $k$ -tori. For example, if  $G = \mathrm{GL}_n$  then by Exercise 4.4.6(i) the maximal  $k$ -tori in  $G$  are in bijective correspondence with maximal finite étale commutative  $k$ -subalgebras of  $\mathrm{Mat}_n(k)$ . In particular, two maximal  $k$ -tori are  $G(k)$ -conjugate if and only if the corresponding maximal finite étale commutative  $k$ -subalgebras of  $\mathrm{Mat}_n(k)$  are  $\mathrm{GL}_n(k)$ -conjugate. Hence, if such  $k$ -subalgebras are not abstractly  $k$ -isomorphic then their corresponding maximal  $k$ -tori are not  $G(k)$ -conjugate. For example, non-isomorphic degree- $n$  finite separable extension fields of  $k$  yield such algebras.

**Remark A.1.4.** By the classical theory,  $G_{\bar{k}}$  has no nontrivial tori if and only if  $G_{\bar{k}}$  is unipotent. Hence, Grothendieck's theorem implies that every smooth connected affine  $k$ -group is *either* unipotent or contains a nontrivial  $k$ -torus. If all  $k$ -tori in  $G$  are *central* then for a maximal  $k$ -torus  $T$  the quotient  $G/T$  is unipotent (as  $(G/T)_{\bar{k}} = G_{\bar{k}}/T_{\bar{k}}$  contains no nontrivial torus). Hence, in such cases  $G$  is solvable. Thus, in the non-solvable case there are always  $k$ -tori  $S$  whose scheme-theoretic centralizer  $Z_G(S)$  (which is smooth, by Lemma 2.2.4, and connected by the classical theory over  $\bar{k}$ ) has lower dimension than  $G$ . This enables us to construct nontrivial smooth subquotients over  $k$  in *non-solvable* connected linear algebraic  $k$ -groups when trying to prove general theorems.

**Definition A.1.5.** For a maximal  $k$ -torus  $T$  in a smooth connected affine  $k$ -group  $G$ , the associated *Cartan  $k$ -subgroup*  $C \subset G$  is  $C = Z_G(T)$ , the scheme-theoretic centralizer.

Cartan  $k$ -subgroups are always smooth (Lemma 2.2.4) and connected (by the classical theory). Since  $T$  is central in its Cartan  $C$ , it follows that  $T$  is the *unique* maximal  $k$ -torus in  $C$ . (Indeed, if there exists another then the  $k$ -subgroup it generates along with the central  $T \subset C$  would be a bigger  $k$ -torus.) We have  $C_{\bar{k}} = Z_{G_{\bar{k}}}(T_{\bar{k}})$  since the formation of scheme-theoretic centralizers commutes with base change, and over  $\bar{k}$  all maximal tori are conjugate. Hence, over  $\bar{k}$  the Cartan subgroups are conjugate, so the dimension of a Cartan  $k$ -subgroup is both independent of the choice of Cartan  $k$ -subgroup and invariant under extension of the ground field. This number is called the *nilpotent rank* of  $G$  in [SGA3, XII, 1.0], and the *rank* of  $G$  in [Bo91, 12.2]. For example, if  $G$  is reductive then  $C = T$ . That is, in a connected reductive group the Cartan subgroups are precisely the maximal tori.

**Remark A.1.6.** It is a difficult theorem that in any smooth connected affine group  $G$  over any field  $k$ , all maximal  *$k$ -split* tori are *rationally* conjugate, i.e. conjugate under  $G(k)$ . This is [Bo91, 20.9(i)] for reductive  $G$ . The general case was announced without proof by Borel and Tits, and is proved in [CGP, C.2.3]. The dimension of a maximal *split*  $k$ -torus is thus also an invariant, called the  *$k$ -rank* of  $G$  (and mainly of interest in the reductive case).

**A.2. Start of proof of main result.** For the proof of Theorem A.1.1 we will largely focus on the case in which  $k$  is *infinite*, which ensures that the subset  $k^n \subset \mathbf{A}_{\bar{k}}^n$  is Zariski-dense, and thus  $\mathfrak{g} = \text{Lie}(G)$  is Zariski-dense in  $\mathfrak{g}_{\bar{k}}$ . The case of finite  $k$  requires a completely different argument; see Exercise 6.5.7.

We first treat the “easy” case in which  $G_{\bar{k}}$  has a *central* maximal torus  $S$ . (The argument in this case will work over all  $k$ , even finite fields.) Since all maximal tori are  $G(\bar{k})$ -conjugate, there exists a *unique* maximal  $\bar{k}$ -torus  $S \subset G_{\bar{k}}$ . Our problem is to produce one defined over  $k$ . This is rather elementary over perfect fields via Galois descent, but here is a uniform method using group schemes that applies over all fields; this technique will be useful later on as well.

Let  $Z = Z_G^0$ , the identity component of the scheme-theoretic center of  $G$ . (See Lemma 2.2.4 or Exercise 2.4.4(ii) for the existence of  $Z_G$ .) Since the formation of the center and of the identity component commute with extension of the ground field, we have  $S \subset (Z_{\bar{k}})_{\text{red}}$  as a maximal torus in the smooth commutative affine  $\bar{k}$ -group  $(Z_{\bar{k}})_{\text{red}}$ . By the structure of smooth connected commutative affine  $\bar{k}$ -groups, it follows that  $(Z_{\bar{k}})_{\text{red}} = S \times U$  for a smooth connected unipotent  $\bar{k}$ -group  $U$ . For any  $n$  not divisible by  $\text{char}(k)$ , consider the torsion subgroup  $Z[n]$ . This is a commutative affine  $k$ -group of finite type. The derivative of  $[n] : Z \rightarrow Z$  is  $n : \text{Lie}(Z) \rightarrow \text{Lie}(Z)$  (as for any commutative  $k$ -group scheme), so  $\text{Lie}(Z[n])$  is killed by  $n \in k^\times$ . Thus  $\text{Lie}(Z[n]) = 0$ , so  $Z[n]$  is finite étale over  $k$ .

This implies that

$$Z[n]_{\bar{k}} = Z_{\bar{k}}[n] \supset (Z_{\bar{k}})_{\text{red}}[n] \supset Z[n]_{\bar{k}},$$

so  $Z[n]_{\bar{k}} = (Z_{\bar{k}})_{\text{red}}[n]$ . Since  $U$  is unipotent,  $U[n] = 0$ . Hence,  $Z[n]_{\bar{k}} = S[n]$ .

Set  $H = (\bigcup_n Z[n])^0 \subset G$ , where the union is taken over  $n$  not divisible by  $\text{char}(k)$ . This is a smooth connected closed  $k$ -subgroup of  $G$ .

**Lemma A.2.1.** *The  $k$ -group  $H$  is a torus descending  $S$ .*

*Proof.* By Galois descent, the formation of  $H$  commutes with scalar extension to  $k_s$ , so we can assume  $k = k_s$ . Hence, the finite étale groups  $Z[n]$  are constant, so  $H$  is the identity component of the Zariski closure of a set of  $k$ -points. It follows that the formation of  $H$  commutes with any further extension of the ground field, so

$$H_{\bar{k}} = \overline{\left(\bigcup_n Z_{\bar{k}}[n]\right)^0} = \overline{\left(\bigcup_n S[n]\right)^0} = S$$

where the final equality uses that in any  $k$ -torus, the collection of  $n$ -torsion subgroups for  $n$  not divisible by  $\text{char}(k)$  is dense (as we see by working over  $\bar{k}$  and checking for  $\text{GL}_1$  by hand).  $\square$

Now we turn to the hard case, when  $G_{\bar{k}}$  does not have a central maximal torus. In particular, there must exist a non-central  $S = \text{GL}_1 \hookrightarrow G_{\bar{k}}$ . We are going to handle these cases using induction on  $\dim G$ . (Note that the case  $\dim G = 1$  is trivial.)

**Lemma A.2.2.** *It suffices to prove that  $G$  contains a nontrivial  $k$ -torus  $M$ .*

*Proof.* Suppose there exists a nontrivial  $k$ -torus  $M \subset G$ . Consider  $Z_G(M)$ , which is a smooth connected  $k$ -subgroup of  $G$ . The maximal tori of  $Z_G(M)_{\bar{k}} = Z_{G_{\bar{k}}}(M_{\bar{k}})$  must have the same dimension as those of  $G_{\bar{k}}$ , as can be seen by considering one containing  $M_{\bar{k}}$ . Thus, if we can find a  $k$ -torus in  $Z_G(M)$  that remains maximal as such after extension of the ground field to  $\bar{k}$  then the  $\bar{k}$ -fiber of such a torus must also be maximal in  $G_{\bar{k}}$  for dimension reasons. Hence, it suffices to prove Theorem A.1.1 for  $Z_G(M)$ .

Consider  $Z_G(M)/M$ . Since  $M$  was assumed to be nontrivial, this has strictly smaller dimension (even if  $Z_G(M) = G$ , which might have happened). Hence, by dimension induction, there exists a  $k$ -torus  $\bar{T} \subset Z_G(M)/M$  which is geometrically maximal. Let  $T$  be the scheme-theoretic preimage of  $\bar{T}$  in  $Z_G(M)$ . Since  $M$  is smooth and connected, the quotient map  $Z_G(M) \rightarrow Z_G(M)/M$  is smooth, so  $T$  is a smooth connected closed  $k$ -subgroup of  $G$ . It sits in a short exact sequence of  $k$ -groups

$$1 \rightarrow M \rightarrow T \rightarrow \bar{T} \rightarrow 1.$$

Since  $M$  and  $\bar{T}$  are tori and  $T$  is smooth and connected, by the structure theory for connected solvable  $\bar{k}$ -groups it follows that  $T$  is a torus.

We claim  $T$  is geometrically maximal in  $Z_G(M)$ . First note that any maximal torus  $T'$  in  $Z_G(M)_{\bar{k}}$  must contain the central  $M_{\bar{k}}$  (since otherwise the subgroup  $T'M_{\bar{k}}$  would be a bigger torus). Thus, maximality of  $T'$  in  $Z_G(M)_{\bar{k}}$  is equivalent to the maximality of the quotient  $T'/M_{\bar{k}}$  in  $(Z_G(M)/M)_{\bar{k}}$ . Since  $T_{\bar{k}}/M = \bar{T}_{\bar{k}}$ ,  $T_{\bar{k}}$  is maximal in  $Z_G(M)_{\bar{k}}$  and thus in  $G_{\bar{k}}$ .  $\square$

Now we need to find such an  $M$ . The basic idea for infinite  $k$  is to use  $\text{Lie}(S) = \mathfrak{gl}_1 \subset \mathfrak{g}_{\bar{k}}$  and the Zariski-density of  $\mathfrak{g}$  in  $\mathfrak{g}_{\bar{k}}$  (infinite  $k$ !) to create a nonzero  $X \in \mathfrak{g}$  that is “semisimple” and such that the  $k$ -subgroup  $Z_G(X)^0 \subset G$  is a lower-dimensional smooth subgroup in which the maximal  $\bar{k}$ -tori are maximal in  $G_{\bar{k}}$ . Then a geometrically maximal  $k$ -torus in  $Z_G(X)^0$  will do this job. (Below we will define what we mean by “semisimple” for elements of  $\mathfrak{g}_{\bar{k}}$  relative to  $G_{\bar{k}}$ . This is a Lie-theoretic version of Jordan decomposition for linear algebraic groups.) The motivation is that whereas it is hard to construct tori over  $k$ , it is much easier to use Zariski-density arguments in  $\mathfrak{g}_{\bar{k}}$  to create semisimple elements in  $\mathfrak{g}$ . Those will serve as a substitute for tori to carry out a centralizer trick and apply dimension induction.

**Remark A.2.3.** Beware that non-centrality of  $S$  in  $G_{\bar{k}}$  does not imply non-centrality of  $\text{Lie}(S)$  in  $\mathfrak{g}_{\bar{k}}$  when  $\text{char}(k) > 0$ . For example, if  $S$  is the diagonal torus in  $\text{SL}_2$  and  $\text{char}(k) = 2$  then  $\text{Lie}(S)$  coincides with the Lie algebra of the central  $\mu_2$ , so it is central in  $\mathfrak{sl}_2(k)$  (as is also clear by inspection).

**A.3. The case of infinite  $k$ .** Now we assume  $k$  is infinite, but otherwise arbitrary. Consider the following hypothesis:

( $\star$ )                    there exists a non-central semisimple element  $X \in \mathfrak{g}$ .

To make sense of this, we now define the concept “non-central, semisimple” in  $\mathfrak{g}$ . The definition of “semisimple” will involve  $G$ . This is not surprising, since the same 1-dimensional Lie algebra  $k$  arises for both  $\mathbf{G}_a$  and  $\mathbf{G}_m$ , and in the first case we want to declare all elements

of the Lie algebra to be nilpotent (since unipotent subgroups of  $\mathrm{GL}_N$  have all elements in their Lie algebra nilpotent inside  $\mathfrak{gl}_N$ , by the Lie–Kolchin theorem) and in the latter case we want to declare all elements of the Lie algebra to be semisimple. (Observe that the same issue is relevant in the study of connected Lie groups over  $\mathbf{R}$  and  $\mathbf{C}$ : the exponential map relates additive and multiplicative groups, and it is very far from an isomorphism in the complex-analytic case.)

We now briefly digress for a review of Lie algebras of smooth linear algebraic groups  $G$  over general fields  $k$ . The *center* of a Lie algebra  $\mathfrak{g}$  is the kernel of the adjoint action

$$\mathrm{ad} : \mathfrak{g} \rightarrow \mathrm{End}(\mathfrak{g}), X \mapsto [X, -].$$

In [Bo91, §4.1–§4.4], a general “Jordan decomposition” is constructed as follows in  $\mathfrak{g}_{\bar{k}}$ , with  $\mathfrak{g} = \mathrm{Lie}(G)$ . Choose a closed  $k$ -subgroup inclusion  $G \hookrightarrow \mathrm{GL}_N$ , and consider the resulting inclusion of Lie algebras  $\mathfrak{g} \hookrightarrow \mathfrak{gl}_N$  over  $k$ . For any  $X \in \mathfrak{g}_{\bar{k}}$  we have an additive Jordan decomposition  $X = X_s + X_n$  in  $\mathfrak{gl}_N(\bar{k}) = \mathrm{Mat}_N(\bar{k})$ . In particular,  $[X_s, X_n] = 0$ . The elements  $X_s, X_n \in \mathfrak{gl}_N(\bar{k})$  lie in  $\mathfrak{g}_{\bar{k}}$  and as such are independent of the chosen embedding  $G \hookrightarrow \mathrm{GL}_N$  (proved similarly to the construction of Jordan decomposition in  $G(\bar{k})$ ). Likewise, the decomposition  $X = X_s + X_n$  is functorial in  $G$  (not  $\mathfrak{g}!$ ), so  $\mathrm{ad}(X_s) = \mathrm{ad}(X)_s$  and  $\mathrm{ad}(X_n) = \mathrm{ad}(X)_n$ .

**Definition A.3.1.** An element  $X \in \mathfrak{g}$  is *semisimple* if  $X = X_s$  and *nilpotent* if  $X = X_n$ .

**Remark A.3.2.** Note that we are *not* claiming that  $\mathrm{ad}(X)$  alone detects the semisimplicity or nilpotence, nor that the definition is being made intrinsically to  $\mathfrak{g}$ . The definitions of semisimplicity and nilpotence involve the  $k$ -group  $G$ . By definition, these concepts are preserved under passage from  $\mathfrak{g}$  to  $\mathfrak{g}_{\bar{k}}$  (and as with algebraic groups, the Jordan components of  $X \in \mathfrak{g}$  are generally only rational over the perfect closure of  $k$ ). One can develop these concepts intrinsically to the Lie algebra, but we will not discuss it here; see [Sel, V.7.2].

If  $p = \mathrm{char}(k) > 0$ , then upon choosing a faithful representation  $\rho : G \hookrightarrow \mathrm{GL}_N$ , the resulting inclusion  $\mathfrak{g} \hookrightarrow \mathfrak{gl}_N$  makes the  $p$ -power map  $A \mapsto A^p$  on  $\mathfrak{gl}_N = \mathrm{Mat}_N(k)$  (*not* the  $p$ -power map on matrix entries) induce the structure of a  *$p$ -Lie algebra* on  $\mathfrak{g}$ . This is a map of sets  $\mathfrak{g} \rightarrow \mathfrak{g}$ , denoted  $X \mapsto X^{[p]}$ , that satisfies  $(cX)^{[p]} = c^p X^{[p]}$ ,  $\mathrm{ad}(X^{[p]}) = \mathrm{ad}(X)^p$  (a computation in  $\mathfrak{gl}_N$ ), and a certain identity for  $(X + Y)^{[p]} - X^{[p]} - Y^{[p]}$ . It has the intrinsic description  $D \mapsto D^p$  from the space of left-invariant derivations to itself (so it is independent of  $\rho$ ), and is functorial in  $G$ . For further details on  $p$ -Lie algebras, see [Bo91, §3.1], [CGP, A.7] (especially [CGP, A.7.13]), and [SGA3, VII<sub>A</sub>, §5]; e.g., the interaction of  $p$ -Lie algebra structures and scalar extension is addressed in [SGA3, VII<sub>A</sub>, 5.3.3bis].

**Remark A.3.3.** In characteristic  $p > 0$ , if  $X \in \mathfrak{g}$  is nilpotent then  $X^{[p^r]} = 0$  for  $r \gg 0$ . This is very important below, and follows from a computation in the special case of  $\mathfrak{gl}_N$ .

Returning to our original problem over infinite  $k$ , let us verify hypothesis  $(\star)$  in characteristic zero. The non-central  $S = \mathrm{GL}_1 \hookrightarrow G_{\bar{k}}$  gives an action of  $S$  on  $\mathfrak{g}_{\bar{k}}$  (via the adjoint

action of  $G_{\bar{k}}$  on  $\mathfrak{g}_{\bar{k}}$ , and this decomposes as a direct sum of weight spaces

$$\mathfrak{g}_{\bar{k}} = \bigoplus \mathfrak{g}_{\chi_i}.$$

The  $S$ -action is described by the weights  $n_i$ , where  $\chi_i(t) = t^{n_i}$ .

**Lemma A.3.4.** *There is at least one nontrivial weight.*

*Proof.* The centralizer  $Z_{G_{\bar{k}}}(S)$  is a smooth (connected) subgroup of  $G_{\bar{k}}$ , and by functorial consideration with the dual numbers we see that  $\text{Lie}(Z_{G_{\bar{k}}}(S)) = \mathfrak{g}_{\bar{k}}^S$  is the subspace of  $S$ -invariants in  $\mathfrak{g}_{\bar{k}}$ . Thus, if  $S$  acts trivially on  $\mathfrak{g}_{\bar{k}}$  then  $Z_{G_{\bar{k}}}(S)$  has Lie algebra with full dimension, forcing  $Z_{G_{\bar{k}}}(S) = G_{\bar{k}}$  by smoothness, connectedness, and dimension reasons. This says that  $S$  is central in  $G_{\bar{k}}$ , which is contrary to our hypotheses on  $S$ .  $\square$

If we choose a  $\bar{k}$ -basis  $Y$  for  $\text{Lie}(S)$  then the element  $Y \in \mathfrak{g}_{\bar{k}}$  is semisimple since any  $G_{\bar{k}} \hookrightarrow \text{GL}_N$  carries  $S$  into a torus and hence carries  $\text{Lie}(S)$  onto a semisimple subalgebra of  $\mathfrak{gl}_N = \text{Mat}_N(k)$ . By Lemma A.3.4, some weight is nonzero. Thus, in characteristic zero (or more generally if  $\text{char}(k) \nmid n_i$  for some  $i$ ) we know moreover that  $\text{ad}(Y)$  is nonzero. Hence,  $Y$  is semisimple and in characteristic 0 is *non-central*.

This does not establish  $(\star)$  when  $\text{char}(k) = 0$ , since  $Y \in \mathfrak{g}_{\bar{k}}$  and we seek a non-central semisimple element of  $\mathfrak{g}$ . To remedy this, consider the characteristic polynomial  $f(X, t)$  of  $\text{ad}(X)$  for “generic”  $X \in \mathfrak{g}$ , as a polynomial in  $k[\mathfrak{g}^*][t]$ . Working in  $\bar{k}[\mathfrak{g}^*][t] = \bar{k}[\mathfrak{g}_{\bar{k}}^*][t]$ , the existence of the non-central semisimple element  $Y$  as established above (when  $\text{char}(k) = 0$ ) shows that  $f(X, t) \neq t^{\dim \mathfrak{g}}$ . In other words, there are lower-order (in  $t$ ) coefficients in  $k[\mathfrak{g}^*]$  which are nonzero as functions on  $\mathfrak{g}_{\bar{k}}$ . Since the subset  $\mathfrak{g} \subset \mathfrak{g}_{\bar{k}}$  is Zariski-dense (as  $k$  is infinite) it follows that there exists  $X \in \mathfrak{g}$  such that  $f(X, t) \in k[t]$  is not equal to  $t^{\dim \mathfrak{g}}$ . In particular,  $\text{ad}(X)$  is not nilpotent, so  $\text{ad}(X)_s$  is nonzero. Since  $\text{ad}(X_s) = \text{ad}(X)_s \neq 0$ ,  $X_s$  is noncentral and semisimple in  $\mathfrak{g}_{\bar{k}}$ . When  $k$  is perfect, such as a field of characteristic 0, the Jordan decomposition is rational over the ground field. Thus,  $X_s$  satisfies the requirements in  $(\star)$ .

**A.4. Hypothesis  $(\star)$  for  $G$  implies the existence of a nontrivial  $k$ -torus.** Now we assume there exists  $X \in \mathfrak{g}$  that is noncentral and semisimple. We will show (for infinite  $k$  of any characteristic) that there exists a smooth  $k$ -subgroup  $G' \subset G$  which is a proper subgroup (and hence  $\dim G' < \dim G$ ) such that  $\mathfrak{g}' := \text{Lie}(G')$  contains a nonzero semisimple element of  $\mathfrak{g}$ . This implies that  $G'_{\bar{k}}$  is not unipotent (for if it were unipotent then its Lie algebra would be nilpotent inside  $\mathfrak{g}_{\bar{k}}$ ). By dimension induction,  $G'$  contains a geometrically maximal  $k$ -torus. Since  $G'_{\bar{k}}$  is not unipotent, this means  $G'$  (and hence  $G!$ ) contains a *nontrivial*  $k$ -torus, which is all we need (by Lemma A.2.2).

Granting  $(\star)$ , it is very easy to finish the proof, as follows. Consider the scheme theoretic centralizer  $Z_G(X)$  of  $X$  (for the action  $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$ ). By Cartier’s theorem in characteristic 0, or the semisimplicity of  $X$  and [Bo91, 9.1] in any characteristic,  $Z_G(X)$  is smooth. We must have  $Z_G(X) \neq G$ . Indeed, assume  $Z_G(X) = G$ , so  $X$  is fixed by  $\text{Ad}(G)$ . By differentiating,  $\text{ad}(X) = 0$  on  $\mathfrak{g}$ . But  $X$  is non-central in  $\mathfrak{g}$ , so this is a contradiction.

Thus  $Z_G(X)$  is a smooth subgroup of  $G$  distinct from  $G$ , and its Lie algebra contains the nonzero semisimple  $X$ . This does the job as required above, so we are done in characteristic 0.

For the remainder of the proof, assume  $\text{char}(k) = p > 0$ . We will make essential use of  $p$ -Lie algebras, due to two facts: (i)  $p$ -Lie subalgebras  $\mathfrak{h} \subset \mathfrak{g}$  are in functorial bijection with infinitesimal  $k$ -subgroup schemes  $H \subset G$  of height  $\leq 1$  (meaning  $a^p = 0$  for all nilpotent functions  $a$  on  $H$ ) via  $H \mapsto \text{Lie}(H)$ , and (ii)  $\mathfrak{h}$  is commutative if and only if  $H$  is commutative. The idea behind the proofs of these facts is to emulate classical Lie-theoretic arguments by using Taylor series truncated in degrees  $< p$ . This makes it possible to reconstruct  $H$  from  $\text{Lie}(H)$  via the functor  $\mathfrak{h} \rightsquigarrow \text{Spec } U_p(\mathfrak{h})^*$ , where  $U_p(\mathfrak{h})$  denotes the *restricted universal enveloping algebra* (carrying the  $p$ -operation on  $\mathfrak{h}$  over to the  $p$ -power map on the associative algebra  $U_p(\mathfrak{h})$ ; see [SGA3, VII<sub>A</sub>, 5.3]). A precise statement is given in [SGA3, VII<sub>A</sub>, 7.2.7.4]:

**Theorem A.4.1.** *For a commutative  $\mathbf{F}_p$ -algebra  $B$ , the functor  $H \rightsquigarrow \text{Lie}_p(H)$  is an equivalence between the category of finite locally free  $B$ -group schemes whose augmentation ideal is killed by the  $p$ -power map and the category of finite locally free  $p$ -Lie algebras over  $B$ .*

*In particular, if  $k$  is a field of characteristic  $p > 0$  and  $G$  is a  $k$ -group scheme of finite type, then for any  $H$  of height  $\leq 1$  the  $p$ -Lie algebra functor defines a bijection*

$$\begin{aligned} \text{Hom}_k(H, G) &= \text{Hom}_k(H, \ker F_{G/k}) \simeq \text{Hom}(\text{Lie}_p(H), \text{Lie}_p(\ker F_{G/k})) \\ &= \text{Hom}(\text{Lie}_p(H), \text{Lie}_p(G)). \end{aligned}$$

In this result,  $F_{G/k} : G \rightarrow G^{(p)}$  denotes the relative Frobenius morphism, discussed in Exercise 1.6.7 over fields and in [CGP, A.3] over  $\mathbf{F}_p$ -algebras. (For  $\text{GL}_n$  it is the  $p$ -power map on matrix entries, and in general it is functorial in  $G$ .) By Nakayama's Lemma, a map  $f : H \rightarrow G$  from an infinitesimal  $H$  is a closed immersion if and only if  $\text{Lie}(f)$  is injective.

**Remark A.4.2.** The proof of Theorem A.4.1 rests on general arguments with  $p$ -Lie algebras in [SGA3, VII<sub>A</sub>, §4-§5], and a key ingredient is that the natural identification of  $\mathfrak{h}$  with  $\text{Tan}_e(\text{Spec } U_p(\mathfrak{h})^*)$  respects the  $p$ -Lie algebra structures. This compatibility rests on a functorial description of the  $p$ -Lie algebra structure arising from a group scheme (and the explicit description of  $U_p(\mathfrak{h})$ ). Such a functorial description is proved in [CGP, A.7.5, A.7.13] (and is proved in related but more abstract terms in [SGA3, VII<sub>A</sub>, §6]).

In the special case of *commutative*  $k$ -groups whose augmentation ideal is killed by the  $p$ -power map, an elementary proof of the equivalence with finite-dimensional *commutative*  $p$ -Lie algebras over  $k$  is given in the proof of the unique Theorem in [Mum, §14] via a method which works over any field (even though [Mum] always assumes the ground field is algebraically closed). But beware that the commutative case is not enough for us, since we need the final bijection among Hom's in Theorem A.4.1, and that rests on using the  $k$ -group scheme  $\ker F_{G/k}$  which is generally non-commutative.

As an illustration of the usefulness of Theorem A.4.1, we now give an alternative construction of a nontrivial  $k$ -torus when  $\text{char}(k) = p > 0$  and  $(\star)$  holds, bypassing the smoothness

of  $Z_G(X)$ . This also provides an opportunity to present some arguments that will be useful in our treatment of the cases when  $(\star)$  fails.

Let  $\mathfrak{h} = \text{Span}_k(\{X^{[p^i]}\}_{i \geq 0})$ . This is manifestly closed under the map  $v \mapsto v^{[p]}$ . Moreover, the  $X^{[p^i]}$  all commute with one another. (Proof: use an embedding  $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n$  arising from a  $k$ -group inclusion of  $G$  into  $\text{GL}_n$ , and the description of the  $p$ -operation on  $\mathfrak{gl}_n = \text{Mat}_n(k)$  as  $A \mapsto A^p$ .) Thus  $\mathfrak{h}$  is a commutative  $p$ -Lie subalgebra of  $\mathfrak{g}$ . A linear combination of commuting semisimple operators is semisimple. Moreover the  $p$ th power of a nonzero semisimple operator is nonzero. Hence,  $v \mapsto v^{[p]}$  has trivial kernel on  $\mathfrak{h}$ . It is a general fact in Frobenius-semilinear algebra that if  $V$  is a finite-dimensional vector space over a perfect field  $F$  of characteristic  $p$  and if  $\phi : V \rightarrow V$  is a Frobenius-semilinear endomorphism then there exists a unique decomposition  $V = V_{\text{ss}} \oplus V_{\text{n}}$  such that  $\phi$  is nilpotent on  $V_{\text{n}}$  and  $(V_{\text{ss}})_{\overline{F}}$  admits a basis of “ $\phi$ -fixed vectors” ( $\phi(v) = v$ ). We will only need this over an algebraically closed field, in which case it is proved in the Corollary at the end of [Mum, §14].

**Lemma A.4.3.** *The  $k$ -subgroup scheme  $Z_G(\mathfrak{h})$  in  $G$  is smooth.*

*Proof.* We may assume  $k = \overline{k}$ , as smoothness can be detected over  $\overline{k}$  and the formation of scheme theoretic centralizers commutes with base change. Now using Theorem A.4.1, let  $H \subset G$  be the infinitesimal  $k$ -subgroup scheme whose Lie algebra is  $\mathfrak{h} \subseteq \mathfrak{g}$ .

As observed above,  $\mathfrak{h}$  splits as a direct sum of  $(\cdot)^{[p]}$ -eigenlines,

$$\mathfrak{h} = \bigoplus kX_i, \quad X_i^{[p]} = X_i.$$

Thus,  $H$  is a power of the order- $p$  infinitesimal commutative  $k$ -subgroup corresponding to the  $p$ -Lie algebra  $kv$  with  $v^{[p]} = v$ . But there are only *two* 1-dimensional  $p$ -Lie algebras over  $k$ : the one with  $v^{[p]} = 0$  and the one with  $v^{[p]} = v$  for some  $k$ -basis  $v$ . (Indeed, if  $v^{[p]} = cv$  for some  $c \in k^\times$  then by replacing  $v$  with  $w = av$  where  $a^{p-1} = c$  we get  $w^{[p]} = w$ .) Hence, there are exactly two commutative infinitesimal order- $p$  groups over an algebraically closed field, so the non-isomorphic  $\mu_p$  and  $\alpha_p$  must be these two possibilities. That is,  $H$  is a power of either  $\mu_p$  or  $\alpha_p$ .

We claim that  $H$  is a power of  $\mu_p$ . To prove this, we will use the  $p$ -Lie algebra structure. The embeddings  $\alpha_p \hookrightarrow \mathbf{G}_a$  and  $\mu_p \hookrightarrow \mathbf{G}_m$  induce isomorphisms on  $p$ -Lie algebras, and the nonzero invariant derivations on  $\mathbf{G}_a$  and  $\mathbf{G}_m$  are given by  $\partial_t$  and  $t\partial_t$  respectively. Taking  $p$ -th powers of these derivations computes the  $(\cdot)^{[p]}$ -map on them, and clearly  $\partial_t^p = 0$  and  $(t\partial_t)^p = t\partial_t$ . Hence, the  $p$ -operation on  $\alpha_p$  vanishes and on  $\mu_p$  is non-vanishing. Thus, the condition  $v^{[p]} = v$  forces  $H = \mu_p^N$  for some  $N$ .

By Lemma 2.2.4,  $Z_G(H)$  is smooth. To conclude the proof, it will suffice to show that the evident inclusion  $Z_G(H) \subseteq Z_G(\mathfrak{h})$  as  $k$ -subgroup schemes of  $G$  is an equality.

Theorem A.4.1 provides more: if  $R$  is any  $k$ -algebra, then the  $p$ -Lie functor defines a bijective correspondence between the sets of  $R$ -group maps  $H_R \rightarrow G_R$  and  $p$ -Lie algebra maps  $\mathfrak{h}_R \rightarrow \mathfrak{g}_R$ . Hence, by Yoneda’s lemma,  $Z_G(H) = Z_G(\mathfrak{h})$  because to check this equality of  $k$ -subgroup schemes of  $G$  it suffices to compare  $R$ -points for arbitrary  $k$ -algebras  $R$ .  $\square$

As in the characteristic zero case, since  $\mathfrak{h}$  contains noncentral elements of  $\mathfrak{g}$ , it follows that  $Z_G(\mathfrak{h}) \neq G$ . And as we saw above, this guarantees the existence of a nontrivial  $k$ -torus in  $G$ , by dimension induction (applied to the identity component  $Z_G(\mathfrak{h})^0$ ).

We have already completed the proof of Theorem A.1.1 in characteristic zero, since  $(\star)$  always holds in characteristic 0, and more generally we have completed it over any  $k$  whatsoever for  $G$  that satisfy  $(\star)$  when the conclusion of Theorem A.1.1 is known over  $k$  in all lower dimensions (as we may always assume, since we argue by induction on  $\dim G$ ).

**A.5. The case  $\text{char}(k) = p > 0$  and  $(\star)$  fails.** Now the idea is to find a central infinitesimal  $k$ -subgroup  $M \subset G$  such that  $G/M$  satisfies  $(\star)$ . We will then lift the result from  $G/M$  back to  $G$  when such an  $M$  exists, and if no such  $M$  exists then we will use a different method to produce a nontrivial  $k$ -torus in  $G$ .

**Lemma A.5.1.** *Regardless of whether  $(\star)$  holds (but still assuming, as we have been, that  $G_{\bar{k}}$  has a noncentral  $\text{GL}_1$ ), there exists a nonzero semisimple element  $X \in \mathfrak{g}$ .*

*Proof.* Arguing as at the end of §A.3, and using the infinitude of the field  $k$ , there exists  $X_0 \in \mathfrak{g}$  such that  $\text{ad}(X_0)$  is not nilpotent. Consider the additive Jordan decomposition  $X_0 = (X_0)_s + (X_0)_n$  in  $\mathfrak{g}_{\bar{k}}$  as a sum of commuting semisimple and nilpotent elements. These components of  $X_0$  are defined over the perfect closure of  $k$ , by Galois descent. For  $r \gg 0$  we see that  $X := X_0^{[p^r]} = ((X_0)_s)^{[p^r]}$ . This is nonzero and semisimple, and if  $r \gg 0$  then  $X \in \mathfrak{g}$  (since  $(X_0)_s$  is rational over the perfect closure of  $k$ ).  $\square$

Obviously if  $(\star)$  fails for  $G$  then every semisimple element of  $\mathfrak{g}$  is central. Assume this is the case. Set

$$\mathfrak{m} = \text{Span}_k(\text{all semisimple } X \in \mathfrak{g}) \subset \mathfrak{g};$$

this is nonzero due to Lemma A.5.1. Since all the semisimple elements of  $\mathfrak{g}$  are central,  $\mathfrak{m}$  is a commutative Lie subalgebra of  $\mathfrak{g}$ . The  $p$ th power of a semisimple element of  $\text{Mat}_N(\bar{k})$  is semisimple, so  $\mathfrak{m}$  is  $(\cdot)^{[p]}$ -stable. Thus,  $\mathfrak{m}$  is a  $p$ -Lie subalgebra, so we can exponentiate it to an infinitesimal commutative subgroup  $M \subset G$ . Since a linear combination of commuting semisimple elements in  $\text{Mat}_N(\bar{k})$  is semisimple,  $\mathfrak{m}$  consists only of semisimple elements. This implies that  $(\cdot)^{[p]}$  has vanishing kernel on  $\mathfrak{m}_{\bar{k}}$ . Thus, as we saw in the proof of Lemma A.4.3, it follows that  $M_{\bar{k}} = \mu_p^N$  for some  $N > 0$ .

**Lemma A.5.2.** *The  $k$ -subgroup scheme  $M$  in  $G$  is central.*

*Proof.* Let  $V \subset \mathfrak{g}_{k_s}$  be the  $k_s$ -span of all semisimple central elements of  $\mathfrak{g}_{k_s}$ . Manifestly we have  $\mathfrak{m}_{k_s} \subset V$ . Let  $\Gamma = \text{Gal}(k_s/k)$ . Since  $V$  is  $\Gamma$ -stable, by Galois descent we have  $V = (V^\Gamma)_{k_s}$ . Since  $V^\Gamma \subset \mathfrak{m}$ , this gives  $V = \mathfrak{m}_{k_s}$ . By inspection, it is clear that  $V$  is stable under the action of  $G(k_s)$  on  $\mathfrak{g}_{k_s}$ . But  $G(k_s)$  is Zariski-dense in  $G_{k_s}$ , so  $G_{k_s}$  preserves  $V = \mathfrak{m}_{k_s} \subset \mathfrak{g}_{k_s}$  under the adjoint action. Hence,  $G$  preserves  $\mathfrak{m}$ , so  $M$  is normal in  $G$ . Thus, it is central by Theorem 2.3.1 or Exercise 7.3.1.  $\square$

Now consider the central purely inseparable  $k$ -isogeny  $\pi : G \rightarrow G' := G/M$ . Note that  $G'$  is smooth and connected of the same dimension as  $G$ , and even contains a non-central

torus  $\pi_{\bar{k}}(S)$  over  $\bar{k}$  (as  $\pi$  is bijective on  $\bar{k}$ -points). Does  $G'$  satisfy  $(\star)$ ? If it does not, then we can run through the same procedure all over again to get a nontrivial central  $M' \subset G'$  such  $M'_k \simeq \mu_p^{N'}$ , and can then consider the composite purely inseparable  $k$ -isogeny

$$G \rightarrow G/M = G' \rightarrow G'/M'.$$

This is not so bad, since the kernel  $E$  of this composite map is necessarily an infinitesimal multiplicative type subgroup, by the following lemma, so it is *central* in  $G$  due to normality and Theorem 2.3.1 (or Exercise 7.3.1):

**Lemma A.5.3.** *If*

$$1 \rightarrow M' \rightarrow E \rightarrow M \rightarrow 1$$

*is a short exact sequence of finite  $k$ -group schemes with  $M$  and  $M'$  multiplicative infinitesimal  $k$ -groups then so is  $E$ ; in particular,  $E$  is commutative.*

*Proof.* We may assume  $k = \bar{k}$ . The infinitesimal nature of  $M$  and  $M'$  implies that  $E(k) = 1$ , so  $E$  is infinitesimal (hence connected). The normality of  $M'$  in  $E$  implies that the conjugation action of  $E$  on  $M'$  is classified by a  $k$ -group homomorphism from the connected  $E$  to the étale automorphism group scheme of  $M'$ . This classifying map must be trivial, so  $M'$  is *central* in  $E$ . Since  $M = E/M'$  is commutative, the functorial commutator  $E \times E \rightarrow E$  factors through a  $k$ -scheme morphism

$$[\cdot, \cdot] : M \times M = (E/M') \times (E/M') \rightarrow M'$$

which is seen to be bi-additive by thinking about  $M = E/M'$  in terms of fppf quotient sheaves. In other words, this bi-additive pairing corresponds (in two ways!) to a  $k$ -group homomorphism

$$M \rightarrow \underline{\mathrm{Hom}}(M, M'),$$

where the target is the affine finite type  $k$ -scheme classifying group scheme homomorphisms (over  $k$ -algebras). By Cartier duality, this Hom-scheme is étale, so the map to it from  $M$  must be *trivial*. This shows that  $E$  has trivial commutator, so  $E$  is commutative.

With commutativity of  $E$  established, it makes sense to apply Cartier duality  $\mathbf{D}(\cdot)$  to our original short exact sequence. This duality operation is contravariant and preserves exact sequences (since it is order-preserving and carries right-exact sequences to left-exact sequences), so we get an exact sequence

$$1 \rightarrow \mathbf{D}(M) \rightarrow \mathbf{D}(E) \rightarrow \mathbf{D}(M') \rightarrow 1.$$

The outer terms are finite constant groups of  $p$ -power order, so the middle one must be too. Hence,  $E = \mathbf{D}(\mathbf{D}(E))$  is also multiplicative, as desired.  $\square$

Returning to our setup of interest, by Lemma A.5.3 the composite isogeny

$$G \rightarrow G/M = G' \rightarrow G'/M'$$

is a quotient by a central multiplicative infinitesimal  $k$ -group. Now we're in position to wrap things up in positive characteristic (when  $k$  is infinite, arguing by induction on  $\dim G$ ).

First, we handle the case when the above process keeps going on *forever*. This provides a *strictly increasing* sequence  $M_1 \subset M_2 \subset \dots$  of central multiplicative infinitesimal  $k$ -subgroups of  $G$ . This is all happening inside the  $k$ -subgroup scheme  $Z_G$ , so it forces  $Z_G$  to not be finite (as otherwise there would be an upper bound on the  $k$ -dimensions of the coordinate rings of the  $M_j$ 's). Since  $(Z_G)_{\bar{k}}^0 / ((Z_G)_{\bar{k}}^0)_{\text{red}}$  is a finite infinitesimal group scheme, for large enough  $j$  the map to this from  $(M_j)_{\bar{k}}$  must have nontrivial kernel. In other words, the smooth connected commutative group  $((Z_G)_{\bar{k}}^0)_{\text{red}}$  contains a nontrivial infinitesimal subgroup that is multiplicative. The group  $((Z_G)_{\bar{k}}^0)_{\text{red}}$  therefore cannot be unipotent (since a smooth unipotent group cannot contain  $\mu_p$ ), so it must contain a nontrivial torus! We conclude by the same argument with  $Z_G[n]$ 's as in §A.2 (using  $n$  not divisible by  $\text{char}(k) = p$ ) that  $Z_G$  contains a nontrivial  $k$ -torus, so we win.

There remains the more interesting case when the preceding process does eventually stop, so we wind up with a central quotient map

$$G \rightarrow G/M$$

by a multiplicative infinitesimal  $k$ -subgroup  $M$  such that  $G/M$  satisfies  $(\star)$ ; beware that now  $M_{\bar{k}}$  is merely a product of several  $\mu_{p^{n_i}}$ 's, not necessarily a power of  $\mu_p$ . We therefore get a nontrivial  $k$ -torus  $T$  in  $G/M$ , so if  $E \subset G$  denotes its scheme-theoretic preimage then there exists a short exact sequence of  $k$ -group schemes

$$1 \rightarrow M \rightarrow E \rightarrow T \rightarrow 1$$

with  $M$  central in  $E$ . We will be done (for infinite  $k$ ) if any such  $E$  contains a nontrivial  $k$ -torus. This is the content of the following lemma.

**Lemma A.5.4.** *For any field  $k$  of characteristic  $p > 0$  and short exact sequence of  $k$ -groups*

$$1 \rightarrow M \rightarrow E \rightarrow T \rightarrow 1$$

*with a central multiplicative infinitesimal  $k$ -subgroup  $M$  in  $E$  and a nontrivial  $k$ -torus  $T$ , there exists a nontrivial  $k$ -torus in  $E$ .*

*Proof.* Certainly  $E_{\bar{k}}$  is connected, since  $T$  and  $M$  are connected. The commutator map on  $E$  factors through a bi-additive pairing  $T \times T \rightarrow M$  which is trivial since  $T$  is smooth and  $M$  is infinitesimal. Hence,  $E$  is commutative. The map  $E_{\bar{k}} \rightarrow T_{\bar{k}}$  is bijective on  $\bar{k}$ -points, so  $(E_{\bar{k}})_{\text{red}}$  is a nontrivial smooth connected commutative  $\bar{k}$ -group. It is therefore a direct product of a torus and a smooth connected unipotent group, and the unipotent part must be trivial (since  $T_{\bar{k}}$  is a torus). Hence,  $(E_{\bar{k}})_{\text{red}}$  is a nontrivial torus. Now we can play the usual game: since  $E$  is commutative, the identity component of the Zariski-closure of the  $k$ -subgroup schemes  $E[n]$  for  $n$  not divisible by  $p$  is a  $k$ -torus in  $E$  which maps onto  $(E_{\bar{k}})_{\text{red}}$  and hence is nontrivial.  $\square$

## APPENDIX B. GROUPS OF MULTIPLICATIVE TYPE

**B.1. Basic definitions and properties.** For a scheme  $S$  and finitely generated  $\mathbf{Z}$ -module  $M$ ,  $D_S(M)$  denotes the  $S$ -group  $\mathrm{Spec}(\mathcal{O}_S[M])$  representing the functor  $\underline{\mathrm{Hom}}_{S\text{-gp}}(M_S, \mathbf{G}_m)$  of characters of the constant  $S$ -group  $M_S$ . (This is denoted  $D(M)_S$  in [Oes, I, 5.1]; it is the base change to  $S$  of the analogous group scheme over  $\mathrm{Spec} \mathbf{Z}$ . See [Oes, I, 5.2] for the proof that it represents the functor of characters of  $M_S$ .)

**Definition B.1.1.** A group scheme  $G \rightarrow S$  is of *multiplicative type* if there is an fppf covering  $\{S_i\}$  of  $S$  such that  $G_{S_i} \simeq D_{S_i}(M_i)$  for a finitely generated abelian group  $M_i$  for each  $i$ .

By fppf descent, a multiplicative type  $S$ -group  $G$  is fppf over  $S$ . Such a  $G$  is *split* (or *diagonalizable*) over  $S$  if  $G \simeq D_S(M)$  for some  $M$ ; see [Oes, I] for the theory of such groups, and [Oes, I, 5.2] for the anti-equivalence between the categories of split  $S$ -groups of multiplicative type and constant commutative  $S$ -groups with finitely generated geometric fibers.

In [Oes] the basic theory of multiplicative type groups  $G \rightarrow S$  is developed under weaker conditions (following [SGA3]):  $M$  is not required to be finitely generated,  $G$  is fpqc and affine over  $S$  but may not be of finite presentation, and diagonalizability is required only fpqc-locally on  $S$ . The proofs of all results in [Oes] that we cite below work verbatim under our finiteness restrictions, due to our insistence on fppf-local diagonalizability in the definition. (In Corollary B.4.2 we show that fpqc-local diagonalizability recovers Definition B.1.1 for fppf group schemes. Thus, our multiplicative type groups are precisely those of [Oes] and [SGA3] with finite type structural morphism; we never use this fact.)

**Remark B.1.2.** In [Oes, I, 5.4] it is noted that for any fppf closed  $S$ -subgroup  $H$  of  $D_S(M)$  there is a unique partition  $\{S_N\}_{N \subset M}$  of  $S$  into pairwise disjoint open and closed subschemes  $S_N$  indexed by the subgroups  $N \subset M$ , with  $H|_{S_N} = D_S(M/N)$  inside  $D_S(M)$ . The special case  $S = \mathrm{Spec} k$  for a field  $k$  is proved there ( $M$  may not be torsion-free, so  $D_k(M)$  may be disconnected or non-smooth), and an alternative proof in this case under our finiteness hypotheses on  $M$  is given in Exercise 2.4.1. To settle the case of arbitrary  $S$ , by “spreading out” considerations (under our finiteness hypotheses) we may assume  $S$  is local, and even noetherian. This case is deduced from the field case in the proof of Corollary B.3.3 below (which provides a generalization for all  $S$ -groups of multiplicative type).

We refer the reader to [Oes, II, 2.1] for several notions of “local triviality” for multiplicative type groups (isotriviality, quasi-isotriviality, etc.) For many applications, it is important that multiplicative type groups are split étale-locally on the base (quasi-isotriviality). This will be proved in Proposition B.3.4 and rests on the following lemma.

**Lemma B.1.3.** *Let  $G$  be an affine  $S$ -group scheme of finite presentation and  $H$  an  $S$ -group of multiplicative type. Any monic homomorphism  $j : H \rightarrow G$  is necessarily a closed immersion.*

This lemma is useful in constructions with fiberwise maximal tori in reductive group schemes, and it eliminates ambiguity about the meaning of “subgroup of multiplicative type”

for homomorphisms from groups of multiplicative type: working sheaf-theoretically (or in terms of group functors) with monomorphisms is equivalent to working algebro-geometrically with closed immersions.

*Proof.* This result is [SGA3, IX, 2.5] without finite presentation hypotheses, and it is also a special case of [SGA3, VIII, 7.13(b)] (relaxing affineness to separatedness). Here is an alternative direct argument under our hypotheses.

We may pass to the case of noetherian  $S$ , and it suffices to show that  $j$  is proper (since a finitely presented proper monomorphism is a closed immersion [EGA, IV<sub>3</sub>, 8.11.5]). By the valuative criterion for properness, we may express the problem in terms of points valued in a discrete valuation ring  $R$  and its fraction field  $K$  over  $S$ , so we may reduce to the case that  $S = \text{Spec } R$  and it is harmless to make a local injective base change  $R \rightarrow R'$  to another discrete valuation ring. Thus, we can assume that  $R$  is henselian. We claim that such an  $R'$  may be found so that  $H_{R'}$  is split. Pick an fppf cover  $S' \rightarrow S$  that splits  $H$ . As for any fppf cover of an affine scheme, there is an affine flat quasi-finite surjection  $S'' \rightarrow S$  admitting a map  $S'' \rightarrow S'$  over  $S$  [EGA, IV<sub>4</sub>, 17.16.2]. Since  $R$  is henselian local, by [EGA, IV<sub>4</sub>, 18.5.11(c)] the affine flat quasi-finite cover  $S''$  of  $S$  contains an open and closed local subscheme that is  $S$ -finite. This subscheme is finite flat over  $S$  and non-empty, so the normalization of its underlying reduced scheme has the form  $\text{Spec } R'$  for a discrete valuation ring  $R'$  such that  $H_{R'}$  is split. By renaming  $R'$  as  $R$ , now  $H$  is split.

Letting  $k$  be the residue field of  $R$ , the maps  $j_K$  and  $j_k$  are monomorphisms between finite type group schemes over a field, so they are closed immersions. Let  $H'$  be the schematic closure of  $H$  in  $G$ . This is an  $R$ -flat closed subgroup since  $R$  is Dedekind, and it is commutative since the  $K$ -group  $H'_K = H_K$  is commutative. We may replace  $G$  with  $H'$ , so now  $G$  is commutative. For any  $n \geq 1$ , the monomorphism  $H[n] \rightarrow G$  is proper since  $H[n]$  is finite, so it is a closed immersion. It is harmless to pass to  $G/H[n]$  and  $H/H[n]$ . Taking  $n$  to be divisible by the order of the torsion subgroup of the finitely generated abelian group that is “dual” to the split  $H$ , the geometric fibers of  $H/H[n]$  are smooth (i.e., tori), so  $H \simeq \mathbf{G}_m^r$  for some  $r \geq 0$ .

We can assume that  $k$  is algebraically closed and  $H \neq 1$ . Let  $T = \mathbf{G}_m$  be the first factor of  $H = \mathbf{G}_m^r$  and let  $T'$  be the closure of  $T$  in  $G$ . If we can prove that  $T' = T$  then we may pass to the quotient by  $T$  and induct on  $r$ . Thus, we may assume  $H = \mathbf{G}_m$ , so  $G$  also has 1-dimensional fibers. In particular, the closed immersion  $j_k$  must identify  $H_k$  with  $(G_k)_{\text{red}}^0$ . For  $N > 0$  relatively prime to the order of the finite group scheme  $G_k/H_k$ , a snake lemma argument shows that  $N : G_k \rightarrow G_k$  is a quotient map with kernel  $H_k[N]$ , so it is flat. Hence, by the fibral flatness criterion,  $N : G \rightarrow G$  is flat, so  $G[N]$  is flat and hence  $G[N] = H[N]$  as closed subschemes of  $G$  due to the equality of their generic fibers in  $G_K = H_K$ .

The translation action by  $H = \mathbf{G}_m$  on  $G$  defines a  $\mathbf{Z}$ -grading  $\bigoplus_{n \in \mathbf{Z}} A_n$  of the coordinate ring  $A$  of  $G$  [Oes, III, 1.5]. (Explicitly, this grading extends the natural one on the coordinate ring  $A_K$  of  $G_K = H_K = \mathbf{G}_m$ .) The quotient map  $A \rightarrow \mathcal{O}(G[N])$  between flat  $R$ -modules is injective on each  $A_n$  for  $n < N$  because it is so over  $K$  (as  $G_K = \mathbf{G}_m$ ). It follows that each  $A_n$  is *finitely generated* over  $R$ , and thus is finite free of rank 1 because we can compute

the rank over  $K$ . The map  $A \rightarrow \mathcal{O}(H) = R[t, 1/t]$  respects the  $\mathbf{Z}$ -gradings and induces a surjection on special fibers (as  $j_k$  is a closed immersion), so the induced maps between the rank-1 graded parts are isomorphisms. Hence,  $A \rightarrow \mathcal{O}(H)$  is an isomorphism.  $\square$

**Remark B.1.4.** Beware that even over a discrete valuation ring, there are examples of monic homomorphisms  $f$  between smooth affine groups with connected fibers such that  $f$  is not an immersion! See [SGA3, XVI, 1.1(c)] for some explicit examples (and [SGA3, VIII, §7] for further discussion). The contrast with the case  $S = \text{Spec } k$  for a field  $k$  is that in such cases homomorphisms between smooth affine  $S$ -groups are always faithfully flat onto a closed image (ultimately due to the “locally closed” property of  $G$ -orbits over fields, which has no good analogue in comparable generality in the relative case).

**Lemma B.1.5.** *Let  $k$  be a field. A  $k$ -group  $H$  of multiplicative type splits over a finite separable extension of  $k$ .*

*Proof.* By direct limit considerations, it suffices to prove that  $H_{k_s}$  is split. Thus, we may and do assume  $k = k_s$  and aim to prove that  $H \simeq D_k(M)$  for a finitely generated abelian group  $M$ . Since any fppf cover of  $\text{Spec } k$  acquires a rational point over a finite extension  $k'/k$ , there is a finitely generated abelian group  $M$  such that  $H$  and  $D_k(M)$  become isomorphic over  $k'$ .

The functor  $I = \underline{\text{Isom}}(H, D_k(M))$  is an fppf sheaf on the category of  $k$ -schemes, so it is an  $\text{Aut}(M)$ -torsor because this can be checked upon restriction to the category of  $k'$ -schemes. More specifically, the restriction of  $I$  over  $k'$  is represented by a constant  $k'$ -scheme, and that constant scheme is equipped with an evident descent datum (arising from  $I$ ) relative to  $k'/k$ . Although  $I_{k'}$  is generally not affine, nor even quasi-compact, the descent to  $k$  is effective because fppf descent is effective for affine schemes and in our descent problem the “equivalence classes” inside  $I_{k'}$  are open and closed subschemes that are  $k$ -finite. Hence,  $I$  is represented by a  $k$ -scheme that is non-empty and étale (as it becomes constant and non-empty over  $k'$ ), so  $I$  is constant (since  $k$  is separably closed).  $\square$

**B.2. Hochschild cohomology.** The key to the infinitesimal properties of groups of multiplicative type is the vanishing of their higher Hochschild cohomology over an affine base. A general introduction to Hochschild cohomology  $H^i(G, \mathcal{F})$  for flat affine group schemes  $G \rightarrow S$  acting linearly on quasi-coherent  $\mathcal{O}_S$ -modules  $\mathcal{F}$  over an affine  $S$  is given in [Oes, III, §3]. Loosely speaking, Hochschild cohomology is a scheme-theoretic version of ordinary group cohomology. This theory is explained (with proofs) in [Oes, III, §3]. Some applications below require a more “functorial” description of Hochschild cohomology, so we now review the basic setup using a variation on the formulation given in [Oes, III, §3].

Let  $M$  be a commutative group functor on the category of  $S$ -schemes, equipped with an action by an  $S$ -group scheme  $G$ . A case of much interest is the functor  $S' \rightsquigarrow \Gamma(S', \mathcal{F}_{S'})$  arising from a *quasi-coherent  $G$ -module*, by which we mean a quasi-coherent  $\mathcal{O}_S$ -module  $\mathcal{F}$  equipped with a linear  $G$ -action; see [Oes, III, 1.2]. (This amounts to an  $\mathcal{O}_{S'}$ -linear action of  $G(S')$  on  $\mathcal{F}_{S'}$  functorially in  $S'$ . By consideration of the “universal point”  $\text{id}_G : G \rightarrow G$ , it is equivalent to specifying an  $\mathcal{O}_G$ -linear automorphism of  $\mathcal{F}_G$ .) For  $n \geq 0$ , let  $C^n(G, M)$  be the abelian

group of natural transformations of set-valued functors  $c : G^n \rightarrow M$  on the category of  $S$ -schemes (i.e., compatible systems of maps of sets  $G(T)^n \rightarrow M(T)$ , or equivalently the abelian group  $M(G^n)$ ). For example, if  $S = \text{Spec } A$  is affine and  $G = \text{Spec}(B)$  is affine and  $\mathcal{F} = \tilde{V}$  for an  $A$ -linear  $B$ -comodule  $V$  then the element  $v \otimes b_1 \otimes \cdots \otimes b_n \in V \otimes B^{\otimes n} = \Gamma(G^n, \mathcal{F}_{G^n})$  corresponds to  $c : G^n \rightarrow \mathcal{F}$  defined functorially by  $(g_1, \dots, g_n) \mapsto (\prod g_i^*(b_i))v_{S'}$  on  $S'$ -valued points for any  $S$ -scheme  $S'$ .

The groups  $C^n(G, M) = M(G^n)$  form a complex  $C^\bullet(G, M)$  in the habitual manner, via

$$dc : (g_0, \dots, g_n) = g_0 \cdot c(g_1, \dots, g_n) + \sum_{i=1}^n (-1)^i c(g_0, \dots, g_{i-1}g_i, \dots, g_n) + (-1)^{n+1} c(g_0, \dots, g_{n-1}).$$

We define the Hochschild cohomology

$$H^n(G, M) = H^n(C^\bullet(G, M)).$$

For example,  $H^0(G, M)$  is the group  $M(S)^G$  of  $m \in M(S)$  that are  $G$ -invariant in the sense that  $g.m_{S'} = m_{S'}$  in  $M(S')$  for any  $S' \rightarrow S$  and  $g \in G(S')$ . More concretely,  $G$ -invariance means that the pullback  $m_G \in M(G)$  is invariant under the universal point  $\text{id}_G \in G(G)$ .

**Proposition B.2.1.** *If  $f : S' = \text{Spec } A' \rightarrow \text{Spec } A = S$  is a flat map of affine schemes and  $G$  is  $S$ -affine then for any quasi-coherent  $G$ -module  $\mathcal{F}$  the natural map  $A' \otimes_A H^n(G, \mathcal{F}) \rightarrow H^n(G_{A'}, \mathcal{F}_{A'})$  is an isomorphism for all  $n \geq 0$ .*

*Proof.* Since each  $G^n$  is affine, the natural map of complexes  $A' \otimes_A C^\bullet(G, \mathcal{F}) \rightarrow C^\bullet(G_{S'}, \mathcal{F}_{S'})$  is identified in degree  $n$  with the natural map  $A' \otimes_A V_{G^n} \rightarrow (A' \otimes_A V)_{G_{A'}^n}$ , with  $\tilde{V} = \mathcal{F}$ . This map is visibly an isomorphism. Passing to homology in degree  $n$  recovers the map of interest as an isomorphism because the exact functor  $A' \otimes_A (\cdot)$  commutes with the formation of homology in the evident manner.  $\square$

Assume  $G$  is  $S$ -affine and  $S$ -flat, and that  $S$  is also affine (so each  $G^n$  is affine and  $S$ -flat). The functor  $\mathcal{F} \rightsquigarrow C^n(G, \mathcal{F})$  on quasi-coherent  $G$ -modules is exact for each  $n$ . Thus,  $H^\bullet(G, \mathcal{F})$  is  $\delta$ -functorial in such  $\mathcal{F}$  via the snake lemma. We shall prove that the category of quasi-coherent  $G$ -modules has enough injectives and that the derived functor of  $\mathcal{F} \rightsquigarrow \mathcal{F}(S)^G$  on this category is  $H^\bullet(G, \cdot)$ .

For  $S$ -affine  $G$ , a right adjoint “Ind” to the forgetful functor from quasi-coherent  $G$ -modules to quasi-coherent  $\mathcal{O}_S$ -modules is given by defining  $\text{Ind}(\mathcal{G})(S')$  (for any  $S$ -scheme  $S'$ ) to be  $\Gamma(G_{S'}, \mathcal{G}_{G_{S'}})$  equipped with the left  $G(S')$ -action  $(g.f)(x) = f(xg)$  for points  $x$  of  $G_{S'}$ . Explicitly, as in [Oes, III, 3.2, Ex.],  $\text{Ind}(\mathcal{G})$  is the quasi-coherent pushforward of  $\mathcal{G}_G$  along the affine map  $G \rightarrow S$ , equipped with a natural  $\mathcal{O}_S$ -linear  $G$ -action. The  $G$ -action amounts to an  $\mathcal{O}_G$ -linear automorphism of the pullback of  $\text{Ind}(\mathcal{G})$  along  $G \rightarrow S$ . By general nonsense, Ind carries injectives to injectives, and monomorphisms to monomorphisms. Moreover, the adjunction morphism  $\mathcal{F} \rightarrow \text{Ind}(\mathcal{F})$  is given by  $m \mapsto (g \mapsto g.m)$ ; it has trivial kernel since  $f \mapsto f(1)$  is a retraction. Thus, the category of quasi-coherent  $G$ -modules on  $S$  has enough injectives (since the category of quasi-coherent  $\mathcal{O}_S$ -modules does; recall that  $S$  is affine).

**Lemma B.2.2.** *For affine  $S$  and  $S$ -flat  $S$ -affine  $G$ , the  $\delta$ -functor  $H^\bullet(G, \cdot)$  on quasi-coherent  $G$ -modules is the right derived functor of  $\mathcal{F} \rightsquigarrow \mathcal{F}(S)^G$ .*

*Proof.* An elegant argument in [Oes, III, 3.2, Rem.] (inspired by the proof of acyclicity of induced modules for ordinary group cohomology) shows that  $H^i(G, \text{Ind}(\mathcal{G})) = 0$  for any quasi-coherent  $\mathcal{O}_S$ -module  $\mathcal{G}$  and  $i > 0$ . For any injective  $\mathcal{F}$  in the category of quasi-coherent  $G$ -modules, the inclusion  $\mathcal{F} \rightarrow \text{Ind}(\mathcal{F})$  splits off  $\mathcal{F}$  as a  $G$ -equivariant direct summand. But  $\text{Ind}(\mathcal{F})$  is acyclic for Hochschild cohomology, so  $\mathcal{F}$  is as well. We conclude that  $H^\bullet(G, \cdot)$  is erasable on the category of quasi-coherent  $G$ -modules, so on this category it is the desired right derived functor.  $\square$

A fundamental consequence (as in [Oes, III, 3.3]) is that for any *multiplicative type* group  $G$  over an affine  $S = \text{Spec } A$ ,  $H^i(G, \cdot) = 0$  on quasi-coherent objects for  $i > 0$ . Indeed, by Proposition B.2.1 it suffices to prove this after a faithfully flat affine base change on  $A$ , so we may assume  $G = D_S(M)$ . Thus, quasi-coherent  $G$ -modules are “the same” as  $M$ -graded quasi-coherent  $\mathcal{O}_S$ -modules  $\mathcal{F} = \bigoplus_{m \in M} \mathcal{F}_m$  [Oes, III, 1.5], so  $H^0(G, \mathcal{F}) = \mathcal{F}_0(S)$ . This is exact in such  $\mathcal{F}$ , so its higher derived functors vanish.

An important application of Hochschild cohomology is the construction of obstructions to deformations of homomorphisms between group schemes. This is inspired by the use of low-degree group cohomology to classify group extensions, and comes out as follows.

**Proposition B.2.3.** *Consider a scheme  $S$  and an exact sequence of group functors*

$$1 \rightarrow M \rightarrow G \rightarrow G' \rightarrow 1$$

*for the fppc topology, with  $M$  commutative. Use this exact sequence to make  $G'$  act on  $M$  via the left  $G$ -conjugation action on  $M$ .*

*Let  $H$  be an  $S$ -group scheme, and fix an  $S$ -homomorphism  $f' : H \rightarrow G'$ . Make  $H$  act on  $M$  through composition with  $f'$ . Assume that  $f'$  admits a lifting to an  $H$ -valued point of  $G$ . There is a canonically associated class  $c(f') \in H^2(H, M)$  that vanishes if and only if  $f'$  lifts to an  $S$ -homomorphism  $f : H \rightarrow G$ . If such an  $f$  exists, the set of such lifts taken up to conjugation by  $M(S)$  on  $G$  is a principal homogeneous space for the group  $H^1(H, M)$ .*

The result holds with the same proof using any topology for which representable functors are sheaves (e.g., Zariski, étale, fppf). See [SGA3, III, 1.2.2] for further generality.

*Proof.* Fix an  $f \in G(H)$  lifting  $f'$ . The obstruction to  $f$  being a homomorphism is the vanishing of the map of  $c = c_f \in M(H \times H)$  defined by  $c(h_0, h_1) = f(h_0 h_1) f(h_1)^{-1} f(h_0)^{-1}$ . The action of any  $h_0$  on  $c(h_1, h_2)$  is induced by  $f(h_0)$ -conjugation, so

$$\begin{aligned} h_0.c(h_1, h_2) &= f(h_0)(f(h_1 h_2) f(h_2)^{-1} f(h_1)^{-1}) f(h_0)^{-1} \\ &= c(h_0, h_1 h_2)^{-1} c(h_0 h_1, h_2) c(h_0, h_1) \\ &= c(h_0 h_1, h_2) - c(h_0, h_1 h_2) + c(h_0, h_1). \end{aligned}$$

In other words, the element  $c_f \in M(H \times H)$  is a Hochschild 2-cocycle (where  $M$  is equipped with its natural  $H$ -action through  $f'$ ). Fixing one choice of  $f$ , all choices have exactly the

form  $m \cdot f : h \mapsto m(h) \cdot f(h)$  for  $m \in M(H)$ , and

$$\begin{aligned} c_{m \cdot f}(h_0, h_1) &= m(h_0 h_1) f(h_0 h_1) f(h_1)^{-1} f(h_0)^{-1} (f(h_0) m(h_1) f(h_0)^{-1}) m(h_0)^{-1} \\ &= m(h_0 h_1) + c_f(h_0, h_1) - h_0 \cdot m(h_1) - m(h_1). \end{aligned}$$

Thus, the class of  $c_f$  in  $H^2(H, M)$  only depends on  $f'$  and not the choice of  $f$ , and as we vary through all  $f$  this 2-cocycle exhausts exactly the members of its cohomology class. Thus, this class vanishes if and only if we can choose  $f$  so that  $c_f = 0$ , which is to say that  $f$  is an  $S$ -homomorphism.

Assume that an  $S$ -homomorphism  $f$  lifting  $f'$  exists. Fix one such choice of  $f$ . The preceding calculation shows that the possible choices are precisely  $h \mapsto m(h) f(h)$  where  $m$  is a Hochschild 1-cocycle on  $H$  with values in  $M$ . Applying conjugation to such an  $f$  by some  $m_0 \in M(S)$  replaces  $f$  with the lifting

$$h \mapsto m_0 \cdot f(h) \cdot m_0^{-1} = m_0 \cdot (f(h) m_0^{-1} f(h)^{-1}) \cdot f(h) = (m_0 - h \cdot m_0) \cdot f(h)$$

due to the definition of the  $H$ -action on  $M$  (through the  $G'$ -action on  $M$  induced by  $G$ -conjugation). It follows that  $H^1(H, M)$  acts simply transitively on the set of  $M(S)$ -conjugacy classes of  $S$ -homomorphisms  $f$  lifting  $f'$ .  $\square$

**Corollary B.2.4.** *Let  $G \rightarrow S$  be a smooth group over an affine scheme  $S = \text{Spec } A$ , and let  $H$  be an affine  $S$ -group. Let  $J$  be a square-zero ideal in  $A$ ,  $A_0 = A/J$ ,  $S_0 = \text{Spec } A_0$ ,  $G_0 = G \bmod J$ , and  $H_0 = H \bmod J$ . Fix an  $S_0$ -homomorphism  $f_0 : H_0 \rightarrow G_0$ , and let  $H_0$  act on  $\text{Lie}(G_0)$  via  $\text{Ad}_{G_0} \circ f_0$ .*

*There is a canonically associated class  $c(f_0) \in H^2(H_0, \text{Lie}(G_0) \otimes J)$  whose vanishing is necessary and sufficient for  $f_0$  to lift to an  $S$ -homomorphism  $f : H \rightarrow G$ . If such an  $f$  exists, the set of such lifts taken up to conjugation by the action of  $\ker(G(S) \rightarrow G(S_0))$  is a principal homogeneous space for the group  $H^1(H_0, \text{Lie}(G_0) \otimes J)$ .*

See [SGA3, III, 2.2, 2.3] for generalizations.

*Proof.* Let  $i : S_0 \rightarrow S$  be the canonical closed immersion, so  $i_*(G_0)$  is the group functor  $S' \rightsquigarrow G_0(S'_0)$  on  $S$ -schemes. An  $S_0$ -homomorphism  $f_0 : H_0 \rightarrow G_0$  corresponds to an  $S$ -group functor homomorphism  $f'_0 : H \rightarrow i_*(G_0)$ . An  $S$ -homomorphism  $f : H \rightarrow G$  lifts  $f'_0$  if and only if  $f \bmod J = f_0$ . Thus, we focus on lifting  $f'_0$ .

By the smoothness of  $G$ , the natural homomorphism  $q : G \rightarrow i_*(G_0)$  is surjective for the Zariski topology (and hence for any finer topology). More specifically, since  $G$  is smooth and  $H_0 \hookrightarrow H$  is defined by a square-zero ideal on the affine scheme  $H$ ,  $f_0$  admits a lifting  $f'_0$  through  $q$  as a scheme morphism. It also follows from the smoothness of  $G$  that  $\ker q$  is the group functor on  $S$ -schemes associated to the quasi-coherent  $\mathcal{O}_{S_0}$ -module  $\text{Lie}(G_0) \otimes J$ . (This uses that  $J\mathcal{O}_G = J \otimes_{A_0} \mathcal{O}_{G_0}$ , a consequence of the  $A$ -flatness of  $G$ .) The conjugation action by  $G$  on the commutative  $\ker q = \text{Lie}(G_0) \otimes J$  factors through an action by  $i_*(G_0)$ , and this “is” the adjoint action of  $G_0$  on  $\text{Lie}(G_0)$ . Thus, Proposition B.2.3 applies to the exact sequence

$$1 \rightarrow \text{Lie}(G_0) \otimes J \rightarrow G \rightarrow i_*(G_0) \rightarrow 1.$$

By quasi-coherence of the kernel, naturally  $H^i(H, \text{Lie}(G_0) \otimes J) = H^i(H_0, \text{Lie}(G_0) \otimes J)$ .  $\square$

**B.3. Deformation theory.** To relativize results established over fields, it is important to have “fibrational criteria” for properties of morphisms (such as flatness, smoothness, etc.) as well as deformation-theoretic results concerning the obstructions to lifting problems. In the direction of fibrational criteria, we often need the “fibrational isomorphism criterion”:

**Lemma B.3.1.** *Let  $h : Y \rightarrow Y'$  be a map between finitely presented schemes over a scheme  $S$ , and assume that  $Y$  is  $S$ -flat. If  $h_s$  is an isomorphism for all  $s \in S$  then  $h$  is an isomorphism.*

*Proof.* This is part of [EGA, IV<sub>4</sub>, 17.9.5], but see Exercise 3.4.2 for another proof.  $\square$

**Theorem B.3.2.** *Let  $(A, \mathfrak{m})$  be a complete local noetherian ring with residue field  $k$ ,  $S = \text{Spec } A$ ,  $G$  an affine  $S$ -group of finite type, and  $H$  an  $S$ -group of multiplicative type that splits over a finite étale cover of  $S$ . Let  $S_n = \text{Spec } A/\mathfrak{m}^{n+1}$ .*

(1) *The natural map*

$$(B.3.1) \quad \text{Hom}_{S\text{-gp}}(H, G) \rightarrow \varprojlim \text{Hom}_{S_n\text{-gp}}(H_{S_n}, G_{S_n})$$

*is bijective.*

(2) *If  $G$  is  $S$ -flat and the special fiber  $G_0$  is of multiplicative type then the map*

$$\text{Hom}_{S\text{-gp}}(H, G) \rightarrow \text{Hom}_{k\text{-gp}}(H_0, G_0)$$

*is bijective and any isomorphism  $j_0 : H_0 \simeq G_0$  uniquely lifts to an open and closed immersion  $j : H \rightarrow G$ .*

The splitting hypothesis on  $H$  is temporary in the sense that it will be shown to always hold (see Proposition B.3.4). Also, the flatness hypothesis on  $G$  in (2) cannot be dropped: for an integer  $d > 1$  let  $H = (\mathbf{Z}/d\mathbf{Z})_R$  over a discrete valuation ring  $R$  such that  $d \in R^\times$  and let  $G$  be the reduced closed complement of the open non-identity locus in the generic fiber.

*Proof.* Let  $S' = \text{Spec } A'$  be a finite étale cover of  $S$  that splits  $H$ . We may assume  $S'$  is connected and Galois over  $S$ , so  $A'$  is a complete local noetherian ring with maximal ideal  $\mathfrak{m}A'$  and descent from  $S'$  to  $S$  can be expressed in terms of actions by  $\Gamma = \text{Aut}(S'/S)$ . The same  $\Gamma$  works for descent from  $S'_n$  to  $S_n$ . Thus, if the analogue of (B.3.1) over  $S'$  is bijective then  $\Gamma$ -equivariance considerations show that (B.3.1) is bijective. The same holds for part (2). We therefore may and do assume  $H = D_S(M)$  for a finitely generated abelian group  $M$ . Upon expressing the problem in (1) in terms of maps of Hopf algebras, it is solved by a clever use of the interaction between completions and tensor products beyond the module-finite setting. See the proof of [SGA3, IX, 7.1] for this important calculation.

We now prove part (2). In such cases, the bijectivity of “passage to the special fiber” on homomorphisms is reduced to showing that the reduction map

$$\text{Hom}_{S_n\text{-gp}}(D_{S_n}(M), G_{S_n}) \rightarrow \text{Hom}_{k\text{-gp}}(D_k(M), G_0)$$

is bijective. That is, upon renaming  $S$  as  $S_n$ , we may assume  $A$  is artinian local ring. By descent we may replace  $A$  with a finite étale extension so that  $G_0 \simeq D_k(N)$  for a finitely generated

abelian group  $N$ . Since  $G$  is an *infinitesimal* flat deformation of  $D_k(N)$ , by the deformation theory of split multiplicative type groups we claim that  $G$  must be of multiplicative type and even that  $G \simeq D_S(N)$ .

To be precise, by induction on  $n$  it suffices to show that if  $J$  is a square-zero ideal in a ring  $R$  and  $G = \text{Spec } A$  is an fppf affine  $R$ -group such that over  $R_0 = R/J$  there is a group isomorphism  $f_0 : G_0 := G \bmod J \simeq D_{R_0}(N)$  for a finitely generated abelian group  $N$  then  $f_0$  lifts to an  $R$ -group isomorphism  $f : G \simeq D_R(N)$ . Since  $H^2(G_0, \cdot) = 0$  on quasi-coherent  $G_0$ -modules (as  $G_0$  is of multiplicative type), by Corollary B.2.4  $f_0$  lifts to an  $S$ -homomorphism  $f$ . But any such lift must be an isomorphism because  $f_0$  is an isomorphism and the source and target of  $f$  are fppf over  $S$ . (A more explicit construction of  $f$  is given in [Oes, IV, §1], directly building an obstruction in a degree-2 Hochschild cohomology group for the multiplicative type  $G_0$ .) The desired bijectivity in (2) now follows via duality for diagonalizable groups (of finite type).

Finally, with general complete local noetherian  $A$ , it remains to show that if  $G_0$  is of multiplicative type and  $G$  is flat then for any isomorphism  $j_0 : D_k(M) \rightarrow G_0$  the unique  $S$ -homomorphism  $j : D_S(M) \rightarrow G$  lifting  $j_0$  is an open and closed immersion. The preceding argument over the  $S_n$ 's shows that  $j_n = j \bmod \mathfrak{m}^{n+1}$  is an isomorphism for all  $n \geq 0$ , so  $j$  is formally flat along the identity section and hence is flat near the identity section. But  $j_s$  must be flat for all  $s \in S$  since a homomorphism between finite type groups over a field is flat if it is flat near the identity (use translation considerations on a geometric fiber), so by the fibral flatness criterion  $j$  is flat.

Consider the kernel  $K = \ker j$ , a flat closed  $S$ -subgroup of  $D_S(M)$  with  $K_0 = 0$ . It suffices to prove  $K = 0$ . Indeed, then  $j$  will be a monomorphism and so even a closed immersion (Lemma B.1.3), and a flat closed immersion between noetherian schemes is an open immersion. To prove that the flat  $S$ -group  $K$  is trivial, observe that each fiber  $K_s$  ( $s \in S$ ) is of multiplicative type, due to being a closed subgroup scheme of  $D_s(M)$  (see Exercise 2.4.1). For  $n \geq 1$  the  $n$ -torsion  $K[n]$  is a finite  $S$ -group scheme (as it is closed in  $D_S(M)[n] = D_S(M/nM)$ ) with special fiber  $K[n]_0 = K_0[n] = 0$ , so by Nakayama's Lemma we have  $K[n] = 0$ . Hence, for each  $s \in S$  the torsion  $K_s[n]$  vanishes for all  $n \geq 1$ , so the multiplicative type group  $K_s$  over  $k(s)$  vanishes [Oes, II, 3.2]. In other words, the identity section  $e : S \rightarrow K$  is an isomorphism on fibers over  $S$ , so it is an isomorphism by the fibral isomorphism criterion (Lemma B.3.1).  $\square$

**Corollary B.3.3.** *Let  $S$  be a scheme and  $H'$  an  $S$ -group of multiplicative type. Any fppf closed subgroup  $H \subset H'$  is of multiplicative type.*

*Proof.* Passing to an fppf cover, we may assume that  $H' = D_S(M)$  for a finitely generated abelian group  $M$ . We may reduce to the case when  $S = \text{Spec } A$  for a ring  $A$  that is noetherian, and then local; let  $k$  be the residue field. Exercise 2.4.1 provides a subgroup  $N \subset M$  such that  $H_k = D_k(M/N)$  inside  $D_k(M)$ . We claim that  $H = D_S(M/N)$  inside  $D_S(M)$ . It suffices to check this equality of closed subschemes after the fpqc base change to  $\widehat{A}$ , so we may assume  $A$  is complete. Thus, by Theorem B.3.2 the isomorphism  $D_k(M/N) \simeq H_k$

uniquely lifts to an abstract  $S$ -homomorphism  $j : D_S(M/N) \rightarrow H$  that is moreover an open and closed immersion. The composition of  $j$  with the inclusion  $H \hookrightarrow H' = D_S(M)$  is an  $S$ -homomorphism  $D_S(M/N) \rightarrow D_S(M)$  that reduces to the canonical inclusion over the closed point, so by duality for diagonalizable groups it is the canonical inclusion over the entire connected base  $S$ . In other words,  $j$  is a containment inside  $H' = D_S(M)$ ; i.e.,  $H$  contains  $D_S(M/N)$  as an open and closed subscheme inside  $D_S(M)$ . Now we can pass to the quotients by  $D_S(M/N)$  to reduce to the case that  $H_k$  is the trivial  $k$ -group and the identity section of  $H$  is an open and closed immersion. In this case we will prove that  $H$  is the trivial  $S$ -group. Since the fppf group scheme  $H \rightarrow S$  has an identity section that is an open and closed immersion, its fibers are étale, so  $H$  is  $S$ -étale.

We claim that  $H \rightarrow S$  is killed by some integer  $n > 0$ . Since  $S$  is noetherian, there is an  $n > 0$  that is a multiple of all fiber-degrees for  $H \rightarrow S$ . Thus,  $n$  kills each finite étale fiber group  $H_s$ . The locus of equality for the two maps  $[n], 0 : H \rightrightarrows H$  is an open and closed subscheme since  $H \rightarrow S$  is étale and separated, so  $n$  kills the  $S$ -group  $H$ . We conclude that the closed subgroup  $H \subset H'$  is contained in the  $S$ -finite  $H'[n]$ , so the  $S$ -étale  $H$  is  $S$ -finite and therefore  $H \rightarrow S$  has constant fiber rank (by connectedness of  $S$ ). This rank must be 1, due to triviality of  $H_k$ . But the identity section  $e : S \rightarrow H$  is an open and closed immersion, so it is surjective and therefore an isomorphism as desired.  $\square$

**Proposition B.3.4.** *Let  $H \rightarrow S$  be an  $S$ -group scheme that becomes diagonalizable (of finite type) fppf-locally on  $S$ . Then  $H$  is diagonalizable étale-locally on  $S$ ; i.e.,  $H$  is quasi-isotrivial. More specifically, the functor  $H \rightsquigarrow \underline{\mathrm{Hom}}_{S\text{-gp}}(H, \mathbf{G}_m)$  is an anti-equivalence between the category of  $S$ -groups of multiplicative type and the category of locally constant abelian étale sheaves on  $S$  whose geometric fibers are finitely generated abelian groups.*

*If  $S = \mathrm{Spec} A$  for a henselian local  $A$  then  $H$  splits over a finite étale cover of  $S$ .*

This result is used very often, generally without comment. For example, it implies that the splitting hypothesis on  $H$  in Theorem B.3.2 is always satisfied.

*Proof.* The final assertion concerning local henselian  $S$  is a consequence of the rest because any étale cover of such an  $S$  has a refinement that is finite étale over  $S$  (due to the equivalent characterizations of henselian local schemes in [EGA, IV<sub>4</sub>, 18.5.11(a),(c)]).

Now using general  $S$ , the group  $H$  is commutative of finite presentation (by fppf descent). By standard limit arguments (including the descent of quasi-compact fppf coverings Zariski-locally on the base), we may assume  $S = \mathrm{Spec} A$  is local noetherian, and even strictly henselian. Consider the special fiber  $H_s$ , a group scheme of multiplicative type over the separably closed  $k(s)$ . By Lemma B.1.5, there is an isomorphism  $j_s : H_s \simeq D_s(M)$  for a finitely generated abelian group  $M$ . Let  $\widehat{S} = \mathrm{Spec} \widehat{A}$ . The map  $j_s$  lifts to an open and closed immersion of  $\widehat{S}$ -groups  $j : \mathcal{H} = D_{\widehat{S}}(M) \hookrightarrow \widehat{H} := H_{\widehat{S}}$  (see Theorem B.3.2). We will prove that  $j$  is an isomorphism, and then then descend it to an isomorphism  $D_S(M) \simeq H$ .

The fppf-local hypothesis on  $H$  is preserved by base change, so there is an fppf cover  $S' \rightarrow \widehat{S}$  such that  $H_{S'} \simeq D_{S'}(M')$  for a finitely generated abelian group  $M'$ . Localize  $S'$  at a point over  $s$ , so  $S' \rightarrow \widehat{S}$  is a local flat map (hence fpqc). The map  $j_{S'} : D_{S'}(M) \rightarrow D_{S'}(M')$  must

arise from a map  $u : M' \rightarrow M$  since  $M$  and  $M'$  are finitely generated (and  $S'$  is connected), and passage to the special fiber implies that  $u$  is an isomorphism (since  $D_s(u_s) = j_s$ ). Hence,  $j_{S'} = D_{S'}(u)$  is an isomorphism, so  $j$  is an isomorphism (by fpqc descent).

Any descent of  $j$  to an  $S$ -homomorphism  $D_S(M) \rightarrow H$  is necessarily an isomorphism (by fpqc descent), so it suffices to prove that the natural map

$$\mathrm{Hom}_{S\text{-gp}}(D_S(M), H) \rightarrow \mathrm{Hom}_{\widehat{S}\text{-gp}}(D_{\widehat{S}}(M), H_{\widehat{S}})$$

is bijective. Injectivity is clear, and for surjectivity we will use fpqc descent for morphisms (inspired by the proof of [SGA3, X, 4.3]).

Since  $H$  splits over an fppf covering of  $S$ , for each  $n \geq 1$  the map  $n : H \rightarrow H$  is finite flat because it can be checked over an fppf covering where  $H$  becomes diagonalizable. Hence, each  $H[n]$  is a commutative *finite*  $S$ -group of multiplicative type with étale Cartier dual. Consider a map  $\widehat{f} : D_{\widehat{S}}(M) \rightarrow H_{\widehat{S}}$ . For each  $n \geq 1$ , the induced map  $\widehat{f}_n$  between  $n$ -torsion subgroups uniquely descends to an  $S$ -homomorphism  $f_n : D_S(M)[n] \rightarrow H[n]$  because we can apply Cartier duality and use that  $E \rightsquigarrow E_{\widehat{S}}$  is an equivalence between the categories of finite étale schemes over  $S$  and  $\widehat{S}$  (as  $A$  is henselian local). To descend  $\widehat{f}$  to an  $S$ -homomorphism, by fpqc descent it is equivalent to check the equality of the pullback maps

$$p_1^*(\widehat{f}), p_2^*(\widehat{f}) : D_{\widehat{S} \times_S \widehat{S}}(M) \rightrightarrows H_{\widehat{S} \times_S \widehat{S}}.$$

A homomorphism from a multiplicative type group to a separated group scheme is determined by its restrictions to the  $n$ -torsion subgroups for all  $n \geq 1$  [Oes, II, 3.2]. By applying this over the (typically non-noetherian!) base  $\widehat{S} \times_S \widehat{S}$ , the desired equality  $p_1^*(\widehat{f}) = p_2^*(\widehat{f})$  is reduced to the same with  $D_S(M)$  replaced by  $D_S(M)[n] = D_S(M/nM)$  for every  $n \geq 1$ . Thus, we may assume that  $M$  is killed by some  $n \geq 1$ . Since  $\widehat{f}_n$  descends, we are done.  $\square$

**Corollary B.3.5.** *Let  $S = \mathrm{Spec} A$  for a complete local noetherian ring  $(A, \mathfrak{m})$  with residue field  $k$ . Let  $G$  be a smooth affine  $S$ -group and  $H$  an  $S$ -group of multiplicative type. Any homomorphism  $f_0 : H_0 \rightarrow G_0$  between special fibers lifts to an  $S$ -homomorphism  $f : H \rightarrow G$ , and if  $f' : H \rightarrow G$  is another such lift then  $f$  and  $f'$  are conjugate under  $\ker(G(S) \rightarrow G(k))$ .*

This corollary is a special case of [SGA3, X, 7.3].

*Proof.* Since  $G$  is affine, the natural map  $G(S) \rightarrow \varprojlim G(S_n)$  is bijective. The smoothness of  $G$  implies that  $G(S_{n+1}) \rightarrow G(S_n)$  is surjective for all  $n \geq 0$ . In view of the bijectivity of (B.3.1) (which is applicable, due to Proposition B.3.4), to prove the existence of  $f$  and its uniqueness up to  $G(S)$ -conjugacy it suffices to show that for each  $n \geq 1$  the maps

$$\mathrm{Hom}_{S_{n+1}\text{-gp}}(H_{S_{n+1}}, G_{S_{n+1}}) \rightarrow \mathrm{Hom}_{S_n\text{-gp}}(H_{S_n}, G_{S_n})$$

are surjective and the fibers are a single orbit under conjugation by  $\ker(G(S_{n+1}) \rightarrow G(S_n))$ .

Let  $J_n = \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$ . Since  $G$  is  $S$ -smooth,  $H$  is  $S$ -affine, and  $S_n \hookrightarrow S_{n+1}$  is defined by the square-zero ideal  $J_n$ , by Corollary B.2.4 the obstructions to surjectivity lie in degree-2 Hochschild cohomology for  $H_{S_n}$  with coefficients in a quasi-coherent  $H_{S_n}$ -module over  $S_n$ . Since  $H_{S_n}$  is of multiplicative type, this cohomology vanishes [Oes, III, 3.3]. Applying

Corollary B.2.4 once more, the obstruction to the transitivity of the conjugation action of  $\ker(G(S_{n+1}) \rightarrow G(S_n))$  on the non-empty set of homomorphisms  $H_{S_{n+1}} \rightarrow G_{S_{n+1}}$  lifting a given homomorphism  $f_n : H_{S_n} \rightarrow G_{S_n}$  lies in a degree-1 Hochschild cohomology group for  $H_{S_n}$  with quasi-coherent coefficients, so again the obstruction vanishes.  $\square$

As an illustration of Corollary B.3.5, if  $G = \mathrm{GL}_n$  and  $H_0 = \mathbf{G}_m$  then it just says that a decomposition of  $k(s)^n$  into a direct sum of subspaces can be lifted to a decomposition of  $A^n$  into a direct sum of finite free submodules.

**Corollary B.3.6.** *Let  $S$  be a normal scheme. Every  $S$ -group  $H \rightarrow S$  of multiplicative type is locally isotrivial: there is a Zariski-open covering  $\{U_i\}$  of  $S$  such that each  $H|_{U_i}$  is isotrivial: it splits over a finite étale cover of  $U_i$ . If  $S$  is irreducible (e.g., connected and locally noetherian) then  $H \rightarrow S$  is isotrivial.*

*In particular, for irreducible normal  $S$  and a geometric point  $\bar{s}$  of  $S$ , the functor  $H \rightsquigarrow \underline{\mathrm{Hom}}_{S\text{-gp}}(H, \mathbf{G}_m)_{\bar{s}}$  is an anti-equivalence from the category of multiplicative type  $S$ -groups to the category of discrete  $\pi_1(S, \bar{s})$ -modules that are finitely generated as abelian groups.*

As an example, for connected locally noetherian normal  $S$ , the category of  $S$ -tori is anti-equivalent to the category of discrete  $\pi_1(S, \bar{s})$ -representations on  $\mathbf{Z}$ -lattices (generalizing the classical case  $S = \mathrm{Spec} k$  for a field  $k$ ). For instance, all  $\mathbf{Z}$ -tori are split because  $\pi_1(\mathrm{Spec} \mathbf{Z}) = 1$  (Hermite–Minkowski). Although a connected normal scheme is irreducible in the locally noetherian case, irreducibility can fail in the non-noetherian affine case; see Exercise 2.4.12.

*Proof.* We call an étale sheaf on a scheme *isotrivial* if it becomes constant over a finite étale cover. By Proposition B.3.4, our task is to prove that if  $\mathcal{F}$  is a finite type locally constant abelian étale sheaf on a normal scheme  $S$  then:  $\mathcal{F}$  is isotrivial if  $S$  is irreducible, and in general  $\mathcal{F}$  is isotrivial Zariski-locally on  $S$ . Each  $\mathrm{Spec} \mathcal{O}_{S,s}$  is normal and irreducible, and any finite étale cover of  $\mathrm{Spec} \mathcal{O}_{S,s}$  spreads out to a finite étale cover of an open neighborhood of  $s$  in  $S$ . Thus, by the local constancy and finite type hypotheses on  $\mathcal{F}$ , the isotriviality Zariski-locally on  $S$  is reduced to the isotriviality for each  $\mathcal{F}|_{\mathrm{Spec} \mathcal{O}_{S,s}}$ .

Now we may and do assume that  $S$  is irreducible. Every connected finite étale  $S$ -scheme  $S'$  is also irreducible. Indeed, the generic points of  $S'$  lie over the unique generic point of the irreducible  $S$ , so there are only finitely many of them. Hence, there are only finitely many irreducible components of  $S'$ . These components are pairwise disjoint (since  $S'$  is normal), so by *finiteness* each is open and closed. By connectedness,  $S'$  must therefore be irreducible.

Since étale maps are open for the Zariski topology, and  $\mathcal{F}$  becomes constant over the constituents of an étale cover of  $S$ , the isomorphism type of the geometric stalk  $\mathcal{F}_{\bar{s}}$  is locally constant in  $s$  for the Zariski topology on  $S$ . Thus, by connectedness of  $S$ , there exists a finitely generated abelian group  $M$  so that  $\mathcal{F}_{S'} \simeq \underline{M}_{S'}$  for some étale cover  $S' \rightarrow S$ . For each  $n \geq 1$ , we have  $(\mathcal{F}/n\mathcal{F})_{S'} \simeq \underline{M/nM}_{S'}$ . The constant  $S'$ -group  $(M/nM)_{S'}$  is finite étale (especially *affine*) over  $S'$ , so the descent datum on it relative to  $S' \rightarrow S$  arising from  $\mathcal{F}/n\mathcal{F}$  is effective. Hence,  $\mathcal{F}/n\mathcal{F}$  is represented by a finite étale  $S$ -group  $G_n \rightarrow S$ . Since a finite étale map has open and closed image, the connectedness of  $S$  and constancy of the

fiber degree of a finite étale  $S$ -scheme  $E$  forces any such  $E$  to “disconnect” at most finitely many times; i.e.,  $E$  is a disjoint union of finitely many *connected* finite étale  $S$ -schemes. By choosing a single connected Galois finite étale cover of  $S$  that dominates all connected components of such an  $E$ , we can split  $E$  using a connected finite étale cover  $S' \rightarrow S$ .

Fix  $n \geq 3$  divisible by the exponent of  $M_{\text{tor}}$  and apply base change to a connected finite étale cover  $S' \rightarrow S$  that splits  $G_n$ , so  $G_n$  is a constant  $S$ -group. We shall prove that  $\mathcal{F}$  is constant. First consider the special case that  $S = \text{Spec } k$  for a field  $k$ . The category of étale abelian sheaves on  $S$  is the category of discrete  $\text{Gal}(k_s/k)$ -modules, so  $\mathcal{F}$  is identified with an action on  $M$  by a finite Galois group  $\text{Gal}(k'/k)$ . To prove that  $\mathcal{F}$  is constant we need to prove the triviality of this action. The constancy of  $\mathcal{F}/n\mathcal{F}$  implies that the  $\text{Gal}(k'/k)$ -action factors through a finite subgroup of  $\Gamma = \ker(\text{Aut}(M) \rightarrow \text{Aut}(M/nM))$ , so it suffices to prove that  $\Gamma$  is torsion-free.

Recall the classical fact that for any integer  $d > 0$ , the kernel of  $\text{GL}_d(\mathbf{Z}) \rightarrow \text{GL}_d(\mathbf{Z}/n\mathbf{Z})$  is torsion-free for  $n \geq 3$ . (This is most efficiently proved by observing that a torsion element in the kernel has eigenvalues that are roots of unity, and considering  $p$ -adic logarithms for  $p|n$ .) Since  $M' := M/M_{\text{tor}} \simeq \mathbf{Z}^d$  for some  $d$ , and  $M'/nM'$  is a quotient of  $M/nM$ , any finite-order  $\gamma \in \Gamma$  is trivial on  $M'$ ; i.e.,  $\gamma(m) = m + h(m)$  for some  $h : M \rightarrow M_{\text{tor}}$ . The hypothesis  $\gamma \equiv \text{id} \pmod{nM}$  implies that  $h$  is valued in  $nM \cap M_{\text{tor}} \subset (nM)_{\text{tor}}$ . Non-canonically  $M \simeq M' \oplus M_{\text{tor}}$ , so  $nM$  is torsion-free. This forces  $h = 0$ , so  $\gamma = \text{id}$  as desired. For general  $S$ , we conclude that  $\mathcal{F}_s$  over  $\text{Spec } k(s)$  is constant for each  $s \in S$ .

Next consider the case  $S = \text{Spec } A$  for an integrally closed *local* domain  $A$ . Let  $K$  be the fraction field of  $A$ . Fix an isomorphism  $f_\eta : \underline{M}_K \simeq \mathcal{F}_K$ . Let  $S' \rightarrow S$  be an étale cover such that  $\underline{M}_{S'} \simeq \mathcal{F}|_{S'}$ . We may assume  $S' = \text{Spec } A'$  is affine, so  $A'$  has only finitely many minimal primes (due to the finiteness of  $S'_K$ ). The finitely many irreducible components of  $S'$  are pairwise disjoint, so each is open. Hence, there are only finitely many connected components of  $S'$  and they are irreducible. At least one of these has non-empty special fiber, and so must cover  $S$  (as its open image in the *local*  $S$  contains the closed point and so is full). Hence, we may assume  $A'$  is a domain, so  $S'_K$  is the generic point  $\eta'$  of  $S'$ . By the connectedness of  $S'$  and constancy of  $\mathcal{F}_{S'}$ ,  $(f_\eta)_{\eta'}$  uniquely extends to an  $S'$ -isomorphism  $\theta : \underline{M}_{S'} \simeq \mathcal{F}_{S'}$ . For  $S'' = S' \times_S S'$ , the descent datum  $p_1^*(\mathcal{F}_{S'}) \simeq p_2^*(\mathcal{F}_{S'})$  is identified via  $\theta$  with an  $S''$ -group isomorphism  $\varphi : M_{S''} \simeq M_{S''}$  whose restriction over  $S''_K = \text{Spec}(K' \otimes_K K')$  is the *identity* map (due to the descent of  $\theta_{S'_K} = \theta_{\eta'}$  to  $f_\eta$ ). But  $S''$  is a disjoint union of finitely many connected components, each of which meets  $S''_K$ , so  $\varphi$  is the identity. Thus,  $f_\eta$  extends to an isomorphism  $\underline{M}_S \simeq \mathcal{F}$ .

Finally, we treat the general irreducible case. Fix an isomorphism  $f_\eta : \underline{M}_\eta \simeq \mathcal{F}_\eta$ . Applying the preceding over the local rings of  $S$  implies that  $\mathcal{F}$  is constant Zariski-locally on  $S$ . Let  $\{U_i\}$  be a covering of the irreducible  $S$  by non-empty open subschemes so that there are isomorphisms  $f_i : \underline{M}_{U_i} \simeq \mathcal{F}|_{U_i}$ . Each  $U_i$  is connected, so we may uniquely choose each  $f_i$  such that  $(f_i)_\eta = f_\eta$ . The overlaps  $U_i \cap U_j$  are irreducible with generic point  $\eta$ , so  $f_i$  and  $f_j$  coincide over  $U_i \cap U_j$ . Hence, the  $f_i$  glue to an isomorphism  $\underline{M}_S \simeq \mathcal{F}$ .  $\square$

**Remark B.3.7.** A scheme  $S$  is *unibranched* if  $\mathrm{Spec} \mathcal{O}_{S,s}^h$  is irreducible for all  $s \in S$ . For irreducible  $S$ , it is equivalent that  $S_{\mathrm{red}}$  has normalization  $S' \rightarrow S_{\mathrm{red}}$  that is radiciel (in which case the integral surjective morphism  $S' \rightarrow S$  is radiciel). Pullback along a radiciel integral surjection defines an equivalence between étale sites, so (by the anti-equivalence in Proposition B.3.4) Corollary B.3.6 is valid with “normal” replaced by “unibranched”. This fact is also noted at the end of [Oes, II, 2.1]. (See [Oes, IV, §2] for an elegant general discussion of the “topological invariance” of the theory of multiplicative type groups without finiteness hypotheses.)

**B.4. Fibral criteria.** For a finite type group scheme  $G_0$  over a field  $k$  and any extension field  $K/k$ ,  $G_0$  is of multiplicative type if and only if  $(G_0)_K$  is. Indeed, this is a problem involving compatible algebraic closures of  $k$  and  $K$ , so it suffices to prove that  $G_0 \simeq D_k(M)$  for a finitely generated abelian group  $M$  if and only if  $(G_0)_K \simeq D_K(M)$ . But  $D_K(M) = (D_k(M))_K$ , so the equivalence is clear (as  $G_0$  and  $D_k(M)$  are finite type over  $k$ ); see the proof of Proposition 3.2.2 for an illustration of the general “spreading out and specialization” technique for descending results from an algebraically closed field to an algebraically closed subfield. We conclude that when we consider whether or not the fibers of an fppf affine group scheme  $G \rightarrow S$  are of multiplicative type, it does not matter if we consider the actual fibers  $G_s$  or associated geometric fibers  $G_{\bar{s}}$  for geometric points  $\bar{s}$  of  $S$ .

**Theorem B.4.1.** *Let  $S$  be a scheme and  $H$  an fppf affine  $S$ -group. Then  $H$  is of multiplicative type if and only if its geometric fibers are of multiplicative type and the order of  $H_s[n]$  is locally constant in  $s$  for each  $n \geq 1$ . The fibral torsion condition can be dropped if  $H \rightarrow S$  has connected fibers; e.g., tori.*

Before we prove Theorem B.4.1, we note that the torsion-order hypothesis holds if we assume that the “type” of each  $H_s$  (i.e., the isomorphism class of the character group of the geometric fiber at  $s$ ) is locally constant in  $s$ . Local constancy of the “type” is used in the formulation of the fibral criterion in [SGA3, X, 4.8]. In the absence of a fibral connectedness condition it is necessary to impose some local constancy hypothesis on the type or at least its torsion levels, even if we assume  $H$  is commutative.

For example, suppose  $S = \mathrm{Spec} R$  for a discrete valuation ring  $R$  and  $d > 1$  is an integer not divisible by the residue characteristic. Consider the fppf affine  $R$ -group  $H$  obtained by removing the closed non-identity locus from the special fiber of the constant group  $(\mathbf{Z}/d\mathbf{Z})_R$ . This is quasi-finite flat and not finite flat (due to jumping of fiber-degree), so it is not of multiplicative type but its fibers are of multiplicative type.

*Proof.* The necessity is clear, and for the proof of sufficiency we know that each fiber  $H_s$  is of multiplicative type since we have already noted that the *algebraic* geometric fibers  $H_{\bar{s}}$  (using an algebraic closure of  $k(s)$ ) must be of multiplicative type.

For any  $s \in S$ , any étale cover of  $\mathrm{Spec} \mathcal{O}_{S,s}$  admits an affine refinement and an affine étale cover of  $\mathrm{Spec} \mathcal{O}_{S,s}$  spreads out to an étale cover over an open neighborhood of  $s$  in  $S$ . Hence, it suffices to treat the case when  $S = \mathrm{Spec} A$  for a local ring  $A$ . We may also assume that

$A$  is strictly henselian (since a strict henselization of  $A$  is a directed union of local-étale extensions of  $A$ ). Let  $k$  be the separably closed residue field of  $A$ . By Lemma B.1.5 applied to the special fiber, there is an isomorphism  $j_k : D_k(M) \simeq H_k$  for a finitely generated abelian group  $M$ .

We claim that  $j_k$  lifts to an  $A$ -homomorphism  $j : D_A(M) \rightarrow H$  that is an open and closed immersion. Granting this, we prove that  $j$  is an isomorphism as follows. If  $H \rightarrow S$  has connected fibers then  $j$  is fiberwise surjective and hence an isomorphism (as it is an open immersion). Suppose instead that the order of  $H_s[n]$  is locally constant in  $s \in S$  for each  $n \geq 1$ . By the connectedness of  $S$  this order must be constant, so  $D_s(M)[n]$  and  $H_s[n]$  have the same order for all  $s \in S$  due to comparison of orders of the closed fibers. Thus, the open and closed immersion  $j_s : D_s(M) \hookrightarrow H_s$  between multiplicative type groups at each  $s \in S$  is an isomorphism on  $n$ -torsion for all  $n \geq 1$ , so  $j_s$  is an isomorphism [Oes, II, 3.2]. Thus, the open immersion  $j$  is surjective, so it is an isomorphism.

To construct  $j$  lifting  $j_k$ , we only use the weaker hypothesis that  $H_k$  is of multiplicative type. The reason for weakening the hypothesis is that if we express  $A$  as a directed union of its strictly henselian local noetherian subrings  $\{A_i\}$  (with local inclusion maps) then a descent of  $H$  to an fppf affine  $A_i$ -group  $H_i$  for large  $i$  inherits the multiplicative type hypothesis for its special fiber but the same for other fibers seems hard to control in the limit process. In this way, we may assume that  $A$  is also noetherian. By Theorem B.3.2, the isomorphism  $j_k : D_k(M) \simeq H_k$  uniquely lifts to an open and closed immersion

$$\widehat{j} : (D_A(M))_{\widehat{A}} = D_{\widehat{A}}(M) \hookrightarrow H_{\widehat{A}}.$$

In particular,  $\widehat{j}$  is an isomorphism between infinitesimal special fibers (so  $H$  has commutative infinitesimal fibers). Using fpqc descent for morphisms as at the end of the proof of Proposition B.3.4, we shall prove that  $\widehat{j}$  descends to an  $A$ -morphism  $j : D_A(M) \rightarrow H$ ; such a descent is necessarily a homomorphism and an open and closed immersion (thereby establishing what we need), by general results on fpqc descent for properties of morphisms.

To perform for descent for  $\widehat{j}$ , it suffices to show that the two pullback homomorphisms

$$p_1^*(\widehat{j}), p_2^*(\widehat{j}) : D_A(M)_{\widehat{A} \otimes_A \widehat{A}} \rightrightarrows H_{\widehat{A} \otimes_A \widehat{A}}$$

over the (typically non-noetherian) ring  $\widehat{A} \otimes_A \widehat{A}$  coincide. Arguing via [Oes, II, 3.2] as near the end of the proof of Proposition B.3.4, it suffices to check equality on  $n$ -torsion for each  $n \geq 1$ , which is exactly descent for the restriction  $\widehat{j}_n$  of  $\widehat{j}$  to  $D_A(M/nM)_{\widehat{A}}$  for each  $n \geq 1$ .

The scheme morphism  $f_n : H \rightarrow H$  given by  $h \mapsto h^n$  is quasi-finite flat (as may be checked on fibers, using that each  $H_s$  is of multiplicative type) and a homomorphism on infinitesimal special fibers, so  $H[n] := f_n^{-1}(1)$  is a quasi-finite flat closed subscheme of  $H$  that is a subgroup scheme on infinitesimal special fibers. By the structure theorem for quasi-finite separated morphisms [EGA, IV<sub>4</sub>, 18.5.11(a),(c)], any quasi-finite separated scheme over a henselian local base  $Z$  is uniquely the disjoint union of a  $Z$ -finite open and closed subscheme and an open and closed subscheme with empty special fiber. Let  $H[n]_{\text{fin}}$  denote the resulting “finite part” of  $H[n]$ . This is a *subgroup* scheme of  $H$ : due to the finiteness of  $H[n]_{\text{fin}}$  over the

base (and the separatedness of  $H$ ) it suffices to check the subgroup property on infinitesimal special fibers, where it is clear.

The finite flat group schemes  $H[n]_{\text{fin}}$  are *commutative* because we may check on the infinitesimal special fibers (where even  $H$  becomes commutative). Their Cartier duals are étale, as this may be checked on the special fiber. Moreover, since the formation of the “finite part” commutes with local henselian base change, we have  $(H[n]_{\text{fin}})_{\widehat{A}} = (H_{\widehat{A}}[n])_{\text{fin}}$ . Since  $D_A(M/nM)$  is  $A$ -finite,  $\widehat{j}_n$  factors through  $(H_{\widehat{A}}[n])_{\text{fin}} = (H[n]_{\text{fin}})_{\widehat{A}}$ . But  $D_A(M/nM)$  and  $H[n]_{\text{fin}}$  are finite flat commutative  $A$ -groups whose Cartier duals are étale (and hence *constant*), so any homomorphism between their special fibers uniquely lifts. The same holds over  $\widehat{A}$ , so comparison through the common special fibers over  $A$  and  $\widehat{A}$  implies that  $\widehat{j}_n$  descends to an  $A$ -homomorphism  $D_A(M/nM) \rightarrow H[n]_{\text{fin}} \subset H$  for each  $n \geq 1$ . This completes the proof that  $\widehat{j}$  descends, and hence the proof that  $H$  is of multiplicative type.  $\square$

**Corollary B.4.2.** *Let  $S$  be a scheme,  $H \rightarrow S$  an affine fppf group scheme.*

- (1) *If  $H$  becomes multiplicative type fpqc-locally on  $S$  then it is of multiplicative type.*
- (2) *Consider a short exact sequence*

$$1 \rightarrow H' \rightarrow H \rightarrow H'' \rightarrow 1$$

*of fppf affine  $S$ -groups with  $H'$  and  $H''$  of multiplicative type. If each  $H_s$  is either connected or commutative then  $H$  is of multiplicative type.*

Assertion (1) shows that the notion of “multiplicative type” used in [SGA3] (with fpqc-local triviality) coincides with the notion that we are using (with fppf-local triviality) in the finite type case. Also, there must be some fibral hypothesis on  $H$  in (2), since the finite constant group over a field of characteristic 0 associated to a non-commutative solvable group of order  $p^3$  for a prime  $p$  is an extension of one multiplicative type group by another.

*Proof.* The validity of (1) is immediate from Theorem B.4.1: the fpqc-locality hypothesis ensures that each  $H_s$  is of multiplicative type and controls the condition on local constancy in  $s$  for the order of  $H_s[n]$  for each  $n \geq 1$ .

For the proof of (2) we may assume  $S = \text{Spec } A$  for a strictly henselian local noetherian ring  $A$ , so  $H' = D_S(M')$  and  $H'' = D_S(M'')$  for finitely generated abelian groups  $M'$  and  $M''$  (Proposition B.3.4). Granting that the special fiber  $H_k$  is of multiplicative type, we may conclude as follows (without using either of the commutativity or connectedness hypotheses on the fibers of  $H$ ). The  $k$ -group  $H_k$  has the form  $D_k(M)$  for an abelian group  $M$  that is an extension of  $M'$  by  $M''$  (dual to the given exact sequence on special fibers), and the proof of Theorem B.4.1 shows that the isomorphism  $D_k(M) \simeq H_k$  uniquely lifts to a homomorphism  $j : D_S(M) \rightarrow H$  that is moreover an open and closed immersion. The composition of  $j$  with the canonical homomorphism  $D_S(M') \hookrightarrow D_S(M)$  must be the inclusion  $D_S(M') = H' \hookrightarrow H$  because the two maps agree on special fibers (and we can appeal to the uniqueness for lifting in Theorem B.3.2 over the fpqc cover  $\widehat{A}$ ). In other words, the open and closed subgroup  $D_S(M) \subset H$  contains  $H' = D_S(M')$ . This forces  $D_S(M) = H$  because passing to quotients

by  $D_S(M')$  gives a chain of inclusions

$$D_S(M)/D_S(M') \subset H/D_S(M') \subset H'' = D_S(M'')$$

whose composition is the equality induced by the identification of  $M'$  with  $M/M''$ .

It remains to prove (2) when  $S = \text{Spec } k$  for a field  $k$ , and we may assume  $k$  is algebraically closed. First we show that if  $H$  is connected (so  $H''$  is also connected) then  $H$  must be commutative. The automorphism functor of  $H'$  is represented by a constant  $k$ -group (via duality for diagonalizable finite type  $k$ -groups), so the conjugation action on  $H'$  by the connected  $H$  must be trivial. That is,  $H'$  is central in  $H$ . The commutator morphism for  $H$  therefore factors through a bi-additive pairing  $H'' \times H'' \rightarrow H'$  whose vanishing is equivalent to the commutativity of  $H$ . To prove that the pairing vanishes it suffices to prove that the only homomorphism of group functors  $f : H'' \rightarrow \underline{\text{Hom}}_{k\text{-gp}}(H'', H')$  is the trivial one. This Hom-functor is represented by a constant  $k$ -group since  $H''$  and  $H'$  are diagonalizable finite type  $k$ -groups and  $H''$  is connected, so  $f$  must vanish.

Now working with commutative  $H$  in general, note that the subgroup  $T := H_{\text{red}}^0$  is smooth and connected without  $\mathbf{G}_a$  as a subgroup, so it is a torus by the classical theory. Let  $T'' \subset H_{\text{red}}^0$  be the image of  $T$  and let  $G' = H' \cap T$ , so  $1 \rightarrow H'/G' \rightarrow H/T \rightarrow H''/T'' \rightarrow 1$  is a short exact sequence of finite commutative  $k$ -group schemes. Both finite quotients  $H''/T''$  and  $H'/G'$  are of multiplicative type (since  $H'$  and  $H''$  are; use Exercise 2.4.1), so by Cartier duality for finite commutative  $k$ -groups we see that  $H/T$  has *étale* Cartier dual. Double duality then implies that  $H/T$  is of multiplicative type. In other words,  $H$  is a commutative affine finite type extension of  $D_k(M)$  by a torus  $T$  for a finite abelian group  $M$ . It suffices to show that any such extension splits, which is to say that the abelian group  $\text{Ext}_k^1(D_k(M), T)$  of isomorphism classes of such  $k$ -group extensions vanishes. Bi-additivity of this Ext-functor reduces the problem to the case  $T = \mathbf{G}_m$  and  $M = \mathbf{Z}/n\mathbf{Z}$  for an integer  $n \geq 1$ .

If  $E$  is a commutative affine finite type  $k$ -group extension of  $\mu_n$  by  $\mathbf{G}_m$ , the fppf surjectivity of  $[n] : \mathbf{G}_m \rightarrow \mathbf{G}_m$  implies (via the snake lemma in the abelian category of commutative fppf group sheaves over  $k$ ) that the sequence

$$(B.4.1) \quad 1 \rightarrow \mu_n \rightarrow E[n] \rightarrow \mu_n \rightarrow 1$$

is *short exact* for the fppf topology. It suffices to show that this splits. Applying Cartier duality to (B.4.1), the dual of  $E[n]$  must be *étale*, so it is the constant  $k$ -group corresponding to an  $n$ -torsion finite abelian group that is an extension of  $\mathbf{Z}/n\mathbf{Z}$  by  $\mathbf{Z}/n\mathbf{Z}$ . Any such exact sequence of  $\mathbf{Z}/n\mathbf{Z}$ -modules splits, so by double duality the sequence (B.4.1) also splits.  $\square$

## APPENDIX C. ORTHOGONAL GROUP SCHEMES

**C.1. Basic definitions and smoothness results.** Let  $V$  be a vector bundle of constant rank  $n \geq 1$  over a scheme  $S$ , and  $L$  a line bundle on  $S$ . A *quadratic form*  $q : V \rightarrow L$  is a map of sheaves of sets such that  $q(cv) = c^2q(v)$  (so  $q$  is compatible with Zariski localization on  $S$ ) and the symmetric map  $B_q : V \times V \rightarrow L$  defined by

$$B_q(x, y) = q(x + y) - q(x) - q(y)$$

is  $\mathcal{O}_S$ -bilinear. We call any such  $(V, L, q)$  a *quadratic triple*. Using local trivializations of  $V$  and  $L$ ,  $q(x) = \sum_{i \leq j} a_{ij}x_i x_j$ . There is an evident notion of base change for quadratic triples.

Assume  $q$  is fiberwise non-zero over  $S$ , so  $(q = 0) \subset \mathbf{P}(V^*)$  is an  $S$ -flat hypersurface with fibers of dimension  $n - 2$  (understood to be empty when  $n = 1$ ). By Exercise 1.6.9 (and trivial considerations when  $n = 1$ ), this is smooth precisely when for each  $s \in S$  one of the following holds: (i)  $B_{q_s}$  is non-degenerate and either  $\text{char}(k(s)) \neq 2$  or  $\text{char}(k(s)) = 2$  with  $n$  even, (ii) the defect  $\delta_{q_s}$  is 1,  $q_s|_{V_s^\perp} \neq 0$ , and  $\text{char}(k(s)) = 2$  with  $n$  odd. (By Exercise 1.6.9,  $\delta_{q_s} \equiv \dim V_s$  when  $\text{char}(k(s)) = 2$ .) In such cases we say  $(V, L, q)$  is *non-degenerate* (the terminology *ordinary* is used in [SGA7, XII, §1]). Case (ii) is the “defect-1” case at  $s$ .

A *quadratic space* is a pair  $(V, q)$  where  $q : V \rightarrow \mathcal{O}_S$  is a non-degenerate  $\mathcal{O}_S$ -valued quadratic form. (When we need to consider pairs  $(V, q)$  with  $q$  possibly not non-degenerate, we may call  $(V, q)$  a *possibly degenerate* quadratic space.) The evident notion of isomorphism for a non-degenerate quadratic triple  $(V, \mathcal{O}_S, q)$  corresponds to the classical notion of *similarity* (compatibility up to a unit scaling on the form) for quadratic spaces. We will work with quadratic spaces rather than non-degenerate quadratic triples below, at least once we begin needing to consider Clifford algebras, due to difficulties with Clifford constructions for  $L$ -valued  $q$  when there is not a given trivialization of  $L^{\otimes 2}$ .

**Remark C.1.1.** Our notion of “non-degeneracy” is often called “regularity” or “semi-regularity” (especially for odd  $n$  when 2 is not a unit on  $S$ ); see [Knus, IV, 3.1]. In the study of quadratic forms  $q$  over a domain  $A$ , such as the ring of integers in a number field or a discrete valuation ring, the phrase “non-degenerate” is often used to mean “non-degenerate over the fraction field”. Indeed, non-degeneracy over  $A$  in the sense defined above is rather more restrictive. Non-degenerate examples over  $\mathbf{Z}$  (in our restrictive sense) include the quadratic spaces arising from even unimodular lattices, such as the  $E_8$  and Leech lattices.

For a non-degenerate quadratic triple  $(V, L, q)$ , clearly the functor

$$S' \rightsquigarrow \{g \in \text{GL}(V_{S'}) \mid q_{S'}(gx) = q_{S'}(x) \text{ for all } x \in V_{S'}\}$$

on  $S$ -schemes is represented by a finitely presented closed  $S$ -subgroup  $\text{O}(q)$  of  $\text{GL}(V)$ . We call it the *orthogonal group* of  $(V, q)$ . This is  $\mu_2$  if  $n = 1$ . Define the *naive special orthogonal group* to be

$$\text{SO}'(q) := \ker(\det : \text{O}(q) \rightarrow \mathbf{G}_m)$$

(so  $\text{SO}'(q) = 1$  if  $n = 1$ ); we say “naive” because this is the wrong notion for non-degenerate  $(V, q)$  when  $n$  is even and 2 is not a unit on  $S$ . The *special orthogonal group*  $\text{SO}(q)$  will

be defined shortly in a characteristic-free way, using Clifford algebras when  $n$  is even. (The distinction between even and odd  $n$  when defining  $\mathrm{SO}(q)$  in terms of  $\mathrm{O}(q)$  is natural, because we will see that  $\mathrm{O}(q)/\mathrm{SO}(q)$  is  $\mu_2$  for odd  $n$  but  $(\mathbf{Z}/2\mathbf{Z})_S$  for even  $n$ . Also, if  $n \geq 3$  then  $\mathrm{SO}(q)_{\bar{s}}$  will be connected semisimple of type  $B_m$  for  $n = 2m + 1$  and type  $D_m$  for  $n = 2m$ .)

**Definition C.1.2.** Let  $S = \mathrm{Spec} \mathbf{Z}$ . The *standard split* quadratic form  $q_n$  on  $V = \mathbf{Z}^n$  is as follows, depending on the parity of  $n \geq 1$ :

$$(C.1.1) \quad q_{2m} = \sum_{i=1}^m x_{2i-1}x_{2i}, \quad q_{2m+1} = x_0^2 + \sum_{i=1}^m x_{2i-1}x_{2i}$$

(so  $q_1 = x_0^2$ ). We define  $\mathrm{O}_n = \mathrm{O}(q_n)$  and  $\mathrm{SO}'_n = \mathrm{SO}'(q_n)$ .

It is elementary to check that  $(\mathbf{Z}^n, q_n)$  is non-degenerate. We do not define a notion of “split” for general  $L$ -valued non-degenerate  $q$  because for odd  $n$  this turns out not to be an interesting concept except essentially for cases when  $L$  is trivial.

**Remark C.1.3.** In some references (such as [Bo69, 11.16]) the quadratic forms

$$x_1x_{2m} + \cdots + x_mx_{m+1}, \quad x_0x_{2m} + \cdots + x_{m-1}x_{m+1} + x_m^2$$

are preferred instead of (C.1.1), since for such  $q$  the split group  $\mathrm{SO}(q)$  admits a Borel subgroup contained in the upper triangular Borel subgroup of  $\mathrm{SL}_n$ .

There is a convenient characterization of non-degeneracy for  $(V, L, q)$  when  $L$  and  $V$  are globally free, using the unit condition on values of a polynomial associated to  $q$  (depending on the parity of  $n$ ), as follows. Suppose there is a chosen isomorphism  $L \simeq \mathcal{O}_S$  and ordered  $\mathcal{O}_S$ -basis  $\mathbf{e} = \{e_1, \dots, e_n\}$  of  $V$  (as may be arranged by Zariski-localization on  $S$ ). Let  $[B_q]_{\mathbf{e}}$  be the matrix  $(B_q(e_i, e_j))$  that computes  $B_q$  relative to  $\mathbf{e}$ . The determinant  $\mathrm{disc}_{\mathbf{e}}(q) = \det([B_q]_{\mathbf{e}})$  is the *discriminant* of  $q$  relative to  $\mathbf{e}$  (and the chosen trivialization of  $L$ ). If  $\mathbf{e}'$  is a second ordered  $\mathcal{O}_S$ -basis of  $V$  then  $\mathrm{disc}_{\mathbf{e}'}(q) = u^2 \mathrm{disc}_{\mathbf{e}}(q)$  for the unit  $u$  given by the determinant of the matrix that converts  $\mathbf{e}'$ -coordinates into  $\mathbf{e}$ -coordinates. In more intrinsic terms,  $\mathrm{disc}_{\mathbf{e}}(q)$  computes the induced linear map  $\wedge^n(V) \rightarrow \wedge^n(V^*)$  arising from the linear map  $V \rightarrow V^*$  defined by  $v \mapsto B_q(v, \cdot) = B_q(\cdot, v)$  (when using the bases  $e_1 \wedge \cdots \wedge e_n$  and  $e_1^* \wedge \cdots \wedge e_n^*$ ). The condition  $\mathrm{disc}_{\mathbf{e}}(q) \in \mathbf{G}_m(S)$  is independent of the choice of  $\mathbf{e}$ , and it expresses exactly the property that  $B_q$  is a perfect pairing on  $V$ . Hence, if  $n$  is even or if  $n$  is odd and 2 is a unit on  $S$  then the condition  $\mathrm{disc}_{\mathbf{e}}(q) \in \mathbf{G}_m$  is equivalent to the non-degeneracy of  $q$ .

To handle the case of odd  $n$  in a characteristic-free way, it is convenient to introduce a modification of the discriminant that was independently discovered by Grothendieck and M. Kneser. This involves a “universal” construction:

**Proposition C.1.4.** Let  $n \geq 1$  be odd and  $Q = \sum_{i < j} A_{ij}x_ix_j \in \mathbf{Z}[A_{ij}][x_1, \dots, x_n]$  the universal quadratic form in  $n$  variables.

- (1) The polynomial  $\mathrm{disc}'(Q) := (1/2)\mathrm{disc}(Q) \in (1/2)\mathbf{Z}[A_{ij}]$  lies in  $\mathbf{Z}[A_{ij}]$ .

- (2) Over the ring  $\mathbf{Z}[A_{ij}, C_{hk}][1/\det(C_{hk})]$  consider the universal linear change of coordinates  $x_h = \sum_k C_{kh}x'_k$  dual to the universal change of basis  $e_h = \sum_k C_{hk}e'_k$ . The quadratic form  $Q'$  in  $x'_1, \dots, x'_n$  obtained from  $Q(x_1, \dots, x_n)$  satisfies  $\text{disc}'(Q') = \text{disc}(Q) \det(C_{hk})^2$ .
- (3) If  $R$  is any commutative ring and  $q = \sum_{i \leq j} a_{ij}x_i x_j$  is a quadratic form in  $n$  variables over  $R$  then  $\text{disc}'(q) := (\text{disc}'(Q))(a_{ij})$  changes by a unit square under linear change of variables and it lies in  $R^\times$  if and only if  $q$  is non-degenerate over  $\text{Spec } R$ .

We call  $\text{disc}'(q)$  the *half-discriminant* of  $q$  (since when  $2 \in R^\times$ , it is  $(1/2)\text{disc}(q)$ ). Computation of the half-discriminant when  $2 \notin R^\times$  (especially when  $2$  is a zero-divisor in  $R$ ) requires lifting  $q$  to a ring in which  $2$  is not a zero-divisor (e.g., if  $R = \mathbf{F}_2$  then we can work with a lift of  $q$  over  $\mathbf{Z}$ ), but it will nonetheless be theoretically useful. The non-degeneracy of  $q_{2m+1}$  in characteristic 2 shows that for odd  $n$  the universal half-discriminant in characteristic 2 is not identically zero.

*Proof.* For (1) it suffices to show that over the fraction field  $\mathbf{F}_2(A_{ij})$  of  $\mathbf{Z}[A_{ij}]/(2)$  the quadratic form  $Q$  has vanishing discriminant. Over any field  $k$  of characteristic 2, a quadratic form  $q$  on a finite-dimensional vector space  $V$  satisfies  $\delta_q \equiv \dim V \pmod{2}$  (Exercise 1.6.9), so the defect  $\delta_q$  is positive when  $\dim V$  is odd. Since  $\delta_q > 0$  precisely when  $\text{disc}(q) = 0$ , part (1) is proved. Part (2) is obvious, since we can multiply both sides by 2 to reduce to the known case of the usual discriminant.

To prove part (3), we may assume  $R = k$  is an algebraically closed field and (by (2)) it is harmless to apply a linear change of variables. The case  $\text{char}(k) \neq 2$  is trivial, so we can assume  $\text{char}(k) = 2$ . First we show that  $\text{disc}'(q) \in k^\times$  if  $q$  is non-degenerate. For such  $q$ , by Exercise 1.6.9(ii) we may apply a linear change of variables so that  $q = x_0^2 + \sum_{i=1}^m x_i x_{i+m}$  with  $n = 2m + 1$ . This arises by scalar extension from the quadratic form  $q_n$  over  $\mathbf{Z}$  given by the same formula, so  $\text{disc}'(q)$  is the image of  $\text{disc}'(q_n) \in \mathbf{Z}$ . Clearly  $\text{disc}(q_n) = 2(-1)^m$ , so  $\text{disc}'(q_n) = (-1)^m$ . Hence,  $\text{disc}'(q) = (-1)^m \in k^\times$ .

For the converse, we assume  $q$  is degenerate and shall prove that  $\text{disc}'(q) = 0$  in  $k$ . In accordance with the definition of  $\text{disc}'(q)$ , we can compute it by working with a lift of  $q$  over the ring  $W(k)$  of Witt vectors (in which 2 is not a zero divisor). It is harmless to first apply a preliminary linear change of variables over  $k$ . Consider the defect space  $V^\perp$  on which  $B_q$  vanishes. This has odd dimension, and  $q|_{V^\perp}$  is the square of a linear form (since  $k$  is algebraically closed of characteristic 2). The degeneracy of  $q$  implies that either  $\dim V^\perp \geq 3$  or  $\dim V^\perp = 1$  with  $q|_{V^\perp} = 0$ . Either way,  $q|_{V^\perp}$  can be expressed in terms of fewer than  $\dim V^\perp$  variables relative to a suitable basis, so  $q$  can be expressed in fewer than  $n$  variables after a linear change of coordinates. Hence, there is a lift  $\tilde{q}$  of  $q$  to a quadratic form over  $W(k)$  that can be expressed in fewer than  $n$  variables after a linear change of coordinates over  $W(k)$ , so the discriminant  $\text{disc}(\tilde{q})$  attached to  $(W(k)^n, \tilde{q})$  vanishes (as we may check over the field  $W(k)[1/2]$  of characteristic 0). Since  $\text{disc}'(q)$  is the reduction of  $\text{disc}'(\tilde{q}) = (1/2)\text{disc}(\tilde{q}) = 0$ , we are done.  $\square$

It is important to note that the discriminant and half-discriminant are attached to a quadratic triple  $(V, L, q)$  equipped with global bases of  $V$  and  $L$ , and not to a “bare” degree-2 homogeneous polynomial. For example, to define the discriminant (or half-discriminant) of the quadratic form  $x_0^2 + x_1x_2$  over  $\mathbf{Z}$ , it is necessary to specify whether this is viewed as a quadratic form on  $\mathbf{Z}^3$  or  $\mathbf{Z}^n$  for some  $n > 3$ . The convenience of the half-discriminant is demonstrated by its role in the proof of:

**Theorem C.1.5.** *Let  $(V, L, q)$  be a non-degenerate quadratic triple of rank  $n \geq 1$  over a scheme  $S$ . The  $S$ -group  $O(q)$  is smooth if and only if  $n$  is even or  $n$  is odd with 2 a unit on  $S$ , and the  $S$ -group  $SO'(q)$  is smooth if and only if  $n$  is odd or  $n$  is even with 2 a unit on  $S$ .*

*Proof.* We use the following *smoothness criterion*: if  $X$  and  $Y$  are  $S$ -schemes locally of finite presentation such that  $X$  is  $S$ -flat, then an  $S$ -morphism  $f : X \rightarrow Y$  is smooth if and only if  $f_s : X_s \rightarrow Y_s$  is smooth for all  $s \in S$ . To prove the criterion it is only necessary to show that  $f$  is flat when each  $f_s$  is flat, and this follows from the  $S$ -flatness of  $X$  (as part of the fibral flatness criterion [EGA, IV<sub>3</sub>, 11.3.10]).

**Step 1.** First we treat the case of orthogonal group schemes. The  $S$ -scheme  $\text{Quad}(V, L)$  of  $L$ -valued quadratic forms on  $V$  is represented by a smooth  $S$ -scheme that is an affine space of relative dimension  $n(n+1)/2$  Zariski-locally over  $S$ , and the subfunctor of such forms that are non-degenerate is represented by an open subscheme  $Y \subset \text{Quad}(V, L)$  given Zariski-locally over  $S$  by the non-vanishing of the discriminant or half-discriminant depending on the parity of  $n$ . Thus,  $Y \rightarrow S$  is smooth and surjective (since  $q_n$  is non-degenerate over any field). There is an evident right action of  $X := \text{GL}(V)$  on  $Y$  over  $S$  via  $(Q, g) \mapsto Q \circ g$ , and the orbit map  $f : X \rightarrow Y$  through  $q \in Y(S)$  is surjective because over any algebraically field  $k$  the non-degenerate quadratic forms on  $k^n$  are a single  $\text{GL}_n(k)$ -orbit (Exercise 1.6.9(ii),(iii)). The fiber of  $f$  over a geometric point of  $Y$  is the orthogonal group scheme for the corresponding quadratic form, so by the smoothness criterion we may assume  $S = \text{Spec } k$  for an algebraically closed field  $k$ .

Now  $X$  and  $Y$  are  $k$ -smooth and irreducible of dimensions  $n^2$  and  $n(n+1)/2$ , so the smoothness of  $f$  is equivalent to the smoothness of its fibers, and is also equivalent to the surjectivity of the maps  $\text{Tan}_g(X) \rightarrow \text{Tan}_{q \circ g}(Y)$  for  $g \in X(k)$ . By homogeneity it is equivalent to verify such surjectivity for a single  $g$ , such as  $g = 1$ , and this in turn is equivalent to  $\text{Tan}_1(f)$  having kernel of dimension  $n^2 - n(n+1)/2 = n(n-1)/2$ . But  $f(1) = q$  and  $f^{-1}(q) = O(q)$ , so the kernel of  $\text{Tan}_1(f)$  is  $\text{Tan}_1(O(q))$ . Thus, the case of orthogonal group schemes is reduced to showing that for any quadratic form  $q : V \rightarrow k$ ,  $\dim \text{Tan}_1(O(q)) = n(n-1)/2$  precisely when  $B_q$  is non-degenerate.

Computing with the algebra  $k[\epsilon]$  of dual numbers, the subspace

$$\text{Tan}_1(O(q)) \subset \text{Tan}_1(\text{GL}(V)) = \text{End}(V)$$

consists of those  $T : V \rightarrow V$  such that  $q((1 + \epsilon T)(v)) = q(v)$  on  $V_{k[\epsilon]}$ . Since  $q(v + \epsilon T(v)) = q(v) + \epsilon B_q(\bar{v}, T(\bar{v}))$  for  $\bar{v} = v \bmod \epsilon$ , the necessary and sufficient condition on  $T$  is that  $B_q(x, T(x)) = 0$  for all  $x \in V$ , which is to say that the bilinear form  $B_q(\cdot, T(\cdot))$  on  $V$  is alternating. Let  $V' = V/V^\perp$ , so  $B_q$  induces a non-degenerate bilinear form  $B'_q$  on  $V'$ . When

$B_q(v, T(w))$  is alternating it is skew-symmetric, so in such cases  $T$  must preserve  $V^\perp$  (because  $B_q(v, T(w)) = 0$  for  $v \in V^\perp$  and any  $w \in V$ ).

Thus,  $\text{Tan}_1(\text{O}(q))$  is the space of  $T$  that preserve  $V^\perp$  and whose induced endomorphism  $T'$  of  $V'$  makes  $B'_q(v', T'(w'))$  alternating. Since  $T'$  determines  $T$  up to precisely translation by  $\text{Hom}(V, V^\perp)$ , by the non-degeneracy of  $B'_q$  on  $V'$  we obtain a short exact sequence

$$0 \rightarrow \text{Hom}(V, V^\perp) \rightarrow \text{Tan}_1(\text{O}(q)) \rightarrow \text{Alt}^2(V/V^\perp) \rightarrow 0$$

(the second map is  $T \mapsto B'_q(\cdot, T'(\cdot))$ ). Letting  $\delta = \dim V^\perp$  denote the defect, we find that

$$\dim \text{Tan}_1(\text{O}(q)) = n\delta + (n - \delta)(n - \delta - 1)/2 = n(n - 1)/2 + (\delta^2 + \delta)/2.$$

This coincides with  $n(n - 1)/2$  precisely when  $\delta = 0$ .

**Step 2.** Now consider the case of  $\text{SO}'(q)$ . By Zariski-localization on  $S$ , we may assume  $L = \mathcal{O}_S$  and may choose an ordered basis of  $V$ . For any quadratic form  $Q$  on  $V_{S'}$  for an  $S$ -scheme  $S'$ , relative to the chosen basis let  $D(Q)$  denote  $\text{disc}(Q)$  when  $n$  is even and  $\text{disc}'(Q)$  when  $n$  is odd. For any  $S$ -scheme  $S'$  and any quadratic form  $Q$  on  $V_{S'}$  we have  $D(Q(gx)) = (\det g)^2 D(Q(x))$  for  $g \in \text{GL}(V_{S'})$ . Indeed, this is obvious when  $n$  is even and follows from reduction to the  $\mathbf{Z}$ -flat universal case for the half-discriminant when  $n$  is odd. In particular, orthogonal group schemes for non-degenerate quadratic triples  $(V, L, q)$  lie in  $\det^{-1}(\mu_2)$ . Since the determinant map  $\text{O}(q) \rightarrow \mu_2$  restricts to the identity on the central  $\mu_2$  when  $n$  is odd, we get  $\text{O}(q) = \mu_2 \times \text{SO}'(q)$  for odd  $n$ . This settles the case of odd  $n$  when 2 is a unit, but below we will give a characteristic-free argument for odd  $n$  that does not ignore characteristic 2.

Let  $X' = \text{SL}(V)$  and let  $Y'$  be the scheme of non-degenerate  $L$ -valued quadratic forms  $Q$  on  $V$  such that  $D(Q) = D(q)$ , so  $X'$  acts on  $Y'$ . Let  $f' : X' \rightarrow Y'$  be the orbit map through  $q \in Y'(S)$ . Note that  $\text{SO}'(q) = f'^{-1}(q)$ , so this is  $S$ -smooth provided that  $f'$  is smooth. Since  $X'$  is  $S$ -flat (even smooth), by the smoothness criterion it is necessary and sufficient to treat the case when  $S = \text{Spec } k$  for an algebraically closed field  $k$ .

We claim that  $f'$  is surjective. It suffices to work with  $k$ -points. By Exercise 1.6.9, Proposition C.1.4(3), and the behavior of  $D(Q)$  under a linear change of coordinates on  $V$ , any non-degenerate  $Q$  on  $k^n$  with  $D(Q) = D(q)$  has the form  $Q = q \circ g$  for some  $g \in \text{GL}_n(k)$  satisfying  $(\det g)^2 = 1$ . Writing  $n = 2m$  or  $n = 2m + 1$ ,  $q = q_n \circ \gamma$  with  $D(q) = (\det \gamma)^2 (-1)^m$  and  $Q = q_n \circ \gamma'$  with  $D(Q) = (\det \gamma')^2 (-1)^m$ . Hence,  $\det \gamma' = \pm \det \gamma$ , so it suffices to check that  $\text{O}_n(k)$  contains an element with determinant  $-1$ . This is obvious by direct inspection of  $q_n$  depending on the parity of  $n$ .

It is harmless to scale  $q$  by some  $c \in k^\times$ , so we may now assume  $D(q) = D(q_n)$ . Thus,  $q_n$  lies in the level set of  $q$ , so to determine whether or not the surjective  $f'$  is smooth we may assume  $q = q_n$ . Since  $f'$  is either everywhere smooth or nowhere smooth, it suffices to check whether or not smoothness holds at  $g = 1 \in \text{SL}_n(k) = X'(k)$ .

**Step 3.** We first treat the case of odd  $n$ , and then even  $n$ . Assume  $n$  is odd. We claim that the level set  $Y'$  is smooth (so smoothness of  $f'$  can be checked via surjectivity on tangent spaces). This amounts to checking that the equation  $D(Q) = D(q_n)$  defining

$Y'$  in the smooth scheme  $Y$  of all non-degenerate quadratic forms on  $V$  is non-constant to first order at  $q_n$ . That is, relative to suitable linear coordinates on  $V$ , we claim that the polynomial  $D(q_n + \epsilon Q) - D(q_n)$  in a varying  $Q \in \text{Quad}(k^n)$  is nonzero. For  $Q = cx_0^2$  we have  $D(q_n + \epsilon Q) - D(q_n) = \epsilon c$  by base change from the universal case over  $\mathbf{Z}[C]$ . This proves the smoothness of  $Y'$ .

Since  $X' = \text{SL}(V)$  is a smooth hypersurface in  $X = \text{GL}(V)$ , the difference in tangent space dimensions at  $1 \in X'(k)$  and  $q_n \in Y'(k)$  is  $n^2 - n(n+1)/2 = n(n-1)/2$ . Thus, surjectivity of  $f'$  on tangent spaces is equivalent to the kernel  $\text{Tan}_1(\text{SO}'(q))$  of  $\text{Tan}_1(f')$  having dimension  $n(n-1)/2$ . But  $\text{O}(q) = \mu_2 \times \text{SO}'(q)$  due to the oddness of  $n$ , so we want  $\text{Tan}_1(\text{O}(q))$  to have dimension  $\dim \text{Tan}_1(\mu_2) + n(n-1)/2$ ; i.e.,  $n(n-1)/2$  when  $\text{char}(k) \neq 2$  and  $1 + n(n-1)/2$  in the defect-1 case in characteristic 2. These were both established in the analysis of smoothness for orthogonal group schemes.

Now assume that  $n = 2m$  is even. We claim that the hypersurface  $Y'$  in  $Y$  is smooth when  $\text{char}(k) \neq 2$  (in which case the desired surjectivity on tangent spaces can be proved exactly as for odd  $n$ ), and that it is nowhere smooth when  $\text{char}(k) = 2$ . Such nowhere smoothness would force all fibers of  $f'$  to be non-smooth (since  $X'$  is smooth), which is to say that  $\text{SO}'(q)$  is non-smooth in characteristic 2 when  $n$  is even.

Away from characteristic 2 we can diagonalize  $q$  as a sum of squares of all variables, so the smoothness proof for odd  $n$  carries over. It remains to show that in characteristic 2, the polynomial function  $\text{disc}(q_n + \epsilon Q) = \det([B_{q_n}] + \epsilon[B_Q])$  of  $Q = \sum_{i \leq j} a_{ij}x_i x_j \in \text{Quad}(k^n)$  is equal to the constant  $\text{disc}(q_n) = (-1)^m$  in  $k[\epsilon]$ . The  $ii$ -entry of  $[B_{q_n}] + \epsilon[B_Q]$  is  $2a_{ii}\epsilon = 0$ , the  $(i, i+m)$  and  $(i+m, i)$ -entries are  $1 + a_{i, i+m}\epsilon$ , and all other off-diagonal entries are of the form  $a_{ij}\epsilon$ . Thus, by inspection, all terms in the sum of products that defines the discriminant as a determinant are divisible by  $\epsilon^2 = 0$  except for the term  $(-1)^m \prod_{i=1}^m (1 + a_{i, i+m}\epsilon)^2$ . Since  $\text{char}(k) = 2$ , this term is equal to  $(-1)^m = \text{disc}(q_n)$ .  $\square$

An alternative proof of the smoothness of orthogonal group schemes for even  $n$  is to use Lemma C.2.1 below to pass to the case of  $q_n$ , for which a direct counting of equations does the job. This alternative method is worked out in [DG, III, §5, 2.3], based on an equation-counting smoothness criterion in [DG, II, §5, 2.7]. We have chosen to use the preceding argument based on another smoothness criterion because it treats even and odd  $n$  on an equal footing (and adapts to other groups; see [GY]).

**C.2. Clifford algebras and special orthogonal groups.** Consider a quadratic form  $q : V \rightarrow \mathcal{O}_S$  with  $V$  of rank  $n \geq 1$ . The *Clifford algebra*  $C(V, q)$  is the quotient of the tensor algebra of  $V$  by the relation  $x^{\otimes 2} = q(x)$  for local sections  $x$  of  $V$ . This inherits a natural  $\mathbf{Z}/2\mathbf{Z}$ -grading (as a direct sum of an “even” part and “odd” part) from the  $\mathbf{Z}$ -grading on the tensor algebra. By considering expansions relative to a local basis of  $V$  we see that  $C(V, q)$  is a finitely generated  $\mathcal{O}_S$ -module. For  $q = 0$ , this is the exterior algebra of  $V$ . We are primarily interested in the case of non-degenerate  $q$ .

There are versions of the Clifford construction for (possibly degenerate)  $L$ -valued  $q$ , but numerous complications arise; see [Au, 1.8], [BK], and [PS, §4] for further discussion of this.

An alternative reference on Clifford algebras and related quadratic invariants is [Knus, IV], at least for an affine base. The Dickson invariant that we will use for even  $n$  goes back to Dickson [Di, p. 206] over finite fields, and Arf [Arf] over general fields of characteristic 2.

**Lemma C.2.1.** *If  $(V, q)$  is a quadratic space of rank  $n \geq 1$  over a scheme  $S$  then it is isomorphic to  $(\mathcal{O}_S^n, q_n)$  fppf-locally on  $S$ . If  $n$  is odd or 2 is a unit on  $S$  then it suffices to use the étale topology rather than the fppf topology.*

Keep in mind that a “quadratic space” is understood to be non-degenerate (in the fiberwise sense) unless we say otherwise.

*Proof.* In [SGA7, XII, Prop. 1.2] the smoothness of  $(q = 0)$  is used to prove the following variant by induction on  $n$ :  $q$  is an étale form of  $q_n$  when  $n$  is even and is an étale form of  $ux_0^2 + q_{2m}$  when  $n = 2m + 1$  is odd, with  $u$  a unit on the base. Once the induction is finished, we are done when  $n$  is even and we need to extract a square root of  $u$  when  $n$  is odd. This requires working fppf-locally for odd  $n$  when 2 is not a unit on the base.  $\square$

Lemma C.2.1 is very useful for reducing problems with general (non-degenerate) quadratic spaces to the case of  $q_n$  over  $\mathbf{Z}$ , as we shall now see. In what follows, we fix a quadratic space  $(V, q)$  of rank  $n \geq 1$  over a scheme  $S$ .

**Proposition C.2.2.** *Assume  $n$  is even. The  $\mathcal{O}_S$ -algebra  $C(V, q)$  and its even part  $C_0(V, q)$  are respectively isomorphic, fppf-locally on  $S$ , to  $\text{Mat}_{2^{n/2}}(\mathcal{O})$  and a product of two matrix algebras. In particular, they are vector bundles whose quasi-coherent centers are respectively equal to  $\mathcal{O}_S$  and a rank-2 finite étale  $\mathcal{O}_S$ -algebra  $Z_q$ . Moreover, the natural map  $V \rightarrow C(V, q)$  is a subbundle inclusion and  $C_0(V, q)$  is the centralizer of  $Z_q$  in  $C(V, q)$ .*

It follows that  $C(V, q)$  is an Azumaya algebra over  $S$  (and  $C_0(V, q)$  is an Azumaya algebra over a degree-2 finite étale cover of  $S$ ). The notion of Azumaya algebra will only arise in Example C.6.3.

*Proof.* By Lemma C.2.1 we may assume that  $V$  admits a basis  $\{e_i\}$  identifying  $q$  with  $q_n$ . In  $C(V, q)$  we have  $v'v = -vv' + B_q(v, v')$  for  $v, v' \in V$ , so  $C(V, q)$  is spanned by the  $2^n$  products  $e_J = e_{j_1} \cdots e_{j_h}$  for subsets  $J = \{j_1, \dots, j_h\} \subset \{1, \dots, n\}$  (with  $j_1 < \dots < j_h$ , and  $e_\emptyset = 1$  for  $h = 0$ ). Thus, if we construct a surjection from  $C(V, q)$  onto  $\text{Mat}_{2^{n/2}}(\mathcal{O})$  then it must be an isomorphism (and the  $e_J$  must be a basis of  $C(V, q)$ ).

Since  $n$  is even, there are complementary isotropic free subbundles  $W, W' \subset V$  of rank  $n/2$ , in perfect duality via  $B_q$ ; e.g.,  $W = \text{span}\{e_1, \dots, e_{n/2}\}$  and  $W' = \text{span}\{e_{1+n/2}, \dots, e_n\}$ . Let  $A := C(W, q|_W) = C(W, 0) = \wedge^\bullet(W)$ , with even and odd parts  $A_+ = \bigoplus \wedge^{2j}(W)$  and  $A_- = \bigoplus \wedge^{2j+1}(W)$ , so  $A$  is a vector bundle of rank  $2^{n/2}$ . The endomorphism algebra  $\mathcal{E}nd(A)$  has a  $\mathbf{Z}/2\mathbf{Z}$ -grading: an endomorphism of  $A = A_+ \oplus A_-$  is *even* if it carries  $A_\pm$  into  $A_\pm$  and *odd* if it carries  $A_\pm$  into  $A_\mp$ . We will construct a surjective algebra homomorphism  $\rho : C(V, q) \rightarrow \mathcal{E}nd(A) \simeq \text{Mat}_{2^{n/2}}(\mathcal{O})$  respecting the  $\mathbf{Z}/2\mathbf{Z}$ -grading, so  $\rho$  is an isomorphism and carries  $C_0(V, q)$  onto  $\mathcal{E}nd(A_+) \times \mathcal{E}nd(A_-)$ , completing the proof. (See [KO] for a generalization of the isomorphism  $\rho$  to the case of certain non-split  $(V, q)$ .)

To build  $\rho$ , by the universal property of the Clifford algebra we just need to define a linear map  $L : W \oplus W' = V \rightarrow \mathcal{E}nd(A)$  such that  $L(v) \circ L(v)$  is multiplication by  $q(v)$ . Define  $L(w) = w \wedge (\cdot)$  and  $L(w')$  to be the contraction operator

$$\delta_{w'} : w_1 \wedge \cdots \wedge w_j \mapsto \sum_{i=1}^j (-1)^{i-1} B_q(w_i, w') w_1 \wedge \cdots \widehat{w}_i \cdots \wedge w_j$$

(the unique anti-derivation of  $A$  satisfying coinciding with  $B_q(\cdot, w')$  on  $W \subset A_-$ ). Induction on  $j$  (or direct calculation) gives  $\delta_{w'} \circ \delta_{w'} = 0$ , and clearly  $L(w) \circ L(w) = 0$ , so

$$L(w + w') \circ L(w + w') = L(w) \circ L(w') + L(w') \circ L(w).$$

This is multiplication by  $B_q(w, w') = q(w + w')$  because

$$(L(w') \circ L(w))(x) = \delta_{w'}(w \wedge x) = \delta_{w'}(w)x - w \wedge \delta_{w'}(x) = B_q(w, w')x - (L(w) \circ L(w'))(x).$$

The resulting map of algebras  $\rho : C(V, q) \rightarrow \mathcal{E}nd(A)$  respects the  $\mathbf{Z}/2\mathbf{Z}$ -gradings since  $w \wedge (\cdot)$  and  $\delta_{w'}$  are odd endomorphisms. To prove that  $\rho$  is surjective it suffices to check on fibers. The maps  $\rho_s$  are isomorphisms by the classical theory over fields (see the proof of [Chev97, II.2.1], or [Knus, IV, 2.1.1]).  $\square$

**Remark C.2.3.** In the special case  $q = q_{2m}$ , the computation of the center  $Z_q$  of  $C_0(V, q)$  as  $\mathcal{O}_S \times \mathcal{O}_S$  via an explicit description of a fiberwise nontrivial idempotent  $z$  is given in [DG, III, 5.2.4]. (Generalizations for non-split  $q$  are given in [Knus, IV, 2.3.1, 4.8.5].) We now give such a calculations. Consider the standard basis  $\{e_1, \dots, e_{2m}\}$ , so  $\{e_{2i-1}, e_{2i}\}$  for  $1 \leq i \leq m$  is a collection of pairwise orthogonal bases of standard hyperbolic planes. To compute a  $z$  in terms of products among the  $e_i$ 's, we may and do work over  $\mathbf{Z}$ .

In  $C_0(V, q)$ , for each such pair  $\{e, e'\} = \{e_{2i-1}, e_{2i}\}$  we have

$$1 = q(e + e') = (e + e')^2 = e^2 + (ee' + e'e) + e'^2 = ee' + e'e,$$

so  $(ee')^2 = e(e'e)e' = e(1 - ee')e' = ee'$ . Hence, the elements  $w_i = e_{2i-1}e_{2i}$  pairwise commute (signs cancel in pairs) and  $(1 - 2w_i)^2 = 1$ , so  $w = \prod_{i=1}^m (1 - 2w_i)$  satisfies  $w^2 = 1$ . Direct calculation shows that  $e_j$  commutes with  $1 - 2w_i$  when  $j \notin \{2i - 1, 2i\}$  whereas  $e_j$  anti-commutes with  $1 - 2w_i$  if  $j \in \{2i - 1, 2i\}$ , so  $e_j$  anti-commutes with  $w$ . Hence, all products  $e_j e_{j'}$  with  $j < j'$  commute with  $w$ , so  $w$  is central in  $C_0(V, q)$ .

Over  $\mathbf{Z}[1/2]$  we define

$$z = (1/2)(1 - w) = \sum_i w_i - 2 \sum_{i < i'} w_i w_{i'} + 4 \sum_{i < i' < i''} w_i w_{i'} w_{i''} + \cdots + (-2)^{m-1} \prod_{i=1}^m w_i.$$

By inspection,  $z$  lies in  $C_0(V, q)$  over  $\mathbf{Z}$  and modulo every prime it is distinct from 0 and 1. Hence,  $\mathbf{Z} \oplus \mathbf{Z}z$  is a direct summand of  $C_0(V, q)$  that is a central subalgebra, so it is  $Z_q$ . Moreover, by computing over  $\mathbf{Z}[1/2]$  we see that  $z^2 = z$ , so  $z$  is a fiberwise nontrivial idempotent of  $Z_q$  (i.e.,  $Z_q = \mathbf{Z} \cdot z \times \mathbf{Z} \cdot (1 - z)$ ).

The structure of  $C(V, q)$  and  $C_0(V, q)$  for odd  $n$  is “opposite” that for even  $n$ :

**Proposition C.2.4.** *Assume  $n = 2m + 1$  is odd. The even part  $C_0(V, q)$  is isomorphic, fppf-locally on  $S$ , to  $\text{Mat}_{2^m}(\mathcal{O})$ . The center  $Z_q$  of  $C(V, q)$  is a  $\mathbf{Z}/2\mathbf{Z}$ -graded finite locally free  $\mathcal{O}_S$ -module of rank 2 whose degree-0 part is  $\mathcal{O}_S$  and whose degree-1 part  $Z_q^1$  is an invertible sheaf that satisfies  $Z_q^1 \otimes Z_q^1 \simeq Z_q^0 = \mathcal{O}_S$  via multiplication. In particular, as an  $\mathcal{O}_S$ -algebra  $Z_q$  is locally generated by the square root of a unit, so over  $S[1/2]$  it is finite étale of rank 2.*

*The natural  $\mathbf{Z}/2\mathbf{Z}$ -graded multiplication map  $Z_q \otimes_{\mathcal{O}_S} C_0(V, q) \rightarrow C(V, q)$  is an isomorphism. In particular,  $C(V, q)$  is a locally free  $\mathcal{O}_S$ -module of rank  $2^n$  that is isomorphic to  $\text{Mat}_{2^m}(Z_q)$  as a  $Z_q$ -algebra fppf-locally on  $S$ . Moreover,  $V \rightarrow C(V, q)$  is a subbundle inclusion.*

This result implies that  $C_0(V, q)$  is an Azumaya algebra over  $S$  whereas  $C(V, q)$  is an Azumaya algebra over a degree-2 finite fppf cover of  $S$ .

*Proof.* First consider the special case  $n = 1$  (i.e.,  $m = 0$ ), so by Zariski localization  $q = ux^2$  for a unit  $u$  on  $S = \text{Spec } R$ . Then  $C(V, q) = R[z]/(z^2 - u) = R \oplus Rz$  with degree-0 part  $R$  and degree-1 part  $Rz$ . This settles all of the assertions in this case, so now we may and do assume  $n \geq 3$  (i.e.,  $m \geq 1$ ).

Since the assertions imply that the center is a local direct summand, these assertions are preserved by base change on  $S$  and descend through an fppf base change. Hence, by Lemma C.2.1 it suffices to treat the case  $q = q_n$  over any  $S$ . This is the orthogonal direct sum of  $x_0^2$  and  $q_{2m}$ , so we will be able to apply the known results for even rank  $2m \geq 2$  from Proposition C.2.2. More specifically, if  $(V', q')$  and  $(V'', q'')$  are (possibly degenerate) quadratic spaces over a scheme  $S$  then the Clifford algebra of their orthogonal sum  $(V' \oplus V'', q' \perp q'')$  is naturally isomorphic as a  $\mathbf{Z}/2\mathbf{Z}$ -graded algebra to the “super-graded tensor product”  $C(V', q') \otimes' C(V'', q'')$  which is the ordinary tensor product module equipped with the algebra structure defined by the requirement  $(1 \otimes a'')(a' \otimes 1) = (-1)^{\deg(a')\deg(a'')}(a' \otimes a'')$  for homogeneous  $a'$  and  $a''$ . (See [Knus, IV, 1.3.1].) Hence,  $C(q_n) = C(x_0^2) \otimes' C(q_{2m})$ , so  $C(q_n)$  is a free  $\mathcal{O}_S$ -module of rank  $2^{1+2m} = 2^n$ . In particular, the even and odd parts of  $C(V, q) = C(q_n)$  are locally free of finite rank and  $V \rightarrow C(V, q)$  is a subbundle inclusion.

Multiplication by the standard basis vector  $e_0$  swaps the even and odd parts of  $C(V, q)$  (since  $e_0^2 = q(e_0) = 1$ ), so each of these parts has rank  $2^{n-1} = 2^{2m}$ . For any  $w$  in the span  $W$  of  $\{e_1, \dots, e_{2m}\}$  we have  $(e_0 w)(e_0 w) = -e_0^2 w^2 = -q_{2m}(w)$ . Thus,  $w \mapsto e_0 w \in C_0(q_n)$  extends to a homomorphism  $f : C(W, -q_{2m}) \rightarrow C_0(q_n)$ . The map  $f$  is fiberwise injective, since the geometric fibers of  $C(W, -q_{2m})$  are simple algebras with rank  $2^{2m}$  (in fact,  $\text{Mat}_{2^m}$ ), so  $f$  is an isomorphism on fibers over  $S$ . The algebra  $C(W, -q_{2m})$  is a locally free module of rank  $2^{2m}$  (in fact, fppf-locally on the base it becomes  $\text{Mat}_{2^m}(\mathcal{O})$ , by Proposition C.2.2), so  $f$  is an isomorphism of algebras. In particular,  $C_0(V, q)$  has the asserted matrix algebra structure fppf-locally on  $S$ .

Consider the element  $z = e_0 \prod_{i=1}^m (1 - 2e_{2i-1}e_{2i}) \in C_1(V, q)$ . Computing as in Remark C.2.3, if  $j > 0$  then  $e_j$  anti-commutes with  $\prod_{i=1}^m (1 - 2e_{2i-1}e_{2i})$ , yet such  $e_j$  also anti-commutes with  $e_0$ , so  $e_j$  commutes with  $z$ . It is likewise clear that  $z$  commutes with  $e_0$  (since the anti-commutation of  $e_0$  with both  $e_{2i-1}$  and  $e_{2i}$  implies that  $e_0$  commutes with  $e_{2i-1}e_{2i}$  for all  $i$ ), so  $z$  commutes with every  $e_j$  ( $j \geq 0$ ). Likewise,  $z^2 = 1$ . Thus,  $\{1, z\}$  spans a  $\mathbf{Z}/2\mathbf{Z}$ -graded

central subalgebra  $Z$  of  $C(V, q)$  that is a subbundle of rank 2 with odd part spanned by  $z$ , so  $Z^1 \otimes Z^1 \simeq \mathcal{O}_S$  via multiplication. Since  $z$  is a unit, the natural  $\mathbf{Z}/2\mathbf{Z}$ -graded algebra map  $Z \otimes C_0(V, q) \rightarrow C(V, q)$  defined by multiplication is fiberwise injective and hence (for rank reasons) an isomorphism. But  $C_0(V, q)$  has trivial center, so  $Z$  is the center of  $C(V, q)$ .  $\square$

Assume  $n$  is even. The natural action of  $O(q)$  on  $C(V, q)$  preserves the  $\mathbf{Z}/2\mathbf{Z}$ -grading and hence induces an action of  $O(q)$  on  $C_0(V, q)$ , so we obtain an action of  $O(q)$  on the finite étale center  $Z_q$  of  $C_0(V, q)$ . The automorphism scheme  $\underline{\text{Aut}}_{Z_q/\mathcal{O}_S}$  is uniquely isomorphic to  $(\mathbf{Z}/2\mathbf{Z})_S$  since  $Z_q$  is finite étale of rank 2 over  $\mathcal{O}_S$ . Thus, for even  $n$  we get a homomorphism

$$D_q : O(q) \rightarrow (\mathbf{Z}/2\mathbf{Z})_S$$

that is compatible with isomorphisms in the quadratic space  $(V, q)$ , and its formation commutes with any base change on  $S$ . This is called the *Dickson invariant* (and is discussed in detail in [Knus, IV, §5]).

To analyze properties of the Dickson invariant for even  $n$ , it is convenient to introduce a certain closed subgroup of the  $S$ -group  $C(V, q)^\times$  of units of the Clifford algebra. The following definition for even  $n$  will later be generalized to odd  $n$ , for which some additional complications arise.

**Definition C.2.5.** For even  $n$ , the *Clifford group*  $\text{GPin}(q)$  is the closed  $S$ -subgroup scheme

$$(C.2.1) \quad \text{GPin}(q) := \{u \in C(V, q)^\times \mid uVu^{-1} = V\}$$

of units of the Clifford algebra that normalize  $V$  inside  $C(V, q)$ .

Consider  $(V, q)$  with even rank  $n$ . Since the center of  $C(V, q)$  is  $\mathcal{O}_S$ , an elementary calculation (given in [Chev97, II.3.2] over fields, but working verbatim over any  $S$ ) shows that every point of  $\text{GPin}(q)$  is “locally homogeneous”: Zariski-locally on the base, it is in either the even or odd part of  $C(V, q)$  locally on the base. (This is false for odd  $n$ , in view of the structure of  $Z_q$  in Proposition C.2.4.) Thus, the  $S$ -group  $\text{GPin}(q)$  agrees with the “Clifford group” as defined in [Knus, IV, 6.1]. For odd  $n$  we will have to force local homogeneity into the definition of the Clifford group in order to get the right notion for such  $n$ ; see §C.4.

There is a natural action by  $\text{GPin}(q)$  on  $V$  via conjugation, and the resulting homomorphism  $\text{GPin}(q) \rightarrow \text{GL}(V)$  lands in  $O(q)$  because for  $v \in V$  and  $u \in \text{GPin}(q)$  we have  $q(uvu^{-1}) = (uvu^{-1})^2 = uv^2u^{-1} = q(v)$  in  $C(V, q)$ . However, this action has a drawback: the intervention of an unpleasant sign for the conjugation action by the dense open non-vanishing locus  $U = \{q \neq 0\} \subset V$ . To be precise, any  $u \in U$  satisfies  $u^2 = q(u) \in \mathbf{G}_m$  in  $C(V, q)$ , so  $u \in C(V, q)^\times$  and  $uv + vu = B_q(u, v)$  for any  $v \in V$ . Hence,

$$(C.2.2) \quad uvu^{-1} = -v + (B_q(u, v)/q(u))u \in V,$$

so  $U \subset \text{GPin}(q)$  but the conjugation action on  $V$  by any  $u \in U$  is the *negative* of reflection through  $u$  relative to  $q$ . Thus, following Atiyah–Bott–Shapiro (and [Knus, IV, 6.1]), we define a representation

$$\pi_q : \text{GPin}(q) \rightarrow \text{GL}(V)$$

by

$$(C.2.3) \quad \pi_q(u)(v) = (-1)^{\deg_q(u)} uvu^{-1}$$

where  $\deg_q : \text{GPin}(q) \rightarrow (\mathbf{Z}/2\mathbf{Z})_S$  is the restriction of the degree on the subalgebra  $C(V, q)_{\text{lh}}$  of locally homogenous sections of the  $\mathbf{Z}/2\mathbf{Z}$ -graded Clifford algebra  $C(V, q)$ .

In other words,  $\pi_q$  is the twist of the conjugation action on  $V$  against the  $S$ -homomorphism  $\text{GPin}(q) \rightarrow \mu_2$  defined by  $u \mapsto (-1)^{\deg_q(u)}$ . Since quadratic forms are  $\mu_2$ -invariant, the representation  $\pi_q$  lands in  $O(q)$  as well, so  $\pi_q$  extends to an action of the Clifford group  $\text{GPin}(q)$  on the entire  $\mathbf{Z}/2\mathbf{Z}$ -graded algebra  $C(V, q)$ . This action coincides with the ordinary conjugation action when restricted to the intersection of  $\text{GPin}(q)$  with the even subalgebra  $C_0(V, q)$ . (The representation  $u : x \mapsto (-1)^{\deg_q(u)} u x u^{-1}$  on  $C(V, q)_{\text{lh}}$  by the group  $C(V, q)_{\text{lh}}^\times$  of locally homogeneous units is denoted as  $u \mapsto i_u$  in [Knus, IV, 6.1]. We will not use it.)

Since  $Z_q \subset C_0(V, q)$ , the action of  $\text{GPin}(q)$  on  $C(V, q)$  via  $\pi_q$  on  $V$  restricts to ordinary conjugation on  $Z_q$ . But  $C_0(V, q)$  is the centralizer of  $Z_q$  in  $C(V, q)$  (Proposition C.2.2), so  $D_q \circ \pi_q : \text{GPin}(q) \rightarrow (\mathbf{Z}/2\mathbf{Z})_S$  computes the restriction to  $\text{GPin}(q)$  of the  $\mathbf{Z}/2\mathbf{Z}$ -grading of  $C(V, q)$  (for even  $n$ ).

**Proposition C.2.6.** *Assume  $n$  is even. The map  $\pi_q : \text{GPin}(q) \rightarrow O(q)$  is a smooth surjection with kernel  $\mathbf{G}_m$  and  $D_q : O(q) \rightarrow (\mathbf{Z}/2\mathbf{Z})_S$  is a smooth surjection.*

*In particular, the affine  $S$ -group  $\text{GPin}(q)$  is smooth.*

*Proof.* Since  $O(q)$  is smooth (as  $n$  is even; see Theorem C.1.5), to prove that  $D_q$  is a smooth surjection it suffices to check surjectivity on geometric fibers. We have already noted that  $D_q \circ \pi_q$  computes the degree on locally homogeneous sections of  $C(V, q)$ , so for the assertion concerning  $D_q$  it suffices to check that on geometric fibers  $\text{GPin}(q)$  does not consist entirely of even elements. But as we saw above, the Zariski-dense open locus  $\{q \neq 0\} \subset V$  viewed in the odd part of  $C(V, q)$  consists of units that lie in the Clifford group  $\text{GPin}(q)$ .

Now we turn to the assertion that  $\pi_q$  is a smooth surjection with kernel  $\mathbf{G}_m$ . The kernel of  $D_q \circ \pi_q$  consists of the points of  $\text{GPin}(q)$  in the even part of  $C(V, q)$ , and this even part acts on  $V$  through ordinary conjugation under  $\pi_q$ , so  $\ker \pi_q$  is the intersection of  $\text{GPin}(q)$  with the part of  $C_0(V, q)$  that centralizes  $V$  inside  $C(V, q)$ . But  $V$  generates  $C(V, q)$  as an algebra, so the centralizer of  $V$  inside  $C(V, q)$  is the center of  $C(V, q)$ . This center is  $\mathcal{O}_S$  since  $n$  is even, so  $\ker \pi_q = \mathbf{G}_m$ . In view of the smoothness of this kernel, to show that  $\pi_q$  is a smooth surjection it suffices to prove that  $\pi_q$  is surjective fppf-locally on the base.

By applying a preliminary base change on  $S$  and renaming the new base as  $S$ , it suffices to show that for any  $g \in O(q)(S)$ , fppf-locally on  $S$  there exists a point  $u$  of  $\text{GPin}(q)$  satisfying  $\pi_q(u) = g$ . Consider the automorphism  $[-g]$  of the  $\mathcal{O}_S$ -algebra  $C(V, q)$  induced by  $-g \in O(q)(S)$ . We claim that this automorphism is inner, fppf-locally (even Zariski-locally) on  $S$ . Since the quotient  $S$ -group  $C(V, q)^\times / \mathbf{G}_m$  is a subfunctor of the automorphism scheme of the algebra  $C(V, q)$  via conjugation, it suffices to show that this quotient coincides with the automorphism scheme. The problem is fppf-local on  $S$ , so we can assume  $q = q_{2m}$ , in which case  $C(V, q)$  is a matrix algebra over  $\mathcal{O}_S$ . But for any  $N \geq 1$ , the natural map of

finite type  $\mathbf{Z}$ -groups  $\mathrm{PGL}_N \rightarrow \underline{\mathrm{Aut}}(\mathrm{Mat}_N)$  is an isomorphism (“relative Skolem–Noether”) by Exercise 5.4.5(i), so  $\mathrm{C}(V, q)^\times / \mathbf{G}_m = \underline{\mathrm{Aut}}(\mathrm{C}(V, q))$ .

Since  $\mathbf{G}_m$ -torsors for the fppf topology are automatically torsors for the Zariski topology, we may now arrange by Zariski localization on  $S$  that there exists  $u \in \mathrm{C}(V, q)^\times$  such that  $[-g](x) = u x u^{-1}$  for all  $x \in \mathrm{C}(V, q)$ . Setting  $x = v \in V$  gives  $[-g](x) = -g(v)$ , so  $u \in \mathrm{GPin}(q)$  and  $\pi_q(u) = g$ .  $\square$

We can finally define special orthogonal groups, depending on the parity of  $n$ .

**Definition C.2.7.** Let  $(V, q)$  be a quadratic space of rank  $n \geq 1$  over a scheme  $S$ . The *special orthogonal group*  $\mathrm{SO}(q)$  is  $\mathrm{SO}'(q) = \ker(\det|_{\mathrm{O}(q)})$  when  $n$  is odd and  $\ker D_q$  when  $n$  is even. For any  $n \geq 1$ ,  $\mathrm{SO}_n := \mathrm{SO}(q_n)$ .

For odd  $n$  the definition of  $\mathrm{SO}(q)$  makes sense for any non-degenerate triple  $(V, L, q)$ , and to achieve the same generality for even  $n$  (for which we have not discussed Clifford constructions in the  $L$ -valued case), we just need to work Zariski-locally on the base to trivialize  $L$  and observe that  $\mathbf{G}_m$ -scaling of  $q$  has no effect on the construction. By definition,  $\mathrm{SO}(q)$  is a closed subgroup of  $\mathrm{O}(q)$ , and it is also an open subscheme of  $\mathrm{O}(q)$  when  $n$  is even. (In contrast, we will see that  $\mathrm{SO}_{2m+1}$  is not an open subscheme of  $\mathrm{O}_{2m+1}$  over  $\mathbf{Z}$  because  $\mathrm{O}(q) = \mathrm{SO}(q) \times \mu_2$  for odd  $n$  via the central  $\mu_2 \subset \mathrm{GL}(V)$ , and over  $\mathrm{Spec} \mathbf{Z}$  the identity section of  $\mu_2$  is not an open immersion.)

The group  $\mathrm{SO}'(q)$  is not of any real interest when  $n$  is even and 2 is not a unit on the base (and we will show that it coincides with  $\mathrm{SO}(q)$  in all other cases). The only reason we define  $\mathrm{SO}'(q)$  for all  $n$  is because it is the first thing that comes to mind when trying to generalize the theory over  $\mathbf{Z}[1/2]$  to work over  $\mathbf{Z}$ . We will see that  $\mathrm{SO}'_{2m}$  is not  $\mathbf{Z}_{(2)}$ -flat. (Example: Consider  $m = 1$  and  $S = \mathrm{Spec} \mathbf{Z}_{(2)}$ . We have  $\mathrm{O}_2 = \mathbf{G}_m \times (\mathbf{Z}/2\mathbf{Z})$  and  $\mathrm{SO}_2 = \mathbf{G}_m$ , whereas  $\mathrm{SO}'_2$  is the reduced closed subscheme of  $\mathrm{O}_2$  obtained by removing the open non-identity component in the generic fiber.)

Here are the main properties we shall prove for the “good” groups associated to quadratic spaces  $(V, q)$ .

**Theorem C.2.8.** *The group  $\mathrm{SO}(q)$  is smooth of relative dimension  $n(n-1)/2$  with connected fibers. Its functorial center is trivial for odd  $n$  and equals the central  $\mu_2 \subset \mathrm{O}(q)$  for even  $n$ .*

*For  $n \geq 3$ ,  $\mathrm{O}(q)$  has functorial center represented by the central  $\mu_2$ , whereas the functorial center of  $\mathrm{SO}'(q)$  coincides with that of  $\mathrm{SO}(q)$ .*

**Remark C.2.9.** Observe that via the Dickson invariant  $D_q$ , for even  $n$  we have  $\mathrm{O}(q)/\mathrm{SO}(q) = (\mathbf{Z}/2\mathbf{Z})_S$ , so Theorem C.2.8 implies that  $\#\pi_0(\mathrm{O}(q)_s) = 2$  for all  $s \in S$ . In contrast, for odd  $n$  multiplication against the central  $\mu_2 \subset \mathrm{O}(q)$  defines an isomorphism  $\mu_2 \times \mathrm{SO}(q) \simeq \mathrm{O}(q)$  because  $\det : \mathrm{O}(q) \rightarrow \mathbf{G}_m$  factors through  $\mu_2$  (due to the half-discriminant, as we saw in the proof of Theorem C.1.5). Thus, if  $n$  is odd then  $\mathrm{O}(q)$  is fppf over  $S$  with  $\mathrm{O}(q)/\mathrm{SO}(q) = \mu_2$  (so  $\mathrm{O}(q)_s$  is connected and non-smooth when  $\mathrm{char}(k(s)) = 2$ ).

Since the case of odd rank requires special care in residue characteristic 2, we shall first analyze even rank, where we can use characteristic-free arguments.

**C.3. Connectedness and center.** Let  $(V, q)$  be a quadratic space of even rank  $n \geq 2$ . By Theorem C.1.5,  $O(q)$  is smooth. By Proposition C.2.6,  $SO(q) = \ker D_q$  is therefore also smooth. Less evident is the connectedness of  $SO(q_s)$  for  $s \in S$ :

**Proposition C.3.1.** *If  $n$  is even then  $SO(q) \rightarrow S$  has connected fibers.*

*Proof.* We proceed by 2-step induction on  $n$ , and we can assume  $S = \text{Spec } k$  for an algebraically closed field  $k$ . Without loss of generality,  $q = q_n$ . In view of the surjectivity of the Dickson invariant, it is equivalent to show that  $O(q)$  has exactly 2 (equivalently, at most 2) connected components. Since  $q_2 = xy$ , clearly  $O_2 = \mathbf{G}_m \amalg \mathbf{G}_m \iota$  for  $\iota = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Now assume  $n \geq 4$  and that the result is known for  $n - 2$ .

Since  $q$  is not a square (as  $n > 1$ ), it is straightforward to check that the smooth affine hypersurface  $H = \{q = 1\}$  is irreducible. The points in  $H(k)$  correspond to isometric embeddings  $(k, x^2) \hookrightarrow (V, q)$ . (Note that  $(k, x^2)$  is degenerate when  $\text{char}(k) = 2$ .) By Witt’s extension theorem [Chev97, I.4.1], if  $Q : W \rightarrow K$  is a quadratic form on a finite-dimensional vector space over a field  $K$  and  $B_Q$  is non-degenerate (so  $\dim W$  is even when  $\text{char}(K) = 2$ ) then  $O(Q)(K)$  acts transitively on the set of isometric embeddings of a fixed (possibly degenerate) quadratic space into  $(W, Q)$ . Hence,  $O(q)(k)$  acts transitively on  $H(k)$ , so the orbit map  $O(q) \rightarrow H$  through  $e_2$  is surjective with stabilizer  $G' := \text{Stab}_{e_2}(O(q))$  that preserves the orthogonal complement  $V' := e_2^\perp = \sum_{i \geq 2} ke_i$ . Since  $H$  is connected, it follows that  $O(q)^0$  acts transitively on  $H$  too, so  $\#\pi_0(O(q)) \leq \#\pi_0(G')$ . Hence, it suffices to show that  $G'$  has 2 connected components.

The action map  $G' \rightarrow \text{GL}(V')$  has kernel consisting of automorphisms of  $V$  that fix  $\{e_2, \dots, e_n\}$  pointwise and preserve  $q$ . Such automorphisms must preserve the orthogonal complement  $ke_1 + ke_2$  of the span of  $\{e_3, \dots, e_n\}$ , so it is an elementary calculation that the scheme-theoretic kernel of  $G' \rightarrow \text{GL}(V')$  is trivial. Let  $W = (ke_1 + ke_2)^\perp = \sum_{i \geq 3} ke_i$ . For any  $w \in W$  we have  $q(ce_2 + w) = c^2q(e_2) + B_q(ce_2, w) + q(w) = q(w)$ , so relative to the ordered basis  $\{e_2, \dots, e_n\}$  of  $V'$  the map  $G' \rightarrow \text{GL}(V')$  is an isomorphism onto the subgroup of  $(n - 1) \times (n - 1)$  matrices whose left column is  $(1, 0, \dots, 0)$ , top row has arbitrary entries beyond the initial 1, and lower right  $(n - 2) \times (n - 2)$  block is in  $O(q|_W)$ . In other words,  $G'$  is an extension of  $O(q|_W)$  by  $\mathbf{G}_a^{n-2}$ . By induction we know that  $O(q|_W)$  has exactly 2 connected components, so the same holds for  $G'$ .  $\square$

**Corollary C.3.2.** *Assume  $n$  is even. Let  $f_S : (\mathbf{Z}/2\mathbf{Z})_S \rightarrow \mu_2$  be the unique  $S$ -homomorphism satisfying  $f(1) = -1$ . The determinant map  $\det : O(q) \rightarrow \mu_2$  coincides with  $f_S \circ D_q$ . In particular,  $\det$  kills  $SO(q)$ .*

*The inclusion  $SO(q) \subset SO'(q) = \ker(\det)$  is an equality over  $S[1/2]$ , and  $SO'(q) \hookrightarrow O(q)$  is an equality on fibers at points in characteristic 2.*

*Proof.* By Lemma C.2.1, we may pass to the case  $q = q_n$  over  $\mathbf{Z}$ . The equality  $\det = f_{\mathbf{Z}} \circ D_{q_n}$  can then be checked over  $\mathbf{Q}$ , where it is immediate from the connectedness of  $SO_n$  over  $\mathbf{Q}$  and the nontriviality of  $\det$  on  $O_n$  over  $\mathbf{Q}$ . Since  $f_{\mathbf{Z}[1/2]}$  is an isomorphism, we get the equality of  $SO_n$  and  $SO'_n$  over  $\mathbf{Z}[1/2]$ . Over  $\mathbf{F}_2$ , the smooth group  $O_n$  must be killed by the determinant map into the infinitesimal  $\mu_2$ , so  $SO'_n = O_n$  over  $\mathbf{F}_2$ .  $\square$

**Remark C.3.3.** Assume  $n$  is even. Consider the element  $g \in O_n(\mathbf{Z})$  that swaps  $e_1$  and  $e_2$  while leaving the other  $e_i$  invariant. The section  $D_q(g)$  of the constant  $\mathbf{Z}$ -group  $\mathbf{Z}/2\mathbf{Z}$  is equal to 1 mod 2 since it suffices to check this on a single geometric fiber (and at any fiber away from characteristic 2 it is clear, as  $SO_n$  coincides with  $SO'_n$  over  $\mathbf{Z}[1/2]$ ). Thus, the Dickson invariant  $D_q : O(q) \rightarrow (\mathbf{Z}/2\mathbf{Z})_S$  splits as a semi-direct product when  $q = q_n$ .

The induced map  $H^1(S_{\text{ét}}, O_n) \rightarrow H^1(S_{\text{ét}}, \mathbf{Z}/2\mathbf{Z})$  assigns to every non-degenerate  $(V, q)$  of rank  $n$  over  $S$  (taken up to isomorphism) a degree-2 finite étale cover of  $S$ . This is the *discriminant* of  $(V, q)$ . If  $S$  is a  $\mathbf{Z}[1/2]$ -scheme (so  $(\mathbf{Z}/2\mathbf{Z})_S = \mu_2$ ) and  $\text{Pic}(S) = 1$  then it recovers the class in  $H^1(S, \mu_2) = \mathbf{G}_m(S)/\mathbf{G}_m(S)^2$  of the unit  $c$  such that the induced quadratic form on the top exterior power  $\det V$  of  $V$  is locally  $cx^2$ . (Concretely, if  $V$  is globally free then this is represented by the unit  $\det[B_q]$ , where  $[B_q]$  is the matrix of  $B_q$  relative to a basis for  $V$ ; see [Knus, IV, 4.1.1, 5.3.2] for affine  $S$ .) If  $S$  is an  $\mathbf{F}_2$ -scheme then it recovers the *pseudo-discriminant*, also called the *Arf invariant* when  $S = \text{Spec } k$  for a field  $k/\mathbf{F}_2$ ; see [Knus, IV, 4.7] for an explicit formula when  $S$  is affine and  $V$  is globally free.

**Corollary C.3.4.** *For even  $n$ , the  $\mathbf{Z}$ -group  $SO'_n$  is reduced and the open and closed subscheme  $SO_n \hookrightarrow SO'_n$  has complement equal to the non-identity component of  $(O_n)_{\mathbf{F}_2}$ . In particular,  $SO'_n$  is not  $\mathbf{Z}$ -flat when  $n$  is even.*

*Proof.* Corollary C.3.2 gives the result over  $\mathbf{Z}[1/2]$ , as well as the topological description of the  $\mathbf{F}_2$ -fiber. It remains to show that  $SO'_n$  is reduced. It is harmless to pass to the quotient by the smooth normal subgroup  $SO_n$ , so under the identification of  $O_n/SO_n$  with the constant group  $\mathbf{Z}/2\mathbf{Z}$  via the Dickson invariant we see that  $G = SO'_n/SO_n$  is identified with the kernel of the unique homomorphism of  $\mathbf{Z}$ -groups  $f : \mathbf{Z}/2\mathbf{Z} \rightarrow \mu_2$  sending 1 to  $-1$ . As a map from the  $\mathbf{Z}$ -group  $(\mathbf{Z}/2\mathbf{Z})_{\mathbf{Z}} = \text{Spec } \mathbf{Z}[t]/(t^2 - t)$  to  $\mu_2 = \text{Spec } \mathbf{Z}[\zeta]/(\zeta^2 - 1)$ , it is given by  $\zeta - 1 \mapsto -2t$  on coordinate rings, so the kernel  $G$  is  $\text{Spec } \mathbf{Z}[t]/(-2t, t^2 - t)$ . This is the disjoint union of the identity section and a single  $\mathbf{F}_2$ -point in the  $\mathbf{F}_2$ -fiber.  $\square$

Now assume  $n$  is odd, so  $SO(q) = SO'(q)$ . This is smooth by Theorem C.1.5. Also, as we noted in Remark C.2.9, consideration of the half-discriminant shows that  $\det : O(q) \rightarrow \mathbf{G}_m$  factors through  $\mu_2$ , so by the oddness of  $n$  the determinant splits off the central  $\mu_2 \subset O(q)$  as a direct factor:  $O(q) = \mu_2 \times SO(q)$ . In particular,  $SO(q) \rightarrow S$  has fibers of dimension  $n(n - 1)/2$ , since the dimension of  $\text{Tan}_1(SO'(q_s))$  was computed in the proof of Theorem C.1.5. The fibral connectedness will be proved by induction on the odd  $n$ :

**Proposition C.3.5.** *Let  $(V, q)$  be a quadratic space over a field  $k$ , with  $n = \dim V$  odd. The group  $SO(q)$  is connected.*

*Proof.* We may assume  $k$  is algebraically closed and  $q = q_n$ . The case  $n = 1$  is trivial, so we assume  $n \geq 3$ . We treat characteristic 2 separately from other characteristics, due to the appearance of the defect space  $V^\perp = ke_0$  in characteristic 2.

First assume  $\text{char}(k) \neq 2$ , so the symmetric bilinear form  $B_q$  is non-degenerate and  $\mu_2 = (\mathbf{Z}/2\mathbf{Z})_k$ . Since  $\mu_2 \times SO(q) = O(q)$ ,  $O(q)$  has at least 2 connected components. It has exactly 2 such components if and only if  $SO(q)$  is connected. Since  $n > 1$ , the hypersurface

$H = \{q = 1\}$  is irreducible, and exactly as in the proof of Proposition C.3.1 we may apply Witt’s extension theorem (valid for odd  $n$  since  $\text{char}(k) \neq 2$ ) to deduce that the action of  $O(q)$  on  $H$  is transitive. The orthogonal complement  $V'$  of  $e_0$  is spanned by  $\{e_1, \dots, e_{2m}\}$  since  $\text{char}(k) \neq 2$ , and it is preserved by  $\text{Stab}_{e_0}(O(q))$ . It is straightforward to check that the action of this stabilizer on  $V'$  defines an isomorphism onto  $O(q|_{V'}) \simeq O_{2m}$ . We already know that  $O_{2m}$  has 2 connected components whereas  $H$  is irreducible, so it follows that  $O(q)$  has at most 2 connected components (hence exactly 2 such components). This settles the case  $\text{char}(k) \neq 2$ .

Now assume  $\text{char}(k) = 2$ . Since  $O(q) = \mu_2 \times \text{SO}(q)$ , the connectedness of  $\text{SO}(q)$  is equivalent to the connectedness of  $O(q)$ . The non-vanishing defect space obstructs induction using the action on  $H$ , so instead we will use an entirely different procedure that is specific to characteristic 2 (cf. [Bo91, 23.6]). When  $\text{char}(k) \neq 2$  we passed to the hyperplane  $ke_0^\perp$  spanned by  $e_1, \dots, e_{2m}$  on which  $B_q$  restricted to a non-degenerate symmetric bilinear form. In characteristic 2, we will use the quotient  $V' := V/V^\perp = V/ke_0$  rather than a hyperplane. More specifically, because  $V^\perp$  is the defect space,  $B_q$  factors through a non-degenerate symmetric bilinear form  $B'_q$  on  $V'$ . This has an extra property: it is *alternating*, since  $\text{char}(k) = 2$ . Any point of  $O(q)$  preserves the defect space, and the induced automorphism of  $V'$  preserves  $\overline{B}_q$ . This defines a homomorphism

$$h : O_{2m+1} = O(q) \rightarrow \text{Sp}(V', B'_q) \simeq \text{Sp}_{2m}$$

from our orthogonal group to a symplectic group.

The kernel is seen by calculation to be  $\mu_2 \times \alpha_2^{n-1}$  (along the top row of matrices, with  $\mu_2$  in the upper left), where  $\mu_2$  acts on the Frobenius kernel  $\alpha_2^{n-1} \subset \mathbf{G}_a^{n-1}$  by the usual diagonal scaling action. Indeed, rather explicitly, since a point  $T$  of  $O(q)$  must restrict to an automorphism of the quadratic space  $V^\perp = (ke_0, x_0^2)$ , it has the block form

$$T = \begin{pmatrix} \zeta & \alpha & \alpha' \\ 0 & A & B \\ 0 & C & D \end{pmatrix}$$

for  $m \times m$  matrices  $A, B, C, D$ , a point  $\zeta$  of  $\mu_2$ , and ordered  $m$ -tuples  $\alpha$  and  $\alpha'$ . Writing a typical ordered  $(2m + 1)$ -tuple as  $(x_0, x, x')$  for ordered  $m$ -tuples  $x$  and  $x'$ , we see that

$$q(T(x_0, x, x')) = x_0^2 + \langle \alpha, x \rangle^2 + \langle \alpha', x' \rangle^2 + B'_q(Ax + By, Cx + Dy),$$

where  $\langle \cdot, \cdot \rangle$  is the standard bilinear form  $(w, z) \mapsto \sum w_j z_j$ . Setting this equal to  $q(x_0, x, x')$  then imposes equations on  $\alpha, \alpha', A, B, C, D$  that define the closed subscheme  $O(q) \subset \text{GL}_n$ . This not only implies the asserted description of  $\ker h$ , but also shows that  $h$  is surjective.

We have shown that  $h$  is surjective with infinitesimal fibers, so it suffices to prove the connectedness of symplectic groups over fields. This is proved by an inductive fibration argument (using lower-dimensional symplectic spaces).  $\square$

**Remark C.3.6.** Assume  $n$  is odd. In the proof of Proposition C.3.5, over any field  $k$  of characteristic 2 we constructed a surjective homomorphism  $h : O(q) \rightarrow \text{Sp}(V', B'_q)$  with

$(\ker h)_{\bar{k}} = \mu_2 \times \alpha_2^{n-1}$ . This kernel meets the kernel  $\mathrm{SO}(q)$  of the determinant map on  $\mathrm{O}(q)$  in  $\alpha_2^{n-1}$  over  $\bar{k}$ , so by smoothness of  $\mathrm{SO}(q)$  we obtain a purely inseparable isogeny  $\mathrm{SO}(q) \rightarrow \mathrm{Sp}(V', B'_q)$  with kernel that is a form of  $\alpha_2^{n-1}$  (and hence is isomorphic to  $\alpha_2^{n-1}$ , as  $\alpha_p^N$  has automorphism scheme  $\mathrm{GL}_N$ ). This “unipotent isogeny” is a source of many weird phenomena related to algebraic groups in characteristic 2 (e.g., see [CP, A.3]).

Special orthogonal groups in  $2m + 1$  variables are type  $B_m$  and symplectic groups in  $2m$  variables are type  $C_m$ , and these types are distinct for  $m \geq 3$ . (For  $m = 1$  and  $m = 2$  the types coincide; see Example C.6.2 and Example C.6.5 respectively.) In characteristics distinct from 2 and 3 there are no isogenies between (absolutely simple) connected semisimple groups of different types. In characteristic 2 we have just built “exceptional” isogenies between  $B_m$  and  $C_m$  for all  $m \geq 3$ . See [SGA3, XXI, 7.5] for further details.

The remaining task for  $\mathrm{SO}(q)$  and  $\mathrm{O}(q)$  (and  $\mathrm{SO}'(q)$ ) is to determine the functorial center. Since  $\mathrm{O}(q)$  is commutative when  $n \leq 2$ , we will assume  $n \geq 3$ . For odd  $n$ , the central  $\mu_2$  in  $\mathrm{O}(q)$  has trivial intersection with  $\mathrm{SO}'(q)$ , and hence with  $\mathrm{SO}(q)$ . If  $n$  is even then the central  $\mu_2$  is contained in  $\mathrm{SO}'(q)$ , and we claim that it also lies in  $\mathrm{SO}(q)$ . In other words, for even  $n$  we claim that the Dickson invariant  $D_q : \mathrm{O}(q) \rightarrow (\mathbf{Z}/2\mathbf{Z})_S$  kills the central  $\mu_2$ . It suffices to treat the case of  $q = q_n$  over  $\mathbf{Z}$ , in which case we just need to show that the *only* homomorphism of  $\mathbf{Z}$ -groups  $\mu_2 \rightarrow \mathbf{Z}/2\mathbf{Z}$  is the trivial one. By flatness, to prove such triviality it suffices to check after localization to  $\mathbf{Z}_{(2)}$ . But over the local base  $\mathrm{Spec} \mathbf{Z}_{(2)}$  the scheme  $\mu_2$  is connected and thus it must be killed by a homomorphism into a constant group.

**Proposition C.3.7.** *Assume  $n \geq 3$ . The functorial centers of  $\mathrm{SO}(q)$  and  $\mathrm{SO}'(q)$  coincide. This common center is represented by  $\mu_2$  in the central  $\mathbf{G}_m \subset \mathrm{GL}(V)$  when  $n$  is even, and it is trivial when  $n$  is odd.*

*Proof.* By Lemma C.2.1, it suffices to treat  $q_n$  over  $S = \mathrm{Spec} k$  for any ring  $k$ . We will use a method similar to the treatment of  $Z_{\mathrm{Sp}_{2n}}$  in Exercise 2.4.6: exhibit a specific torus  $T$  that we show to be its own centralizer in  $G := \mathrm{SO}'(q)$  (so  $T$  its own centralizer in  $\mathrm{SO}(q)$ ) and then we will look for the center inside this  $T$ .

First suppose  $n = 2m$ , so relative to a suitable ordered basis  $\{e_1, e'_1, \dots, e_m, e'_m\}$  we have  $q = \sum_{i=1}^m x_i x'_i$ . In this case we identify  $\mathrm{GL}_1^m$  with a  $k$ -subgroup  $T$  of  $\mathrm{SO}'(q)$  via

$$j : (t_1, \dots, t_m) \rightarrow (t_1, 1/t_1, \dots, t_m, 1/t_m).$$

The action by  $T$  on  $k^n = k^{2m}$  has each standard basis line as a weight space for a collection of  $2m$  fiberwise distinct characters over  $\mathrm{Spec} k$ . Hence,  $Z_{\mathrm{GL}_n}(T)$  is the diagonal torus in  $\mathrm{GL}_n$ , so clearly  $Z_{\mathrm{O}(q_n)}(T) = T$  and hence  $Z_G(T) = T$ .

Now suppose  $n = 2m + 1$  with  $m \geq 1$ . Pick a basis  $\{e_0, e_1, e'_1, \dots, e_m, e'_m\}$  relative to which

$$q = x_0^2 + \sum_{i=1}^m x_i x'_i.$$

If we define  $T$  in the same way (using the span of  $e_1, e'_1, \dots, e_m, e'_m$ ) then the same analysis gives the same result:  $T$  is its own scheme-theoretic centralizer in  $\mathrm{SO}'(q)$ . The point is that

there is no difficulty created by  $e_0$  because we are requiring the determinant to be 1. (If we try the same argument with  $O(q)$  then the centralizer of  $T$  is  $\mu_2 \times T$ .)

We are now in position to identify the center of  $SO'(q)$  when  $n \geq 3$ . First we assume  $n \geq 4$  (i.e.,  $m \geq 2$ ). In terms of the ordered bases as above, consider the automorphism  $g_i$  obtained by swapping the ordered pairs  $(e_i, e'_i)$  and  $(e'_1, e_1)$  for  $1 < i \leq m$ . (Such  $i$  exist precisely because  $m \geq 2$ .) These automorphisms  $g_i$  lie in  $SO'(q)$  since the determinant is  $(-1) \cdot (-1) = 1$ , and a point of  $T$  centralizes  $g_i$  if and only if  $t_1 = t_i$ . Letting  $i$  vary, we conclude that the center of  $SO'(q)$  is contained in the “scalar” subgroup  $GL_1 \hookrightarrow T$  given by  $t_1 = \cdots = t_m$ . This obviously holds when  $n = 3$  as well.

Letting  $\lambda$  denote the common value of the  $t_j$ , to constrain it further we consider more points of  $SO'(q)$  that it must centralize. First assume  $m \geq 2$ . Consider the automorphism  $f$  of  $V$  which acts on the plane  $ke_i \oplus ke'_i$  by the matrix  $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  for exactly *two* values  $i_0, i_1 \in \{1, \dots, m\}$  (and leaves all other basis vectors invariant), so  $\det f = 1$ . Clearly  $f$  preserves  $q$ , so  $f$  lies in  $SO'(q)(k)$ . But  $f$ -conjugation of  $t \in T$  viewed in  $SO'(q)$  (or  $GL(V)$ ) swaps the entries  $t_i$  and  $1/t_i$  for  $i \in \{i_0, i_1\}$ . Thus, the centralizing property forces  $\lambda \in \mu_2$ , so  $Z_{SO'(q)}$  is contained in the central  $\mu_2$  in  $GL(V)$  when  $n \geq 4$ . This inclusion is an equality for even  $n \geq 4$ , since the central  $\mu_2 \subset GL(V)$  is contained in  $SO'(q)$  for even  $n$ . If  $n$  is odd then the central  $\mu_2$  in  $GL(V)$  meets  $SO'(q) = SO(q)$  in 1, so  $Z_{SO'(q)} = 1$  for odd  $n \geq 5$ .

Next, we prove that  $Z_{SO_3} = 1$ . The action of  $PGL_2$  on  $\mathfrak{sl}_2$  via conjugation defines an isomorphism  $PGL_2 \simeq SO_3$ ; see the self-contained calculations in Example C.6.2. By Exercise 2.4.6(ii) the scheme-theoretic center of  $PGL_r$  is trivial for any  $r \geq 2$  (and for  $PGL_2$  it can be verified by direct calculation), so  $SO_3$  has trivial center.

We have settled the case of odd  $n \geq 3$ , and for even  $n \geq 4$  we have proved that  $SO'(q)$  has center  $\mu_2$  that also lies in  $SO(q)$ . It remains to show, assuming  $n \geq 4$  is even, that the functorial center of  $SO(q)$  is no larger than this  $\mu_2$ . We may and do assume  $q = q_n$ . The torus  $T$  constructed above in  $SO'_n$  lies in the open and closed subgroup  $SO_n$  for topological reasons, and  $Z_{SO_n}(T) = T$  since  $T$  has been shown to be its own centralizer in  $SO'_n$ . Thus, it suffices to show that the central  $\mu_2$  is the kernel of the adjoint action of  $T$  on  $\text{Lie}(SO_n) = \text{Lie}(O_n)$ . The determination of the weight space decomposition for  $T$  acting on  $\text{Lie}(O_n)$  is classical, from which the kernel is easily seen to be the diagonal  $\mu_2$ .  $\square$

**Corollary C.3.8.** *For  $n \geq 3$ , the functorial center of  $O(q)$  is represented by the central  $\mu_2$ .*

*Proof.* If  $n$  is odd then the identification  $O(q) = \mu_2 \times SO(q)$  yields the result since  $SO(q)$  has trivial functorial center for such  $n$ . Now suppose that  $n$  is even. In this case the open and closed subgroup  $SO(q)$  contains the central  $\mu_2$  is its functorial center. To prove that  $\mu_2$  is the functorial center of  $O(q)$  we again pass to the case  $q = q_n$ . It suffices to check that the diagonal torus  $T$  in  $SO_n$  is its own centralizer in  $O_n$ , and this was shown in the proof of Proposition C.3.7.  $\square$

**C.4. Spin groups and related group schemes.** For any  $n \geq 1$  we shall use unit groups of Clifford algebras to “explicitly” construct a degree-2 central extension  $\text{Spin}(q)$  of  $SO(q)$  by  $\mu_2$  that is the simply connected central cover when  $n \geq 3$ . For even  $n$  the Clifford algebra

has center  $\mathcal{O}_S$  (Proposition C.2.2) whereas for odd  $n$  the center is a finite locally free  $\mathcal{O}_S$ -algebra of rank 2 that is non-étale in characteristic 2 (Proposition C.2.4). Thus, we shall first consider even  $n$ . In such cases we will use the Clifford group  $\mathrm{GPin}(q)$  from Definition C.2.5 to construct  $\mathrm{Spin}(q) \rightarrow \mathrm{SO}(q)$  by relativizing arguments over fields in [Chev97, II.3.5]. (In [Chev97, II.3.5], spin groups are called *reduced Clifford groups*.)

Assume  $n = 2m \geq 2$  is even. The  $S$ -group  $\mathrm{GPin}(q)$  is a central extension of  $\mathrm{O}(q)$  by  $\mathbf{G}_m$  (Proposition C.2.6), so  $\mathrm{GSpin}(q) := \mathrm{GPin}(q)^0$  is a central extension of  $\mathrm{SO}(q)$  by  $\mathbf{G}_m$  and

$$(C.4.1) \quad \mathrm{GPin}(q)/\mathrm{GSpin}(q) \simeq \mathrm{O}(q)/\mathrm{SO}(q) = (\mathbf{Z}/2\mathbf{Z})_S$$

for even  $n$ . In view of the link between  $D_q$  and the  $\mathbf{Z}/2\mathbf{Z}$ -grading on  $\mathrm{GPin}(q)$  at noted immediately before Proposition C.2.6, we see that  $\mathrm{GSpin}(q) = \mathrm{C}_0(V, q) \cap \mathrm{GPin}(q)$  (so  $\mathrm{GSpin}(q)$  is open and closed in  $\mathrm{GPin}(q)$ ). For this reason,  $\mathrm{GSpin}(q)$  is called the *even Clifford group*.

The extension structure on  $\mathrm{GSpin}(q)$  implies that it is a smooth affine  $S$ -group with connected reductive fibers, so it is a reductive  $S$ -group. For  $n = 2$ , if  $Z_q$  denotes the degree-2 finite étale center of  $\mathrm{C}_0(V, q)$  then the containment  $\mathrm{R}_{Z_q/S}(\mathbf{G}_m) \subset \mathrm{GSpin}(q)$  of smooth affine  $S$ -groups is an equality for fibral connectedness and dimension reasons. When  $n = 2m \geq 4$  the  $S$ -group  $\mathrm{SO}(q)$  is semisimple (with geometric fibers of type  $\mathrm{D}_m$ , where  $\mathrm{D}_2 := \mathrm{A}_1 \times \mathrm{A}_1$ ), so the derived group

$$\mathrm{Spin}(q) := \mathcal{D}(\mathrm{GSpin}(q))$$

(in the sense of Theorem 5.3.1) is a central extension of  $\mathrm{SO}(q)$  by an  $S$ -subgroup  $\mu \subset \mathbf{G}_m$  that must be finite fppf (Proposition 6.1.10). Since we saw that  $\mathrm{GSpin}(q)$  is an extension of  $\mathrm{SO}(q)$  by  $\mathbf{G}_m$ , it follows that  $\mathrm{GSpin}(q) = \mathbf{G}_m \times^\mu \mathrm{Spin}(q)$ .

**Lemma C.4.1.** *For even  $n \geq 4$ , the intersection  $\mu = \mathbf{G}_m \cap \mathrm{Spin}(q)$  is equal to  $\mu_2 \subset \mathbf{G}_m$ .*

*Proof.* We shall use the anti-involution  $\alpha$  of  $\mathrm{C}(V, q)$  induced by the anti-involution of the tensor algebra of  $V$  via  $v_1 \otimes \cdots \otimes v_j \mapsto (-1)^j v_j \otimes \cdots \otimes v_1$ . Note that  $\alpha$  restricts to negation on  $V$  and the identity on the center  $\mathcal{O}_S$ , and the intervention of the sign is not seen on the even subalgebra  $\mathrm{C}_0(V, q)$ . (The effect of  $\alpha$  on the rank-2 étale central  $\mathcal{O}_S$ -algebra  $Z_q$  in  $\mathrm{C}_0(V, q)$  is the identity when  $4|n$  and is the unique fiberwise nontrivial algebra automorphism when  $n \equiv 2 \pmod{4}$ . We will not need this, but to prove it we can pass to the case  $q = q_n$  over  $\mathbf{Z}$  and then use the decomposition  $Z_{q_n} = \mathcal{O}_S z \times \mathcal{O}_S(1 - z)$  in Remark C.2.3 for a fiberwise nontrivial idempotent  $z$  described over  $\mathbf{Z}[1/2]$  in terms of an auxiliary explicit element  $w$  to show that  $\alpha(z) = 1 - z$  over  $\mathbf{Z}$  by checking that  $\alpha(w) = (-1)^{n/2} w$ .)

For  $u \in \mathrm{GPin}(q)$ , the operator  $h = \pi_q(u) \in \mathrm{O}(q)$  as in (C.2.3) satisfies  $uv = \varepsilon_u h(v)u$  in  $\mathrm{C}(V, q)$  for all  $v \in V$ , where  $\varepsilon_u = (-1)^{\deg_q(u)}$  as in (C.2.3). Applying  $\alpha$  gives  $-v\alpha(u) = \varepsilon_u \alpha(u)(-h(v))$ . Hence,  $\alpha(u)uv = \varepsilon_u \alpha(u)h(v)u = v\alpha(u)u$ , so the point  $\alpha(u)u$  in  $\mathrm{GPin}(q)$  centralizes  $V$  in  $\mathrm{C}(V, q)$  and thus is central in  $\mathrm{C}(V, q)$ . In other words,  $\alpha(u)u \in \mathbf{G}_m$ . Since  $\alpha$  is an anti-automorphism, so

$$\alpha(uu')uu' = \alpha(u')\alpha(u)uu' = (\alpha(u)u)(\alpha(u')u'),$$

the map  $u \mapsto \alpha(u)u$  is an  $S$ -homomorphism

$$(C.4.2) \quad \nu_q : \mathbf{GPin}(q) \rightarrow \mathbf{G}_m$$

(called the *Clifford norm*) whose restriction to the central  $\mathbf{G}_m$  is  $t \mapsto t^2$ . The Clifford norm must kill the semisimple  $\mathrm{Spin}(q)$ , so  $\mu := \mathbf{G}_m \cap \mathrm{Spin}(q) \subset \mu_2$ . We claim that this inclusion between finite fppf  $S$ -groups is an isomorphism. It suffices to check on fibers.

That is, with  $S = \mathrm{Spec} k$  for a field  $k$ , we must rule out the possibility  $\mathbf{G}_m \cap \mathrm{Spin}(q) = 1$ . If this happens then  $\pi_q : \mathrm{Spin}(q) \rightarrow \mathrm{SO}(q)$  is an isomorphism. This map is equivariant for the natural actions by  $\mathrm{O}(q) = \mathbf{GPin}(q)/\mathbf{G}_m$  on each side, so if  $\pi_q$  were an isomorphism then  $\mathrm{Spin}(q)$  would contain a  $\mu_2$  that is centralized by  $\mathbf{GPin}(q)$  (lifting  $Z_{\mathrm{O}(q)} = \mu_2 \subset \mathrm{SO}(q)$ ) and is not contained in the central  $\mathbf{G}_m \subset \mathrm{C}(V, q)$ . But  $\mathbf{GPin}(q)$  generates the  $k$ -algebra  $\mathrm{C}(V, q)$  (since  $\mathbf{GPin}(q)$  contains the Zariski-dense open  $U = \{q \neq 0\} \subset V$ ) and this algebra has center  $k$ , so  $\mathbf{G}_m = Z_{\mathbf{GPin}(q)}$ . Hence, no such  $\mu_2$  subgroup can exist.  $\square$

**Remark C.4.2.** In [Chev97, II.3.5], the Clifford norm is defined by replacing  $\alpha$  with the anti-involution  $\alpha'$  defined without the sign  $(-1)^j$ . This yields a different Clifford norm. We choose the convention for  $\alpha$  extending negation rather than the identity on  $V$  so that its effect on the center  $Z_q \subset \mathrm{C}(V, q)$  is as described in the proof of Lemma C.4.1; this choice agrees with [Knus, IV, 6.1]. For odd  $n$ , the same definitions of  $\alpha$  and  $\nu_q$  as in the preceding proof carry over, but in such cases  $Z_q$  is a rank-2 finite flat  $\mathcal{O}_S$ -algebra that is not contained in the even part. This leads to some complications when  $n$  is odd.

We conclude that for even  $n = 2m \geq 4$ , the semisimple  $S$ -group  $\mathrm{Spin}(q)$  is a central extension of  $\mathrm{SO}(q)$  by  $\mu_2$ . Since  $\#Z_{\mathrm{Spin}(q)} = 2 \cdot \#Z_{\mathrm{SO}(q)} = 4$  and the fundamental group for  $\mathrm{D}_m$  ( $m \geq 2$ ) has order 4,  $\mathrm{Spin}(q)$  must be simply connected. Thus,  $\mathrm{Spin}(q)$  is the simply connected central cover of  $\mathrm{SO}(q)$ . (By construction, the central pushout of  $\mathrm{Spin}(q)$  along  $\mu_2 \hookrightarrow \mathbf{G}_m$  is the relative identity component  $\mathrm{GSpin}(q)$  of  $\mathbf{GPin}(q)$  inside  $\mathrm{C}(V, q)^\times$ .)

Since  $\mathrm{GSpin}(q) = \mathbf{G}_m \times^{\mu_2} \mathrm{Spin}(q)$ , we see that  $\mathrm{Spin}(q)$  is the kernel of the Clifford norm  $\nu_q : \mathrm{GSpin}(q) \rightarrow \mathbf{G}_m$ . This description of  $\mathrm{Spin}(q)$  via the Clifford norm on  $\mathrm{GSpin}(q)$  makes sense even when  $n = 2$  (whereas the “derived group” definition of the spin group for  $n \geq 4$  is not suitable, due to the commutativity of  $\mathrm{GSpin}(q) = \mathrm{R}_{Z_q/S}(\mathbf{G}_m)$  when  $n = 2$ ). Using it as the *definition* of  $\mathrm{Spin}(q)$  when  $n = 2$ , direct calculation with  $q_2$  shows that if  $n = 2$  then the Clifford norm  $\nu_q : \mathrm{GSpin}(q) = \mathrm{R}_{Z_q/S}(\mathbf{G}_m) \rightarrow \mathbf{G}_m$  arises from the  $\mathcal{O}_S$ -algebra norm on  $\mathcal{O}_{Z_q}$ . Thus,  $\mathrm{Spin}(q)$  is the group of norm-1 units in the rank-2 torus  $\mathrm{R}_{Z_q/S}(\mathbf{G}_m)$  and the resulting map  $\mathrm{Spin}(q) \rightarrow \mathrm{SO}(q)$  is a degree-2 isogeny with kernel equal to  $\mu_2 \subset \mathbf{G}_m$ , so Lemma C.4.1 also holds for  $n = 2$ . Likewise, by inspection we still have  $\mathrm{GSpin}(q) = \mathbf{G}_m \times^{\mu_2} \mathrm{Spin}(q)$  when  $n = 2$  as well.

Now we turn to spin groups and other related groups for *odd*  $n = 2m + 1 \geq 1$ . As in the case of even  $n$ , we abuse notation by letting  $\mathrm{C}(V, q)^\times$  denote the “ $S$ -group scheme of units of  $\mathrm{C}(V, q)$ ”. Recall from the discussion following Definition C.2.5 that for even  $n$ , all points of this unit group are locally homogeneous. For odd  $n$  this fails, as we can already see with the  $\mathbf{Z}/2\mathbf{Z}$ -graded center, so the definition of the Clifford group  $\mathbf{GPin}(q)$  for odd  $n$  will require the insertion of a local homogeneity condition that is automatically satisfied for even  $n$ .

The Zariski-closed subgroup scheme  $C(V, q)_{\text{lh}}^\times$  of locally homogeneous units in  $C(V, q)^\times$  meets the unit group  $Z_q^\times = R_{Z_q/S}(\mathbf{G}_m)$  in the group  $(Z_q^\times)_{\text{lh}}$  of locally homogeneous units of  $Z_q$ . The  $S$ -group  $(Z_q^\times)_{\text{lh}}$  is an extension of  $(\mathbf{Z}/2\mathbf{Z})_S$  by  $\mathbf{G}_m$ , consisting of sections that locally lie in either  $\mathbf{G}_m$  or the  $\mathbf{G}_m$ -torsor of local generators of the degree-1 line of  $Z_q$ . Thus, the isomorphism  $Z_q \otimes_{\sigma_S} C_0(V, q) = C(V, q)$  from Proposition C.2.4 shows that  $C(V, q)_{\text{lh}}^\times$  is an extension of  $(\mathbf{Z}/2\mathbf{Z})_S$  by  $C_0(V, q)^\times$  for the Zariski topology. In particular,  $C(V, q)_{\text{lh}}^\times$  is smooth with relative identity component  $C_0(V, q)^\times$  that is a form of  $\text{GL}_{2m}$  ( $n = 2m + 1$ ). Note that  $Z_q^\times$  is a rank-2 torus over  $S[1/2]$  but it has non-reductive fibers at points in characteristic 2. The local homogeneity condition in  $(Z_q^\times)_{\text{lh}}$  eliminates the intervention of such non-reductivity in what follows.

**Definition C.4.3.** For odd  $n \geq 1$ , the *naive Clifford group* over  $S$  is

$$\text{GPin}'(q) = \{u \in C(V, q)^\times \mid uVu^{-1} = V\}$$

and the *Clifford group* over  $S$  is

$$\text{GPin}(q) = \text{GPin}'(q) \cap C(V, q)_{\text{lh}}^\times = \text{GPin}'(q) \cap C(V, q)_{\text{lh}}.$$

**Remark C.4.4.** Note that for even  $n$ , Definition C.4.3 makes sense and we have seen in the discussion after Definition C.2.5 that the two resulting groups coincide. For odd  $n$  they are fiberwise distinct (see Lemma C.4.5). It is  $\text{GPin}'(q)$  rather than  $\text{GPin}(q)$  that is called the *Clifford group* in [Chev97, II, §3]. For odd  $n$  both groups will yield the same “Spin” and “GSpin” groups, but the associated “Pin groups” (see §C.5) will only agree when  $n \equiv 3 \pmod 4$  and  $S$  is a  $\mathbf{Z}[1/2]$ -scheme. In [Chev97, II, §3], the case of odd  $n$  is essentially defined out of consideration in characteristic 2 and Pin groups are not considered.

By the same calculation as with even  $n$  in the proof of Proposition C.2.6, for odd  $n$  and any point  $u$  of  $\text{GPin}'(q)$  the induced linear automorphism  $v \mapsto uvu^{-1}$  of  $V$  lies in  $\text{O}(q)$ , and  $Z_q^\times \subset \ker(\text{GPin}'(q) \rightarrow \text{O}(q) = \mu_2 \times \text{SO}(q))$ .

**Lemma C.4.5.** *For odd  $n$ , the inclusion  $Z_q^\times \times^{(Z_q^\times)_{\text{lh}}} \text{GPin}(q) \subset \text{GPin}'(q)$  is an equality. In particular,  $\text{GPin}'(q)/\text{GPin}(q)$  is smooth with geometric fiber  $\mathbf{G}_m$  away from characteristic 2 and  $\mathbf{G}_a$  in characteristic 2.*

*Proof.* Let  $u'$  be a point of  $\text{GPin}'(q)$  valued in some  $S$ -scheme  $S'$ , so we may rename  $S'$  as  $S$ . Consider the global decomposition  $u'_+ + u'_-$  of  $u'$  as a sum of even and odd parts in  $C(V, q)$ . We claim that Zariski-locally on  $S$ , at least one of  $u'_+$  or  $u'_-$  is a unit in  $C(V, q)$ . To prove this we may and do assume  $S = \text{Spec } k$  for an algebraically closed field  $k$ . By the classical theory over fields (see [Chev97, II.3.2]),  $u' = zu$  where  $u$  is an element of  $\text{GPin}(q)$  and  $z \in Z_q$ . It follows that  $z \in Z_q^\times$ . Since  $C(V, q) = Z_q \otimes C_0(V, q)$ , we can scale  $z$  and  $u$  by reciprocal elements of  $Z_q^1 - \{0\}$  if necessary to arrange that  $u$  has degree 0. In particular,  $u'_\pm = (z_\pm)u$ , where  $z_\pm$  are the homogeneous components of  $z$ . But a nonzero homogeneous element of  $Z_q$  is a *unit*, and  $u$  lies in  $\text{GPin}(q)$ , so one of  $u_\pm$  is a unit in  $C(V, q)$ .

Over the original  $S$ , locally scale  $u'$  by a generator of  $Z_q^1$  if necessary so that  $u'_+$  is a unit. For every local section  $v$  of  $V$ , the local section  $\tilde{v} := u'vu'^{-1}$  of  $V$  satisfies  $u'_+v + u'_-v = u'v = \tilde{v}u' = \tilde{v}u'_+ + \tilde{v}u'_-$ . Comparing odd-degree terms on both sides gives that  $u'_+ \in \text{GPin}(q)$ , so we may assume  $u'_+ = 1$ . Writing  $u = 1 + x$  with  $x$  of degree 1, the relation  $uv = \tilde{v}u$  for  $v \in V$  and  $\tilde{v} := uvu^{-1} \in V$  says  $v + xv = \tilde{v} + \tilde{v}x$ . Comparing odd-degree parts implies  $\tilde{v} = v$ , so  $u$  centralizes  $V$  and hence is central in  $C(V, q)$  (i.e.,  $u \in Z_q^\times$ ).  $\square$

Fix an odd  $n \geq 1$ . The locally constant degree homomorphism  $\text{deg}_q : \text{GPin}(q) \rightarrow (\mathbf{Z}/2\mathbf{Z})_S$  has restriction to  $(Z_q^\times)_{\text{lh}}$  that kills the degree-0 part  $\mathbf{G}_m$  and induces the unique isomorphism  $(Z_q^\times)_{\text{lh}}/\mathbf{G}_m \simeq (\mathbf{Z}/2\mathbf{Z})_S$ . Hence, the subgroup

$$\text{GSpin}(q) := \ker(\text{deg}_q) \subset \text{GPin}(q)$$

is open and closed, with  $\text{GPin}(q)/\text{GSpin}(q) = (\mathbf{Z}/2\mathbf{Z})_S$ . Since  $(Z_q^\times)_{\text{lh}} \cap \text{GSpin}(q) = \mathbf{G}_m$ , we have

$$(Z_q^\times)_{\text{lh}} \times^{\mathbf{G}_m} \text{GSpin}(q) = \text{GPin}(q).$$

In particular,

$$\text{GPin}(q)/\text{GSpin}(q) = (Z_q^\times)_{\text{lh}}/\mathbf{G}_m = (\mathbf{Z}/2\mathbf{Z})_S$$

for odd  $n$ , just as we saw for even  $n$  in (C.4.1).

For odd  $n$ , the calculation (C.2.2) carries over without change and so leads us to define  $\pi_q : \text{GPin}(q) \rightarrow \text{GL}(V)$  by  $\pi_q(u)(v) = (-1)^{\text{deg}_q(u)}uvu^{-1}$  as for even  $n$  (so for  $u \in V \subset C_1(V, q)$  such that  $q(u)$  is a unit, the automorphism  $\pi_q(u) : V \simeq V$  is the reflection through  $u$  relative to  $q$ , not the negative of the reflection). This is the twist of the conjugation action by the  $\mu_2$ -valued character given by exponentiating  $\text{deg}_q$ . Since  $\mu_2$ -scaling has no effect on  $q$ , we see that  $\pi_q$  is valued in  $\text{O}(q)$ . Also, since  $\text{GSpin}(q) \subset C_0(V, q)$  by definition, the restriction of  $\pi_q$  to  $\text{GSpin}(q)$  is the action on  $V$  through conjugation inside  $C(V, q)$ .

**Proposition C.4.6.** *Fix an odd  $n \geq 1$ .*

- (1) *The representation  $\text{GPin}'(q) \rightarrow \text{GL}(V)$  defined by the conjugation action  $u.v = uvu^{-1}$  on  $V$  inside  $C(V, q)$  is valued in  $\text{SO}(q)$ , and defines a diagram of  $S$ -groups*

$$1 \rightarrow Z_q^\times \rightarrow \text{GPin}'(q) \rightarrow \text{SO}(q) \rightarrow 1$$

*that is short exact for the Zariski topology. In particular,  $\text{GPin}'(q)$  is  $S$ -smooth with connected fibers that are reductive over  $S[1/2]$  and non-reductive in characteristic 2.*

- (2) *The homomorphism  $\pi_q : \text{GSpin}(q) \rightarrow \text{O}(q) = \mu_2 \times \text{SO}(q)$  factors through  $\text{SO}(q)$  and defines a diagram of  $S$ -groups*

$$1 \rightarrow \mathbf{G}_m \rightarrow \text{GSpin}(q) \rightarrow \text{SO}(q) \rightarrow 1$$

*that is short exact for the Zariski topology on the category of  $S$ -schemes. In particular,  $\text{GSpin}(q)$  is  $S$ -smooth with connected reductive fibers.*

(3) *The natural map*

$$(\mathbf{Z}/2\mathbf{Z})_S \times \mathrm{SO}(q) = ((Z_q^\times)_{\mathrm{lh}}/\mathbf{G}_m) \times (\mathrm{GSpin}(q)/\mathbf{G}_m) \rightarrow \mathrm{GPin}(q)/\mathbf{G}_m$$

is an isomorphism (so  $\mathrm{GPin}(q)$  is  $S$ -smooth). The composition of its inverse with the canonical homomorphism  $(\mathbf{Z}/2\mathbf{Z})_S \rightarrow \mu_2$  recovers the homomorphism  $\mathrm{GPin}(q)/\mathbf{G}_m \rightarrow \mu_2 \times \mathrm{SO}(q) = \mathrm{O}(q)$  induced by  $\pi_q$ . In particular, over  $S[1/2]$  the  $S$ -group  $\mathrm{GPin}(q)$  is an extension of  $\mathrm{O}(q)$  by  $\mathbf{G}_m$ .

Keep in mind that the representation  $\pi_q$  of  $\mathrm{GPin}(q)$  on  $V$  is the twist of the conjugation action by the quadratic character  $u \mapsto (-1)^{\deg_q(u)}$ .

*Proof.* Rather than directly show that  $\mathrm{GPin}'(q)$  is carried into  $\mathrm{SO}(q)$  as desired in (1), we will first show that all points  $g$  of  $\mathrm{SO}(q)$  Zariski-locally lift to  $\mathrm{GPin}'(q)$  under the conjugation action on  $V$  (without the quadratic twist). Since  $\mathrm{SO}(q) \subset \mathrm{O}(q)$ , by the functoriality of Clifford algebras there exists a unique algebra automorphism  $[g]$  of  $\mathrm{C}(V, q)$  extending  $g$  on  $V$ . Consider the induced automorphism of  $Z_q$ . We claim that this is trivial, due to the condition that  $g$  lies in  $\mathrm{SO}(q)$ . The construction  $g \mapsto [g]|_{Z_q}$  defines a homomorphism of  $S$ -groups  $\mathrm{SO}(q) \rightarrow \underline{\mathrm{Aut}}_{Z_q/S}$  and we are claiming that this is trivial. By Lemma C.2.1 it suffices to consider  $q = q_n$  over  $S = \mathrm{Spec} \mathbf{Z}$ . Since  $\mathrm{SO}_n$  is  $\mathbf{Z}$ -flat, it suffices to treat the problem over  $\mathbf{Z}[1/2]$ , or even over  $\mathbf{Q}$ . We have reduced to the case when  $Z_q$  is étale over  $S$ , so its automorphism scheme is  $(\mathbf{Z}/2\mathbf{Z})_S$ . Since  $\mathrm{SO}(q) \rightarrow S$  is smooth with connected fibers,  $\mathrm{Hom}_{S\text{-gp}}(\mathrm{SO}(q), (\mathbf{Z}/2\mathbf{Z})_S) = 1$ .

Now returning to the original base  $S$ , since  $[g]$  is an automorphism of the  $Z_q$ -algebra  $\mathrm{C}(V, q)$  that becomes a matrix algebra over  $Z_q$  fppf-locally on  $S$ , by the relative Skolem–Noether theorem (as in the proof of Proposition C.2.6) we may Zariski-localize on the base so that  $[g]$  is inner. That is, there exists a unit  $u$  of  $\mathrm{C}(V, q)$  such that  $[g](x) = u x u^{-1}$  for all points  $x$  of  $\mathrm{C}(V, q)$ . Clearly  $u$  is a point of  $\mathrm{GPin}'(q)$  that lifts  $g$ .

Since we have already proved that all points of  $\mathrm{SO}(q)$  Zariski-locally lift into  $\mathrm{GPin}'(q)$ , and  $\mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q)$ , to show that  $\mathrm{GPin}'(q)$  is carried into  $\mathrm{SO}(q)$  it suffices to check that if  $\zeta \in \mu_2(S')$  for an  $S$ -scheme  $S'$  and it arises from  $u \in \mathrm{GPin}'(q)(S')$  then  $\zeta = 1$ . We may and do rename  $S'$  as  $S$ . The automorphism  $v \mapsto \zeta v$  of  $V$  uniquely extends to an algebra automorphism  $f$  of  $\mathrm{C}(V, q)$  that is the identity on  $\mathrm{C}_0(V, q)$  and is multiplication by  $\zeta$  on  $\mathrm{C}_1(V, q)$ . By hypothesis, the inner automorphism  $x \mapsto u x u^{-1}$  of  $\mathrm{C}(V, q)$  agrees with  $f$  on  $V$ , so it agrees with  $f$  (since  $V$  generates  $\mathrm{C}(V, q)$  as an algebra). In particular,  $f$  is trivial on the  $\mathbf{Z}/2\mathbf{Z}$ -graded central subalgebra  $Z_q$ . But the odd part  $Z_q^1$  is an invertible sheaf on which  $f$  acts as multiplication by  $\zeta$ , so  $\zeta = 1$ . This completes the proof of (1).

The representation  $\pi_q$  of  $\mathrm{GSpin}(q) = \mathrm{GPin}'(q) \cap \mathrm{C}_0(V, q)^\times$  on  $V$  is via conjugation inside  $\mathrm{C}(V, q)$  (the quadratic twist is trivial in even degree), so since  $Z_q^\times$  has degree-0 part  $\mathbf{G}_m$  we can deduce (2) from (1) once we show that the quotient map  $\mathrm{GPin}'(q) \rightarrow \mathrm{SO}(q)$  in (1) carries  $\mathrm{GSpin}(q)$  onto  $\mathrm{SO}(q)$  locally for the Zariski topology on  $S$ . By Lemma C.4.5,  $\mathrm{GPin}(q)$  is carried onto  $\mathrm{SO}(q)$  under the Zariski-quotient map in (1). Since scaling by a Zariski-local

generator of  $Z_q^1$  swaps  $\mathrm{GSpin}(q)$  with the fiber of  $\mathrm{GPin}(q) \rightarrow (\mathbf{Z}/2\mathbf{Z})_S$  over 1, it follows that  $\mathrm{GSpin}(q)$  is also carried onto  $\mathrm{SO}(q)$  as required in (2).

The initial isomorphism assertion in (3) is now clear, and for the rest it remains to analyze the  $S$ -homomorphism

$$(Z_q^\times)_{\mathrm{lh}} \xrightarrow{\pi_q} \mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q).$$

The kernel is  $\mathbf{G}_m$ , so it induces an  $S$ -homomorphism

$$(\mathbf{Z}/2\mathbf{Z})_S = (Z_q^\times)_{\mathrm{lh}}/\mathbf{G}_m \rightarrow \mu_2 \times \mathrm{SO}(q).$$

Our problem is to show that the second component of this map is trivial and that the first component is the canonical  $S$ -homomorphism  $(\mathbf{Z}/2\mathbf{Z})_S \rightarrow \mu_2$ . In other words, we have to show that a local generator of  $Z_q^1$  is carried to  $-1 \in \mathrm{O}(q)(S)$  under  $\pi_q$ . But this is obvious, since  $\pi_q$  on  $\mathrm{GPin}(q)$  is defined as the quadratic twist of conjugation inside  $\mathrm{C}(V, q)$  against  $u \mapsto (-1)^{\deg_q(u)}$ .  $\square$

For odd  $n$ , Proposition C.4.6 shows that  $\mathrm{GSpin}(q)$  is the relative identity component of the smooth  $S$ -group  $\mathrm{GPin}(q)$  (which we took as the *definition* of  $\mathrm{GSpin}(q)$  for even  $n$ ), and our definition  $\mathrm{GSpin}(q) = \mathrm{C}_0(V, q) \cap \mathrm{GPin}(q)$  for odd  $n$  was earlier seen to be valid for even  $n$ . Hence, our descriptions of the relationships between  $\mathrm{GPin}(q)$  and  $\mathrm{GSpin}(q)$  for even and odd  $n$  are consistent.

The fibral connectedness of the naive Clifford group  $\mathrm{GPin}'(q)$  for odd  $n$  forces us to use the grading if we wish to define  $\mathrm{GSpin}'(q)$  for any  $n \geq 1$ : it is  $\mathrm{C}_0(V, q) \cap \mathrm{GPin}'(q)$ . But this is  $\mathrm{GSpin}(q)$ , so it provides nothing new.

**Corollary C.4.7.** *Assume  $n \geq 1$  is odd. The subgroup  $\mathrm{GSpin}(q) \subset \mathrm{GPin}(q)$  is a central extension of  $\mathrm{SO}(q)$  by  $\mathbf{G}_m$ , and*

$$\mathrm{GPin}(q) = (Z_q^\times)_{\mathrm{lh}} \times^{\mathbf{G}_m} \mathrm{GSpin}(q), \quad \mathrm{GPin}'(q) = Z_q^\times \times^{\mathbf{G}_m} \mathrm{GSpin}(q).$$

This result is a relative version of [Chev97, II.3.2].

*Proof.* Since  $Z_q^\times$  has degree-0 part  $\mathbf{G}_m$  and local bases of  $Z_q^1$  are units, Lemma C.4.5 and Proposition C.4.6 yield the assertions immediately.  $\square$

For any  $n \geq 1$ , the *main anti-involution*  $\alpha$  of  $\mathrm{C}(V, q)$  is defined as in the proof of Lemma C.4.1: it is the unique anti-involution that restricts to negation on  $V$ , which is to say that it carries  $v_1 \cdots v_r$  to  $(-1)^r v_r \cdots v_1$ . For any  $u \in \mathrm{GPin}'(q)$ , the same calculation as in the proof of Lemma C.4.1 (but setting  $\varepsilon_u = 1$  and  $h(v) = uvu^{-1}$ ) shows that  $\alpha(u)u \in Z_q^\times$  and that the resulting the  $S$ -morphism

$$\nu'_q : \mathrm{GPin}'(q) \rightarrow Z_q^\times$$

is a homomorphism of  $S$ -groups; it is called the *naive Clifford norm*. When  $n$  is even, so  $Z_q = \mathcal{O}_S$ ,  $\nu'_q$  restricts to the squaring map on  $Z_q^\times$ . When  $n$  is odd, we have:

**Lemma C.4.8.** *For odd  $n \geq 1$ ,  $\alpha : Z_q \rightarrow Z_q$  is the identity map when  $n \equiv 3 \pmod{4}$  and is the canonical algebra involution  $z \mapsto \mathrm{Tr}_{Z_q/\mathcal{O}_S}(z) - z$  when  $n \equiv 1 \pmod{4}$ . In particular,  $\nu'_q$*

is the squaring map on  $Z_q^\times$  when  $n \equiv 3 \pmod 4$  and it is the restriction of the algebra norm  $Z_q \rightarrow \mathcal{O}_S$  when  $n \equiv 1 \pmod 4$ .

*Proof.* By working fppf-locally on  $S$ , we may assume  $q = q_n$  with  $n = 2m + 1$ . Hence, if  $\{e_0, \dots, e_{2m}\}$  denotes the standard basis, then we saw in the proof of Proposition C.2.4 that  $Z_q = \mathcal{O}_S \oplus \mathcal{O}_S z_0 = \mathcal{O}_S[t]/(t^2 - 1)$ , where

$$z_0 = e_0 \prod_{i=1}^m (1 - 2e_{2i-1}e_{2i})$$

and  $e_{2i-1}e_{2i} + e_{2i}e_{2i-1} = 1$ . The main anti-involution  $\alpha$  carries  $e_j$  to  $-e_j$ , so  $\alpha(1 - 2e_{2i-1}e_{2i}) = -(1 - 2e_{2i-1}e_{2i})$ . Hence,  $\alpha(z_0) = (-1)^{m+1}z_0$ , so  $\alpha$  is the identity on  $Z_q$  when  $m$  is odd and is the canonical involution  $x \mapsto \text{Tr}_{Z_q/\mathcal{O}_S}(x) - x$  when  $m$  is even.  $\square$

Similar calculations (using Remark C.2.3) show that for even  $n$ , the effect of  $\alpha$  on the rank-2 finite étale center of  $C_0(V, q)$  is the identity when  $4|n$  and is the unique fiberwise nontrivial automorphism when  $n \equiv 2 \pmod 4$ .

We are primarily interested in the restriction  $\nu_q$  of  $\nu'_q$  to the subgroup  $\text{GPin}(q)$  of locally homogeneous points  $u$  for any  $n \geq 1$ . For such  $u$ ,  $\nu_q(u)$  is the product of the locally homogeneous points  $u$  and  $\alpha(u)$  of the same degree, so  $\nu_q(u)$  is valued in the degree-0 part  $\mathbf{G}_m$  of  $Z_q^\times$ . This is the *Clifford norm*

$$(C.4.3) \quad \nu_q : \text{GPin}(q) \rightarrow \mathbf{G}_m$$

for any  $n \geq 1$ .

Now assume  $n = 2m + 1 \geq 3$  (so  $m \geq 1$ ). The group  $\text{SO}(q)$  is semisimple of type  $B_m$ , so the derived group  $\text{Spin}(q) := \mathcal{D}(\text{GSpin}(q))$  is a semisimple  $S$ -group that is a central extension of  $\text{SO}(q)$  by a finite fppf subgroup of  $\mathbf{G}_m$  (Proposition 6.1.10). The group  $\text{SO}(q)$  is adjoint since  $n$  is odd, and calculations with  $B_m$  show that the simply connected central cover of  $\text{SO}(q)$  has degree 2. We can now adapt arguments in the proof of Lemma C.4.1 to show that the central isogeny  $\pi_q : \text{Spin}(q) \rightarrow \text{SO}(q)$  has kernel equal to  $\mu_2 \subset \mathbf{G}_m$ , so this is the simply connected central cover:

**Proposition C.4.9.** *Assume  $n \geq 3$  is odd. The map  $\pi_q$  identifies  $\text{Spin}(q)$  with the simply connected central cover of  $\text{SO}(q)$ .*

*Proof.* By Lemma C.2.1, it suffices to treat the case  $q = q_n$  over  $\mathbf{Z}$ . The finite multiplicative type kernel  $\ker \pi_q$  must be flat and so has constant degree that must be 1 or 2, so we just have to rule out the possibility of degree 1. Equivalently, it suffices to prove that over a field  $k$  of characteristic  $\neq 2$  (or even just algebraically closed of characteristic 0), the central subgroup  $\mu_2 \subset \mathbf{G}_m$  in  $\text{GSpin}(q)$  is contained in the derived group  $\text{Spin}(q)$ .

The first step is to reduce to the case  $n = 3$ . We may assume  $q = q_n$  with  $n = 2m + 1 \geq 3$ . Let  $V_3 \subset V = k^n$  be the span of  $\{e_0, e_1, e_2\}$  (so  $q|_{V_3} = q_3$ ) and let  $V'$  be the span of  $\{e_3, \dots, e_{2m}\}$  (so  $V' = 0$  if  $n = 3$ ). The inclusion  $(V_3, q_3) \hookrightarrow (V, q)$  induces an injective homomorphism of  $\mathbf{Z}/2\mathbf{Z}$ -graded algebras  $j : C(q_3) \rightarrow C(V, q)$ . Since  $V_3$  is orthogonal to  $V'$ ,

the *even* subalgebra  $C_0(q_3)$  centralizes  $C(V', q|_{V'}) \subset C(V, q)$ . Hence, a unit in  $C_0(q_3)^\times$  whose conjugation action on  $C_0(q_3)$  preserves  $V_3$  is carried into the group  $\mathrm{GSpin}(q)$  of even units in  $C(V, q)$  whose conjugation action on  $C(V, q)$  preserves  $V$ . In other words,  $j$  carries  $\mathrm{GSpin}(q_3)$  into  $\mathrm{GSpin}(q)$ , so on derived groups it carries  $\mathrm{Spin}(q_3)$  into  $\mathrm{Spin}(q)$ . But  $j$  carries  $\mathbf{G}_m$  to  $\mathbf{G}_m$  via the identity map, so to prove that  $\mu_2 \subset \mathrm{Spin}(q)$  it suffices to treat the case of  $q_3$ . That is, we may and do now assume  $n = 3$ .

If  $\mathrm{Spin}(q)$  does not contain the central  $\mu_2$  then the natural map  $\mathrm{Spin}(q) \rightarrow \mathrm{SO}(q)$  is an isomorphism. The inverse would provide a section to  $\mathrm{GSpin}(q) \rightarrow \mathrm{SO}(q)$  that identifies  $\mathrm{GSpin}(q)$  with  $\mathbf{G}_m \times \mathrm{SO}(q)$ . In particular, the kernel of the Clifford norm  $\nu_q : \mathrm{GSpin}(q) \rightarrow \mathbf{G}_m$  would equal  $\mu_2 \times \mathrm{SO}(q)$ , which is disconnected (since  $\mathrm{char}(k) \neq 2$ ). Hence, it suffices to prove that the kernel of  $\nu_q$  on  $\mathrm{GSpin}(q)$  is irreducible.

For the standard basis  $\{e_0, e_1, e_2\}$  of  $V_3$ , let  $e = e_0e_1, e' = e_0e_2, e'' = e_1e_2$ , so  $\{1, e, e', e''\}$  is a  $k$ -basis of  $C_0(q_3)$ . We claim that the inclusion  $\mathrm{GSpin}(q_3) \subset C_0(q_3)^\times$  between  $k$ -groups is an equality. Since  $\mathrm{GSpin}(q_3)$  is smooth and connected of dimension 4 (a central extension of  $\mathrm{SO}_3$  by  $\mathbf{G}_m$ ), it suffices to show the same for  $C_0(q_3)^\times$ . For any finite-dimensional associative algebra  $A$  over a field  $F$ , the associated  $F$ -group  $\underline{A}^\times$  of units is defined by the non-vanishing on the affine space  $\underline{A}$  of the determinant of the left multiplication action of  $A$  on itself. In particular,  $\underline{A}^\times$  is smooth and connected of dimension equal to  $\dim_F A$ . Hence,  $C_0(q_3)^\times$  is smooth and connected of dimension 4 as desired.

The algebra structure on  $C_0(q_3)$  is determined by the relations

$$(C.4.4) \quad e^2 = e'^2 = 0, e''^2 = e'', ee' = -e'', e'e = e'' - 1, e'e'' = e', e''e' = 0, ee'' = 0, e''e = e,$$

and the restriction to  $C_0(q_3)$  of the main anti-involution  $\alpha$  of  $C(q_3)$  is given by

$$u = t + ae + be' + ce'' \mapsto (t + c) - ae - be' - ce'' =: u^*.$$

Using (C.4.4) we compute that  $\nu(u) := u^*u = ab + t^2 + ct$ . By inspection,  $\nu - 1$  is an irreducible polynomial in  $k[t, a, b, c]$  and hence its non-unit restriction over the open subset  $\mathrm{GSpin}(q_3)$  in the affine 4-space  $C_0(q_3)$  (which has zero locus  $\mathrm{Spin}(q_3)$ ) has irreducible zero locus. □

Fix an odd  $n \geq 1$ . If  $n \geq 3$  then  $\mu_2 \subset \mathrm{Spin}(q)$  and this must exhaust the order-2 center, so  $\mathrm{GSpin}(q) = \mathrm{Spin}(q) \times^{\mu_2} \mathbf{G}_m$  (by Proposition C.4.6(2)) and hence Corollary C.4.7 gives

$$\mathrm{GPin}(q) = \mathrm{Spin}(q) \times^{\mu_2} (Z_q^\times)_{\mathrm{th}}, \quad \mathrm{GPin}'(q) = \mathrm{Spin}(q) \times^{\mu_2} Z_q^\times.$$

These hold for  $n = 1$  by *defining*  $\mathrm{Spin}(q) := \mu_2$  inside  $\mathrm{GSpin}(q) = \mathbf{G}_m$  when  $n = 1$ .

**C.5. Pin groups.** For any  $n \geq 1$ , the *Pin group*  $\mathrm{Pin}(q)$  is the kernel of the Clifford norm  $\nu_q : \mathrm{GPin}(q) \rightarrow \mathbf{G}_m$  defined by  $u \mapsto \alpha(u)u$  as in (C.4.3). For odd  $n \geq 1$ , the *naive Pin group*  $\mathrm{Pin}'(q)$  is the kernel of the naive Clifford norm  $\nu'_q : \mathrm{GPin}'(q) \rightarrow Z_q^\times$  (so  $\mathrm{Pin}(q) \subset \mathrm{Pin}'(q)$ ).

**Remark C.5.1.** If we use the modified Clifford norm as in Remark C.4.2 then we obtain a subgroup of  $\mathrm{GPin}(q)$  distinct from  $\mathrm{Pin}(q)$  in general.

Suppose  $n$  is even, so (as we saw early in §C.4)  $\mathrm{GSpin}(q)$  is an open and closed subgroup of  $\mathrm{GPin}(q)$ . The kernel of the Clifford norm on  $\mathrm{GSpin}(q)$  is the smooth  $S$ -group  $\mathrm{Spin}(q)$  (as we saw below Remark C.4.2), so  $\mathrm{Spin}(q)$  is an open and closed subgroup of  $\mathrm{Pin}(q)$ . But  $\mathrm{GPin}(q)$  maps onto  $\mathrm{O}(q)$  with kernel  $\mathbf{G}_m$ , and this  $\mathbf{G}_m$  subgroup is carried onto  $\mathbf{G}_m$  under  $\nu_q$  via  $t \mapsto t^2$ , so the kernel  $\mathrm{Pin}(q)$  of  $\nu_q$  fits into an fppf central extension

$$1 \rightarrow \mu_2 \rightarrow \mathrm{Pin}(q) \rightarrow \mathrm{O}(q) \rightarrow 1.$$

The subgroup  $\mathrm{Spin}(q) \subset \mathrm{Pin}(q)$  is compatibly a central extension of  $\mathrm{SO}(q)$  by  $\mu_2$  (by Lemma C.4.1 when  $n \geq 4$ , and direct arguments with the definition of  $\mathrm{Spin}(q)$  when  $n = 2$ ), so  $\mathrm{Pin}(q)/\mathrm{Spin}(q) = \mathrm{O}(q)/\mathrm{SO}(q) = (\mathbf{Z}/2\mathbf{Z})_S$  when  $n$  is even. Hence,  $\mathrm{Pin}(q)$  is a smooth affine  $S$ -group for even  $n$ . For odd  $n$  the analogous assertions will be more subtle to prove, due to complications in characteristic 2 (as we shall see).

Next, suppose  $n \geq 1$  is odd. By Lemma C.4.8, if  $n \equiv 3 \pmod{4}$  then

$$(C.5.1) \quad \mathrm{Pin}(q) = \mathrm{Spin}(q) \times^{\mu_2} (Z_q^\times)_{\mathrm{lh}}[2], \quad \mathrm{Pin}'(q) = \mathrm{Spin}(q) \times^{\mu_2} Z_q^\times[2],$$

whereas if  $n \equiv 1 \pmod{4}$  then

$$(C.5.2) \quad \mathrm{Pin}(q) = \mathrm{Spin}(q) \times^{\mu_2} \ker(\mathrm{N}_{Z_q/S}|_{(Z_q^\times)_{\mathrm{lh}}}), \quad \mathrm{Pin}'(q) = \mathrm{Spin}(q) \times^{\mu_2} \ker(\mathrm{N}_{Z_q/S}|_{Z_q^\times}).$$

Hence, if  $n \equiv 3 \pmod{4}$  then  $\mathrm{Pin}(q)/\mathrm{Spin}(q) = (Z_q^\times)_{\mathrm{lh}}[2]/\mu_2$  and  $\mathrm{Pin}'(q)/\mathrm{Spin}(q) = Z_q^\times[2]/\mu_2$  whereas if  $n \equiv 1 \pmod{4}$  then  $\mathrm{Pin}(q)/\mathrm{Spin}(q) = \ker(\mathrm{N}_{Z_q/S}|_{(Z_q^\times)_{\mathrm{lh}}})/\mu_2$  and  $\mathrm{Pin}'(q)/\mathrm{Spin}(q) = \ker(\mathrm{N}_{Z_q/S}|_{Z_q^\times})/\mu_2$ .

The norm map  $\mathrm{N}_{Z_q/S} : Z_q^\times \rightarrow \mathbf{G}_m$  is an fppf surjection whose kernel is a torus over  $S[1/2]$  and has geometric fiber  $\mathbf{G}_a$  in characteristic 2. Hence, if  $n \equiv 1 \pmod{4}$  then  $\mathrm{Pin}'(q)$  is smooth. We will see later that if  $n \equiv 3 \pmod{4}$  then  $\mathrm{Pin}'(q_n)$  is not even  $\mathbf{Z}$ -flat (due to dimension-jumping of the fiber at  $\mathrm{Spec} \mathbf{F}_2$ ).

**Proposition C.5.2.** *Fix an odd  $n \geq 1$ . The quotient  $\mathrm{Pin}(q)/\mathrm{Spin}(q)$  is identified with  $(\mathbf{Z}/2\mathbf{Z})_S$  (so  $\mathrm{Pin}(q)$  is smooth) and there is a natural central extension*

$$1 \rightarrow \mu_2 \rightarrow \mathrm{Pin}(q) \rightarrow (\mathbf{Z}/2\mathbf{Z})_S \times \mathrm{SO}(q) \rightarrow 1$$

whose restriction over  $S[1/2]$  recovers  $\pi_q : \mathrm{Pin}(q) \rightarrow \mathrm{O}(q)$  as a central extension by  $\mu_2$ .

The quotient  $\mathrm{Pin}'(q)/\mathrm{Spin}(q)$  has geometric fiber  $\mathbf{G}_a$  in characteristic 2, and its restriction over  $S[1/2]$  is  $(\mathbf{Z}/2\mathbf{Z})_S$  when  $n \equiv 3 \pmod{4}$  and is a rank-1 torus when  $n \equiv 1 \pmod{4}$ . In particular, for odd  $n$ ,  $\mathrm{Pin}(q) = \mathrm{Pin}'(q)$  if and only if  $n \equiv 3 \pmod{4}$  and  $S$  is a  $\mathbf{Z}[1/2]$ -scheme.

In view of the preceding observations for even  $n$ , we conclude that  $\mathrm{Pin}(q)/\mathrm{Spin}(q) = (\mathbf{Z}/2\mathbf{Z})_S$  for all  $n$ . (This is also proved in [Knus, IV, (6.4.1)].) Since  $\mathrm{O}(q)/\mathrm{SO}(q) = \mu_2$  for odd  $n$ , this explains why  $\pi_q : \mathrm{Pin}(q) \rightarrow \mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q)$  is not an fppf quotient map on characteristic-2 fibers when  $n$  is odd.

*Proof.* First assume  $n \equiv 3 \pmod{4}$ . The exact sequence

$$1 \rightarrow \mathbf{G}_m \rightarrow (Z_q^\times)_{\mathrm{lh}} \rightarrow (\mathbf{Z}/2\mathbf{Z})_S \rightarrow 1$$

for the fppf topology (for any odd  $n$ ) identifies  $(Z_q^\times)_{\text{lh}}[2]/\mu_2$  with  $(\mathbf{Z}/2\mathbf{Z})_S$ , so if  $n \equiv 3 \pmod 4$  then  $\text{Pin}(q)/\text{Spin}(q) = (\mathbf{Z}/2\mathbf{Z})_S$  by (C.5.1). Likewise,  $\text{Pin}'(q)/\text{Spin}(q) = Z_q^\times[2]/\mu_2 = (Z_q^\times/\mathbf{G}_m)[2]$ , and over  $S[1/2]$  this is uniquely isomorphic to the constant group  $(\mathbf{Z}/2\mathbf{Z})$  but its geometric fibers in characteristic 2 are  $\mathbf{G}_a$ .

Next assume  $n \equiv 1 \pmod 4$ . Under the identification of  $(Z_{q_n}^\times)_{\text{lh}}$  with  $\mathbf{G}_m \times (\mathbf{Z}/2\mathbf{Z})_S$  as in the proof of Lemma C.4.8,  $N_{Z_{q_n}/S} : (Z_{q_n}^\times)_{\text{lh}} \rightarrow \mathbf{G}_m$  is identified with the product of the squaring map on  $\mathbf{G}_m$  and the canonical map  $(\mathbf{Z}/2\mathbf{Z})_S \rightarrow \mu_2 \hookrightarrow \mathbf{G}_m$ , so  $\ker(N_{Z_{q_n}/S}|_{(Z_{q_n}^\times)_{\text{lh}}})$  is identified with the subgroup of  $\mathbf{G}_m \times (\mathbf{Z}/2\mathbf{Z})_S$  consisting of points  $(\zeta, c)$  such that  $\zeta^2 = (-1)^c$ . Hence, the finite subgroup scheme

$$\text{Pin}(q)/\text{Spin}(q) \subset (\mu_4/\mu_2) \times (\mathbf{Z}/2\mathbf{Z})_S \simeq \mu_2 \times (\mathbf{Z}/2\mathbf{Z})_S$$

is (uniquely) isomorphic to  $(\mathbf{Z}/2\mathbf{Z})_S$  since  $\mu_4/\mu_2 \simeq \mu_2$  via  $\zeta \mapsto \zeta^2$ . In contrast,  $\text{Pin}'(q)/\text{Spin}(q) = \ker(N_{Z_q/S}|_{Z_q^\times})/\mu_2$  is a rank-1 torus over  $S[1/2]$  and has geometric fiber  $\mathbf{G}_a$  in characteristic 2 since  $Z_q$  is étale of rank 2 over  $S[1/2]$  and has geometric fiber algebra  $k[x]/(x^2 - 1)$  in characteristic 2.  $\square$

**Remark C.5.3.** The jumping of fiber dimension implies that for  $n \equiv 3 \pmod 4$  the naive Pin group  $\text{Pin}'_n := \text{Pin}'(q_n)$  over  $\mathbf{Z}$  is not flat at  $\text{Spec } \mathbf{F}_2$  (whereas we have seen that  $\text{Pin}'_n$  is smooth when  $n \equiv 1 \pmod 4$ ).

**C.6. Accidental isomorphisms.** The study of (special) orthogonal groups provides many examples of *accidental isomorphisms* between low-dimensional members of distinct “infinite families” of algebraic groups. This is analogous to the isomorphisms between small members of distinct “infinite families” of finite groups. For example, using the hyperplane  $H = \{\sum x_i = 0\} \subset \mathbf{A}^n$  over  $\mathbf{Z}$ , the quadratic form  $q = \sum x_i^2$  on  $H$  over  $\mathbf{Z}[1/2]$ , and the line  $L = \{x_1 = \dots = x_n\} \subset H$  over  $\mathbf{F}_p$  with  $p|n$ , the  $S_n$ -action on  $\mathbf{A}^n$  yields isomorphisms:

- $S_3 \simeq \text{SL}_2(\mathbf{F}_2)$  (use  $H \subset \mathbf{A}^3$ ),
- $S_4 \simeq \text{PGL}_2(\mathbf{F}_3)$  (identify  $\mathbf{P}_{\mathbf{F}_3}^1$  with the smooth conic  $\{q = 0\}$  in  $\mathbf{P}(H^*)$  for  $H \subset \mathbf{A}_{\mathbf{F}_3}^4$ ),
- $S_5 \simeq \text{PGL}_2(\mathbf{F}_5)$  (identify  $\mathbf{P}_{\mathbf{F}_5}^1$  with the smooth conic  $\{\bar{q} = 0\}$  in the plane  $\mathbf{P}((H/L)^*)$ , where  $H \subset \mathbf{A}_{\mathbf{F}_5}^5$  and  $L$  is the defect line for  $q$  on  $H$ ),
- $A_6 \simeq \text{SL}_2(\mathbf{F}_9)/\langle -1 \rangle$  (for  $H \subset \mathbf{A}_{\mathbf{F}_3}^6$  and the defect line  $L$  of  $q$  on  $H$ ,  $(H/L, \bar{q}) \simeq x^2 + y^2 + z^2 - t^2$  and the  $S_6$ -action on the projective quadric  $\{\bar{q} = 0\} \subset \mathbf{P}((H/L)^*)$  defines a map  $S_6 \rightarrow \text{O}(\bar{q})(\mathbf{F}_3)/\langle -1 \rangle \subset (\text{O}(\bar{q})/\mu_2)(\mathbf{F}_3)$  that must be injective and carry the simple  $A_6$  into the normal subgroup  $\text{Spin}(\bar{q})(\mathbf{F}_3)/Z_{\text{Spin}(\bar{q})}(\mathbf{F}_3)$  that is identified with  $\text{SL}_2(\mathbf{F}_9)/\langle -1 \rangle$  in Remark C.6.4),
- $S_6 \simeq \text{Sp}_4(\mathbf{F}_2)$  (use  $(H/L, \psi)$  where  $H \subset \mathbf{A}_{\mathbf{F}_2}^6$ ,  $L$  is the defect line for  $Q = \sum_{i < j} x_i x_j$  on  $H$ , and  $\psi$  is induced by the symplectic form  $B_Q(x, y) = \sum_{i \neq j} x_i y_j$ ),
- $S_8 \simeq \text{O}_6(\mathbf{F}_2)$  (use the non-degenerate quadratic space  $(H/L, \bar{Q})$  where  $H \subset \mathbf{A}_{\mathbf{F}_2}^8$ ,  $Q = \sum_{i < j} x_i x_j$ , and  $L$  is the defect line for  $Q|_H$ ).

When accidental isomorphisms among algebraic groups are applied to rational points over *finite* fields one obtains some of the accidental isomorphisms among small finite simple

groups. This is seen in the discussion of  $A_6$  above. As another illustration, in Example C.6.6 we will see that  $\mathrm{SL}_4/\mu_2 \simeq \mathrm{SO}_6$  as  $\mathbf{Z}$ -groups, so by passing to  $\mathbf{F}_2$ -points and using the vanishing of  $H^1(\mathbf{F}_2, \mu_2)$  (fppf Kummer theory) we see that  $\mathrm{SL}_4(\mathbf{F}_2) = \mathrm{SO}_6(\mathbf{F}_2)$ , an index-2 subgroup of  $\mathrm{O}_6(\mathbf{F}_2) = S_8$ . The only such subgroup is  $A_8$ , so  $\mathrm{SL}_4(\mathbf{F}_2) = A_8$ .

The case of  $A_6$  above can be pushed a bit further. First, we note that  $S_6 \not\cong \mathrm{PGL}_2(\mathbf{F}_9)$ , despite the isomorphism between their index-2 perfect commutator subgroups, since  $S_6$  has no element of order 8. More interesting is that we can interpret the nontrivial outer automorphism of  $S_6$ . (For  $n > 2$ ,  $\mathrm{Out}(S_n)$  is trivial for  $n \neq 6$  and has order 2 for  $n = 6$ .) This rests on an isomorphism associated to any non-degenerate quadratic space  $(W, Q)$  of rank 4 over a finite field  $k$ : by Example C.6.3 below, for some degree-2 étale  $k$ -algebra  $k'$ ,

$$\mathrm{O}(Q)/\mu_2 \simeq \mathrm{Aut}_{\mathbf{R}_{k'/k}(\mathbf{P}_{k'}^1)/k} = \mathbf{R}_{k'/k}(\mathrm{PGL}_2) \rtimes \langle -1 \rangle$$

by using the nontrivial  $k$ -automorphism of  $k'$  to define the action of  $\langle -1 \rangle$ . On identity components this gives  $\mathrm{SO}(Q)/\mu_2 \simeq \mathbf{R}_{k'/k}(\mathrm{PGL}_2)$ , so if  $Q$  is not  $k$ -split (so  $\mathrm{SO}(Q)/\mu_2$  is not  $k$ -split) then  $k'$  must be a field. We conclude that  $S_6$  is a subgroup of  $\mathrm{PGL}_2(\mathbf{F}_9) \rtimes \langle -1 \rangle$  such that  $A_6$  is the index-4 image of  $\mathrm{SL}_2(\mathbf{F}_9)$ . Thus,  $S_6$  is an index-2 subgroup of  $\mathrm{PGL}_2(\mathbf{F}_9) \rtimes \langle -1 \rangle$  (it is the unique one distinct from  $\mathrm{PGL}_2(\mathbf{F}_9)$  that does not contain the Galois involution). This index-2 inclusion defines the nontrivial outer automorphism of  $S_6$ .

Just as isomorphisms among small finite groups are due to the limited possibilities for finite groups of small size, accidental isomorphisms between low-dimensional semisimple groups are due to a limitation in the possibilities for a “small” case of the root datum that governs the (geometric) isomorphism class of a connected semisimple group. We now work out the accidental isomorphisms for  $\mathrm{SO}_n$  with  $2 \leq n \leq 6$ ; for  $3 \leq n \leq 6$  these correspond to equalities between distinct classical infinite families of root systems, as we will explain. (See [KMRT, IV, §15] for further discussion over fields, using algebras with involution, and [Knus, V] for further discussion over rings.) In what follows,  $(V, q)$  is a quadratic space (so  $q$  is  $\mathcal{O}_S$ -valued and non-degenerate); some aspects go through with little change for  $L$ -valued  $q$ , especially when the odd part of the Clifford algebra does not intervene.

**Example C.6.1.** Suppose  $n = 2$ . In this case, Lemma C.2.1 and calculations with  $q_2 = xy$  imply that  $\mathrm{SO}(q)$  is a rank-1 torus and  $Z := \{q = 0\} = \mathrm{Proj}(\mathrm{Sym}(V)/(q)) \subset \mathbf{P}(V^*)$  is a degree-2 finite étale  $S$ -scheme. (There is also a natural isomorphism between  $\mathrm{GSpin}(q)$  and the Weil restriction torus  $\mathbf{R}_{Z/S}(\mathbf{G}_m)$ .)

By using the left action of  $\mathrm{PGL}(V)$  on  $\mathbf{P}(V^*)$  we can interpret the torus  $\mathrm{SO}(q)/\mu_2$  in an interesting way, as follows. Consideration of the relative homogenous coordinate ring shows that the  $\mathrm{PGL}(V)$ -stabilizer  $G_q$  of  $Z$  is  $\mathrm{GO}(q)/\mathbf{G}_m = \mathrm{O}(q)/(\mathrm{O}(q) \cap \mathbf{G}_m)$ . Calculation with  $q_2$  shows that the intersection  $\mathrm{O}(q) \cap \mathbf{G}_m$  with the central  $\mathbf{G}_m$  in  $\mathrm{GL}(V)$  is  $\mu_2$ . We claim that the resulting map

$$\alpha_q : \mathrm{O}(q) \twoheadrightarrow \mathrm{O}(q)/\mu_2 = G_q \rightarrow \underline{\mathrm{Aut}}_S(Z) = (\mathbf{Z}/2\mathbf{Z})_S$$

is the Dickson invariant (e.g., it is fiberwise nontrivial). It would then follow that the subgroup  $\mathrm{SO}(q)/\mu_2 \subset \mathrm{SL}(V)/\mu_2 = \mathrm{PGL}(V)$  is identified with the kernel of  $G_q \rightarrow \underline{\mathrm{Aut}}_S(Z)$ .

To check the comparison with the Dickson invariant we may assume  $q = q_2$  over  $\mathbf{Z}$ . Since

$$O_2 = T \amalg T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ for } T = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \right\} = \mathbf{G}_m,$$

it is clear that  $\alpha_q$  is fiberwise nontrivial. Hence, connectedness considerations over  $\mathbf{Q}$  imply  $\alpha_q = D_q$ . (We also recover from the explicit description of  $O_2$  that if  $\text{char}(k(s)) = 2$  then  $\text{SO}'(q_s)$  coincides with the disconnected  $O(q_s)$ , since visibly  $\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1 \equiv 1 \pmod{2}$ .)

**Example C.6.2.** Suppose  $n = 3$ . This case corresponds to the equality  $A_1 = B_1$  for adjoint groups. Consider the zero scheme  $C_q = \{q = 0\} \subset \mathbf{P}(V^*)$ . This is a smooth conic in a  $\mathbf{P}^2$ -bundle (as we see by fppf-local calculation with  $q_3$ ), so it is a  $\mathbf{P}^1$ -bundle. The automorphism scheme  $\text{Aut}_{C_q/S}$  is therefore a form of  $\text{Aut}_{\mathbf{P}^1/S} = \text{PGL}_2$ . Under the action of  $\text{PGL}(V)$  on  $\mathbf{P}(V^*)$ , the stabilizer  $G_q$  of  $C_q$  is  $\text{GO}(q)/\mathbf{G}_m = \text{SO}(q)$ , and we claim that the action map

$$\alpha_q : \text{SO}(q) \rightarrow \text{Aut}_{C_q/S}$$

to a form of  $\text{PGL}_2$  is an isomorphism. (In the split case, this is a map  $\text{SO}_3 \rightarrow \text{PGL}_2$ .)

The source and target of  $\alpha_q$  are smooth with fibers that are connected of the same dimension, and it is easy to check that  $\alpha_q$  is injective on geometric points (since a smooth plane conic over an algebraically closed field contains many triples of points in general position). Thus,  $\alpha_q$  is a purely inseparable isogeny on fibers over each  $s \in S$ , so by Proposition 6.1.10 (and Definition 3.3.8) it is finite flat as an  $S$ -morphism. To prove that  $\alpha_q$  has constant degree 1, we may first pass to  $q = q_3$  over  $\mathbf{Z}$  (by Lemma C.2.1) and then check over  $\mathbf{Q}$ . A purely inseparable isogeny in characteristic 0 is an isomorphism, so we are done.

In the special case  $q = q_3$ , an isomorphism in the opposite direction can be described by the following alternative procedure. Consider the linear “conjugation” action of  $\text{PGL}_2 = \text{GL}_2/\mathbf{G}_m$  on the rank-3 affine space  $\mathfrak{sl}_2$ . This action preserves the non-degenerate quadratic form  $Q$  on  $\mathfrak{sl}_2$  given by the determinant. Explicitly,  $Q \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = -(x^2 + yz)$  is, up to sign, the quadratic form  $q_3$ . Preservation of  $q_3$  is the same as that of  $-q_3$ , so the sign does not affect the orthogonal group. Thus, we obtain a homomorphism  $\text{PGL}_2 \rightarrow O_3 = \mu_2 \times \text{SO}_3$  over  $\mathbf{Z}$  with trivial kernel. By computing over  $\mathbf{Q}$ , the map to the  $\mu_2$ -factor must be trivial. Thus, the map  $\text{PGL}_2 \rightarrow O_3$  factors through  $\text{SO}_3$ . Since  $\text{PGL}_2$  is smooth and fiberwise connected of dimension 3, it follows that the monic map  $\text{PGL}_2 \rightarrow \text{SO}_3$  is an isomorphism on fibers and hence is an isomorphism (Lemma B.3.1). We leave it to the interested reader to relate this isomorphism to  $\alpha_{q_3}^{-1}$  (relative to a suitable isomorphism  $C_{q_3} \simeq \mathbf{P}^1$ ).

**Example C.6.3.** Suppose  $n = 4$ . This case corresponds to the equality of root systems  $D_2 = A_1 \times A_1$ . More specifically, in this case  $\text{SO}(q)$  is not “absolutely simple”; i.e., on geometric fibers it contains nontrivial smooth connected proper normal subgroups. (This is the only  $n \geq 3$  for which that happens.) In more concrete terms, we claim that

$$(C.6.1) \quad (\text{SL}_2 \times \text{SL}_2)/M \simeq \text{SO}_4$$

with  $M = \mu_2$  diagonally embedded in the evident central manner. As in Example C.6.2, there is also a geometrically-defined isomorphism in the opposite direction from  $\mathrm{SO}(q)/\mu_2$  onto an adjoint semisimple group of type  $A_1 \times A_1$  for *any*  $q$ .

First we explain the concrete isomorphism (C.6.1) for  $q = q_4$ , since the relative geometry underlying the isomorphism in the opposite direction for general  $q$  is more complicated than in Example C.6.2. Apply a sign to the third standard coordinate to convert  $q_4$  into  $Q = x_1x_2 - x_3x_4$ , which we recognize as the determinant of a  $2 \times 2$  matrix. The group  $\mathrm{SL}_2$  acts on the rank-4 space of such matrices in two evident commuting ways, via  $(g, g').x = gxg'^{-1}$ , and these actions preserve the determinant by the definition of  $\mathrm{SL}_2$ . This defines a homomorphism  $\mathrm{SL}_2 \times \mathrm{SL}_2 \rightarrow \mathrm{SO}'(Q) \simeq \mathrm{SO}'_4$  whose kernel is easily seen to be  $M$ . This map visibly lands in  $\mathrm{SO}_4$  since  $\mathrm{SL}_2$  is fiberwise connected and  $\mathrm{O}_4/\mathrm{SO}_4 = \mathbf{Z}/2\mathbf{Z}$ . Hence, we obtain a monomorphism  $(\mathrm{SL}_2 \times \mathrm{SL}_2)/M \rightarrow \mathrm{SO}_4$  that must be an isomorphism on fibers (as both sides have smooth connected fibers of the same dimension), and therefore an isomorphism.

Passing to general  $q$  with  $n = 4$ , we now build an isomorphism  $\varphi : \mathrm{SO}(q)/\mu_2 \simeq \mathrm{R}_{S'/S}(G')$  for a canonically associated degree-2 finite étale cover  $S' \rightarrow S$  and  $S'$ -form  $G'$  of  $\mathrm{PGL}_2$ . (For  $q = q_4$  we will have  $S' = S \amalg S$  and  $G' = \mathrm{PGL}_2$  over  $S'$ , yielding a canonical isomorphism

$$\mathrm{SO}_4/\mu_2 \simeq \mathrm{PGL}_2 \times \mathrm{PGL}_2$$

that the interested reader can relate to the isomorphism of adjoint groups induced by (C.6.1).) There are two methods to construct  $(S' \rightarrow S, G', \varphi)$ : an algebraic method via Clifford algebras and a geometric method via automorphism schemes. The algebraic method is simpler, so we explain that one first (see [Knus, V, 4.4] for a related discussion).

By Proposition C.2.2 (and its proof), the even part  $\mathrm{C}_0(V, q)$  of the Clifford algebra  $\mathrm{C}(V, q)$  is a quaternion algebra (i.e., rank-4 Azumaya algebra) over a rank-2 finite étale  $\mathcal{O}_S$ -algebra  $Z_q$ . Denote by  $\mathrm{C}_0(V, q)^\times$  the associated unit group over  $Z_q$ . This is an inner form of  $\mathrm{GL}_2$  over  $Z_q$ , and  $\mathrm{GSpin}(q)$  is a reductive closed subgroup of  $\mathrm{R}_{Z_q/S}(\mathrm{C}_0(V, q)^\times)$ . The derived group  $\mathrm{Spin}(q)$  has fibers of dimension 6, yet the reductive  $\mathrm{R}_{Z_q/S}(\mathrm{C}_0(V, q)^\times)$  also has derived group with fibers of dimension 6, so the containment

$$\mathrm{Spin}(q) \subset \mathcal{D}(\mathrm{R}_{Z_q/S}(\mathrm{C}_0(V, q)^\times))$$

is an equality. Passing to adjoint quotients, we get

$$\mathrm{SO}(q)/\mu_2 = \mathrm{Spin}(q)/Z_{\mathrm{Spin}(q)} \simeq \mathrm{R}_{Z_q/S}(\mathrm{C}_0(V, q)^\times/\mathbf{G}_m).$$

Thus, we take  $S' = Z_q$  and  $G' = \mathrm{C}_0(V, q)^\times/\mathbf{G}_m$ .

The geometric method rests on the zero scheme  $\Sigma_q = \{q = 0\} \subset \mathbf{P}(V^*)$ , a smooth proper  $S$ -scheme with geometric fibers over  $S$  given by a ruled quadric  $\mathbf{P}^1 \times \mathbf{P}^1$  in  $\mathbf{P}^3$ . The  $\mathrm{PGL}(V)$ -stabilizer of  $\Sigma_q$  is  $\mathrm{GO}(q)/\mathbf{G}_m = \mathrm{O}(q)/\mu_2$ , so we get an action map

$$f_q : \mathrm{O}(q)/\mu_2 \rightarrow \mathrm{Aut}_{\Sigma_q/S}.$$

We claim that for any smooth proper map  $X \rightarrow S$  whose geometric fibers are  $\mathbf{P}^1 \times \mathbf{P}^1$  (e.g.,  $\Sigma_q$ ), there is a unique triple  $(S' \rightarrow S, C', \varphi)$  consisting of a degree-2 finite étale cover  $S' \rightarrow S$ , a  $\mathbf{P}^1$ -bundle  $C' \rightarrow S'$ , and an  $S$ -isomorphism  $\varphi : X \simeq \mathrm{R}_{S'/S}(C')$ . Here, “unique” means that

if  $(S'', C'', \psi)$  is second such triple then there is a unique  $S$ -isomorphism  $\alpha : S' \simeq S''$  and isomorphism  $C' \simeq C''$  over  $\alpha$  such that the induced  $S$ -isomorphism  $R_{S'/S}(C') \simeq R_{S''/S}(C'')$  coincides with  $\psi \circ \varphi^{-1}$ .

To prove the existence and uniqueness of  $(S', C', \varphi)$ , by limit arguments and the uniqueness assertion we may assume  $S = \text{Spec } A$  for a noetherian local ring  $A$ . Uniqueness allows us to work fppf-locally, so we may increase the residue field  $k$  by a finite amount to make the special fiber isomorphic to  $\mathbf{P}_k^1 \times \mathbf{P}_k^1$ . Since  $H^1(\mathbf{P}_k^1 \times \mathbf{P}_k^1, \mathcal{O}) = 0$ , by standard cohomological and deformation theory arguments the Isom-functor  $\underline{\text{Isom}}(X, \mathbf{P}^1 \times \mathbf{P}^1)$  is formally smooth. Hence, by formal GAGA this functor has an  $\widehat{A}$ -point. In view of the uniqueness assertion, by fpqc descent it suffices to solve the existence and uniqueness problems over  $\widehat{A}$ , so we may assume  $X = \mathbf{P}^1 \times \mathbf{P}^1$ . It therefore suffices to prove that the map of  $\mathbf{Z}$ -groups

$$(\text{PGL}_2 \times \text{PGL}_2) \rtimes (\mathbf{Z}/2\mathbf{Z}) \rightarrow \text{Aut}_{\mathbf{P}^1 \times \mathbf{P}^1}$$

is an isomorphism, and this is part of Exercise 1.6.3(iv).

Now that  $(S_q, C_q, f_q)$  has been constructed, we have an  $S$ -homomorphism

$$(C.6.2) \quad \text{O}(q)/\mu_2 \rightarrow \text{Aut}_{R_{S_q/S}(C_q)/S}.$$

This automorphism scheme is an étale form of  $(\text{PGL}_2 \times \text{PGL}_2) \rtimes (\mathbf{Z}/2\mathbf{Z})_S$ , so it is a smooth affine  $S$ -group and the natural map  $R_{S_q/S}(\text{Aut}_{C_q/S_q}) \rightarrow \text{Aut}_{R_{S_q/S}(C_q)/S}$  is an open and closed immersion onto the open relative identity component. Thus, we get an induced  $S$ -homomorphism

$$(C.6.3) \quad \text{SO}(q)/\mu_2 \rightarrow R_{S_q/S}(\text{Aut}_{C_q/S_q})$$

between adjoint semisimple  $S$ -groups with the same constant fiber dimension. To prove it is an isomorphism we can argue similarly to the treatment of  $\alpha_q$  in Example C.6.2 (by passing to  $q_4$  over  $\mathbf{Z}$ ). It also then follows that (C.6.2) is an isomorphism.

To link the algebraic and geometric methods, we claim that the composite  $S$ -group isomorphism

$$R_{S_q/S}(\text{Aut}_{C_q/S_q}) \simeq \text{SO}(q)/\mu_2 \simeq R_{Z_q/S}(C_0(V, q)^\times / \mathbf{G}_m)$$

arises from a unique pair  $(\beta_q, h_q)$  consisting of an  $S$ -isomorphism  $\beta_q : S_q \simeq Z_q$  and isomorphism  $h_q : \text{Aut}_{C_q/S_q} \simeq C_0(V, q)^\times / \mathbf{G}_m$  over  $\beta_q$ . Since  $\text{PGL}_2 \times \text{PGL}_2$  is its own automorphism scheme, the claim follows from a descent argument similar to the general construction of  $(S' \rightarrow S, C', \varphi)$  in Example C.6.3. Thus,  $S_q \rightarrow S$  corresponds to the center of  $C_0(V, q)$ , and the map  $\delta : H^1(Z_q, \text{PGL}_2) \rightarrow \text{Br}(Z_q)[2]$  carries the class of  $C_q$  to the class of the quaternion algebra  $C_0(V, q)$ . However,  $\ker \delta$  can be nontrivial (e.g., when  $S$  is Dedekind and  $\text{Pic}(Z_q) \neq 1$ ), and working with cohomology only keeps track of torsors and Azumaya algebras up to an equivalence. Hence, it is better to express the link between geometry and algebra via the canonical  $\beta_q$  and  $h_q$ .

**Remark C.6.4.** The abstract considerations in Example C.6.3 become concrete when  $S = \text{Spec } k$  for a finite field  $k$ . Since central simple algebras over finite fields are matrix algebras, and  $\mathbf{P}^1$ -bundles over finite fields are trivial, we get an isomorphism  $\text{SO}(q)/\mu_2 \simeq R_{k'/k}(\text{PGL}_2)$

for a quadratic étale  $k$ -algebra  $k'$ . Assume that  $q$  is non-split, so  $\mathrm{SO}(q)$  is not  $k$ -split and hence  $k'$  must be a field. The isomorphism  $\mathrm{SO}(q)/\mu_2 \simeq \mathrm{R}_{k'/k}(\mathrm{PGL}_2)$  lifts uniquely to an isomorphism  $\mathrm{Spin}(q) \simeq \mathrm{R}_{k'/k}(\mathrm{SL}_2)$  between the simply connected central covers, so

$$\mathrm{Spin}(q)(k)/Z_{\mathrm{Spin}(q)}(k) \simeq \mathrm{R}_{k'/k}(\mathrm{SL}_2)(k)/\mathrm{R}_{k'/k}(\mu_2)(k) = \mathrm{SL}_2(k')/\mu_2(k')$$

for non-split  $q$  of rank 4 over a finite field  $k$  (with  $k'$  a quadratic extension field).

**Example C.6.5.** Suppose  $n = 5$ . This case corresponds to the accidental isomorphism  $\mathrm{B}_2 = \mathrm{C}_2$  for adjoint groups, which says that  $\mathrm{SO}_5$  is isomorphic to the quotient of  $\mathrm{Sp}_4$  by its center  $\mu_2$ . We will construct this isomorphism even in the absence of a “split” condition on the groups (so this incorporates some descent data, in addition to an equality of root systems). To that end, consider a rank-4 symplectic space  $(V, \omega_0)$  with  $\omega_0 \in \wedge^2(V)^* = \wedge^2(V^*)$  the given symplectic form. The rank-6 vector bundle  $\wedge^2(V)$  contains the rank-5 subbundle  $W$  of sections killed by  $\omega_0$ , and on  $\wedge^2(V)$  there exists a natural non-degenerate quadratic form  $q$  valued in  $L = \det(V)$  defined by  $q(\omega, \eta) = \omega \wedge \eta$ . The action of  $\mathrm{SL}(V)$  clearly preserves  $q$ , the restriction  $q|_W$  is non-degenerate (by calculation), and  $\mathrm{Sp}(\omega_0)$  preserves  $W$  (due to the definition of  $W$ ). Thus, the  $\mathrm{Sp}(\omega_0)$ -action on  $W$  defines a homomorphism  $\mathrm{Sp}(\omega_0) \rightarrow \mathrm{O}(q|_W)$  that factors through  $\mathrm{SO}(q|_W)$  (pass to the standard symplectic space  $(V, \omega_0)$  of rank 4 over  $\mathbf{Z}$  and consider connectedness over  $\mathbf{Q}$ ), and it kills the center  $\mu_2$ .

We claim that the resulting map  $h : \mathrm{Sp}_4/\mu_2 \rightarrow \mathrm{SO}_5$  is an isomorphism. A computation shows that  $\ker h$  has trivial intersection with the “diagonal” maximal torus, so  $\ker h$  is quasi-finite and hence  $h$  is surjective for fibral dimension reasons. This forces  $h$  to be finite flat (Proposition 6.1.10), so  $h$  has locally constant fiber degree that we claim is 1. By base change, it suffices to treat the standard symplectic space of rank 4 over  $\mathbf{Z}$ , for which it suffices to compute the fiber degree over  $\mathbf{Q}$ . In characteristic 0, isogenies between connected semisimple groups of adjoint type are necessarily isomorphisms.

**Example C.6.6.** Suppose  $n = 6$ . This case corresponds to the equality  $\mathrm{D}_3 = \mathrm{A}_3$  for groups that are neither adjoint nor simply connected: we claim that  $\mathrm{SO}_6$  is the quotient of  $\mathrm{SL}_4$  by the subgroup  $\mu_2$  in the central  $\mu_4$ . To construct the isomorphism, we again use the natural action of  $\mathrm{SL}(V)$  on the rank-6 bundle  $\wedge^2(V)$  equipped with the non-degenerate quadratic form  $q(\omega, \eta) = \omega \wedge \eta$  valued in the line bundle  $\det V$ . The homomorphism  $\mathrm{SL}(V) \rightarrow \mathrm{O}(q)$  defined in this way clearly kills the central  $\mu_2$ , and it factors through  $\mathrm{SO}(q)$  (as  $\mathrm{O}(q)/\mathrm{SO}(q) = \mathbf{Z}/2\mathbf{Z}$ ).

To prove that the resulting map  $h : \mathrm{SL}(V)/\mu_2 \rightarrow \mathrm{SO}(q)$  is an isomorphism, by Lemma C.2.1 we may pass to  $q_6$  over  $\mathbf{Z}$ . As in Example C.6.5, we reduce to the isomorphism problem over  $\mathbf{Q}$ . Isogenies between smooth connected groups in characteristic 0 are always central, and  $\mu_4/\mu_2$  is not killed by  $h$ , so we are done.

APPENDIX D. PROOF OF EXISTENCE THEOREM OVER  $\mathbf{C}$ 

In this appendix, we prove the Existence Theorem over  $\mathbf{C}$ . The main difficulty is the construction of connected semisimple groups whose semisimple root datum is simply connected and has irreducible underlying root system. (See Definition 1.3.11 and Lemma 6.3.1.) Over any algebraically closed field  $k$  of characteristic 0, algebraic methods can easily settle the case of semisimple reduced root data that are adjoint (see Proposition D.1.1). From that case, one can bootstrap to simply connected semisimple reduced root data via an existence theorem for *projective* representations (modeled on the Borel–Weil construction for semisimple complex Lie groups). This method is explained in [BIBLE, 15.3, 23.1], and is cited in [SGA3, XXV, 1.4]. In this appendix, we present an alternative analytic argument via covering space methods over the ground field  $\mathbf{C}$ , linking up the analytic viewpoint with algebraic techniques. The references we shall use are [BtD] for the “algebraicity” of compact Lie groups and [Ho65] for general facts related to complex Lie groups and maximal compact subgroups of Lie groups with a finite component group. An elegant summary (without proofs) of much of this analytic background is given in [Ser01, VIII].

But first, we sketch another algebraic proof based on enveloping algebras, which can be extracted from [Ho70] and was pointed out by P. Polo. Let  $\mathfrak{g}$  be a semisimple Lie algebra over  $k$  having the desired root system (this exists, by [Hum72, 18.4(a)]). Consider the universal enveloping algebra  $U(\mathfrak{g})$ , an associative  $k$ -algebra whose representation theory coincides with that of  $\mathfrak{g}$ . As  $J$  varies through the 2-sided ideals of finite codimension in  $U(\mathfrak{g})$ , the dual spaces  $(U(\mathfrak{g})/J)^*$  exhaust the  $k$ -algebra generated by the “matrix coefficients” (on  $U(\mathfrak{g})$ ) of the finite-dimensional representations of  $\mathfrak{g}$ . This equips  $H(\mathfrak{g}) := \varinjlim (U(\mathfrak{g})/J)^*$  with a natural commutative  $k$ -algebra structure and a compatible Hopf algebra structure that defines an affine  $k$ -group scheme  $G(\mathfrak{g}) = \text{Spec } H(\mathfrak{g})$  (see [Ho70, §3]). Every dominant integral weight for  $\mathfrak{g}$  is a  $\mathbf{Z}_{\geq 0}$ -linear combination of the finitely many fundamental weights (dual basis to the coroots), so by highest-weight theory for  $\mathfrak{g}$  the  $k$ -algebra  $H(\mathfrak{g})$  is finitely generated (see the end of [Ho59, §5]). The linear dual of  $U(\mathfrak{g})$  is naturally a  $k$ -algebra that contains  $H(\mathfrak{g})$  as a subalgebra, so  $H(\mathfrak{g})$  is a domain since  $U(\mathfrak{g})^*$  is a formal power series ring (see [Ho59, §2]). Hence,  $G(\mathfrak{g})$  is connected and smooth.

By [Ho70, 3.1] (or [Ho70, 6.1]),  $\mathfrak{g}$  is identified with  $\text{Lie}(G(\mathfrak{g}))$  (so  $G(\mathfrak{g})$  is semisimple; see Exercise 6.5.6). But every finite-dimensional  $\mathfrak{g}$ -representation is naturally an  $H(\mathfrak{g})$ -comodule, which is to say a  $G(\mathfrak{g})$ -representation, and this is inverse to applying the Lie functor to a  $G(\mathfrak{g})$ -representation. Since every representation  $\mathfrak{g} \rightarrow \mathfrak{gl}(V)$  has just been “integrated” to a representation  $G(\mathfrak{g}) \rightarrow \text{GL}(V)$ , an argument with highest-weight theory for  $\mathfrak{g}$  (as in the proof of “ $\Rightarrow$ ” in Proposition D.4.1) shows that  $G(\mathfrak{g})$  has a simply connected root datum.

**D.1. Preliminary steps.** By the self-contained (and rather formal) Lemma 6.3.1, to prove the Existence Theorem over  $\mathbf{C}$  it suffices to consider semisimple root data  $R$  that are simply connected. We will approach such  $R$  through preliminary consideration of the adjoint case, beginning with a basic existence result over any algebraically closed field  $k$  with  $\text{char}(k) = 0$ .

**Proposition D.1.1.** *Let  $R = (X, \Phi, X^\vee, \Phi^\vee)$  be an adjoint semisimple reduced root datum. There exists a connected semisimple  $k$ -group  $(\mathcal{G}, \mathcal{T})$  such that  $R(\mathcal{G}, \mathcal{T}) \simeq R$ .*

*Proof.* By [Hum72, 18.4(a)], there exists a semisimple Lie algebra  $\mathfrak{g}$  over  $k$  and a Cartan subalgebra  $\mathfrak{t}$  such that the root system  $\Phi(\mathfrak{g}, \mathfrak{t})$  (see [Hum72, 8.5, 15.3]) coincides with  $(X_{\mathbf{Q}}, \Phi)$ . By Ado’s theorem [Bou1, I, §7.3, Thm. 2], there exists an injective map of Lie algebras  $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n$  over  $k$ . Since  $\mathfrak{g}$  is its own derived subalgebra (due to semisimplicity) and  $\text{char}(k) = 0$ , it is an “algebraic” subalgebra of  $\mathfrak{gl}_n$  in the sense that there exists a (unique) connected linear algebraic  $k$ -subgroup  $G \subset \text{GL}_n$  satisfying  $\text{Lie}(G) = \mathfrak{g}$  inside  $\mathfrak{gl}_n$  [Bo91, 7.9]. The  $k$ -group  $G$  is necessarily semisimple, since the Lie algebra  $\text{Lie}(\mathcal{R}(G))$  of the radical is a solvable ideal in the semisimple  $\mathfrak{g}$  and hence vanishes (so  $\mathcal{R}(G) = 1$ ).

Let  $T \subset G$  be a maximal torus, so the abelian subalgebra  $\text{Lie}(T) \subset \mathfrak{g}$  is its own Lie-theoretic centralizer (as  $T = Z_G(T)$  and  $\text{char}(k) = 0$ ). Hence,  $\text{Lie}(T)$  is a Cartan subalgebra of the semisimple Lie algebra  $\mathfrak{g}$  (i.e., it is maximal among commutative subalgebras whose elements have semisimple adjoint action on  $\mathfrak{g}$ ). All Cartan subalgebras of  $\mathfrak{g}$  are in the same  $\text{Aut}(\mathfrak{g})$ -orbit [Hum72, 16.2], so we get an isomorphism of root systems

$$(X(T)_{\mathbf{Q}}, \Phi(G, T)) = \Phi(\mathfrak{g}, \text{Lie}(T)) \simeq \Phi(\mathfrak{g}, \mathfrak{t}) = (X_{\mathbf{Q}}, \Phi).$$

This identifies the Weyl groups  $W(\Phi(G, T))$  and  $W(\Phi)$  since the root system determines the Weyl group (without reference to a Euclidean structure). The roots determine the coroots via the axioms on the reflections  $s_a : v \mapsto v - \langle v, a^\vee \rangle a$  [CGP, Lemma 3.2.4], so the isomorphism  $X(T)_{\mathbf{Q}} \simeq X_{\mathbf{Q}}$  identifies  $\Phi(G, T)^\vee$  with  $\Phi^\vee$  as well.

Since  $X(T/Z_G) = \mathbf{Z}\Phi(G, T)$  by Corollary 3.3.6(1), and  $\mathbf{Z}\Phi = X$  by our hypothesis that  $R$  is adjoint, the equality  $\Phi(G, T) = \Phi$  via  $X(T)_{\mathbf{Q}} \simeq X_{\mathbf{Q}}$  forces the compatible identification  $X^*(T/Z_G) \simeq X$ . This yields an isomorphism of root data  $R(G/Z_G, T/Z_G) \simeq (X, \Phi, X^\vee, \Phi^\vee) = R$ . Set  $(\mathcal{G}, \mathcal{T}) = (G/Z_G, T/Z_G)$ .  $\square$

Fix a semisimple reduced root datum  $R = (X, \Phi, X^\vee, \Phi^\vee)$  that is simply connected. Let  $R^{\text{ad}} = (\mathbf{Z}\Phi, \Phi, (\mathbf{Z}\Phi)^*, \Phi^\vee)$  denote the associated adjoint semisimple root datum. Let  $k$  be an algebraically closed field of characteristic 0. By Proposition D.1.1, there exists a connected semisimple  $k$ -group  $(G, T)$  such that  $R(G, T) \simeq R^{\text{ad}}$ . We seek a connected semisimple  $k$ -group  $G'$  and an isogeny  $h : G' \rightarrow G$  such that for the maximal torus  $T' = h^{-1}(T)$  in  $G'$  there exists a (necessarily unique) isomorphism  $R(G', T') \simeq R$  over the isomorphism  $R(G, T) \simeq R^{\text{ad}}$  via the central isogenies  $R(G', T') \rightarrow R(G, T)$  and  $R \rightarrow R^{\text{ad}}$ . Over  $\mathbf{C}$  we can build covers via topology, so now we set  $k = \mathbf{C}$  and pass to the analytic theory.

**D.2. Compact and complex Lie groups.** For a connected semisimple  $\mathbf{C}$ -group  $G$ , the group  $G(\mathbf{C})$  is connected for the analytic topology (since for an open cell  $\Omega \subset G$  around 1,  $\Omega(\mathbf{C})$  is visibly connected and  $g\Omega(\mathbf{C})$  meets  $\Omega(\mathbf{C})$  for any  $g \in G(\mathbf{C})$ , due to the irreducibility of  $G$ ). Thus,  $G(\mathbf{C})$  is a connected Lie group. To build isogenous covers of  $G$ , we will use maximal compact subgroups of  $G(\mathbf{C})$  and topological facts from the theory of Lie groups.

**Proposition D.2.1.** *The functor  $G \rightsquigarrow G(\mathbf{C})$  from connected reductive  $\mathbf{C}$ -groups to complex Lie groups is fully faithful. More generally, if  $G$  is a connected reductive  $\mathbf{C}$ -group and  $G'$*

is a linear algebraic  $\mathbf{C}$ -group then every analytic homomorphism  $f : G(\mathbf{C}) \rightarrow G'(\mathbf{C})$  arises from a unique  $\mathbf{C}$ -homomorphism  $G \rightarrow G'$ .

The essential image of the functor in Proposition D.2.1 is identified in Example D.3.3.

*Proof.* First we treat semisimple  $G$ , and then we allow a central torus. Assume  $G$  is semisimple, so the Lie algebra  $\mathfrak{g}$  is semisimple (Exercise 6.5.6). The graph  $\Gamma_f$  is a connected closed complex Lie subgroup of  $G(\mathbf{C}) \times G'(\mathbf{C})$ , so its Lie algebra is a subalgebra of  $\mathfrak{g} \oplus \mathfrak{g}'$  that projects isomorphically onto the semisimple  $\mathfrak{g}$  and hence is its own derived subalgebra. But in characteristic 0, a Lie subalgebra  $\mathfrak{h}$  of the Lie algebra of a linear algebraic group arises from a connected linear algebraic subgroup provided that  $\mathfrak{h}$  is its own derived subalgebra [Bo91, 7.9]. We thereby get a connected linear algebraic subgroup  $H \subset G \times G'$  such that  $\text{Lie}(H) = \text{Lie}(\Gamma_f)$  inside  $\mathfrak{g} \oplus \mathfrak{g}'$ . The connected closed Lie subgroups  $H(\mathbf{C})$  and  $\Gamma_f$  in  $G(\mathbf{C}) \times G'(\mathbf{C})$  must therefore be equal. Thus, the algebraic projection  $H \rightarrow G$  induces a holomorphic isomorphism on  $\mathbf{C}$ -points, so it is an isomorphism. It follows that the  $\mathbf{C}$ -subgroup  $H$  in  $G \times G'$  is the graph of a  $\mathbf{C}$ -homomorphism  $G \rightarrow G'$ , and this clearly analytifies to  $f$ .

Now consider the general case, so  $G = (Z \times \mathcal{D}(G))/\mu$  for the maximal central torus  $Z$ , connected semisimple derived group  $\mathcal{D}(G)$ , and a finite central subgroup  $\mu \subset Z \times \mathcal{D}(G)$ . The algebraic theory of quotients by finite subgroups over  $\mathbf{C}$  analytifies to the analogous theory on the analytic side, so it suffices to treat the cases of  $T$  and  $\mathcal{D}(G)$ . We have already settled  $\mathcal{D}(G)$ , so we may now assume  $G = T$ , and more specifically  $G = \mathbf{G}_m$ . Our problem is to prove the algebraicity of any holomorphic homomorphism  $\mathbf{C}^\times \rightarrow G'(\mathbf{C})$ . Using a faithful representation  $G' \hookrightarrow \text{GL}_n$ , we are reduced to checking the algebraicity of any holomorphic action of  $\mathbf{C}^\times$  on a finite-dimensional  $\mathbf{C}$ -vector space  $V$ . The action of each finite subgroup  $\mathbf{C}^\times[n] = \mu_n$  diagonalizes, and the  $\mathbf{C}^\times$ -action preserves each  $\mu_n$ -isotypic subspace of  $V$ . Thus, by induction on  $\dim V$  we can assume that every  $\mu_n$  acts through a character  $\chi_n : \mu_n \rightarrow \mathbf{C}^\times$ . This says that the given holomorphic map  $\rho : \mathbf{C}^\times \rightarrow \text{GL}(V)$  carries each  $\mu_n$  into the central  $Z = \mathbf{C}^\times$ , so  $\rho$  does as well (since the complex-analytic subgroup  $\rho^{-1}(Z) \subset \mathbf{C}^\times$  contains all  $\mu_n$ , so it is infinite and thus exhausts the connected 1-dimensional  $\mathbf{C}^\times$ ). In other words, we are reduced to showing that a holomorphic homomorphism  $\chi : \mathbf{C}^\times \rightarrow \mathbf{C}^\times$  must be  $\chi(z) = z^n$  for some  $n \in \mathbf{Z}$ . Since  $\mathbf{C}^\times = \mathbf{C}/2\pi i\mathbf{Z}$  with  $\mathbf{C}$  simply connected, this clearly holds.  $\square$

**Remark D.2.2.** The analogue of Proposition D.2.1 over  $\mathbf{R}$  fails, even for semisimple groups whose set of  $\mathbf{R}$ -points is connected. For example, the natural isogeny  $\text{SL}_{2n+1} \rightarrow \text{PGL}_{2n+1}$  of degree  $2n + 1$  induces an isomorphism on  $\mathbf{R}$ -points, the inverse of which is non-algebraic.

**Corollary D.2.3.** *For a connected reductive  $\mathbf{C}$ -group  $G$ , the finite-dimensional holomorphic representations of  $G(\mathbf{C})$  are completely reducible.*

*Proof.* By Proposition D.2.1, it suffices to prove the analogous result for the finite-dimensional algebraic representations of  $G$ . As in the proof of Proposition D.2.1, we immediately reduce to the separate cases of tori and semisimple  $G$ . The case of tori is clear (weight spaces), so we can assume that  $G$  is semisimple. Let  $\rho : G \rightarrow \text{GL}(V)$  be a finite-dimensional representation, and let  $\mathfrak{g} \rightarrow \text{End}(V)$  be the associated Lie algebra representation. Since we are in

characteristic 0 and  $G$  is connected, a linear subspace of  $V$  is  $G$ -stable if and only if it is  $\mathfrak{g}$ -stable. Hence, it is sufficient to prove complete reducibility of the representation theory of  $\mathfrak{g}$ . This in turn follows from the fact that  $\mathfrak{g}$  is semisimple (Exercise 6.5.6).  $\square$

The mechanism by which we will keep track of algebraicity when working on the analytic side is to use maximal compact subgroups of complex Lie groups and the “algebraicity” of the theory of compact Lie groups:

**Theorem D.2.4** (Chevalley–Tannaka). *The functor  $H \rightsquigarrow H(\mathbf{R})$  is an equivalence from the category of  $\mathbf{R}$ -anisotropic reductive  $\mathbf{R}$ -groups whose connected components contain  $\mathbf{R}$ -points to the category of compact Lie groups.*

*The  $\mathbf{R}$ -group  $H$  is connected if and only if the Lie group  $H(\mathbf{R})$  is connected, and  $H$  is semisimple if and only if the Lie group  $H(\mathbf{R})$  has semisimple Lie algebra.*

*Proof.* The semisimplicity criterion for  $H$  follows from Exercise 6.5.6 since  $\text{Lie}(H(\mathbf{R})) = \text{Lie}(H)$  (which is semisimple over  $\mathbf{R}$  if and only if the complex Lie algebra  $\text{Lie}(H)_{\mathbf{C}} = \text{Lie}(H_{\mathbf{C}})$  is semisimple). It is not at all obvious that  $H(\mathbf{R})$  is compact when  $H$  is  $\mathbf{R}$ -anisotropic, nor that  $H(\mathbf{R})$  is connected when  $H$  is moreover connected. We begin by proving the compactness of  $H(\mathbf{R})$ , by a variant of the method used by G. Prasad in his elementary proof of the analogous result over non-archimedean local fields in [Pr].

Fix a faithful finite-dimensional representation  $\rho : H \hookrightarrow \text{GL}(V)$  over  $\mathbf{R}$  (i.e.,  $\ker \rho = 1$ ). By Exercise 1.6.10,  $V_{\mathbf{C}}$  is completely reducible as an  $H_{\mathbf{C}}$ -representation. Let  $\bigoplus W_j$  be a decomposition of  $V_{\mathbf{C}}$  into a direct sum of irreducible subrepresentations of  $H(\mathbf{C})$ . Since  $H(\mathbf{R})$  is Zariski-dense in  $H$ , it is also Zariski-dense in  $H_{\mathbf{C}}$ . Thus, each  $W_j$  is irreducible as a representation of  $H(\mathbf{R})$ . If  $H(\mathbf{R})$  has compact image in each  $\text{GL}(W_j)$  then  $H(\mathbf{R})$  is compact. The image  $H_j$  of  $H$  in  $\text{R}_{\mathbf{C}/\mathbf{R}}(\text{GL}(W_j))$  is necessarily  $\mathbf{R}$ -anisotropic, and  $H_j(\mathbf{R})$  must be Zariski-dense in  $H_j$  since even the image of  $H(\mathbf{R})$  in  $H_j$  is Zariski-dense. We may therefore replace  $H$  with each  $H_j$  to reduce to the case that  $H$  is a closed subgroup of  $\text{R}_{\mathbf{C}/\mathbf{R}}(\text{GL}(W))$  with  $W$  irreducible as a  $\mathbf{C}$ -linear representation of  $H(\mathbf{R})$ .

For any  $h \in H(\mathbf{R})$ , all eigenvalues  $\lambda$  for  $h$  on  $W$  must satisfy  $|\lambda| = 1$ . Indeed, Jordan decomposition in  $H(\mathbf{R})$  allows us to consider only semisimple  $h$ , and it is harmless to replace  $h$  with  $h^n$  for  $n > 0$ . Thus, we can assume that  $h$  lies in the identity component of its commutative semisimple Zariski closure. That identity component must be an  $\mathbf{R}$ -torus  $T$ . All  $\mathbf{R}$ -tori in  $H$  are  $\mathbf{R}$ -anisotropic and hence have a compact group of  $\mathbf{R}$ -points, so the eigenvalues of the integral powers of  $h$  are bounded in  $\mathbf{C}$ . This forces the eigenvalues to lie on the unit circle.

Since  $H(\mathbf{R})$  acts irreducibly on  $W$ , its image in  $\text{GL}(W)$  must generate the  $\mathbf{C}$ -algebra  $\text{End}(W)$  (Burnside), so  $H(\mathbf{R})$  spans  $\text{End}(W)$  over  $\mathbf{C}$ . Choose  $\{h_j\}$  in  $H(\mathbf{R})$  that is a  $\mathbf{C}$ -basis of  $\text{End}(W)$ , and consider the dual basis  $\{L_j\}$  under the perfect  $\mathbf{C}$ -bilinear trace pairing  $(T, T') = \text{Tr}(T \circ T')$  on  $\text{End}(W)$ . The eigenvalue condition forces  $|\text{Tr}(hh_j)| \leq n := \dim W$  for all  $h \in H(\mathbf{R})$ , so (exactly as in the proof of finite generation of integer rings of number fields) the coefficients of  $\rho(h)$  relative to  $\{L_j\}$  are bounded in  $\mathbf{C}$  independently of  $h$ . Hence, the closed set  $\rho(H(\mathbf{R}))$  in  $\text{GL}(W)$  is bounded in  $\text{End}(W)$ . It is also closed in  $\text{End}(W)$  because

$|\det(\rho(h))| = 1$  for all  $h \in H(\mathbf{R})$  (due to the eigenvalues of  $\rho(h)$  lying on the unit circle), so  $\rho(H(\mathbf{R}))$  is compact.

To establish the desired equivalence of categories between anisotropic reductive  $\mathbf{R}$ -groups and compact Lie groups (including the connectedness aspect), we shall now proceed in reverse by constructing a functor from compact Lie groups to reductive  $\mathbf{R}$ -groups. This rests on the “real representation algebra”  $R(K)$  of a compact Lie group  $K$ : the  $\mathbf{R}$ -algebra generated by the matrix entries of the finite-dimensional continuous representations of  $K$  over  $\mathbf{R}$ . Let  $R(K, \mathbf{C})$  denote the analogue defined using continuous linear representations over  $\mathbf{C}$ . The operation of scalar extension of  $\mathbf{R}$ -linear representations to  $\mathbf{C}$ -linear representations defines an  $\mathbf{R}$ -algebra injection  $R(K) \rightarrow R(K, \mathbf{C})$ . There is a natural involution  $\iota$  of  $R(K, \mathbf{C})$  over complex conjugation on  $\mathbf{C}$  (using scalar extension through complex conjugation on  $\mathbf{C}$ -linear representations of  $K$ ), and this acts trivially on  $R(K)$ . Galois descent gives that  $R(K, \mathbf{C}) = \mathbf{C} \otimes_{\mathbf{R}} R(K, \mathbf{C})^\iota$ , so the natural map of  $\mathbf{C}$ -algebras

$$(D.2.1) \quad \mathbf{C} \otimes_{\mathbf{R}} R(K) \rightarrow R(K, \mathbf{C})$$

is injective. But every  $\mathbf{C}$ -linear representation can be viewed as an  $\mathbf{R}$ -linear representation, so (via a choice of  $i = \sqrt{-1} \in \mathbf{C}$ ) if  $f = u + iv \in R(K, \mathbf{C})$  with  $\mathbf{R}$ -valued  $u$  and  $v$  then  $u, v \in R(K)$ . In other words, (D.2.1) is an isomorphism.

By [BtD, III, 4.1] (which rests on the Peter–Weyl theorem), there exists a faithful representation  $\rho : K \hookrightarrow \mathrm{GL}_n(\mathbf{R})$ . Upon choosing one, we get an  $\mathbf{R}$ -algebra map  $\mathbf{R}[x_{ij}][1/\det] \rightarrow R(K)$ . The representation algebra is generated by the matrix functions from any single faithful representation [BtD, III, 1.4(iii), 2.7(i)], so this map is surjective. In particular,  $R(K)$  is finitely generated over  $\mathbf{R}$ , and likewise  $R(K, \mathbf{C})$  is finitely generated over  $\mathbf{C}$ . There is a natural Hopf algebra structure on  $R(K)$  that is contravariantly functorial in  $K$  (see [BtD, III, §7]), and by Tannaka duality [BtD, III, 7.15] the natural map  $K \rightarrow (\mathrm{Spec} R(K))(\mathbf{R})$  defined via evaluation is an isomorphism of Lie groups. The linear algebraic  $\mathbf{R}$ -group  $K^{\mathrm{alg}} := \mathrm{Spec} R(K)$  may be disconnected, but we can detect its connected components using  $\mathbf{R}$ -points:

**Lemma D.2.5.** *The locus  $K = K^{\mathrm{alg}}(\mathbf{R})$  is Zariski dense in  $K^{\mathrm{alg}}$ , and  $(K^{\mathrm{alg}})^0$  is  $\mathbf{R}$ -anisotropic reductive. In particular, each connected component of  $K^{\mathrm{alg}}$  contains an  $\mathbf{R}$ -point and hence is geometrically connected over  $\mathbf{R}$ .*

*Proof.* By definition  $R(K)$  is a subalgebra of the  $\mathbf{R}$ -algebra of continuous  $\mathbf{R}$ -valued functions on  $K$ , so every element of the finite type  $\mathbf{R}$ -algebra  $R(K)$  is uniquely determined by its restriction to the set  $K$  of  $\mathbf{R}$ -points. This exactly encodes the Zariski density property in the affine algebraic  $\mathbf{R}$ -scheme  $K^{\mathrm{alg}}$ .

To prove reductivity, assume to the contrary. Any nontrivial connected unipotent group over a perfect field contains the additive group as a closed subgroup, so  $K^{\mathrm{alg}}$  contains  $\mathbf{G}_a$  as a closed  $\mathbf{R}$ -subgroup. Hence,  $\mathbf{R} = \mathbf{G}_a(\mathbf{R})$  is a closed subgroup of the group  $K^{\mathrm{alg}}(\mathbf{R}) = K$  that is compact, a contradiction. The same reasoning shows that the connected reductive group  $(K^{\mathrm{alg}})^0$  is  $\mathbf{R}$ -anisotropic.  $\square$

Every continuous homomorphism  $f : K \rightarrow K'$  between compact Lie groups gives rise to a map of Hopf algebras  $R(K') \rightarrow R(K)$  via composition of  $K'$ -representations with  $f$ . The corresponding  $\mathbf{R}$ -group homomorphism  $f^{\text{alg}} : K^{\text{alg}} \rightarrow K'^{\text{alg}}$  recovers  $f$  on  $\mathbf{R}$ -points, and it is uniquely determined by this condition since  $K$  is Zariski-dense in  $K^{\text{alg}}$ . In particular, for any  $\mathbf{R}$ -group map  $\varphi : K^{\text{alg}} \rightarrow K'^{\text{alg}}$ , necessarily  $\varphi = f^{\text{alg}}$  where  $f$  is the restriction of  $\varphi$  to  $\mathbf{R}$ -points. Also, if  $K$  is a closed subgroup of  $K'$  then  $R(K') \rightarrow R(K)$  is surjective [BtD, III, 4.3], so  $f^{\text{alg}}$  a closed immersion when  $f$  is injective.

**Lemma D.2.6.**  $K^0 = (K^{\text{alg}})^0(\mathbf{R})$  and  $(K^{\text{alg}})^0 = (K^0)^{\text{alg}}$ .

*Proof.* Let  $j : K^0 \rightarrow K$  be the natural map, so  $j^{\text{alg}}$  is a closed immersion since  $j$  is injective. The dimensions of  $(K^0)^{\text{alg}}$  and  $K^{\text{alg}}$  coincide because the Lie algebra of a linear algebraic  $\mathbf{R}$ -group can be naturally computed using the Lie group of  $\mathbf{R}$ -points. Thus,  $j^{\text{alg}}$  is an open and closed immersion, so the identity component of  $K^{\text{alg}}$  is the same as that of  $(K^0)^{\text{alg}}$ .

We claim that  $(K^0)^{\text{alg}}$  is the identity component of  $K^{\text{alg}}$ ; i.e.,  $(K^0)^{\text{alg}}$  is connected. Since  $(K^{\text{alg}})^0(\mathbf{R})$  is an open and closed subgroup of  $K^{\text{alg}}(\mathbf{R}) = K$ , it contains  $K^0$ . But we just saw that  $(K^{\text{alg}})^0 \subset (K^0)^{\text{alg}}$ , so

$$K^0 \subset (K^{\text{alg}})^0(\mathbf{R}) \subset (K^0)^{\text{alg}}(\mathbf{R}) = K^0,$$

forcing the containment  $(K^{\text{alg}})^0 \subset (K^0)^{\text{alg}}$  to be an equality on  $\mathbf{R}$ -points and hence an equality of  $\mathbf{R}$ -groups (due to the Zariski-density of  $\mathbf{R}$ -points in  $K^{\text{alg}}$  and  $(K^0)^{\text{alg}}$ ).  $\square$

To summarize, we have defined a fully faithful functor  $K \rightsquigarrow K^{\text{alg}}$  from the category of compact Lie groups into the category of anisotropic reductive  $\mathbf{R}$ -groups all of whose connected components contain  $\mathbf{R}$ -points, and this functor is compatible with identity components (i.e.,  $K$  is connected if and only if  $K^{\text{alg}}$  is connected).

It remains to show that if  $G$  is an anisotropic reductive  $\mathbf{R}$ -group whose connected components have  $\mathbf{R}$ -points then for the compact Lie group  $K := G(\mathbf{R})$  (which is Zariski-dense in  $G$ ) there exists an  $\mathbf{R}$ -group isomorphism  $G \simeq K^{\text{alg}}$  extending the equality of  $\mathbf{R}$ -points. (Such an isomorphism is unique if it exists, due to the Zariski-density of  $\mathbf{R}$ -points in these  $\mathbf{R}$ -groups, and likewise it is necessarily functorial in  $G$ ).

Since  $\mathcal{O}(G)$  is exhausted by finite-dimensional  $G$ -stable  $\mathbf{R}$ -subspaces, we get a restriction map from  $\mathcal{O}(G)$  to the real representation algebra  $R(K)$ . This map is injective, due to the Zariski density of  $K$  in  $G$ . To complete the proof of Theorem D.2.4, we just have to check that the map  $\mathcal{O}(G) \hookrightarrow R(K) = \mathcal{O}(K^{\text{alg}})$  of  $\mathbf{R}$ -algebras is an isomorphism of Hopf algebras.

The corresponding  $\mathbf{R}$ -scheme map  $K^{\text{alg}} \rightarrow G$  induces a group isomorphism between  $\mathbf{R}$ -points, so it is an  $\mathbf{R}$ -group map. In other words,  $\mathcal{O}(G) \hookrightarrow R(K)$  is a map of Hopf algebras. To check surjectivity, we note that a choice of faithful representation  $G \hookrightarrow \text{GL}_n$  over  $\mathbf{R}$  defines a faithful representation  $K \hookrightarrow \text{GL}_n(\mathbf{R})$ . The induced pullback map  $\mathcal{O}(\text{GL}_n) \rightarrow R(K)$  is surjective, as we noted in the discussion preceding Lemma D.2.5, so this forces the map  $\mathcal{O}(G) \hookrightarrow R(K)$  to be surjective.  $\square$

Now it is time to review some more facts from the theory of compact Lie groups. For a connected compact Lie group  $K$ , let  $K^{\text{alg}}$  denote the associated  $\mathbf{R}$ -anisotropic connected

reductive  $\mathbf{R}$ -group via Theorem D.2.4. We will use this algebraic structure to establish a finiteness property for the universal cover of  $K$  when  $K$  is sufficiently non-commutative. Recall that for any connected Lie group  $H$ , the universal covering space  $q : \tilde{H} \rightarrow H$  (equipped with a marked point over the identity of  $H$ ) has a unique compatible structure of real Lie group, and the discrete closed normal subgroup  $\ker q = \pi_1(H, 1)$  is central (as in any connected Lie group). This recovers the fact that  $\pi_1(H, 1)$  is abelian, so  $\ker q = H_1(H, \mathbf{Z})$ . In the compact case this universal covering has finite degree:

**Proposition D.2.7** (Weyl). *If  $K$  is a connected compact Lie group and  $\mathrm{Lie}(K)$  is semisimple then the universal cover  $\tilde{K} \rightarrow K$  is a finite-degree covering. In particular,  $\tilde{K}$  is compact.*

The hypothesis on the Lie algebra cannot be dropped; consider  $K = S^1$ .

*Proof.* The kernel of the covering map is the group  $H_1(K, \mathbf{Z})$  that is finitely generated (as  $K$  is a compact manifold), so this group is finite if and only if its finite-order quotients have bounded cardinality. In other words, we seek an upper bound on the degree of isogenies  $f : K' \rightarrow K$  from connected compact Lie groups  $K'$  onto the given group  $K$ . By Theorem D.2.4,  $f$  “algebraizes” to an isogeny of connected semisimple  $\mathbf{R}$ -groups  $K'^{\mathrm{alg}} \rightarrow K^{\mathrm{alg}}$ . By extending scalars to  $\mathbf{C}$ , we are reduced to checking that for any connected semisimple  $\mathbf{C}$ -group  $G$  there exists an upper bound on the degree of all connected isogenous covers  $f : G' \rightarrow G$ .

Such a bound is obtained from the root datum of  $G$ , as follows. Let  $T \subset G$  be a maximal torus and  $T' = f^{-1}(T)$  the associated maximal torus in  $G'$ , so the centrality of  $\ker f$  forces  $\ker f = \ker(T' \rightarrow T)$ . Hence, to bound the degree of  $f$  it is equivalent to bound the index of the inclusion of lattices  $X(T) \rightarrow X(T')$ . The centrality of  $\ker f$  implies that the isomorphism  $X(T)_{\mathbf{Q}} \simeq X(T')_{\mathbf{Q}}$  identifies  $\Phi(G, T)$  with  $\Phi(G', T')$  and the dual isomorphism identifies  $\Phi(G, T)^{\vee}$  with  $\Phi(G', T')^{\vee}$ . (This is seen most concretely by using that  $f$  even identifies  $\mathfrak{g}'$  with  $\mathfrak{g}$ , as we are in characteristic 0.) Thus, by (1.3.2) we have

$$\mathbf{Z}\Phi(G, T) \subset X(T) \subset X(T') \subset (\mathbf{Z}\Phi(G, T)^{\vee})^*.$$

Hence, an upper bound on  $\deg f$  is the index of  $\mathbf{Z}\Phi(G, T)$  in  $(\mathbf{Z}\Phi(G, T)^{\vee})^*$  (i.e., the absolute determinant of the non-degenerate pairing between  $\mathbf{Z}\Phi(G, T)$  and  $\mathbf{Z}\Phi(G, T)^{\vee}$ ).  $\square$

The following result lies very deep but is fundamental in the general theory.

**Theorem D.2.8.** *Let  $\mathcal{H}$  be a Lie group with finite component group. Every compact subgroup of  $\mathcal{H}$  is contained in a maximal one, and all maximal compact subgroups  $K$  of  $\mathcal{H}$  are conjugate to each other. Moreover, for any  $K$  there are closed vector subgroups  $V_1, \dots, V_n \subset \mathcal{H}$  such that the multiplication map  $K \times V_1 \times \dots \times V_n \rightarrow \mathcal{H}$  is a  $C^\infty$  isomorphism. In particular,  $K$  is a deformation retract of  $\mathcal{H}$ , so  $\pi_0(K) = \pi_0(\mathcal{H})$  and  $\pi_1(K, 1) = \pi_1(\mathcal{H}, 1)$ .*

*Proof.* This is [Ho65, XV, 3.1]. (We only need the case of connected  $\mathcal{H}$ . The connected case does not seem to formally imply the general case.)  $\square$

**D.3. Complexification.** To go further, we need to link up compact Lie groups and complex Lie groups. For any Lie group  $H$ , the *complexification* of  $H$  is an initial object among  $C^\infty$  homomorphisms from  $H$  to complex Lie groups; i.e., it is a  $C^\infty$  homomorphism  $j_H : H \rightarrow H_{\mathbf{C}}$  to a complex Lie group such that any  $C^\infty$  homomorphism  $f : H \rightarrow \mathcal{H}$  to a complex Lie group has the form  $F \circ j_H$  for a unique holomorphic homomorphism  $F : H_{\mathbf{C}} \rightarrow \mathcal{H}$ . We emphasize that this definition is intrinsic to the  $C^\infty$  and holomorphic theories, having *no reliance on the algebraic theory*. In particular, if  $H = G(\mathbf{R})$  for a linear algebraic  $\mathbf{R}$ -group  $G$  then it is not at all clear whether or not the canonical map  $H \rightarrow G(\mathbf{C})$  is a complexification in the sense we have defined it, nor is it clear if  $\ker j_H = 1$  (or if  $\dim \ker j_H = 0$ ).

Here is a basic example of complexifications (whose verification is easy): if we denote by  $\exp_{\mathbf{R}}(\mathfrak{h})$  the unique connected and simply connected real Lie group  $H$  such that  $\text{Lie}(H)$  is equal to a given real Lie algebra  $\mathfrak{h}$ , and similarly for complex Lie algebras, then  $\exp_{\mathbf{R}}(\mathfrak{h}) \rightarrow \exp_{\mathbf{C}}(\mathfrak{h}_{\mathbf{C}})$  is a complexification. In the commutative case, this says that if  $V$  is a real vector space then the inclusion  $V \hookrightarrow \mathbf{C} \otimes_{\mathbf{R}} V$  is a complexification.

If a Lie group  $H$  admits a complexification  $H \rightarrow H_{\mathbf{C}}$  and  $\Lambda$  is a discrete central subgroup of  $H$  whose image  $\Lambda'$  in  $H_{\mathbf{C}}$  is discrete then  $H/\Lambda \rightarrow H_{\mathbf{C}}/\Lambda'$  is a complexification. (For example, the inclusion  $S^1 = \mathbf{R}/\mathbf{Z} \hookrightarrow \mathbf{C}/\mathbf{Z} \simeq \mathbf{C}/2\pi i\mathbf{Z} \simeq \mathbf{C}^\times$  given by  $\theta \mapsto \exp(2\pi i\theta)$ , where  $i^2 = -1$ , is a complexification.) A formal argument using the settled simply connected case (and a pushout construction when  $H$  is disconnected) shows that a complexification exists for any  $H$  [Bou1, III, 6.10, Prop. 20] (also see [Ho65, XVII, §5]). The construction gives that  $H_{\mathbf{C}}$  is connected when  $H$  is connected and that the  $\mathbf{C}$ -linearization  $\mathbf{C} \otimes_{\mathbf{R}} \text{Lie}(H) \rightarrow \text{Lie}(H_{\mathbf{C}})$  is surjective, but in general  $\ker j_H$  can have positive dimension, as the following example from [OV, Ch. 1, §4.1, Thm. 7.2] shows.

**Example D.3.1.** Since  $\text{SL}_2(\mathbf{R})$  has maximal compact subgroup  $\text{SO}_2(\mathbf{R}) = S^1$ , the universal cover  $G$  of  $\text{SL}_2(\mathbf{R})$  fits into an exact sequence

$$1 \rightarrow \mathbf{Z} \rightarrow G \rightarrow \text{SL}_2(\mathbf{R}) \rightarrow 1$$

and the induced natural map  $G \rightarrow \text{SL}_2(\mathbf{C})$  is the complexification of  $G$  (since  $\text{SL}_2(\mathbf{C})$  is simply connected; i.e.,  $\text{SL}_2(\mathbf{C}) = \exp_{\mathbf{C}}(\mathfrak{sl}_2(\mathbf{C}))$ ). Consider the central pushout  $H$  of  $G$  along an injection  $j : \mathbf{Z} \hookrightarrow S^1$ , so  $H = (S^1 \times G)/\mathbf{Z}$ . Then  $H_{\mathbf{C}}$  is the quotient of  $\mathbf{C}^\times \times \text{SL}_2(\mathbf{C})$  modulo the unique minimal complex Lie subgroup containing the central subgroup  $j(\mathbf{Z})$  in the first factor. In other words,  $H_{\mathbf{C}} = \text{SL}_2(\mathbf{C})$ , so  $\ker(H \rightarrow H_{\mathbf{C}}) = S^1$ .

Fortunately, in the compact case the pathological situation  $\dim \ker j_H > 0$  does not arise:

**Proposition D.3.2.** *If  $K$  is a compact Lie group then  $j_K : K = K^{\text{alg}}(\mathbf{R}) \rightarrow K^{\text{alg}}(\mathbf{C}) =: K^{\text{an}}$  is a complexification and  $K$  is a maximal compact subgroup of  $K^{\text{an}}$ . In particular,  $j_K$  is a closed embedding and the  $\mathbf{C}$ -linearization of  $\text{Lie}(j_K)$  is an isomorphism (so  $\text{Lie}(K_{\mathbf{C}})$  is semisimple when  $K$  has finite center).*

See Example D.4.2 for a discussion of examples and counterexamples related to complexification beyond the compact case. The main difficulty in the proof of Proposition D.3.2 is that we are demanding the universal property for maps into arbitrary complex Lie groups;

the weaker version of Proposition D.3.2 that only asserts the universal property relative to maps to complex Lie groups arising from linear algebraic groups over  $\mathbf{C}$  (which is entirely sufficient for our needs) is much easier to prove (see [BtD, III, 8.6]).

*Proof.* Every connected component of  $K^{\text{an}}$  meets  $K = K^{\text{alg}}(\mathbf{R})$  (as the connected components of  $K^{\text{alg}}$  have  $\mathbf{R}$ -points and are geometrically connected), and  $(K^{\text{alg}})^0 = (K^0)^{\text{alg}}$ . Thus, by a translation argument it suffices to treat the connected case, so now assume  $K$  is connected.

Let  $K'$  correspond to  $\mathcal{D}(K^{\text{alg}})$ , and let  $T$  correspond to the maximal central torus in  $K^{\text{alg}}$ , so  $K'$  and  $T$  are connected and  $K' \times T \rightarrow K$  is surjective with finite central kernel  $\mu$ . (Here we have used that  $K$  is connected.) It is clear that the finite  $\mathbf{C}$ -group associated to the finite abelian group  $\mu$  is a complexification of  $\mu$ , and that a complexification of  $K = (K' \times T)/\mu$  is given by  $(K'_{\mathbf{C}} \times T_{\mathbf{C}})/\mu_{\mathbf{C}}$  where  $\mu_{\mathbf{C}}$  is a finite  $\mathbf{C}$ -group. Thus, it suffices to treat  $K'$  and  $T$  rather than  $K$ . The case of  $T = (S^1)^r$  is elementary, so now we may assume that  $K^{\text{alg}}$  is semisimple. Equivalently,  $\text{Lie}(K)$  is semisimple.

Let  $j_K : K \rightarrow K_{\mathbf{C}}$  be the complexification. The natural map  $j : K \rightarrow K^{\text{an}}$  uniquely factors as  $f \circ j_K$  for a holomorphic map  $f : K_{\mathbf{C}} \rightarrow K^{\text{an}}$  between connected complex Lie groups, and  $f$  is an isomorphism on Lie algebras since  $\mathbf{C} \otimes_{\mathbf{R}} \text{Lie}(K) \rightarrow \text{Lie}(K^{\text{an}})$  is an isomorphism and  $\mathbf{C} \otimes_{\mathbf{R}} \text{Lie}(K) \rightarrow \text{Lie}(K_{\mathbf{C}})$  via  $j_K$  is surjective. Hence, the  $\mathbf{C}$ -linearization of  $\text{Lie}(j_K)$  is an isomorphism and  $f$  is a quotient map modulo a discrete subgroup. Thus, the real Lie group  $K_{\mathbf{C}}$  has dimension  $2 \dim K$  and to show that  $f$  is an isomorphism it suffices to show that  $f$  is injective. Note that  $j_K$  is necessarily injective (and so is a closed embedding) since we have factored it through the injective map  $K \rightarrow K^{\text{an}}$ .

By Proposition D.2.7, the universal covering  $\pi : \tilde{K} \rightarrow K$  has finite degree. Let  $q : \tilde{K}^{\text{alg}} \rightarrow K^{\text{alg}}$  be the corresponding map between  $\mathbf{R}$ -anisotropic connected reductive  $\mathbf{R}$ -groups, and let  $\mu = \ker q$  be its finite central kernel, so  $K^{\text{an}} = \tilde{K}^{\text{an}}/\mu(\mathbf{C})$ . The induced map  $\pi_{\mathbf{C}} : \tilde{K}_{\mathbf{C}} \rightarrow K_{\mathbf{C}}$  is an isomorphism on Lie algebras, so it is surjective between the connected complexifications and has discrete central kernel. If we can handle the simply connected case (i.e., if  $\tilde{K}_{\mathbf{C}} \rightarrow \tilde{K}^{\text{an}}$  is an isomorphism) then  $\pi_{\mathbf{C}}$  has degree dividing  $\deg \pi^{\text{an}} = \#\mu(\mathbf{C})$  with equality if and only if  $K_{\mathbf{C}} \rightarrow K^{\text{an}}$  is an isomorphism.

Thus, to reduce to the case that  $K$  is simply connected we just need to show that  $\ker \pi_{\mathbf{C}}$  has size at least  $\#\mu(\mathbf{C})$ . But if  $T \subset K$  is a maximal (compact) torus and  $\tilde{T} = \pi^{-1}(T)$  is the preimage maximal torus in  $K$  then  $\ker \pi = \ker(\tilde{T} \rightarrow T)$  and hence  $\ker \pi_{\mathbf{C}}$  has size at least that of  $\ker(\tilde{T}_{\mathbf{C}} \rightarrow T_{\mathbf{C}})$ . Compact tori are powers of  $S^1$ , so by inspection  $T_{\mathbf{C}} \rightarrow T^{\text{an}}$  is an isomorphism and similarly for  $\tilde{T}$ . Thus,  $\tilde{T}_{\mathbf{C}} \rightarrow T_{\mathbf{C}}$  is the map on  $\mathbf{C}$ -points arising from the map  $\tilde{T}^{\text{alg}} \rightarrow T^{\text{alg}}$  induced by  $q : \tilde{K}^{\text{alg}} \rightarrow K^{\text{alg}}$  between maximal  $\mathbf{R}$ -tori. This map between maximal  $\mathbf{R}$ -tori has kernel  $\mu$  as well, so  $\mu(\mathbf{C})$  is identified with the kernel of  $\tilde{T}_{\mathbf{C}} \rightarrow T_{\mathbf{C}}$ . Hence, now we can assume  $K$  is simply connected.

In view of how the complexification is constructed in the simply connected case, it suffices (for the isomorphism property of  $f$ ) to show that if  $K$  is simply connected then so is  $K^{\text{an}}$ . The group  $K^{\text{alg}}(\mathbf{R}) = K$  is a compact subgroup of  $K^{\text{an}}$ , and any connected Lie group has a deformation retract onto its maximal compact subgroups (and so has their homotopy type).

It now suffices to prove that if  $K$  is *any* connected compact Lie group (not necessarily simply connected) then it is maximal as a compact subgroup of  $K^{\text{an}}$ . Since the maximal compact subgroups of  $K^{\text{an}}$  are connected, it suffices to show that the common dimension of the maximal compact subgroups of  $K^{\text{an}}$  is  $\dim K$ .

To control the possibilities for the maximal compact subgroups of  $K^{\text{an}}$ , we need to consider its finite-dimensional holomorphic representations. The group  $K^{\text{an}}$  has a faithful finite-dimensional holomorphic representation (using a faithful representation of the  $\mathbf{C}$ -group  $K_{\mathbf{C}}^{\text{alg}}$ ) and the finite-dimensional holomorphic representations of  $K^{\text{an}}$  are completely reducible (Corollary D.2.3). Thus, the connected complex Lie group  $K^{\text{an}}$  is “reductive” in the sense of [Ho65, XVII, §5]. It then follows from [Ho65, XVII, 5.3] that  $K^{\text{an}}$  is the complexification of its maximal compact subgroups. That is, if  $\mathcal{K}$  is a maximal compact subgroup of  $K^{\text{an}}$  then the map  $\mathcal{K}_{\mathbf{C}} \rightarrow K^{\text{an}}$  induced by the inclusion  $\mathcal{K} \hookrightarrow K^{\text{an}}$  is an isomorphism. But we already know that complexification of a compact Lie group has twice the real dimension in general, so applying that to the abstract  $\mathcal{K}$  and using the equality  $\mathcal{K}_{\mathbf{C}} = K^{\text{an}} = K^{\text{alg}}(\mathbf{C})$  gives that  $\dim \mathcal{K} = (1/2) \dim_{\mathbf{R}} K^{\text{alg}}(\mathbf{C}) = \dim K^{\text{alg}} = \dim K$ .  $\square$

**Example D.3.3.** In the proof of Proposition D.3.2 we saw that every connected compact Lie group is a maximal compact subgroup of its complexification, so the same holds without a connectedness requirement. Turning this around, let  $K$  be the maximal compact subgroup of a complex Lie group  $\mathcal{G}$  with finite component group, and assume  $\text{Lie}(\mathcal{G})$  is semisimple (over  $\mathbf{R}$  or  $\mathbf{C}$ ; these are equivalent). The inclusion  $j : K \hookrightarrow \mathcal{G}$  factors through a unique holomorphic homomorphism  $K_{\mathbf{C}} \rightarrow \mathcal{G}$ , and we claim that this latter map is an isomorphism.

In other words, we claim that every complex Lie group with semisimple Lie algebra and finite component group is the complexification of its maximal compact subgroups. (In particular, using Proposition D.2.1,  $G \rightsquigarrow G(\mathbf{C})$  is an equivalence from connected semisimple  $\mathbf{C}$ -groups to connected complex Lie groups with semisimple Lie algebra. The analogous equivalence for connected reductive  $\mathbf{C}$ -groups is onto the category of connected complex Lie groups  $\mathcal{H}$  such that  $\text{Lie}(\mathcal{H})$  is reductive and  $Z_{\mathcal{H}}^0 \simeq (\mathbf{C}^{\times})^r$  for some  $r \geq 0$ .)

To prove that  $\mathcal{G}$  is the complexification of  $K$ , we use the criterion in [Ho65, XVII, 5.3]: it suffices to prove that  $\mathcal{G}$  has a faithful finite-dimensional representation (i.e., an injective holomorphic homomorphism  $\mathcal{G} \rightarrow \text{GL}(V)$  for a finite-dimensional  $\mathbf{C}$ -vector space  $V$ ) and that the holomorphic finite-dimensional representations of  $\mathcal{G}$  are completely reducible. Complete reducibility is inherited from semisimplicity of  $\text{Lie}(\mathcal{G})$  (and finiteness of  $\pi_0(\mathcal{G})$ ), as in the proof of Corollary D.2.3. The existence of a faithful finite-dimensional representation of  $\mathcal{G}$  lies deeper (and its analogue in the  $\mathbf{R}$ -theory is false; e.g., the universal cover of  $\text{SL}_2(\mathbf{R})$ ). We only need the  $\mathbf{C}$ -points of a connected semisimple  $\mathbf{C}$ -group, for which the existence of a faithful finite-dimensional representation is obvious. The general case is [Ho65, XVII, 3.2].

**D.4. Construction of covers and applications.** Returning to the connected semisimple  $\mathbf{C}$ -group  $G$  at the end of §D.1, we now use §D.2–§D.3 to build an isogeny  $G' \rightarrow G$  from a connected semisimple  $\mathbf{C}$ -group  $G'$  such that the semisimple root datum of  $G'$  is simply connected. Let  $H$  be the connected Lie group  $G(\mathbf{C})$  whose Lie algebra is semisimple, and let

$K$  be a maximal compact subgroup of  $H$ , so  $K^{\text{alg}}$  is an  $\mathbf{R}$ -anisotropic connected reductive  $\mathbf{R}$ -group. By Proposition D.3.2 and Example D.3.3,  $K^{\text{alg}}(\mathbf{C}) \simeq H = G(\mathbf{C})$ . It follows that  $K^{\text{alg}}(\mathbf{C})$  has finite center, so the connected reductive  $\mathbf{R}$ -group  $K^{\text{alg}}$  is semisimple. By Proposition D.2.1, the analytic isomorphism  $K^{\text{alg}}(\mathbf{C}) \simeq G(\mathbf{C})$  arises from a  $\mathbf{C}$ -group isomorphism  $K_{\mathbf{C}}^{\text{alg}} \simeq G$ ; i.e.,  $K^{\text{alg}}$  is an  $\mathbf{R}$ -descent of  $G$  (the “compact form”).

Since  $K^{\text{alg}}$  is an  $\mathbf{R}$ -descent of  $G$ , we will now convert our problem for isogenous covers of  $G$  over  $\mathbf{C}$  into an analogous problem for isogenous covers of  $K^{\text{alg}}$  over  $\mathbf{R}$ . Inspired by Theorem D.2.4, we will make constructions in the category of compact connected Lie groups, and then pass back to the algebraic theory over  $\mathbf{R}$  (and finally extend scalars to  $\mathbf{C}$ ). To this end, we will use the connected compact universal covering space  $\tilde{K} \rightarrow K$  (see Proposition D.2.7).

Passing to the corresponding  $\mathbf{R}$ -anisotropic connected semisimple  $\mathbf{R}$ -groups, we get an isogeny  $\tilde{K}^{\text{alg}} \rightarrow K^{\text{alg}}$ . Extending scalars to  $\mathbf{C}$  defines an isogeny

$$f : G' := \tilde{K}_{\mathbf{C}}^{\text{alg}} \rightarrow K_{\mathbf{C}}^{\text{alg}} = G$$

between connected semisimple  $\mathbf{C}$ -groups. Let  $T' = f^{-1}(T)$  be the maximal torus of  $G'$  corresponding to the choice of maximal torus  $T$  of  $G$ , so we get an isogeny of root data

$$R(G', T') \rightarrow R(G, T) = R^{\text{ad}}.$$

We will prove that  $R(G', T')$  is simply connected (i.e., the coroots span the cocharacter group of  $T'$ ), so  $R(G', T') = R$  compatibly with the initial identification  $R(G, T) = R^{\text{ad}}$ . This will complete the proof of the Existence Theorem over  $\mathbf{C}$ .

We saw in the proof of Proposition D.3.2 that the Lie group  $G'(\mathbf{C}) = \tilde{K}^{\text{alg}}(\mathbf{C})$  contains  $\tilde{K} = \tilde{K}^{\text{alg}}(\mathbf{R})$  as a maximal compact subgroup, so  $G'(\mathbf{C})$  inherits the simply connectedness property from  $\tilde{K}$  (Theorem D.2.8). Our problem is therefore reduced to verifying a relationship between combinatorial and topological notions of being “simply connected”:

**Proposition D.4.1.** *If  $(G', T')$  is a connected semisimple  $\mathbf{C}$ -group then  $G'(\mathbf{C})$  is topologically simply connected if and only if  $R(G', T')$  is simply connected in the sense of root data.*

*Proof.* We will use an “existence theorem” (for highest-weight representations) in the representation theory of semisimple Lie algebras over  $\mathbf{C}$  and exponentiation of Lie algebra representations to Lie group representations in the topologically simply connected case.

Note that the  $\mathbf{Q}$ -vector space  $(\mathbf{Z}\Phi'^{\vee})_{\mathbf{Q}} = X_*(T')_{\mathbf{Q}}$  has complexification that is naturally identified with  $\mathfrak{t} = \text{Lie}(T')$  (by identifying  $\lambda \in \text{Hom}(\mathbf{G}_m, T')$  with  $d\lambda := \text{Lie}(\lambda)(z\partial_z) \in \mathfrak{t}$ ). The set  $\Phi'^{\vee}$  of coroots spans  $X_*(T')$  over  $\mathbf{Z}$  provided that every  $\mathbf{Z}$ -linear form  $\ell : \mathbf{Z}\Phi'^{\vee} \rightarrow \mathbf{Z}$  is  $\mathbf{Z}$ -valued on  $X_*(T')$  (i.e., arises from  $X(T')$ ) when  $\ell$  is viewed as a  $\mathbf{Q}$ -linear form on  $X_*(T')_{\mathbf{Q}}$ , or equivalently when  $\ell$  is viewed as a  $\mathbf{C}$ -linear form on  $X_*(T')_{\mathbf{C}} = \mathfrak{t}$ . This integrality property of  $\ell$  is invariant with respect to the action of  $W_{G'}(T')$ , so to check it always holds for a particular  $(G', T')$  there is no loss of generality in picking a positive system of roots  $\Phi'^+$  and imposing the additional requirement on the element  $\ell \in X(T)_{\mathbf{Q}}$  that it lies in the associated Weyl chamber of  $X(T)_{\mathbf{R}}$  (i.e.,  $\langle \ell, a^{\vee} \rangle \geq 0$  for all  $a \in \Phi'^+$ ).

Letting  $d\ell = \text{Lie}(\ell) : \mathfrak{t}' \rightarrow \text{Lie}(\mathbf{G}_m) = \mathbf{C}$  be the associated linear form on  $\mathfrak{t}'$ , we have  $\langle d\ell, d(a^\vee) \rangle = \langle \ell, a^\vee \rangle \in \mathbf{Z}_{\geq 0}$  for all  $a \in \Phi' = \Phi(\mathfrak{g}', \mathfrak{t}')$ . Thus, by the existence theorem for highest-weight representations of semisimple Lie algebras over  $\mathbf{C}$  (Theorem 1.5.7), there exists a (unique) irreducible representation

$$(D.4.1) \quad \mathfrak{g}' \rightarrow \mathfrak{gl}(V) = \text{Lie}(\text{GL}(V))$$

having  $d\ell$  as its highest weight (relative to the Cartan subalgebra  $\mathfrak{t}'$  and  $\Phi'^+ \subset \Phi(\mathfrak{g}', \mathfrak{t}')$ ).

Now assume that  $G'(\mathbf{C})$  is simply connected, so every finite-dimensional representation of  $\mathfrak{g}'$  over  $\mathbf{C}$  exponentiates to a holomorphic representation of  $G'(\mathbf{C})$  on the same vector space. By Proposition D.2.1, any finite-dimensional holomorphic representation of  $G'(\mathbf{C})$  arises from a unique algebraic representation of  $G'$  on the same vector space, due to the semisimplicity of  $G'$ . Thus, (D.4.1) arises from a  $\mathbf{C}$ -group representation  $\rho : G' \rightarrow \text{GL}(V)$ , and the highest weight vector  $v \in V$  for  $(\mathfrak{g}', \mathfrak{t}')$  (which is unique up to  $\mathbf{C}^\times$ -scaling) is a  $T'$ -eigenvector since it is a  $\mathfrak{t}'$ -eigenvector. The corresponding weight homomorphism  $w : T' \rightarrow \mathbf{G}_m$  via the  $T'$ -action on  $\mathbf{C}v$  induces  $d\ell$  on Lie algebras, so  $w = \ell$  in  $X(T')_{\mathbf{C}}$ . In particular,  $\ell \in X(T')$ . That is,  $X(T') = (\mathbf{Z}\Phi'^\vee)^*$ , or equivalently  $X_*(T') = \mathbf{Z}\Phi'^\vee$ , so  $R(G', T')$  is simply connected.

For the converse, assume that  $R(G', T')$  is simply connected. Let  $K$  be a maximal compact subgroup of  $G'(\mathbf{C})$ , so  $K_{\mathbf{C}}^{\text{alg}} \simeq G'(\mathbf{C})$  by Proposition D.3.2 and Example D.3.3. Hence,  $K^{\text{alg}}$  is an  $\mathbf{R}$ -descent of  $G'$ , by Proposition D.2.1. Consider the finite-degree universal cover  $f : \tilde{K} \rightarrow K$ . In the proof of Proposition D.3.2 we saw that  $\tilde{K}^{\text{alg}}(\mathbf{C})$  is simply connected in the topological sense. Since  $f$  “algebraizes” to an isogeny  $\tilde{K}^{\text{alg}} \rightarrow K^{\text{alg}}$  between connected semisimple  $\mathbf{R}$ -groups,  $\tilde{K}^{\text{alg}}(\mathbf{C})$  is the universal cover of  $K^{\text{alg}}(\mathbf{C}) = G'(\mathbf{C})$ . (This covering map is the complexification of  $f$ .) The description of the effect of central isogenies at the level of root data (see Example 6.1.9) shows that any isogeny onto  $G'$  from a connected semisimple  $\mathbf{C}$ -group is an isomorphism because  $R(G', T')$  is simply connected. Thus,  $f_{\mathbf{C}}$  must be an isomorphism, so  $G'(\mathbf{C})$  is topologically simply connected.  $\square$

**Example D.4.2.** As an application of our work with complex Lie groups and the Existence Theorem over  $\mathbf{C}$ , we now relate the Lie group notion of complexification to the algebraic notion of scalar extension from  $\mathbf{R}$  to  $\mathbf{C}$  for Lie groups arising from semisimple  $\mathbf{R}$ -groups, going beyond the  $\mathbf{R}$ -anisotropic case. This is not used elsewhere in these notes.

Let  $H$  be a connected semisimple  $\mathbf{R}$ -group (so  $H_{\mathbf{C}}$  denotes the associated semisimple  $\mathbf{C}$ -group, not to be confused with the complexification  $H(\mathbf{R})_{\mathbf{C}}$  of the Lie group  $H(\mathbf{R})$ ). Even when  $H(\mathbf{R})$  is connected, it can happen that the natural map  $H(\mathbf{R}) \rightarrow H(\mathbf{C})$  is not the complexification. For example, the isomorphism  $\text{SL}_3(\mathbf{R}) \simeq \text{PGL}_3(\mathbf{R})$  provides a homomorphism  $\text{PGL}_3(\mathbf{R}) \rightarrow \text{SL}_3(\mathbf{C})$  to a degree-3 connected cover of  $\text{PGL}_3(\mathbf{C})$ . This problem disappears if we require the root datum for  $H_{\mathbf{C}}$  to be simply connected, as we now explain.

Consider any homomorphism  $f : H(\mathbf{R}) \rightarrow \mathcal{G}$  to a complex Lie group  $\mathcal{G}$ . The map  $\text{Lie}(f)$  is a homomorphism from  $\mathfrak{h} = \text{Lie}(H)$  into the underlying real Lie algebra of  $\text{Lie}(\mathcal{G})$ , so it linearizes to a map of complex Lie algebras  $\tilde{f} : \mathfrak{h}_{\mathbf{C}} \rightarrow \text{Lie}(\mathcal{G})$ . But  $\mathfrak{h}_{\mathbf{C}} = \text{Lie}(H)_{\mathbf{C}} = \text{Lie}(H_{\mathbf{C}}) = \text{Lie}(H(\mathbf{C}))$ , and  $H(\mathbf{C})$  is simply connected in the topological sense by Proposition D.4.1 (whose proof relied on our work in the proof of the Existence Theorem over  $\mathbf{C}$ ). Hence,

$\tilde{f}$  exponentiates to a holomorphic homomorphism  $F : H(\mathbf{C}) \rightarrow \mathcal{G}$ . The maps  $F \circ j, f : H(\mathbf{R}) \rightrightarrows \mathcal{G}$  agree on Lie algebras by construction of  $F$ , so they coincide on  $H(\mathbf{R})^0$ . Likewise, by Lie algebra considerations,  $F$  is uniquely determined on  $H(\mathbf{R})^0$  by the equality  $F \circ j = f$ .

It remains to prove that  $H(\mathbf{R})$  is connected when  $H_{\mathbf{C}}$  has a simply connected root datum. The connectedness of  $H(\mathbf{R})$  in such cases is a deep result of E. Cartan, originally proved by Riemannian geometry (going through the theory of compact groups, for which the main connectedness ingredient is proved in [He, VII, Thm. 8.2]).

Here is a sketch of a proof of Cartan's connectedness theorem using an algebraic connectedness result of Steinberg. The real Lie group underlying  $H(\mathbf{C})$  is connected and has an involution  $\theta$  (complex conjugation) with fixed-point locus  $H(\mathbf{R})$ . By a result of Mostow [Mos, §6], there exists a  $\theta$ -stable maximal compact subgroup  $K'$  of  $H(\mathbf{C})$  such that  $K := K' \cap H(\mathbf{R})$  is a maximal compact subgroup of  $H(\mathbf{R})$ . (Mostow's proof gives more:  $K' \mapsto K' \cap H(\mathbf{R})$  is a bijection from the set of  $\theta$ -stable maximal compact subgroups of  $H(\mathbf{C})$  to the set of maximal compact subgroups of  $H(\mathbf{R})$ , with  $H$  any connected reductive  $\mathbf{R}$ -group.)

Fix such a  $K'$ , so  $K'$  is connected (as  $H(\mathbf{C})$  is connected) and  $\pi_0(H(\mathbf{R})) = \pi_0(K)$  for the maximal compact subgroup  $K = K' \cap H(\mathbf{R}) = K'^{\theta}$  in  $H(\mathbf{R})$ . To prove the connectedness of  $H(\mathbf{R})$ , it suffices to prove the connectedness of  $K$ , so we are reduced to proving more generally that the fixed-point locus of *any* involution  $\theta$  of  $K'$  is connected. Note that the  $\mathbf{C}$ -group  $(K'^{\text{alg}})_{\mathbf{C}} = H_{\mathbf{C}}$  has a simply connected root datum.

The involution  $\theta$  of the connected compact Lie group  $K'$  arises from an involution  $\theta^{\text{alg}}$  of  $K'^{\text{alg}}$ , so  $(K'^{\text{alg}})^{\theta^{\text{alg}}}$  is a closed  $\mathbf{R}$ -subgroup of  $K'^{\text{alg}}$  whose group of  $\mathbf{R}$ -points is  $K'^{\theta}$ . The fixed-point subgroup for an involution of a connected reductive  $\mathbf{R}$ -group has reductive identity component (see [PY02, 2.2], or [PY02, 2.4] for an algebraic proof), so  $(K'^{\text{alg}})^{\theta^{\text{alg}}}$  has  $\mathbf{R}$ -anisotropic reductive identity component. (This identity component may not be semisimple; e.g., for  $K' = \text{SU}_2$  we have  $K = S^1$ .) Thus, by Theorem D.2.4, if  $(K'^{\text{alg}})^{\theta^{\text{alg}}}$  is connected for the Zariski topology then its group  $K$  of  $\mathbf{R}$ -points is connected for the analytic topology.

We are reduced to an algebraic assertion over  $\mathbf{R}$ : for any  $\mathbf{R}$ -anisotropic connected semisimple  $\mathbf{R}$ -group  $G$  such that the semisimple  $\mathbf{C}$ -group  $G_{\mathbf{C}}$  has a simply connected root datum (e.g,  $G = K'^{\text{alg}}$ ) and any involution  $\iota$  of  $G$  (e.g.,  $\theta^{\text{alg}}$ ), the linear algebraic  $\mathbf{R}$ -subgroup  $G^{\iota}$  of  $G$  is connected for the Zariski topology. We can reformulate this assertion over any field  $k$ : if  $G$  is a connected semisimple  $k$ -group such that the root datum for  $G_{\bar{k}}$  is simply connected and if  $\theta$  is an automorphism of  $G$  that is semisimple (i.e., the induced automorphism of the coordinate ring is semisimple, such as an involution when  $\text{char}(k) \neq 2$ ), then the closed fixed-point scheme  $G^{\theta}$  is connected for the Zariski topology. It suffices to consider algebraically closed  $k$  and to work with  $k$ -valued points, in which case the connectedness assertion is a theorem of Steinberg [St, 8.1] (which was partially motivated by the desire for an algebraic version of Cartan's connectedness theorem).

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