SURVEY OF KISIN'S PAPER *CRYSTALLINE REPRESENTATIONS AND F-CRYSTALS*

by

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1. Introduction

In p-adic Hodge theory there are fully faithful functors from certain categories of p-adic representations of the Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$ of a p-adic field K to certain categories of semi-linear algebra structures on finite-dimensional vector spaces in characteristic 0. For example, semistable representations give rise to weakly admissible filtered (φ, N)-modules, and Fontaine conjectured that this is an equivalence of categories. For many purposes (such as in Galois deformation theory with artinian coefficients) it is useful to have a finer theory in which p-adic vector spaces are replaced with lattices or torsion modules. Fontaine and Laffaille gave such a theory in the early 1980's under stringent restrictions on the Hodge-Tate weights and absolute ramification in K. The aim of these lectures on integral p-adic Hodge theory is to explain a more recent theory, due largely to Breuil and Kisin, that has no ramification or weight restrictions. We are essentially giving a survey of [**11**], to which the reader should turn for more details. (If we omit discussion of a proof of a result, this should not be interpreted to mean that the proof is easy; rather, it may only mean that the techniques of the proof are a digression from the topics that seem most essential for us to discuss.)

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1.1. Notation. — We fix a perfect field k of characteristic $p > 0$, and as usual we denote by $W := W(k)$ the ring of Witt vectors of k and by $K_0 := \mathrm{Frac}(W)$ the fraction field of W. Let K be a finite and totally ramified extension of K_0, and fix a uniformizer $\pi \in \mathscr{O}_K$. Let $|\cdot| : K \to \mathbf{R}_{\geq 0}$ be the unique normalized p-adic absolute value on K satisfying $|p| = p^{-1}$. We fix once and for all a choice $\overline{K}$ of algebraic closure of K, and again denote by $|\cdot|$ the unique extension to $\overline{K}$ of $|\cdot|$ on K. Let $E \in W[u]$ be the minimal polynomial of π over K_0, and denote by Δ the rigid-analytic open unit disc over K_0; recall that the points of Δ are identified with the orbits of $\mathrm{Gal}(\overline{K}/K_0)$ acting on the set

$$\{x \in \overline{K} \bigm| |x| < 1\}.$$

2. First Lecture

We shall begin by introducing several categories of "linear algebra data." Among these is a certain category of vector bundles (with extra structure, depending on the uniformizer $\pi \in \mathscr{O}_K$) over the rigid-analytic open unit disc Δ over K_0. Our aim is to sketch the proof of the equivalence of this category with the category $\mathrm{MF}_K^{\varphi,N,\mathrm{Fil}\geq 0}$ of filtered (φ, N)-modules over K whose filtration is effective (i.e., $\mathrm{Fil}^0(M) = M$, or equivalently the associated graded module over K has its grading supported in non-negative degrees). Roughly speaking, the idea behind the construction of this equivalence is to show that any (effective) filtered (φ, N)-module D can be naturally "promoted" to a vector bundle $\mathscr{M}$ over Δ, with D recovered as the "fiber of $\mathscr{M}$ to the origin." See Theorem 2.2.1 for a precise statement. Using Kedlaya's theory of slopes [**10**] (as a black box), we then explain how to translate the condition that D be weakly admissible into a certain condition ("slope zero") on $\mathscr{M}$. This description will motivate the introducion another category of "integral" linear algebra data that enables us to study broad classes of interesting p-adic Galois representations in the next two lectures.

2.1. Modules with φ and connection. — Fix a choice of coordinate u on Δ and let $\mathscr{O} \subseteq K_0[\![u]\!]$ be the K_0-algebra of rigid-analytic functions on Δ. For $0 < r < 1$ (and r always understood to lie in the value group $p^{\mathbf{Q}} = |\overline{K}^{\times}|$), the closed disc $\Delta_r := \{|u| \leq r\}$ is an admissible open subspace of Δ, and the ring of rigid-analytic functions $\mathscr{O}_r$ on Δ_r is equipped with the supremum norm

$$\|f\|_r := \sup_{x \in \Delta_r} |f(x)| < \infty.$$

These norms make $\mathscr{O}$ into a Fréchet space (i.e. we topologize $\mathscr{O}$ by uniform convergence on the Δ_r's for $r \to 1^-$). Concretely, $\mathscr{O}$ is the K_0-subalgebra of $K_0[\![u]\!]$ consisting of power series that converge on every closed subdisc of Δ with radius $r < 1$.

If we denote by $\varphi : W \to W$ the Frobenius automorphism of W (lifting the Frobenius automorphism $\alpha \mapsto \alpha^p$ of the perfect field k), then φ naturally extends to an endomorphism $\varphi_{\mathscr{O}} : \mathscr{O} \to \mathscr{O}$ over φ by

$$\varphi_{\mathscr{O}}\left(\sum_{n\geq 0} a_n u^n\right) = \sum_{n\geq 0} \varphi(a_n) u^{np}.$$

Note that $\varphi_{\mathscr{O}}$ is finite and faithfully flat with degree p.

We will denote by λ the infinite product

$$\lambda := \prod_{n\geq 0} \varphi_{\mathscr{O}}^n \left(\frac{E(u)}{E(0)}\right), \tag{2.1.1}$$

which converges (uniformly on closed subdiscs) on Δ. (In fact, if $s(u) \in W[\![u]\!]\left[\frac{1}{p}\right] \subseteq \mathscr{O}$ has constant term 1, then the product $\prod_{n\geq 0} \varphi_{\mathscr{O}}^n(s)$ converges in $\mathscr{O}$ [**9**, Rem. 4.5].) Note that λ depends on the choice of uniformizer π, and that the zeroes of λ in the closed unit disc are precisely the p^nth roots of the zeroes of $E^{(\varphi^n)}$ for all $n \geq 0$, where $h^{(\varphi^n)}(u) = \sum_{m\geq 0} \varphi^n(c_m) u^m$ for $h = \sum c_m u^m \in \mathscr{O}$. We easily calculate

$$\varphi_{\mathscr{O}}(\lambda) = \prod_{n\geq 0} \varphi_{\mathscr{O}}^{n+1} \left(\frac{E(u)}{E(0)}\right) = \left(\frac{E(0)}{E(u)}\right)\lambda, \tag{2.1.2}$$

so in particular $\varphi_{\mathscr{O}}(1/\lambda) = \frac{E(u)/E(0)}{\lambda}$ and hence $\varphi_{\mathscr{O}}$ naturally acts on the ring $\mathscr{O}\left[\frac{1}{\lambda}\right]$.

Example 2.1.1. — Let ζ_p be a primitive pth root of unity and suppose $K = K_0(\zeta_p)$. Choose $\pi = \zeta_p - 1$ and extend φ to $\mathscr{O}$ via $\varphi_{\mathscr{O}}(u) := (u+1)^p - 1$. We claim that with these choices, the analogous definition of λ gives

$$\lambda = \frac{\log(1+u)}{u}.$$

Indeed, we have $E(u) = \Phi_p(u+1)$, and hence (keeping in mind that we have also modified the definition of $\varphi_{\mathscr{O}}$) we get

$$\lambda = \lim_{N\to\infty} \prod_{n=0}^{N} \frac{\Phi_{p^n}(u+1)}{p} = \frac{1}{u} \cdot \lim_{N\to\infty} \frac{(u+1)^{p^N} - 1}{p^N} = \frac{\log(1+u)}{u},$$

where the final equality follows from the binomial theorem and simple p-adic estimates on the explicit binomial coefficients (to also recover uniform convergence on each Δ_r).

Definition 2.1.2. — Define the differential operator $N_\nabla : \mathscr{O} \to u\mathscr{O} \subseteq \mathscr{O}$ by $N_\nabla := -\lambda u \frac{d}{du}$.

The minus sign in this definition is due to the fact that $\lambda(0) = 1$, and we cannot say more to justify this sign intervention at the outset other than that it makes certain calculations later in the theory (for semistable non-crystalline representations) work out well, such as [**11**, Prop. 1.7.8]. A straightforward calculation (using (2.1.2)) shows that the relation

$$N_\nabla \circ \varphi_{\mathscr{O}} = p\frac{E(u)}{E(0)}\varphi_{\mathscr{O}} \circ N_\nabla \tag{2.1.3}$$

holds, which at $u = 0$ recovers the familiar relation "$N\varphi = p\varphi N$" between Frobenius and monodromy operators in p-adic Hodge theory. Thus, we may think of the operators N_∇ and $\varphi_{\mathscr{O}}$ as deformations of the usual N and φ.

Since K is discretely-valued, every invertible sheaf on Δ is trivial. (Indeed, for $c \in K^\times$ with $0 < |c| < 1$, the Dedekind coordinate ring of each of the exhausting discs $\{|t| \leq |c|^{1/n}\}$ is a UFD and hence has trivial Picard group. A line bundle on Δ therefore admits *compatible* trivializations on the Δ_r's (and hence is globally trivial) via an infinite product trick used in the proof of [**2**, 1.3.3]; the discreteness of $|K^\times|$ implies the exponentially decaying coefficient-estimates which ensure the convergence of the intervening infinite products.) In particular, every effective divisor on Δ is the divisor of an analytic function (which is false for more general K [**7**, Ex. 2.7.8]), so $\mathscr{O}$ is a *Bezout domain*; i.e. every finitely generated ideal is principal. In general $\mathscr{O}$ is not noetherian. For example, let $\{x_n\}$ be a collection of K-points of Δ with $|x_n| \to 1$ and let the nonzero $f_r \in \mathscr{O}$ have divisor $\sum_{n\leq r}[x_n] + \sum_{n>r} 2[x_n]$. If the ideal $(f_r)_{r\geq 1}$ is finitely generated then by the Bezout property it must be principal, (g), and g must have divisor $\sum_n [x_n]$. But such a g obviously does not lie in $(f_r)_{r\geq 1}$, so we get the non-noetherian claim for $\mathscr{O}$. Nonetheless, the Bezout property for $\mathscr{O}$ ensures that coherent $\mathscr{O}$-modules behave much as if they were modules over a principal ideal domain:

Lemma 2.1.3. — *Let $\mathscr{M}$ be free $\mathscr{O}$-module of finite rank, and $\mathscr{N} \subseteq \mathscr{M}$ an arbitrary submodule. The following are equivalent:*

1. *$\mathscr{N}$ is a closed submodule of $\mathscr{M}$,*
2. *$\mathscr{N}$ is finitely generated as an $\mathscr{O}$-module,*
3. *$\mathscr{N}$ is a free $\mathscr{O}$-module of finite rank.*

Proof. — See [**11**, Lemma 1.1.4]. ■

We remark that the implication (1) $\implies$ (3) will be especially useful for our purposes. With these preliminaries out of the way, we can now define the first category of "linear algebra data" over $\mathscr{O}$ that we shall consider.

Definition 2.1.4. — Let $\mathrm{Mod}^{\varphi}_{/\mathscr{O}}$ be the category whose objects are pairs $(\mathscr{M}, \varphi_{\mathscr{M}})$ consisting of a finite free $\mathscr{O}$-module $\mathscr{M}$ and an endomorphism $\varphi_{\mathscr{M}}$ of $\mathscr{M}$ satisfying the following two conditions:

1. The map $\varphi_{\mathscr{M}} : \mathscr{M} \to \mathscr{M}$ is $\varphi_{\mathscr{O}}$-semilinear and injective.
2. The cokernel $\operatorname{coker}(1 \otimes \varphi_{\mathscr{M}})$ of the $\mathscr{O}$-linearization of $\varphi_{\mathscr{M}}$ is killed by a power E^h for some integer $h \geq 0$.

Morphisms in $\mathrm{Mod}^{\varphi}_{/\mathscr{O}}$ are $\mathscr{O}$-module homomorphisms that are φ-equivariant. We will abbreviate condition (2) by saying that the pair $(\mathscr{M}, \varphi_{\mathscr{M}})$ has *finite E-height*. The least integer h that works in (2) is the *E-height of $\mathscr{M}$*.

Observe that a $\varphi_{\mathscr{O}}$-semilinear operator $\varphi_{\mathscr{M}} : \mathscr{M} \to \mathscr{M}$ is injective if its $\mathscr{O}$-linearization

$$1 \otimes \varphi_{\mathscr{M}} : \varphi_{\mathscr{O}}^{*}\mathscr{M} = \mathscr{O} \otimes_{\mathscr{O}, \varphi_{\mathscr{O}}} \mathscr{M} \to \mathscr{M}$$

is injective, and this latter injectivity is equivalent to $\operatorname{coker}(1 \otimes \varphi_{\mathscr{M}})$ having nonzero $\mathscr{O}$-annihilator. If condition (1) is satisfied and $\operatorname{ann}_{\mathscr{O}}(\operatorname{coker}(1 \otimes \varphi_{\mathscr{M}})) \neq 0$ then by arguing in terms of vector bundles we see that the cokernel of $1 \otimes \varphi_{\mathscr{M}}$ (which corresponds to a coherent sheaf on Δ that is killed by a nonzero element of $\mathscr{O}$) has discrete support in Δ. Geometrically, the condition (2) says that the cokernel of $1 \otimes \varphi_{\mathscr{M}}$ is supported in the single point $\pi \in \Delta$ (recall that points of Δ correspond to $\mathrm{Gal}(\overline{K}/K_0)$-orbits of points $x \in \overline{K}$ with $|x| < 1$).

We can enhance the category $\mathrm{Mod}^{\varphi}_{/\mathscr{O}}$ by equipping a module in $\mathrm{Mod}^{\varphi}_{/\mathscr{O}}$ with the data of a *monodromy operator* over the differential operator $N_{\nabla} : \mathscr{O} \to \mathscr{O}$. This gives rise to the following category:

Definition 2.1.5. — Let $\mathrm{Mod}^{\varphi, N_{\nabla}}_{/\mathscr{O}}$ be the category whose objects are triples $(\mathscr{M}, \varphi_{\mathscr{M}}, N_{\nabla}^{\mathscr{M}})$ where

1. the pair $(\mathscr{M}, \varphi_{\mathscr{M}})$ is an object of $\mathrm{Mod}^{\varphi}_{/\mathscr{O}}$,
2. $N_{\nabla}^{\mathscr{M}} : \mathscr{M} \to \mathscr{M}$ is a K_0-linear endomorphism of $\mathscr{M}$ satisfying the relations:
   (a) for every $f \in \mathscr{O}$ and $m \in \mathscr{M}$,

$$N_{\nabla}^{\mathscr{M}}(fm) = N_{\nabla}(f)m + fN_{\nabla}^{\mathscr{M}}(m),$$

   (b)

$$N_{\nabla}^{\mathscr{M}} \circ \varphi_{\mathscr{M}} = p\frac{E(u)}{E(0)}\varphi_{\mathscr{M}} \circ N_{\nabla}^{\mathscr{M}},$$

and whose morphisms are $\mathscr{O}$-module homomorphisms that are compatible with the additional structures.

Remark 2.1.6. — Given $N_{\nabla}^{\mathscr{M}} : \mathscr{M} \to \mathscr{M}$, we obtain a map

$$\nabla : \mathscr{M}\left[\frac{1}{\lambda u}\right] \to \mathscr{M}\left[\frac{1}{\lambda u}\right] \otimes_{\mathscr{O}} \Omega^1_{\Delta/K_0}$$

by defining

$$\nabla(m) := -\frac{1}{\lambda} N_{\nabla}^{\mathscr{M}}(m) \frac{du}{u},$$

where the sign is due to the appearance of the sign in the definition of the operator N_∇ on $\mathscr{O}$. The condition (2a) ensures that ∇ satisfies the Leibnitz rule, and so is a meromorphic connection on $\mathscr{M}$ with at most simple poles supported in the zero locus of λu, and a straightforward calculation shows that the condition (2b) guarantees that ∇ is compatible with evident actions of $\varphi_{\mathscr{M}}$. Moreover, we can reverse this construction, and associate a monodromy operator $N_{\nabla}^{\mathscr{M}}$ on $\mathscr{M}$ to any $\varphi_{\mathscr{M}}$-compatible meromorphic connection on $\mathscr{M}$ with at most simple poles supported in the zero locus of λu. Note that at $u = 0$, the relation (2b) recovers the familiar relation "$N\varphi = p\varphi N$" between Frobenius and monodromy operators in p-adic Hodge theory.

Observe that both the categories $\mathrm{Mod}_{/\mathscr{O}}^{\varphi}$ and $\mathrm{Mod}_{/\mathscr{O}}^{\varphi, N_\nabla}$ have evident notions of exactness and tensor product, and the forgetful functor from the second of these two categories to the first is neither fully faithful nor essentially surjective (but in Lemma 2.4.2 we will establish full faithfulness on the full subcategory of triples $(\mathscr{M}, \varphi_{\mathscr{M}}, N_{\nabla}^{\mathscr{M}})$ such that $N_{\nabla}^{\mathscr{M}}(\mathscr{M}) \subseteq u\mathscr{M}$). Also, neither $\mathrm{Mod}_{/\mathscr{O}}^{\varphi}$ nor $\mathrm{Mod}_{/\mathscr{O}}^{\varphi, N_\nabla}$ is an abelian category, as the cokernel of a morphism of finite free $\mathscr{O}$-modules need not be free.

2.2. The equivalence of categories. — In this subsection, we will sketch the proof of the following remarkable result:

Theorem 2.2.1. — *There are exact tensor-compatible functors*

$$\mathrm{MF}_K^{\varphi, N, \mathrm{Fil}\geq 0} \underset{\underline{D}}{\overset{\underline{\mathscr{M}}}{\rightleftarrows}} \mathrm{Mod}_{/\mathscr{O}}^{\varphi, N_\nabla}$$

and natural isomorphisms of functors

$$\underline{\mathscr{M}} \circ \underline{D} \xrightarrow{\simeq} \mathrm{id} \quad \textit{and} \quad \underline{D} \circ \underline{\mathscr{M}} \xrightarrow{\simeq} \mathrm{id}.$$

Remarks 2.2.2. — Recall that each object of $\mathrm{MF}_K^{\varphi, N, \mathrm{Fil}\geq 0}$ is equipped with a descending, exhaustive, and separated filtration by K-subspaces. The notion of exactness in this category includes the filtration data (in the sense that an *exact sequence* of finite-dimensional filtered vector spaces is an exact sequence of vector spaces such that the natural subspace and quotient filtrations on the common kernel and image at each stage coincide). Hence, $\mathrm{MF}_K^{\varphi, N, \mathrm{Fil}\geq 0}$ is not an abelian category since maps with vanishing kernel and cokernel may fail to be filtration-compatible in the reverse direction.

The definitions of $\underline{\mathscr{M}}$ and $\underline{D}$ as module-valued functors, as well as the construction of the natural transformations as in Theorem 2.2.1, will not use N_∇.

For example, the definition of $\underline{D}(\mathscr{M})$ as a K_0-vector space does not use the data of $N_\nabla^{\mathscr{M}}$ and the definition of $\underline{\mathscr{M}}(D)$ in $\mathrm{Mod}^\varphi_{/\mathscr{O}}$ comes before its N_∇-structure is defined. Moreover, once $\underline{\mathscr{M}}$ and $\underline{D}$ have been defined, it turns out to be easy to show that for any $\mathscr{M} \in \mathrm{Mod}^\varphi_{/\mathscr{O}}$ there is a natural map of vector bundles over Δ

$$\underline{\mathscr{M}} \circ \underline{D}(\mathscr{M}) \to \mathscr{M}$$

that is an isomorphism away from the point $\pi \in \Delta$. That this latter map is an isomorphism on π-stalks (and hence is an isomorphism) crucially uses the operator $N_\nabla^{\mathscr{M}}$.

Rather than give the proof of Theorem 2.2.1, we will content ourselves with giving the definitions of $\underline{\mathscr{M}}$ and $\underline{D}$. Moreover, we will only define $\underline{\mathscr{M}}$ on objects D with $N_D = 0$, as this simplifies the exposition. For a complete discussion, see [**11**, Theorem 1.2.15].

Let D be an object of $\mathrm{MF}_K^{\varphi,N,\mathrm{Fil}\geq 0}$ and denote by $\mathrm{Fil}^j D_K$ the jth filtered piece of $D_K = K \otimes_{K_0} D$. As we just noted above, to simplify the exposition of the construction of $\underline{\mathscr{M}}(D)$, we shall assume $N_D = 0$. We will define $\underline{\mathscr{M}}(D)$ as a certain $\mathscr{O}$-submodule of $\mathscr{O}\left[\frac{1}{\lambda}\right] \otimes_{K_0} D$ by imposing "polar conditions" at specific points in Δ. Roughly, we can think of elements of $\mathscr{O}\left[\frac{1}{\lambda}\right] \otimes_{K_0} D$ as certain meromorphic D-valued functions on Δ with poles supported in the divisor of λ, and we will use the additional data on D (Frobenius and filtration) to restrict the order of poles that we allow for elements of $\underline{\mathscr{M}}(D)$.

For each integer $n \geq 0$, let x_n be the point of Δ corresponding to the (irreducible) Eisenstein polynomial $E(u^{p^n}) \in K_0[u]$ (so x_n corresponds to the $\mathrm{Gal}(\overline{K}/K_0)$-conjugacy class of a choice of $\pi_n := \sqrt[p^n]{\pi} \in \overline{K}$). If $\mathscr{O}^\wedge_{\Delta,x_n}$ denotes the complete local ring of Δ at x_n, then the specialization map

$$\mathscr{O}^\wedge_{\Delta,x_n} \to K_0(\pi_n)$$

sending a function to its value at x_n realizes $K_0(\pi_n)$ as the residue field of $\mathscr{O}^\wedge_{\Delta,x_n}$. It follows that $\mathscr{O}^\wedge_{\Delta,x_n}$ is a complete equicharacteristic discrete valuation ring with maximal ideal $(u-\pi_n)\mathscr{O}^\wedge_{\Delta,x_n}$; i.e. we have a K_0-algebra isomorphism

$$\mathscr{O}^\wedge_{\Delta,x_n} \simeq K_0(\pi_n)[\![u-\pi_n]\!]$$

of $\mathscr{O}^\wedge_{\Delta,x_n}$ with the ring of x_n-centered power series over $K_0(\pi_n)$. Since

$$K_0(\pi_n) \supseteq K_0(\pi_0) = K,$$

we see that $\mathscr{O}^\wedge_{\Delta,x_n}$ uniquely contains K over K_0.

Denote by $\varphi_W : \mathscr{O} \to \mathscr{O}$ the "Frobenius operator" given by acting only on coefficients:

$$\varphi_W\left(\sum_{n\geq 0} a_n u^n\right) := \sum_{n\geq 0} \varphi(a_n) u^n,$$

so in particular φ_W is bijective and $\varphi_{\mathscr{O}}$ is the composition of φ_W with the pth power map $u \mapsto u^p$. From this description and the product formula (2.1.1) defining λ, we see that $\varphi_W^{-n}(\lambda)$ has a simple zero at each zero of $\varphi_W^{-n} \circ \varphi_{\mathscr{O}}^n(E(u)/E(0)) = E(u^{p^n})/E(0)$ in $\overline{K}$, and so as a function on Δ it has a simple zero at $x_n \in \Delta$. We conclude that that under the natural localization map

$$\mathscr{O} \to \mathscr{O}^{\wedge}_{\Delta,x_n} \tag{2.2.1}$$

the element $\varphi_W^{-n}(\lambda) \in \mathscr{O}$ maps to a uniformizer. Recalling that $\varphi_D : D \to D$ is bijective, the composite map

$$\mathscr{O} \otimes_{K_0} D \xrightarrow[\varphi_W^{-n} \otimes \varphi_D^{-n}]{\simeq} \mathscr{O} \otimes_{K_0} D \xrightarrow[(2.2.1)\otimes 1]{} \mathscr{O}^{\wedge}_{\Delta,x_n} \otimes_{K_0} D = \mathscr{O}^{\wedge}_{\Delta,x_n} \otimes_K D_K$$

thus induces a map

$$\iota_n : \mathscr{O}\left[\tfrac{1}{\lambda}\right] \otimes_{K_0} D \longrightarrow \mathscr{O}^{\wedge}_{\Delta,x_n}\left[\tfrac{1}{u-\pi_n}\right] \otimes_K D_K.$$

Concretely, up to the intervention of the isomorphism $\varphi_W^{-n} \otimes \varphi_D^{-n}$, the map ι_n is nothing more than the map sending a D-valued meromorphic function on Δ to its Laurent expansion at $x_n \in \Delta$.

Define

$$\underline{\mathscr{M}}(D) := \left\{ \delta \in \mathscr{O}\left[\frac{1}{\lambda}\right] \otimes_{K_0} D \;\middle|\; \iota_n(\delta) \in \sum_{j \in \mathbf{Z}} (u-\pi_n)^{-j} \operatorname{Fil}^j D_K \text{ for all } n \geq 0 \right\}.$$

Observe that the sum occurring in the definition of $\underline{\mathscr{M}}(D)$ is a *finite* sum, as $\operatorname{Fil}^j D_K = D_K$ for all $j < 0$ (D is an object of $\mathrm{MF}_K^{\varphi,N,\mathrm{Fil}\geq 0}$) and $\operatorname{Fil}^j D_K = 0$ for all j sufficiently large (the filtration on D_K is separated). Thus, this sum really makes sense as a "finite" condition on the polar part of δ at x_n.

Remark 2.2.3. — Let A be any ring and let N_1 and N_2 be A-modules endowed with decreasing filtrations. Suppose that the filtration on N_2 is finite, exhaustive, and separated in the sense that $\operatorname{Fil}^j N_2 = N_2$ for $j \ll 0$ and $\operatorname{Fil}^j N_2 = 0$ for $j \gg 0$. The tensor product $N_1 \otimes_A N_2$ has a natural filtration given by

$$\operatorname{Fil}^j(N_1 \otimes_A N_2) := \sum_{m+n=j} \operatorname{image}((\operatorname{Fil}^m N_1) \otimes_A (\operatorname{Fil}^n N_2) \to N_1 \otimes_A N_2),$$

and this sum is finite because of the hypotheses on the filtration on N_2 and the fact that the filtration on N_1 is decreasing. We apply this with $A = K$, $N_2 = D_K$, and N_1 equal to the fraction field $\mathscr{O}^{\wedge}_{\Delta,x_n}\left[\frac{1}{u-\pi_n}\right]$ of the complete local ring $\mathscr{O}^{\wedge}_{\Delta,x_n}$ endowed with its natural $(u-\pi_n)$-adic filtration. The sum occurring in the definition of $\underline{\mathscr{M}}(D)$ is the $\mathscr{O}^{\wedge}_{\Delta,x_n}\left[\frac{1}{u-\pi_n}\right]$-module

$$\mathrm{Fil}^0\left(\mathscr{O}^{\wedge}_{\Delta,x_n}\left[\frac{1}{u-\pi_n}\right]\otimes_K D_K\right).$$

If $h \geq 0$ is any integer with $\mathrm{Fil}^{h+1} D_K = 0$, then it is easy to see that we have $(u-\pi_n)^h\underline{\mathscr{M}}(D) \subseteq \mathscr{O}\otimes_{K_0} D$, and so (since $\iota_n(\lambda)$ is $(u-\pi_n)$ times a unit in $\mathscr{O}^{\wedge}_{\Delta,x_n}$)

$$\underline{\mathscr{M}}(D) \subseteq \lambda^{-h}\mathscr{O}\otimes_{K_0} D.$$

Moreover, one readily checks from consideration of finite-tailed Laurent expansions that $\underline{\mathscr{M}}(D)$ is a closed submodule of $\lambda^{-h}\mathscr{O}\otimes_{K_0} D$ (because the membership condition at each x_n in the definition of $\underline{\mathscr{M}}(D)$ is obviously a closed condition on $\lambda^{-h}\mathscr{O}\otimes_{K_0} D$). Thus, by Lemma 2.1.3, we conclude that $\underline{\mathscr{M}}(D)$ is a finite free $\mathscr{O}$-module.

From the computation (2.1.2) we have seen that $\varphi_{\mathscr{O}}$ acts on $\mathscr{O}\left[\frac{1}{\lambda}\right]$, and an easy calculation shows that N_∇ (see Definition 2.1.2) also acts on $\mathscr{O}\left[\frac{1}{\lambda}\right]$. We define operators $\varphi_{\underline{\mathscr{M}}(D)}$ and $N_\nabla^{\underline{\mathscr{M}}(D)}$ on $\mathscr{O}\left[\frac{1}{\lambda}\right]\otimes_{K_0} D$ by the formulae

$$\varphi_{\underline{\mathscr{M}}(D)} := \varphi_{\mathscr{O}}\otimes\varphi_D \quad\text{and}\quad N_\nabla^{\underline{\mathscr{M}}(D)} := N_\nabla\otimes 1.$$

The relation (2.1.3) ensures that $\varphi_{\underline{\mathscr{M}}(D)}$ and $N_\nabla^{\underline{\mathscr{M}}(D)}$ satisfy the desired "deformation" (Definition 2.1.5(2b)) of the usual Frobenius and monodromy relation $N\varphi = p\varphi N$, and one easily calculates using Definition 2.1.2 that $N_\nabla^{\underline{\mathscr{M}}(D)}$ satisfies the Leibnitz rule (2a). We remark that the above constructions can be generalized to allow for $N_D \neq 0$. (Beware that the definition of $\underline{\mathscr{M}}(D)$ must be changed if $N_D \neq 0$.) The following lemma makes no assumptions on N_D (although we have only explained the definition of $\underline{\mathscr{M}}(D)$ when $N_D = 0$).

Lemma 2.2.4. — *The operators $\varphi_{\underline{\mathscr{M}}(D)}$ and $N_\nabla^{\underline{\mathscr{M}}(D)}$ preserve the submodule $\underline{\mathscr{M}}(D)$ of $\mathscr{O}\left[\frac{1}{\lambda}\right]\otimes_{K_0} D$. Moreover, the $\mathscr{O}$-linear map*

$$1\otimes\varphi_{\underline{\mathscr{M}}(D)} : \varphi^*_{\underline{\mathscr{M}}(D)}\underline{\mathscr{M}}(D) \to \underline{\mathscr{M}}(D)$$

is injective, and has cokernel isomorphic to

$$\bigoplus_{i\geq 0}\left(\mathscr{O}/E(u)^i\mathscr{O}\right)^{h_i},$$

where $h_i = \dim_K \mathrm{gr}^i D_K$ (so $\underline{\mathscr{M}}(D) \neq 0$ if $D \neq 0$).

Proof. — This is essentially [**11**, Lemma 1.2.2]. ■

It follows at once from Lemma 2.2.4 that $\varphi_{\underline{\mathscr{M}}(D)}$ and $N_\nabla^{\underline{\mathscr{M}}(D)}$ make $\underline{\mathscr{M}}(D)$ into an object of $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N_\nabla}$ and that if $D \neq 0$ then the E-height of $\underline{\mathscr{M}}(D)$ is bounded above by the largest i for which h_i is nonzero. We have thus defined the functor

$\underline{\mathscr{M}}$ on objects. To define $\underline{\mathscr{M}}$ on morphisms (assuming the vanishing of the N_D's, which is the only case in which we have explained how to define $\underline{\mathscr{M}}(D)$), one must check that for any morphism $\alpha : D \to D'$ of effective filtered (φ, N)-modules such that $N_D = 0$ and $N_{D'} = 0$, the map

$$1 \otimes \alpha : \mathscr{O}\left[\frac{1}{\lambda}\right] \otimes_{K_0} D \to \mathscr{O}\left[\frac{1}{\lambda}\right] \otimes_{K_0} D$$

restricts to a morphism $\underline{\mathscr{M}}(\alpha) : \underline{\mathscr{M}}(D) \to \underline{\mathscr{M}}(D)$ of (φ, N_∇)-modules over $\mathscr{O}$; this is an easy exercise using the definition of $\underline{\mathscr{M}}$ and the fact that α respects filtrations.

We will define

$$\underline{D} : \mathrm{Mod}_{/\mathscr{O}}^{\varphi, N_\nabla} \to \mathrm{MF}_K^{\varphi, N, \mathrm{Fil}\geq 0}$$

by sending a (φ, N_∇)-module given by the data $(\mathscr{M}, \varphi_{\mathscr{M}}, N_\nabla^{\mathscr{M}})$ to its fiber at the origin of the disc:

$$\underline{D}(\mathscr{M}) := \mathscr{M}/u\mathscr{M}.$$

(Similarly, the functor $\underline{D}$ takes a morphism to its specialization at $u = 0$.) We equip $\underline{D}(\mathscr{M})$ with Frobenius and monodromy operators

$$\varphi := \varphi_{\mathscr{M}} \mod u \quad \text{and} \quad N := N_\nabla^{\mathscr{M}} \mod u.$$

Observe that $\mathscr{M}/u\mathscr{M}$ is a finite-dimensional K_0-vector space, and that $N\varphi = p\varphi N$ thanks to Definition 2.1.5(2b).

In order to show that $\underline{D}(\mathscr{M}) := \mathscr{M}/u\mathscr{M}$ is an object of $\mathrm{MF}_K^{\varphi, N, \mathrm{Fil}\geq 0}$, we must equip the K-vector space $\underline{D}(\mathscr{M})_K$ with an effective filtration. To do this, we proceed as follows. Recall that we have normalized $|\cdot|$ on $\overline{K}$ by $|p| = 1/p$. For any $r \in (|\pi|, 1)$ that is in the value group $p^{\mathbf{Q}}$ of the absolute value on $\overline{K}^\times$, "specialization at π" defines a map

$$(2.2.2)\quad \underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O}_r \to \underline{D}(\mathscr{M}) \otimes_{K_0} (\mathscr{O}_r/E(u)\mathscr{O}_r) = \underline{D}(\mathscr{M}) \otimes_{K_0} K = \underline{D}(\mathscr{M})_K$$

(recall from §2.1 that $\mathscr{O}_r$ is the ring of rigid-analytic functions on the closed rigid-analytic disc Δ_r of radius r over K_0 centered at the origin).

For any $r \in p^{\mathbf{Q}}$ with $r < 1$ we write $(\cdot)|_{\Delta_r}$ to denote the functor $(\cdot) \otimes_{\mathscr{O}} \mathscr{O}_r$ from $\mathscr{O}$-modules to $\mathscr{O}_r$-modules. If $|\pi| < r < |\pi|^{1/p}$ then we will define an infinite descending filtration on the left side of (2.2.2) by $\mathscr{O}_r$-submodules. The K-linear pushforward of this filtration will be the desired filtration on $\underline{D}(\mathscr{M})_K$; it is independent of the choice of such r. The definition of this filtration of $\underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O}_r$ by $\mathscr{O}_r$-submodules requires:

Lemma 2.2.5. — *Let $\mathscr{M}$ be any object of $\mathrm{Mod}_{/\mathscr{O}}^{\varphi}$ with E-height h.*

1. *There exists a unique $\mathscr{O}$-linear and φ-compatible map $\xi = \xi_{\mathscr{M}}$*

$$\begin{tikzcd} \underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O} \arrow[r, "\xi"] \arrow[loop below, "\varphi_{\underline{D}(\mathscr{M})}\otimes\varphi_{\mathscr{O}}"] & \mathscr{M} \arrow[loop below, "\varphi_{\mathscr{M}}"] \end{tikzcd}$$

with the property that

$$\xi \mod u = \mathrm{id}_{\underline{D}(\mathscr{M})}.$$

2. *The map ξ is injective, and* $\mathrm{coker}(\xi)$ *is killed by λ^h.*
3. *If $r \in (|\pi|, |\pi|^{1/p})$ is in the value group of $\overline{K}^{\times}$, then $\xi\big|_{\Delta_r}$ has the same image in $\mathscr{M}\big|_{\Delta_r}$ as does the linearization*

$$1 \otimes \varphi_{\mathscr{M}} : \varphi_{\mathscr{O}}^{*}\mathscr{M} \to \mathscr{M}$$

over Δ_r.

Before sketching the proof of Lemma 2.2.5, let us apply it to define a filtration on $\underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O}_r$. It follows at once from (2) that $\xi\left[\frac{1}{\lambda}\right]$ is an isomorphism. Moreover, (3) readily implies that for r as in the Lemma, $\xi\big|_{\Delta_r}$ is an isomorphism away from every $\pi_n \in \Delta_r$ and that $\xi\big|_{\Delta_r}$ induces an isomorphism

$$\underline{D}(\mathscr{M}) \otimes \mathscr{O}_r \xrightarrow{\simeq} (1 \otimes \varphi_{\mathscr{M}})(\varphi_{\mathscr{O}}^{*}\mathscr{M})\big|_{\Delta_r}. \tag{2.2.3}$$

The right side of (2.2.3) is naturally filtered by its intersections with the $E^i\mathscr{M}\big|_{\Delta_r}$. That is, we define

$$\mathrm{Fil}^i(1 \otimes \varphi_{\mathscr{M}})(\varphi_{\mathscr{O}}^{*}\mathscr{M})\big|_{\Delta_r} := (1 \otimes \varphi_{\mathscr{M}})(\varphi_{\mathscr{O}}^{*}\mathscr{M})\big|_{\Delta_r} \cap E^i\mathscr{M}\big|_{\Delta_r}.$$

Since $\mathscr{O}_r$ is Dedekind, each Fil^i is a finite free $\mathscr{O}_r$-module. Via (2.2.3), we get a filtration on $\underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O}_r$; the image of this filtration under (2.2.2) is the desired K-linear filtration on $\underline{D}(\mathscr{M})_K$. Obviously this filtration is independent of r.

Remark 2.2.6. — Note that the definition of E-height implies that

$$\mathrm{Fil}^i(1 \otimes \varphi_{\mathscr{M}})(\varphi_{\mathscr{O}}^{*}\mathscr{M})\big|_{\Delta_r} = E^i\mathscr{M}\big|_{\Delta_r}$$

for $i \geq h$; in particular, for $i \geq h$ the map $\xi\big|_{\Delta_r}$ induces an isomorphism

$$\mathrm{Fil}^i(\underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O}_r) \simeq \underline{D}(\mathscr{M}) \otimes_{K_0} E^{i-h}\mathscr{O}_r.$$

Specializing at π shows that $\mathrm{Fil}^i(\underline{D}(\mathscr{M})_K) = 0$ for all $i \geq h+1$.

Proof of Lemma 2.2.5. — The uniqueness aspect of the lemma is a standard "φ-argument" that we omit. For the existence of ξ, note that the data of ξ is equivalent to a K_0-linear section

$$s : \underline{D}(\mathscr{M}) \to \mathscr{M}$$

to the natural surjection such that

$$\varphi_{\mathscr{M}} \circ s = s \circ \varphi_{\underline{D}(\mathscr{M})}.$$

Begin by choosing *any* K_0-section $s_0 : \underline{D}(\mathscr{M}) \to \mathscr{M}$. We would like to define

$$s = \lim_{n\to\infty} \varphi_{\mathscr{M}}^n s_0 \varphi_{\underline{D}(\mathscr{M})}^{-n}$$

pointwise on $\underline{D}(\mathscr{M})$. To see that this limit does indeed converge pointwise, one works on a fixed lattice $\mathscr{L} \subseteq \underline{D}(\mathscr{M})$ and makes p-adic estimates in $\mathscr{L}$ and in $\mathscr{M}\big|_{\Delta_\rho}$ for all $\rho \in (0,1) \cap p^{\mathbf{Q}}$. By construction s is φ-compatible, so ξ exists, establishing (1).

To prove (2) and (3), we fix $r \in (|\pi|, |\pi|^{1/p}) \cap p^{\mathbf{Q}}$ and proceed as follows. Since $\xi \mod u$ is an isomorphism, it follows that $\xi\big|_{\Delta_{r^{p^i}}}$ is an isomorphism for some (possibly large) $i \geq 1$. By devissage, we will get to the case $i = 1$. If $i > 1$, then consider the following diagram of finite $\mathscr{O}$-modules (which we think of as coherent sheaves over Δ):

(2.2.4)
$$\begin{tikzcd} \varphi_{\mathscr{O}}^* \left(\underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O}\right) \arrow[r, "\varphi_{\mathscr{O}}^*(\xi)"] \arrow[d, "\simeq"'] & \varphi_{\mathscr{O}}^*(\mathscr{M}) \arrow[d, hook, "1\otimes\varphi_{\mathscr{M}}"] \\ \underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O} \arrow[r, "\xi"'] & \mathscr{M} \end{tikzcd}$$

Due to the fact that ξ is φ-compatible, this diagram commutes. Moreover, since the cokernel of the right vertical map $1 \otimes \varphi_{\mathscr{M}}$ is killed by E^h, where h is the E-height of $\mathscr{M}$, we see that this map is an isomorphism away from the point $\pi \in \Delta$; in particular, it is an isomorphism over $\Delta_{r^p} \supseteq \Delta_{r^{p^{i-1}}}$ since $|\pi| > r^p$.

Since $\varphi_{\mathscr{O}}^{-1}(\Delta_{r^{p^i}}) = \Delta_{r^{p^{i-1}}}$ and ξ is an isomorphism over $\Delta_{r^{p^i}}$ by hypothesis, we deduce that the top arrow $\varphi_{\mathscr{O}}^*(\xi)$ is an isomorphism over $\Delta_{r^{p^{i-1}}}$. The right vertical map is also an isomorphism over $\Delta_{r^{p^{i-1}}}$, so the bottom arrow ξ must be an isomorphism over $\Delta_{r^{p^{i-1}}}$ as well. It follows by descending induction that ξ is an isomorphism over Δ_{r^p}, and hence $\varphi_{\mathscr{O}}^*(\xi)$ is an isomorphism over Δ_r. Thus, $\xi\big|_{\Delta_r}$ is injective, so ξ is injective by analytic continuation (any element of the kernel of the $\mathscr{O}$-module map ξ must vanish over Δ_r, and therefore vanishes identically on Δ). The diagram (2.2.4) also shows that $\xi\big|_{\Delta_r}$ and $(1 \otimes \varphi_{\mathscr{M}})\big|_{\Delta_r}$ have the same image. Finally, coker $\left(\xi\big|_{\Delta_r}\right)$ is killed by E^h, as this is true of coker$(1 \otimes \varphi_{\mathscr{M}})$ (by definition), so running the above analysis of the diagram (2.2.4) in reverse shows that $\varphi_{\mathscr{O}}^n(E^h)$ kills $\operatorname{coker}(\xi|_{\Delta_{r^{1/p^n}}})$ for all $n \geq 0$, and hence λ^h kills $\operatorname{coker}(\xi)$. ■

2.3. Slopes and weak admissibility. — We now recall Kedlaya's theory of slopes [**10**] and apply it to translate weak admissibility across the equivalence of categories in Theorem 2.2.1. Kedlaya's theory works over a certain extension of $\mathscr{O}$, the *Robba ring*:

$$\mathscr{R} := \varinjlim_{r \to 1^-} \mathscr{O}_{\{r<|u|<1\}},$$

where $\mathscr{O}_{\{r<|u|<1\}}$ denotes the ring functions on the rigid-analytic (open) annulus $\{r < |u| < 1\}$ over K_0. Observe that the transition maps in the direct limit are injective, thanks to analytic continuation, and it follows in particular that $\mathscr{O}$ is naturally a subring of $\mathscr{R}$. One easily identifies $\mathscr{R}$ as a certain set of formal Laurent series over K_0. The ring $\mathscr{R}$ is equipped with a Frobenius endomorphism

$$\varphi_{\mathscr{R}} : \mathscr{R} \to \mathscr{R}$$

restricting to $\varphi_{\mathscr{O}}$ on $\mathscr{O}$; one easily checks that $\varphi_{\mathscr{R}}$ is faithfully flat.

The *bounded Robba ring* is the ring

$$\mathscr{R}^{\mathrm{b}} := \varinjlim_{r \to 1^-} \mathscr{O}^{\mathrm{bnd}}_{\{r<|u|<1\}},$$

where $\mathscr{O}^{\mathrm{bnd}}_{\{r<|u|<1\}}$ denotes the subring of $\mathscr{O}_{\{r<|u|<1\}}$ consisting of those functions which are bounded. We also define

$$\mathscr{R}^{\mathrm{int}} := \left\{ \sum_{n \in \mathbf{Z}} a_n u^n \in \mathscr{R} \,\middle|\, a_n \in W \text{ for all } n \in \mathbf{Z} \right\}; \tag{2.3.1}$$

this is a henselian discrete valuation ring with uniformizer p.

Observe that $\mathscr{R}^{\mathrm{b}} = \operatorname{Frac}(\mathscr{R}^{\mathrm{int}})$, so in particular $\mathscr{R}^{\mathrm{b}}$ is a field. In fact, the nonzero elements of $\mathscr{R}^{\mathrm{b}}$ are *precisely* the units of $\mathscr{R}$. Moreover, since $\mathscr{R}^{\mathrm{int}}$ is henselian, roots of polynomials with coefficients in $\mathscr{R}^{\mathrm{b}}$ have canonical p-adic ordinals.

Example 2.3.1. — As E is a polynomial in u with W-coefficients, we clearly have $E \in \mathscr{R}^{\mathrm{int}} \subseteq \mathscr{R}^{\mathrm{b}}$. Since the leading coefficient of E is a unit in W, we see that $\frac{1}{p}E \in \mathscr{R}^{\mathrm{b}}$ is *not* in $\mathscr{R}^{\mathrm{int}}$. It follows that the p-adic ordinal of E is 0, so $E \in (\mathscr{R}^{\mathrm{int}})^\times$.

Definition 2.3.2. — Let $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$ be the category whose objects are pairs (M, φ_M) with M a finite free $\mathscr{R}$-module and

$$\varphi_M : M \to M$$

a $\varphi_{\mathscr{R}}$-semilinear endomorphism whose $\mathscr{R}$-linearization $1 \otimes \varphi_M : \varphi_{\mathscr{R}}^* M \to M$ is an isomorphism. Morphisms in $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$ are φ-compatible morphisms of $\mathscr{R}$-modules. We define the category $\mathrm{Mod}^{\varphi}_{/\mathscr{R}^{\mathrm{b}}}$ similarly.

Warning 2.3.3. — The natural inclusion map $\mathscr{R}^{\mathrm{b}} \hookrightarrow \mathscr{R}$ allows us to view any $\mathscr{R}$-module as an $\mathscr{R}^{\mathrm{b}}$-module. However, $\mathscr{R}$ is *not* finitely generated as an $\mathscr{R}^{\mathrm{b}}$-module (since $\mathscr{R}^b$ is a field but the domain $\mathscr{R}$ is not), so the induced restriction functor from the category of $\mathscr{R}$-modules to $\mathscr{R}^{\mathrm{b}}$-modules does *not* restrict to a functor from $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$ to $\mathrm{Mod}^{\varphi}_{/\mathscr{R}^{\mathrm{b}}}$.

The following example will play a crucial role in what follows:

Example 2.3.4. — Let $(\mathscr{M}, \varphi_{\mathscr{M}}) \in \mathrm{Mod}^{\varphi}_{/\mathscr{O}}$. We claim that the $\mathscr{R}$-module $\mathscr{M}_{\mathscr{R}} := \mathscr{M} \otimes_{\mathscr{O}} \mathscr{R}$ equipped with $\varphi_{\mathscr{M}_{\mathscr{R}}} := \varphi_{\mathscr{M}} \otimes \varphi_{\mathscr{R}}$ is an object of $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$. Obviously the $\mathscr{R}$-module $\mathscr{M}_{\mathscr{R}}$ is free. Since $\mathscr{O} \to \mathscr{R}$ is flat and

$$1 \otimes \varphi_{\mathscr{M}} : \varphi_{\mathscr{O}}^* \mathscr{M} \to \mathscr{M}$$

is injective with cokernel killed by a power of E, we see that the $\mathscr{R}$-linearization of $\varphi_{\mathscr{M}_{\mathscr{R}}}$ is an isomorphism, as E is a unit in $\mathscr{R}$ (even in $\mathscr{R}^{\mathrm{int}}$) by Example 2.3.1.

Definition 2.3.5. — Let (M, φ_M) be a nonzero object of $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$. We say that (M, φ_M) is *pure of slope zero* if it descends to an object of $\mathrm{Mod}^{\varphi}_{/\mathscr{R}^{\mathrm{b}}}$ such that the matrix of φ on the descent has all eigenvalues with p-adic ordinal 0. By a suitable twisting procedure [**10**, Def. 1.6.1] we define *pure of slope* s similarly, for any $s \in \mathbf{Q}$. If $(\mathscr{M}, \varphi_{\mathscr{M}})$ is a nonzero object of $\mathrm{Mod}^{\varphi}_{/\mathscr{O}}$, we say that $(\mathscr{M}, \varphi_{\mathscr{M}})$ is *pure of slope* s if $(\mathscr{M}_{\mathscr{R}}, \varphi_{\mathscr{M}_{\mathscr{R}}}) \in \mathrm{Mod}^{\varphi}_{/\mathscr{R}}$ is.

Remarks 2.3.6. —
1. The notion of "pure of some slope s" is well-behaved with respect to tensor and exterior products; see [**10**, Cor. 1.6.4] (whose proof also applies to exterior products).
2. The condition "pure of slope zero" is equivalent to the existence of a φ_M-stable $\mathscr{R}^{\mathrm{int}}$-lattice $L \subseteq M$ with the property that the matrix of φ_M acting on L is invertible. This follows by a lattice-saturation argument with the linearization of φ_M viewed over a sufficiently large finite extension of the fraction field $\mathscr{R}^b$ of the henselian discrete valuation ring $\mathscr{R}^{\mathrm{int}}$ (where "sufficiently large" means large enough to contain certain eigenvalues).
3. Since $(\mathscr{R}^{\mathrm{b}})^{\times} = \mathscr{R}^{\times}$, a linear map $M' \to M$ of finite free $\mathscr{R}^{\mathrm{b}}$-modules is a direct summand (respectively surjective) if and only if the scalar extension $M'_{\mathscr{R}} \to M_{\mathscr{R}}$ to $\mathscr{R}$ is a direct summand (respectively surjective).

The following important theorem of Kedlaya [**10**, Thm. 1.7.1] elucidates the structure of $\mathscr{R}$-modules.

***Theorem 2.3.7* (Kedlaya).** — *For any nonzero object* (M, φ_M) *of* $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$, *there exists a unique filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$$

in $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$ *such that each successive quotient* M_i/M_{i-1} *is a nonzero object of* $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$ *that is pure of slope* s_i, *with the rational numbers* s_i *satisfying*

$$s_1 < s_2 < \cdots < s_r.$$

The filtration on a nonzero object (M, φ_M) guaranteed by Kedlaya's theorem is called the *slope filtration* of M. Given a nonzero object $\mathscr{M}$ of $\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathscr{O}}$, we know that $(\mathscr{M}_{\mathscr{R}}, \varphi_{\mathscr{M}_{\mathscr{R}}})$ is a nonzero object of $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$, and it is natural to ask if the slope filtration on $(\mathscr{M}_{\mathscr{R}}, \varphi_{\mathscr{M}_{\mathscr{R}}})$ has an interpretation purely in terms of $\mathscr{M}$ in the category $\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathscr{O}}$. This is indeed the case, as it is possible to use φ and N_∇ to obtain the following refinement of Theorem 2.3.7:

Theorem 2.3.8. — *Let* $\mathscr{M}$ *be a nonzero object of* $\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathscr{O}}$. *There exists a unique filtration*

$$0 = \mathscr{M}_0 \subseteq \mathscr{M}_1 \subseteq \cdots \subseteq \mathscr{M}_r = \mathscr{M} \tag{2.3.2}$$

in $\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathscr{O}}$ *whose successive quotients* $\mathscr{M}_i/\mathscr{M}_{i-1}$ *are nonzero objects of* $\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathscr{O}}$ *such that* (2.3.2) *descends the slope filtration of* $\mathscr{M}_{\mathscr{R}}$.

We are now able to translate the condition of weak admissibility for a filtered (φ, N)-module across the equivalence of categories in Theorem 2.2.1.

Theorem 2.3.9. — *A nonzero object* D *of* $\mathrm{MF}^{\varphi,N,\mathrm{Fil}\geq 0}_K$ *is weakly admissible if and only if the nonzero* $\underline{\mathscr{M}}(D)$ *is pure of slope zero.*

Proof. — Since $\underline{\mathscr{M}}$ is an exact covariant tensor-compatible functor, we have

$$\wedge^i \underline{\mathscr{M}}(D) \simeq \underline{\mathscr{M}}(\wedge^i D)$$

for all $i \geq 0$. But $\underline{\mathscr{M}}$ is an equivalence, so therefore it preserves rank (using the characterization of rank in terms of exterior algebra). It follows that

$$\det \underline{\mathscr{M}}(D) \simeq \underline{\mathscr{M}}(\det D).$$

Recalling that

$$t_N(D) = t_N(\det D) \text{ and } t_H(D) = t_H(\det D), \tag{2.3.3}$$

we are motivated to first treat the case that $\dim_{K_0} D = 1$.

If $\dim_{K_0} D = 1$ then since N is nilpotent, we must have $N = 0$. Setting $h := t_H(D)$, by the definition of $t_H(D)$ we have $\mathrm{Fil}^j D_K = D_K$ for all $j \leq h$ and

$\mathrm{Fil}^j D_K = 0$ for all $j \geq h+1$. Thus, from the definition of $\underline{\mathscr{M}}(D)$ (given in §2.2) we easily see that

$$\underline{\mathscr{M}}(D) = \lambda^{-h}(\mathscr{O} \otimes_{K_0} D).$$

If we select a K_0-basis $\mathbf{e}$ of D, then $\varphi_D(\mathbf{e}) = \alpha\mathbf{e}$ for some $\alpha \in K_0^\times$; by the definition of $t_N(D)$, we have $\mathrm{ord}_p(\alpha) = t_N(D)$. Viewing $\mathbf{e}$ as a $\mathscr{O}$-basis of $\mathscr{O} \otimes_{K_0} D$, we calculate (using (2.1.2))

$$\varphi_{\underline{\mathscr{M}}(D)}(\lambda^{-h}e) = \varphi_{\mathscr{O}}(\lambda)^{-h}\alpha\mathbf{e} = \left(\frac{E(u)}{E(0)}\right)^h \alpha(\lambda^{-h}\mathbf{e}). \tag{2.3.4}$$

Now $E(u) \in (\mathscr{R}^{\mathrm{int}})^\times$ by Example 2.3.1, and $E(0) \in p \cdot W^\times \subseteq p \cdot (\mathscr{R}^{\mathrm{int}})^\times$ so $(E(u)/E(0))^h \in p^{-t_H(D)} \cdot (\mathscr{R}^{\mathrm{int}})^\times$ by the definition of h. Since $\alpha \in p^{t_N(D)} \cdot (\mathscr{R}^{\mathrm{int}})^\times$, we conclude from (2.3.4) that $\underline{\mathscr{M}}(D)$ is pure of slope $t_N(D) - t_H(D)$ (by the definition of "pure slope"). This settles the case that D has rank 1.

It now follows formally from the properties of $\underline{\mathscr{M}}$ (such as det-compatibility) and of slopes, and the identities (2.3.3), that a nonzero D is weakly admissible when $\underline{\mathscr{M}}(D)$ is pure of slope zero.

For the converse, suppose that D is nonzero and weakly admissible. By Theorem 2.3.8, the slope filtration of $\underline{\mathscr{M}}(D)_{\mathscr{R}}$ descends to

$$0 = \mathscr{M}_0 \subseteq \mathscr{M}_1 \subseteq \cdots \subseteq \mathscr{M}_r = \underline{\mathscr{M}}(D) \tag{2.3.5}$$

in $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N_\nabla}$ with nonzero $\mathscr{M}_i/\mathscr{M}_{i-1} \in \mathrm{Mod}_{/\mathscr{O}}^{\varphi,N_\nabla}$ pure of slope $s_i \in \mathbf{Q}$ such that

$$s_1 < s_2 < \cdots < s_r.$$

Our goal is to show that $r = 1$ and $s_1 = 0$.

Set $d_i := \mathrm{rk}_{\mathscr{O}}\, \mathscr{M}_i/\mathscr{M}_{i-1}$ and note that $d_i \geq 1$. Since $\wedge^{d_i}(\mathscr{M}_i/\mathscr{M}_{i-1})$ is pure of slope $s_i d_i$ by the proof of [**10**, Cor. 1.6.4], it follows that $\det \underline{\mathscr{M}}(D) \simeq \otimes \det(\mathscr{M}_i/\mathscr{M}_{i-1})$ is pure of slope $\sum_i s_i d_i$. On the other hand, we deduce from our calculations in the rank-1 case that $\underline{\mathscr{M}}(\det(D))$ is pure of slope

$$t_N(\det D) - t_H(\det D) = t_N(D) - t_H(D) = 0$$

by (2.3.3) and the weak admissibility hypothesis. Since $\det \underline{\mathscr{M}}(D) = \underline{\mathscr{M}}(\det D)$ as observed before, we conclude that

$$\sum_i s_i d_i = 0. \tag{2.3.6}$$

As $s_1 < s_2 < \cdots < s_r$, in order to show what we want ($r = 1$ and $s_1 = 0$) it is therefore enough to show that $s_1 \geq 0$.

Since $\underline{\mathscr{M}}$ is an equivalence of categories by Theorem 2.2.1, corresponding to the nonzero subobject $\mathscr{M}_1$ of $\underline{\mathscr{M}}(D)$ (in $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N_\nabla}$) is a nonzero subobject D_1 of D (in $\mathrm{MF}_K^{\varphi,N}$) with

$$\mathscr{M}_1 = \underline{\mathscr{M}}(D_1).$$

We have calculated that $\det \mathscr{M}_1$ is pure of slope $s_1 d_1$, so since $\det \mathscr{M}_1 = \underline{\mathscr{M}}(\det D_1)$, which is pure of slope $t_N(D_1) - t_H(D_1)$ (again by the rank-1 case), we conclude that

$$s_1 d_1 = t_N(D_1) - t_H(D_1) \geq 0$$

as D_1 is a nonzero subobject of the weakly admissible filtered (φ, N)-module D (and therefore $t_N(D_1) - t_H(D_1) \geq 0$ by the *definition* of weak-admissibility). This gives $s_1 \geq 0$, as required. ■

2.4. Integral Theory. — We now describe a certain "integral theory" that will be used in the next lecture to study semi-stable Galois representations. To motivate this theory, we first define a new category of linear algebra data.

Definition 2.4.1. — Let $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N}$ be the category whose objects are triples $(\mathscr{M}, \varphi_{\mathscr{M}}, N)$ where

1. the pair $(\mathscr{M}, \varphi_{\mathscr{M}})$ is an object of $\mathrm{Mod}_{/\mathscr{O}}^{\varphi}$,
2. $N : \mathscr{M}/u\mathscr{M} \to \mathscr{M}/u\mathscr{M}$ is a K_0-linear endomorphism satisfying
$$N\varphi = p\varphi N,$$
where $\varphi := \varphi_{\mathscr{M}} \mod u$.

Morphisms in $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N}$ are $\mathscr{O}$-module homomorphisms compatible with $\varphi_{\mathscr{M}}$ and N.

Note that $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N}$ is defined exactly like $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N_\nabla}$, except that we only impose a monodromy operator "at the origin." Denote by $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N_\nabla,0}$ and $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N,0}$ the full subcategories of $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N_\nabla}$ and $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N}$, respectively, consisting of those objects that are 0 or of pure slope zero (where $\mathscr{M}$ is said to be pure of slope zero if $\mathscr{M} \otimes_{\mathscr{O}} \mathscr{R}$ is; cf. Definition 2.3.5). There is a natural "forgetful" functor

$$\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N_\nabla} \to \mathrm{Mod}_{/\mathscr{O}}^{\varphi,N} \tag{2.4.1}$$

defined by sending the triple $(\mathscr{M}, \varphi_{\mathscr{M}}, N_\nabla)$ to the triple $(\mathscr{M}, \varphi_{\mathscr{M}}, N_\nabla \mod u)$. Using the quasi-inverse equivalences of categories $\underline{\mathscr{M}}$ and $\underline{D}$, one proves (see [**11**, Lemma 1.3.10(2)]):

Lemma 2.4.2. — *The functor* (2.4.1) *is fully faithful.*

By Theorems 2.2.1 and 2.3.9, we obtain an exact, fully faithful tensor-functor

$$^{\mathrm{w.a.}}\mathrm{MF}_K^{\varphi,N,\mathrm{Fil}\geq 0} \xrightarrow[\underline{\mathscr{M}}]{\simeq} \mathrm{Mod}_{/\mathscr{O}}^{\varphi,N_\nabla,0} \xhookrightarrow[(2.4.1)]{} \mathrm{Mod}_{/\mathscr{O}}^{\varphi,N,0}. \tag{2.4.2}$$

The purpose of the "integral" theory that we will introduce is to describe the category $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N,0}$ and the essential image of (2.4.2) in more useful terms. Before we embark on this task, let us remark that by the exactness of $\underline{D}$, the "inverse" to (2.4.2) on its essential image is also exact.

Let $\mathfrak{S} := W[\![u]\!]$, and denote by $\varphi_{\mathfrak{S}}$ the unique semi-linear extension of the Frobenius endomorphism of W to $\mathfrak{S}$ that satisfies $\varphi_{\mathfrak{S}}(u) = u^p$. We now define analogues of $\mathrm{Mod}^{\varphi}_{/\mathscr{O}}$ and $\mathrm{Mod}^{\varphi,N}_{/\mathscr{O}}$ using $\mathfrak{S}$-modules.

Definition 2.4.3. — Let $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ be the category whose objects are pairs $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ where:

1. $\mathfrak{M}$ is a finite free $\mathfrak{S}$-module and $\varphi_{\mathfrak{M}}$ is a $\varphi_{\mathfrak{S}}$-semilinear endomorphism,
2. $\mathfrak{M}$ is of finite E-height in the sense that the cokernel of the $\mathfrak{S}$-linearization
$$1 \otimes \varphi_{\mathfrak{M}} : \varphi_{\mathfrak{S}}^{*}\mathfrak{M} \to \mathfrak{M}$$
is killed by some power E^h of E (so $1 \otimes \varphi_{\mathfrak{S}}$ is injective, and hence so is $\varphi_{\mathfrak{M}}$).

Morphisms in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ are φ-equivariant morphisms of $\mathfrak{S}$-modules.

As usual, we enhance the category $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ by adding a "monodromy operator":

Definition 2.4.4. — Let $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$ be the category whose objects are triples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, N_{\mathfrak{M}})$ where:

1. the pair $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is an object of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$,
2. $N_{\mathfrak{M}}$ is a K_0-linear endomorphism of $(\mathfrak{M}/u\mathfrak{M}) \otimes_W K_0$ which satisfies
$$N_{\mathfrak{M}} \circ \overline{\varphi}_{\mathfrak{M}} = p\overline{\varphi}_{\mathfrak{M}} \circ N_{\mathfrak{M}}$$
(with $\overline{\varphi}_{\mathfrak{M}} := \varphi_{\mathfrak{M}} \mod u$).

Morphisms in $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$ are morphisms in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ compatible with N mod u.

Remark 2.4.5. — Note that the definition of $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$ parallels that of $\mathrm{Mod}^{\varphi,N}_{/\mathscr{O}}$, except that we only impose $N_{\mathfrak{M}}$ on $(\mathfrak{M}/u\mathfrak{M}) \otimes_W K_0$ and *not* on $\mathfrak{M}/u\mathfrak{M}$. (This lack of integrality conditions on $N_{\mathfrak{M}}$ is solely because it is unclear if Lemma 2.4.7 is true with an integrality requirement on $N_{\mathfrak{M}}$.) Further, observe that $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ embeds as a full subcategory of $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$ by taking $N_{\mathfrak{M}} = 0$. We will not need the category $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ until the next lecture.

Remark 2.4.6. — Observe that $\mathfrak{S}\left[\frac{1}{p}\right] = \mathscr{O}^{\mathrm{bnd}}$ (the subring of functions on the open unit rigid-analytic disc that are bounded), and that the natural inclusion
$$\mathfrak{S}\left[\frac{1}{p}\right] \to \mathscr{O}$$
is faithfully flat. Moreover, it follows at once from the definition (2.3.1) of $\mathscr{R}^{\mathrm{int}}$ that we have a natural inclusion
$$\mathfrak{S}_{(p)} \to \mathscr{R}^{\mathrm{int}}$$

which is moreover faithfully flat, as it is a local extension of discrete valuation rings.

For the convenience of the reader, we summarize the relationships between the various rings considered above in the following diagram:

(2.4.3)
$$\begin{tikzcd}
\mathfrak{S} := W[\![u]\!] \arrow[r, hook] \arrow[d, hook] & \mathfrak{S}\left[\frac{1}{p}\right] \arrow[r, equal] & \mathscr{O}^{\mathrm{bnd}} \arrow[r, hook] \arrow[d, hook] & \mathscr{O} \arrow[d, hook] \\
\mathfrak{S}_{(p)} \arrow[r, hook] & \mathscr{R}^{\mathrm{int}} \arrow[r, hook] & \mathrm{Frac}(\mathscr{R}^{\mathrm{int}}) = \mathscr{R}^{\mathrm{b}} \arrow[r, hook] & \mathscr{R} \\
 & & \mathscr{R}^{\mathrm{b}} \setminus \{0\} \arrow[r, equal] \arrow[u, hook] & \mathscr{R}^{\times} \arrow[u, hook]
\end{tikzcd}$$

Let $\mathfrak{M}$ be any object of $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$. Then $\mathscr{M} := \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}$ is easily seen to be an object of $\mathrm{Mod}^{\varphi,N}_{/\mathscr{O}}$. In fact, since the natural inclusion $\mathfrak{S} \hookrightarrow \mathscr{O} \hookrightarrow \mathscr{R}$ has image in $\mathscr{R}^{\mathrm{int}}$ and $E \in (\mathscr{R}^{\mathrm{int}})^{\times}$, it follows from Remark 2.3.6(2) that $\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}$ is pure of slope zero if $\mathfrak{M} \neq 0$. Since p is invertible in $\mathscr{O}$, the resulting functor $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \to \mathrm{Mod}^{\varphi,N}_{/\mathscr{O}}$ factors through the p-isogeny category, so we obtain a functor

$$\Theta : \mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes \mathbf{Q}_p \to \mathrm{Mod}^{\varphi,N,0}_{/\mathscr{O}} \qquad \mathfrak{M} \mapsto \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}. \tag{2.4.4}$$

that respects tensor products and is exact.

Lemma 2.4.7. — *The functor* Θ *of* (2.4.4) *is an equivalence of categories.*

Proof. — We just explain how to functorially (up to p-isogeny) equip any object $\mathscr{M}$ of $\mathrm{Mod}^{\varphi,N,0}_{/\mathscr{O}}$ with a $\mathfrak{S}$-structure, and refer the reader to the proof of [**11**, Lemma 1.3.13] for the complete argument. The key algebraic inputs are:

$$\mathscr{R}^{\mathrm{b}} \cap \mathscr{O} = \mathscr{O}^{\mathrm{bnd}} = \mathfrak{S}\left[\frac{1}{p}\right] \quad \text{and} \quad \mathscr{R}^{\mathrm{int}} \cap \mathscr{O} = \mathfrak{S},$$

where both intersections are taken inside of the Robba ring $\mathscr{R}$; see (2.4.3). The idea to exploit this is the following: by definition of pure slope zero (Definition 2.3.5 and Remark 2.3.6(2)), there is a descent of $\mathscr{M}_{\mathscr{R}} := \mathscr{M} \otimes_{\mathscr{O}} \mathscr{R} \in \mathrm{Mod}^{\varphi}_{/\mathscr{R}}$ to an object $\mathscr{M}_{\mathscr{R}^{\mathrm{b}}}$ of $\mathrm{Mod}^{\varphi}_{/\mathscr{R}^{\mathrm{b}}}$ with a φ-stable $\mathscr{R}^{\mathrm{int}}$-lattice $\mathscr{L} \subseteq \mathscr{M}_{\mathscr{R}^{\mathrm{b}}}$. We "glue" the $\mathscr{O}$-module $\mathscr{M}$ and the $\mathscr{R}^{\mathrm{int}}$-lattice $\mathscr{L}$ to get a module $\mathfrak{M}$ over $\mathscr{O} \cap \mathscr{R}^{\mathrm{int}} = \mathfrak{S}$.

To be more precise, $\mathscr{M}_{\mathscr{R}^{\mathrm{b}}}$ is functorial in $\mathscr{M}_{\mathscr{R}}$ [**10**, Prop. 1.5.5], and under the isomorphism

$$\mathscr{M}_{\mathscr{R}^{\mathrm{b}}} \otimes_{\mathscr{R}^{\mathrm{b}}} \mathscr{R} \simeq \mathscr{M}_{\mathscr{R}}$$

there exists a subset of $\mathscr{M}_{\mathscr{R}}$ that is both an $\mathscr{O}$-basis of $\mathscr{M}$ and an $\mathscr{R}^{\mathrm{b}}$-basis of $\mathscr{M}_{\mathscr{R}^{\mathrm{b}}}$. Indeed, if we choose an $\mathscr{O}$-basis $\{v_i\}$ of $\mathscr{M}$ and an $\mathscr{R}^{\mathrm{b}}$-basis $\{w_j\}$ of $\mathscr{M}_{\mathscr{R}^{\mathrm{b}}}$ then each is an $\mathscr{R}$-basis of $\mathscr{M}_{\mathscr{R}}$, so there is an invertible matrix A over $\mathscr{R}$

carrying $\{v_i\}$ to $\{w_j\}$. By the first part of [**8**, Proposition 6.5], we can express A as a product (in either order) of an invertible matrix over $\mathscr{O}$ and an invertible matrix over $\mathscr{R}^{\mathrm{b}}$, so by using such factor matrices to change the respective choices of $\{v_i\}$ over $\mathscr{O}$ and $\{w_j\}$ over $\mathscr{R}^{\mathrm{b}}$ we get the asserted "common basis". It follows that

$$\mathscr{M}^{\mathrm{b}} := \mathscr{M} \cap \mathscr{M}_{\mathscr{R}^{\mathrm{b}}} \subseteq \mathscr{M}_{\mathscr{R}}$$

is a φ-stable, finite free $\mathscr{R}^{\mathrm{b}} \cap \mathscr{O} = \mathfrak{S}\left[\frac{1}{p}\right]$-module descending $(\mathscr{M}, \varphi_{\mathscr{M}})$. This shows that Θ is fully faithful, since for any object $\mathfrak{M}$ of $\mathrm{Mod}_{/\mathfrak{S}}^{\varphi,N}$, the object $\mathscr{M} := \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}$ satisfies

$$\mathscr{M}^{\mathrm{b}} = \mathfrak{M}\left[\frac{1}{p}\right],$$

so we recover $\mathfrak{M}$ up to p-isogeny from $\mathscr{M}$.

Now for any object $\mathscr{M}$ of $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N}$, the $\mathscr{R}^{\mathrm{int}}$-lattice $\mathscr{L}$ inside $\mathscr{M}_{\mathscr{R}^{\mathrm{b}}}$ allows one to equip $\mathscr{M}^{\mathrm{b}} = \mathscr{M} \cap \mathscr{M}_{\mathscr{R}^{\mathrm{b}}}$ with the desired $\mathfrak{S}$-structure (up to p-isogeny); see the proof of [**11**, Lemma 1.3.13] for the details. ∎

Using the fully faithful functor (2.4.2) and the "inverse" of Θ in (2.4.4), we have:

Corollary 2.4.8. — *There exists an exact and fully faithful tensor functor*

$$\widetilde{\Theta} : {}^{\mathrm{w.a}}\mathrm{MF}_K^{\varphi,N,\mathrm{Fil}\geq 0} \hookrightarrow \mathrm{Mod}_{/\mathfrak{S}}^{\varphi,N} \otimes \mathbf{Q}_p. \tag{2.4.5}$$

Thus, for any object D of ${}^{\mathrm{w.a}}\mathrm{MF}_K^{\varphi,N,\mathrm{Fil}\geq 0}$, there is a canonical $\mathfrak{S}$-structure on $\underline{\mathscr{M}}(D)$ up to p-isogeny. For example, in the next section, we will be particularly interested in the case that

$$D = \underline{D}_{\mathrm{st}*}(V) := \left(B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V\right)^{G_K}$$

for some object V of $\mathrm{Rep}_{G_K}^{\mathrm{st}}$.

We now wish to describe the essential image of $\widetilde{\Theta}$. To do this, we must answer the following question: for which objects $\mathfrak{M}$ of $\mathrm{Mod}_{/\mathfrak{S}}^{\varphi,N}$ does the object $\mathscr{M} := \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}$ of $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N,0}$ admit an operator $N_\nabla^{\mathscr{M}}$ as in Definition 2.1.5(2) that lifts $N_{\mathfrak{M}} \otimes 1$ on $(\mathfrak{M}/u\mathfrak{M}) \otimes_W K_0 = \mathscr{M}/u\mathscr{M}$ and makes the triple $(\mathscr{M}, \varphi_{\mathscr{M}}, N_\nabla^{\mathscr{M}})$ into an object of $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N_\nabla}$?

Thanks to Lemma 2.2.5, for *any* object $\mathscr{M}$ of $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N}$ we have an injective map of finite free $\mathscr{O}$-modules

$$\xi : \underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O} \hookrightarrow \mathscr{M}$$

with cokernel killed by λ^h (where h is the E-height of $\mathscr{M}$), so in particular ξ is an isomorphism after inverting λ. Therefore, there exists a unique connection

$$\nabla_{\mathscr{M}} : \mathscr{M}\left[\frac{1}{\lambda u}\right] \to \mathscr{M}\left[\frac{1}{\lambda u}\right] \otimes_{\mathscr{O}} \Omega^1_{\Delta/K_0} \tag{2.4.6}$$

satisfying $\nabla_{\mathscr{M}}(d) = -N(d)\frac{du}{u}$ for all $d \in \underline{D}(\mathscr{M})$. Moreover, $\nabla_{\mathscr{M}}$ commutes with $\varphi_{\mathscr{M}}$ and has poles of order at most h supported on the zeroes of λ, and at worst a simple pole at $u = 0$.

Defining $N_{\nabla}^{\mathscr{M}} : \mathscr{M}[1/\lambda u] \to \mathscr{M}[1/\lambda u]$ by the relation

$$\nabla_{\mathscr{M}}(m) = -\frac{1}{\lambda} N_{\nabla}^{\mathscr{M}}(m)\frac{du}{u} \tag{2.4.7}$$

for all $m \in \mathscr{M}$, as in Remark 2.1.6, gives the only *possible* $N_{\nabla}^{\mathscr{M}}$ for $\mathscr{M} = \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}$ as above. In case $\mathscr{M}$ has $\mathscr{O}$-rank 1, it follows from a calculation (see the proof of [**11**, 1.3.10(3)]) that $\nabla_{\mathscr{M}}$ has at worst simple poles; that is, $N_{\nabla}^{\mathscr{M}}$ carries $\mathscr{M}$ into itself in the rank-1 case. Thus:

Corollary 2.4.9. — *Let $\mathfrak{M}$ be an object of $\mathrm{Mod}_{/\mathfrak{S}}^{\varphi,N} \otimes \mathbf{Q}_p$ and let $\mathscr{M} := \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}$ be the corresponding object of $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N,0}$. Then $\mathfrak{M}$ is in the image of $\widetilde{\Theta}$ if and only if the connection $\nabla_{\mathscr{M}}$ as in* (2.4.6) *has at worst simple poles* (*equivalently, if and only if the operator $N_{\nabla}^{\mathscr{M}}$ defined by* (2.4.7) *is holomorphic*)*. In particular, any such $\mathfrak{M}$ with $\mathfrak{S}$-rank* 1 *is in the image of $\widetilde{\Theta}$.*

3. Second Lecture

This lecture introduces the category of $\mathfrak{S}$-modules, roughly an integral version of the category of vector bundles with connection from §2, and we set up a fully faithful functor from the category of effective weakly admissible filtered (φ, N)-modules to the isogeny category of $\mathfrak{S}$-modules (and we describe the essential image). In the reverse direction we construct a fully faithful functor from the category of $\mathfrak{S}$-modules into the category of G_{K_∞}-stable lattices in semistable G_K-representations, where K_∞/K is generated by compatible p-power roots of a uniformizer π of $\mathscr{O}_K$. As applications, we obtain a proof of the conjecture of Fontaine that the natural fully faithful functor from semistable representations to weakly admissible modules is an equivalence and we obtain a proof of the conjecture of Breuil that restriction from crystalline G_K-modules to underlying G_{K_∞}-modules is fully faithful. We also use $\mathfrak{S}$-modules to describe the category of all G_{K_∞}-stable lattices in crystalline representations of G_K.

We begin by using the fully faithful tensor-compatible functor

$$\begin{array}{ccc} {}^{\mathrm{w.a.}}\mathrm{MF}_K^{\varphi,N,\mathrm{Fil}\geq 0} & \xhookrightarrow{\widetilde{\Theta}} & \mathrm{Mod}_{/\mathfrak{S}}^{\varphi,N} \otimes \mathbf{Q}_p \\ & & \\ D & \longmapsto & \text{“}\mathfrak{S}\text{-structures on } \underline{\mathscr{M}}(D)\text{”} \end{array}$$

to study $\mathrm{Rep}_{G_K}^{\mathrm{st}}$. Let us first introduce the following notation: fix a profinite group H (e.g., $H = G_K$) and let

$\mathrm{Rep}_H^{\mathrm{tor}}$ = category of continuous H-representations on finite abelian p-groups,

$\mathrm{Rep}_{H/\mathbf{Z}_p}$ = category of continuous H-representations on finite free $\mathbf{Z}_p$-modules,

Rep_H = category of continuous H-representations on finite dimensional $\mathbf{Q}_p$-vector spaces.

Morphisms in each category are the obvious ones; observe that Rep_H is the p-isogeny category of $\mathrm{Rep}_{H/\mathbf{Z}_p}$.

Recall that K/K_0 is a totally ramified extension of $K_0 = \mathrm{Frac}(W)$ with uniformizer $\pi \in \mathscr{O}_K$. Choose a compatible sequence of p-power roots of π:

$$\pi_n := \sqrt[p^n]{\pi} \in \overline{K} \quad (\pi_0 = \pi),$$

set $K_\infty := \cup K_0(\pi_n) \subseteq \overline{K}$, and let $G_{K_\infty} := \mathrm{Gal}(\overline{K}/K_\infty) \subseteq G_K$.

The main goals of this lecture are:

1. Show that weakly admissible implies admissible; i.e. that if D is a nonzero object of ${}^{\mathrm{w.a.}}\mathrm{MF}_K^{\varphi,N}$ then
$$D = \underline{D}_{\mathrm{st}*}(V) := \left(B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V\right)^{G_K}$$
for some object V of $\mathrm{Rep}_{G_K}^{\mathrm{st}}$.
2. Show that the restriction of the natural functor
$$\mathrm{Rep}_{G_K} \to \mathrm{Rep}_{G_{K_\infty}}$$
to the subcategory of crystalline representations $\mathrm{Rep}_{G_K}^{\mathrm{cris}} \subseteq \mathrm{Rep}_{G_K}$ is fully faithful, and describe G_{K_∞}-stable $\mathbf{Z}_p$-lattices in crystalline p-adic representations of G_K using $\mathrm{Mod}_{/\mathfrak{S}}^{\varphi}$ (recall from Definition 2.4.3 that this is “the category $\mathrm{Mod}_{/\mathfrak{S}}^{\varphi,N}$ with $N = 0$”).

3.1. Étale φ-modules. — The starting point is a fundamental theory of Fontaine that describes the entire category $\mathrm{Rep}_{G_{K_\infty}}$ in terms of semilinear algebra data over an extension of $\mathfrak{S}$. Recall the following definition from Breuil's lectures:

Definition 3.1.1. — Let R be the $\mathbf{F}_p$-algebra

$$R := \varprojlim_{c \mapsto c^p} \mathscr{O}_{\overline{K}}/p\mathscr{O}_{\overline{K}}.$$

This is easily seen to be a perfect domain. (Note also that $\mathscr{O}_{\overline{K}}/p\mathscr{O}_{\overline{K}} = \mathscr{O}_{\mathbf{C}_K}/p\mathscr{O}_{\mathbf{C}_K}$.)

Remark 3.1.2. — There is a natural map

$$\varprojlim_{c \mapsto c^p} \mathscr{O}_{\mathbf{C}_K} \to \varprojlim_{c \mapsto c^p} \mathscr{O}_{\overline{K}}/p\mathscr{O}_{\overline{K}} = R$$

given by reduction modulo p at each finite level. This map is a multiplicative bijection (cf. [**5**, §1.2.2]). We can therefore think of any element $x \in R$ as a sequence $(x^{(m)})_{m \geq 0}$ of elements of $\mathscr{O}_{\mathbf{C}_K}$ satisfying $(x^{(m)})^p = x^{(m-1)}$ for all $m \geq 1$. In particular, $x = 0$ in R if and only if $x^{(0)} = 0$ in $\mathscr{O}_{\mathbf{C}_K}$. The ring structure of these sequences in characteristic 0 is given by $(x+y)^{(m)} = \lim_{n\to\infty}(x^{(n+m)}+y^{(n+m)})^{p^n} \in \mathscr{O}_{\mathbf{C}_K}$. The motivation for considering R is probably the desire to find a Witt-type construction of a complete discrete valuation ring with residue field $\mathbf{C}_K$.

Defining $|(x^{(m)})_{m\geq 0}|_R := |x^{(0)}|_{\mathbf{C}_K}$ makes R into a complete valuation ring of characteristic p; evidently R contains $\overline{k}$, so $W(\overline{k}) \subseteq W(R) \subseteq W(\mathrm{Frac}(R))$. Moreover, $\mathrm{Frac}(R)$ is algebraically closed (this is not obvious, and is part of the general Fontaine–Wintenberger theory of norm fields but also admits a direct proof), and G_K acts by isometries (with respect to $|\cdot|_R$) on R.

Example 3.1.3. — Let $\underline{\pi} := (\pi_n)_{n\geq 0} \in R$. The isotropy subgroup of $\underline{\pi}$ in G_K is clearly G_{K_∞}.

There is a natural map

$$\begin{tikzcd} \mathfrak{S} = W[\![u]\!] \arrow[loop above, "\varphi_{\mathfrak{S}}"] \arrow[r, hook] & W(R) \arrow[loop above, "\mathrm{Frob}_R"] \end{tikzcd} \tag{3.1.1}$$

$$\textstyle\sum_{n\geq 0} a_n u^n \longmapsto \sum_{n\geq 0} a_n [\underline{\pi}]^n$$

which is G_{K_∞}-invariant (by Example 3.1.3) and φ-compatible (as $[\underline{\pi}^p] = [\underline{\pi}]^p$). Since $\underline{\pi} \in \mathrm{Frac}\, R$ is nonzero we have $[\underline{\pi}] \in W(\mathrm{Frac}\, R)^\times$, so (3.1.1) extends to a map

$$\mathfrak{S}\left[\tfrac{1}{u}\right] \hookrightarrow W(\mathrm{Frac}\, R)\ . \tag{3.1.2}$$

The source of this map is a Dedekind domain in which (p) is a prime ideal and the target is a complete discrete valuation ring with uniformizer p, so (3.1.2) gives a map

$$j : \mathfrak{S}\left[\tfrac{1}{u}\right]^{\wedge}_{(p)} \hookrightarrow W(\mathrm{Frac}\, R)$$

that fits into a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{\mathscr{E}} := \mathfrak{S}\left[\tfrac{1}{u}\right]^{\wedge}_{(p)} \arrow[r, hook, "j"] \arrow[d] & W(\mathrm{Frac}\, R) \arrow[d] \\
k(\!(u)\!) \arrow[r] & \mathrm{Frac}\, R
\end{tikzcd} \tag{3.1.3}$$

(with both horizontal maps defined by sending u to $[\underline{\pi}]$, and the bottom map over k). Since $\mathrm{Frac}(R)$ is algebraically closed, the bottom side of the diagram provides a separable closure of $k(\!(u)\!)$ in $\mathrm{Frac}(R)$.

Clearly, $\mathscr{O}_{\mathscr{E}}$ in (3.1.3) is a complete discrete valuation ring with uniformizer p, and it has a "Frobenius endomorphism" $\varphi_{\mathscr{O}_{\mathscr{E}}}$ induced by $\varphi_{\mathfrak{S}}$; due to the φ-equivariance of (3.1.1), the horizontal maps in (3.1.3) are φ-compatible. Let $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}/\mathscr{O}_{\mathscr{E}}$ be the maximal unramified extension of $\mathscr{O}_{\mathscr{E}}$ with respect to the separable closure $k(\!(u)\!)_{\mathrm{sep}} \subseteq \mathrm{Frac}(R)$ of $k(\!(u)\!)$. We define

$$\mathscr{E} := \mathrm{Frac}(\mathscr{O}_{\mathscr{E}}) \quad \text{and} \quad \mathscr{E}^{\mathrm{un}} := \mathrm{Frac}(\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}).$$

By the universal property of the strict henselization $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ of $\mathscr{O}_{\mathscr{E}}$, there exists a unique map

$$\widetilde{j} : \mathscr{O}_{\mathscr{E}}^{\mathrm{un}} \hookrightarrow W(\mathrm{Frac}\, R)$$

over j which lifts the inclusion $k(\!(u)\!)_{\mathrm{sep}} \hookrightarrow \mathrm{Frac}\, R$ on residue fields. We thus obtain a commutative diagram

$$\begin{tikzcd}
\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}[1/p] = \widehat{\mathscr{E}^{\mathrm{un}}} \arrow[rr, hook] & & W(\mathrm{Frac}\, R)[1/p] \arrow[dd, hook] \\
\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \arrow[u, hook] \arrow[drr, hook] & & \\
\mathscr{O}_{\mathscr{E}}^{\mathrm{un}} \arrow[u, hook] \arrow[rr, hook, "\widetilde{j}"] & & W(\mathrm{Frac}\, R) \\
\mathscr{O}_{\mathscr{E}} \arrow[u, hook] \arrow[urr, hook, "j"'] & &
\end{tikzcd}$$

The unicity of $\widetilde{j}$ implies that the G_{K_∞}-action on $W(\mathrm{Frac}\, R)$ over $\mathscr{O}_{\mathscr{E}}$ preserves the subring $\widetilde{j}(\mathscr{O}_{\mathscr{E}}^{\mathrm{un}})$.

Remark 3.1.4. — The natural map $\mathfrak{S} \to \mathscr{O}_{\mathscr{E}}$ is flat, and the natural map $\mathfrak{S}_{(p)} \to \mathscr{O}_{\mathscr{E}}$ is faithfully flat. Moreover, since $E \equiv u^e \mod p$ ($e = [K : K_0]$), we see that $E \in \mathscr{O}_{\mathscr{E}}^{\times}$ because $\mathscr{O}_{\mathscr{E}}$ is a discrete valuation ring having residue field $k(\!(u)\!)$.

The following important theorem (a special case of a general result in the Fontaine–Wintenberger theory of norm fields) will allow us to study G_{K_∞}-representations via characteristic-p methods:

Theorem 3.1.5. — *The natural action of G_{K_∞} on $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ via the inclusion $\widetilde{j}$ induces an isomorphism of topological groups*

$$G_{K_\infty} \xrightarrow{\simeq} \mathrm{Aut}(\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}/\mathscr{O}_{\mathscr{E}}) \simeq G_{k(\!(u)\!)}.$$

Definition 3.1.6. — An *étale φ-module* is a finitely generated $\mathscr{O}_{\mathscr{E}}$-module M equipped with a $\varphi_{\mathscr{O}_{\mathscr{E}}}$-semilinear map $M \to M$ whose $\mathscr{O}_{\mathscr{E}}$-linearization $\varphi^*_{\mathscr{O}_{\mathscr{E}}} M \to M$ is an isomorphism. We denote by $\mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}}$ the category whose objects are étale φ-modules that are free over $\mathscr{O}_{\mathscr{E}}$, and by $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{/\mathscr{O}_{\mathscr{E}}}$ the category of étale φ-modules that are killed by a power of p. Morphisms in these categories are φ-compatible maps of $\mathscr{O}_{\mathscr{E}}$-modules.

Example 3.1.7. — It follows easily from Remark 3.1.4 that for any object $\mathfrak{M}$ of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$, the scalar extension $M := \mathscr{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ is an object of $\mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}}$.

We define contravariant functors

$$\mathrm{Mod}^{\varphi,\mathrm{tor}}_{/\mathscr{O}_{\mathscr{E}}} \underset{\underline{D}^*_{\mathscr{O}_{\mathscr{E}}}}{\overset{\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}}{\rightleftarrows}} \mathrm{Rep}^{\mathrm{tor}}_{G_{K_\infty}}$$

by

$$\begin{aligned}\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}(M) &:= \mathrm{Hom}_{\mathscr{O}_{\mathscr{E}},\varphi}\left(M, \mathscr{E}^{\mathrm{un}}/\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}\right)\\ \underline{D}^*_{\mathscr{O}_{\mathscr{E}}}(T) &:= \mathrm{Hom}_{G_{K_\infty}}\left(T, \mathscr{E}^{\mathrm{un}}/\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}\right).\end{aligned}$$

Similarly, we define

$$\underline{V}^*_{\mathscr{O}_{\mathscr{E}}} : \mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}} \longrightarrow \mathrm{Rep}_{G_{K_\infty}/\mathbf{Z}_p}$$

by

$$\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}(M) := \mathrm{Hom}_{\mathscr{O}_{\mathscr{E}},\varphi}\left(M, \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}\right),$$

and likewise $\underline{D}^*_{\mathscr{O}_{\mathscr{E}}} : \mathrm{Rep}_{G_{K_\infty}/\mathbf{Z}_p} \to \mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}}$ is defined by

$$\underline{D}^*_{\mathscr{O}_{\mathscr{E}}}(T) := \mathrm{Hom}_{G_{K_\infty}}(T, \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}).$$

(It is not obvious that $\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}(M)$ and $\underline{D}^*_{\mathscr{O}_{\mathscr{E}}}(T)$ lie in the asserted target categories.) It will always be clear from the context on what category of objects (torsion or finite free) we are considering the functors $\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}$ and $\underline{D}^*_{\mathscr{O}_{\mathscr{E}}}$.

***Lemma 3.1.8* (Fontaine).** — *With the notation above, we have the following properties.*

1. *The functors $\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}$ and $\underline{D}^*_{\mathscr{O}_{\mathscr{E}}}$ between torsion categories are quasi-inverse equivalences of categories that are exact and tensor-compatible and preserve "invariant factors": i.e. we have as modules*
$$M \simeq \bigoplus_i \mathscr{O}_{\mathscr{E}}/p^{n_i}\mathscr{O}_{\mathscr{E}} \quad \text{if and only if} \quad \underline{V}^*_{\mathscr{O}_{\mathscr{E}}}(M) \simeq \bigoplus_i \mathbf{Z}/p^{n_i}\mathbf{Z}.$$
2. *The contravariant functors $\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}$ and $\underline{D}^*_{\mathscr{O}_{\mathscr{E}}}$ between the "finite free" module categories are rank-preserving, exact, tensor-compatible quasi-inverse equivalences of categories.*

Proof. — This is a special case of [**4**, 1.2.4, 1.2.6, 1.2.7] (which works with $k(\!(u)\!)$ replaced by an arbitrary field of characteristic p). ∎

Remark 3.1.9. — There are dualities on $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{/\mathscr{O}_{\mathscr{E}}}$, $\mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}}$ and on $\mathrm{Rep}^{\mathrm{tor}}_{G_{K_\infty}}$, $\mathrm{Rep}_{G_{K_\infty}/\mathbf{Z}_p}$ given by $\mathrm{Hom}_{\mathscr{O}_{\mathscr{E}},\varphi}(-,\mathscr{E}/\mathscr{O}_{\mathscr{E}})$, $\mathrm{Hom}_{\mathscr{O}_{\mathscr{E}},\varphi}(-,\mathscr{O}_{\mathscr{E}})$, $\mathrm{Hom}_{G_{K_\infty}}(-,\mathbf{Q}_p/\mathbf{Z}_p)$, and $\mathrm{Hom}_{\mathbf{Z}_p[G_{K_\infty}]}(-,\mathbf{Z}_p)$ respectively. The functors $\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}$ and $\underline{D}^*_{\mathscr{O}_{\mathscr{E}}}$ naturally commute with these dualities. Moreover, we have a covariant analogue of $\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}$ defined by
$$\underline{V}_{\mathscr{O}_{\mathscr{E}}*}(M) := \begin{cases} \left(M \otimes_{\mathscr{O}_{\mathscr{E}}} \mathscr{O}^{\mathrm{un}}_{\mathscr{E}}\right)^{\varphi=\mathrm{id}} & \text{if } M \in \mathrm{Mod}^{\varphi,\mathrm{tor}}_{/\mathscr{O}_{\mathscr{E}}}, \\ \left(M \otimes_{\mathscr{O}_{\mathscr{E}}} \widehat{\mathscr{O}^{\mathrm{un}}_{\mathscr{E}}}\right)^{\varphi=\mathrm{id}} & \text{if } M \in \mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}}. \end{cases}$$

Our aim is to adapt Fontaine's theory of étale φ-modules to study $\mathfrak{S}$-modules (rather than $\mathscr{O}_{\mathscr{E}}$-modules), and to find an analogue of Lemma 3.1.8 describing the essential image of $\mathrm{Rep}^{\mathrm{cris}}_{G_K}$ in $\mathrm{Rep}_{G_{K_\infty}}$ in terms of $\mathfrak{S}$-modules and certain linear algebra data on them. To do this, we will replace $\mathscr{O}^{\mathrm{un}}_{\mathscr{E}}$ and $\widehat{\mathscr{O}^{\mathrm{un}}_{\mathscr{E}}}$ with
$$\mathfrak{S}^{\mathrm{un}} := \mathscr{O}_{\mathscr{E}^{\mathrm{un}}} \cap W(R) \subseteq W(\mathrm{Frac}\, R),$$
$$\widehat{\mathfrak{S}^{\mathrm{un}}} := \widehat{\mathscr{O}_{\mathscr{E}^{\mathrm{un}}}} \cap W(R) \subseteq W(\mathrm{Frac}\, R).$$

Note that $\mathfrak{S} \subseteq \mathscr{O}_{\mathscr{E}} \cap W(R)$ (so $\mathfrak{S}^{\mathrm{un}}$ is a flat $\mathfrak{S}$-module) and that $\widehat{\mathfrak{S}^{\mathrm{un}}}$ is isomorphic to the p-adic completion of $\mathfrak{S}^{\mathrm{un}}$ (justifying the notation), as $W(R)$ is p-adically separated and complete.

Warning 3.1.10. — Unlike the case of modules over the discrete valuation ring $\mathscr{O}_{\mathscr{E}}$, finitely generated p-power torsion $\mathfrak{S}$-modules need *not* be isomorphic to a direct sum of modules of the form $\mathfrak{S}/p^n\mathfrak{S}$ (for example, let $I \subseteq \mathfrak{S}$ be the ideal $I = (p^2 - u, u^2)$, and consider the module $\mathfrak{S}/I$). The correct analogue of "finitely generated p-power torsion $\mathscr{O}_{\mathscr{E}}$-module" in this context turns out to be a finite p-power torsion $\mathfrak{S}$-module of projective dimension at most 1.

3.2. $\mathfrak{S}$-modules and G_{K_∞}-representations. — Recall the definition (Definition 2.4.3) of the category $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$. We will treat this category as an analogue of $\mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}}$. We now define the $\mathfrak{S}$-module analogue of the category $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{/\mathscr{O}_{\mathscr{E}}}$:

Definition 3.2.1. — Let $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{/\mathfrak{S}}$ be the category whose objects are finite $\mathfrak{S}$-modules $\mathfrak{M}$ such that:

1. $\mathfrak{M}$ is killed by some power of p and $\operatorname{projdim}\mathfrak{M} \leq 1$,
2. there is a $\varphi_{\mathfrak{S}}$-semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that the $\mathfrak{S}$-linearization
$$1 \otimes \varphi_{\mathfrak{M}} : \varphi^*_{\mathfrak{S}}\mathfrak{M} \to \mathfrak{M}$$
is injective and has cokernel killed by some power E^h of E (so $\varphi_{\mathfrak{M}}$ is injective).

Morphisms in $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{/\mathfrak{S}}$ are φ-compatible maps of $\mathfrak{S}$-modules.

Observe that if $\mathfrak{M}$ is a direct sum of $\mathfrak{S}$-modules of the type $\mathfrak{S}/p^n\mathfrak{S}$ then any $\varphi_{\mathfrak{S}}$-semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ has $\mathfrak{S}$-linearization that is automatically injective since the image of E in $(\mathfrak{S}/p\mathfrak{S})[1/u] = k(\!(u)\!)$ is a unit. Although not every object $\mathfrak{M}$ of $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{/\mathfrak{S}}$ is a direct sum of objects of the form $\mathfrak{S}/p^n\mathfrak{S}$, we do have:

Lemma 3.2.2. — *Every object $\mathfrak{M}$ of $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{/\mathfrak{S}}$ is a successive extension of objects that are free over $\mathfrak{S}/p\mathfrak{S}$.*

Proof. — See the proof of [**11**, Lemma 2.3.2], and note that although that proof assumes that the cokernel of $1 \otimes \varphi_{\mathfrak{M}} : \varphi^*_{\mathfrak{S}}\mathfrak{M} \to \mathfrak{M}$ is killed by E, the same argument works with any power E^h of E. ■

Using the (obvious) fact that
$$\mathfrak{S}/p\mathfrak{S} = k[\![u]\!] \subseteq k(\!(u)\!) = \mathscr{O}_{\mathscr{E}}/p\mathscr{O}_{\mathscr{E}},$$
calculations of Fontaine give:

Lemma 3.2.3. — *Let $\mathfrak{M}$ be any object of $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{/\mathfrak{S}}$. Then there is a natural isomorphism of $\mathbf{Z}_p[G_{K_\infty}]$-modules*
$$\underline{V}^*_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}\left(\mathfrak{M}, \mathfrak{S}^{\mathrm{un}}\left[\frac{1}{p}\right]/\mathfrak{S}^{\mathrm{un}}\right) \xrightarrow{\simeq} \underline{V}^*_{\mathscr{O}_{\mathscr{E}}}(\mathscr{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{M})\ .$$

It follows immediately from Lemma 3.1.8 and Remark 3.1.4 that $\underline{V}^*_{\mathfrak{S}}$ is exact, commutes with tensor products, and
$$\text{if}\quad \mathfrak{M} \simeq \bigoplus_i \mathfrak{S}/p^{n_i}\mathfrak{S} \quad\text{then}\quad \underline{V}^*_{\mathfrak{S}}(\mathfrak{M}) \simeq \bigoplus_i \mathbf{Z}/p^{n_i}\mathbf{Z}.$$

Passing to inverse limits gives:

Corollary 3.2.4. — *We have:*

1. *Let* $\mathfrak{M}$ *be any object of* $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$. *Then*
$$\underline{V}^*_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}\left(\mathfrak{M}, \widehat{\mathfrak{S}^{\mathrm{un}}}\right)$$
is a finite free $\mathbf{Z}_p$*-module of rank equal to* $\mathrm{rk}_{\mathfrak{S}}(\mathfrak{M})$ *and the natural map of* $\mathbf{Z}_p[G_{K_\infty}]$*-modules*
$$\underline{V}^*_{\mathfrak{S}}(\mathfrak{M}) \longrightarrow \underline{V}^*_{\mathscr{O}_{\mathscr{E}}}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathscr{E}})$$
obtained by extending scalars to $\mathscr{O}_{\mathscr{E}}$ *is an isomorphism.*
2. *Let* $\underline{V}^*_{\mathfrak{S}} : \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \to \mathrm{Rep}_{G_{K_\infty}/\mathbf{Z}_p}$ *be the functor defined in* (1). *For all* $n \geq 1$, *there are natural isomorphisms*
$$\underline{V}^*_{\mathfrak{S}}(\mathfrak{M})/(p^n) \xrightarrow{\simeq} \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}/p^n\mathfrak{M}, \mathfrak{S}^{\mathrm{un}}/p^n\mathfrak{S}^{\mathrm{un}}) \xrightarrow{\simeq} \underline{V}^*_{\mathfrak{S}}(\mathfrak{M}/p^n\mathfrak{M}).$$

Thus, the functor $\underline{V}^*_{\mathfrak{S}}$ on the category $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ is exact and commutes with tensor products.

Remark 3.2.5. — For any object $\mathfrak{M}$ of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$, the $\mathbf{Z}_p[G_{K_\infty}]$-module
$$\underline{V}_{\mathfrak{S}*}(\mathfrak{M}) := \left(\mathfrak{M} \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}^{\mathrm{un}}}\right)^{\varphi=1}$$
satisfies
$$\underline{V}_{\mathfrak{S}*}(\mathfrak{M})^{\vee} \simeq \underline{V}^*_{\mathfrak{S}}(\mathfrak{M}^{\vee}).$$

Recall from Lemma 3.1.8 that the functor $\underline{D}^*_{\mathscr{O}_{\mathscr{E}}}$ provides a quasi-inverse to $\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}$. The $\mathfrak{S}$-module analogue of this is:

Lemma 3.2.6. — *Let* $\mathfrak{M}$ *be an object of* $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ *with* $\mathfrak{S}$*-rank equal to* d, *and define*
$$\mathfrak{M}' := \mathrm{Hom}_{\mathbf{Z}_p[G_{K_\infty}]}\left(\underline{V}^*_{\mathfrak{S}}(\mathfrak{M}), \widehat{\mathfrak{S}^{\mathrm{un}}}\right).$$
Then $\mathfrak{M}'$ *is a finite-free* $\mathfrak{S}$*-module of rank* d, *and the natural map* $\mathfrak{M} \to \mathfrak{M}'$ *is injective.*

Using Corollary 3.2.4 and Lemma 3.2.6, we can now prove:

Proposition 3.2.7. — *The functor* $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \to \mathrm{Mod}^{\varphi}_{\mathscr{O}_{\mathscr{E}}}$ *given by*
$$\mathfrak{M} \mapsto \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathscr{E}} \tag{3.2.1}$$
(*see Example* 3.1.7) *is fully faithful.*

The proof of Proposition 3.2.7 is a straightforward adaptation of the "gluing argument" in the proof of Lemma 2.4.7, replacing $\mathscr{R}$ and $\mathscr{O}$ with $\mathscr{O}_{\mathscr{E}}$ and $\mathfrak{S}$. However, it requires one extra ingredient:

Lemma 3.2.8. — *Let $h : \mathfrak{M}' \to \mathfrak{M}$ be a morphism in* $\mathrm{Mod}^{\varphi}_{\mathfrak{S}}$. *If*

$$h \otimes 1 : \mathfrak{M}' \otimes_{\mathfrak{S}} \mathscr{O}_{\mathscr{E}} \to \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathscr{E}}$$

is an isomorphism, then so is h.

Proof. — Since $\mathfrak{M}'$ and $\mathfrak{M}$ must have the same $\mathfrak{S}$-rank, we can pass to determinants to reduce to the case that $\mathfrak{M}'$ and $\mathfrak{M}$ both have rank 1. Let $\mathscr{M}' := \Theta(\mathfrak{M}') = \mathfrak{M}' \otimes_{\mathfrak{S}} \mathscr{O}$ and $\mathscr{M} := \Theta(\mathfrak{M}) = \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}$ be the corresponding objects of $\mathrm{Mod}^{\varphi,N,0}_{/\mathscr{O}}$ under the equivalence Θ of (2.4.4). The map h thus induces a nonzero map between rank-1 $\mathscr{O}$-modules

$$h \otimes 1 : \mathscr{M}' \to \mathscr{M}. \tag{3.2.2}$$

By the equivalence of categories of Theorem 2.2.1, this map corresponds to a nonzero map

$$\underline{D}(h \otimes 1) : \underline{D}(\mathscr{M}') \to \underline{D}(\mathscr{M})$$

of rank-1 objects of $\mathrm{MF}^{\varphi,N,\mathrm{Fil}\geq 0}_{K}$. By the final part of Corollary 2.4.9, these filtered (φ, N)-modules are weakly admissible. But a 1-dimensional weakly admissible filtered (φ, N)-module has its unique filtration jump determined by its slope, so any nonzero map between such rank-1 objects is not only a K_0-linear φ-compatible isomorphism, but also respects the filtrations in *both* directions. (This is not true without the weak admissibility property!) Hence, $\underline{D}(h \otimes 1)$ is an isomorphism. Since $\underline{D}$ is an equivalence, it follows that (3.2.2) is an isomorphism, whence as $\mathfrak{S}\left[\frac{1}{p}\right] \to \mathscr{O}$ is faithfully flat by Remark 2.4.6, the map

$$h\left[\frac{1}{p}\right] : \mathfrak{M}'\left[\frac{1}{p}\right] \to \mathfrak{M}\left[\frac{1}{p}\right]$$

is an isomorphism as well. To conclude that h itself is an isomorphism, it remains to show that it is an isomorphism over $\mathfrak{S}_{(p)}$ since $\mathfrak{S}$ is a normal noetherian domain. But $\mathfrak{S}_{(p)} \to \mathscr{O}_{\mathscr{E}}$ is faithfully flat by Remark 3.1.4, so this is clear. ∎

Recall [**6**, 5.5.2 (iii)] that the functor

$$\begin{aligned} \underline{D}_{\mathrm{cris}*} : \mathrm{Rep}^{\mathrm{cris},\leq 0}_{G_K} &\longrightarrow {}^{\mathrm{w.a.}}\mathrm{MF}^{\varphi,\mathrm{Fil}\geq 0}_{K} \\ V &\longmapsto \left(B_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V\right)^{G_K} \end{aligned}$$

is fully faithful (with inverse given by the restriction of $\underline{V}_{\mathrm{cris}*}$ to the image of $\underline{D}_{\mathrm{cris}*}$). Combining this with Corollary 2.4.8, we obtain a fully faithful functor

$$\mathrm{Rep}^{\mathrm{cris},\leq 0}_{G_K} \xhookrightarrow[\underline{D}_{\mathrm{cris}*}]{} {}^{\mathrm{w.a.}}\mathrm{MF}^{\varphi,\mathrm{Fil}\geq 0}_{K} \xhookrightarrow[\widetilde{\Theta}]{} \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbf{Q}_p. \tag{3.2.3}$$

On the other hand, by Proposition 3.2.7 and Lemma 3.1.8(2), we have a fully faithful functor

$$\text{(3.2.4)}\qquad \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbf{Q}_p \underset{(3.2.1)}{\hookrightarrow} \mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}} \otimes \mathbf{Q}_p \xrightarrow[\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}]{\simeq} \left(\mathrm{Rep}_{G_{K_\infty}/\mathbf{Z}_p}\right) \otimes \mathbf{Q}_p \simeq \mathrm{Rep}_{G_{K_\infty}}$$

(which coincides with the functor $\underline{V}^*_{\mathfrak{S}}$ on p-isogeny categories thanks to Corollary 3.2.4(1)). An object in the essential image of (3.2.4) is called a *p-adic G_{K_∞}-representation with finite E-height*.

We will see later that $\underline{D}_{\mathrm{cris}*}$ is an equivalence of categories (i.e. weakly admissible implies admissible) and that the composite functor

$$\mathrm{Rep}^{\mathrm{cris},\leq 0}_{G_K} \underset{(3.2.3)}{\hookrightarrow} \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbf{Q}_p \underset{(3.2.4)}{\hookrightarrow} \mathrm{Rep}_{G_{K_\infty}}$$

coincides with the "restriction functor" $\mathrm{Rep}_{G_K} \to \mathrm{Rep}_{G_{K_\infty}}$ evaluated on crystalline representations.

Using $\mathfrak{S}$-modules, we can now describe G_{K_∞}-stable $\mathbf{Z}_p$-lattices in p-adic G_{K_∞}-representations of finite E-height:

Lemma 3.2.9. — *Fix an object $\mathfrak{M}$ of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ with $\mathfrak{S}$-rank at most d, let $V := \underline{V}^*_{\mathfrak{S}}(\mathfrak{M}) \otimes \mathbf{Q}_p$ be the corresponding d-dimensional object of $\mathrm{Rep}_{G_{K_\infty}}$, and set $\mathscr{M}_{\mathscr{E}} := \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{E}$. Then the functor*

$$\underline{V}^*_{\mathfrak{S}} : \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \longrightarrow \mathrm{Rep}_{G_{K_\infty}/\mathbf{Z}_p}$$

restricts to a bijection between objects $\mathfrak{N}$ of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ that are contained in $\mathscr{M}_{\mathscr{E}}$ and have $\mathfrak{S}$-rank d, and G_{K_∞}-stable $\mathbf{Z}_p$-lattices $L \subseteq V$ with rank d.

The proof shows that the E-height of $\mathfrak{N}$ as in the lemma is independent of $\mathfrak{N}$ (and is equal to the E-height of $\mathfrak{M}$).

Idea of proof. — By Fontaine's theory (Lemma 3.1.8(2)), for any G_{K_∞}-stable $\mathbf{Z}_p$-lattice $L \subseteq V$ there is a unique object $\mathscr{N}$ of $\mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}}$ that is contained in $\mathscr{M}_{\mathscr{E}}$ with full $\mathscr{O}_{\mathscr{E}}$-rank and satisfies

$$L = \underline{V}^*_{\mathscr{O}_{\mathscr{E}}}(\mathscr{N})$$

(recall $\mathscr{N}$ is given explicitly by $\mathscr{N} = \mathrm{Hom}_{\mathbf{Z}_p[G_{K_\infty}]}\left(L, \widehat{\mathscr{O}_{\mathscr{E}^{\mathrm{un}}}}\right)$). The key idea is to adapt the gluing method used in the proof of Lemma 2.4.7, using Corollary 3.2.4 and Proposition 3.2.7 to show that

$$\mathfrak{N} := \mathscr{N} \cap \mathfrak{M}\left[\frac{1}{p}\right] \subseteq \mathscr{M}_{\mathscr{E}}$$

is an object of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ (e.g., it is finite and free over $\mathfrak{S}$) and that the natural map

$$\mathfrak{N} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathscr{E}} \to \mathscr{N}$$

is an isomorphism. See the proof of [**11**, Lemma 2.1.15] for the details. ■

3.3. Applications to semistable and crystalline representations. — Recall that the ring of p-adic periods $B_{\mathrm{st}} = B_{\mathrm{st},K}$ depends on our choice of uniformizer π [**5**, §3–4], and that the functor

$$\underline{D}_{\mathrm{st}*} : \mathrm{Rep}_{G_K} \longrightarrow \mathrm{MF}_K^{\varphi,N}$$

$$V \longmapsto \left(V \otimes_{\mathbf{Q}_p} B_{\mathrm{st}}\right)^{G_K}$$

has restriction to $\mathrm{Rep}^{\mathrm{st}}_{G_K}$ that is fully faithful and has image in the subcategory ${}^{\mathrm{w.a}}\mathrm{MF}_K^{\varphi,N}$ of weakly admissible filtered (φ, N)-modules [**6**, 5.3.5 (iii)]. On the full subcategory $\mathrm{Rep}^{\mathrm{st},\leq 0}_{G_K}$ of representations having non-positive Hodge-Tate weights, the functor $\underline{D}_{\mathrm{st}*}$ has image contained in the subcategory ${}^{\mathrm{w.a}}\mathrm{MF}_K^{\varphi,N,\mathrm{Fil}\geq 0}$. We have the following diagram of categories, in which all sub-diagrams commute, except possibly the large rectangle near the bottom:

$$\begin{tikzcd}
\mathrm{Rep}^{\mathrm{st},\leq 0}_{G_K} \arrow[r, hook, "\underline{D}_{\mathrm{st}}"] & {}^{\mathrm{w.a}}\mathrm{MF}_K^{\varphi,N,\mathrm{Fil}\geq 0} \arrow[r, "\simeq", "\mathrm{Thm.2.3.9}"'] & \mathrm{Mod}^{\varphi,N_\nabla,0}_{/\mathscr{O}} \arrow[r, hook, "(2.4.1)"] \arrow[drr, bend left] & \mathrm{Mod}^{\varphi,N,0}_{/\mathscr{O}} & \\
\mathrm{Rep}^{\mathrm{cris},\leq 0}_{G_K} \arrow[r, hook, "\underline{D}_{\mathrm{cris}}"] \arrow[u, hook] \arrow[d, "\mathrm{res}"'] & {}^{\mathrm{w.a}}\mathrm{MF}_K^{\varphi,\mathrm{Fil}\geq 0} \arrow[r, "\simeq", "\mathrm{Thm.2.3.9}"'] \arrow[u, hook] & \mathrm{Mod}^{\varphi,N_\nabla,0,N=0}_{/\mathscr{O}} \arrow[r, hook] \arrow[u, hook] & \mathrm{Mod}^{\varphi,0}_{/\mathscr{O}} \arrow[u, hook, "(\dagger)"] & \mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes \mathbf{Q}_p \arrow[ul, "\simeq", "(2.4.4)"'] \\
\mathrm{Rep}_{G_{K_\infty}} & \mathrm{Mod}^{\varphi}_{\mathscr{O}_{\mathscr{E}}} \otimes \mathbf{Q}_p \arrow[l, "\simeq"', "\underline{V}_{\mathscr{O}_{\mathscr{E}}*}"] & & \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbf{Q}_p \arrow[ll, hook', "(3.2.7)"'] \arrow[u, "(2.4.7)", "\simeq"'] \arrow[ur, hook, "(\dagger)"'] \arrow[lll, bend left, "\underline{V}_{\mathfrak{S}*}"]
\end{tikzcd}$$

Note that if we start at $\mathrm{Rep}^{\mathrm{cris},\leq 0}_{G_K}$ and move around the large rectangle in the bottom of the diagram in the clockwise direction, then we obtain a fully faithful embedding $\mathrm{Rep}^{\mathrm{cris},\leq 0}_{G_K} \hookrightarrow \mathrm{Rep}_{G_{K_\infty}}$. If we know that this rectangle commutes, we thus deduce a conjecture of Breuil:

Corollary 3.3.1. — *The natural restriction map*

$$\mathrm{res} : \mathrm{Rep}^{\mathrm{cris}}_{G_K} \to \mathrm{Rep}_{G_{K_\infty}}$$

is fully faithful.

Remarks 3.3.2. — Before sketching the proof of Breuil's conjecture, we make the following remarks:

1. Recall that the essential image of the curving map in the upper right corner of the diagram is described by Corollary 2.4.9.
2. The two maps labeled (†) in the diagram are *not* essentially surjective. This prevents us from generalizing Corollary 3.3.1 to the case of semistable representations (and rightly so: the analogous conjecture for semistable representations is false, as one sees by using Tate elliptic curves).

To prove Corollary 3.3.1, we first observe that after twisting by $\mathbf{Q}_p(-n)$ for large enough n, it is enough to show that the restriction map

$$\mathrm{Rep}_{G_K}^{\mathrm{cris},\leq 0} \to \mathrm{Rep}_{G_{K_\infty}}$$

is fully faithful. As noted above, this follows if we can show that the large rectangle in the diagram commutes. Such commutativity follows once we know that the entire outside edge of the diagram commutes. Using the fact that

$$\underline{V}_{\mathrm{st}*} : {}^{\mathrm{w.a}}\mathrm{MF}_K^{\varphi,N,\mathrm{Fil}\geq 0} \to \mathrm{Rep}_{G_K}^{\mathrm{st},\leq 0}$$

is quasi-inverse to $\underline{D}_{\mathrm{st}*}$ on the essential image of $\underline{D}_{\mathrm{st}*}$, it therefore suffices to prove the commutativity of

$$\begin{tikzcd} {}^{\mathrm{w.a}}\mathrm{MF}_K^{\varphi,N,\mathrm{Fil}\geq 0} \arrow[r, hook, "\widetilde{\Theta}"] \arrow[d, "\underline{V}_{\mathrm{st}*}"] & \mathrm{Mod}_{/\mathfrak{S}}^{\varphi,N} \otimes \mathbf{Q}_p \arrow[d, "(\underline{V}_{\mathfrak{S}})_* \otimes \mathbf{Q}_p"] \\ \mathrm{Rep}_{G_K}^{\mathrm{st},\leq 0} \arrow[r, "\mathrm{res}"'] & \mathrm{Rep}_{G_{K_\infty}} \end{tikzcd} \tag{3.3.1}$$

where the right side "forgets N" and the top horizontal arrow is (2.4.5). Note that we do not yet know that the left side is an equivalence, since we have not yet proved "weakly admissible implies admissible".

To show that (3.3.1) commutes, let D be any object of ${}^{\mathrm{w.a}}\mathrm{MF}_K^{\varphi,N,\mathrm{Fil}\geq 0}$. Let $\mathscr{M} := \underline{\mathscr{M}}(D)$ be the corresponding object of $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N_\nabla,0}$ (via Theorems 2.2.1 and 2.3.9), and choose an object $\mathfrak{M}$ of $\mathrm{Mod}_{/\mathfrak{S}}^{\varphi,N}$ such that $\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O} \simeq \mathscr{M}$ in $\mathrm{Mod}_{/\mathscr{O}}^{\varphi,N}$ (via the equivalence of Lemma 2.4.7), so $\mathfrak{M} = \widetilde{\Theta}(D)$. Recall that $\mathfrak{M}$ is functorial in D, up to p-isogeny. The commutativity of (3.3.1) follows immediately from the following statement (by dualizing in $\mathrm{Rep}_{G_{K_\infty}}$; cf. Remark 3.1.9):

Proposition 3.3.3. — *With the notation above, there is a natural* $\mathbf{Q}_p[G_{K_\infty}]$*-linear isomorphism*

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathfrak{S},\varphi}\left(\mathfrak{M},\widehat{\mathfrak{S}^{\mathrm{un}}}\right)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{Fil},\varphi,N}\left(D,B_{\mathrm{st}}^{+}\right)\\
\| & & \|\\
\underline{V}^{*}_{\mathfrak{S}}(\mathfrak{M})\otimes_{\mathbf{Z}_p}\mathbf{Q}_p & & \mathrm{Hom}_{\mathrm{Fil},\varphi,N}\left(D,B_{\mathrm{st}}\right)\\
 & & \|\\
 & & \underline{V}^{*}_{\mathrm{st}}(D)
\end{array}$$

Before we explain the proof of this proposition, note that by Corollary 3.2.4(1) we have

$$\dim_{\mathbf{Q}_p} V^{*}_{\mathfrak{S}}(\mathfrak{M})\otimes\mathbf{Q}_p=\mathrm{rk}_{\mathfrak{S}}(\mathfrak{M})=\dim_{K_0}\mathscr{M}/u\mathscr{M}=\dim_{K_0}D,$$

and so as a nice consequence of Proposition 3.3.3, which says in particular that $\dim_{\mathbf{Q}_p}\underline{V}^{*}_{\mathrm{st}}(D)=\dim_{\mathbf{Q}_p}V^{*}_{\mathfrak{S}}(\mathfrak{M})\otimes\mathbf{Q}_p$, we deduce that $\dim_{\mathbf{Q}_p}\underline{V}^{*}_{\mathrm{st}}(D)=\dim_{K_0}D$. Thus, by weak admissibility of D and an argument of Fontaine and Colmez [**3**, Proposition 4.5], the natural map

$$B_{\mathrm{st}}\otimes_{\mathbf{Q}_p}\underline{V}^{*}_{\mathrm{st}}(D)\to B_{\mathrm{st}}\otimes_{K_0}D$$

is an isomorphism. Hence, D is admissible by [**6**, Proposition 5.3.6].

Remark 3.3.4. — Since D above was any object in ${}^{\mathrm{w.a}}\mathrm{MF}_{K}^{\varphi,N,\mathrm{Fil}\geq 0}$, this shows that "weakly admissible implies admissible" in full generality, as we can always shift the filtration to be effective.

Proof of Proposition 3.3.3. — By [**3**, Proposition 4.5], if D is weakly admissible and $\dim_{\mathbf{Q}_p}\underline{V}^{*}_{\mathrm{st}}(D)\geq\dim_{K_0}D$ then D is admissible, and conversely. Thus, it suffices to construct a natural $\mathbf{Q}_p[G_{K_\infty}]$-linear *injection*

$$\underline{V}^{*}_{\mathfrak{S}}(\mathfrak{M})\otimes\mathbf{Q}_p\hookrightarrow\underline{V}^{*}_{\mathrm{st}}(D)=\mathrm{Hom}_{\varphi,\mathrm{Fil},N}(D,B_{\mathrm{st}}).\tag{3.3.2}$$

We will just do this in the case that $N_D=0$, as it contains the essential ideas for the general case (see the proof of [**11**, Proposition 2.1.5] for the details in the general case).

Recall that $B_{\mathrm{cris}}\otimes_{K_0}K$ is equipped with a filtration via its inclusion into the discretely-valued field B_{dR}, and that a K_0-linear map $D\to B_{\mathrm{cris}}$ is *compatible with filtrations* if the extension of scalars $D_K\to B_{\mathrm{cris}}\otimes K$ respects filtrations (i.e. if the composite $D_K\to B_{\mathrm{dR}}$ is compatible with filtrations). Defining

$$\underline{V}^{*}_{\mathrm{cris}}:=\mathrm{Hom}_{\varphi,\mathrm{Fil}}\left(-,B_{\mathrm{cris}}\right)$$

and replacing $B_{\rm st}$ with $(B_{\rm st})^{N=0} = B_{\rm cris}$ and $\underline{V}^*_{\rm st}$ with $\underline{V}^*_{\rm cris}$ in (3.3.2), we have $\underline{V}^*_{\rm cris}(D) = \underline{V}^*_{\rm st}(D)$ since $N_D = 0$, so our aim is to construct a natural $\mathbf{Q}_p[G_{K_\infty}]$-linear injection
(3.3.3)

$$\underline{V}^*_{\mathfrak{S}}(\mathfrak{M}) \otimes \mathbf{Q}_p = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \widehat{\mathfrak{S}^{\rm un}}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \hookrightarrow \mathrm{Hom}_{\varphi,\mathrm{Fil}}(D, B^+_{\rm cris}) = \underline{V}^*_{\rm cris}(D)$$

(the final equality using the effectivity of the filtration on D_K).

An element of $\mathscr{O}$ has a Taylor expansion $\sum c_m u^m$ with $c_m \in K_0 = W(k)[1/p]$ and $|c_m| r^m \to 0$ for all $0 < r < 1$. For $p^{-1/(p-1)} < r_0 < 1$ we have $|m!|/r_0^m \to 0$, so

$$|m! c_m| = (|c_m| r_0^m) \cdot \frac{|m!|}{r_0^m} \to 0.$$

Thus, by [**5**, 4.1.3] there is a natural map of K_0-algebras

$$\mathscr{O} \to B^+_{\rm cris} \tag{3.3.4}$$

extending the natural map $\mathfrak{S} \hookrightarrow W(R) \subseteq B^+_{\rm cris}$ defined by "evaluation at $[\underline{\pi}]$" (i.e. $u \mapsto [\underline{\pi}]$). Using the the natural topologies on $\mathscr{O}$ and $B^+_{\rm cris}$, one checks that this map is moreover continuous, and since $K_0[u]$ is dense in $\mathscr{O}$ it is the unique such continuous K_0-algebra map. Since $\mathfrak{S}\left[\frac{1}{p}\right]$ is dense in $\mathscr{O}$, the map (3.3.4) is also φ-compatible, as this is true of

$$\mathfrak{S}\left[\frac{1}{p}\right] \hookrightarrow W(R)\left[\frac{1}{p}\right]$$

(thanks to the relation $[\underline{\pi}^p] = [\underline{\pi}]^p$).

We will define the map (3.3.3) as the composite of two maps. First, recalling that $\mathscr{M} := \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}$, consider the map

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \widehat{\mathfrak{S}^{\rm un}}) \longrightarrow \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, B^+_{\rm cris}) = \mathrm{Hom}_{\mathscr{O},\varphi}(\mathscr{M}, B^+_{\rm cris}) \tag{3.3.5}$$

defined by composition with the natural map $\widehat{\mathfrak{S}^{\rm un}} \hookrightarrow W(R) \subseteq B^+_{\rm cris}$. This map is obviously injective.

Second, we consider the map

$$\mathrm{Hom}_{\mathscr{O},\varphi}(\mathscr{M}, B^+_{\rm cris}) \longrightarrow \mathrm{Hom}_{\mathscr{O},\varphi}(D \otimes_{K_0} \mathscr{O}, B^+_{\rm cris}) = \mathrm{Hom}_{\varphi}(D, B^+_{\rm cris})\ . \tag{3.3.6}$$

given by composition with the φ-compatible $\mathscr{O}$-linear map

$$\xi : D \otimes_{K_0} \mathscr{O} = \underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O} \hookrightarrow \mathscr{M}$$

of Lemma 2.2.5. We claim that (3.3.6) is injective with image contained in

$$\mathrm{Hom}_{\varphi,\mathrm{Fil}}(D, B^+_{\rm cris}) \subseteq \mathrm{Hom}_{\varphi}(D, B^+_{\rm cris}).$$

To verify these claims, we proceed as follows.

Obviously $E(u) = (u - \pi)G(u)$ in $K[u]$, for some $G(u) \in K[u]$ with $G(\pi) \neq 0$. It follows that the map

$$\mathfrak{S} \to B^+_{\mathrm{cris}} \subseteq B^+_{\mathrm{dR}} \tag{3.3.7}$$

(defined by $u \mapsto [\underline{\pi}]$) carries E to

$$E([\underline{\pi}]) = ([\underline{\pi}] - \pi) \cdot G([\underline{\pi}]).$$

As $[\underline{\pi}] - \pi$ is a uniformizer of B^+_{dR} and $G(\pi) \neq 0$, we see that $G([\underline{\pi}]) \in (B^+_{\mathrm{dR}})^\times$ and hence that $E([\underline{\pi}])$ is a uniformizer of B^+_{dR}. Therefore, (3.3.7) induces a local map of local K_0-algebras

$$\mathfrak{S}\left[\frac{1}{p}\right]_{(E)} \to B^+_{\mathrm{dR}}.$$

Passing to completions (and recalling that B^+_{dR} is a complete discrete valuation ring) we see that the map (3.3.4) extends to a K_0-algebra map

$$\mathscr{O}^\wedge_{\Delta,\pi} = \mathfrak{S}\left[\frac{1}{p}\right]^\wedge_{(E)} \to B^+_{\mathrm{dR}}$$

which is even a map of K-algebras, as can be seen (via Hensel's Lemma) by examining the induced map on residue fields.

Thus, given an $\mathscr{O}$-linear map $\mathscr{M} \to B^+_{\mathrm{cris}} \subseteq B^+_{\mathrm{dR}}$, the map

$$(1 \otimes \varphi_{\mathscr{M}})(\varphi^*_{\mathscr{O}}\mathscr{M}) \to B^+_{\mathrm{dR}}$$

induced by restriction carries $(1 \otimes \varphi_{\mathscr{M}})(\varphi^*_{\mathscr{O}}\mathscr{M}) \cap E^i\mathscr{M}$ into $\mathrm{Fil}^i\, B^+_{\mathrm{dR}}$, and hence is compatible with these filtrations. Moreover, $\xi : \underline{D}(\mathscr{M}) \otimes_{K_0} \mathscr{O} \hookrightarrow \mathscr{M}$ is φ-compatible and so has image landing in

$$\varphi_{\mathscr{M}}(\mathscr{M}) \subseteq (1 \otimes \varphi_{\mathscr{M}})(\varphi^*_{\mathscr{O}}\mathscr{M}).$$

But [**11**, 1.2.12(4)] gives that the induced map

$$\mathscr{O}^\wedge_{\Delta,\pi} \otimes_K \underline{D}(\mathscr{M})_K = \mathscr{O}^\wedge_{\Delta,\pi} \otimes_{K_0} \underline{D}(\mathscr{M}) \longrightarrow \mathscr{O}^\wedge_{\Delta,\pi} \otimes_{\mathscr{O}} (1 \otimes \varphi_{\mathscr{M}})(\varphi^*_{\mathscr{O}}\mathscr{M})$$

is a filtration-compatible isomorphism (where the filtrations are the usual tensor-filtrations; cf. Remark 2.2.3). It follows at once that (3.3.6) has image contained in $\mathrm{Hom}_{\varphi,\mathrm{Fil}}(D, B^+_{\mathrm{cris}})$; moreover the resulting map

$$\mathrm{Hom}_{\mathscr{O},\varphi}(\mathscr{M}, B^+_{\mathrm{cris}}) \longrightarrow \mathrm{Hom}_{\varphi,\mathrm{Fil}}(D, B^+_{\mathrm{cris}}) \tag{3.3.8}$$

is injective since the injective map

$$(1 \otimes \varphi_{\mathscr{M}})(\varphi^*_{\mathscr{O}}\mathscr{M}) \hookrightarrow \mathscr{M}$$

has cokernel killed by some E^h and $E([\underline{\pi}]) \in B^+_{\mathrm{dR}}$ is a nonzero element of a domain.

Composing the injective maps (3.3.8) and (3.3.5) gives a $\mathbf{Q}_p$-linear injective map

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \widehat{\mathfrak{S}^{\mathrm{un}}}) \hookrightarrow \mathrm{Hom}_{\varphi,\mathrm{Fil}}(D, B^+_{\mathrm{cris}}) .$$

This map is moreover $\mathbf{Q}_p[G_{K_\infty}]$-linear because the action of G_{K_∞} on B^+_{cris} leaves the map $\mathscr{O} \to B^+_{\mathrm{cris}}$ invariant, as this holds on $\mathfrak{S} \subseteq \mathscr{O}$ due to the fact that $[\underline{\pi}]$ is G_{K_∞}-invariant (Example 3.1.3). This gives the desired map (3.3.3). ∎

4. Third Lecture

We keep our notation from the previous two sections. In this final lecture, we will apply the theory of $\mathfrak{S}$-modules developed in Lecture 2 to the study of p-divisible groups and finite flat group schemes over $\mathscr{O}_K$. (For us, a *finite flat group scheme* is a finite, commutative, locally free group scheme with constant p-power order.) We will also discuss applications to torsion and lattice representations of G_K in the context of the earlier work of Fontaine and Laffaille, and we study the restriction from G_K to G_{K_∞} for representations arising from finite flat group schemes over $\mathscr{O}_K$. This builds on ideas and results of Breuil.

4.1. Divided powers and Grothendieck-Messing theory. — Classical Dieudonné theory classifies p-divisible groups over the perfect field k. We wish to find an analogue of this classification over $\mathscr{O}_K$. To do this, we will use Grothendieck-Messing theory as our starting point, and to review this we begin with the concept of a divided power structure on an ideal in a ring.

Definition 4.1.1. — Let I be an ideal in a commutative ring A. A *PD-structure* on I is a collection of maps

$$\gamma_n : I \to I \qquad n \geq 0$$

such that the γ_n satisfy the "obvious" properties of $t^n/n!$ in characteristic zero:

1. For all $x \in I$, we have $\gamma_0(x) = 1$, $\gamma_1(x) = x$, and $\gamma_n(x) \in I$ for $n \geq 1$.
2. For all $x, y \in I$ and all $n \geq 0$,
$$\gamma_n(x+y) = \sum_{i+j=n} \gamma_i(x)\gamma_j(y).$$
3. If $a \in A$ and $x \in I$ then $\gamma_n(ax) = a^n\gamma_n(x)$ for all $n \geq 0$.
4. For all $x \in I$ and all $m, n \geq 0$,
$$\gamma_m(x)\gamma_n(x) = \frac{(m+n)!}{m!n!}\gamma_{m+n}(x).$$

5. For all $x \in I$ and all $m, n \geq 0$,

$$\gamma_n(\gamma_m(x)) = \frac{(mn)!}{(m!)^n n!}\gamma_{mn}(x)$$

Remark 4.1.2. — The "PD" standard for *puissances-divisée*–literally "divided powers." Note that the combinatorial coefficients appearing in (4) and (5) are in fact integers and hence can be viewed in a unique way as elements of A.

If $I \subseteq A$ is an ideal in a commutative ring that is equipped with a PD-structure $\{\gamma_n\}$ then for $x \in I$ we will often write $x^{[n]}$ for $\gamma_n(x)$.

Example 4.1.3. — It follows easily from (4) and (1) that $n!\gamma_n(x) = x^n$ for all n and all $x \in I$, so when A is $\mathbf{Z}$-flat there is at most one PD-structure on any ideal I of A: $\gamma_n(x) = x^n/n!$. At the other extreme, if A is a $\mathbf{Z}/N\mathbf{Z}$-algebra for some $N \geq 1$ and $I \subseteq A$ admits a PD-structure then $x^N = 0$ for all $x \in I$.

We say that a PD-structure $\{\gamma_n\}$ on I is *PD-nilpotent* if the ideal $I^{[n]}$ generated by all products $x_1^{[i_1]} \cdots x_r^{[i_r]}$ with $\sum i_r \geq n$ is zero for some (and hence all) sufficiently large n. This forces $I^n = 0$.

For $I \subseteq \mathscr{O}_K$ the maximal ideal, a PD-structure exists on I if and only if the absolute ramification index e satisfies $e \leq p - 1$. On the other hand, the ideal $p\mathscr{O}_K$ *always* has a PD-structure, as $\gamma_n(py) = (p^n/n!) \cdot y^n$ with $p^n/n! \in \mathbf{Z}_p$.

In general, there can be many choices of PD-structure $\{\gamma_n\}$ on an ideal I.

Example 4.1.4. — Recall that A_{cris} is $\mathbf{Z}_p$-flat and comes equipped with a canonical surjection $A_{\mathrm{cris}} \twoheadrightarrow \mathscr{O}_{\mathbf{C}_K}$. The kernel of $\mathrm{Fil}^1 A_{\mathrm{cris}}$ of this surjection has a (necessarily unique) PD-structure.

***Theorem 4.1.5* (Grothendieck-Messing)**. — *Let A_0 be a ring in which p is nilpotent and let G_0 be a p-divisible group over A_0. For any surjection $h : A \twoheadrightarrow A_0$ such that $I := \ker h$ is endowed with a PD-structure $\{\gamma_n\}$ and some power I^N vanishes, there is attached a finite locally free A-module*

$$\mathbf{D}(G_0)(A) = \mathbf{D}(G_0)\left(A \twoheadrightarrow A_0, \{\gamma_n\}\right)$$

with $\mathrm{rk}_A(\mathbf{D}(G_0)(A)) = \mathrm{ht}\, G_0$. This association is contravariant in G_0 and commutes with PD-base change in A (i.e. base change that respects h and the divided power structure on $\ker h$).

Moreover, the locally free A_0-module $\mathrm{Lie}(G_0)$ is naturally a quotient of $\mathbf{D}(G_0)(A_0)$, and if $\{\gamma_n\}$ is PD-nilpotent then there is an equivalence of categories between the category $\mathrm{Def}(G_0, A)$ of deformations of G_0 to A and the category of locally free quotients $\mathbf{D}(G_0)(A) \twoheadrightarrow \mathscr{E}$ lifting $\mathbf{D}(G_0)(A_0) \twoheadrightarrow \mathrm{Lie}(G_0)$.

Remarks 4.1.6. — The classification of deformations at the end of the theorem can also be formulated in terms of subbundles rather than quotients. Colloquially speaking, we may say that in order to lift G_0 through a nilpotent divided power thickening A of A_0, it is equivalent to lift its Lie algebra to a locally free quotient of $\mathbf{D}(G_0)(A)$.

1. The equivalence of categories at the end of the theorem associates to any deformation G of G_0 to A the module $\mathrm{Lie}(G)$, which is naturally a quotient of $\mathbf{D}(G_0)(A)$.
2. This equivalence also works for deforming maps of p-divisible groups $G_0 \to H_0$. That is, a map $f_0 : G_0 \to H_0$ has at most one lift to a map $f : G \to H$, and f exists if and only if $\mathbf{D}(f_0) : \mathbf{D}(G_0)(A) \to \mathbf{D}(H_0)(A)$ is compatible with the quotients associated to the liftings G and H of G_0 and H_0 respectively.
3. If $p > 2$ then $p^n/n! \to 0$ in $\mathbf{Z}_p$, whereas $2^{2^j}/(2^j)! \in 2\mathbf{Z}_2^\times$ for all $j \geq 0$. It follows that the (unique) PD-structure on the ideal (p) in $\mathbf{Z}_p$ is topologically PD-nilpotent for $p > 2$ but not for $p = 2$.
4. The right way to state Theorem 4.1.5 is to use the language of *crystals*. In this terminology, $\mathbf{D}$ is a contravariant functor from the category of p-divisible groups over a base scheme S on which p is locally nilpotent to the category of crystals in locally free $\mathscr{O}_S$-modules.
5. By taking projective limits, the Grothendieck-Messing Theorem has an analogue for A_0 merely p-adically separated and complete (for example, $A_0 = \mathscr{O}_K$) and $A \twoheadrightarrow A_0$ any surjection of p-adically separated and complete rings with kernel $I \subseteq A$ that is topologically nilpotent (resp. endowed with topologically PD-nilpotent divided powers).

4.2. S-modules. — In earlier work of Breuil, a certain ring S played a vital role in the description of p-divisible groups and finite flat group schemes over $\mathscr{O}_K$. Breuil's method began by studying finite flat group schemes over $\mathscr{O}_K$ in terms of S-modules, and then gave a theory for p-divisible groups by passage to the limit. Kisin provided an approach in the other direction, using Grothendieck–Messing theory to derive Breuil's description of p-divisible groups via S-modules without any preliminary work at finite level, and then used this to deduce a classification for p-divisible groups and finite flat group schemes (with $\mathfrak{S}$-modules rather than S-modules). We now introduce Breuil's ring S.

Let $W[u]\left[\frac{E(u)^i}{i!}\right]_{i\geq 1}$ be the subring of $K_0[u]$ generated over $W[u]$ by the set $\{E^i/i!\}_{i\geq 1}$ (this is the *divided power envelope* of $W[u]$ with respect to the ideal $E(u)W[u]$). Clearly this ring is W-flat. Further, there is an evident surjective map

$$W[u]\left[\frac{E(u)^i}{i!}\right]_{i\geq 1} \twoheadrightarrow \mathscr{O}_K \tag{4.2.1}$$

defined via $u \mapsto \pi$. with kernel generated by all $E^i/i!$. Let S be the p-adic completion of $W[u]\left[\frac{E(u)^i}{i!}\right]_{i\geq 1}$ and let $\mathrm{Fil}^1 S \subseteq S$ be the ideal that is (topologically) generated by all $E^i/i!$. We view S as a topological ring via its (separated and complete) p-adic topology. The ring S is local and W-flat (but *not* noetherian), and the map (4.2.1) induces an isomorphism

$$S/\,\mathrm{Fil}^1 S \xrightarrow{\simeq} \mathscr{O}_K \ .$$

Moreover, there is a unique continuous map $\varphi_S : S \to S$ restricting to the Frobenius endomorphism of W and satisfying $\varphi_S(u) = u^p$. Note that $\varphi_S(\mathrm{Fil}^1 S) \subseteq pS$ and $\varphi_S \bmod pS = \mathrm{Frob}_{S/pS}$.

The ideal $\mathrm{Fil}^1 S$ admits (topologically PD-nilpotent) divided powers, so for any p-divisible group G over $\mathscr{O}_K$ with Cartier dual G^* we get a finite free (as S is local) S-module

$$\begin{aligned}\underline{\mathscr{M}}(G) &:= \mathbf{D}(G^*)(S \twoheadrightarrow \mathscr{O}_K)\\ &= \varprojlim_N \mathbf{D}(G^* \bmod p^N)(S/p^N S \twoheadrightarrow \mathscr{O}_K/p^N\mathscr{O}_K)\end{aligned}$$

with $\mathrm{rk}_S\, \underline{\mathscr{M}}(G) = \mathrm{ht}(G)$. Here, the kernel of $S/p^N S \twoheadrightarrow \mathscr{O}_K/p^N\mathscr{O}_K$ is given the PD-structure induced by that on $\mathrm{Fil}^1 S$, and $\underline{\mathscr{M}}(G)$ is contravariant in G.

Since the ideal

$$\mathrm{Fil}^1 S + pS = \ker(S \twoheadrightarrow \mathscr{O}_K/p\mathscr{O}_K)$$

is also equipped with topologically PD-nilpotent divided powers if $p > 2$, and the formation of $\mathbf{D}$ is compatible with base change (i.e., it is a crystal), by setting $G_0 = G \bmod p$ we also have

$$\underline{\mathscr{M}}(G) := \mathbf{D}(G_0^*)(S \twoheadrightarrow \mathscr{O}_K/p\mathscr{O}_K)$$

if $p > 2$. This shows, in particular, that $\underline{\mathscr{M}}(G)$ depends contravariantly functorially on G_0 if $p > 2$. With some more work (see [**11**, Lemma A.2]), for all p the S-module $\underline{\mathscr{M}}(G)$ can naturally be made into an object of the following category that was introduced by Breuil.

Definition 4.2.1. — Let $\mathrm{BT}^{\varphi}_{/S}$ be the category of finite free S-modules $\mathscr{M}$ that are equipped with an S-submodule $\mathrm{Fil}^1 \mathscr{M} \subseteq \mathscr{M}$ and a φ_S-semilinear map $\varphi_{\mathscr{M}} : \mathrm{Fil}^1 \mathscr{M} \to \mathscr{M}$ such that

1. $(\mathrm{Fil}^1 S)\cdot \mathscr{M} \subseteq \mathrm{Fil}^1 \mathscr{M}$,
2. the finitely generated $S/\,\mathrm{Fil}^1 S \simeq \mathscr{O}_K$-module $\mathscr{M}/\,\mathrm{Fil}^1 \mathscr{M}$ is free,
3. the subset $\varphi_{\mathscr{M}}(\mathrm{Fil}^1 \mathscr{M})$ spans $\mathscr{M}$ over S.

Morphisms are S-module homomorphisms that are compatible with $\varphi_{\mathscr{M}}$ and Fil^1. A three-term sequence of objects of $\mathrm{BT}^{\varphi}_{/S}$ is said to be a *short exact sequence* if the sequences of S-modules and Fil^1's are both short exact.

Example 4.2.2. — We give the two "canonical" examples of S-modules arising from p-divisible groups over $\mathscr{O}_K$ via the functor $\underline{\mathscr{M}}$. Both examples follow from unraveling definitions (including the construction of the crystal $\mathbf{D}$ in terms of a universal vector extension).

For $G = \mathbf{G}_m[p^\infty] = \varinjlim_n \mathbf{G}_m[p^n]$ we have

$$\underline{\mathscr{M}}(G) = S, \quad \mathrm{Fil}^1 \underline{\mathscr{M}}(G) = \mathrm{Fil}^1 S, \quad \text{and} \quad \varphi_{\underline{\mathscr{M}}(G)} = \frac{\varphi_S}{p} : \mathrm{Fil}^1 S \to S.$$

Meanwhile, for $G = \mathbf{Q}_p/\mathbf{Z}_p = \varinjlim \frac{1}{p^n}\mathbf{Z}/\mathbf{Z}$ we have

$$\underline{\mathscr{M}}(G) = S, \quad \mathrm{Fil}^1 \underline{\mathscr{M}}(G) = S, \quad \text{and} \quad \varphi_{\underline{\mathscr{M}}(G)} = \varphi_S : S \to S.$$

Example 4.2.3. — The classical contravariant Dieudonné module $D(G_0)$ of $G_0 = G \mod \pi$ (equipped with its F and V operators) can be recovered from $\underline{\mathscr{M}}(G)$; for example, the underlying $W(k)$-module of $D(G_0)$ is the scalar extension of $\underline{\mathscr{M}}(G)$ along the map $S \twoheadrightarrow S/uS = W$ followed by scalar extension by the inverse of the Frobenius automorphism of W. In particular, G is connected if and only if $m \mapsto \varphi_{\underline{\mathscr{M}}(G)}(E(u)m)$ viewed on $\underline{\mathscr{M}}(G)/u\underline{\mathscr{M}}(G)$ is topologically nilpotent for the p-adic topology. (This evaluation of $\varphi_{\underline{\mathscr{M}}(G)}$ makes sense since $E(u)m \in (\mathrm{Fil}^1 S) \cdot \mathscr{M} \subseteq \mathrm{Fil}^1 \mathscr{M}$ for any $\mathscr{M}$ in $\mathrm{BT}^{\varphi}_{/S}$.) Thus, we say $\mathscr{M}$ in $\mathrm{BT}^{\varphi}_{/S}$ is *connected* if $m \mapsto \varphi_{\mathscr{M}}(E(u)m)$ on $\mathscr{M}/u\mathscr{M}$ is topologically nilpotent for the p-adic topology.

Using results for p-torsion groups, Breuil proved (for $p > 2$) the following theorem classifying p-divisible groups over $\mathscr{O}_K$.

Proposition 4.2.4. — *If $p > 2$ then the contravariant functor $\underline{\mathscr{M}}$ from the category of p-divisible groups over $\mathscr{O}_K$ to the category $\mathrm{BT}^{\varphi}_{/S}$ is an exact equivalence of categories with exact quasi-inverse. The same statement holds for $p = 2$ working only with connected objects.*

Proof. — For $p > 2$, one uses Grothendieck-Messing theory (Theorem 4.1.5) to "lift" from $\mathscr{O}_K/\pi^i\mathscr{O}_K$ to $\mathscr{O}_K/\pi^{i+1}\mathscr{O}_K$, beginning with the analogous statement for p-divisible groups over k as furnished by classical Dieudonné theory. For $p = 2$, one must adapt this method using Zink's theory of windows [**12**]. ∎

Lemma 4.2.5. — *If $p > 2$ then $\mathrm{Hom}_{S,\varphi,\mathrm{Fil}}(\underline{\mathscr{M}}(G), A_{\mathrm{cris}})$ is a finite free $\mathbf{Z}_p$-module, and there is a natural $\mathbf{Z}_p[G_{K_\infty}]$-linear isomorphism*

$$T_p G \xrightarrow{\simeq} \mathrm{Hom}_{S,\varphi,\mathrm{Fil}}(\underline{\mathscr{M}}(G), A_{\mathrm{cris}}) .$$

The same holds for $p = 2$ provided that G is connected.

Proof. — We only address the case $p > 2$. There is a unique map of W-algebras $S \to A_{\mathrm{cris}}$ such that $u \mapsto [\underline{\pi}]$ (and hence $E^i/i! \in S$ maps to $E([\underline{\pi}])^i/i!$). Since G_{K_∞} acts trivially on S and is equal to the isotropy subgroup of $\underline{\pi} \in R$ (Example 3.1.3), this map is G_{K_∞}-equivariant. Furthermore, the diagram

$$\begin{tikzcd} S \arrow[r, two heads] \arrow[d] & \mathscr{O}_K \arrow[d, hook] \\ A_{\mathrm{cris}} \arrow[r, two heads, "\theta"'] & \mathscr{O}_{\mathbf{C}_K} \end{tikzcd}$$

commutes, so by the "crystal" condition we have a natural isomorphism

$$\mathbf{D}(G^*_{\mathscr{O}_{\mathbf{C}_K}})(A_{\mathrm{cris}}) \simeq \mathbf{D}(G^*)(S) \otimes_S A_{\mathrm{cris}} \tag{4.2.2}$$

$$= \underline{\mathscr{M}}(G) \otimes_S A_{\mathrm{cris}}. \tag{4.2.3}$$

Thus, since $\mathbf{D}(G)$ is covariant in G, we get a $\mathbf{Z}_p$-linear map

$$\begin{aligned} T_pG := \mathrm{Hom}_{\mathbf{C}_K}(\mathbf{Q}_p/\mathbf{Z}_p, G_{\mathbf{C}_K}) &= \mathrm{Hom}_{\mathscr{O}_{\mathbf{C}_K}}(\mathbf{Q}_p/\mathbf{Z}_p, G_{\mathscr{O}_{\mathbf{C}_K}}) \\ &\xrightarrow{\mathbf{D}((\cdot)^*)} \mathrm{Hom}_{A_{\mathrm{cris}}}\Big(\mathbf{D}(G^*_{\mathscr{O}_{\mathbf{C}_K}})(A_{\mathrm{cris}}), \mathbf{D}(\mathbf{G}_m[p^\infty])(A_{\mathrm{cris}})\Big) \\ &= \mathrm{Hom}_S(\underline{\mathscr{M}}(G), A_{\mathrm{cris}}) \end{aligned}$$

and one checks that this map lands in the submodule $\mathrm{Hom}_{S,\varphi,\mathrm{Fil}}(\underline{\mathscr{M}}(G), A_{\mathrm{cris}})$. Here, the last equality above uses both the identification $\mathbf{D}(\mathbf{G}_m[p^\infty])(A_{\mathrm{cris}}) \simeq A_{\mathrm{cris}}$ of Example 4.2.2 and the isomorphism (4.2.3). Since $S \to A_{\mathrm{cris}}$ is G_{K_∞}-equivariant, the map

$$T_pG \longrightarrow \mathrm{Hom}_{S,\varphi,\mathrm{Fil}}(\underline{\mathscr{M}}(G), A_{\mathrm{cris}}) \tag{4.2.4}$$

thus obtained is $\mathbf{Z}_p[G_{K_\infty}]$-linear. When $G = \mathbf{G}_m[p^\infty]$ and $p > 2$, the map (4.2.4) is seen to be an isomorphism by direct calculation, using Example 4.2.2 and the fact that $A_{\mathrm{cris}}^{\varphi=1,\mathrm{Fil}\geq 0} = \mathbf{Z}_p$; this case of (4.2.4) is not an isomorphism if $p = 2$. Provided $p > 2$, combining this special isomorphism with the Cartier duality between G and G^* yields that (4.2.4) is an isomorphism for any G when $p > 2$. (This is an instance of a duality argument due to Faltings.) The isomorphism claim for $p = 2$ requires more work. ∎

4.3. From S to $\mathfrak{S}$. — Let $\mathfrak{S} = W[\![u]\!]$ be as in §2.4. We have a unique $W[u]$-linear map $\mathfrak{S} \to S$, and the diagram

$$\begin{tikzcd} W(R) \arrow[r, phantom, "\subseteq"] & A_{\mathrm{cris}} \\ \mathfrak{S} \arrow[u, hook] \arrow[r] & S \arrow[u] \end{tikzcd}$$

commutes. Denote by $\varphi : \mathfrak{S} \to S$ the composite map

$$\mathfrak{S} \longrightarrow S \xrightarrow{\varphi_S} S$$

and for any object $\mathfrak{M}$ of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ define

$$\mathscr{M} := S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}.$$

If $E\text{-ht}(\mathfrak{M}) \leq 1$ then (see [**11**, 2.2.3]) $\mathscr{M}$ can be naturally made into an object of $\mathrm{BT}^{\varphi}_{/S}$ for $p > 2$ (and one has an analogue using Zink's theory of windows when $p = 2$ if $\mathfrak{M}$ is *connected* in the sense that $\varphi_{\mathfrak{M}}$ on $\mathfrak{M}$ is topologically nilpotent). This motivates the following definition.

Definition 4.3.1. — Denote by $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ the full subcategory of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ consisting of those $\mathfrak{S}$-modules $\mathfrak{M}$ that have E-height at most 1.

For $p > 2$ Breuil showed that the functor

$$\mathrm{BT}^{\varphi}_{/\mathfrak{S}} \longrightarrow \mathrm{BT}^{\varphi}_{/S}$$

$$\mathfrak{M} \longmapsto S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$$

is exact and fully faithful; as before, when $p = 2$ one has an analogue for connected objects $\mathfrak{M}$ of $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$. By Proposition 4.2.4, we have an anti-equivalence of categories

$$\mathrm{BT}^{\varphi}_{/S} \xleftarrow{\simeq} \{p\text{-divisible groups over } \mathscr{O}_K\}$$

that is exact with exact quasi-inverse for $p > 2$ (and a similar result for $p = 2$ using connected objects), so we get a contravariant and fully faithful functor

$$\underline{G} : \mathrm{BT}^{\varphi}_{/\mathfrak{S}} \longrightarrow \{p\text{-divisible groups over } \mathscr{O}_K\}$$

when $p > 2$ (and a similar functor working with connected objects when $p = 2$). Using Dieudonné theory over k, one shows that a 3-term complex in $\mathrm{BT}^{\varphi}_{\mathfrak{S}}$ is short exact if and only if its image under $\underline{G}$ is short exact.

Theorem 4.3.2. — *For $p > 2$, the functor $\underline{G}$ is an equivalence of categories. The same statement holds for $p = 2$ if we work with connected objects.*

Proof. — We only discuss the case $p > 2$ (and the case $p = 2$ is treated in [**12**]). We will construct a contravariant functor

$$\underline{\mathfrak{M}} : \{p\text{-divisible groups over } \mathscr{O}_K\} \longrightarrow \mathrm{BT}^{\varphi}_{/\mathfrak{S}}$$

for any p, and will show that this functor is quasi-inverse to $\underline{G}$ when $p > 2$.

Since the Tate module $V_p(G) := T_p(G) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is an object of $\mathrm{Rep}^{\mathrm{cris}}_{G_K}$ with Hodge-Tate weights in $\{0, 1\}$, it is in the image of the functor

$$\underline{V}^*_{\mathrm{cris}} : {}^{\mathrm{w.a.}}\mathrm{MF}^{\varphi,\mathrm{Fil}\geq 0}_K \longrightarrow \mathrm{Rep}^{\mathrm{cris}}_{G_K} .$$

Thus, using the fully faithful functor

$$\widetilde{\Theta} : {}^{\mathrm{w.a.}}\mathrm{MF}^{\varphi,\mathrm{Fil}\geq 0}_K \hookrightarrow \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbf{Q}_p$$

of §3, corresponding to the representation $V_p(G)$ is an $\mathfrak{S}$-module $\mathfrak{M}$, uniquely determined by and functorial in G up to p-isogeny. Moreover, we have $E\text{-ht}(\mathfrak{M}) \leq 1$. If h is the height of G then by Lemma 3.2.9, the functor

$$\underline{V}^*_{\mathfrak{S}} : \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \longrightarrow \mathrm{Rep}_{G_{K_\infty}/\mathbf{Z}_p}$$

induces a one to one correspondence between G_{K_∞}-stable lattices $L \subseteq V_p(G)$ with rank h and objects $\mathfrak{N}$ of $\mathrm{Mod}^{\varphi}_{\mathfrak{S}}$ that are contained in $\mathscr{M}_{\mathscr{E}} := \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{E}$ and have $\mathfrak{S}$-rank h. Furthermore, since the functor

$$\mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \longrightarrow \mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}} \simeq \mathrm{Rep}_{G_{K_\infty}}$$

is fully faithful by Proposition 3.2.7, we see that $\mathfrak{N}$ is functorial in and uniquely determined by L. Taking $L = T_p(G)$ thus gives an object $\mathfrak{N}$ of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ with $\underline{V}^*_{\mathfrak{S}}(\mathfrak{N}) = T_p(G)$; by our discussion $\mathfrak{N}$ is contravariant in G and we define

$$\underline{\mathfrak{M}}(G) := \mathfrak{N}.$$

To show that $\underline{\mathfrak{M}} \circ \underline{G} \simeq \mathrm{id}$ for $p > 2$, one uses Lemma 4.2.5 to reduce to comparing divisibility by p in $\widehat{\mathfrak{S}^{\mathrm{un}}}$ and A_{cris}; this comparison works if $p > 2$.

To show that $\underline{G} \circ \underline{\mathfrak{M}} \simeq \mathrm{id}$ for $p > 2$, one must construct an isomorphism of p-divisible groups. Using Tate's theorem, the crystalline property of the representations arising from p-divisible groups, and the full-faithfulness of $\mathrm{Rep}^{\mathrm{cris}}_{G_K} \to \mathrm{Rep}_{G_{K_\infty}}$ (Corollary 3.3.1), one reduces this to a problem with $\mathbf{Z}_p[G_{K_\infty}]$-modules, again solved by Lemma 4.2.5. ■

4.4. Finite flat group schemes and strongly divisible lattices. — For an isogeny $f : \Gamma_1 \to \Gamma_2$ between p-divisible groups over $\mathscr{O}_K$, $\ker f$ is a finite flat group scheme. Conversely, Oort showed that every finite flat group scheme G over $\mathscr{O}_K$ arises in this way. (Raynaud proved a stronger result, using abelian schemes instead of p-divisible groups [**1**, 3.1.1].) If the Cartier dual G^* is connected then we may arrange that Γ_1^* and Γ_2^* are connected as well.

The anti-equivalence of categories

$$\mathrm{BT}^{\varphi}_{/\mathfrak{S}} \simeq \{p\text{-divisible groups over } \mathscr{O}_K\}$$

of Theorem 4.3.2 for $p > 2$ and its "connected" analogue for $p = 2$ motivate the following definition due to Breuil.

Definition 4.4.1. — Let $(\mathrm{Mod}\,/\mathfrak{S})$ be the category of pairs $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ in $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{/\mathfrak{S}}$ such that $E\text{-ht}(\mathfrak{M}) \leq 1$.

We can also define the full subcategory of "connected" objects by requiring $\varphi_{\mathfrak{M}}$ to be nilpotent.

Example 4.4.2. — If $\mathfrak{M}$ is an object of $(\mathrm{Mod}\,/\mathfrak{S})$, then $\mathscr{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ is an object of $\mathrm{Mod}^{\mathrm{tor}}_{/\mathscr{O}_{\mathscr{E}}}$ (Definition 3.1.6); indeed, the image of $E \in \mathfrak{S}$ under the natural map $\mathfrak{S} \to \mathscr{O}_{\mathscr{E}}$ lands in $\mathscr{O}_{\mathscr{E}}^{\times}$ by Remark 3.1.4.

Objects in $(\mathrm{Mod}\,/\mathfrak{S})$ are precisely cokernels of maps in $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ that are isomorphisms in the isogeny category, so (with more work for $p = 2$) we get the following result, which was conjectured by Breuil and proved by him in some cases.

Theorem 4.4.3. — *If $p > 2$ then there is an anti-equivalence of categories between* $(\mathrm{Mod}\,/\mathfrak{S})$ *and the category of finite flat group schemes over $\mathscr{O}_K$. For $p = 2$, one has such an equivalence working with connected objects in* $(\mathrm{Mod}\,/\mathfrak{S})$ *and connected finite flat group schemes.*

Remark 4.4.4. — These equivalences are compatible with the ones for p-divisible groups. Thus, if the finite flat group scheme G over $\mathscr{O}_K$ corresponds to the object $\mathfrak{M}$ of $(\mathrm{Mod}\,/\mathfrak{S})$, then we have an isomorphism of G_{K_∞}-modules $G(\overline{K}) \simeq \underline{V}^*_{\mathfrak{S}}(\mathfrak{M})$, since the analogous statement holds for p-divisible groups (as one sees via the proof of Theorem 4.3.2).

Definition 4.4.5. — We say that an object T of $\mathrm{Rep}^{\mathrm{tor}}_{G_K}$ is *flat* (resp. *connected*) if $T \simeq G(\overline{K})$ for some finite flat (resp. finite flat and connected) group scheme G over $\mathscr{O}_K$.

Corollary 4.4.6. — *The natural restriction functor*

$$\mathrm{Rep}^{\mathrm{tor}}_{G_K} \longrightarrow \mathrm{Rep}^{\mathrm{tor}}_{G_{K_\infty}}$$

is fully faithful on flat (*respectively connected*) *representations if $p > 2$* (*respectively $p = 2$*).

Proof. — This proof is due to Breuil. We only treat the cases $p > 2$. Using the equivalence of categories $\mathrm{Rep}^{\mathrm{tor}}_{G_{K_\infty}} \simeq \mathrm{Mod}^{\varphi,\mathrm{tor}}_{\mathscr{O}_{\mathscr{E}}}$ of Lemma 3.1.8(1) and the fact that the diagram

$$\begin{tikzcd}
\mathrm{Rep}^{\mathrm{tor}}_{G_K} \arrow[rr] & & \mathrm{Rep}^{\mathrm{tor}}_{G_{K_\infty}} \\
\left\{\begin{matrix}\text{finite flat group}\\ \text{schemes over } \mathscr{O}_K\end{matrix}\right\} \arrow[u, "G(\overline{K})"] \arrow[r, "\simeq", "(4.4.3)"'] & (\mathrm{Mod}\,/\mathfrak{S}) \arrow[r, "(\cdot)\otimes_{\mathfrak{S}}\mathscr{O}_{\mathscr{E}}"'] & \mathrm{Mod}^{\varphi,\mathrm{tor}}_{\mathscr{O}_{\mathscr{E}}} \arrow[u, "\underline{V}^*_{\mathscr{O}_{\mathscr{E}}}", "\simeq"']
\end{tikzcd}$$

commutes (due to Lemma 3.2.3 and Remark 4.4.4), it suffices to prove the following statement. Let T_1 and T_2 be flat representations and let G_1 and G_2 be the corresponding finite flat group schemes over $\mathscr{O}_K$, so $T_1 \simeq G_1(\overline{K})$ and $T_2 \simeq G_2(\overline{K})$ (Definition 4.4.5). Denote by $\mathfrak{M}_1$ and $\mathfrak{M}_2$ the objects of $(\mathrm{Mod}\,/\mathfrak{S})$ corresponding to G_1, G_2 via Theorem 4.4.3 and let $\mathscr{M}_i = \mathscr{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_i$ for $i = 1, 2$ be the corresponding objects of $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{\mathscr{O}_{\mathscr{E}}}$. If $h : \mathscr{M}_1 \to \mathscr{M}_2$ is a morphism in $\mathrm{Mod}^{\varphi,\mathrm{tor}}_{\mathscr{O}_{\mathscr{E}}}$ then, after possibly modifying the $\mathfrak{M}_i$ without changing the generic fibers $(G_i)_K$ (so the Galois representations $G_i(\overline{K})$ remain unaffected), there is a morphism $\mathfrak{M}_1 \to \mathfrak{M}_2$ inducing h after extending scalars to $\mathscr{O}_{\mathscr{E}}$.

Due to Lemma 3.2.2, every object of $(\mathrm{Mod}\,/\mathfrak{S})$ has a filtration with successive quotients that are isomorphic to $\oplus_j \mathfrak{S}/p\mathfrak{S}$, so by a standard devissage we may restrict to the case that each $\mathfrak{M}_i$ is killed by p. In this situation, the natural map

$$\mathfrak{M}_i \longrightarrow \mathscr{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_i = k(\!(u)\!) \otimes_{k[\![u]\!]} \mathfrak{M}_i$$

is injective, so

$$\mathfrak{M}'_2 := \mathfrak{M}_2 + h(\mathfrak{M}_2) \subseteq \mathscr{M}_2$$

makes sense and is a φ-stable (as h is φ-equivariant) $\mathfrak{S}$-submodule of $\mathscr{M}_2$. Moreover, $\mathfrak{M}_2$ is clearly an object of $(\mathrm{Mod}\,/\mathfrak{S})$ and so corresponds to a finite flat group scheme G'_2 over $\mathscr{O}_K$. Since $\mathfrak{M}_2$ and $\mathfrak{M}'_2$ are both φ-stable lattices in $\mathscr{M}_2$, one shows that $(G_2)_K \simeq (G'_2)_K$. The map h then restricts to a map $h' : \mathfrak{M}_1 \to \mathfrak{M}'_2$ that induces h after extending scalars to $\mathscr{O}_{\mathscr{E}}$; this is the desired map. ∎

Now we turn to strongly divisible lattices and Fontaine-Laffaille modules. For the remainder of this section, we assume that $K = K_0$ and we take $\pi = p$, so $E = u - p$.

Definition 4.4.7. — Let D be an object of ${}^{\mathrm{w.a.}}\mathrm{MF}^{\varphi,\mathrm{Fil}\geq 0}_K$. A *strongly divisible lattice* in D is a W-lattice $L \subseteq D$ such that

1. $\varphi_D(L \cap \mathrm{Fil}^i D) \subseteq p^i D$ for all $i \geq 0$ (so $\varphi_D(L) \subseteq L$ by taking $i = 0$),
2. $\sum_{i\geq 0} p^{-i}\varphi_D(L \cap \mathrm{Fil}^i D) = L$.

We set $\mathrm{Fil}^i L = L \cap \mathrm{Fil}^i D$, and we say that L is *connected* if $\varphi_D : L \to L$ is topologically nilpotent for the p-adic topology.

Theorem 4.4.8. — *There are exact quasi-inverse anti-equivalences between the category of strongly divisible lattices L with $\mathrm{Fil}^p\, L = 0$ and the category of $\mathbf{Z}_p[G_K]$-lattices Λ in crystalline G_K-representations with Hodge-Tate weights in the set $\{0, \dots, p-1\}$.*

Proof. — Let V be a crystalline G_K-representation with Hodge-Tate weights in $\{0, \dots, p-1\}$ and let $\Lambda \subseteq V$ be a G_K-stable $\mathbf{Z}_p$-lattice. Because of Corollary 3.3.1 and Lemma 3.2.9, the lattice Λ corresponds to a unique object $\mathfrak{M}$ of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ such that $\underline{V}^*_{\mathfrak{S}}(\mathfrak{M}) \simeq \Lambda$ (as G_{K_∞}-representations); moreover, $\mathfrak{M}$ is functorial in Λ. Letting $D := D^*_{\mathrm{cris}}(V)$, we have that $\mathrm{Fil}^0 D = D$ and $\mathrm{Fil}^p D = 0$ due to the condition on the Hodge-Tate weights of the crystalline representation V, and there is a natural injection

$$D \hookrightarrow \mathscr{O} \otimes_{K_0} D \xhookrightarrow{\xi} \mathscr{O} \otimes_{\mathfrak{S}} \mathfrak{M} \hookrightarrow S\left[\tfrac{1}{p}\right] \otimes_{\mathfrak{S}} \mathfrak{M}\ ,$$

so twisting by Frobenius defines a natural injection

$$D \underset{1\otimes\varphi_D}{\overset{\simeq}{\longleftarrow}} \varphi^*(D) \hookrightarrow S\left[\tfrac{1}{p}\right] \otimes_{\mathfrak{S},\varphi} \mathfrak{M} = S\left[\tfrac{1}{p}\right] \otimes_{\mathfrak{S}} \varphi^*_{\mathfrak{S}}\mathfrak{M}\ .$$

Viewing D as a K_0-submodule of $S\left[\tfrac{1}{p}\right] \otimes_{\mathfrak{S},\varphi} \mathfrak{M}$ in this way, we define

$$L := D \cap (S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}) \subseteq S\left[\frac{1}{p}\right] \otimes_{\mathfrak{S},\varphi} \mathfrak{M}.$$

Clearly L is a φ-stable W-lattice in D. We claim that L is strongly divisible. Indeed, this follows from the fact that $p | \varphi_S(E)$ in S and $D = D^*_{\mathrm{cris}}(V)$ with V
1 crystalline. (1) Furthermore, one shows that the association $\Lambda \rightsquigarrow L$ is exact by using the filtration bounds (in $\{0, \dots, p-1\}$) to deduce that ξ as above induces
2 an isomorphism $L \simeq \mathfrak{M}/u\mathfrak{M}$. (2)

Conversely, let L be any strongly divisible lattice in an object D of ${}^{\mathrm{w.a.}}\mathrm{MF}_K^{\varphi,\mathrm{Fil}\geq 0}$ with $\mathrm{Fil}^p L := L \cap \mathrm{Fil}^p D = 0$ (so $\mathrm{Fil}^p D = 0$). Note that $\mathrm{Fil}^0 L = L$ since $\mathrm{Fil}^0 D = D$. We set

$$\Lambda := \mathrm{Hom}_{\varphi,\mathrm{Fil}}(L, A_{\mathrm{cris}}).$$

This is a G_K-stable lattice in the crystalline representation $\underline{V}^*_{\mathrm{cris}}(D)$ with $D = L[1/p]$. One shows that $L \rightsquigarrow \Lambda$ is exact and quasi-inverse to the other functor
3 built above. (3)

One shows that these associations are quasi-inverse. ∎

(1) need to say more

(2) must say more here.

(3) should say more here.

We now wish to apply this theory to torsion representations. In order to do this, we need a torsion replacement for strongly divisible lattices:

Definition 4.4.9. — A *Fontaine-Laffaille module* over W is a finite length W-module M equipped with a finite and separated decreasing filtration $\{\mathrm{Fil}^i M\}$ and φ-semilinear endomorphisms $\varphi_M^i : \mathrm{Fil}^i M \to M$ such that

1. the map $p\varphi_M^{i+1} : \mathrm{Fil}^{i+1} M \to M$ coincides with the restriction of φ_M^i to $\mathrm{Fil}_M^{i+1} \subseteq \mathrm{Fil}_M^i$,
2. $\sum_i \varphi_M^i(\mathrm{Fil}^i M) = M$,
3. $\mathrm{Fil}^0 M = M$.

We say M is *connected* if $\varphi_M^0 : M \to M$ is nilpotent.

Example 4.4.10. — If L is a strongly divisible lattice, then for each $n > 0$ we obtain a Fontaine-Laffaille module M by setting $M = L/p^n L$, taking $\mathrm{Fil}^i M$ to be the image of $\mathrm{Fil}^i L$ under the natural quotient map, and letting $\varphi_M^i := p^{-i}\varphi_L$. This is connected if and only if L is connected.

More generally, if $L' \to L$ is an isogeny of strongly divisible lattices, then L/L' has a natural structure of Fontaine-Laffaille module (and it is connected if L is connected).

Lemma 4.4.11. — *Let M be any Fontaine-Laffaille module with a one-step filtration (i.e. there is some $i_0 \geq 0$ such that $\mathrm{Fil}^i M = M$ for all $i \leq i_0$ and $\mathrm{Fil}^i M = 0$ for all $i > i_0$). Then there exists an isogeny of strongly divisible lattices $L' \to L$ with cokernel M.* [4] 4

By using such presentations and the functoriality and exactness properties of strongly divisible lattices, we get:

Theorem 4.4.12. — *Consider the contravariant functor*

$$M \rightsquigarrow \mathrm{Hom}_{\mathrm{Fil},\varphi}(M, A_{\mathrm{cris}} \otimes (\mathbf{Q}_p/\mathbf{Z}_p))$$

from the category of Fontaine-Laffaille modules M with one-step filtration that satisfies $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^p M = 0$ to the category of p-power torsion discrete G_K-modules. If $p > 2$ this is an exact and fully faithful functor into the category $\mathrm{Rep}_{G_K}^{\mathrm{tor}}$ (i.e., image objects are finite abelian groups). If $p = 2$, the same statement holds if one restricts to connected Fontaine-Laffaille modules. [5] 5

[4] should sketch proof or give reference

[5] should sketch proof or give reference

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