Workshop on group schemes and p-divisible groups: Homework 1.

1. Let S be a scheme, and G and G' group schemes over S.

(i) Using Yoneda's Lemma and group theory, show that the identity section and inversion morphism for G are uniquely determined by the multiplication morphism and that if $f : G \to G'$ is an S-morphism that respects the multiplication law then f is a morphism of group schemes. Unwind the *proof* of Yoneda's Lemma to derive at least one of these facts by writing down some diagrams of morphisms (using well-chosen "test objects").

(ii) Prove that if $f : G' \to H$ is a map of S-groups then $\ker f$ is a normal subgroup of G' that is moreover closed if H is S-separated.

(iii) If $S = \mathrm{Spec}(k)$ for an algebraically closed field k and G and G' are locally of finite type and smooth over k with G a closed k-subgroup of G' then deduce that G is a normal k-subgroup of G' if $G(k)$ is a normal subgroup of $G'(k)$. Give a counterexample if the smoothness condition is dropped.

2. Let R be a ring, and work below with R-groups.

(i) Prove that there are no non-trivial homomorphisms from $\mathbf{G}_m$ to $\mathbf{G}_a$.

(ii) If R is reduced, prove that there are no non-trivial homomorphisms from $\mathbf{G}_a$ to $\mathbf{G}_m$.

(iii) If $\varepsilon \in R$ is nonzero and $\varepsilon^2 = 0$, use it to construct a non-trivial homomorphism from $\mathbf{G}_a$ to $\mathbf{G}_m$. If moreover R is an $\mathbf{F}_p$-algebra, construct an automorphism of $\mathbf{G}_a$ not given by $R^\times$-scaling.

3. Let k be a field of characteristic $p > 0$.

(i) For all $n \geq 1$ prove that α_{p^n} and μ_{p^n} are isomorphic as k-schemes, but not as k-groups.

(ii) If a group scheme G acts on a group scheme H over a base scheme S (via group automorphisms of H), define the notion of *semi-direct product* $G \ltimes H$ as an S-group. Make a non-commutative semi-direct product $\mu_p \ltimes \alpha_p$ over $\mathbf{F}_p$.

(iii) Construct a short exact sequence $0 \to \alpha_p \to \alpha_{p^2} \to \alpha_p \to 0$ over $\mathbf{F}_p$ and prove that it is not split even scheme-theoretically (let alone as a semi-direct product) over any extension field.

4. Let k be a field of characteristic $p > 0$, and let $F : \mathbf{G}_a \to \mathbf{G}_a$ be the k-group map given functorially by the pth-power map on $\mathbf{G}_a(X) = \Gamma(X, \mathscr{O}_X)$ for k-schemes X.

(i) Prove that $\mathrm{End}_k(\mathbf{G}_a)$ is a non-commutative (for $k \neq \mathbf{F}_p$) polynomial ring $k\{F\}$ with the relation $Fc = c^pF$. That is, prove that the "additive polynomials" over k are precisely $\sum c_j T^{p^j}$.

(ii) Prove $\ker(F^n) = \alpha_{p^n}$ and $\ker(F - 1) \simeq \mathbf{Z}/p\mathbf{Z}$. What is $\ker(F^n - 1)$?

(iii) Show that if K is a field of characteristic 0, then $\mathrm{End}_K(\mathbf{G}_a)$ consists of scalar multiplications.

5. Prove "by hand" that the diagram $0 \to \mu_N \to \mu_{NN'} \xrightarrow{\zeta^N} \mu_{N'} \to 0$ is a short exact sequence. Use Cartier duality to give a second proof by pure thought.

6. Let p be a prime, and let $W(X, Y) = ((X + Y)^p - X^p - Y^p)/p \in \mathbf{Z}[X, Y]$.

(i) Prove that for the $\mathbf{F}_p$-scheme $G = \mathrm{Spec}(\mathbf{F}_p[X, Y]/(X^p, Y^p))$ with composition law

$$(x, y) \cdot (x', y') = (x + x', y + y' + W(x, x'))$$

is a commutative group scheme structure. What is inversion?

(ii) Prove $G \simeq \alpha_{p^2}^\vee$ and describe the sequence dual to Exercise 3(iii) via self-duality of α_p.

7. (i) Prove that the group functor $X \mapsto \mathrm{GL}_n(\Gamma(X, \mathscr{O}_X))$ on the category of schemes is represented by the scheme $\mathrm{GL}_n = \mathrm{Spec}(\mathbf{Z}[t_{ij}][1/\det])$ (with $1 \leq i, j \leq n$). What is its Hopf algebra structure?

(ii) Do the same for SL_n, and prove that both GL_n and SL_n are flat over $\mathrm{Spec}\,\mathbf{Z}$ with geometric fibers that are connected and smooth. (For smoothness of geometric fibers for SL_n, find the dimension of the tangent space at the identity by using the dual numbers.)

(*iii*) Write the ring map corresponding to the $\mathbf{Z}$-group map $\det : \mathrm{GL}_n \to \mathbf{G}_m$, and use the irreducibility of $\det(t_{ij})$ over any field (proof?) to deduce that the only group scheme maps from GL_n to $\mathbf{G}_m$ over a field are $\det^r$ for $r \in \mathbf{Z}$.

(*iv*) What is the scheme-theoretic intersection of SL_n and the diagonally embedded closed subgroup $\mathbf{G}_m \hookrightarrow \mathrm{GL}_n$? Do this functorially and algebraically.

8. Let k be a field, $\overline{k}/k$ an algebraic closure, and G a locally finite type k-group.

(*i*) Prove that a group scheme is separated if and only if its identity section is a closed immersion (Hint: identity section is base change of the diagonal); deduce that G is k-separated.

(*ii*) If $G_{\overline{k}}$ is smooth, prove that for any subgroup $\Gamma \subseteq G(k)$ the Zariski closure of Γ in G is a closed k-subgroup of G whose formation commutes with extension of the base field.

(*iii*) If k is perfect, prove that G_{red} is a (closed) k-subgroup of G. Can you find a counterexample if k is not perfect? For any field k with $\mathrm{char}(k) = p > 0$, show that the natural semidirect product $G = \mathbf{G}_m \ltimes \alpha_p$ has $G_{\mathrm{red}} = \mathbf{G}_m$ a non-normal k-subgroup of G.

(*iv*) Prove that a connected k-scheme X that is locally of finite type is geometrically connected if $X(k)$ is non-empty. (Hint: Use local finiteness of the set of irreducible components to reduce to the quasi-compact case, and show that $K \otimes_k \Gamma(X, \mathscr{O}_X) \to \Gamma(X_K, \mathscr{O}_{X_K})$ is an isomorphism for any K/k in this case. Then chase idempotents and study fibers of the map $k' \otimes_k X \to X$ for finite separable k'/k. Actually, with better technique the locally finite type hypothesis can be dropped; see EGA IV_2, 4.5.13) Deduce via (*iii*) over $\overline{k}$ that if G is connected then it is irreducible and in fact quasi-compact. (Hint: Use $(g_1, g_2) \mapsto (g_1 g_2, g_2)$ to prove that $m : G \times G \to G$ is flat, hence open, so $m(U \times U)$ is open in G for any open $U \subseteq G$.) Can you build a non-quasi-compact and non-separated flat group locally of finite type over the spectrum of a discrete valuation ring?

(*v*) If G^0 is the identity component of G, prove that G^0 is an open and closed normal subgroup of G. Can you use Galois descent to define an étale "component group" G/G^0 such that $G \to G/G^0$ has kernel G^0 and is universal for k-group maps from G to étale k-groups? (It may be easier to just consider the case when G is quasi-compact, in which case G/G^0 is a finite k-group.)

9. Let S be a scheme and X an S-scheme.

(*i*) Prove that for any S-scheme T, the presheaf of sets $U \mapsto X(U) = \mathrm{Hom}_S(U, X)$ on T is a sheaf for the Zariski topology. Deduce that if X and Y are S-schemes and X_{aff} and Y_{aff} denote the resulting functors on the category of affine schemes over S (equipped with S-morphisms), then any natural transformation $X_{\mathrm{aff}} \to Y_{\mathrm{aff}}$ arises from a unique map of S-schemes $X \to Y$. Hence, we may functorially work with schemes as covariant functors on rings (including structure maps to a base scheme), and in particular we may speak of a functor on rings being *representable* by a scheme.

(*ii*) Use rings with non-trivial Picard group to show that the covariant group functor $R \mapsto \mathrm{GL}_n(R)/R^\times$ does not satisfy the "Zariski sheaf" property in (*i*) for $n > 1$, and so it is not representable by a scheme.

(*iii*) Prove that if a covariant functor F on rings is representable by a scheme, then for any ring R and finite group $G \subseteq \mathrm{Aut}(R)$ the natural map $F(R^G) \to F(R)^G$ is a bijection. Deduce via the possible non-triviality of $k^\times/k^{\times n} \simeq \mathrm{H}^1(k_{\mathrm{s}}/k, \mu_n)$ for fields k with $\mathrm{char}(k) \nmid n$ that the functor $R \mapsto \mathrm{SL}_n(R)/\mu_n(R)$ is not representatable by a scheme for $n > 1$.

10. The *miracle flatness theorem* (§23 in Matsumura's *Commutative Ring Theory*) says that if $A \to B$ is a local map between local noetherian rings with A regular and B Cohen-Macaulay (e.g., B regular) then the dimension formula $\dim B = \dim A + \dim(B/\mathfrak{m}_A B)$ implies flatness. (The converse holds without regularity and CM conditions.)

Prove that if $f : G \to G'$ is a surjective map between locally finite type group schemes over a field k with $G_{\overline{k}}$ and $G'_{\overline{k}}$ smooth then f is faithfully flat.

11. (*i*) If $A \to A'$ is faithfully flat and M is an A-module, prove that $M \to M' = A' \otimes_A M$ is an isomorphism onto the A-submodule of elements $m' \in M'$ that satisfy $p_1^*(m') = p_2^*(m')$, where

$$p_1^*, p_2^* : M' \rightrightarrows A' \otimes_A A' \otimes_A M$$

are determined by $a' \otimes m \mapsto a' \otimes 1 \otimes m, 1 \otimes a' \otimes m$. (Hint: First assume $A \to A'$ has a section with kernel ideal I, and use the resulting decomposition $A' = A \oplus I$ as A-modules. Then use faithfully flat base change by *itself* to reduce to this case via the diagonal section.) In particular, $a' \in A'$ lies in A if and only if $a' \otimes 1 = 1 \otimes a'$ in $A' \otimes_A A'$; give a counterexample if "faithful" is dropped.

(*ii*) If $\{\operatorname{Spec} A_i\}$ is a finite open affine covering of $\operatorname{Spec} A$ prove that $A' = \prod A_i$ is faithfully flat over A and express (*i*) in terms of gluing for the Zariski topology. Likewise, if K/k is a finite Galois extension of fields with Galois group G then use the isomorphism $K \otimes_k K \simeq \prod_{g \in G} K$ defined by $a \otimes b \mapsto (g(a)b)$ to express (*i*) as the statement $(K \otimes_k V)^G = V$ for any k-vector space V (with G acting on $K \otimes_k V$ through the left tensor factor).

(*iii*) Let S be a scheme and $f : X' \to X$ a faithfully flat and quasi-compact S-map. Let

$$p_1, p_2 : R = X' \times_X X' \rightrightarrows X'$$

be the two projections. By working Zariski-locally on X and X' (i.e., use the first part of Exercise 9(*i*)) and using that a finite disjoint union of affines is affine, prove that if Y is an S-scheme then $Y(X) = \operatorname{Hom}_S(X, Y)$ naturally injects into $Y(X')$ and is identified with the subset of elements with the same image under both maps $p_i^* : Y(X') \rightrightarrows Y(R)$. How does this recover Exercise 9(*i*) as a special case? (Suggestion: First take $S = \operatorname{Spec} \mathbf{Z}$. Then track S-compatibility by taking $Y = S$.)

(*iv*) In the setup of Exercise 10, construct an isomorphism $G \times_{G'} G \simeq G \times \ker(f)$ as k-schemes, and deduce that if $G \to Z$ is a map of k-schemes that is invariant under translation by $\ker(f)$ then it uniquely factors through $f : G \to G'$.

(*v*) Use (*iii*) to derive a new proof that Cartier duality carries short exact sequences to short exact sequences. (Hint: recall that finite – even proper – monomorphisms are closed immersions.)

12. Let $\mathrm{PGL}_n = \operatorname{Spec}(\mathbf{Z}[t_{ij}]_{(\det)}) = \mathrm{D}_+(\det) \subseteq \mathbf{P}^{n^2-1}$ with $0 \le i, j \le n-1$.

(*i*) Check that $\mathbf{Z}[t_{ij}]_{(\det)}$ is a Hopf subalgebra of the coordinate ring of GL_n, and prove that the natural map $\mathrm{GL}_n \to \mathrm{PGL}_n$ is a faithfully flat homomorphism with kernel given by the diagonally embedded $\mathbf{G}_m \hookrightarrow \mathrm{GL}_n$.

(*ii*) Functorially identify $\mathrm{GL}_n(R)/R^\times$ with a subgroup of $\mathrm{PGL}_n(R)$, and show that for any $M \in \mathrm{PGL}_n(R)$ there is a Zariski-open covering $\{\operatorname{Spec} R_{r_i}\}$ of $\operatorname{Spec} R$ (with $r_i \in R$) such that the image of M in $\mathrm{PGL}_n(R_{r_i})$) is in the subgroup $\mathrm{GL}_n(R_{r_i})/R_{r_i}^\times$. Prove also that for $n > 1$, $\mathrm{GL}_n(R)/R^\times = \mathrm{PGL}_n(R)$ if $\operatorname{Pic}(R)$ is trivial (e.g., R local).

(*iii*) Formulate a universal property for $\mathrm{GL}_n \to \mathrm{PGL}_n$ akin to Exercise 11(*iv*).

(*iv*) Prove that the composite homomorphism $\mathrm{SL}_n \to \mathrm{PGL}_n$ is faithfully flat (use Exercise 10 on fibers over $\operatorname{Spec} \mathbf{Z}$, and also the fibral flatness criterion), and that its kernel is the diagonally embedded μ_n; also formulate a universal property of this homomorphism (suggesting that one should say $\mathrm{PSL}_n = \mathrm{PGL}_n$). Show also that $\mathrm{SL}_n(R)/\mu_n(R)$ is naturally a subgroup of $\mathrm{PGL}_n(R)$, with equality for a local ring R if and only if $R^\times = (R^\times)^n$.