DESCENT FOR COHERENT SHEAVES ON RIGID-ANALYTIC SPACES

BRIAN CONRAD

Let k be a field complete with respect to a non-trivial non-archimedean valuation. Throughout, we work with rigid-analytic spaces over k. The purpose of these notes is to prove:

Theorem. *Let $f : S' \to S$ be a faithfully flat quasi-compact map of rigid spaces. The functor from coherent sheaves on S to coherent sheaves on S' equipped with descent data is an equivalence of categories.*

If one assumes k to have a discrete valuation, then descent theory for coherent sheaves is an old result of Gabber. Gabber's method was extended to the general case by Bosch and Görtz [BG]. Our method is rather different from theirs (though both approaches do use Raynaud's theory of formal models [BL1], [BL2], we use less of this theory). We think that our approach may be of independent interest, because in contrast with all other known examples of descent (at least to this author) the method of proof is to show non-vanishing of the "descent module" *before* one proves effectivity of descent, and to invoke noetherian induction to obtain the effectivity. Further geometric examples of *fpqc* descent in the rigid-analytic context are discussed in [C].

Proof. We refer to the lucid exposition in [BLR1, §6.1] for the algebraic version of descent theory, which will play a crucial role in what we do. Faithfulness is obvious (without even requiring quasi-compactness of f), and so for the rest we may work locally on S. Thus, we can assume S is affinoid. Arguing exactly as in the algebraic case, since faithfulness doesn't require quasi-compactness of f we can reduce to the case where S' is affinoid too. In particular, f is now (quasi-)separated and quasi-compact. Thus, using Raynaud's theory of formal models [BL1], [BL2] we can run through the localization argument again to reduce to the case in which $S' \to S$ is the "generic fiber" of a map of formal affines $\mathrm{Spf}(\mathscr{A}') \to \mathrm{Spf}(\mathscr{A})$ where $\mathscr{A} \to \mathscr{A}'$ is a faithfully flat map of topologically finitely presented and flat R-algebras (R being the valuation ring of k). Let $A \to A'$ denote the corresponding faithfully flat map of k-affinoids.

For the full faithfulness, the theory of formal models for coherent sheaves permits us (after running through the affine localization argument a second time on the level of formal models) to use the scheme-theoretic argument, once we establish the analogue of [BLR1, 6.1/2]: if $\mathscr{M}$ is a coherent $\mathscr{A}$-module, then the standard complex of $\mathscr{A}$-modules

$$0 \to \mathscr{M} \to \mathscr{M}\widehat{\otimes}_{\mathscr{A}}\mathscr{A}' \to \mathscr{M}\,\widehat{\otimes}_{\mathscr{A}}\mathscr{A}'\widehat{\otimes}_{\mathscr{A}}\mathscr{A}' \tag{0.0.1}$$

is exact. But this sequence is the inverse limit of the corresponding sequences of ordinary tensor products modulo $\mathfrak{I}^n$ for all $n \geq 1$ (with $\mathfrak{I}$ an ideal of definition of R), so by left exactness of inverse limits we reduce to the standard algebraic exactness for the faithfully flat ring extension $\mathscr{A}/\mathfrak{I}^n\mathscr{A} \to \mathscr{A}'/\mathfrak{I}^n\mathscr{A}'$ and the module $\mathscr{M}/\mathfrak{I}^n\mathscr{M}$. The *effectivity* of descent in the coherent sheaf case is much more interesting.

The point of difficulty is that the descent data isomorphism

$$(\mathscr{M}'\,\widehat{\otimes}_{\mathscr{A}}\mathscr{A}') \otimes_R k \simeq (\mathscr{A}'\widehat{\otimes}_{\mathscr{A}}\mathscr{M}') \otimes_R k$$

need not respect the "integral structure", so we cannot trivially reduce to the standard effectivity proof for modules by working modulo powers of an ideal of definition. Instead, we will use a rather curious indirect method to push through the usual commutative algebra argument with completed tensor products on the k-affinoid level.

Date: Jan. 24, 2003.

1991 *Mathematics Subject Classification.* Primary 14G22; Secondary 14H52.

This research was partially supported by a grant from the NSF and the Clay Mathematics Institute.

Let $M' = \mathscr{M}' \otimes_R k$ be the finite A'-module corresponding to $\mathscr{F}'$, so we have two continuous A-linear maps

$$p_2, \varphi \circ p_1 : M' \rightrightarrows A' \widehat{\otimes}_A M',$$

with p_2 the canonical map, p_1 the canonical map to $M' \widehat{\otimes}_A A'$, and

$$\varphi : M' \widehat{\otimes}_A A' \simeq A' \widehat{\otimes}_A M'$$

the "mysterious" $A' \widehat{\otimes}_A A'$-linear descent data isomorphism (here and below, we give finite modules over affinoids their unique Banach module topologies). For convenience we will define M'' to be the $A' \widehat{\otimes}_A A'$-module $A' \widehat{\otimes}_A M'$ and we define the continuous A-linear map

$$\delta = p_2 - \varphi \circ p_1 : M' \to M'' \tag{0.0.2}$$

to be the difference of the two A-linear maps just indicated. Finally, define the closed Banach A-submodule $M \subseteq M'$ by the left exact sequence

$$0 \to M \to M' \to M''$$

We wish to prove that M is a finitely generated A-module and that the resulting A'-linear map

$$A' \otimes_A M \simeq A' \widehat{\otimes}_A M \to M'$$

(that automatically respects the descent data) is an isomorphism. A priori, M could be 0 or could fail to be A-finite. In any case, the appearance of completed tensor products prevents a trivial application of Grothendieck's proof from the algebraic case. Surprisingly, we will have to prove $M \neq 0$ whenever $M' \neq 0$ somewhat *prior* to being able to prove that M is A-finite and a descent of M'.

For ease of notation in what follows, we let $A^{(n)}$ denote the n-fold completed tensor product of A' over A (with $A^{(0)} = A$) and we let $M^{(n)}$ denote the $A^{(n)}$-module $A^{(n-1)} \widehat{\otimes}_A M'$. We wish to study that the standard *Čech-descent complex*

$$M^{(1)} \to M^{(2)} \to M^{(3)} \to \dots$$

where the A-linear map $\mathrm{d}^n_{M^\bullet} : M^{(n)} \to M^{(n+1)}$ is given by the usual $(n+1)$-fold "alternating difference" formula involving the descent data:

$$\mathrm{d}^n_{M^\bullet}(a'_1 \widehat{\otimes} \dots \widehat{\otimes} a'_n \widehat{\otimes} m') = \sum_{i=1}^{n+1} (-1)^{i+1} (a'_1 \widehat{\otimes} \dots \widehat{\otimes} \delta(a'_i \widehat{\otimes} m') \widehat{\otimes} \dots \widehat{\otimes} a'_n)$$

(with the understanding that in the ith term, we move the "M'-part" back out to the far right). We will write $M^{(\bullet)}$ as shorthand for this complex. The complex $M^{(\bullet)}$ admits the evident "inclusion" augmentation from M in degree 0. We will present the proof as a series of 8 steps.

Step 1: *If the descent is effective, then M must be the finite A-module descent and the A-module Čech complex $M^{(\bullet)}$ is a resolution of M via the augmentation.*

Since we have already established full faithfulness for descent of coherent sheaves (using the crutch of formal models in (0.0.1)), we just have to show that for any *finite* A-module N, the augmented Čech-descent complex attached to this N is exact. By choosing a coherent R-flat $\mathscr{A}$-module $\mathscr{N}$ that is a formal model for N, our complex of interest is the result of applying the localization functor $k \otimes_R (\cdot)$ to the analogous complex of $\mathscr{A}$-modules

$$0 \to \mathscr{N} \to \mathscr{N}^{(\bullet)} \tag{0.0.3}$$

It suffices to show that (0.0.3) is exact. Modulo any π^n, (0.0.3) is the Čech-descent complex for the $\mathscr{A}/\pi^n\mathscr{A}$-module $\mathscr{N}/\pi^n\mathscr{N}$ relative to the ordinary faithfully flat covering $\mathrm{Spec}(\mathscr{A}'/\pi^n\mathscr{A}') \to \mathrm{Spec}(\mathscr{A}/\pi^n\mathscr{A})$. Thus, (0.0.3) modulo π^m is exact for all m by usual descent theory. Since the transition maps in these inverse systems are surjective and all $\mathscr{N}^{(n)}$'s are R-flat and π-adically separated and complete, the usual Mittag-Leffler argument gives exactness in the limit. This completes Step 1.

Step 2: *If $0 \to M'_1 \to M' \to M'_2 \to 0$ is an exact sequence of finite A'-modules with descent data and if descent is effective for M'_1 and M'_2, then the sequence of A-module kernels*

$$0 \to M_1 \to M \to M_2 \to 0$$

is exact (so M is a finite A-module) and descent is effective for M'.

Consider the commutative diagram of complexes of A-modules

(0.0.4)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_1' & \longrightarrow & M_1'' & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & M' & \longrightarrow & M'' & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_2 & \longrightarrow & M_2' & \longrightarrow & M_2'' & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The effectivity hypothesis, coupled with Step 1, implies that the top and bottom rows are exact. By hypothesis, the second column is a short exact sequence of finite A'-modules. Each successive column in (0.0.4) is obtained from this by applications of $A^{(n+1)}\widehat{\otimes}_{A^{(n)}}(\cdot)$ for $n = 1, 2, \ldots$. Since this functor takes short exact sequences of finite $A^{(n)}$-modules to short exact sequences of finite $A^{(n+1)}$-modules, we get the short exactness of all columns after the first. Now the snake lemma ensures that the first column of kernels in (0.0.4) is short exact. In particular, M is a *finite* A-module.

It now makes sense to consider the commutative diagram of *finite* A'-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' \otimes_A M_1 & \longrightarrow & A' \otimes_A M & \longrightarrow & A' \otimes_A M_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1' & \longrightarrow & M' & \longrightarrow & M_2' & \longrightarrow & 0
\end{array}$$

Both rows are short exact and the left and right vertical maps are isomorphisms by hypothesis, so the middle vertical map is an isomorphism, visibly compatible with the descent data (due to how M was *defined*). This completes Step 2.

Now comes a strengthening of Step 1 in which we make no finiteness hypotheses on M or effectivity hypotheses on the descent.

Step 3: *The Čech-descent complex $M^{(\bullet)}$ is exact.*

Before we verify the exactness, we need to review some basic formulas. Consider a faithfully map $B \to B'$ of k-affinoids. The examples to keep in mind are $A^{(n)} \to A^{(n)}\widehat{\otimes}_A A'$ (via $x \mapsto x\widehat{\otimes}1$) for $n \geq 1$. Recall that the Čech-descent complex

$$B' \to B'\widehat{\otimes}_B B' \to \ldots$$

has continuous B-linear differential $\mathrm{d}^n : B^{(n)} \to B^{(n+1)}$ for $n \geq 1$ determined by

$$\mathrm{d}^n(b_1\widehat{\otimes}\ldots\widehat{\otimes}b_n) = \sum_{i=1}^{n+1}(-1)^{i+1}b_1\widehat{\otimes}\ldots\widehat{\otimes}1\widehat{\otimes}\ldots\widehat{\otimes}b_n$$

(where the ith term has a 1 in the ith slot). When we are given a B-linear section $s : B' \twoheadrightarrow B$, this complex is exact because we can use s to make an explicit homotopy between 0 and the identity. The standard choice for defining the homotopy maps $k^n : B^{(n+1)} \to B^{(n)}$ is

$$k^n(b_1\widehat{\otimes}\ldots\widehat{\otimes}b_{n+1}) = s(b_1)b_2\widehat{\otimes}\ldots\widehat{\otimes}b_{n+1}$$

for $n \geq 1$, but for our purposes it is more convenient to use $s^n : B^{(n+1)} \to B^{(n)}$ defined by $s^0 = s$ and

(0.0.5)
$$s^n(b_1\widehat{\otimes}\ldots\widehat{\otimes}b_{n+1}) = (-1)^n s(b_{n+1})b_1\widehat{\otimes}\ldots\widehat{\otimes}b_n$$

for $n \geq 1$ (the identity $\mathrm{d}^{n-1} \circ s^{n-1} + s^n \circ \mathrm{d}^n = \mathrm{id}$ is easily checked for all $n \geq 0$). The point is that when $n \geq 1$, the homotopy map s^n has no "interaction" with the first factor b_1 of the tensor product (in contrast to the more standard k^n). This will ensure a certain compatibility later on.

Whenever we are given a Čech-descent complex $N^{(\bullet)}$ for $\mathrm{Sp}(B') \to \mathrm{Sp}(B)$ attached to descent data on a finite B'-module N' such that the descent is effective, with N' descending to a finite B-module N, we get a B'-linear isomorphism $B'\widehat{\otimes}_B N \simeq N'$ that induces a *canonical* B-linear identification of $N^{(\bullet)}$ with $B^{(\bullet)}\widehat{\otimes}_B N$ as augmented complexes, the latter being exact by Step 1. If we are given the specification of a section s as above, then descent for coherent sheaves *is* effective (with the same proof as in the algebraic case, via pullback along the section), so in such cases if we let N denote the descended module then the B-linear $s^\bullet\widehat{\otimes}\mathrm{id}_N$, with $s^\bullet$ as defined in (0.0.5), induces an explicit B-linear homotopy between 0 and the identity on $N^{(\bullet)}$. This latter formula is hard to "see" in terms of $N^{(\bullet)}$ alone, as it rests on the effectivity of the descent.

Returning to our original situation, consider the first quadrant *commutative* diagram of Banach A-modules

(0.0.6)
$$\begin{array}{ccccccc}
 & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A'\widehat{\otimes}_A A'\widehat{\otimes}_A M^{(n)} & \xrightarrow{1\widehat{\otimes}1\widehat{\otimes}\mathrm{d}^n_{M^\bullet}} & A'\widehat{\otimes}_A A'\widehat{\otimes}_A M^{(n+1)} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle p_2-p_1} & & \uparrow{\scriptstyle p_2-p_1} & & \\
\cdots & \longrightarrow & A'\widehat{\otimes}_A M^{(n)} & \xrightarrow{1\widehat{\otimes}\mathrm{d}^n_{M^\bullet}} & A'\widehat{\otimes}_A M^{(n+1)} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & \longrightarrow & M^{(n)} & \xrightarrow[\mathrm{d}^n_{M^\bullet}]{} & M^{(n+1)} & \longrightarrow & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array}$$

with $n \geq 1$ and maps defined as follows. The bottom (or 0th) row is the A-linear Čech-descent complex $M^{(\bullet)}$ whose exactness we wish to prove. For $q \geq 1$, the $A^{(q)}$-linear qth row is the *completed tensor product* extension of scalars by $A^{(q-1)} \to A'\widehat{\otimes}_A A^{(q-1)} = A^{(q)}$ on the $A^{(q-1)}$-linear $(q-1)$th row. Finally, the nth column is the result of applying $(\cdot)\widehat{\otimes}_A M^{(n)}$ to the standard augmented A-linear Čech complex $A \to A^{(\bullet)}$ (associated to the *fpqc* covering $\mathrm{Sp}(A') \to \mathrm{Sp}(A)$). The commutativity of (0.0.6) follows from A-linear functoriality. Using the isomorphism

$$A^{(q)}\widehat{\otimes}_A M^{(n)} \simeq A^{(q)}\widehat{\otimes}_A A^{(n)}\widehat{\otimes}_{A^{(n)}} M^{(n)} \simeq (A'\widehat{\otimes}_A A^{(n)})^{\widehat{\otimes} q}\widehat{\otimes}_{A^{(n)}} M^{(n)}$$

(in which the q-fold tensor product in the right term is taken over $A^{(n)}$), the nth column in (0.0.6) is transformed into the augmented Čech-descent complex for the finite $A^{(n)}$-module $M^{(n)}$ relative to the *fpqc* covering

$$\mathrm{Sp}(A^{(n+1)}) = \mathrm{Sp}(A'\widehat{\otimes}_A A^{(n)}) \to \mathrm{Sp}(A^{(n)}).$$

Thus, by Step 1, each column in (0.0.6) is exact.

For $q \geq 0$, base change compatibility of descent theory implies that the qth row in (0.0.6) is exactly the $A^{(q)}$-linear Čech-descent complex for the finite $A^{(q+1)}$-module $A^{(q)}\widehat{\otimes}_A M'$ with respect to the *fpqc* covering $\mathrm{Sp}(A^{(q+1)}) = \mathrm{Sp}(A^{(q)}\widehat{\otimes}_A A') \to \mathrm{Sp}(A^{(q)})$. Consider the section s_q that corresponds to the k-affinoid algebra map

(0.0.7) $$s_q(a_1\widehat{\otimes}\dots\widehat{\otimes}a_{q+1}) = a_1\widehat{\otimes}\dots\widehat{\otimes}a_q a_{q+1}$$

(geometrically, this is $(x_1,\dots,x_q) \mapsto (x_1,\dots,x_q,x_q)$). Following the homotopy convention (0.0.5), we get a homotopy $s_q^\bullet$ between 0 and the identity for the qth row of (0.0.6). More specifically, if $x \in A^{(q)}\widehat{\otimes}_A M^{(n)}$ is killed by the horizontal d^n for some $n \geq 2$, then $x = \mathrm{d}^{n-1} \circ s_q^{n-1}(x)$, with $s_q^{n-1}(x) \in A^{(q)}\widehat{\otimes}_A M^{(n-1)}$. This explicates the *exactness* of the qth row of (0.0.6).

By a standard spectral sequence argument, the exactness of the bottom row of (0.0.6) is equivalent to the exactness of the induced sequence of horizontal kernels along the left side in rows above the bottom row. If we let K^q denote the kernel at the far left of the qth row, then K^q is a finite $A^{(q)}$-module and $A'\widehat{\otimes}_A K^\bullet \simeq A^{(\bullet)}\widehat{\otimes}_A M'$ is an exact sequence (it is just the first column of (0.0.6), in rows ≥ 1). If we could deduce the exactness of $K^\bullet$ from this, then we'd be done. However, since the completed tensor products in $A'\widehat{\otimes}_A K^\bullet$ cannot be replaced with ordinary tensor products (as K^q is rarely A-finite), we cannot use algebraic faithful flatness to descend exactness to $K^\bullet$. Thus, we use the following slightly indirect procedure that carefully analyzes the above explication of the exactness of some rows in (0.0.6) via the homotopies $s_\bullet$.

Doing a simple diagram chase through the bottom three rows of (0.0.6) with the help of these homotopies and the exactness of the columns, it follows that $M^{(\bullet)}$ is exact in degree n (with $n \geq 2$) as long as the diagram

(0.0.8)
$$\begin{array}{ccc} A'\widehat{\otimes}_A A'\widehat{\otimes}_A M^{(n-1)} & \xleftarrow{s_2^{n-1}} & A'\widehat{\otimes}_A A'\widehat{\otimes}_A M^{(n)} \\ \uparrow{\scriptstyle p_2-p_1} & & \uparrow{\scriptstyle p_2-p_1} \\ A'\widehat{\otimes}_A M^{(n-1)} & \xleftarrow[s_1^{n-1}]{} & A'\widehat{\otimes}_A M^{(n)} \end{array}$$

commutes, where we recall that the horizontal maps are defined in terms of our homotopy conventions (depending on the sections s_1 and s_2) and the effectivity of *fpqc* descent in the presence of a section.

Unwinding the effectivity of descent, the descended modules drop out (via functoriality) and we are left with checking the commutativity of the diagram

$$\begin{array}{ccccccc} (A')^{\widehat{\otimes}_A(n+1)} & \xrightarrow{\simeq} & (A'\widehat{\otimes}_A A'\widehat{\otimes}_A A')^{\widehat{\otimes}_{A'\widehat{\otimes}_A A'}(n-1)} & \xleftarrow{s_2^{n-1}} & (A'\widehat{\otimes}_A A'\widehat{\otimes}_A A')^{\widehat{\otimes}_{A'\widehat{\otimes}_A A'}n} & \xleftarrow{\simeq} & (A')^{\widehat{\otimes}_A(n+2)} \\ \uparrow{\scriptstyle p_2-p_1} & & & & & & \uparrow{\scriptstyle p_2-p_1} \\ (A')^{\widehat{\otimes}_A n} & \xrightarrow{\simeq} & (A'\widehat{\otimes}_A A')^{\widehat{\otimes}_{A'}(n-1)} & \xleftarrow{s_1^{n-1}} & (A'\widehat{\otimes}_A A')^{\widehat{\otimes}_{A'}n} & \xleftarrow[\simeq]{} & (A')^{\widehat{\otimes}_A(n+1)} \end{array}$$

Recalling (0.0.5) and (0.0.7), checking the commutativity of this diagram is a simple computation that we illustrate as follows:

$$\begin{array}{ccc} (-1)^{n-1}(1\widehat{\otimes}a_1 - a_1\widehat{\otimes}1)\widehat{\otimes}a_2\widehat{\otimes}\dots\widehat{\otimes}a_n a_{n+1} & \longleftarrow & (1\widehat{\otimes}a_1 - a_1\widehat{\otimes}1)\widehat{\otimes}a_2\widehat{\otimes}\dots\widehat{\otimes}a_{n+1} \\ \uparrow & & \uparrow \\ (-1)^{n-1}a_1\widehat{\otimes}a_2\widehat{\otimes}\dots\widehat{\otimes}a_n a_{n+1} & \longleftarrow & a_1\widehat{\otimes}a_2\widehat{\otimes}\dots\widehat{\otimes}a_{n+1} \end{array}$$

The point is that since $n+1 \geq 3$, the products $a_n a_{n+1}$ do not interfere with a_1.

Step 4: *If* $M' \neq 0$ *then* $M \neq 0$.

In other words, if δ in (0.0.2) is injective then we claim $M' = 0$. Beware that it is *not* obvious that $1\widehat{\otimes}\delta$ is necessarily injective. We will eventually deduce this with the help of Step 3. It seems rather hard to directly prove Step 4. We shall instead prove the contrapositive using formal models and the Banach Open Mapping Theorem.

As we noted already, the A-linear δ in (0.0.2) might not take the R-flat $\mathscr{M}' \subseteq M'$ into the R-flat $\mathscr{M}'' \subseteq M''$. However, if we choose an ideal of definition $\mathfrak{I} = (\pi)$ for R and let $\Delta = \pi^n\delta$ for suitably large n, then Δ is injective and takes $\mathscr{M}'$ into $\mathscr{M}''$. Consider the exact sequence of R-modules

$$0 \to \mathscr{M}' \xrightarrow{\Delta} \mathscr{M}'' \to \mathscr{N} \to 0$$

Thus, by R-flatness of $\mathscr{M}''$ we have exact sequences

$$0 \to \mathrm{Tor}_R^1(R/\mathfrak{I}^m, \mathscr{N}) \to \mathscr{M}'/\mathfrak{I}^m\mathscr{M}' \to \mathscr{M}''/\mathfrak{I}^m\mathscr{M}''$$

for all m. Moreover, since torsion-free R-modules are flat, we have an isomorphism

$$\mathrm{Tor}^i_R(\cdot, \mathscr{N}_{\mathrm{tors}}) \simeq \mathrm{Tor}^i_R(\cdot, \mathscr{N})$$

for all $i > 0$, where $\mathscr{N}_{\mathrm{tors}}$ denotes the torsion R-submodule of $\mathscr{N}$. Thus, we have an exact sequence

$$0 \to \mathrm{Tor}^1_R(R/\mathfrak{I}^m, \mathscr{N}_{\mathrm{tors}}) \to \mathscr{M}'/\mathfrak{I}^m\mathscr{M}' \to \mathscr{M}''/\mathfrak{I}^m\mathscr{M}'' \tag{0.0.9}$$

The advantage of using $\mathscr{N}_{\mathrm{tors}}$ to compute higher Tor's is that we can actually produce a single power of π that annihilates it! In order to establish this, we note that by Step 3, the map $\delta : M^{(1)} \to M^{(2)}$ has *closed* image (namely, the kernel of a suitable continuous map $M^{(2)} \to M^{(3)}$).

Since δ is a continuous injection between k-Banach spaces, the closedness of its image implies (by the Banach open mapping theorem) that δ must be a closed embedding. But the source and target of δ are countable type k-Banach spaces (i.e., have dense subspaces of countable dimension), so it follows from [BGR, 2.7.1/4] that δ admits a *continuous* k-linear splitting. Multiplying this by a suitable power of π, we conclude that there is an R-module map $\mathscr{M}'' \to \mathscr{M}'$ such that the composite

$$\mathscr{M}' \xrightarrow{\Delta} \mathscr{M}'' \to \mathscr{M}'$$

is multiplication by some π^r. Since these R-modules are all flat, it follows that the torsion submodule of the cokernel $\mathscr{N}$ of Δ is annihilated by $\mathfrak{I}^r$. Thus, the first term in (0.0.9) is annhilated by $\mathfrak{I}^r$ for all m. If we apply the faithfully flat base change $\mathscr{A}/\mathfrak{I}^m\mathscr{A} \to \mathscr{A}'/\mathfrak{I}^m\mathscr{A}'$ to (0.0.9) and pass to the inverse limit on m, we arrive at an exact sequence

$$0 \to \mathscr{K} \to \mathscr{A}'\widehat{\otimes}_{\mathscr{A}}\mathscr{M}' \to \mathscr{A}'\widehat{\otimes}_{\mathscr{A}}\mathscr{M}''$$

in which $\mathscr{K}$ is killed by $\mathfrak{I}^r$ (as it is an inverse limit of R-modules killed by $\mathfrak{I}^r$). Hence, upon applying the localization functor $k \otimes_R (\cdot)$, we get an *injection* that is exactly the map

$$1\widehat{\otimes}\delta : A'\widehat{\otimes}_A M' \to A'\widehat{\otimes}_A M''$$

The map $1\widehat{\otimes}\delta$ is exactly the analogue of δ after we apply the base change $\mathrm{Sp}(A') \to \mathrm{Sp}(A)$ throughout our original descent data situation. But as we noted in the proof of Step 3, since after such a base change the structure map through which we want to do descent acquires a section, descent for coherent sheaves in such a situation is *always* effective (with the descended module exactly the kernel of the first map in the Čech-descent complex, by Step 1). We conclude from the vanishing of the kernel of $1\widehat{\otimes}\delta$ that the $A'\widehat{\otimes}_A A'$-module $A'\widehat{\otimes}_A M'$ is the base change (via second projection) of the A'-module 0. That is, $A'\widehat{\otimes}_A M' = 0$. But we have an isomorphism

$$A'\widehat{\otimes}_A M' \simeq (A'\widehat{\otimes}_A A')\widehat{\otimes}_{A'} M' \simeq (A'\widehat{\otimes}_A A') \otimes_{A'} M'$$

since M' is a finite A'-module. Since $A'\widehat{\otimes}_A A'$ is a faithfully flat A'-algebra, it follows that $M' = 0$, as desired. In other words, we have proven that when $M' \neq 0$ then $M \neq 0$.

Step 5: *Reduction to the case where M spans M' as an A'-module.*

We may suppose $M' \neq 0$, so by Step 4 we have $M \neq 0$. Let $M'_1 \neq 0$ be the A'-span of M inside of the finite A'-module M'. Due to the very definition of M, it follows that M'_1 inherits descent data from M' in an evident sense (since M has the same A-linear map to M'' under either pullback from M', due to invariance under the descent data action, and M''_1 is simply the $(A'\widehat{\otimes}_A A')$-span of M inside of M''). Defining $M'_2 = M'/M'_1$, Step 2 reduces us to the analysis of M'_1 and M'_2. Note that M_1 is an A-submodule of M'_1 which is nothing other than M, so M'_1 is the A'-span of M_1. If $M'_2 = 0$ then $M' = M'_1$ is the A'-span of M, so we'd be done. If $M'_2 \neq 0$ then $M_2 \neq 0$ by Step 4, so we can repeat the process to get a *non-zero* A'-submodule M'_3 of M'_2 (that corresponds to an A'-submodule of M' *strictly containing* M'_1) and the game continues.

As we keep encountering non-zero A'-modules, we build up a strictly increasing chain of A'-submodules of M' beginning with M'_1. Since M' is a finite A'-module and A' is noetherian, this process must eventually stop, at which point we have achieved a descent-data-stable filtration of M' by A'-modules whose successive quotients satisfy the condition in our reduction step. Thanks to Step 2, this completes Step 5.

Step 6: *Reduction to the case where a single element of M spans M' as an A'-module.*

(Beware that we *still* have not yet shown that M is even a *finite* A-module.)

Since M' is spanned by M as an A'-module by Step 5, we can let $n \geq 1$ denote the minimal positive number of elements of M that suffice to span M' over A'. Let $m_1, \ldots m_n \in M$ be A'-module generators of M'. Defining $M_1' = A'm_1$ and $M_2' = M'/M_1'$, it is obvious that M_1', M_2' are compatible with the descent data and that the images of the $(n-1)$ elements $m_2, \ldots, m_n$ in M_2 span M_2' as an A'-module. By induction on n and Step 2, we're done with Step 6.

Now that a single element of M spans M' as an A'-module, we have $M' \simeq A'/I'$ as an abstract A'-module, where $I' = \mathrm{ann}_{A'}(M')$.

Step 7: *The ideal I' of A' descends uniquely to an ideal I of A.*

The uniqueness is clear, so we are only concerned with existence. Since the formation of the annihilator ideal of a coherent sheaf commutes with flat base change, the descent data on M' induces compatible descent data on I'. More specifically, we have the literal equality

$$I' \widehat{\otimes}_A A' = A' \widehat{\otimes}_A I' \tag{0.0.10}$$

inside of $A' \widehat{\otimes}_A A'$. Let $\mathscr{I}' = \ker(\mathscr{A}' \to A'/I')$, so $\mathscr{A}'/\mathscr{I}'$ is R-flat. By [BL1, 1.2(c)], $\mathscr{I}'$ is automatically a coherent ideal in the coherent ring $\mathscr{A}'$, so $\mathscr{A}'/\mathscr{I}'$ is an R-flat formal model of A'/I'. In particular, the injection $\mathscr{I}' \to \mathscr{A}'$ remains injective modulo all powers of π.

We conclude that the two closed immersion

$$\mathrm{Spf}((\mathscr{A}'/\mathscr{I}') \widehat{\otimes}_{\mathscr{A}} \mathscr{A}'), \mathrm{Spf}(\mathscr{A}' \widehat{\otimes}_{\mathscr{A}} (\mathscr{A}'/\mathscr{I}')) \hookrightarrow \mathrm{Spf}(\mathscr{A}' \widehat{\otimes}_{\mathscr{A}} \mathscr{A}') \tag{0.0.11}$$

are $\mathrm{Spf}(R)$-flat closed formal subschemes that are formal models for the closed rigid analytic subspaces

$$\mathrm{Sp}((A'/I') \widehat{\otimes}_A A'), \mathrm{Sp}(A' \widehat{\otimes}_A (A'/I')) \hookrightarrow \mathrm{Sp}(A' \widehat{\otimes}_A A')$$

But (0.0.10) implies that these latter closed subspaces literally coincide, whence by R-flatness we conclude that the formal closed subschemes (0.0.11) literally coincide. In other words, the defining ideals

$$\mathscr{I}' \widehat{\otimes}_{\mathscr{A}} \mathscr{A}', \mathscr{A}' \widehat{\otimes}_{\mathscr{A}} \mathscr{I}' \hookrightarrow \mathscr{A}' \widehat{\otimes}_{\mathscr{A}} \mathscr{A}'$$

are exactly the same. Note that the injectivity of these two maps into $\mathscr{A}' \widehat{\otimes}_{\mathscr{A}} \mathscr{A}'$ follows from the fact that $\mathscr{A}/\mathfrak{I}^m \mathscr{A} \to \mathscr{A}'/\mathfrak{I}^m \mathscr{A}'$ is flat for all m and $\mathscr{I}' \to \mathscr{A}$ has R-flat cokernel (and hence remains injective modulo every power of $\mathfrak{I}$). Passing to quotients by $\mathfrak{I}^m$, it follows that we have an equality

$$(\mathscr{I}'/\mathfrak{I}^m \mathscr{I}') \otimes_{\mathscr{A}/\mathfrak{I}^m \mathscr{A}} (\mathscr{A}'/\mathfrak{I}^m \mathscr{A}') = (\mathscr{A}'/\mathfrak{I}^m \mathscr{A}') \otimes_{\mathscr{A}/\mathfrak{I}^m \mathscr{A}} (\mathscr{I}'/\mathfrak{I}^m \mathscr{I}')$$

inside of $(\mathscr{A}'/\mathfrak{I}^m \mathscr{A}') \otimes_{\mathscr{A}/\mathfrak{I}^m \mathscr{A}} (\mathscr{A}'/\mathfrak{I}^m \mathscr{A}')$ for all m. But by usual faithfully flat descent for $\mathscr{A}/\mathfrak{I}^m \mathscr{A} \to \mathscr{A}'/\mathfrak{I}^m \mathscr{A}'$, it follows that there exists a unique *finitely generated* ideal $\mathscr{I}_m \subseteq \mathscr{A}/\mathfrak{I}^m \mathscr{A}$ that induces the finitely generated ideal $\mathscr{I}'/\mathfrak{I}^m \mathscr{I}' \subseteq \mathscr{A}'/\mathfrak{I}^m \mathscr{A}'$ under base change. Also, the cokernel of $\mathscr{A}/\mathfrak{I}^m \mathscr{A}$ by $\mathscr{I}_m$ is $R/\mathfrak{I}^m$-flat (as this can be checked after faithfully flat base change to $\mathscr{A}'/\mathfrak{I}^m \mathscr{A}'$), so $\mathscr{I}_m/\mathfrak{I}^{m-1} \mathscr{I}_m \to \mathscr{A}/\mathfrak{I}^{m-1} \mathscr{A}$ is injective. The uniqueness of descent then implies that this must be the ideal $\mathscr{I}_{m-1}$.

To summarize, we have finitely generated ideals $\mathscr{I}_m \subseteq \mathscr{A}/\mathfrak{I}^m \mathscr{A}$ compatible with change in m, so passage to the limit (and a reworking of [EGA, 0_{I}, §7] without noetherian hypotheses, as in [Gu, Appendix]) yields a *coherent* ideal

$$\mathscr{I} \stackrel{\mathrm{def}}{=} \varprojlim \mathscr{I}_m \subseteq \mathscr{A}$$

whose mod $\mathfrak{I}^m$ reduction is $\mathscr{I}_m$ for all $m \geq 1$. Since the $\mathfrak{I}$-adic topology on $\mathscr{A}$ induces that on $\mathscr{I}$ by [BL1, 1.2(a)], the topologically faithfully flat base change $\mathscr{A} \to \mathscr{A}'$ consequently induces an identification

$$\mathscr{A}' \widehat{\otimes}_{\mathscr{A}} \mathscr{I} = \mathscr{I}'$$

inside of $\mathscr{A}'$. Now passing to the affinoid world by localization, the ideal $I = k \otimes_R \mathscr{I} \subseteq A$ induces I' under base change to A'. This finishes Step 7.

Step 8: *Completion of the proof.*

It is clear that we can now replace A with A/I so as to reduce to the case $I' = 0$. That is, we have $M' \simeq A'$ as A'-modules where $1 \in A'$ corresponds to an element of M, which is to say it is an element of M' that is invariant under the descent data. It is then automatic that carrying the descent data across the isomorphism $M' \simeq A'$ must induce on $A'\widehat{\otimes}_A A' \simeq M''$ exactly the *usual* trivially effective descent data structure for the finite A-module A relative to the faithfully flat covering $\mathrm{Sp}(A') \to \mathrm{Sp}(A)$. Translating back to M', the original descent is effective. ■

References

[BGR] S. Bosch, U. Günzter, R. Remmert, *Non-Archimedean analysis*, Springer-Verlag, 1984.

[BL1] S. Bosch, W. Lütkebohmert, *Formal and rigid geometry* I, Math. Annalen, **295** (1993), pp. 291–317.

[BL2] S. Bosch, W. Lütkebohmert, *Formal and rigid geometry* II, Math. Annalen, **296** (1993), pp. 403–429.

[BLR1] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models*, Springer-Verlag, 1984.

[BG] S. Bosch, U. Görtz, *Coherent modules and their descent on relative rigid spaces*, J. Reine Angew. Math. **495** (1998), pp. 119–134.

[C] B. Conrad, *Ampleness and descent on rigid-analytic spaces*, in preparation.

[EGA] J. Dieudonné, A. Grothendieck, *Éléments de géométrie algébrique*, Publ. Math. IHES, **4, 8, 11, 17, 20, 24, 28, 32**, (1960-7).

[Gu] W. Gubler, *Local heights on subvarieties over non-Archimedean fields*, J. Reine Angew. Math. **498** (1998), pp. 61–113.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

E-mail address: bdconrad@math.lsa.umich.edu