THE KEEL–MORI THEOREM VIA STACKS

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1. Introduction

Let $\mathscr{X}$ be an Artin stack (always assumed to have quasi-compact and separated diagonal over $\operatorname{Spec} \mathbf{Z}$; cf. [2, §1.3]). A *coarse moduli space* for $\mathscr{X}$ is a map $\pi : \mathscr{X} \to X$ to an algebraic space such that (i) π is initial among maps from $\mathscr{X}$ to algebraic spaces (note that the category of maps from $\mathscr{X}$ to an algebraic space is discrete), and (ii) for every algebraically closed field k the map $[\mathscr{X}(k)] \to X(k)$ is bijective (where $[\mathscr{X}(k)]$ denotes the set of isomorphism classes of objects in the small category $\mathscr{X}(k)$). If $\mathscr{X}$ is equipped with a map to a scheme S then X has a unique compatible map to S, and so it is equivalent to require the universal property for algebraic spaces over S.

In [5], Keel and Mori used a close study of groupoids to prove that if $\mathscr{X}$ is locally of finite type over a locally noetherian scheme S and its inertia stack $I_S(\mathscr{X}) = \mathscr{X} \times_{\mathscr{X} \times_S \mathscr{X}} \mathscr{X}$ is finite over $\mathscr{X}$ then there exists a coarse moduli space $\pi : \mathscr{X} \to X$ with X locally of finite type over S (and separated over S when $\mathscr{X}$ is separated over S). They also proved that π is a proper universal homeomorphism, that for any flat (locally finite type) base change $X' \to X$ in the category of algebraic spaces the map $\pi \times_X X'$ is a coarse moduli space, and that $\mathscr{O}_X \to \pi_*(\mathscr{O}_{\mathscr{X}})$ is an isomorphism. The finiteness hypothesis on $I_S(\mathscr{X})$ is weaker than finiteness of $\Delta_{\mathscr{X}/S}$ and is stronger than quasi-finiteness of $\Delta_{\mathscr{X}/S}$.

The purpose of this note is to explain how to systematically use stacks instead of groupoids to give a more transparent version of the Keel–Mori method and to eliminate noetherian assumptions. Since the formation of coarse spaces does not generally commute with (non-flat) base change, it does not seem possible to immediately reduce existence problems to the locally noetherian case. If we wish to avoid separatedness hypotheses on $\mathscr{X}$ over S but want $\pi : \mathscr{X} \to X$ to be a universal homeomorphism then it is unreasonable to consider $\mathscr{X}$ for which $\Delta_{\mathscr{X}/S}$ is not quasi-finite (e.g., the non-separated stack $\mathscr{Q} = \mathbf{A}^1/\mathbf{G}_m$ that classifies line bundles equipped with a section has k-points specializing to other k-points and so it cannot admit a coarse moduli space $\mathscr{Q} \to Q$ that is a universal homeomorphism).

Theorem 1.1 (Keel–Mori)**.** *Let S be a scheme and let $\mathscr{X}$ be an Artin stack that is locally of finite presentation over S and has finite inertia stack $I_S(\mathscr{X})$. There exists a coarse moduli space $\pi : \mathscr{X} \to X$, and it satisfies the following additional properties:*

1. *The structure map $X \to S$ is separated if $\mathscr{X} \to S$ is separated, and it is locally of finite type if S is locally noetherian.*
2. *The map π is proper and quasi-finite.*

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Moreover, if $X' \to X$ is a flat map of algebraic spaces then $\pi' : \mathscr{X}' = \mathscr{X} \times_X X' \to X'$ is a coarse moduli space.

Note that the natural map $\mathscr{O}_X \to \pi_*(\mathscr{O}_{\mathscr{X}})$ on the étale site of X is an isomorphism. Indeed, for any étale map $U \to X$ with U an algebraic space we have that $\mathscr{X} \times_X U \to U$ is a coarse moduli space, and applying the universal mapping property with respect to morphisms to $\mathbf{A}^1_{\mathbf{Z}}$ then gives $\mathscr{O}_X(U) \simeq (\pi_*\mathscr{O}_{\mathscr{X}})(U)$. In case S is locally noetherian we get a converse result: if $p : \mathscr{X} \to Y$ is a proper quasi-finite map to an algebraic space and $\mathscr{O}_Y \simeq p_*(\mathscr{O}_{\mathscr{X}})$ then the induced map $\overline{p} : X \to Y$ is an isomorphism. (Indeed, $\overline{p}$ is certainly locally of finite type since $X \to S$ is, and so $\overline{p}$ is proper and quasi-finite because π is a proper homeomorphism and p is proper. Thus, $\overline{p}$ is finite by [6, A.2] and thus is an isomorphism since $\mathscr{O}_Y \to \overline{p}_*(\mathscr{O}_X)$ is an isomorphism; cf. [8, 2.6(*iii*)].)

In §2 we review how to reduce the proof of Theorem 1.1 to the case of a special class of Artin stacks. The case of stacks admitting a finite locally free scheme cover is addressed in §3, where the aim is to check that certain quotient schemes also satisfy a universal property with respect to arbitrary algebraic spaces. In §4 we give our stack-theoretic variant on the Keel–Mori method to prove Theorem 1.1 in the case of a locally noetherian base scheme. Limit arguments are used in §5 to handle the case of a general base scheme, where the key point is to control the étale property through a limit process. At the end (Corollary 5.2), we show that if $\Delta_{\mathscr{X}/S}$ is quasi-finite then finiteness of the inertia stack is equivalent to existence of a coarse moduli space whose classifying map is separated. In view of Lemma 4.1, an existence theorem for coarse moduli spaces under weaker diagonal hypotheses than in Theorem 1.1 will therefore require a different approach than that of Keel and Mori.

2. Preliminary reduction steps

Let $\mathscr{X}$ be an Artin stack, and $\{\mathscr{X}_i\}$ an open covering of $\mathscr{X}$ by open substacks for which there exist coarse moduli spaces $\pi_i : \mathscr{X}_i \to X_i$ whose formation is compatible with flat base change (e.g., Zariski-localization), with each π_i a homeomorphism. Forming images and preimages under π_i defines an inclusion-preserving bijection between the set of open substacks of $\mathscr{X}_i$ and the set of open subspaces of X_i, so we can use $\mathscr{X}_i \cap \mathscr{X}_j$ to define open subspaces of X_i and X_j that each serve as a coarse moduli space for $\mathscr{X}_i \cap \mathscr{X}_j$. This provides gluing data on the X_i's to construct a map $\pi : \mathscr{X} \to X$ to an algebraic space such that π recovers the maps π_i over the opens X_i that cover X. Such a π is a coarse moduli space for $\mathscr{X}$, and if $\mathscr{X}$ is a stack over a scheme S then X is S-separated when $\mathscr{X}$ and all X_i are S-separated. The following lemma (due to Grothendieck [3, Exp. V, 7.2], and rediscovered by Keel and Mori [5, Lemma 3.3]) therefore reduces Theorem 1.1 to the case when $\mathscr{X}$ has a quasi-finite, flat, and finitely presented scheme covering.

Lemma 2.1. *Let $\mathscr{X}$ be an Artin stack locally of finite presentation over a scheme S, and assume $\Delta_{\mathscr{X}/S}$ is quasi-finite. There is a covering of $\mathscr{X}$ by open substacks admitting a quasi-finite, flat, and finitely presented scheme covering.*

Proof. Working Zariski-locally on S and $\mathscr{X}$, we can assume S is affine and $\mathscr{X}$ is quasi-compact (hence finitely presented over S), so by direct-limit methods [8, 2.2] we reduce to the noetherian case. The second paragraph of the proof of [8, 2.11] gives a stack-theoretic treatment of the noetherian case (using [3, Exp. V, 7.2]). ■

By Lemma 2.1, for the proof of Theorem 1.1 we may assume $S = \operatorname{Spec} A$ is affine and that there exists a quasi-finite, flat, and finitely presented covering $U \to \mathscr{X}$ by a quasi-projective

S-scheme U. Letting R denote the algebraic space $U \times_{\mathscr{X}} U$, the projections $R \rightrightarrows U$ over S are quasi-finite and flat. The maps $U \to \mathscr{X}$ and $R \rightrightarrows U$ are separated because U is S-separated and $\Delta_{\mathscr{X}/S}$ is separated. Thus, R is a scheme since U is a scheme [6, A.2].

Lemma 2.2. *Let $\mathscr{X}$ be an Artin stack locally of finite presentation over a scheme S such that there is a finitely presented, quasi-finite, and flat cover $V \to \mathscr{X}$ by a separated S-scheme (so $V \to \mathscr{X}$ is separated, and hence schematic).*

There is a representable étale cover $\mathscr{W} \to \mathscr{X}$ by an S-separated Artin stack and an open and closed immersion $Z \hookrightarrow V \times_{\mathscr{X}} \mathscr{W}$ such that $Z \to \mathscr{W}$ is a finite locally free cover. It can also be arranged that for every $v \in V$ there are $k(v)$-points $w \in \mathscr{W}$ and $u \in Z_w$ such that some open $\mathscr{U}' \subseteq \mathscr{W}$ around w has preimage U in Z that is quasi-projective over an open affine in S.

Proof. Let $\mathscr{H}$ be the Hilbert stack $\mathrm{Hilb}_{V/\mathscr{X}}$ whose fiber category over $\mathscr{X}(T)$ for an S-scheme T is the set of closed subschemes of $V \times_{\mathscr{X}} T$ that are finite locally free covers of T; this is an Artin stack locally of finite presentation over $\mathscr{X}$ by descent theory for closed subschemes and the theory of Hilbert functors for quasi-finite, finitely presented, and separated scheme maps. The map $\mathscr{H} \to \mathscr{X}$ is schematic and separated (valuative criterion).

The open étale locus $\mathscr{W}$ in $\mathscr{H}$ for the map $\mathscr{H} \to \mathscr{X}$ is compatible with flat base change on $\mathscr{X}$, and it surjects onto $\mathscr{X}$. Indeed, for $v \in V$ over $x \in \mathscr{X}$, by an infinitesimal deformation argument the map $\mathscr{H} \times_{\mathscr{X}} V \to V$ is etale at the $k(v)$-rational point u in the v-fiber classifying the non-empty finite $k(v)$-scheme $V \times_{\mathscr{X}} \operatorname{Spec} k(v)$ (and likewise for its non-empty open and closed subschemes). Thus, the image $w \in \mathscr{H}_x(k(v))$ of u lies in $\mathscr{W}_x$.

Let $i : Z \to V \times_{\mathscr{X}} \mathscr{W}$ be the universal closed immersion with Z a finite locally free cover of $\mathscr{W}$. Note that $V \times_{\mathscr{X}} \mathscr{W}$ is a scheme because it is separated and locally quasi-finite over V (as $\mathscr{W} \to \mathscr{X}$ is locally quasi-finite and $\mathscr{H} \to \mathscr{X}$ is separated), so Z is an S-separated scheme. Since Z is a finite cover of $\mathscr{W}$, it follows that $\mathscr{W}$ is S-separated. For each $v \in V$, i is an isomorphism on fibers over the $k(v)$-point w of $\mathscr{W}$ as above, and so by the fibral flatness criterion Z_w is in the étale locus of i. This fiber contains the $k(v)$-point u classifying $V \times_{\mathscr{X}} \operatorname{Spec} k(v)$. The non-étale locus of i has closed image in $\mathscr{W}$ that misses w, so we may shrink $\mathscr{W}$ around all such points w (varying v) to make i étale without losing surjectivity of $\mathscr{W} \to \mathscr{X}$. The closed immersion i is now étale and so is an open immersion.

The composite $Z \to V \times_{\mathscr{X}} \mathscr{W} \to V$ is a residually trivial étale neighborhood of each v via the $k(v)$-point $u \in Z_w$. Since the finitely presented map $Z \to V$ is quasi-finite and separated, by Zariski's Main Theorem [4, IV_3, 8.12.6] it is quasi-affine. Thus, for $v \in V$ we can find an open affine $Z' \subseteq Z$ containing Z_w and lying over an open affine in S. The image of $Z - Z'$ in $\mathscr{W}$ is a closed set whose open complement $\mathscr{U}' \subseteq \mathscr{W}$ around w has preimage $U \subseteq Z$ contained in Z', so $U \to S$ is quasi-projective over an open affine in S. ∎

Remark 2.3. A key observation of Keel and Mori is that for $\mathscr{X}$ as in Lemma 2.1, $I_S(\mathscr{X})$ is $\mathscr{X}$-finite if and only if for every point $x \in \mathscr{X}$ there is a representable, quasi-compact, and étale neighborhood $\mathscr{U}' \to \mathscr{X}$ by an Artin stack $\mathscr{U}'$ such that (i) $\mathscr{U}'$ is S-separated and admits a finite locally free scheme cover, and (ii) for all algebraically closed fields k and objects $u' \in \mathscr{U}'(k)$ over $x \in \mathscr{X}(k)$ the injective group map $\mathrm{Aut}_{\mathscr{U}'(k)}(u') \to \mathrm{Aut}_{\mathscr{X}(k)}(x)$ is surjective. This assertion is designed to suppress any explicit mention of how $\mathscr{U}'$ mat have been constructed (e.g., in a Hilbert stack over $\mathscr{X}$), and since it is Zariski-local on $\mathscr{X}$ we can assume for purposes of its verification (by Lemma 2.2) that there is a representable, quasi-compact, and étale cover $\mathscr{U}' \to \mathscr{X}$ such that $\mathscr{U}'$ is S-separated and admits a finite locally free covering by a scheme Z. (If we ignore the condition on automorphism groups, then we may Zariski-locally always build $\mathscr{U}'$ as an S-separated open substack of a Hilbert stack $\mathscr{H} = \mathrm{Hilb}_{V/\mathscr{X}} \to \mathscr{X}$. For a geometric point $x \in \mathscr{X}(k)$ dominated by $u' \in \mathscr{H}(k)$ corresponding to a non-empty closed subscheme $Z_{u'} \hookrightarrow V_x = V \times_{\mathscr{X},x} \mathrm{Spec}\, k$, $\mathrm{Aut}_{\mathscr{X}}(x)$ acts on V_x and the subgroup preserving $Z_{u'}$ is $\mathrm{Aut}_{\mathscr{H}(k)}(u')$; this is $\mathrm{Aut}_{\mathscr{X}(k)}(x)$ if $Z_{u'} = V_x$. Since $V \to \mathscr{X}$ may have varying fiber-degrees, the desired equality of automorphism groups is generally not an open condition on $\mathscr{H}$. If $I_S(\mathscr{X})$ is finite then the assertion we are aiming to show ensures openness in the $\mathscr{X}$-étale locus in $\mathscr{H}$.)

Let $I_{Z,\mathscr{U}'}$ and $I_{Z,\mathscr{X}}$ be the pullbacks of $\Delta_{Z/S}$ along the projections from $Z \times_{\mathscr{U}'} Z$ and $Z \times_{\mathscr{X}} Z$, so $I_{Z,\mathscr{U}'}$ is open and closed in $I_{Z,\mathscr{X}}$ (since $\Delta_{\mathscr{U}'/\mathscr{X}}$ is an open and closed immersion). Since Z is an fpqc cover of $\mathscr{X}$, $I_S(\mathscr{X})$ is finite if and only if the structure map $I_{Z,\mathscr{X}} \to Z$ is finite. For the same reason, since $\mathscr{U}'$ is S-separated (so $I_S(\mathscr{U}')$ is $\mathscr{U}'$-finite) we have that $I_{Z,\mathscr{U}'}$ is Z-finite. If $z \in Z(k)$ lies over $u' \in \mathscr{U}'(k)$ and $x \in \mathscr{X}(k)$ then the open and closed immersion $I_{Z,\mathscr{U}'} \hookrightarrow I_{Z,\mathscr{X}}$ is an equality on z-fibers if and only if $\mathrm{Aut}_{\mathscr{U}'(k)}(u') = \mathrm{Aut}_{\mathscr{X}(k)}(x)$, and this latter condition does not mention the particular z over u'. It follows that if this equality of automorphism groups holds for all $u' \in \mathscr{U}'(k)$ (over any $x \in \mathscr{X}(k)$) then $I_{Z,\mathscr{X}} = I_{Z,\mathscr{U}'}$ and so $I_{Z,\mathscr{X}}$ is Z-finite, whence $I_S(\mathscr{X})$ is finite.

Conversely, if $I_S(\mathscr{X})$ is finite then the preceding argument shows that the locus in Z over which $I_{Z,\mathscr{U}'}$ and $I_{Z,\mathscr{X}}$ have the same fiber is the preimage of a Zariski-open locus in $\mathscr{U}'$ (which in turn has Zariski-open image in $\mathscr{X}$). Hence, it remains to show that for each $x \in \mathscr{X}(k)$ we can make the construction in Lemma 2.2 so that $\mathrm{Aut}_{\mathscr{U}'(k)}(u') = \mathrm{Aut}_{\mathscr{X}(k)}(x)$ for some $u' \in \mathscr{U}'(k)$ over x. In terms of the output of the construction of $\mathscr{U}'$ from a Hilbert stack $\mathrm{Hilb}_{V/\mathscr{X}}$, we take u' classifying the full fiber V_x.

3. Finite locally free scheme covers

In [3, Exp. V, 4.1], Grothendieck implicitly studied coarse moduli *schemes* for Artin stacks with a finite locally free covering by an affine scheme. Related considerations in the finite type case over a noetherian base scheme are worked out in [5, Prop. 5.1, Lemma 6.5], but to handle the non-noetherian case we need to review Grothendieck's results and explain why his quotient construction in the category of schemes also serves as a quotient in the category of algebraic spaces.

Let $\mathscr{X}$ be an Artin stack and assume that there is a finite locally free covering $U \to \mathscr{X}$ by a scheme. Let $R = U \times_{\mathscr{X}} U$ and let $p_1, p_2 : R \rightrightarrows U$ be the projections. Assume that each orbit $p_1(p_2^{-1}(u))$ is contained in an affine open in U; this holds if U is quasi-affine (in the sense of [4, II, §5.1]) or if $\mathscr{X}$ has a map to a scheme S such that every finite subset in each fiber of $U \to S$ is contained in an open affine in U (e.g., $U \to S$ quasi-projective locally on S). Let $p : U \to X$ be the topological quotient space modulo the set-theoretic equivalence relation imposed by R (i.e., $u \sim u'$ if $p_1^{-1}(u)$ meets $p_2^{-1}(u')$), and define $\mathscr{O}_X$ to be the equalizer of the maps $p_1^*, p_2^* : p_*(\mathscr{O}_U) \rightrightarrows q_*(\mathscr{O}_R)$ with $q = p \circ p_1 = p \circ p_2$. By (the proof of) [3, Exp. V, 4.1(i)], X is a scheme and $U \to X$ is an integral affine surjection such that over an open affine $X_0 \subseteq X$ with preimage $U_0 = \mathrm{Spec}\, B$ in U we have $X_0 = \mathrm{Spec}(B^R)$ with $B^R = \{b \in B \,|\, p_1^*(b) = p_2^*(b)\} \subseteq B$. In particular, $B^R \to B$ is integral.

Moreover, $U \to X$ is a categorical quotient for $R \rightrightarrows U$ in the category of all ringed spaces (as well as locally ringed spaces) and for any algebraically closed field k we have that $X(k)$ is the quotient of $U(k)$ modulo the equivalence relation given by the image of $R(k)$ in $U(k) \times U(k)$. The scheme X is quasi-separated (and so it "is" an algebraic space) since $\Delta_{\mathscr{X}/\mathrm{Spec}\,\mathbf{Z}}$ is quasi-compact. The map $\pi : \mathscr{X} \to X$ satisfies the universal mapping property of a coarse moduli space restricted to the category of schemes.

Theorem 3.1. *With notation and hypotheses as above, suppose that $\mathscr{X}$ has a structure of stack over a scheme S. The natural S-map $\pi : \mathscr{X} \to X$ to the scheme quotient is a coarse moduli space and it is separated. Moreover:*

1. *The map π is a universal homeomorphism, and $\pi' : \mathscr{X} \times_X X' \to X'$ is a coarse moduli space for any flat map of algebraic spaces $X' \to X$.*
2. *If $\mathscr{X}$ is locally of finite type over S then π is proper, and if S is locally noetherian then X is locally of finite type over S.*
3. *If $\mathscr{X}$ is separated over S then X is S-separated.*

Proof. The scheme X is S-separated if and only if $\Delta_{X/S}(X)$ is closed in $X \times_S X$, so by surjectivity of π and factoring $\mathscr{X} \times_S \mathscr{X} \to X \times_S X$ as a composite of base changes of π we see that (3) is a consequence of (1). For the rest we can work locally on X and S to arrange that X and S are affine. In particular, U and R are affine, so $U = \operatorname{Spec} B$ and $X = \operatorname{Spec}(B^R)$.

Let us check that π is separated and a universal homeomorphism. The separatedness of π means that the separated and finite type $\Delta_\pi : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is universally closed, and this latter property holds because $\Delta_{U/S}$ is universally closed and $U \to \mathscr{X}$ is a finite cover. Since $U \to X$ is an affine integral surjection and $U \to \mathscr{X}$ surjective, π is a universally closed surjection. Thus, π is a universal homeomorphism if $[\mathscr{X}(k)] \to X(k)$ is bijective for all algebraically closed fields k. Since $[\mathscr{X}(k)]$ is the quotient of $U(k)$ by the equivalence relation induced by the image of $R(k)$ in $U(k) \times U(k)$, bijectivity is immediate.

If $\mathscr{X}$ is finite type over S then U and R are as well, so the integral map $B^R \to B$ is finite type and therefore finite; i.e., π is proper. In such cases with S noetherian, it follows from [1, Prop. 7.8] that $X = \operatorname{Spec}(B^R)$ is of finite type over S. This settles (2).

It remains to prove that π satisfies the universal property to be a coarse moduli space, and that it continues to do so after arbitrary flat base change $X' \to X$ with any algebraic space X'. It suffices to work with affine U, R, and X as above. We fix a flat map $X' \to X$ with an algebraic space X', and the mapping property problem for $\pi \times_X X'$ is to show that if Y is any algebraic space then the co-commutative diagram

$$Y(X') \to Y(X' \times_X U) \rightrightarrows Y(X' \times_X R) \tag{3.1}$$

is exact. Using an étale chart in schemes $X'_1 \rightrightarrows X'_0$ for the algebraic space X', this exactness problem is easily reduced to the case when X' is a scheme. Working Zariski-locally on X' allows us to assume $X' = \operatorname{Spec} C$ with $B^R \to C$ flat. Thus, the exactness of the functor $C \otimes_{B^R} (\cdot)$ gives $C = \{\beta \in C \otimes_{B^R} B \,|\, p_1^*(\beta) = p_2^*(\beta) \text{ in } C \otimes_{B^R} B\}$, so by (the proof of) [3, Exp. V, 4.1(i)] the diagram $X' \times_X R \rightrightarrows X' \times_X U \to X'$ is a quotient diagram in the category of locally ringed spaces. In particular, (3.1) is exact when Y is a scheme.

Since X', $X' \times_X U$, and $X' \times_X R$ are quasi-compact, for the proof of exactness it suffices to treat quasi-compact opens in Y. Thus, we can assume that there is an étale equivalence relation $Y_1 \rightrightarrows Y_0$ with affine Y_0 such that $Y = Y_0/Y_1$; by Zariski's Main Theorem [4, IV_3, 8.12.6], Y_1 is quasi-affine. Let us first prove that $Y(X') \to Y(X' \times_X U)$ is injective (so (3.1) is injective on the left in general). Suppose $\xi, \eta \in Y(X')$ have the same pullback in $Y(X' \times_X U)$. Let $X'' \to X'$ be a quasi-compact étale scheme covering such that $\xi|_{X''}, \eta|_{X''}$ lift to $\xi_0, \eta_0 \in Y_0(X'')$. We want to prove that $(\xi_0, \eta_0) \in Y_0(X'') \times Y_0(X'')$ lies in the subset $Y_1(X'')$. By the settled exactness of (3.1) with Y replaced by the schemes Y_0 and Y_1 (and X' replaced with X''), a simple diagram chase reduces the problem to proving that the ordered pair

$$(\xi_0|_{X'' \times_X U}, \eta_0|_{X'' \times_X U}) \in Y_0(X'' \times_X U) \times Y_0(X'' \times_X U)$$

lies in $Y_1(X'' \times_X U)$. Since the étale presheaf of sets $T \mapsto Y_0(T)/Y_1(T)$ on the category of S-schemes is separated, it is equivalent to show that the elements $\xi|_{X''\times_X U}, \eta|_{X''\times_X U} \in Y(X'' \times_X U)$ coincide. But this equality follows immediately from the equality of $\xi|_{X'\times_X U}$ and $\eta|_{X'\times_X U}$ in $Y(X' \times_X U)$ (recall the initial hypothesis on ξ and η).

With the injectivity at the left of (3.1) now settled in general, we have to prove exactness in the middle. It is again enough to work in the above affine case ($X' = \operatorname{Spec} C$, $U = \operatorname{Spec} B$, $X = \operatorname{Spec}(B^R)$, quasi-compact Y, affine Y_0, quasi-affine Y_1). Choose $y \in Y(X' \times_X U)$ satisfying $p_1^*(y) = p_2^*(y)$ in $Y(X' \times_X R)$. We want $y \in Y(X')$. Since algebraic spaces satisfy the sheaf axiom for the fpqc topology (see [6, A.4], whose proof can be modified to avoid fpqc-sheafification), the settled injectivity in general reduces us to establishing the desired result after pullback to Q and $Q \times_{X'} Q$ for some fpqc morphism of schemes $Q \to X'$.

Using y and the map $p_1^*(y) = p_2^*(y)$, we get a finite locally free groupoid in *quasi-affine* schemes (over the affine scheme Y_0) $(X' \times_X R) \times_Y Y_0 \rightrightarrows (X' \times_X U) \times_Y Y_0$. Let Q be the associated quasi-compact scheme quotient in the category of locally ringed spaces, so Q has a unique compatible structure of Y_0-scheme and by functoriality of such quotients in the category of schemes we get a commutative diagram

$$\begin{array}{ccc} (X' \times_X U) \times_Y Y_0 & \longrightarrow & X' \times_X U \\ \downarrow & & \downarrow \\ Q & \longrightarrow & X' \end{array}$$

with left side that is an affine integral surjective morphism having finite physical geometric fibers and top side that is an étale surjection. In particular, $Q \to X'$ is a quasi-compact surjective morphism. The pullbacks of y and $p_1^*(y) = p_2^*(y)$ along $Q \to X'$ canonically (and compatibly) factor through the étale map $Y_0 \to Y'$, so by the settled case of exactness for (3.1) in the case when the target is a scheme (e.g., Y_0 in the role of Y) we can solve our problem after pullback to the schemes Q and $Q \times_{X'} Q$. Hence, it suffices to prove that $Q \to X'$ is an fpqc morphism; i.e., it is flat. We shall prove that induced maps on strict henselizations of local rings are isomorphisms.

Strict henselization is compatible with finite base change [4, IV_4, 18.8.10], so by expressing an integral extension as a direct limit of finite subextensions and using both the construction of Q and the compatibility of its formation with flat base change, the strict henselization of Q at a geometric point q is the quotient of the diagram obtained from

$$(X' \times_X R) \times_Y Y_0 \rightrightarrows (X' \times_X U) \times_Y Y_0 \tag{3.2}$$

by strict henselization along the finite sets of geometric points over q in the category of Y_0-schemes. Let y_0 be the geometric point of Y_0 induced by $Q \to Y_0$ and the geometric point q of Q. Since $Y_0 \to Y$ is an étale cover, and so induces an isomorphism of strict henselizations at geometric points, in the formation of the strict henselizations in (3.2) over q the effect of the fiber product against Y_0 over Y is eliminated (as we only get the contribution from the y_0-fiber). In other words, we have exactly the "invariant subring" description of the strict henselization of the local ring at q viewed as a geometric point on the scheme quotient X' of $X' \times_X R \rightrightarrows X' \times_X U$. ■

4. Proof of Theorem 1.1 in the locally noetherian case

We initially avoid noetherian assumptions. By §2 we may assume S is affine and $\mathscr{X}$ admits a quasi-finite, flat, and finitely presented cover $p : U \to \mathscr{X}$ by a scheme U that is quasi-projective over S, and that there is a representable, étale, and quasi-compact map of Artin stacks $h : \mathscr{U}' \to \mathscr{X}$ such that $\mathscr{U}'$ is S-separated and admits a finite locally free covering $U \to \mathscr{U}'$ whose composite with h is identified with p. By Zariski's Main Theorem [4, IV$_3$, 8.12.6], the quasi-finite separated U-scheme $R = U \times_{\mathscr{X}} U$ is also quasi-projective over S. Let $p_1, p_2 : R \rightrightarrows U$ be the canonical projections.

Since $I_S(\mathscr{X})$ is finite, by Remark 2.3 we can assume that $\mathrm{Aut}_{\mathscr{U}'(k)}(u') = \mathrm{Aut}_{\mathscr{X}(k)}(x)$ for all $u' \in \mathscr{U}'(k)$ over $x \in \mathscr{X}(k)$ for any algebraically closed field k; we postpone the use of this property until later.

By Theorem 3.1, there is a coarse moduli space U' for $\mathscr{U}'$ such that (i) U' is a scheme, (ii) $\mathscr{U}' \to U'$ is a proper map that is a universal homeomorphism, and (iii) any flat base change on U' is a coarse moduli space for the induced base change on $\mathscr{U}'$. Likewise, if S is locally noetherian then U' is locally of finite type over S, and if $\mathscr{X}$ is S-separated (so $\mathscr{U}'$ is S-separated) then so is U'. Since the Artin stack $\mathscr{R}' := \mathscr{U}' \times_{\mathscr{X}} \mathscr{U}'$ has a finite locally free covering by the scheme R that is quasi-projective over the affine S, there is also a coarse moduli space R' for $\mathscr{R}'$ that is a scheme and enjoys the analogous properties just listed for U'. There are evident S-maps $p'_1, p'_2 : R' \rightrightarrows U'$ compatible with $p_1, p_2 : R \rightrightarrows U$ and the analogous maps $\mathscr{R}' \rightrightarrows \mathscr{U}'$. We do not yet know if $R' \times_{U'} R'$ is a coarse moduli space for $\mathscr{R}' \times_{\mathscr{U}'} \mathscr{R}' = \mathscr{U}' \times_{\mathscr{X}} \mathscr{U}' \times_{\mathscr{X}} \mathscr{U}'$, so it is not evident if the quasi-compact S-map $(p'_1, p'_2) : R' \to U' \times_S U'$ has image on T-points (for S-schemes T) that is an equivalence relation on $U'(T)$. The following lemma has no noetherian hypotheses.

Lemma 4.1. *Assume that $R' \rightrightarrows U'$ is an étale equivalence relation, so there is an algebraic space quotient $X = U'/R'$. The canonical S-map $\pi : \mathscr{X} \to X$ is a coarse moduli space and π is separated and quasi-compact. If $R' \to U' \times_S U'$ is a closed immersion then X is separated over S.*

Proof. The final part is immediate from the rest since $R' = U' \times_X U' \to U' \times_S U'$ is a base change of $\Delta_{X/S}$ by the étale cover $U' \times_S U' \to X \times_S X$. To show that π must be separated, or equivalently that $\mathscr{X} \to \mathscr{X} \times_X \mathscr{X}$ is proper, using the fpqc covering $\mathscr{U}' \to \mathscr{X}$ reduces this to properness of $\mathscr{R}' = \mathscr{U}' \times_{\mathscr{X}} \mathscr{U}' \to \mathscr{U}' \times_X \mathscr{U}'$. But $\mathscr{R}' \to R' = U' \times_X U'$ is a proper surjection and $\mathscr{U}' \to U'$ is separated, so we get the result. By construction, π is quasi-compact. It remains to treat the coarse moduli space property. The composite map $U \to U' \to U'/R' = X$ is R-invariant, so it induces a unique compatible S-map $\pi : \mathscr{X} \to X$. Let $g : \mathscr{X} \to Y$ be a morphism to an algebraic space. For the universal mapping property, we seek a unique morphism $\overline{g} : X \to Y$ such that $g = \overline{g} \circ \pi$.

Let $p : U \to \mathscr{X}$ be the projection, so $f = g \circ p : U \to Y$ is R-invariant and hence the maps $U \times_{\mathscr{U}'} U \rightrightarrows U \to Y$ coincide. Thus, f uniquely factors through $U \to \mathscr{U}'$ and so through the coarse moduli space U' for $\mathscr{U}'$. That is, $f = \overline{f} \circ \pi_0$ for a unique map $\overline{f} : U' \to Y$ and the canonical map $\pi_0 : U \to \mathscr{U}' \to U'$. The composites $\overline{f} \circ p'_1, \overline{f} \circ p'_2 : R' \rightrightarrows Y$ coincide because composition with $R \to \mathscr{R}' \to R'$ carries these to the pair of equal maps $f \circ p_1, f \circ p_2 : R \rightrightarrows Y$. Hence, $\overline{f}$ uniquely factors through the projection $U' \to U'/R' = X$, giving a map $\overline{g} : X \to Y$. To see that $\overline{g} \circ \pi = g$ as maps from $\mathscr{X}$ to Y it suffices to check equality after composition with $p : U \to \mathscr{X}$, and this becomes the equality $\overline{f} \circ \pi_0 = f$.

For uniqueness, if $\overline{g}_1, \overline{g}_2 : X \rightrightarrows Y$ satisfy $\overline{g}_1 \circ \pi = \overline{g}_2 \circ \pi$ then composing with $p : U \to \mathscr{X}$ gives that $(\overline{g}_1 \circ p') \circ \pi_0 = (\overline{g}_2 \circ p') \circ \pi_0$ with $p' : U' \to U'/R' = X$ denoting the quotient map. Hence, by the universal property of $\pi_0 : U \to U'$ with respect to $U \times_{\mathscr{U}'} U$-invariant maps from U we have $\overline{g}_1 \circ p' = \overline{g}_2 \circ p'$. By the quotient property for p', we get $\overline{g}_1 = \overline{g}_2$.

Finally, we have to check that $[\mathscr{X}(k)] \to X(k)$ is bijective for any algebraically closed field k. Since k is algebraically closed, $[\mathscr{X}(k)]$ is the quotient $U(k)/R(k)$ of the set $U(k)$ modulo the equivalence relation induced by the image of $R(k)$ in $U(k) \times U(k)$. Likewise, $X(k) = U'(k)/R'(k)$ since $X = U'/R'$. By the construction of the schemes U' and R' as coarse moduli spaces for $\mathscr{U}'$ and $\mathscr{R}' = \mathscr{U}' \times_{\mathscr{X}} \mathscr{U}'$, for $P = U \times_{\mathscr{U}'} U$ we have

$$U'(k) = U(k)/P(k), \;\; R'(k) = R(k)/(P(k) \times_{U(k)} R(k) \times_{U(k)} P(k))$$

because $P \times_U R \times_U P = P \times_{\mathscr{X}} P$ and $R = U \times_{\mathscr{X}} U$. Since the image of $P(k)$ in $U(k) \times U(k)$ is contained in the image of $R(k)$, the natural map $U(k)/R(k) \to U'(k)/R'(k)$ is bijective. ■

The maps $\mathscr{R}' = \mathscr{U}' \times_{\mathscr{X}} \mathscr{U}' \rightrightarrows \mathscr{U}'$ are representable, étale, separated, and quasi-compact. In particular, for any algebraically closed field k and objects $r' \in \mathscr{R}'(k)$ over $u' \in \mathscr{U}'(k)$ the natural group map $\mathrm{Aut}_{\mathscr{R}'(k)}(r') \to \mathrm{Aut}_{\mathscr{U}'(k)}(u')$ is injective. Now we bring in the hypothesis that $I_S(\mathscr{X})$ is finite, or more specifically the consequence (recorded already) that this allows us to arrange that $\mathrm{Aut}_{\mathscr{U}'(k)}(u') = \mathrm{Aut}_{\mathscr{X}(k)}(x)$ for all such k and $u' \in \mathscr{U}'(k)$ over $x \in \mathscr{X}(k)$. This implies the corresponding equality of automorphism groups for r' over u' as above, and so for locally noetherian S the projections $R' \rightrightarrows U'$ are étale due to:

Theorem 4.2. *Let $\mathscr{Y}' \to \mathscr{Y}$ be a representable S-map between Artin stacks that are locally of finite type over a locally noetherian scheme S, and assume that this map is étale, separated, and quasi-compact. Also assume that there is a finite locally free covering $U \to \mathscr{Y}$ with $U \to S$ quasi-projective locally on S. Let $\mathscr{Y} \to Y$ be the coarse moduli space.*

The stack $\mathscr{Y}'$ also has a finite locally free covering $U' \to \mathscr{Y}'$ with $U' \to S$ quasi-projective locally on S, and if Y' is its coarse moduli space then the induced map $Y' \to Y$ is étale at any geometric point $y' \in Y'(k)$ over $y \in Y(k)$ such that the injective group map $\mathrm{Aut}_{\mathscr{Y}'(k)}(y') \to \mathrm{Aut}_{\mathscr{Y}(k)}(y)$ (well-defined up to conjugation) is surjective.

Proof. By Theorem 3.1(2), $U \to Y$ is a finite surjection and Y is locally of finite type over S. Let $U' = \mathscr{Y}' \times_{\mathscr{Y}} U$, so $U' \to \mathscr{Y}'$ is a finite locally free covering by an algebraic space U' that is quasi-finite and separated over the scheme U. In particular, U' is a scheme and $U' \to S$ is quasi-projective locally over S. Thus, by Theorem 3.1(2) we get a coarse moduli space Y' for $\mathscr{Y}'$ that is a locally finite type S-scheme and the map $U' \to Y'$ is a finite surjection. The given S-map $\mathscr{Y}' \to \mathscr{Y}$ induces an S-map $Y' \to Y$ between the locally finite type coarse moduli schemes, so this latter map is étale in a Zariski-open neighborhood of $y' \in Y'(k)$ over $y \in Y(k)$ if and only if $\mathscr{O}^{\mathrm{sh}}_{Y,y} \to \mathscr{O}^{\mathrm{sh}}_{Y',y'}$ is an isomorphism.

Choose $y' \in Y'(k)$ over $y \in Y(k)$. By the compatibility of strict henselization and finite extension of rings, as well as flat base change compatibility of coarse moduli spaces for stacks admitting suitable finite locally free scheme covers (Theorem 3.1(1)), we may identify $\mathscr{O}^{\mathrm{sh}}_{Y,y}$ as a subring of the product $\prod_{u \mapsto y} \mathscr{O}^{\mathrm{sh}}_{U,u}$ of the strict henselizations of U at the points in $U(k)$ over $y \in Y(k)$. To be precise, $\mathscr{O}^{\mathrm{sh}}_{Y,y}$ is the subring of elements with the same pullbacks under both finite locally free projections $q_1, q_2 : P = U \times_{\mathscr{Y}} U \rightrightarrows U$.

If we pick $u_0 \in U(k)$ over y then, by projection to the u_0-factor, $\mathscr{O}^{\mathrm{sh}}_{Y,y}$ is identified with the subring in $\mathscr{O}^{\mathrm{sh}}_{U,u_0}$ of elements $b \in \mathscr{O}^{\mathrm{sh}}_{U,u_0}$ such that $q_1^*(b) = q_2^*(b)$ in $\mathscr{O}^{\mathrm{sh}}_{P,\xi}$ for all $\xi \in q_1^{-1}(u_0) \cap q_2^{-1}(u_0) \subseteq P(k)$. Likewise, if we define P' to be the scheme $\mathscr{Y}' \times_{\mathscr{Y}} P$ (so $q_1', q_2' : P' \simeq U' \times_{\mathscr{Y}'} U' \rightrightarrows U'$ are finite locally free), then for $u_0' \in U'(k)$ over y' the projection to the u_0'-factor identifies $\mathscr{O}^{\mathrm{sh}}_{Y',y'}$ with the subring of elements $b' \in \mathscr{O}^{\mathrm{sh}}_{U',u_0'}$ satisfying ${q_1'}^*(b') = {q_2'}^*(b')$ in $\mathscr{O}^{\mathrm{sh}}_{P',\xi'}$ for all $\xi' \in {q_1'}^{-1}(u_0') \cap {q_2'}^{-1}(u_0') \subseteq P'(k)$.

Consider y' and y so that $\mathrm{Aut}_{\mathscr{Y}'(k)}(y') \to \mathrm{Aut}_{\mathscr{Y}(k)}(y)$ is an isomorphism. Since $P' = \mathscr{Y}' \times_{\mathscr{Y}} P$ and $U' = \mathscr{Y}' \times_{\mathscr{Y}} U$, the equalities $[\mathscr{Y}'(k)] = Y'(k)$ and $[\mathscr{Y}(k)] = Y(k)$ therefore imply that the surjections $P'(k) \twoheadrightarrow Y'(k) \times_{Y(k)} P(k)$ and $U'(k) \twoheadrightarrow Y'(k) \times_{Y(k)} U(k)$ are injective over $\{y'\} \times P(k)$ and $\{y'\} \times U(k)$ respectively. Thus, choosing u_0 and u_0' as above with u_0' also over u_0 (this can be done), the map $P' \to P$ induces a bijection on k-points from ${q_1'}^{-1}(u_0') \cap {q_2'}^{-1}(u_0')$ to $q_1^{-1}(u_0) \cap q_2^{-1}(u_0)$. But $U' \to U$ and $P' \to P$ are étale because of the hypothesis that $\mathscr{Y}' \to \mathscr{Y}$ is étale, so the natural map $\mathscr{O}^{\mathrm{sh}}_{U,u_0} \to \mathscr{O}^{\mathrm{sh}}_{U_0',u_0'}$ is an isomorphism and likewise for the strictly henselian local rings at ξ' and ξ on P' and P as above. Hence, the naturally induced map $\mathscr{O}^{\mathrm{sh}}_{Y,y} \to \mathscr{O}^{\mathrm{sh}}_{Y',y'}$ via $Y' \to Y$ is an isomorphism. ∎

We now show that the finite type map $\Delta : R' \to U' \times_S U'$ is monic for locally noetherian S. The composite of Δ with the first projection $U' \times_S U' \to U'$ has étale fibers, so Δ has étale fibers and hence is unramified. Thus, by [4, IV$_4$, 17.2.6], Δ is monic if and only if it is injective on k-points for any algebraically closed field k. The problem is to prove that the map $R'(k) \to U'(k) \times U'(k)$ is injective. By the coarse moduli property, this is identified with the map $[(\mathscr{U}' \times_{\mathscr{X}} \mathscr{U}')(k)] = [\mathscr{R}'(k)] \to [\mathscr{U}'(k)] \times [\mathscr{U}'(k)]$ whose injectivity follows from the condition that $\mathscr{U}'(k) \to \mathscr{X}(k)$ induces bijections on automorphism groups.

If $\mathscr{X}$ is S-separated then Δ is even a closed immersion. Indeed, since Δ is a monomorphism we just have to show that it is proper, and this goes as follows. Certainly Δ is finite type, and to see that it is separated we observe that its diagonal is the bottom side of the commutative square

$$\begin{array}{ccc} R & \longrightarrow & R \times_{U \times_S U} R \\ \downarrow & & \downarrow \\ R' & \longrightarrow & R' \times_{U' \times_S U'} R' \end{array}$$

whose left side is surjective, top side is a closed immersion (since the base change $R \to U \times_S U$ of $\Delta_{\mathscr{X}/S}$ is separated), and right side is finite (since $U \to U'$ and $R \to R'$ are finite). Hence, the bottom has closed image and thus is a closed immersion. To prove that Δ is universally closed, we use the commutative square

$$\begin{array}{ccc} R & \longrightarrow & U \times_S U \\ \downarrow & & \downarrow \\ R' & \xrightarrow[\Delta]{} & U' \times_S U' \end{array}$$

whose left side is surjective and top and right sides are universally closed.

Corollary 4.3. *The subfunctor $R' \subseteq U' \times_S U'$ is an equivalence relation.*

Proof. The fiber product $\mathscr{U}' \times_{\mathscr{X}} R' = \mathscr{R}' \times_{\mathscr{U}'} R'$ is a scheme that is étale over R' (via second projection), and the natural map $\mathscr{R}' \times_{\mathscr{U}'} R' \to R' \times_{U'} R'$ of étale R'-schemes is étale. It is also bijective on geometric points (since the second projection $\mathscr{R}' = \mathscr{U}' \times_{\mathscr{X}} \mathscr{U}' \to \mathscr{U}'$ induces a bijection on automorphism groups at geometric points), whence it is an isomorphism. But the scheme $\mathscr{R}' \times_{\mathscr{U}'} R'$ is flat over the coarse moduli space R' for $\mathscr{R}'$, and so by compatibility with flat base change for this coarse moduli space it follows that $\mathscr{R}' \times_{\mathscr{U}'} R'$ is the coarse moduli space for $(\mathscr{R}' \times_{\mathscr{U}'} R') \times_{R'} \mathscr{R}' = \mathscr{R}' \times_{\mathscr{U}'} \mathscr{R}' = \mathscr{U}' \times_{\mathscr{X}} \mathscr{U}' \times_{\mathscr{X}} \mathscr{U}'$. Hence, $R' \times_{U'} R'$ is the coarse moduli space for $\mathscr{R}' \times_{\mathscr{U}'} \mathscr{R}'$. It follows that we may define a unique map $c' : R' \times_{U'} R' \to R'$ compatible with $p_{13} : \mathscr{R}' \times_{\mathscr{U}'} \mathscr{R}' = \mathscr{U}' \times_{\mathscr{X}} \mathscr{U}' \times_{\mathscr{X}} \mathscr{U}' \to \mathscr{U}' \times_{\mathscr{X}} \mathscr{U}' = \mathscr{R}'$.

Thus, the pre-relation $R' \subseteq U' \times_S U'$ is transitive. Symmetry follows via the involution $i' : R' \simeq R'$ induced by the "flip" automorphism $\mathscr{R}' = \mathscr{U}' \times_{\mathscr{X}} \mathscr{U}' \simeq \mathscr{U}' \times_{\mathscr{X}} \mathscr{U}' = \mathscr{R}'$, and reflexivity is obtained via the section $e' : U' \to R'$ induced by $\Delta_{\mathscr{U}'/\mathscr{X}} : \mathscr{U}' \to \mathscr{R}'$. ■

Lemma 4.1 provides a coarse moduli space $\pi : \mathscr{X} \to X = U'/R'$ with separated and quasi-compact π when S is locally noetherian, and so Theorem 1.1 for such S is settled via:

Lemma 4.4. *For locally noetherian S, $\pi : \mathscr{X} \to X$ is proper and quasi-finite.*

Proof. Since $\mathscr{X}$ and X are locally of finite type over S, π is locally of finite type and hence is quasi-finite. To check properness of π, we recall that by [4, II, 5.6.3] a separated map of finite type $T' \to T$ between locally noetherian schemes is proper if the induced map $\mathbf{A}^n_{T'} \to \mathbf{A}^n_T$ is closed for all $n \geq 1$. By Chow's lemma for stacks [7], the same assertion holds if T' is an Artin stack rather than a scheme. Hence, to prove that π is proper it suffices to prove that it induces a closed map on topological spaces after flat base change by the maps $\mathbf{A}^n_{U'} \to X$. Since $U \times_X \mathbf{A}^n_{U'}$ is separated and quasi-finite over $\mathbf{A}^n_{U'}$, hence quasi-projective locally over S, and the formation of the coarse moduli space $\pi : \mathscr{X} \to X$ commutes with flat base change on X by construction, base change from X to such schemes $\mathbf{A}^n_{U'}$ reduces us to proving that π is closed. The closed sets in $\mathscr{X}$ are the images of closed sets $Z \subseteq U$ with $p_1^{-1}(Z) = p_2^{-1}(Z)$ as closed subsets in R (i.e., Z is R-invariant). The image of Z in U' under the affine integral map $U \to U'$ is a closed subset $Z' \subseteq U'$. We want Z' to have closed image in $X = U'/R'$, so it suffices to prove that Z' is R'-invariant.

Pick $z' \in Z'$ and $r' \in R'$ with (say) $p_2'(r') = z'$. We need that $u' = p_1'(r') \in U'$ also lies in Z'. Pick $z \in Z$ over z'. If $R \to R' \times_{p_2', U'} U$ is surjective then we can find $r \in R$ over r' with $p_2(r) = z \in Z$, so in that case $p_1(r) \in U$ also lies in Z (by R-invariance) and thus the image $u' \in U'$ of $p_1(r)$ lies in Z' as desired. As for the surjectivity of the map $R \to R' \times_{p_2', U'} U$, we work at the level of k-points for an algebraically closed field k to reduce to verifying surjectivity of the map $R \to \mathscr{R}' \times_{\mathscr{U}'} U$ (using the second projections $p_2 : R \to U$ and $\mathscr{R}' \to \mathscr{U}'$). This map is identified with the map $U \times_{\mathscr{X}} U \to \mathscr{U}' \times_{\mathscr{X}} U$ that is a base change of the surjection $U \to \mathscr{U}'$, so it is surjective. ■

5. Proof of Theorem 1.1 over a general base scheme

In the absence of noetherian assumptions our working setup is with $S = \operatorname{Spec} A$ for a commutative ring A, so we may write $S = \varprojlim S_\alpha$ with $S_\alpha = \operatorname{Spec} A_\alpha$ for the directed system of noetherian subrings A_α in A. By working Zariski-locally on $\mathscr{X}$, it follows from [8, 2.2] that for some α_0 we can find a finite type Artin stack $\mathscr{X}_{\alpha_0}$ over S_{α_0} that induces $\mathscr{X}$ by the base change $S \to S_{\alpha_0}$, and if $\mathscr{X}$ is S-separated then we can assume $\mathscr{X}_{\alpha_0}$ is S_{α_0}-separated. Since $I_S(\mathscr{X}) = S \times_{S_{\alpha_0}} I_{S_{\alpha_0}}(\mathscr{X}_{\alpha_0})$ is finite over $\mathscr{X} = S \times_{\alpha_0} \mathscr{X}_{\alpha_0}$, by increasing α_0 we can also assume that $I_{S_{\alpha_0}}(\mathscr{X}_{\alpha_0})$ is $\mathscr{X}_{\alpha_0}$-finite. We can apply Lemma 2.2 Zariski-locally on $\mathscr{X}_{\alpha_0}$ to arrange that there is a quasi-finite, flat, finite type covering $\mathscr{U}'_{\alpha_0} \to \mathscr{X}_{\alpha_0}$ by an S_{α_0}-separated Artin stack admitting a finite flat cover $U_{\alpha_0} \to \mathscr{U}'_{\alpha_0}$ by a quasi-projective S_{α_0}-scheme. Define $\mathscr{R}'_{\alpha_0} = \mathscr{U}'_{\alpha_0} \times_{\mathscr{X}_{\alpha_0}} \mathscr{U}'_{\alpha_0}$ and $R_{\alpha_0} = U_{\alpha_0} \times_{\mathscr{X}_{\alpha_0}} U_{\alpha_0}$, and for $\alpha \geq \alpha_0$ (the only α we shall consider) define $\mathscr{X}_\alpha$, $\mathscr{U}'_\alpha$, $\mathscr{R}'_\alpha$, U_α, and R_α by the base change $S_\alpha \to S_{\alpha_0}$ (so $\mathscr{R}'_\alpha = \mathscr{U}'_\alpha \times_{\mathscr{X}_\alpha} \mathscr{U}'_\alpha$). Similarly define $\mathscr{U}'$, $\mathscr{R}'$, U, and R over S, so $U = \varprojlim U_\alpha$ and $R = \varprojlim R_\alpha$ (with affine transition maps). We can choose $\mathscr{U}'_{\alpha_0} \to \mathscr{X}_{\alpha_0}$ so that it induces bijections on automorphism groups at geometric points, so the analogous property holds at each α-level and for the associated stacks over S due to the argument in Remark 2.3.

Let U'_α and R'_α denote the coarse moduli spaces (even schemes) associated to $\mathscr{U}'_\alpha$ and $\mathscr{R}'_\alpha$ respectively, and likewise with U' and R' for $\mathscr{U}'$ and $\mathscr{R}'$ by Theorem 3.1. The formation of these schemes does not generally commute with change in α. By functoriality we get natural inverse systems $\{U'_\alpha\}$ and $\{R'_\alpha\}$, and by the construction of these coarse moduli spaces via "rings of invariants" it follows that the transition maps in these inverse systems are affine and the inverse limits are respectively naturally identified with U' and R'.

By our work in the noetherian case, $R'_\alpha \rightrightarrows U'_\alpha$ over S_α is an étale equivalence relation for each α. Let $\Delta'_\alpha : R'_\alpha \to U'_\alpha \times_{S_\alpha} U'_\alpha$ be the monic diagonal. If $\mathscr{X}$ is S-separated then the Δ'_α are also proper and hence are closed immersions. A map between directed systems of rings is an isomorphism (resp. surjective) on direct limits when it is an isomorphism (resp. surjective) at each level. Thus, passing to the limit on α, the natural map

$$\Delta' = \varprojlim \Delta'_\alpha : R' \to U' \times_S U'$$

has diagonal that is an isomorphism (so Δ' is monic) and if $\mathscr{X}$ is S-separated then Δ' is a closed immersion. The subfunctor $R' \subseteq U' \times_S U'$ is an equivalence relation because each $R'_\alpha \subseteq U'_\alpha \times_{S_\alpha} U'_\alpha$ is an equivalence relation.

To show that the maps $p'_1, p'_2 : R' \rightrightarrows U'$ are étale we cannot pass to direct limits because perhaps the condition of being locally of finite presentation may be lost upon passing to the limit. Since $p'_{2,\alpha} : R'_\alpha \to U'_\alpha$ is étale, the natural map $R'_\alpha \to R'_{\alpha_0} \times_{U'_{\alpha_0}} U'_\alpha$ induced by p'_{2,α_0} and $p'_{2,\alpha}$ is a U'_α-map between étale U'_α-schemes, so it is an étale map between S_α-schemes of finite type. This is an isomorphism if and only if it is bijective on geometric points (by [4, IV_4, 17.9.1]). Such bijectivity follows from:

Lemma 5.1. *The S_α-map $R'_\alpha \to R'_{\alpha_0} \times_{U'_{\alpha_0}} U'_\alpha$ is a universal homeomorphism.*

Proof. Since diagonal maps for schemes are immersions, it is equivalent to prove that this map induces a homeomorphism after any base change on S_α. Although the formation of the coarse moduli scheme in Theorem 3.1 may fail to commute with base change, the base change morphism is always a universal homeomorphism due to the topological description of the quotient space and the fact that affine integral maps are universally closed. Also, the formation of such coarse schemes is compatible with flat base change, so $R'_{\alpha_0} \times_{U'_{\alpha_0}} U'_\alpha$ is the coarse moduli scheme for the quotient stack $R'_{\alpha_0} \times_{U'_{\alpha_0}} \mathscr{U}'_\alpha = R'_{\alpha_0} \times_{U'_{\alpha_0}} \mathscr{U}'_{\alpha_0} \times_{S_{\alpha_0}} S_\alpha$ over S_α. Thus, the natural map $R'_\alpha \times_{U'_\alpha} \mathscr{U}'_\alpha \to R'_{\alpha_0} \times_{U'_{\alpha_0}} U'_\alpha$ is a universal homeomorphism with respect to base change on S_α. But $\mathscr{R}'_\alpha \times_{\mathscr{U}'_\alpha} \mathscr{U}'_\alpha \to R'_\alpha \times_{U'_\alpha} \mathscr{U}'_\alpha$ is a universal homeomorphism with respect to base change on S_α, so the natural map $\mathscr{R}'_{\alpha_0} \times_{\mathscr{U}'_{\alpha_0}} \mathscr{U}'_\alpha \to R'_{\alpha_0} \times_{U'_{\alpha_0}} U'_\alpha$ is a universal homeomorphism with respect to such base change.

Hence, since the formation of the quotient stacks $\mathscr{R}'_\alpha$ and $\mathscr{U}'_\alpha$ commutes with base change on S_α, our universal homeomorphism problem with respect to base change on S_α is identified with the problem for the natural map $\mathscr{R}'_\alpha \to \mathscr{R}'_{\alpha_0} \times_{\mathscr{U}'_{\alpha_0}} \mathscr{U}'_\alpha$. The formation of this final map commutes with any base change on S_α. ■

This lemma and its analogue for $p'_{1,\alpha}$ and p'_{1,α_0} give the crucial result that the natural maps $R'_\alpha \to R'_{\alpha_0} \times_{U'_{\alpha_0}} U'_\alpha$ and $R'_\alpha \to U'_\alpha \times_{U'_{\alpha_0}} R'_{\alpha_0}$ (respectively induced by the second and first projections $R'_\alpha \rightrightarrows U'_\alpha$) are isomorphisms. By passing to the limit on α, we conclude that the natural maps $R' \to R'_{\alpha_0} \times_{U'_{\alpha_0}} U'$ and $R' \to U' \times_{U'_{\alpha_0}} R'_{\alpha_0}$ (respectively induced by the second and first projections $R' \rightrightarrows U'$) are isomorphisms. Hence, the projection $p'_i : R' \to U'$ is a base change of $p'_{i,\alpha_0} : R'_{\alpha_0} \to U'_{\alpha_0}$, so each p'_i is étale since each p'_{i,α_0} is étale. Thus, by

Lemma 4.1 we obtain a quasi-compact separated map $\pi : \mathscr{X} \to X = U'/R'$ that is a coarse moduli space, and X is separated over S if $\mathscr{X}$ is S-separated.

Since the S-scheme maps $U' \to U'_{\alpha_0} \times_{S_{\alpha_0}} S$ and $R' \to R'_{\alpha_0} \times_{S_{\alpha_0}} S$ are universal homeomorphisms with repsect to base change on S, they are universal homeomorphisms of schemes. (Here we use that diagonal maps for scheme morphisms are immersions, exactly as in the proof of Lemma 5.1.) The induced map on algebraic space quotients $X \to X_{\alpha_0} \times_{S_{\alpha_0}} S$ is therefore a universal homeomorphism. But $\mathscr{X} \to \mathscr{X}_{\alpha_0} \times_{S_{\alpha_0}} S$ is an isomorphism since $U \to U_{\alpha_0} \times_{S_{\alpha_0}} S$ and $R \to R_{\alpha_0} \times_{S_{\alpha_0}} S$ are isomorphisms, so we conclude that $\pi : \mathscr{X} \to X$ is a universal homeomorphism because $\pi_{\alpha_0} : \mathscr{X}_{\alpha_0} \to X_{\alpha_0}$ is a universal homeomorphism (via the settled noetherian case). Since $\mathscr{X} \to S$ is locally of finite type, the map π is locally of finite type and hence (by separatedness and quasi-compactness) proper.

The formation of π commutes with flat base change of algebraic spaces $X' \to X$ because the formation of the coarse moduli spaces R' and U' for $\mathscr{R}'$ and $\mathscr{U}'$ commutes with flat base change on R' and U' respectively (Theorem 3.1(1)).

Corollary 5.2. *Let $\mathscr{X}$ be a locally finitely presented Artin stack over a scheme S, and assume $\Delta_{\mathscr{X}/S}$ is quasi-finite. The existence of a coarse moduli space $\pi : \mathscr{X} \to X$ with separated π is equivalent to finiteness of $I_S(\mathscr{X})$.*

Proof. By Theorem 1.1, finiteness of $I_S(\mathscr{X})$ provides such a π. Conversely, if such a separated π exists then let us show that $I_S(\mathscr{X})$ is finite. We use the criterion in Remark 2.3. Let $\mathscr{X}_0 \subseteq \mathscr{X}$ be a quasi-compact open substack over an open affine $S_0 \subseteq S$, and $X_0 \subseteq X$ its quasi-compact open image in X, so there is a quasi-compact étale cover $X'_0 \to X_0$ by an affine scheme. Let $\mathscr{X}'_0 = \mathscr{X}_0 \times_{X_0} X'_0$, so this is separated over X'_0 and thus over S_0. Since $\Delta_{\mathscr{X}'_0/S_0}$ is quasi-finite, the stack $I_{S_0}(\mathscr{X}'_0)$ is therefore finite (over $\mathscr{X}'_0$). Pick $x \in \mathscr{X}_0(k)$ for an algebraically closed field k, and choose $x' \in \mathscr{X}'_0(k)$ over x, so x and x' have the same automorphism group (in their respective categories). By finiteness of $I_{S_0}(\mathscr{X}'_0)$, there is a representable, quasi-compact, separated, and étale neighborhood $\mathscr{U}'$ of $(\mathscr{X}'_0, x')$ such that $\mathscr{U}'$ is separated over S_0, $\mathscr{U}'$ has a finite locally free scheme cover, and $\mathscr{U}' \to \mathscr{X}'_0$ induces bijections on automorphism groups at geometric points. Since $\mathscr{U}' \to \mathscr{X}_0$ is a representable, quasi-compact, and étale neighborhood of $(\mathscr{X}_0, x)$, by using the converse direction in Remark 2.3 we deduce the $\mathscr{X}_0$-finiteness of $I_{S_0}(\mathscr{X}_0) = \mathscr{X}_0 \times_{\mathscr{X}} I_S(\mathscr{X})$. Such $\mathscr{X}_0$'s cover $\mathscr{X}$, so $I_S(\mathscr{X})$ is $\mathscr{X}$-finite. ■

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