Math 632. Čech Cohomology and Alternating Cochains

Let \( \mathcal{U} = \{ U_i \}_{i \in I} \) be an arbitrary open covering of a topological space \( X \) and let \( \mathcal{F} \) a sheaf of abelian groups on \( X \). This data is fixed. Let \( C^\bullet(\mathcal{F}) \) denote the ‘unrestricted’ Čech complex of \( \mathcal{F} \) with respect to \( \mathcal{U} \), namely (for \( n \geq 0 \))

\[
C^n(\mathcal{F}) = \prod_{i \in I^{n+1}} \Gamma(U_i, \mathcal{F}),
\]

where for \( i = (i_0, \ldots, i_n) \in I^{n+1} \) we define \( U_i = U_{i_0} \cap \cdots \cap U_{i_n} \), and the differential maps are given by the habitual formula. The symmetric group \( \Sigma_n \) acts in the usual manner on the set \( I^n = \text{Hom}(\{1, \ldots, m\}, I) \) for each integer \( m \geq 1 \). Using this action, we define \( C^\bullet_n(\mathcal{F}) \) to be the subcomplex consisting of alternating Čech cochains (i.e., those cochains \( (s_i) \in C^n(\mathcal{F}) \) for \( n \geq 0 \) with \( s_{\sigma(i)} = \text{sgn}(\sigma)s_i \) for \( \sigma \in \Sigma_{n+1} \) and \( s_i = 0 \) if some \( i_r = i_s \) for \( r \neq s \)). Fixing a well-ordering on the set \( I \), we define the quotient complex \( \overline{C}^\bullet(\mathcal{F}) \) via

\[
\overline{C}^n(\mathcal{F}) = \prod_{i_0 < \cdots < i_n} \Gamma(U_{i}, \mathcal{F})
\]

(for \( n \geq 0 \)) and again we use the usual differentiation formulas.

The composite map of complexes \( C^\bullet_n(\mathcal{F}) \to C^\bullet(\mathcal{F}) \to \overline{C}^\bullet(\mathcal{F}) \) is clearly an isomorphism, so we get an induced diagram on cohomology \( H^\bullet(\mathcal{U}, \mathcal{F}) \to H^\bullet(\mathcal{F}) \to H^\bullet(\overline{C}^\bullet(\mathcal{F})) \) such that the composite map is an isomorphism. We claim that both intermediate maps are isomorphisms (the first map being canonical in the sense that it makes no reference to a choice of well-ordering). In order to prove this, it is enough to analyze the right-hand map, which is visibly surjective, and to prove that this map is an injection. We will prove this by constructing an appropriate homotopy operator.

More precisely, we will construct maps \( K_n : C^n(\mathcal{F}) \to C^{n-1}(\mathcal{F}) \) for each \( n \geq 1 \) such that

\[
K_{n+1}d_n + d_{n-1}K_n = h_n - 1
\]

for all \( n \geq 1 \), where \( h_n : C^n(\mathcal{F}) \to \overline{C}^n(\mathcal{F}) \simeq C^n_n(\mathcal{F}) \to C^n(\mathcal{F}) \). From this, it follows that the induced composite maps \( H^\bullet(C^\bullet(\mathcal{F})) \to H^\bullet(\overline{C}^\bullet(\mathcal{F})) \simeq H^\bullet(C^\bullet_n(\mathcal{F})) \to H^\bullet(\overline{C}^\bullet(\mathcal{F})) \) are injective in positive degree. In particular, the left maps are injective in positive degree, which is what we wanted to prove (we take care of degree 0 by observing that all of the degree 0 cohomology maps are identified with the identity map on \( \Gamma(X, \mathcal{F}) \), since \( \mathcal{F} \) is a sheaf). Of course, we can suppose at this time that \( X \) is a non-empty space, so \( I \) is a non-empty set. This will come up shortly.

In order to construct the homotopy operator \( k_\bullet \), we will first consider a more formal ‘dual’ situation which clarifies the role of the index set in our manipulations. For each \( n \geq 0 \), define \( C_n(I) \) to be the free abelian group on the set \( I^{n+1} \), with the usual chain complex structure determined by the formula

\[
\partial(i_0, \ldots, i_n) = \sum_{r=0}^n (-1)^r (i_0, \ldots, \hat{i}_r, \ldots, i_n)
\]

(here we are using the canonical injection \( I^{n+1} \hookrightarrow C^n(I) \) of sets for each \( n \geq 0 \) in order to identify the standard ‘basis’ of \( C^n(I) \) with the set \( I^{n+1} \)). This complex is simpler than the Čech complex for a couple of reasons. First of all, many Čech manipulations are purely combinatorial in the index data, so by eliminating the sheaf data which is just “carried along”, the situation is made less cluttered. Also, the objects \( C_n(I) \) are direct sums rather than direct products, so it is easier to construct certain maps in this context.

There is a subcomplex \( j : C_n(I) \hookrightarrow C_n(\mathcal{F}) \), with \( C_n(I) \) the free abelian group on those \( (i_0, \ldots, i_n) \in I^{n+1} \) with \( i_0 < \cdots < i_n \) (this is the same well-ordering on \( I \) which was chosen above). We also have a projection map \( p : C_\bullet(\mathcal{F}) \to C_n(\mathcal{F}) \) which annihilates those \( i \in I^{n+1} \) for which some \( i_r = i_s \) with \( r \neq s \), and otherwise sends \( i \) to \( \text{sgn}(\sigma)(i_{\sigma(0)}, \ldots, i_{\sigma(n)}) \), where \( \sigma \in \Sigma_{n+1} \) is the unique element for which \( \sigma r < \cdots < \sigma n \). Since \( \Sigma_{n+1} \) is generated by transpositions of the form \( (t, t+1) \) for \( t \in \mathbb{Z}/(n+1) \), it is easy to check that \( p \) is a map of complexes. Clearly \( p_j = 1 \), though we won’t use this (but note that this is essentially why the above cohomology map \( H^\bullet(\mathcal{U}, \mathcal{F}) \to H^\bullet(C_n(\mathcal{F})) \) is surjective; our proof of injectivity will make clear the precise sense in which these two facts are related).
Using the ‘augmentation’ $C_0(I) \to \mathbb{Z}$ given by $i \mapsto 1$ for all $i \in I$, we get a complex $C_\bullet(I) \to \mathbb{Z} \to 0$. We claim this complex is exact, so it is a free (or more conceptually, projective) resolution of $\mathbb{Z}$ in the category of abelian groups. Exactness at $\mathbb{Z}$ is clear (recall $I$ is not empty!). To handle the higher degree terms, we define a homotopy operator. Pick any $i \in I$ ($I$ is non-empty!), and define $k_i : C_n(I) \to C_{n+1}(I)$ for each $n \geq 0$ by $k_i(i_0, \ldots, i_n) = (i, i_0, \ldots, i_n)$. Moreover, if we define $C_{-1}(I) = \mathbb{Z}$ and define $\partial_0 : C_0(I) \to C_{-1}(I)$ to be the augmentation map, we can even compatibly define $k_i : C_{-1}(I) \to C_0(I)$ by $1 \mapsto i$.

It is easy to check that $\partial k_i + k_i \partial = 1$ on each $C_n(I)$ for $n \geq 0$. Thus, $C_\bullet(I)$ with the augmentation map is indeed an exact complex. Since $jp : C_\bullet(I) \to C_\bullet(I)$ is a map of complexes respecting the augmentation, it follows from the general theory of projective resolutions that there exist homotopy operators $k_n : C_n(I) \to C_{n+1}(I)$ for each $n \geq 0$ such that $\sum_{i} k_{n-1} = \partial_{n+1} k_{n} + k_{n-1} \partial_{n}$ for all $n \geq 1$. More precisely, we will prove the existence of maps $k_m : C_m(I) \to C_{m+1}(I)$ ($m \geq 0$) such that $\sum_{i} k_{n-1} = \partial_{m+1} k_{m} + k_{m-1} \partial_{m}$ for $m \geq 0$ (where $k_{-1} = 0$), with the extra property that for each $i \in I^{n+1}$ ($n \geq 0$), $k_n(i)$ has support inside of the support of $i$ (we define the support of an element $\sum a_i i \in C_n(I)$ for $n \geq 0$ to be the subset of $I$ consisting of the ‘coordinates’ of those $i \in I^{n+1}$ for which $a_i \neq 0$). We describe this by saying “$k_n$ does not increase support”.

We recursively construct the maps. More precisely, suppose for some $n \geq 0$ that we are given maps $k_m : C_m(I) \to C_{m+1}(I)$ for $0 \leq m \leq n$ which do not increase support, and with

$$j_n p_n - 1 = \partial_{m+1} k_m + k_{m-1} \partial_{m}$$

(we define $k_{-1} = 0$). For example, this is true when $n = 0$ by taking $k_0 = 0$ — note that $j_0 p_0$ is equal to the identity map! Define $k_{n+1}$ by

$$k_{n+1}(i) = k_n(i) \omega(i),$$

where $\omega(i) = j_{n+1} p_{n+1} - \partial_{n+1} k_n$ for $i \in I^{n+2}$. Note that (by the formulas for $\partial$, $j$, $p$, and $\kappa_{i_0}$, and our inductive hypothesis on $k_n$) $k_{n+1}(i)$ has support inside that of $i$. Also, since $j$ and $p$ are complex maps and $j_{n+1} p_{n+1} - \partial_{n+1} k_n = k_{n-1} \partial_{n}$, we easily compute $\partial_{n+1} \omega(i) = k_{n-1} \partial_{n+1}(i) = 0$. Thus, $\omega(i) = \partial_{n+1} k_{n}(i)$ (note the dependence of $\kappa$ on $i$), and it is straightforward to check that $\partial_{n+1} k_{n+1}$ and $j_{n+1} p_{n+1} - 1 - k_{n-1} \partial_{n+1}$ take the same value on each $i \in I^{n+2}$. This completes the construction of the $k_{n}$’s.

Now we make a functorial observation which will allow us to construct the sought-after homotopy operator in Čech cohomology. Fix $n, m \geq 0$ and a group map $f : C_n(I) \to C_m(I)$ with the property that for each $c \in C_n(I)$, the support of $f(c)$ is a subset of the support of $c$ (equivalently, this can be checked on each $i \in I^{n+1}$, and is loosely described by saying that “$f$ doesn’t increase support”). Define a map

$$\tilde{f} : C^m(\mathcal{F}) \to C^n(\mathcal{F})$$

by

$$\tilde{f}(s) = \sum a_i \text{res}^{f}_i (s),$$

where $f(i) = \sum a_i i'$. Here, $i' \in I^{n+1}$ runs through the finitely many elements with support inside of the support of $i$, and

$$\text{res}^{f}_i : \Gamma(U_i, \mathcal{F}) \to \Gamma(U_{i'}, \mathcal{F})$$

is the usual restriction map (note that $U_{i'} \subset U_i' \subset U'$ because of the the support condition on $i'$).

It is easy to check that $\tilde{f}$ is a group map, $\text{id} = \text{id}$, and if $g : C_n(I) \to C_m(I)$, $h : C_m(I) \to C_r(I)$ are two other such group maps which don’t increase support, then $\tilde{f} + g$ and $h \circ \tilde{f}$ also don’t increase support and $\tilde{f} + g = \tilde{f}$ and $h \circ \tilde{f}$ are both defined. Since $\tilde{f} = d_{n-1}$ for each $n \geq 1$ (this is why $\partial_n$ was defined as it was for $n \geq 1$), it follows that $k_n d_n + d_{n-1} k_{n-1} = j_n p_n - 1$ for each $n \geq 1$. But for each $n \geq 1$, $\tilde{j}_n p_n : C^n(\mathcal{F}) \to C^n(\mathcal{F})$ is exactly the map denoted $h_n$ above. Thus, we have the desired homotopy operators $\tilde{K}_n \overset{\text{def}}{=} k_{n-1}$ in each degree $n \geq 1$. 