

THE STRUCTURE OF SOLVABLE GROUPS OVER GENERAL FIELDS

BRIAN CONRAD

Consider a smooth connected solvable group G over a field k . If k is algebraically closed then $G = T \times \mathcal{R}_u(G)$ for any maximal torus T of G [Bo, 10.6(4)]. Over more general k , an analogous such structure can fail to exist. For example, consider an imperfect field k of characteristic $p > 0$ and an element $a \in k - k^p$, so $k' := k(a^{1/p})$ is a degree- p purely inseparable extension of k . Note that $k'_s := k' \otimes_k k_s = k_s(a^{1/p})$ is a separable closure of k' , and $k_s'^p \subset k_s$. The affine Weil restriction $G = \mathbf{R}_{k'/k}(\mathbf{G}_m)$ is an open subscheme of $\mathbf{R}_{k'/k}(\mathbf{A}_{k'}^1) = \mathbf{A}_k^p$, so it is a smooth connected affine k -group of dimension $p > 1$. Loosely speaking, G is “ k'^{\times} viewed as a k -group”. More precisely, for k -algebras R we have $G(R) = (k' \otimes_k R)^{\times}$ functorially in R . This commutative k -group contains an evident 1-dimensional torus $T \simeq \mathbf{G}_m$ corresponding to the subgroup $R^{\times} \subset (k' \otimes_k R)^{\times}$, and G/T is unipotent because $(G/T)(k_s) = (k'_s)^{\times}/(k_s)^{\times}$ is p -torsion. In particular, T is the unique maximal torus of G . Since $G(k_s) = k_s'^{\times}$ has no nontrivial p -torsion, G contains *no* nontrivial unipotent smooth connected k -subgroup. Thus, G is a commutative counterexample over k to the analogue of the semidirect product structure for connected solvable smooth affine groups over \bar{k} .

The appearance of imperfect fields in the preceding counterexample is essential. To explain this, recall that over a general field k , if S is a maximal k -torus in G then $S_{\bar{k}}$ is maximal in $G_{\bar{k}}$ by a theorem of Grothendieck (see [C, A.1.2]). Thus, by the conjugacy of maximal tori in $G_{\bar{k}}$, $G = T \times U$ for a k -torus T and a unipotent smooth connected normal k -subgroup $U \subset G$ if and only if the subgroup $\mathcal{R}_u(G_{\bar{k}}) \subset G_{\bar{k}}$ is defined over k (i.e., descends to a k -subgroup of G). In such cases, the semi-direct product structure holds for G over k using any maximal k -torus T of G (and U is unique: it must be a k -descent of $\mathcal{R}_u(G_{\bar{k}})$). If k is perfect then by Galois descent we may always descend $\mathcal{R}_u(G_{\bar{k}})$ to a k -subgroup of G . The main challenge is the case of imperfect k .

In these notes, we explain Tits’ structure theory for unipotent smooth connected groups over general fields of positive characteristic (especially imperfect fields), and we use it to establish a general structure theorem for solvable smooth connected groups that replaces (and generalizes) the semidirect product structure over perfect k . The bulk of the work is in the unipotent case, for which our exposition is a mild revision of [CGP, App. B] (which develops some fundamental results of Tits, building on earlier work of Rosenlicht, concerning the structure of smooth connected unipotent groups and torus actions on such groups over an arbitrary ground field of positive characteristic).

The results on unipotent groups were presented by Tits in a course at Yale University in 1967, and lecture notes [Ti1] for that course were circulated but never published. Much of the course was concerned with general results on linear algebraic groups that are available now in many standard references (such as [Bo], [Hum], and [Spr]). The original account (with proofs) of Tits’ structure theory of unipotent groups is his unpublished Yale lecture notes, and a summary of the results is given in [Oes, Ch. V]. In some parts we have simply reproduced arguments from Tits’ lecture notes.

Throughout the discussion below, k is an arbitrary field with characteristic $p > 0$.

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1. SUBGROUPS OF VECTOR GROUPS

The *additive group* is denoted \mathbf{G}_a and the *multiplicative group* is denoted \mathbf{G}_m , always with the base ring understood from context.

Definition 1.1. A *vector group* over a field k is a smooth commutative k -group scheme V that admits an isomorphism to \mathbf{G}_a^n for some $n \geq 0$. The \mathbf{G}_m -scaling action arising from such an isomorphism is a *linear structure* on V .

Observe that the \mathbf{G}_m -action on V arising from a linear structure induces the canonical k^\times -action on $\text{Lie}(V)$ (e.g., if $\text{char}(k) = p > 0$ then the composition of such a \mathbf{G}_m -action on V with the p -power map on \mathbf{G}_m does not arise from a linear structure on V when $V \neq 0$).

Example 1.2. If W is a finite-dimensional k -vector space then the *associated vector group* \underline{W} represents the functor $R \rightsquigarrow R \otimes_k W$ on k -algebras and its formation commutes with any extension of the ground field. Explicitly, $\underline{W} = \text{Spec}(\text{Sym}(W^*))$ and it has a unique linear structure relative to which the natural identification of groups $\underline{W}(k) \simeq W$ carries the linear structure over to the k^\times -action on W arising from the k -vector space structure; call this the *canonical* linear structure on \underline{W} . For finite-dimensional k -vector spaces W and W' , the subset $\text{Hom}_k(W, W') \subset \text{Hom}_{k\text{-gp}}(\underline{W}, \underline{W}')$ consists of precisely the k -homomorphisms respecting the canonical linear structures.

When linear structures are specified on a pair of vector groups, a homomorphism respecting them is called *linear*. Over a field of characteristic 0 there is a unique linear structure and all homomorphisms are linear. Over a field with characteristic $p > 0$ the linear structure is not unique in dimension larger than 1 (e.g., $a \cdot (x, y) := (ax + (a - a^p)y^p, ay)$ is a linear structure on \mathbf{G}_a^2 , obtained from the usual one via the non-linear k -group automorphism $(x, y) \mapsto (x + y^p, y)$ of \mathbf{G}_a^2).

A smooth connected solvable k -group G is *k-split* if it admits a composition series whose successive quotients are k -isomorphic to \mathbf{G}_a or \mathbf{G}_m . In the case of tori this is a widely-used notion, and it satisfies convenient properties, such as: (i) every subtorus or quotient torus (over k) of a k -split k -torus is k -split, (ii) every k -torus is an almost direct product of its maximal k -split subtorus and its maximal k -anisotropic subtorus. However, in contrast with the case of tori, it is not true for general smooth connected solvable G that the k -split property is inherited by smooth connected normal k -subgroups:

Example 1.3 (Rosenlicht). Suppose that k is imperfect and choose $a \in k - k^p$. The k -group

$$\mathbf{U} := \{y^p = x - ax^p\}$$

is a k -subgroup of the k -split $G = \mathbf{G}_a^2$ and it becomes isomorphic to \mathbf{G}_a over $k(a^{1/p})$ but there is no non-constant k -morphism $f : \mathbf{A}_k^1 \rightarrow \mathbf{U}$, let alone a k -group isomorphism $\mathbf{G}_a \simeq \mathbf{U}$. Indeed, the regular compactification $\overline{\mathbf{U}}$ of \mathbf{U} has a unique point $\infty_{\mathbf{U}} \in \overline{\mathbf{U}} - \mathbf{U}$, so any non-constant map f extends to a (finite) surjective map $\mathbf{P}_k^1 \rightarrow \overline{\mathbf{U}}$ that must carry $[1, 0]$ to $\infty_{\mathbf{U}}$, an absurdity since $k(\infty_{\mathbf{U}}) = k(a^{1/p}) \neq k$.

Tits introduced an analogue for unipotent k -groups of the notion of anisotropy for tori over a field. This rests on a preliminary understanding of the properties of subgroups of vector groups, so we take up that study now. The main case of interest to us will be imperfect ground fields.

Definition 1.4. A polynomial $f \in k[x_1, \dots, x_n]$ is a *p-polynomial* if every monomial appearing in f has the form $c_{ij}x_i^{p^j}$ for some $c_{ij} \in k$; that is, $f = \sum f_i(x_i)$ with $f_i(x_i) = \sum_j c_{ij}x_i^{p^j} \in k[x_i]$. (In particular, $f_i(0) = 0$ for all i . Together with the identity $f = \sum f_i(x_i)$, this uniquely determines each f_i in terms of f . Note that $f(0) = 0$.)

Proposition 1.5. *A polynomial $f \in k[x_1, \dots, x_n]$ is a p -polynomial if and only if the associated map of k -schemes $\mathbf{G}_a^n \rightarrow \mathbf{G}_a$ is a k -homomorphism.*

Proof. This is elementary and is left to the reader. \square

A nonzero polynomial over k is *separable* if its zero scheme in affine space is generically k -smooth.

Proposition 1.6. *Let $f \in k[x_1, \dots, x_n]$ be a nonzero polynomial such that $f(0) = 0$. Then the subscheme $f^{-1}(0) \subseteq \mathbf{G}_a^n$ is a smooth k -subgroup if and only if f is a separable p -polynomial.*

Proof. The “if” direction is clear. For the converse, we assume that $f^{-1}(0)$ is a smooth k -subgroup and we denote it as G . The smoothness implies that f is separable. To prove that f is a p -polynomial, by Proposition 1.5 it suffices to prove that the associated map of k -schemes $\mathbf{G}_a^n \rightarrow \mathbf{G}_a$ is a k -homomorphism. Without loss of generality, we may assume that k is algebraically closed.

For any $\alpha \in G(k)$, $f(x + \alpha)$ and $f(x)$ have the same zero scheme (namely, G) inside of \mathbf{G}_a^n . Thus, $f(x + \alpha) = c(\alpha)f(x)$ for a unique $c(\alpha) \in k^\times$. Consideration of a highest-degree monomial term appearing in f implies that $c = 1$. Pick $\beta \in k^n$, so $f(\beta + \alpha) - f(\beta) = 0$ for all $\alpha \in G(k)$. Thus $f(\beta + x) - f(\beta)$ vanishes on G , so $f(\beta + x) - f(\beta) = g(\beta)f(x)$ for a unique $g(\beta) \in k$. Consideration of a highest-degree monomial term in f forces $g(\beta) = 1$. \square

Corollary 1.7. *Let $G \subseteq \mathbf{G}_a^n$ be a smooth k -subgroup of codimension 1. Then G is the zero scheme of a separable nonzero p -polynomial in $k[x_1, \dots, x_n]$.*

Proof. Since G is smooth of codimension 1 in \mathbf{G}_a^n , it is the zero scheme of a separable nonzero polynomial $f \in k[x_1, \dots, x_n]$. By Proposition 1.6, f is a p -polynomial. \square

Lemma 1.8. *If $f : U' \rightarrow U$ is a surjective homomorphism between smooth connected unipotent k -groups and U' is k -split then so is U .*

This result is [Bo, 15.4(i)]. We give a proof based on the elementary Proposition 1.6.

Proof. Let $\{U'_i\}$ be a descending composition series of U' over k with successive quotients isomorphic to \mathbf{G}_a . Then the k -groups $U_i = f(U'_i)$ are a composition series for U and U_i/U_{i-1} is a quotient of $U'_i/U'_{i-1} = \mathbf{G}_a$. It therefore suffices to show that for any surjective k -homomorphism $q : \mathbf{G}_a \rightarrow G$ with $G \neq 1$, necessarily $G \simeq \mathbf{G}_a$. Clearly q is an isogeny. If $\ker q$ is not étale then $\ker q$ has nontrivial Frobenius kernel. But the Frobenius kernel of \mathbf{G}_a is α_p , so q factors through $\mathbf{G}_a/\alpha_p \simeq \mathbf{G}_a$. Hence, by induction on $\deg q$ we can assume $\ker q$ is étale. By Proposition 1.6, the smooth k -subgroup $\ker q \subset \mathbf{G}_a$ must be the zero scheme of a 1-variable separable p -polynomial $f = \sum c_j t^{p^j}$ (so $c_0 \neq 0$). But $f : \mathbf{G}_a \rightarrow \mathbf{G}_a$ is then an isogeny and its kernel $\{f = 0\}$ coincides with $\ker q$, so f identifies $G = \mathbf{G}_a/\ker q$ with \mathbf{G}_a . \square

Definition 1.9. If $f = \sum_{i=1}^n f_i(x_i)$ is a p -polynomial over k in n variables with $f_i(0) = 0$ for all i , then the *principal part* of f is the sum of the leading terms of the f_i .

Lemma 1.10. *Let V be a vector group of dimension $n \geq 1$ over k , and let $f : V \rightarrow \mathbf{G}_a$ be a k -homomorphism. Then the following are equivalent:*

- (1) *there exists a non-constant k -scheme morphism $f' : \mathbf{A}_k^1 \rightarrow V$ such that $f \circ f' = 0$;*
- (2) *for every k -group isomorphism $h : \mathbf{G}_a^n \simeq V$, the principal part of the p -polynomial $f \circ h \in k[x_1, \dots, x_n]$ has a nontrivial zero in k ;*
- (3) *there exists a k -group isomorphism $h : \mathbf{G}_a^n \simeq V$ such that $f \circ h$ “only depends on the last $n - 1$ coordinates” (i.e., $\ker(f \circ h)$ contains the first factor of \mathbf{G}_a^n).*

In this lemma, it is not sufficient in (2) to consider just a single choice of h . For example, if k is imperfect and $a \in k - k^p$, then $f = y^p - (x + ax^p)$ has principal part $y^p - ax^p$ with no zeros on $k^2 - \{0\}$. Composing f with the k -automorphism $(x, y) \mapsto (x, y + x^p)$ yields the polynomial $y^p + x^{p^2} - (x + ax^p)$ whose principal part is $y^p + x^{p^2}$, which has zeros on $k^2 - \{0\}$.

Proof. We will show that (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1).

For (1) \Rightarrow (2), assume that (1) holds and let $\varphi = h^{-1} \circ f'$. Let $\varphi_i : \mathbf{G}_a \rightarrow \mathbf{G}_a$ be the i th component of φ , and $a_i t^{s_i}$ denote the leading term of $\varphi_i(t)$, with $s_i = 0$ when $\varphi_i = 0$. For some i we have $s_i > 0$, since some φ_i is non-constant (as φ is non-constant, because of the same for f'). Let $\sum_{i=1}^n c_i x_i^{p^{m_i}}$ be the principal part of $f \circ h$, so

$$0 = f(h(\varphi(t))) = \sum_{i=1}^n c_i a_i^{p^{m_i}} t^{s_i p^{m_i}} + \dots$$

since $f \circ h \circ \varphi = f \circ h \circ h^{-1} \circ f' = f \circ f' = 0$. Let $N = \max_i \{s_i p^{m_i}\} > 0$. Define $b_i = a_i$ if $s_i p^{m_i} = N$ (so $b_i \neq 0$), and $b_i = 0$ if $s_i p^{m_i} < N$. Since the coefficient of the term of degree N in $f(h(\varphi(t)))$ must be zero, we have $\sum_{i=1}^n c_i b_i^{p^{m_i}} = 0$ with $b_i \in k$ and some b_i is nonzero, so (2) holds.

To prove (2) \Rightarrow (3), assume (2) holds and let $h : \mathbf{G}_a^n \simeq V$ be any k -group isomorphism. We may assume $f \neq 0$, so the principal part of $f \circ h$ is nonzero. The proof will proceed by induction on the sum d of the degrees of nonzero terms of the principal part $\sum_{i=1}^n c_i x_i^{p^{m_i}}$ of $f \circ h$. If $c_r = 0$ for some r , we are done by interchanging x_r and x_1 . So we may assume that all c_i are nonzero and, upon permuting the coordinates, that $m_1 \geq \dots \geq m_n \geq 0$. By (2), there exists $(a_1, \dots, a_n) \in k^n - \{0\}$ such that $\sum_{i=1}^n c_i a_i^{p^{m_i}} = 0$. Let $r \geq 0$ be minimal such that $a_r \neq 0$. Define the k -group isomorphism $h' : \mathbf{G}_a^n \simeq \mathbf{G}_a^n$ by $h'(y_1, \dots, y_n) = (x_1, \dots, x_n)$ with

$$\begin{aligned} x_1 &= y_1, \dots, x_{r-1} = y_{r-1}, \\ x_r &= a_r y_r, \quad x_{r+1} = y_{r+1} + a_{r+1} y_r^{p^{m_r - m_{r+1}}}, \dots, x_n = y_n + a_n y_r^{p^{m_r - m_n}} \end{aligned}$$

Thus, $f \circ h \circ h'$ is a p -polynomial with principal part

$$\sum_{i \neq r} c_i y_i^{p^{m_i}} + \sum_{i=1}^n c_i a_i^{p^{m_i}} \cdot y_r^{p^{m_r}} = \sum_{i \neq r} c_i y_i^{p^{m_i}}$$

since $\sum_{i=1}^n c_i a_i^{p^{m_i}} = 0$. The sum of the degrees of the nonzero terms of the principal part of $f \circ h \circ h'$ is strictly smaller than d since $c_r \neq 0$, so the induction hypothesis applies.

Finally, we assume (3) and prove (1). Let $h : \mathbf{G}_a^n \rightarrow V$ be a k -isomorphism such that $\ker(f \circ h)$ contains the first factor of \mathbf{G}_a^n . Define $\varphi : \mathbf{G}_a \rightarrow \mathbf{G}_a^n$ by $\varphi(t) = (t, 0, 0, \dots, 0)$. Finally, let $f' = h \circ \varphi$. Then $f \circ f' = f \circ h \circ \varphi = 0$. \square

Lemma 1.11. *If K/k is a Galois extension and a p -polynomial of the form $\sum_{i=1}^n c_i x_i^{p^{m_i}}$ over k has a zero in $K^n - \{0\}$ then it has a zero in $k^n - \{0\}$.*

Proof. The proof is by induction on n . The terms may be ordered so that $m_1 \geq m_2 \geq \dots$. If $n = 1$, then since $c_1 a_1^{p^{m_1}} = 0$ with $a_1 \in K^\times$ we see that $c_1 = 0$, so $c_1 x_1^{p^{m_1}}$ has a zero in k^\times .

Now suppose $n > 1$ and that $\sum_{i=1}^n c_i a_i^{p^{m_i}} = 0$ with $a_i \in K$ not all zero. Let $a = (a_1, \dots, a_n)$. If $a_n = 0$ then the theorem is true by the induction hypothesis. If $a_n \neq 0$, we may assume $a_n = 1$ by replacing a_i with $a_i/a_n^{p^{m_n - m_i}}$ for all i . For all $\sigma \in \text{Gal}(K/k)$, the point $a - \sigma(a)$ is a zero of $\sum c_i x_i^{p^{m_i}}$. If not all a_i belong to k then $a - \sigma(a) \neq 0$, so since $a_n - \sigma(a_n) = 0$ we may again apply the inductive hypothesis. \square

Lemma 1.12. *Let V be a vector group over k , K/k a Galois extension, and $f : V \rightarrow \mathbf{G}_a$ a k -homomorphism. The equivalent conditions (1), (2), and (3) of Lemma 1.10 hold over K if and only if they hold over k .*

Proof. It is clear that if (1) holds over k , it also holds over K . On the other hand, by Lemma 1.11, (2) is true over k if it is true over K . \square

Lemma 1.13. *Every smooth p -torsion commutative affine k -group G embeds as a k -subgroup of a vector group over k . Moreover, G admits an étale isogeny onto a vector group over k , and if G is connected and $k = \bar{k}$ then G is a vector group over k .*

Proof. We first construct the embedding into a vector group over k , and then at the end use this to make the étale isogeny. Consider the canonical k -subgroup inclusion $G \hookrightarrow \mathbf{R}_{k'/k}(G_{k'})$ for any finite extension field k'/k . Since $\mathbf{R}_{k'/k}(\mathbf{G}_a) \simeq \mathbf{G}_a^{[k':k]}$, it is harmless (for the purpose of finding an embedding into a vector group over k) to replace k with a finite extension. If $G_{\bar{k}}$ embeds as a subgroup of \mathbf{G}_a^N over \bar{k} , the embedding descends to a finite extension k'/k inside of \bar{k} . Hence, for the construction of the embedding into a vector group we can now assume that k is algebraically closed.

The component group G/G^0 is a power of $\mathbf{Z}/p\mathbf{Z}$. Thus, since G is commutative and p -torsion, the connected-étale sequence of G splits. That is, $G = G^0 \times (\mathbf{Z}/p\mathbf{Z})^n$ for some $n \geq 0$. The finite constant k -group $\mathbf{Z}/p\mathbf{Z}$ is a k -subgroup of \mathbf{G}_a , so we can assume that G is connected. We shall prove that G is a vector group. By [Bo, 10.6(2), 10.9], G has a composition series whose successive quotients are \mathbf{G}_a . By induction on $\dim G$, it suffices to prove that a commutative extension U of \mathbf{G}_a by \mathbf{G}_a over k is a split extension if $p \cdot U = 0$.

Let W_2 be the additive k -group of Witt vectors of length 2, so there is a canonical exact sequence of k -groups

$$0 \rightarrow \mathbf{G}_a \rightarrow W_2 \rightarrow \mathbf{G}_a \rightarrow 0.$$

It is a classical fact (see [Ser, Ch. VII.9, Lemma 3]) that every commutative extension U of \mathbf{G}_a by \mathbf{G}_a over k is obtained by pullback of this Witt vector extension along a (unique) k -homomorphism $f : \mathbf{G}_a \rightarrow \mathbf{G}_a$. In other words, there is a unique pullback diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{G}_a & \longrightarrow & U & \longrightarrow & \mathbf{G}_a \longrightarrow 0 \\ & & \parallel & & \downarrow f' & & \downarrow f \\ 0 & \longrightarrow & \mathbf{G}_a & \longrightarrow & W_2 & \longrightarrow & \mathbf{G}_a \longrightarrow 0 \end{array}$$

and we claim that if U is p -torsion then $f = 0$ (so the top row is a split sequence). Clearly $f'(U) \subset W_2[p]$, but the maximal smooth k -subgroup of $W_2[p]$ is the kernel term \mathbf{G}_a along the bottom row. Hence, $f'(U)$ is killed by the quotient map along the bottom row, so $f = 0$.

Now return to the setting of a general ground field k , and fix a k -subgroup inclusion of G into a vector group V , say with codimension c . Choose a linear structure on V (in the sense of Definition 1.1). Then $W \mapsto \text{Lie}(W)$ is a bijection between the set of linear subgroups of V and the set of linear subspaces of $\text{Lie}(V)$. Hence, if we choose W so that $\text{Lie}(W)$ is complementary to $\text{Lie}(G)$ then the natural map $G \rightarrow V/W$ is an isomorphism on Lie algebras, so it is an étale isogeny. Since W is a linear subgroup of V , the quotient V/W is a vector group over k . \square

Proposition 1.14. *Let V_1, \dots, V_n be k -groups isomorphic to \mathbf{G}_a , and let $V = \prod_{i=1}^n V_i$. Let U be a smooth k -subgroup of V such that U_{k_s} is the k_s -subgroup of V_{k_s} generated by images of k_s -scheme morphisms $\mathbf{A}_{k_s}^1 \rightarrow V_{k_s}$ that pass through 0.*

There exists a k -group automorphism $h : V \simeq V$ such that $h(U)$ is the direct product of some of the V_i inside of V . In particular, U is a vector group over k and is a k -group direct factor of V .

Proof. The proof is by induction on n and is trivial for $n = 1$. Now consider $n > 1$. The case $U = V$ is trivial, so we can assume $\dim U \leq n - 1$. First assume that $\dim U = n - 1 > 0$. By Corollary 1.7, U is the kernel of a k -homomorphism $f : V \rightarrow \mathbf{G}_a$. By hypothesis, there exists a non-constant k_s -scheme morphism $\mathbf{A}_{k_s}^1 \rightarrow U_{k_s}$, so by Lemma 1.10 (applied over k_s) and Lemma 1.12 there exists a k -group automorphism h' of V such that $h'(U) \supseteq V_1$. But then $h'(U) = V_1 \times U'$, where U' denotes the projection of $h'(U)$ into $V' = \prod_{i=2}^n V_i$. Applying the induction hypothesis to V' and U' , we are done.

Suppose now that $\dim U < n - 1$, and let U' denote the projection of U into the product V' as defined above. By the inductive hypothesis, after relabelling V_2, \dots, V_n there exists a k -group automorphism $h_1 : V' \rightarrow V'$ such that $h_1(U') = \prod_{i=2}^r V_i$ for some $r < n$. Setting

$$h' = \text{id}_{V_1} \times h_1 : V \simeq V,$$

we then have $h'(U) \subseteq \prod_{i=1}^r V_i$, and we can again apply induction. The proof is now complete. \square

Corollary 1.15. *In a smooth p -torsion commutative affine k -group G , every smooth k -subgroup that is a vector group is a k -group direct factor.*

Proof. This is a consequence of Proposition 1.14, provided that G is a k -subgroup of a vector group. Such an embedding is provided by Lemma 1.13. \square

The following proposition is a useful refinement of Lemma 1.13.

Proposition 1.16. *Let k be an infinite field of characteristic $p > 0$ and let U be a smooth p -torsion commutative affine k -group. Then U is k -isomorphic to a k -subgroup of codimension 1 in a k -vector group. In particular, U is isomorphic (as a k -group) to the zero scheme of a separable nonzero p -polynomial over k .*

This proposition is also true for finite k if U is connected since then U is a vector group; see Corollary 2.8.

Proof. By Lemma 1.13, U can be identified with a k -subgroup of a k -vector group V . Let $m = \dim V - \dim U$. If $m \leq 1$ then we are done by Corollary 1.7, so we assume $m > 1$. We will show that U can be embedded in a k -vector group W with $\dim W = \dim V - 1$, which will complete the argument via induction on m . The vector group W will arise as a quotient of V .

The k -linear subspace $\text{Lie}(U)$ in $\text{Lie}(V)$ has codimension m . Fix a choice of linear structure on V (in the sense of Definition 1.1). Since $m \geq 2$, the Zariski closure $\mathbf{G}_a.U (\subseteq V)$ of the image of the multiplication map $\mathbf{G}_a \times U \rightarrow V$ is a closed subscheme of V with nonzero codimension. By irreducibility of V , the union $\text{Lie}(U) \cup (\mathbf{G}_a.U)$ inside of V is a proper closed subscheme of V .

Since $V(k)$ is Zariski-dense in V (as k is infinite), there exists $v \in V(k)$ with

$$v \notin \text{Lie}(U) \cup (\mathbf{G}_a.U).$$

Let $L \subseteq V$ be the k -subgroup corresponding to the line $kv \subseteq V(k)$. Consider the canonical k -homomorphism $\phi : V \rightarrow W := V/L$, and let $\psi = \phi|_U$. We shall prove $\ker \psi = 1$, from which it follows that ψ identifies U with a k -subgroup of W .

It suffices to show that $\text{Lie}(\psi)$ is injective (so $\ker \psi$ is étale) and that $\psi|_{U(\bar{k})}$ is injective. The map $\text{Lie}(\psi)$ has kernel $L \cap \text{Lie}(U) = \{0\}$, so it is indeed injective. If $\psi|_{U(\bar{k})}$ is not injective then the line L would lie in $\mathbf{G}_a.U$ since $\mathbf{G}_a.U$ is stable under the \mathbf{G}_a -multiplication on V . But the point $v \in L(k)$ does not lie in $(\mathbf{G}_a.U)(\bar{k})$, due to how we chose v , so indeed $\psi|_{U(\bar{k})}$ is injective. \square

2. WOUND UNIPOTENT GROUPS

A smooth connected unipotent k -group U is analogous to an anisotropic torus if U does not contain \mathbf{G}_a as a k -subgroup. This concrete viewpoint is inconvenient for developing a general theory, but eventually we will prove that it gives the right concept. A more convenient definition to get the theory off the ground requires going beyond the category of k -groups, as follows.

Definition 2.1. A smooth connected unipotent k -group U is k -wound if every map of k -schemes $\mathbf{A}_k^1 \rightarrow U$ is a constant map to a point in $U(k)$. Equivalently, $U(k) = U(k[x])$.

By considering translation by k -points, it is equivalent to say that every map of pointed k -schemes $(\mathbf{A}_k^1, 0) \rightarrow (U, 1)$ is constant.

Remark 2.2. An analogous definition for tori using $\mathbf{A}^1 - \{0\}$ recovers the usual notion of anisotropy: if F is any field (possibly of characteristic 0) and T is an F -torus, then the condition $T(F) = T(F[x, 1/x])$ (i.e., the constancy of any F -scheme map $\mathbf{G}_m \rightarrow T$, or equivalently the triviality of any map of pointed F -schemes $(\mathbf{G}_m, 1) \rightarrow (T, 1)$) characterizes F -anisotropy of T .

Indeed, F -anisotropy is equivalent to the vanishing of $\mathrm{Hom}_{F\text{-gp}}(\mathbf{G}_m, T)$, so we just need to check that in general a map of pointed F -schemes $(\mathbf{G}_m, 1) \rightarrow (T, 1)$ is a homomorphism. By extending scalars we may assume $F = \bar{F}$, so T is a power of \mathbf{G}_m , and this reduces us to the case $T = \mathbf{G}_m$. An endomorphism of the pointed F -scheme $(\mathbf{G}_m, 1)$ is the “same” as an element $u \in F[x, 1/x]^\times$ satisfying $u(1) = 1$, and such units are precisely $u = x^n$ for $n \in \mathbf{Z}$.

The main reason that we go beyond the category of k -groups in this definition is due to the intervention of a non-homomorphic conjugation morphism φ' that arises in the proof of Proposition 3.2. The interested reader can easily check that all appearances of maps from \mathbf{A}^1 in §1–§2 can be replaced with homomorphisms from \mathbf{G}_a without affecting the proofs there.

Remark 2.3. The definition of wound makes sense in characteristic 0, where it is only satisfied by $U = 1$ (since a nontrivial smooth connected unipotent group in characteristic 0 contains \mathbf{G}_a as a subgroup over the ground field). Thus, although we only work with ground fields of positive characteristic, it is convenient in practice (for handling some trivialities) to make the convention that “wound” means “trivial” for smooth connected unipotent groups in characteristic 0.

Whereas anisotropy for a torus over a field is insensitive to purely inseparable extension of the ground field but is often lost under a separable algebraic extension of the ground field, the k -wound property will behave in the opposite manner: we will prove that it is insensitive to a separable extension on k (such as scalar extension from a global field to a completion), but it is often lost under a purely inseparable extension on k .

Example 2.4. Assume k is imperfect and choose $a \in k - k^p$. The k -group $\{y^p = x - ax^p\}$ becomes isomorphic to \mathbf{G}_a over the purely inseparable extension $k(a^{1/p})$ but by Example 1.3 it is k -wound. Observe that the isogeny $y : U \rightarrow \mathbf{G}_a$ is étale, so applying an étale isogeny can destroy the wound property. (Although y is étale, its extension to a degree- p finite flat covering $\bar{U} \rightarrow \mathbf{P}_k^1$ between regular compactifications is not étale: explicitly, at the point at infinity the ramification index is 1 but the residue field extension is $k(a^{1/p})/k$.) Hence, for problems involving wound unipotent groups one must be more attentive to the use of isogenies than is usually necessary when working with tori.

Note that the wound k -group U is a k -subgroup of the k -split group \mathbf{G}_a^2 . In the opposite direction, there also exist nontrivial k -split quotients of k -wound groups modulo smooth connected k -subgroups. For instance, in [Oes, Ch. V, 3.5] there is an example over any imperfect field k

of a 2-dimensional k -wound smooth connected p -torsion commutative affine group G admitting a 1-dimensional (necessarily k -wound) smooth connected k -subgroup G' such that $G/G' \simeq \mathbf{G}_a$ as k -groups.

Example 2.5. Assume k is infinite. By Corollary 1.7, smooth p -torsion commutative affine k -groups G are precisely the zero schemes of separable nonzero p -polynomials f over k . Since G is connected if and only if it is geometrically irreducible (as for any k -group scheme of finite type), we see that G is connected if and only if f is irreducible over k , as well as if and only if f is geometrically irreducible over k . Assume G is connected.

If the principal part f_{prin} of f has no zero on $k^n - \{0\}$ then by Lemma 1.10 it follows that G is k -wound. The converse is false, as we saw following the statement of Lemma 1.10. However, if f_{prin} has a zero on $k^n - \{0\}$ then the calculation in the proof of (2) \Rightarrow (3) in Lemma 1.10 (taking h to be the identity map of \mathbf{G}_a^n) shows that we can find a p -polynomial $F \in k[x_1, \dots, x_n]$ having zero scheme k -isomorphic to G as a k -group (so F is geometrically irreducible over k) with the sum of the degrees of the monomials appearing in F_{prin} strictly less than the corresponding sum for f_{prin} . Continuing in this way, we eventually arrive at a choice of f having zero scheme G (as a k -group) such that f_{prin} has no zeros on $k^n - \{0\}$. In this sense, the zero schemes of geometrically irreducible p -polynomials f over k for which f_{prin} has no nontrivial k -rational zero are precisely the p -torsion commutative k -wound groups (up to k -isomorphism).

Theorem 2.6. *Every smooth connected p -torsion commutative affine k -group U is a direct product $U = V \times W$ of a vector group V and a smooth connected unipotent k -group W such that W_{k_s} is k_s -wound. In this decomposition, the subgroup V is uniquely determined: V_{k_s} is generated by the images of k_s -scheme morphisms $\varphi : \mathbf{A}_{k_s}^1 \rightarrow U_{k_s}$ passing through the identity.*

Proof. By Galois descent, there is a unique smooth connected k -subgroup V of U such that V_{k_s} is generated by the images of k_s -scheme morphisms $\varphi : \mathbf{A}_{k_s}^1 \rightarrow U_{k_s}$ that pass through the identity. By Lemma 1.13, we can identify U with a k -subgroup of a vector group over k . Thus, by Proposition 1.14, V is a vector group over k and (by Corollary 1.15) we have $U = V \times W$ as k -groups for some k -subgroup W of U . Since U is a smooth connected unipotent k -group, so is its direct factor W . Clearly, W_{k_s} is k_s -wound (due to the definition of V).

Now we prove that V in this decomposition is unique. Consider any decomposition of k -groups $U = V' \times W'$, where V' is a vector group over k and W' is a smooth connected unipotent k -subgroup of U such that W'_{k_s} is k_s -wound. The image of any k_s -scheme morphism $\varphi : \mathbf{A}_{k_s}^1 \rightarrow U_{k_s}$ passing through the identity is contained in V'_{k_s} because otherwise the composite of φ and the canonical projection $U_{k_s} \rightarrow W'_{k_s}$ would be a non-constant k_s -scheme morphism from $\mathbf{A}_{k_s}^1$ to W'_{k_s} (contradicting that W'_{k_s} is assumed to be k_s -wound). Hence, $V \subseteq V'$, so $V' = V \times V'_1$ with V'_1 the image of the vector group V' under the projection $U \rightarrow W$. Since W_{k_s} is k_s -wound and V' is a vector group, $V'_1 = 0$. That is, $V' = V$. \square

In Theorem 2.6, the group W as an abstract k -group is unique up to isomorphism, since it is identified with the quotient U/V modulo the uniquely determined k -subgroup V . However, the decomposition of U as $V \times W$ is not unique. That is, there may be more than one k -homomorphic section to $U \rightarrow U/V = W$, or in other words $\text{Hom}_k(W, V)$ may be nontrivial. For example, over an imperfect field consider $U = \mathbf{G}_a^2 \times \mathbf{U}$ where \mathbf{U} is as in Example 1.3. Then $\text{Hom}_k(\mathbf{U}, \mathbf{G}_a^2)$ is nontrivial.

Corollary 2.7. *A smooth connected p -torsion commutative affine k -group U is k -wound if and only if U_{k_s} is k_s -wound, and also if and only if there are no nontrivial k -homomorphisms $\mathbf{G}_a \rightarrow U$. The k -group U is a vector group over k if and only if U_{k_s} is a vector group over k_s .*

Proof. This is immediate from Theorem 2.6. \square

Corollary 2.8. *If k is perfect then a smooth connected p -torsion commutative affine k -group is a vector group.*

Proof. By Corollary 2.7, we may assume that k is algebraically closed. This case is part of Lemma 1.13. \square

To get results on k -wound groups beyond the commutative p -torsion case, we need to study smooth connected p -torsion central k -subgroups in a general smooth connected unipotent k -group U . This is taken up in the next section.

Example 2.9. Let k be a field and let G be a commutative smooth connected affine k -group containing no nontrivial unipotent smooth connected k -subgroup. (By Exercise U.6, an example of such a G with $\mathcal{R}_u(G_{\bar{k}}) \neq 1$ is $R_{k'/k}(T')$ for a nontrivial purely inseparable finite extension k'/k and a k' -torus T' .) For the maximal k -torus T in G , consider the smooth connected commutative unipotent quotient $U = G/T$. We claim that U is k -wound. Since G_{k_s} is pseudo-reductive we may assume $k = k_s$, so T is k -split. By definition, we need to prove that any map of k -schemes $f : \mathbf{A}_k^1 \rightarrow U$ is constant.

Consider the pullback $G \times_U \mathbf{A}_k^1$. This is a T -torsor over \mathbf{A}_k^1 , so it is trivial since T is split and $\text{Pic}(\mathbf{A}_k^1) = 1$. A choice of splitting defines a k -scheme morphism $\tilde{f} : \mathbf{A}_k^1 \rightarrow G$ over f , so it suffices to prove that \tilde{f} is constant. Using a translation, we may assume $\tilde{f}(0) = 1$. We claim that for any smooth connected commutative k -group C and any k -scheme morphism $h : \mathbf{A}_k^1 \rightarrow C$ satisfying $h(0) = 1$, the smooth connected k -subgroup of C generated by the image of h is unipotent. Applying this to the commutative pseudo-reductive G would then force $\tilde{f} = 1$, as desired.

To prove our claim concerning C we may assume that k is algebraically closed, so C is a direct product of a torus and a unipotent group. Using projections to factors, it suffices to treat the case $C = \mathbf{G}_m$. In this case h is a nowhere-vanishing polynomial in one variable with value 1 at the origin, so $h = 1$.

3. THE $cckp$ -KERNEL

In a smooth connected unipotent k -group U , any two smooth connected p -torsion central k -subgroups generate a third such subgroup. Hence, the following definition makes sense.

Definition 3.1. The maximal smooth connected p -torsion central k -subgroup of U is the *cckp-kernel*.

Note that if $U \neq 1$ then its *cckp*-kernel is nontrivial, since the latter contains the *cckp*-kernel of the last nontrivial term of the descending central series of U . By Galois descent and specialization (as in the proof of [CGP, 1.1.9(1)]), the formation of the *cckp*-kernel commutes with any separable extension on k . However, its formation generally does *not* commute with purely inseparable extension on k ; see Exercise U.11(ii).

Proposition 3.2. *Let U be a smooth connected unipotent k -group, and let k'/k be a separable extension. Let F denote the *cckp*-kernel of U . Then U is k -wound if and only if $U_{k'}$ is k' -wound, and the quotient U/F is k -wound whenever U is k -wound. Also, the following conditions are equivalent:*

- (1) U is k -wound,
- (2) U does not have a central k -subgroup k -isomorphic to \mathbf{G}_a ,
- (3) the *cckp*-kernel F of U is k -wound.

This proposition implies that U is k -wound if and only if U admits no nontrivial k -homomorphism from \mathbf{G}_a . Such a characterization of the k -wound property is analogous to the characterization of anisotropic tori over a field in terms of homomorphisms from \mathbf{G}_m over the ground field.

Proof. Obviously (1) \Rightarrow (2). By Theorem 2.6, (2) and (3) are equivalent. Also, by specialization (as in the proof of [CGP, 1.1.9(1)]), if U_K is not K -wound for some separable extension K/k then the same holds with K/k taken to be some finite separable extension. Thus, to prove the equivalence of (1), (2), and (3) and the fact that $U_{k'}$ is k' -wound whenever U is k -wound, it suffices to show that if U_{k_s} is not k_s -wound then the $cckp$ -kernel F of U is not k -wound.

Let $\varphi : \mathbf{A}_{k_s}^1 \rightarrow U_{k_s}$ be a non-constant k_s -scheme morphism. Composing with a $U(k_s)$ -translation if necessary, we may assume $\varphi(0) = 1$. We may choose such a φ so that $\varphi(\mathbf{A}_{k_s}^1)$ is central. Indeed, suppose $\varphi(\mathbf{A}_{k_s}^1)$ is non-central, so U is not commutative and there exists $g \in U(k_s)$ not centralizing $\varphi(\mathbf{A}_{k_s}^1)$. The k_s -scheme morphism $\varphi' : \mathbf{A}_{k_s}^1 \rightarrow U_{k_s}$ defined by $\varphi'(x) = g^{-1}\varphi(x)^{-1}g\varphi(x)$ (which is generally not a homomorphism even when φ is a homomorphism) carries 0 to 1, so it is then non-constant, and its image lies in derived group $\mathcal{D}(U_{k_s}) = \mathcal{D}(U)_{k_s}$. The k -subgroup $\mathcal{D}(U)$ has smaller dimension than U and is nontrivial since the smooth connected k -group U is not commutative. Hence, by iteration with the descending central series, the required non-constant φ with $\varphi(\mathbf{A}_{k_s}^1)$ central is eventually obtained. We may also assume that $\varphi(\mathbf{A}_{k_s}^1)$ is p -torsion by replacing the original φ with $p^e \cdot \varphi$ for some $e \geq 0$.

The nontrivial k_s -subgroup generated by $\varphi(\mathbf{A}_{k_s}^1)$ then lies in the $cck_s p$ -kernel of U_{k_s} ; i.e., it lies in F_{k_s} . Thus F_{k_s} is not k_s -wound, so by Corollary 2.7 the k -group F is not k -wound.

It remains to show that if U is k -wound then U/F is k -wound. For this we may, in view of the preceding conclusions, assume that $k = k_s$. Suppose that U is k -wound and U/F is not k -wound. Thus, there exists a central k -subgroup A of U/F that is k -isomorphic to \mathbf{G}_a . Let π denote the canonical homomorphism $U \rightarrow U/F$. The k -subgroup scheme $\pi^{-1}(A)$ in U is an extension of A by F , so it is smooth, connected, and unipotent.

We claim that $\pi^{-1}(A)$ is central in U . If not, we get a non-constant k -scheme morphism $\varphi : \mathbf{A}_k^1 \rightarrow F$ (contradicting that U is k -wound) as follows. Choose $g \in U(k)$ not centralizing $\pi^{-1}(A)$ (recall $k = k_s$), identify \mathbf{G}_a with $A = \pi^{-1}(A)/F$, and define $\varphi : \pi^{-1}(A)/F \rightarrow F$ by $xF \mapsto gxg^{-1}x^{-1}$. Thus, $\pi^{-1}(A)$ is central in U . Similarly, $\pi^{-1}(A)$ is p -torsion because otherwise we would get a non-constant k -scheme morphism $\psi : \mathbf{A}_k^1 \rightarrow F$ via $\psi(xF) = x^p$. We have shown that $\pi^{-1}(A)$ lies in the $cckp$ -kernel F of U , so the given inclusion $F \subseteq \pi^{-1}(A)$ is an equality. Hence, $A = 1$, which is absurd since $A \simeq \mathbf{G}_a$. \square

Corollary 3.3. *Let U be a k -wound smooth connected unipotent k -group. Define the ascending chain of smooth connected normal k -subgroups $\{U_i\}_{i \geq 0}$ as follows: $U_0 = 1$ and U_{i+1}/U_i is the $cckp$ -kernel of the k -wound group U/U_i for all $i \geq 0$. These subgroups are stable under k -group automorphisms of U , their formation commutes with any separable extension of k , and $U_i = U$ for sufficiently large i .*

Moreover, if H is a smooth k -group equipped with an action on U then H carries each U_i back into itself.

Proof. Well-posedness of the definition (e.g., that U/U_1 is k -wound) and compatibility with separable extension on k follow from Proposition 3.2. By dimension considerations, $U_i = U$ for sufficiently large i since the $cckp$ -kernel of a nontrivial smooth connected unipotent k -group is nontrivial.

Finally, if H is a smooth k -group acting on U then we need to prove that H carries each U_i into itself. For this we may extend scalars to k_s , so k is separably closed. Then the H -stability of U_i

is equivalent to the $H(k)$ -stability of U_i , and this latter property is a special case of each U_i being stable under all k -automorphisms of U . \square

As an application of the structure of k -wound groups we can unify the definitions of wound for unipotent groups and anisotropy for tori (see Remark 2.2):

Corollary 3.4. *A unipotent smooth connected k -group U is k -wound if and only if $U(k) = U(k[x, 1/x])$. More generally, if $h \in k[x]$ is nonzero and separable then U is k -wound if and only if $U(k) = U(k[x][1/h])$.*

Proof. The equality $U(k) = U(k[x, 1/h])$ clearly forces U to be k -wound. For the converse, suppose U is k -wound, so U_{k_s} is k_s -wound (Proposition 3.2). Thus, to prove that $U(k[x][1/h]) = U(k)$ we may replace k with k_s (by Galois descent). Hence, now $h = \prod (x - a_i)$ for pairwise distinct $a_i \in k$. For each i , the k -wound property implies $U(k((x - a_i))) = U(k[[x - a_i]])$ by [Oes, V, §8] (whose proof rests on the existence of a composition series for the k -wound U with successive quotients that are commutative p -torsion wound hypersurface groups; see Corollary 3.3, Proposition 1.16, and Example 2.5). Writing $h = (x - a_i)q_i$, inside $k((x - a_i))$ we have $k[x][1/h] \cap k[[x - a_i]] = k[x][1/q_i]$. Thus, $U(k[x][1/h]) = \bigcap_i U(k[x][1/q_i]) = U(k[x])$ since $\gcd_i(q_i) = 1$, and $U(k[x]) = U(k)$. \square

Remark 3.5. It is well-known that if F is a local field and T is an F -torus then $T(F)$ is compact if and only if T is F -anisotropic. (To prove compactness of $T(F)$ for F -anisotropic T , identify $X(T_{F_s})$ with a quotient of a direct sum of copies of the regular representation of $\text{Gal}(F'/F)$ over \mathbf{Z} for a finite Galois splitting field F'/F of T . This identifies T with an F -subgroup of $\mathbf{R}_{F'/F}(\mathbf{G}_m)^N$ for some $N \geq 1$. By F -anisotropy, T lies in $(T_{F'/F}^1)^N$, where $T_{F'/F}^1$ is the F -torus $\ker(\mathbf{R}_{F'/F}(\mathbf{G}_m) \rightarrow \mathbf{G}_m)$ of “norm-1 units”. Since $T_{F'/F}^1(F) = \mathcal{O}_{F'}^\times$, we are done.)

There is a similar equivalence in the unipotent case, as follows. We restrict attention to unipotent smooth connected U over a local function field k , since in characteristic 0 the split condition always holds for unipotent groups and hence compactness cannot hold when the unipotent group is nontrivial. Over such k , the equivalence of k -woundness for U and compactness for $U(k)$ is [Oes, VI, §1] (whose proof ultimately reduces to an explicit calculation with wound hypersurface groups over $k = \mathbf{F}_q((t))$, using the “principal part” criterion at the end of Example 2.5).

Remark 3.6. The separability condition on h in Corollary 3.4 cannot be removed. For example, if $p = 2$ and $a \in k - k^2$ then the k -wound group $U = \{y^2 = x - ax^2\}$ is a smooth plane conic with $U(k) \neq \emptyset$, so U is k -rational. Explicitly, $\text{Spec } k[t, 1/(t^2 - a)] \simeq U$ via $t \mapsto (1/(t^2 - a), t/(t^2 - a))$.

We will now prove a structure theorem that is analogous to the unique presentation of a torus over a field as an extension of an anisotropic torus by a split torus.

Theorem 3.7. *Let U be a unipotent smooth connected k -group. There exists a unique smooth connected normal k -split k -subgroup $U_{\text{split}} \subset U$ such that U/U_{split} is k -wound.*

The subgroup U_{split} contains the image of every k -homomorphism from a k -split smooth connected unipotent k -group into U . Also, the kernel of every k -homomorphism from U into a k -wound smooth connected unipotent k -group contains U_{split} , and the formation of the k -subgroup U_{split} is compatible with any separable extension of k .

Proof. The proof is by induction on $\dim U$. If U is k -wound then $U_{\text{split}} = \{1\}$ satisfies the requirements and is unique as such. Assume that U is not k -wound, and let A be a smooth central k -subgroup isomorphic to \mathbf{G}_a (Proposition 3.2). Let $H = U/A$. By induction, there exists a smooth connected normal k -subgroup H_{split} in H with the desired properties in relation to H (in

the role of U). Let U_{split} be the corresponding subgroup of U containing A . It is k -split, and $U/U_{\text{split}} \simeq H/H_{\text{split}}$ is k -wound.

Let U' be a smooth connected unipotent k -group having a composition series

$$U' = U'_0 \supseteq U'_1 \supseteq \cdots$$

with successive quotients k -isomorphic to \mathbf{G}_a , and let $\varphi : U' \rightarrow U$ be a k -homomorphism. There exists a minimal i such that $\varphi(U'_i) \subseteq U_{\text{split}}$. If $i > 0$ then there is induced a k -homomorphism $\mathbf{G}_a \simeq U'_{i-1}/U'_i \rightarrow U/U_{\text{split}}$ with nontrivial image. This contradicts that U/U_{split} is k -wound. Thus, $i = 0$; i.e., $\varphi(U') \subseteq U_{\text{split}}$. It follows in particular that U_{split} is unique. Also, for any k -homomorphism $\varphi : U \rightarrow U''$ into a k -wound smooth connected unipotent k -group U'' we have $\varphi(U_{\text{split}}) \subseteq U''_{\text{split}} = \{1\}$. This says that $\ker \varphi$ contains U_{split} .

The last assertion of the theorem follows from Proposition 3.2. Indeed, if k'/k is a separable extension and $U' := U_{k'}$ then $(U_{\text{split}})_{k'} \subseteq U'_{\text{split}}$ and the k' -split quotient $U'_{\text{split}}/(U_{\text{split}})_{k'}$ is a k' -subgroup of the k' -group $(U/U_{\text{split}})_{k'}$ that is k' -wound (by Proposition 3.2). This forces $U'_{\text{split}} = (U_{\text{split}})_{k'}$. \square

Example 3.8. An elementary non-commutative example of Theorem 3.7 over any imperfect field k of characteristic $p > 0$ is obtained via a central pushout construction, as follows. Let $U_3 \subset \text{GL}_3$ be the standard upper triangular unipotent subgroup. Its scheme-theoretic center is the group $Z = \mathbf{G}_a$ of $u \in U_3$ that agree with the identity matrix except possibly in the upper-right entry. Viewing U_3 as a central extension of \mathbf{G}_a^2 by Z , let U be the pullback along the inclusion $y : U' \hookrightarrow \mathbf{G}_a^2$ where U' is a 1-dimensional k -wound group as in Example 2.4. A straightforward calculation shows that U is a non-commutative 2-dimensional smooth connected k -subgroup of U_3 that is neither split nor wound (since it contains a central $Z = \mathbf{G}_a$ and admits the wound quotient U'). Thus, $Z = U_{\text{split}}$ and the sequence $1 \rightarrow Z \rightarrow U \rightarrow U/Z \rightarrow 1$ cannot split since Z is the center of U and U/Z is commutative.

Corollary 3.9. *A unipotent smooth connected k -group U is k -split if and only if $U \simeq \mathbf{A}_k^n$ as k -schemes for some $n \geq 0$. It is also equivalent for there to be a dominant k -morphism $V = \mathbf{A}_k^d - Z \rightarrow U$ for a generically smooth closed subscheme $Z \subset \mathbf{A}_k^d$.*

By Remark 3.6, we cannot remove the generic smoothness condition on Z . Also, Corollary 3.9 has no analogue for tori, since any torus T over any field F is unirational (by using an isogeny-splitting of the inclusion of F -tori $T \hookrightarrow \mathbf{R}_{F'/F}(T_{F'})$ for a finite separable splitting field F'/F of T). Finally, we note that although the proof of sufficiency of the second criterion in Corollary 3.9 uses Bertini's Theorem in the affine setting over k_s , the only Z that actually use in later applications is a (possibly empty) union of hyperplane slices in distinct coordinate directions, for which linear algebra works equally well in place of Bertini's Theorem.

Proof. First assume that U is k -split, and let $n = \dim U$. We seek to prove that $U \simeq \mathbf{A}_k^n$ as k -schemes. The cases $n \leq 1$ are obvious, so we may assume $n > 1$. Thus, there is a k -split smooth connected normal k -subgroup $U' \subset U$ such that $U/U' \simeq \mathbf{G}_a$. By induction, $U' \simeq \mathbf{A}_k^{n-1}$ as k -schemes. We claim that the U' -torsor $U \rightarrow \mathbf{G}_a = \mathbf{A}_k^1$ for the étale topology is trivial. More generally, for any affine k -scheme X the cohomology set $H^1(X_{\text{ét}}, U')$ classifying U' -torsors for the étale topology on X is trivial. Indeed, using a composition series for U' over k reduces this to the case of $H^1(X_{\text{ét}}, \mathbf{G}_a)$, which coincides with $H^1(X_{\text{Zar}}, \mathcal{O}) = 0$ (as X is affine). We conclude that as k -schemes, $U \simeq U' \times (U/U') \simeq \mathbf{A}_k^n$, as desired.

For the converse, suppose that there is a dominant k -morphism $f : V = \mathbf{A}_k^d - Z \rightarrow U$ for a generically smooth closed subscheme $Z \subset \mathbf{A}_k^d$. To prove that U is k -split, we may replace U with

the k -wound quotient U/U_{split} from Theorem 3.7 to reduce to the case that U is k -wound. In such cases we seek to prove that $U = 1$, so it suffices to prove that the dominant f is a constant map into $U(k)$. It is harmless to extend scalars to k_s , so $V(k)$ is Zariski-dense in V . Since Z is generically smooth, by Bertini's Theorem over k there exists a dense open locus Ω in the $2(d-1)$ -dimensional quasi-projective variety \mathbf{Gr}_d of affine lines in \mathbf{A}_k^d such that the closed subscheme $Z_K \cap \ell$ in ℓ is 0-dimensional and K -smooth for all K/k and lines ℓ in K^d corresponding to a point in $\Omega(K)$. (If Z is a union of several affine hyperplanes then linear algebra gives the same conclusion, without using Bertini's Theorem.) Such a closed subscheme is K -étale, so for each affine line $\ell \simeq \mathbf{A}_k^1$ corresponding to a point in $\Omega(k)$ the open locus $V \cap \ell$ in ℓ is the complement of the zero locus on ℓ of a separable polynomial. Hence, by Corollary 3.4 and the k -wound hypothesis on U , f has constant restriction to $V \cap \ell$ for all $\ell \in \Omega(k)$.

To prove the constancy of f , it suffices to prove the constancy of f on $V'(k)$ for a dense open $V' \subset V$ (since $k = k_s$). The idea is that for a generic pair of distinct points v and v' in V , the line ℓ joining them should correspond to a point in Ω and hence the constancy of f on $V \cap \ell$ forces $f(v) = f(v')$. To make this idea rigorous, consider the $2d$ -dimensional variety $X = V \times V - \Delta$ of ordered pairs of distinct points in V . There is an evident morphism $X \rightarrow \mathbf{Gr}_d$ assigning to any $(v, v') \in X$ the unique line joining them, and all fibers are 2-dimensional, so for dimension reasons this map is dominant. Hence, there is a dense open locus $X' \subset X$ that is carried into Ω . For all $(v, v') \in X'(k)$, the unique line $\ell \subset k^d$ passing through v and v' corresponds to a point in $\Omega(k)$, so f is constant on $V \cap \ell$. In particular, $f(v) = f(v')$. The projection $\text{pr}_1 : X' \rightarrow V$ is dominant, so its image contains a dense open subset of V . We may choose $v_0 \in V(k)$ in this image, so the open subset $V' := X' \cap (\{v_0\} \times V)$ in V (via pr_2) is non-empty and therefore dense. Clearly $f(v') = f(v_0)$ for all $v' \in V'(k)$. \square

Remark 3.10. The cohomological proof that $U \simeq \mathbf{A}_k^n$ as k -schemes for k -split unipotent smooth connected k -groups U generalizes to show that any k -split solvable smooth connected k -group is k -isomorphic to $\mathbf{A}_k^{n,m} := \mathbf{A}_k^n \times (\mathbf{A}_k^1 - \{0\})^m$ for some $n, m \geq 0$. The only change required to incorporate \mathbf{G}_m -factors into the argument is the observation that the affine schemes $\mathbf{A}^{n,m}$ satisfy $H^1((\mathbf{A}_k^{n,m})_{\text{ét}}, \mathbf{G}_m) = \text{Pic}(\mathbf{A}_k^{n,m}) = 1$ (the first equality following from descent theory for line bundles).

Corollary 3.11. *If G is a k -split solvable smooth connected affine k -group then $\mathcal{D}(G)$ is k -split.*

Proof. By the structure theory over \bar{k} , $\mathcal{D}(G)$ is unipotent. Hence, by Corollary 3.9 it suffices to construct a dominant k -morphism $\mathbf{A}_k^n - Z \rightarrow \mathcal{D}(G)$ for some $n \geq 1$ and some geometrically reduced closed subscheme $Z \subset \mathbf{A}_k^n$. Since the product of several varieties $\mathbf{A}_k^{n_i} - Z_i$ with generically smooth Z_i has the form $\mathbf{A}_k^{\sum n_i} - Z$ for a generically smooth closed subscheme Z , and the geometric points of $\mathcal{D}(G)$ can be expressed as a product of a universally bounded number of commutators (depending on G), by considering such a product morphism for a sufficiently large set of commutators we are reduced to constructing a dominant k -morphism $\mathbf{A}_k^N - Z \rightarrow G$ for some $N \geq 1$ and generically smooth Z . By Remark 3.10 there is a k -scheme isomorphism $\mathbf{A}_k^{n,m} \simeq G$, so we are done. \square

Let G be a smooth connected affine k -group. The k -unipotent radical $\mathcal{R}_{u,k}(G)$ is the maximal normal unipotent smooth connected k -subgroup of G , and the k -split unipotent radical $\mathcal{R}_{us,k}(G)$ is the maximal normal k -split unipotent smooth connected k -subgroup of G . For any extension field K/k clearly $\mathcal{R}_{u,k}(G)_K \subset \mathcal{R}_{u,K}(G_K)$ inside G_K . This inclusion is an equality when K/k is separable [CGP, 1.1.9(1)], but generally not otherwise (e.g., for a nontrivial purely inseparable extension k'/k and G equal to the Weil restriction $R_{k'/k}(\mathbf{G}_m)$ we have $\mathcal{R}_{u,k}(G) = 1$ but $\mathcal{R}_{u,k'}(G_{k'}) = \mathbf{G}_a^{p-1}$; see [CGP, 1.1.3, 1.6.3]).

Corollary 3.12. *For any smooth connected affine k -group G , the inclusion $\mathcal{R}_{us,k}(G) \subset \mathcal{R}_{u,k}(G)_{\text{split}}$ is an equality. In particular, $\mathcal{R}_{u,k}(G)/\mathcal{R}_{us,k}(G)$ is k -wound, and the formation of $\mathcal{R}_{us,k}(G)$ commutes with separable extension on k .*

Proof. By Galois descent, $\mathcal{R}_{us,k_s}(G_{k_s})$ descends to a smooth connected unipotent normal k -subgroup of G . Such a descent is k -split, since the k -split property of smooth connected unipotent k -groups is insensitive to separable extension on k (due to Theorem 3.7). Thus, the descent is contained in $\mathcal{R}_{us,k}(G)$, so the inclusion $\mathcal{R}_{us,k}(G)_{k_s} \subseteq \mathcal{R}_{us,k_s}(G_{k_s})$ is an equality. In other words, the formation of $\mathcal{R}_{us,k}(G)$ is compatible with separable algebraic extension on k . Hence, to prove the compatibility with general separable extension on k and the agreement with the maximal k -split smooth connected k -subgroup of $\mathcal{R}_{u,k}(G)$, we may assume that $k = k_s$. But $\mathcal{R}_{u,k}(G)_{\text{split}}$ is a characteristic k -subgroup of G , so it is normal due to the Zariski-density of $G(k)$ in G when $k = k_s$. This proves that $\mathcal{R}_{u,k}(G)_{\text{split}} \subseteq \mathcal{R}_{us,k}(G)$, so equality holds.

The compatibility of the formation of $\mathcal{R}_{us,k}(G)$ with respect to separable extension on k now follows from such a compatibility for two constructions: the formation of U_{split} in Theorem 3.7, and the formation of $\mathcal{R}_{u,k}(G)$ [CGP, 1.1.9(1)]. \square

4. TORUS ACTIONS ON UNIPOTENT GROUPS

Consider the action of a k -torus T on a smooth connected unipotent k -group U . This induces a linear representation of T on $\text{Lie}(U)$, so if T is k -split then we get a weight space decomposition of $\text{Lie}(U)$. If U is a vector group then it is natural to wonder if this decomposition of $\text{Lie}(U)$ can be lifted to the group U . When $\dim U > 1$, the T -action may not respect an initial choice of linear structure on U (in the sense of Definition 1.1) since $\text{char}(k) = p > 0$, so we first seek a T -equivariant linear structure.

For example, if $U = \mathbf{G}_a^2$ with its usual linear structure and $T = \mathbf{G}_m$ with the action $t.(x, y) = (tx, (t^p - t)x^p + ty)$ then the T -action is not linear and the action on $\text{Lie}(U) = k^2$ has the single weight given by the identity character of T . But note that if we transport the T -action by the additive automorphism $(x, y) \mapsto (x, y + x^p)$ of U then the action becomes $t.(x, y) = (tx, ty)$, which is linear.

Tits proved rather generally that if a k -split T acts on U with only nontrivial weights on $\text{Lie}(U)$, then there are nontrivial constraints on the possibilities for U as a k -group and that (after passing to a suitable characteristic composition series for U) the action can always be described in terms of linear representations of T . To explain his results in this direction, we begin with the following proposition that generalizes Lemma 1.13 by incorporating a torus action.

Proposition 4.1. *Let U be a smooth p -torsion commutative affine k -group equipped with an action by an affine k -group scheme T of finite type. There exists a linear representation of T on a finite dimensional k -vector space V and a T -equivariant isomorphism of U onto a k -subgroup of V .*

Proof. Let $\mathbf{Hom}(U, \mathbf{G}_a)$ be the covariant functor assigning to any k -algebra R the R -module $\text{Hom}_R(U_R, \mathbf{G}_a)$ of R -group morphisms $\phi : U_R \rightarrow \mathbf{G}_a$ (with R -module structure defined via the R -linear structure on the R -group \mathbf{G}_a). There is a natural R -linear injection $\mathbf{Hom}(U, \mathbf{G}_a)(R) \hookrightarrow R[U_R] = R \otimes_k k[U]$ defined by $\phi \mapsto \phi^*(x)$ (where x is the standard coordinate on \mathbf{G}_a), and its image is the R -submodule of “group-like” elements: those f satisfying $m_R^*(f) = f \otimes 1 + 1 \otimes f$ (where $m : U \times U \rightarrow U$ is the group law). This is an R -linear condition on f and is functorial in R , so by k -flatness the R -module of group-like elements over R is J_R where $J \subset k[U]$ is the k -subspace of group-like elements over k . In particular, the natural map $R \otimes_k \text{Hom}(U, \mathbf{G}_a) \rightarrow \text{Hom}_R(U_R, \mathbf{G}_a)$ is an isomorphism.

The (left) T -action on U defines a left T -action on $\mathbf{Hom}(U, \mathbf{G}_a)$ (via $(t.\phi)(u) = \phi(t^{-1}.u)$) making the k -linear inclusion $\mathbf{Hom}(U, \mathbf{G}_a) \hookrightarrow k[U]$ a T -equivariant map. Thus, $\mathbf{Hom}(U, \mathbf{G}_a)$ is the directed union of T -stable finite-dimensional k -subspaces, due to the same property for $k[U]$ [Bo, 1.9–1.10]. By Lemma 1.13 there is a k -subgroup inclusion $j : U \hookrightarrow \mathbf{G}_a^n$ for some $n \geq 1$. Let $W \subset \mathbf{Hom}(U, \mathbf{G}_a)$ be a T -stable finite-dimensional k -subspace containing $j^*(x_1), \dots, j^*(x_n)$. The canonical map $U \rightarrow W^* = \text{Spec}(\text{Sym}(W))$ is a T -equivariant closed immersion that is a k -homomorphism (since W consists of group-like elements in $k[U]$ that generate $k[U]$ as a k -algebra). \square

We now apply our work with wound groups to analyze the structure of smooth connected unipotent k -groups equipped with a sufficiently nontrivial action by a k -torus.

Proposition 4.2. *Let T , U , and V be as in Proposition 4.1, with T a k -torus, and let $V = V_0 \times V'$ be the unique T -equivariant k -linear decomposition of V with $V_0 = V^T$ (so V' is the span of the isotypic k -subspaces for the nontrivial irreducible representations of T over k that occur in V). The product map*

$$\iota : (U \cap \underline{V}_0) \times (U \cap \underline{V}') \rightarrow U$$

is an isomorphism and there is a T -equivariant k -linear decomposition $V' = V'_1 \times V'_2$ of V' and a T -equivariant k -automorphism α of the additive k -group \underline{V} such that

$$\alpha(U) = (\alpha(U) \cap \underline{V}_0) \times \underline{V}'_1.$$

In particular, if $V^T = 0$ then the k -group U is a vector group admitting a T -equivariant linear structure.

Proof. Clearly $\underline{V}_0 = Z_{\underline{V}}(T)$ as k -subgroups of \underline{V} , so $U_0 := U \cap \underline{V}_0$ is $Z_U(T)$. This is smooth since U is smooth. We will first prove that ι is an isomorphism, so $U \cap \underline{V}'$ is smooth.

Since the formation of V' clearly commutes with scalar extension on k , to establish that ι is an isomorphism we may assume k is algebraically closed. Choose $s \in T(k)$ such that for every weight χ of T in V' , $\chi(s) \neq 1$. Consider the k -linear map $f : V \rightarrow V$ defined by $f(v) = s \cdot v - v$. It is obvious that f maps V onto V' with $\ker f = Z_V(s) = V_0$ and that the restriction of f to V' is a linear automorphism. The image $f(U)$ is a smooth k -subgroup of \underline{V}' , and it lies in U due to the T -stability of U inside of \underline{V} . By definition, \underline{V}' has a T -equivariant composition series whose successive quotients are 1-dimensional vector groups with a nontrivial T -action. Hence, all T -stable k -subgroup schemes of \underline{V}' are connected. In particular, $f(U)$ is connected.

Since $U_0 \cap f(U) = 0$ (as $\underline{V}_0 \cap \underline{V}' = 0$), under addition $U_0 \times f(U)$ is a k -subgroup of U . Thus, $f : U \rightarrow f(U)$ is a map onto a k -subgroup of U and the restriction of this map to $f(U)$ is therefore an endomorphism $f(U) \rightarrow f(U)$ with trivial kernel. But $f(U)$ is smooth and connected, so this endomorphism is an automorphism. In other words, $f : U \rightarrow f(U)$ is a projector up to an automorphism of $f(U)$. Since $U \cap \ker f = U \cap \underline{V}_0 = U_0$, this shows that the k -subgroup inclusion $U_0 \times f(U) \hookrightarrow U$ is an isomorphism, so $f(U) = U \cap \underline{V}'$. This completes the proof that ι is an isomorphism.

Let $U' = U \cap \underline{V}'$ and define $V'_1 = \text{Lie}(U')$. Then V'_1 is a T -stable k -linear subspace of V' . Complete reducibility of k -linear representations of T provides a T -stable k -linear complement V'_2 of V'_1 in V' . Using the decomposition $\underline{V}' = \underline{V}'_1 \times \underline{V}'_2$, the projection $U' \rightarrow \underline{V}'_1$ is an isomorphism on Lie algebras, so it is étale. By T -equivariance, the finite étale kernel is T -stable and therefore centralized by the connected T . But $Z_{\underline{V}'}(T) = 0$, so this kernel vanishes. In other words, $U' \rightarrow \underline{V}'_1$ is an isomorphism. It follows that the k -subgroup $U' \subseteq \underline{V}' = \underline{V}'_1 \times \underline{V}'_2$ is the graph of a T -equivariant k -homomorphism $g : \underline{V}'_1 \rightarrow \underline{V}'_2$. The T -equivariant k -automorphism α of \underline{V} may be taken to be the automorphism that is the identity on \underline{V}_0 and is the inverse of the map $(v_1, v_2) \mapsto (v_1, g(v_1) + v_2)$ on $\underline{V}'_1 \times \underline{V}'_2$. \square

Theorem 4.3. *Let T be a k -torus and U a smooth p -torsion commutative affine k -group. Suppose that there is given an action of T on U over k . Then $U = U_0 \times U'$ with $U_0 = Z_U(T)$ and U' a T -stable k -subgroup that is a vector group admitting a linear structure relative to which T acts linearly. Moreover, U' is uniquely determined and is functorial in U .*

Proof. By Propositions 4.1 and 4.2 we get the existence of U' . To prove the uniqueness and functoriality of U' , we may assume $k = k_s$. Under the decomposition of U' into weight spaces relative to a T -equivariant linear structure on U' , all T -weights must be nontrivial due to the definition of U_0 . Hence, the canonical map $T \times U \rightarrow U$ defined by $(t, u) \mapsto t.u - u$ has image U' . This proves the uniqueness and functoriality of U' . \square

If U in Theorem 4.3 is k -wound, then it must coincide with U_0 and so have trivial T -action. This is a special case of the following general consequence of invariance of the wound property with respect to separable extension of the ground field (Proposition 3.2):

Corollary 4.4. *Let T be a k -torus and U a k -wound smooth connected unipotent k -group. The only T -action on U is the trivial one.*

Proof. Our aim is to prove that the k -subgroup scheme $Z_U(T)$ is equal to U . For the k -group $G = U \rtimes T$, we have that the torus centralizer $Z_G(T)$ is equal to $Z_U(T) \rtimes T$. But $Z_G(T)$ is smooth and connected, so the same holds for $Z_U(T)$. Since $Z_U(T)$ is a scheme-theoretic centralizer, $\text{Lie}(Z_U(T))$ is the T -centralizer in $\text{Lie}(U)$. Hence, to prove that $Z_U(T) = U$ it suffices (by smoothness and connectedness of U) to prove that T acts trivially on $\text{Lie}(U)$.

By Proposition 3.2, we may extend scalars to k_s , so T is k -split. Consider the composition series $\{U_i\}$ from Corollary 3.3. This is T -equivariant, and each U_{i+1}/U_i is k -wound, commutative, and p -torsion. The Lie algebras $\text{Lie}(U_i)$ provide a T -equivariant filtration on $\text{Lie}(U)$ whose successive quotients are the $\text{Lie}(U_{i+1}/U_i)$'s. By complete reducibility for the T -action on $\text{Lie}(U)$, to prove triviality of the action it suffices to treat the successive quotients of a T -stable composition series of k -subspaces of $\text{Lie}(U)$. Hence, it suffices to treat each U_{i+1}/U_i separately in place of U , so we may assume that the k -wound U is commutative and p -torsion. Applying the decomposition in Theorem 4.3, we have $U = Z_U(T) \times U'$ where U' is a vector group. Since U is wound, we conclude that $U' = 1$, so the T -action on U is trivial. \square

5. SOLVABLE GROUPS

By Theorem 3.7, if U is a unipotent smooth connected k -group then there is a unique k -split smooth connected k -subgroup U_{split} such that U/U_{split} is k -wound. For tori the analogous assertion using an anisotropic quotient is elementary. Is there a common generalization for solvable smooth connected affine k -groups G ? This rests on the following common generalization of the wound condition in the unipotent case and the anisotropy condition for tori:

Definition 5.1. A solvable smooth connected affine k -group G is k -wound if $G(k) = G(k[x, 1/x])$.

By Remark 2.2, if G is a torus then this definition coincides with k -anisotropy. By Corollary 3.4, if G is unipotent then this coincides with Definition 2.1.

Remark 5.2. A delicate aspect of Definition 5.1 is that it is generally poorly behaved with respect to any nontrivial extension of the ground field. More specifically, in the unipotent case the separable extensions preserve the wound property and the inseparable ones can destroy it, whereas in the torus case the purely inseparable extensions preserve the wound (anisotropy) property and the separable ones can destroy it.

For a k -wound solvable smooth connected affine k -group G , it is obvious that any smooth connected k -subgroup is k -wound and that if G' is a k -split solvable smooth connected affine k -group then $\mathrm{Hom}_{k\text{-gp}}(G', G) = 1$ (e.g., argue by induction on $\dim G'$, using a composition series over k whose successive quotients are \mathbf{G}_a or \mathbf{G}_m). In particular, if we drop the k -wound hypothesis on G then there is at most one k -split smooth connected k -subgroup $G_s \subset G$ such that G/G_s is k -wound.

Since any quotient of a k -split solvable smooth connected affine k -group is k -split, it is elementary that there exists a unique maximal k -split *normal* smooth connected k -subgroup $G_{\mathrm{split}} \subset G$. By [Bo, 15.4(i)], G_{split} is the semidirect product of a k -split torus against a k -split unipotent smooth connected normal k -subgroup of G_{split} . (This will be reproved in the next result.)

The only possibility for G_s is G_{split} , so G_s exists if and only if G/G_{split} is k -wound (in which case G_{split} remains maximal in G even without the normality requirement as a k -subgroups of G). The main result of this section is:

Theorem 5.3. *For any solvable smooth connected affine k -group G , the k -group G/G_{split} is a central extension of a k -wound unipotent group by a k -wound torus (so G/G_{split} is k -wound). In particular, G is k -wound if and only if $G_{\mathrm{split}} = 1$. The k -group G_{split} is the semi-direct product of a maximal k -split torus against a normal k -split unipotent smooth connected k -subgroup.*

The natural map $G \rightarrow G/G_{\mathrm{split}}$ is initial among k -homomorphisms from G to k -wound solvable smooth connected affine k -groups and the natural map $G_{\mathrm{split}} \rightarrow G$ is final among k -homomorphisms to G from k -split smooth connected affine k -groups.

Example 5.4. If F is a perfect field (perhaps of characteristic 0) and G is a solvable smooth connected affine F -group then $G = T \rtimes U$ for an F -torus T and an F -split unipotent smooth connected F -group U . Thus, $G_{\mathrm{split}} := T_{\mathrm{split}} \rtimes U$ is an F -split normal smooth connected F -subgroup such that $G/G_{\mathrm{split}} = T/T_{\mathrm{split}}$ is an F -anisotropic F -torus. It follows that Theorem 5.3 is only interesting when k is imperfect. Likewise, Theorem 5.3 is only nontrivial when $\mathcal{R}_u(G_{\bar{k}})$ is not defined over k as a \bar{k} -subgroup of $G_{\bar{k}}$ (e.g., $G = \mathbf{R}_{k'/k}(\mathbf{G}_m)$ for a nontrivial purely inseparable finite extension k'/k).

Remark 5.5. Although Definition 5.1 goes beyond the category of k -groups (using k -scheme morphisms from $\mathbf{A}_k^1 - \{0\}$), it is natural to wonder if it can be expressed within the category of k -groups, as in the case of tori and unipotent groups. That is, if G is a solvable smooth connected affine k -group and $\mathrm{Hom}_{k\text{-gp}}(\mathbf{G}_a, G) = 1$ and $\mathrm{Hom}_{k\text{-gp}}(\mathbf{G}_m, G) = 1$ (equivalently, $\mathrm{Hom}_{k\text{-gp}}(G', G) = 1$ for all k -split solvable smooth connected affine k -groups G') then is G a k -wound group? This will be immediate once we prove that G/G_{split} is always k -wound.

An obvious but useful reformulation of Definition 5.1 is that the specialization homomorphism $G(k[x, 1/x]) \rightarrow G(k)$ at $x = 1$ has trivial kernel. For example, this immediately implies:

Lemma 5.6. *Let $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$ be an exact sequence of solvable smooth connected k -groups. If G' and G'' are k -wound then so is G .*

The converse of Lemma 5.6 fails in the commutative unipotent case, as we noted in Example 2.4.

Lemma 5.7. *Let U be a k -split unipotent smooth connected k -group, and M a (finite type) k -group scheme of multiplicative type. Any exact sequence of affine finite type k -groups*

$$1 \rightarrow M \rightarrow G \rightarrow U \rightarrow 1$$

is uniquely split: $G = M \times U$ as k -groups.

Proof. By the uniqueness claim and Galois descent, we may and do assume $k = k_s$. Hence, M is Cartier dual to a finitely generated \mathbf{Z} -module (so M is a k -subgroup of a k -torus). The uniqueness

of the splitting amounts to the assertion that $\mathrm{Hom}_{k\text{-gp}}(U, M) = 1$, which is obvious (e.g., use an inclusion of M into a k -torus). For the existence, we first note that G must be a central extension of U by M , since the conjugation action of $G/M = U$ on the commutative normal subgroup M defines a homomorphism from U to the automorphism functor of M , and any such homomorphism is trivial since U is connected whereas $\underline{\mathrm{Aut}}_{M/k}$ is represented by a constant k -group. Thus, we aim to prove the triviality of the pointed set $\mathrm{Ex}_k(U, M)$ of central extensions of U by M (in the category of affine k -group schemes of finite type).

By using a composition series of U over k with successive quotients isomorphic to \mathbf{G}_a , the low-degree δ -functoriality involving $\mathrm{Hom}_{k\text{-gp}}(\cdot, M)$ and $\mathrm{Ex}_k(\cdot, M)$ (or direct bare-hands arguments with exact sequences and splittings thereof) reduces the problem to the case $U = \mathbf{G}_a$. That is, we seek to prove the vanishing of $\mathrm{Ex}_k(\mathbf{G}_a, M)$. Since M is a product of \mathbf{G}_m 's and μ_n 's, by low-degree δ -functoriality considerations in the second variable (rather than the first) it suffices to separately treat the cases $M = \mu_n$ and $M = \mathbf{G}_m$. The Kummer sequence $1 \rightarrow \mu_n \rightarrow \mathbf{G}_m \rightarrow \mathbf{G}_m \rightarrow 1$ and the vanishing of $\mathrm{Hom}_{k\text{-gp}}(\mathbf{G}_a, \mathbf{G}_m)$ reduce us to the special case $M = \mathbf{G}_m$. That is, we want $\mathrm{Ex}(\mathbf{G}_a, \mathbf{G}_m) = 1$.

If G is a central extension of \mathbf{G}_a by \mathbf{G}_m then the commutator of G factors through a bi-additive pairing $\mathbf{G}_a \times \mathbf{G}_a \rightarrow \mathbf{G}_m$. The only such pairing is the trivial one, since $\mathrm{Hom}_{K\text{-gp}}(\mathbf{G}_a, \mathbf{G}_m) = 1$ for any field extension K/k (such as the function field of \mathbf{G}_a). Thus, G must be commutative. By viewing G as a \mathbf{G}_m -torsor over the affine line (for the étale topology, and hence the Zariski topology due to descent theory for line bundles), we see that the quotient map $\pi : G \rightarrow \mathbf{G}_a$ admits a k -scheme section σ . Using translation by a point in $\mathbf{G}_m(k) = (\ker \pi)(k)$ we can arrange that $\sigma(0) = e \in G(k)$. Hence, the resulting identification of G with the pointed k -scheme $(\mathbf{G}_m \times \mathbf{A}_k^1, (1, 0))$ carries the group law on G over to a composition law $(c, x) \cdot (c', x') = (cc'f(x, x'), x + x')$ for a symmetric polynomial $f : \mathbf{A}_k^2 \rightarrow \mathbf{G}_m$ satisfying $f(0, 0) = 1$. The only such f is $f = 1$. \square

Proof of Theorem 5.3. There are no nontrivial k -homomorphisms from a k -split solvable smooth connected affine k -group to a k -wound solvable smooth connected affine k -group, so the only task is to prove that G/G_{split} and G_{split} have the asserted structures.

First consider $H = G_{\mathrm{split}}$. The derived group $\mathcal{D}(H)$ is unipotent (as we may check over \bar{k}) and k -split (Corollary 3.11), and any maximal k -torus of H maps isomorphically onto a maximal k -torus of $H/\mathcal{D}(H)$. Thus, to prove that H is a semidirect product of a maximal k -split torus against a normal k -split unipotent subgroup U (in which case the k -torus H/U is k -split, so all maximal k -tori in H are k -split), we may pass to the k -split commutative $C = H/\mathcal{D}(H)$. This has a unique maximal k -torus T and the quotient $U = C/T$ is k -split unipotent, so by Lemma 5.7 there exists a unique decomposition $C = T \times U$. Thus, T is a quotient of the k -split C , so it is k -split.

It remains to understand the structure of G/G_{split} , which is to say that we can assume $G_{\mathrm{split}} = 1$. By Lemma 5.6 it remains to show that G is a central extension of a k -wound unipotent group by a k -wound torus. Since G is solvable, $\mathcal{D}(G)$ is unipotent (as we may check over \bar{k}). Thus, the formation of $\mathcal{D}(G)_{\mathrm{split}}$ commutes with separable extension on k (even though such extension may ruin the hypothesis that $G_{\mathrm{split}} = 1$). By computing with $G(k_s)$ -conjugation on $\mathcal{D}(G)_{k_s}$, it follows that $\mathcal{D}(G)_{\mathrm{split}}$ is normal in G . But we have arranged that $G_{\mathrm{split}} = 1$, so $\mathcal{D}(G)_{\mathrm{split}} = 1$. Hence, by the structure theory in the unipotent case, $\mathcal{D}(G)$ is k -wound.

Let T be a maximal k -torus in G . Since $\mathcal{D}(G)$ is k -wound unipotent, the conjugation action by T on $\mathcal{D}(G)$ is trivial by Corollary 4.4. But T maps isomorphically onto its image \bar{T} in the commutative $G/\mathcal{D}(G)$ (due to the unipotence of $\mathcal{D}(G)$), so the k -subgroup $T \times \mathcal{D}(G)$ in G is *normal*. Thus, the conjugation action by $G(k_s)$ on the normal k_s -subgroup $T_{k_s} \times \mathcal{D}(G)_{k_s}$ of G_{k_s} must preserve the unique maximal k_s -torus T_{k_s} , so T is normal in G . The connectedness of G then forces T to be

central in G . Since $G_{\text{split}} = 1$, it also follows that $T_{\text{split}} = 1$, which is to say that T is k -anisotropic. Note that the formation of T as the maximal central torus commutes with scalar extension on k , even though such scalar extension may ruin the anisotropy property of T .

The quotient $U = G/T$ now makes sense and is unipotent. It remains to prove that U is k -wound. By the structure theory in the unipotent case, it suffices to show that $U_{\text{split}} = 1$. The preimage G' of U_{split} in G is an extension of U_{split} by T , so by Lemma 5.7 there is a unique k -group decomposition $G' = U_{\text{split}} \times T$. The formation of G' commutes with scalar extension to k_s , as does the formation of $U_{\text{split}} \subset U$, so the same holds for the unique subgroup of G' isomorphically lifting U_{split} . That is, the unique product decomposition of G' commutes with scalar extension to k_s , so consideration of $G(k_s)$ -conjugation on G'_{k_s} shows that U'_{split} is normal in G . But $G_{\text{split}} = 1$, so $U'_{\text{split}} = 1$. \square

In Remark 3.10, we saw that every k -split solvable smooth connected k -group is isomorphic as a k -scheme to $\mathbf{A}_k^{n,m} = \mathbf{A}_k^n \times (\mathbf{A}_k^1 - \{0\})^m$ for some $n, m \geq 0$. Here is a converse result for solvable groups in the spirit of the splitting criterion for unipotent groups in Corollary 3.9.

Corollary 5.8. *A solvable smooth connected k -group G is k -split if and only if there is a dominant k -morphism $f : \mathbf{A}_k^{n,m} \rightarrow G$.*

Proof. The implication “ \Rightarrow ” was shown in Remark 3.10, and for the converse we will use Theorem 5.3. Assuming such an f exists, to prove that G is split we may compose f with the quotient map $G \rightarrow G/G_{\text{split}}$ to reduce to the case that G is k -wound, so G is an extension of a k -wound unipotent smooth connected k -group U by a k -anisotropic torus T . Our aim is to prove that $G = 1$. The composite map $\mathbf{A}_k^{n,m} \rightarrow U$ is dominant, so U is k -split by Corollary 3.9. But U is k -wound, so $U = 1$. That is, $G = T$ is a k -anisotropic torus.

Since the units in $k[x_1^{\pm 1}, \dots, x_m^{\pm 1}]$ are precisely the monomials $\prod x_i^{e_i}$ with $e_i \in \mathbf{Z}$, the same argument as in Remark 2.2 shows that any k -morphism $\mathbf{A}_k^{0,m} = (\mathbf{A}_k^1 - \{0\})^m \rightarrow T$ is a constant map to some $t \in T(k)$. Thus, the case $n = 0$ is settled. The anisotropy has done its work, as it now suffices to show that for any k -torus T whatsoever and any k -morphism $f : \mathbf{A}_k^{n,m} \rightarrow T$, there is a (unique) factorization of f through the projection $\mathbf{A}_k^{n,m} \rightarrow \mathbf{A}_k^{0,m}$. This says that $f^* : k[T] \rightarrow k[\mathbf{A}_k^{n,m}]$ lands in the Laurent polynomial subalgebra $k[\mathbf{A}_k^{0,m}]$, for which it is harmless to check after extending scalars to k_s or even \bar{k} . Now $T = (\mathbf{G}_m)^N$ for some $N \geq 0$, so we are reduced to the case $T = \mathbf{G}_m$. Any unit on $\mathbf{A}_k^{n,m}$ is the pullback of a unit on $\mathbf{A}_k^{0,m}$, so we are done. \square

We end our discussion with some applications to general smooth connected affine k -groups G . Our interest is in variants of the k -subgroups $\mathcal{R}_{u,k}(G)$ and $\mathcal{R}_{us,k}(G)$ considered in Corollary 3.12. Define the k -radical $\mathcal{R}_k(G)$ to be the maximal normal solvable smooth connected k -subgroup of G , and the k -split radical $\mathcal{R}_{s,k}(G)$ to be the maximal normal k -split solvable smooth connected k -subgroup of G . Obviously $G/\mathcal{R}_k(G)$ has trivial k -radical and $G/\mathcal{R}_{s,k}(G)$ has trivial k -split radical. Beware that $G/\mathcal{R}_k(G)$ may *not* be equal to its own derived group (in contrast with $G_{\bar{k}}/\mathcal{R}(G_{\bar{k}})$). Equivalently, there exist G such that $\mathcal{R}_k(G) = 1$ but $G \neq \mathcal{D}(G)$; see [CGP, 11.2.1] for many such G over any imperfect field k .

A *pseudo-reductive* k -group is a smooth connected affine k -group G such that $\mathcal{R}_{u,k}(G) = 1$. By Galois descent, $\mathcal{R}_k(G)_{k_s} = \mathcal{R}_{k_s}(G_{k_s})$ and $\mathcal{R}_{us,k}(G)_{k_s} = \mathcal{R}_{us,k_s}(G_{k_s})$. There is no analogue of these equalities for $\mathcal{R}_{s,k}$.

Proposition 5.9. *Let G be a smooth connected affine k -group. Then $\mathcal{R}_k(G) = 1$ if and only if G is pseudo-reductive and has no nontrivial central k -torus.*

Proof. In either direction, G is pseudo-reductive, so we may and do assume that G is pseudo-reductive. Since pseudo-reductivity is inherited by smooth connected normal k -subgroups (as explained near the beginning of [CGP, 1.1]), $\mathcal{R}_k(G)$ is solvable and pseudo-reductive. But a solvable pseudo-reductive group is commutative [CGP, 1.2.3], so $\mathcal{R}_k(G)$ is commutative. The unique maximal k -torus S in $\mathcal{R}_k(G)$ must be normal in G and hence central (due to the connectedness of G), and $S \neq 1$ if $\mathcal{R}_k(G) \neq 1$ since $\mathcal{R}_k(G)$ cannot be unipotent when it is nontrivial (due to the pseudo-reductivity of $\mathcal{R}_k(G)$). Since any central k -torus in G lies in $\mathcal{R}_k(G)$, S is the maximal central k -torus in G . Thus, $\mathcal{R}_k(G) = 1$ if and only if $S = 1$. \square

Clearly $\mathcal{R}_{s,k}(G) \subset \mathcal{R}_k(G)_{\text{split}}$, and it is natural to wonder if this containment is an equality. It is equivalent to ask if the k -radical of $G/\mathcal{R}_{s,k}(G)$ is k -wound, or if $\mathcal{R}_k(G)_{\text{split}}$ is normal in G . In the proof of the unipotent analogue in Corollary 3.12 it was harmless to extend scalars to k_s , but that technique is not available in the present circumstances. Nonetheless, the equality holds:

Proposition 5.10. *For smooth connected affine k -group G , the inclusion $\mathcal{R}_{s,k}(G) \subset \mathcal{R}_k(G)_{\text{split}}$ is an equality and if $\mathcal{R}_{us,k}(G) = 1$ then $\mathcal{R}_{s,k}(G)$ is the maximal central k -split torus in G .*

In particular, $\mathcal{R}_{s,k}(G) = 1$ if and only if $\mathcal{R}_{us,k}(G) = 1$ and G contains no nontrivial k -split central k -torus.

Proof. We may replace G with $G/\mathcal{R}_{us,k}(G)$ so that $\mathcal{R}_{us,k}(G) = 1$. Let $R := \mathcal{R}_k(G)_{\text{split}}$ and $U = \mathcal{R}_{u,k}(R)$, so U is k -split and R is the semi-direct product of a k -split torus against U . As a first step, we claim that U is normal in G (thereby forcing $U = 1$, since $\mathcal{R}_{us,k}(G) = 1$, so R is a torus).

Since $U_{k_s} = \mathcal{R}_{u,k_s}(R_{k_s})$, the normality of R in $\mathcal{R}_k(G)$ forces U_{k_s} to be stable under $\mathcal{R}_k(G)(k_s)$ -conjugation on R_{k_s} . Hence, U is normal in $\mathcal{R}_k(G)$. But $\mathcal{R}_k(G)/R$ is k -wound, so $U = \mathcal{R}_{us,k}(\mathcal{R}_k(G))$. The formation of \mathcal{R}_k and $\mathcal{R}_{us,k}$ commute with scalar extension to k_s , so U_{k_s} is stable under $G(k_s)$ -conjugation on $\mathcal{R}_k(G)_{k_s}$ and hence U is normal in G as desired. As we noted above, this implies that R is a torus.

The k -split torus R is normal in $\mathcal{R}_k(G)$, hence central in $\mathcal{R}_k(G)$. The maximal central k -torus $T \subset \mathcal{R}_k(G)$ is normal in G (as we may check over k_s), so it is central in G . Hence, R is the maximal central k -split torus in G . In particular, R is normal in G , so $R = \mathcal{R}_{s,k}(G)$. \square

Corollary 5.11. *Let G be a non-solvable smooth connected affine k -group. The following three conditions are equivalent:*

- (1) $G/\mathcal{R}_{s,k}(G)$ contains a nontrivial k -split solvable smooth connected k -subgroup,
- (2) $G/\mathcal{R}_k(G)$ contains a nontrivial k -split solvable smooth connected k -subgroup,
- (3) G contains a proper pseudo-parabolic k -subgroup.

In (1) and (2) it is equivalent to contain \mathbf{G}_m as a non-central k -subgroup.

The notion of pseudo-parabolicity is defined in [CGP, 2.2.1]; it coincides with parabolicity in the connected reductive case [CGP, 2.2.9]. A typical example of a pseudo-parabolic k -subgroup that is not parabolic is $P := R_{k'/k}(P') \subset R_{k'/k}(G') =: G$ for a nontrivial purely inseparable finite extension k'/k and a proper parabolic k' -subgroup P' in a connected reductive k' -group G' . (Such P are precisely the pseudo-parabolic k -subgroups of G , by [CGP, 11.4.4]. The non-parabolicity of P , which is to say the non-properness of $G/P \simeq R_{k'/k}(G'/P')$, follows from [CGP, A.5.6] since $\dim G'/P' > 0$.) By [CGP, 2.2.10], condition (3) is equivalent to the same for the maximal pseudo-reductive quotient $G/\mathcal{R}_{u,k}(G)$, and if G is pseudo-reductive then (3) is equivalent to saying that G has no nontrivial non-central k -split torus [CGP, 2.2.3(1)].

Proof. The kernel $\mathcal{R}_k(G)/\mathcal{R}_{s,k}(G) = \ker(G/\mathcal{R}_{s,k}(G) \rightarrow G/\mathcal{R}_k(G))$ is k -wound since $\mathcal{R}_{s,k}(G) = \mathcal{R}_k(G)_{\text{split}}$ (Proposition 5.10), so a nontrivial k -homomorphism from \mathbf{G}_a or \mathbf{G}_m to $G/\mathcal{R}_{s,k}(G)$ yields a nontrivial composite homomorphism to $G/\mathcal{R}_k(G)$. Hence, (1) implies (2).

To prove that (2) implies (3), we may replace G with $G/\mathcal{R}_{u,k}(G)$, so G is pseudo-reductive. The hypothesis in (2) says that the pseudo-reductive $G/\mathcal{R}_k(G)$ contains \mathbf{G}_a or \mathbf{G}_m as a k -subgroup. By [CGP, C.3.8], if a pseudo-reductive k -group contains \mathbf{G}_a as a k -subgroup then it contains a non-central \mathbf{G}_m as a k -subgroup. Since \mathbf{G}_m as a k -subgroup of $G/\mathcal{R}_k(G)$ cannot be central (as $\mathcal{R}_k(G/\mathcal{R}_k(G)) = 1$), it suffices to prove that if $G/\mathcal{R}_k(G)$ contains a non-central \mathbf{G}_m then so does G . The preimage H in G of such a \mathbf{G}_m is a smooth k -subgroup, so a maximal k -torus T in H must map onto this \mathbf{G}_m . Hence, T contains a k -subgroup \mathbf{G}_m that is not in $\mathcal{R}_k(G)$ and thus is not central in G . The existence of a non-central \mathbf{G}_m in the pseudo-reductive k -group G is equivalent to (3), by [CGP, 2.2.3(2)].

Finally, we show that (3) implies (1). It is harmless to replace G with $G/\mathcal{R}_{us,k}(G)$, so $\mathcal{R}_{us,k}(G) = 1$. Thus, $\mathcal{R}_{s,k}(G)$ is the maximal k -split central k -torus in G (Proposition 5.10), so $G/\mathcal{R}_{s,k}(G)$ contains no non-trivial normal k -split k -tori (as a normal k -split k -torus in $G/\mathcal{R}_{s,k}(G)$ has preimage in G that is a k -split normal k -torus, and such a normal torus must be central due to connectedness of G , contradicting the maximality of $\mathcal{R}_k(G)$). From the definition of pseudo-parabolicity, (3) implies that G contains a non-central \mathbf{G}_m . Its image in $G/\mathcal{R}_{s,k}(G)$ is a non-central k -subgroup isomorphic to \mathbf{G}_m . \square

6. EXERCISES ON UNIPOTENT GROUPS

U.1. Let k be a field, U_n the standard strictly upper-triangular unipotent k -subgroup of GL_n .

(i) Prove that no nontrivial k -group scheme is isomorphic to a closed k -subgroup of both \mathbf{G}_a and \mathbf{G}_m . (If $\mathrm{char}(k) = p > 0$, the key is to prove that μ_p is not a k -subgroup of \mathbf{G}_a .) Deduce that the scheme-theoretic intersection $T \cap U_n$ is trivial for any k -torus T in GL_n .

(ii) Using Lie–Kolchin theorem, prove that $T \cap U = 1$ for any k -torus T and unipotent smooth connected k -group U in a linear algebraic k -group G .

U.2. Let k be a nonzero commutative ring, and let $E = \mathrm{End}_{k\text{-gp}}(\mathbf{G}_a)$, an associative ring containing k via $c.x = cx$ ($c \in k$).

(i) If k is a \mathbf{Q} -algebra, prove $k = E$. Deduce that the functor $\underline{\mathrm{Aut}}_{\mathbf{Q}\text{-gp}}(\mathbf{G}_a^n)$ on \mathbf{Q} -schemes is represented by GL_n .

(ii) If $x \mapsto x^n$ lies in E with $n > 1$, prove that k is a $\mathbf{Z}/n\mathbf{Z}$ -algebra and n is a prime power.

(iii) If k is an \mathbf{F}_p -algebra prove that $E = \{\sum c_j t^{pj} \mid c_j \in k\}$. Now see Exercise U.8(iv).

U.3. Let G be a group of finite type over a field k .

(i) Prove that $(G_{\bar{k}})_{\mathrm{red}}$ is a closed \bar{k} -subgroup of $G_{\bar{k}}$, and prove it is *smooth*. Deduce that the identity component G^0 is *geometrically irreducible*.

(ii) Over any imperfect field k , one can make a non-reduced k -group G such that G_{red} is *not* a k -subgroup. (See [CGP, Ex. A.8.3(i)].) Where does an attempted proof to the contrary get stuck?

(iii) Assume k is imperfect, $\mathrm{char}(k) = p > 0$. For $a \in k - k^p$, define a natural k -group structure on

$$G = \{x_0^p + ax_1^p + \cdots + a^{p-1}x_{p-1}^p = 1\}$$

(hint: consider the kernel of $N_{k(a^{1/p})/k} : R_{k(a^{1/p})/k}(\mathbf{G}_m) \rightarrow \mathbf{G}_m$, using Weil restriction as in Exercise U.4). Show that G is reduced but $G_{\bar{k}}$ is non-reduced, so the reduced k -group G is *not* smooth.

(iv) Prove that the condition $t^n = 1$ defines a finite closed k -subgroup $\mu_n \subseteq \mathbf{G}_m$, and show its preimage G under $\det : \mathrm{GL}_N \rightarrow \mathbf{G}_m$ is a k -subgroup of GL_N . Prove $G^0 = \mathrm{SL}_N$, and for $k = \mathbf{Q}$ and $n = 5$ prove that $G - G^0$ is connected but over \bar{k} has 4 connected components.

U.4. This exercise develops the important concept of *Weil restriction of scalars* in the affine case (see [BLR, 7.6] and [CGP, A.5, A.7] for further information). It is an analogue of viewing a complex manifold as a real manifold with twice the dimension (and “complex points” become “real points”). Let k be a ring, k' a k -algebra that is finite and locally free as a k -module, and X' an affine k' -scheme of finite type. Consider the functor $R_{k'/k}(X') : A \rightsquigarrow X'(k' \otimes_k A)$ on k -algebras.

(i) By considering $X' = \mathbf{A}_{k'}^n$ and then any X' via a closed immersion into an affine space, prove that this functor is represented by an affine k -scheme of finite type, again denoted $R_{k'/k}(X')$. Prove its formation naturally commutes with products in X' , and compute $R_{k'/k}(\mathbf{G}_m)$ inside $R_{k'/k}(\mathbf{A}_{k'}^1)$. What if $k' = 0$? What if $k' = \prod k'_i$ for k -algebras k'_i ?

(ii) Prove $R_{k'/k}(\mathrm{Spec} k') = \mathrm{Spec} k$, and explain why $R_{k'/k}(X')$ is naturally a k -group when X' is a k' -group. In case k is a complete valued field and k' is an extension field equipped with the extended valuation, prove that the k -analytic topology on $R_{k'/k}(X')(k)$ coincides with the k' -analytic topology on $X'(k')$.

(iii) Use the infinitesimal criterion to prove that $R_{k'/k}(X')$ is k -smooth when X' is k' -smooth. (Can you see it directly from the construction?) Warning: if k'/k is a finite extension of fields that is not separable then $R_{k'/k}(X')$ can be empty (resp. disconnected) even when X' is non-empty (resp. geometrically integral)!

(iv) For a k -algebra K , prove that $R_{k'/k}(X')_K \simeq R_{K'/K}(X'_{K'})$ for $K' = k' \otimes_k K$. If k'/k is a separable extension of fields, prove $R_{k'/k}(X')_{k_s} \simeq \prod_{\sigma} \sigma^*(X')$ with σ varying through $\text{Hom}_k(k', k_s)$. Transfer the natural $\text{Gal}(k_s/k)$ -action on the left over to the right and describe it. This recovers Weil's definition of Weil restriction of scalars (in the affine case), from which we see that if k'/k is a separable extension of fields then the functor $R_{k'/k}$ preserves the following conditions on smooth affine groups: connected (see Exercise U.7(ii)), torus, unipotent, reductive, semisimple.

(v) Assume $\text{char}(k) = p > 0$ and let k'/k be a non-trivial purely inseparable extension of fields. For $n > 0$, compute $R_{k'/k}(\text{GL}_n)(\bar{k})$ and rigorously prove that $G = R_{k'/k}(\text{GL}_n)$ is not reductive by relating the geometric unipotent radical to the filtration on $k' \otimes_k \bar{k}$ by powers of its nilpotent maximal ideal. (See Exercise U.6(i) for an elementary approach when $n = 1$.) Use the universal property of Weil restriction to show that G contains no nontrivial unipotent smooth connected normal k -subgroup! (See [CGP, 1.1.3, 1.6.1] for more surprises with non-étale Weil restriction.)

U.5. Let U be a unipotent smooth connected commutative group scheme over a field k , and assume U is p -torsion if $\text{char}(k) = p > 0$.

(i) If $\text{char}(k) > 0$ and U is k -split, use Corollary 1.15 to prove that U is a vector group.

(ii) Assume $\text{char}(k) = 0$. Prove that any short exact sequence $0 \rightarrow \mathbf{G}_a \rightarrow G \rightarrow \mathbf{G}_a \rightarrow 0$ with commutative G is split. Deduce that $U \simeq \mathbf{G}_a^N$, and prove that any action on U by a k -split torus T respects this linear structure. Also prove that every unipotent k -group is k -split.

(iii) Prove that any commutative extension of \mathbf{G}_a by \mathbf{G}_m is uniquely split over k . (Hint: first make a scheme splitting using that $\text{Pic}(\mathbf{G}_a) = 1$.)

U.6. Let k'/k be a purely inseparable finite extension of a field k of characteristic $p > 0$.

(i) Prove that $U = R_{k'/k}(\mathbf{G}_m)/\mathbf{G}_m$ is smooth and connected of dimension $[k' : k] - 1$, and is $[k' : k]$ -torsion. Deduce it is unipotent.

(ii) Prove that $R_{k'/k}(\mathbf{G}_m)(k_s)[p] = 1$. Deduce that $R_{k'/k}(\mathbf{G}_m)$ contains no nontrivial unipotent smooth connected k -subgroup and that U in (i) does not contain \mathbf{G}_a as a k -subgroup! (Hint: use Exercise U.5(iii).) For a salvage, see [CGP, B.1.10]: a p -torsion smooth connected commutative affine group over any field of characteristic $p > 0$ admits an étale isogeny onto a vector group.

(iii) Generalize (i) and (ii) with \mathbf{G}_m replaced by any nontrivial k' -torus T' . (Hint: use the equality $\text{Gal}(k_s/k) = \text{Gal}(k'_s/k')$ to descend T' to a k -torus, and extend scalars to k_s to reduce problems to the case of \mathbf{G}_m .)

U.7. (i) Prove that if a connected scheme X of finite type over a field k has a k -rational point, then $X_{k'} = X \otimes_k k'$ is connected for every finite extension k'/k (hint: $X_{k'} \rightarrow X$ is open and closed; look at a fiber over $X(k)$). Deduce that $X_{k'}$ is connected for *every* extension k'/k (i.e., X is *geometrically connected* over k).

(ii) Prove that if X and Y are geometrically connected of finite type over k , so is $X \times Y$; give a counterexample over $k = \mathbf{Q}$ if “geometrically” is removed. Deduce that if G is a k -group then the identity component G^0 is a k -subgroup whose formation commutes with any extension on k .

U.8. Let X be a scheme over a field k , and $x \in X(k)$. Recall that $\text{Tan}_x(X)$ is identified as a set with the fiber of $X(k[\epsilon]) \rightarrow X(k)$ over x . Let $k[\epsilon, \epsilon'] = k[t, t']/(t, t')^2$, so this is 3-dimensional with basis $\{1, \epsilon, \epsilon'\}$.

(i) For $c \in k$, consider the k -algebra endomorphism of $k[\epsilon]$ defined by $\epsilon \mapsto c\epsilon$. Show that the resulting endomorphism of $X(k[\epsilon])$ over $X(k)$ restricts to scalar multiplication by c on $\text{Tan}_x(X)$.

(ii) Using the two natural quotient maps $k[\epsilon, \epsilon'] \twoheadrightarrow k[\epsilon]$ via killing ϵ or ϵ' , define a natural map

$$X(k[\epsilon, \epsilon']) \rightarrow X(k[\epsilon]) \times_{X(k)} X(k[\epsilon])$$

and prove it is bijective. Using the natural quotient map $k[\epsilon, \epsilon'] \rightarrow k[\epsilon]$ defined by $\epsilon, \epsilon' \mapsto \epsilon$, show that the resulting map

$$X(k[\epsilon]) \times_{X(k)} X(k[\epsilon]) \xrightarrow{\sim} X(k[\epsilon, \epsilon']) \rightarrow X(k[\epsilon])$$

induces addition on $\mathrm{Tan}_x(X)$: the *functor of points* of X encodes the k -linear structure on $\mathrm{Tan}_x(X)$.

(iii) For a k -group (G, e) , relate addition on $\mathrm{Tan}_e(G)$ to the group law as follows: for $m : G \times G \rightarrow G$, show that $\mathrm{Tan}_e(G) \times \mathrm{Tan}_e(G) = \mathrm{Tan}_{(e,e)}(G \times G) \rightarrow \mathrm{Tan}_e(G)$ is addition.

(iv) For a representable functor $F \simeq \mathrm{Hom}_k(\cdot, X)$ on the category of k -schemes, if the natural map $\varinjlim F(A_i) \rightarrow F(\varinjlim A_i)$ is bijective for any directed system of k -algebras $\{A_i\}$ then X is locally of finite type over k . Prove this if X is affine, and for the general case see 8.14.2 in EGA IV₃. In particular, for such F the fibers of $F(k[\epsilon]) \rightarrow F(k)$ are *finite-dimensional* over k when equipped with their natural k -linear structure as given in functorial terms in (i) and (ii). Using this, deduce from Exercise U.2(iii) that $\underline{\mathrm{End}}_{k\text{-gp}}(\mathbf{G}_a)$ and $\underline{\mathrm{Aut}}_{k\text{-gp}}(\mathbf{G}_a^n)$ are *not* representable when $\mathrm{char}(k) = p > 0$.

U.9. Let U be a unipotent smooth connected affine group over a field k with $\mathrm{char}(k) = p > 0$. Let k'/k be a finite purely inseparable extension such that $U_{k'}$ is k' -split.

(i) Choose $n \geq 1$ such that $k'^{p^n} \subset k$. Prove that the n -fold base change $U^{(p^n)}$ through the p^n -power endomorphism of k is k -split.

(ii) For any $m \geq 1$ and any k -group scheme G , prove that the m -fold relative Frobenius morphism $F_{G/k, m} : U \rightarrow G^{(p^m)}$ over k is a k -homomorphism, and that it is a finite flat surjection when G is also smooth (see Exercise 1.6.7 in the lecture notes on reductive group schemes). Use $F_{U/k, n}$ to prove that if $U \neq 1$ then $\mathrm{Hom}_{k\text{-gp}}(U, \mathbf{G}_a) \neq 1$.

U.10. Let U be a unipotent smooth connected group over a field k . Recall from [Bo, 11.5(2), 10.5] that there exists a closed immersion of k -groups $j : U \hookrightarrow U_N$ for some $N > 0$. Assume k is *perfect*.

(i) If G is a k -group scheme of finite type, prove that G_{red} is a smooth k -subgroup of G . (Note by Exercise U.3 that perfectness must be used to prove that G_{red} is a k -subgroup scheme.)

(ii) If $\dim U = 1$, construct a finite Galois extension k'/k such that $U_{k'} \simeq \mathbf{G}_a$. (Hint: by perfectness, \bar{k} is exhausted by subfields that are finite Galois over k). Using the vanishing of $H^1(\mathrm{Gal}(k'/k), k'^{\times})$ to build a k -isomorphism $U \simeq \mathbf{G}_a$.

(iii) Using a “ \mathbf{G}_a composition series” for U_N and induction on $\dim U$ (and (i)), construct a composition series for U consisting of smooth connected k -subgroups such that the successive quotients are 1-dimensional (and unipotent). Deduce with the help of (ii) that U is k -split.

U.11. Let k be an imperfect field of characteristic $p > 0$. Let k''/k be a purely inseparable finite extension such that $k''^{p^2} \subset k$ and $k' := k'' \cap k^{1/p} \neq k''$. Let $U = \mathrm{R}_{k''/k}(\mathbf{G}_m)/\mathbf{G}_m$.

(i) For any smooth connected affine k' -group G' , prove that the natural map $\mathrm{R}_{k'/k}(G')_{k'} \rightarrow G'$ defined functorially on k' -algebras by $G'(k' \otimes_k A') \rightarrow G'(A')$ is a smooth surjection with k' -split unipotent smooth connected kernel. Describe $(U_{k'})_{\mathrm{split}}$ and $U_{k'}/(U_{k'})_{\mathrm{split}}$, and show each is p -torsion and nontrivial. Deduce that $U_{k'} \rightarrow U_{k'}/(U_{k'})_{\mathrm{split}}$ has no k' -homomorphic section, so $U_{k'}$ is *not* a direct product of split and wound k' -groups.

(ii) Show that $(U_{k'})_{\mathrm{split}}$ is the *ckp*-kernel of $U_{k'}$ whereas $\mathrm{R}_{k'/k}(\mathbf{G}_m)/\mathbf{G}_m$ is the *ckp*-kernel of U . (Hint: compute on k'_s -points and k_s -points respectively.) Why does this illustrate failure of the formation of the *ckp*-kernel to commute with non-separable extension on k , and why is the non-smoothness of the p -torsion a necessary condition for any such example?

(iii) Does there exist a unipotent smooth connected k -group that is not an extension of a k -split group by a k -wound group, perhaps even a commutative example?

(iv) Does there exist a non-commutative k -wound group?

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