The Comparison Isomorphisms C_{cris}

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July 9, 2009

Contents

1 Introduction

Let $p > 0$ denote a prime integer and let k be a perfect field of characteristic p. Put $\mathcal{O}_K := W(k)$ and let K be its fraction field. Denote by $\overline{K}$ a fixed algebraic closure of K and set $G_K := \mathrm{Gal}(\overline{K}/K)$. We write B_{cris} for the crystalline period ring defined by J.-M. Fontaine in [Fo]. Recall that B_{cris} is a topological ring, endowed with a continuous action of G_K, an exhaustive decreasing filtration $\mathrm{Fil}^\bullet B_{\mathrm{cris}}$ and a Frobenius operator φ.

Following the paper [AI2], these notes aim at presenting a new proof of the so called *crystalline conjecture* formulated by J.-M. Fontaine in [Fo]. Let X be a smooth proper scheme, geometrically irreducible, of relative dimension d over $\mathcal{O}_K$:

Conjecture 1.1 ([Fo]). *For every $i \geq 0$ there is a canonical and functorial isomorphism commuting with all the additional structures (namely, filtrations, G_K–actions and Frobenii)*

$$\mathrm{H}^i(X^{\mathrm{et}}_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \cong \mathrm{H}^i_{\mathrm{cris}}(X_k/\mathcal{O}_K) \otimes_{\mathcal{O}_K} B_{\mathrm{cris}}.$$

It follows from a classical theorem for crystalline cohomology, see [BO], that $\mathrm{H}^i_{\mathrm{cris}}(X_k/\mathcal{O}_K) \cong \mathrm{H}^i_{\mathrm{dR}}(X/\mathcal{O}_K)$. The latter denotes the hypercohomology of the de Rham complex $0 \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_{X/\mathcal{O}_K} \longrightarrow \Omega^2_{X/\mathcal{O}_K} \longrightarrow \cdots \Omega^d_{X/\mathcal{O}_K} \longrightarrow 0$. The first interpretation provides a Frobenius operator, the second one provides a filtration, the Hodge filtration. For every $n \in \mathbb{N}$ we put $\mathrm{Fil}^n \mathrm{H}^i_{\mathrm{dR}}(X/\mathcal{O}_K) = \mathrm{H}^i_{\mathrm{dR}}(X/\mathcal{O}_K)$ if $n \leq 0$, to be 0 if $n \geq d+1$ and to be the image of the hypercohomology of the complex $0 \longrightarrow \Omega^n_{X/\mathcal{O}_K} \longrightarrow \Omega^2_{X/\mathcal{O}_K} \longrightarrow \cdots \Omega^d_{X/\mathcal{O}_K} \longrightarrow 0$ if $0 \leq n \leq d$. Then, in the statement of the conjecture

1) Frobenius on the left is defined by the identity operator on $\mathrm{H}^i(X^{\mathrm{et}}_{\overline{K}}, \mathbb{Q}_p)$ and by Frobenius on B_{cris}. Frobenius on the right hand side is defined by Frobenius on $\mathrm{H}^i_{\mathrm{cris}}(X_k/\mathcal{O}_K)$ and by Frobenius on B_{cris};

2) the filtration on the left hand side is defined by $\mathrm{H}^i(X^{\mathrm{et}}_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{Fil}^n B_{\mathrm{cris}}$, while on the right hand side it is given by the composite filtration $\mathrm{Fil}^n\big(\mathrm{H}^i_{\mathrm{dR}}(X/\mathcal{O}_K) \otimes_{\mathcal{O}_K} B_{\mathrm{cris}}\big) := \sum_{a+b=n} \mathrm{Fil}^a \mathrm{H}^i_{\mathrm{dR}}(X/\mathcal{O}_K) \otimes_{\mathcal{O}_K} \mathrm{Fil}^b B_{\mathrm{cris}}$;

3) the Galois action on the left hand side is induced by the action of G_K on $\mathrm{H}^i(X^{\mathrm{et}}_{\overline{K}}, \mathbb{Q}_p)$ and on B_{cris}. The Galois action on the right hand side is given by letting G_K act trivially on $\mathrm{H}^i_{\mathrm{cris}}(X_k/\mathcal{O}_K)$ and through its natural action on B_{cris}.

The conjecture is now a theorem, proven by G. Faltings in [F2]. In fact it holds without assuming that K is unramified and with non-constant coefficients. But these assumptions will simplify our considerations. There are various approaches to the proof of the conjecture. One (and the first) is based on ideas of Fontaine and Messing in [FM] using the syntomic cohomology on X; a full proof (for constant coefficients) using these methods was given by T. Tsuji in [T]. There is also an approach (for constant coefficients) based on a comparison isomorphism in K-theory which is due to W. Niziol [N]. We will follow the approach by Faltings, not in its original version but using a certain topology described in [F3]. Our approach, which works also for non constant coefficients, is based on the papers [AI2] and [AB]. The strategy consists in defining a new cohomology theory associated to X and proving that it computes both the left hand side (via the theory of almost étale extensions) and the right hand side of conjecture 1.1. The new inputs, compared to Faltings's original approach, are:

i) we systematically study the underlying sheaf theory of Faltings' topology. Faltings uses Galois cohomology of affines and then patches his computations to global results using hypercoverings;

ii) we introduce certain acyclic resolutions of sheaves of periods on Faltings' topology. This allows to avoid the use of Poincaré duality. In Faltings' original approach one needed to prove Poincaré duality in Faltings' theory and to show that it is compatible both with poicaré duality on crystalline or de Rham cohomology and with Poincaré duality on étale cohomology. We explain how to avoid this.

In order to simplify the proof, we also assume that there exists a morphism $F\colon X \to X$ lifting Frobenius on $\mathcal{O}_K$ and the Frobenius morphism on the special fiber X_k. This is a strong hypothesis. For example, if X is an abelian scheme over $\mathcal{O}_K$ with ordinary reduction, it amounts to require that X is the canonical lift of X_k. In fact, in [AI2] the existence of such a lift of Frobenius is assumed only Zariski locally on X (and this is harmless). On the other hand, this stronger assumption allows us to work with $\mathrm{H}^i_{\mathrm{dR}}(X/\mathcal{O}_K)$ instead of $\mathrm{H}^i_{\mathrm{cris}}(X_k/\mathcal{O}_K)$ with the Frobenius operator induced by F. We can then avoid the use of crystalline cohomology (and its technicalities) completely.

2 The crystalline conjecture with coefficients

We present the statement of the comparison isomorphism conjecture in the case of non constant coefficients. As before $X \to \mathrm{Spec}(\mathcal{O}_K)$ is a proper and smooth scheme, geometrically irreducible. We make no assumption on $\mathcal{O}_K$ but we assume that X is obtained by base change from a scheme defined over $W(k)$ where k is the residue field of $\mathcal{O}_K$ (this is needed in [AI2] not in Faltings' original approach). We consider two categories:

The category $\mathbb{Q}_p - \mathrm{Sh}(X_{\overline{K}}^{\mathrm{et}})$ of $\mathbb{Q}_p$-adic étale sheaves. By a p–adic étale sheaf $\mathbb{L}$ on $X_{\overline{K}}^{\mathrm{et}}$ we mean an inverse system $\mathbb{L} := \{\mathbb{L}_n\} \in \mathrm{Sh}(X_{\overline{K}}^{\mathrm{et}})^{\mathbb{N}}$ such that $\mathbb{L}_n$ is a locally constant and locally free of finite rank sheaf of $\mathbb{Z}/p^n\mathbb{Z}$–modules for the étale topology of X_K and $\mathbb{L}_n = \mathbb{L}_{n+1}/p^n\mathbb{L}_{n+1}$ for every $n \in \mathbb{N}$. Given two such objects $\mathcal{M} = \{\mathcal{M}_n\}_n$ and $\mathbb{L} := \{\mathbb{L}_n\}_n$ a morphism of p-adic étale sheaves $f\colon \mathcal{M} \to \mathbb{L}$ is a collection of morphisms $f_n\colon \mathcal{M}_n \to \mathbb{L}_n$ of sheaves such that $f_n \equiv f_{n+1}$ modulo p^n for every $n \in \mathbb{N}$. The category $\mathbb{Q}_p - \mathrm{Sh}(X_{\overline{K}}^{\mathrm{et}})$ has the p-adic étale sheaves as objects and one defines the morphisms by tensoring the $\mathbb{Z}_p$-module of morphisms as p-adic étale sheaves with $\mathbb{Q}_p$.

The category Fil$-F-$Isoc(X_K/K) of filtered F-isocrystals on X_K. Let $F-\mathrm{Isoc}\big(X_k/W(k)\big)$ be the category of isocrystals $\mathcal{E}$ on $X_k/W(k)$, endowed with a non-degenerate Frobenius $\varphi\colon \varphi^*(\mathcal{E}) \longrightarrow \mathcal{E}$. By isocrystal we mean a crystal of $\mathcal{O}_{X_k/W(k)}$-modules up to isogeny. See also [B3], especially Thm. 2.4.2, for a different point of view in the context of the rigid analytic spaces. To any such object one can associate a coherent $\mathcal{O}_{X_K}$-module $\mathcal{E}_{X_K}$ with integrable, quasi-nilpotent connection $\nabla\colon \mathcal{E}_{X_K} \longrightarrow \mathcal{E}_{X_K} \otimes_{\mathcal{O}_{X_K}} \Omega^1_{X_K/K}$. The category $\mathrm{Fil} - F - \mathrm{Isoc}(X_K/K)$ consists of an F-isocrystal $\mathcal{E}$ on $X_k/W(k)$ and an exhaustive descending filtration $\mathrm{Fil}^\bullet\mathcal{E}_{X_K}$ on $\mathcal{E}_{X_K}$ by $\mathcal{O}_{X_K}$-submodules satisfying Griffith's transversality with respect to the connection ∇ on $\mathcal{E}_{X_K}$ i. e., such that $\nabla\mathrm{Fil}^n\mathcal{E}_{X_K} \subset \mathrm{Fil}^{n-1}\mathcal{E}_{X_K}\otimes_{\mathcal{O}_{X_K}}\Omega^1_{X_K/K}$. Then,

Conjecture 2.1. *Assume that there exist a $\mathbb{Q}_p$-adic étale sheaf $\mathbb{L}$ and a filtered-F-isocrystal $\mathcal{E}$ which are associated. Then, for every $i \geq 0$ there is a canonical and functorial isomorphism commuting with all the additional structures (namely, filtrations, G_K–actions and Frobenii)*

$$\mathrm{H}^i(X_{\overline{K}}^{\mathrm{et}}, \mathbb{L}) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \cong \mathrm{H}^i_{\mathrm{cris}}(X_k/W(k), \mathcal{E}) \otimes_{W(k)} B_{\mathrm{cris}}.$$

Here, the filtration on $\mathrm{H}^i_{\mathrm{cris}}(X_k/W(k), \mathcal{E}) \otimes_{W(k)} B_{\mathrm{cris}}$ is obtained via $\mathrm{H}^i_{\mathrm{cris}}(X_k/W(k), \mathcal{E}) \otimes_{W(k)} B_{\mathrm{cris}} \cong \mathrm{H}^i_{\mathrm{dR}}\big(X_K, \mathcal{E}_{X_K}\big) \otimes_K B_{\mathrm{cris}}$ using the Hodge filtration on $\mathrm{H}^i_{\mathrm{dR}}\big(X_K, \mathcal{E}_{X_K}\big)$. The clarification of what "being associated" means is part of the conjecture. As an example, let $f\colon E \to X$ be a relative elliptic curve or an abelian scheme. Then, one can consider the $\mathbb{Q}_p$-adic étale sheaf $(\mathbb{L}_n)_n$ where $\mathbb{L}_n := E_K[p^n]$ is the groups scheme of p^n-torsion points of E_K. On the other hand, one can consider the filtered F-isocrystal $\mathcal{E}$ defined by $\mathrm{R}^1 f_{\mathrm{cris},*}\mathcal{O}^{\mathrm{cris}}_{E_k/W(k)}$ (the first derived functor in the crystalline sense of the crystal $\mathcal{O}^{\mathrm{cris}}_{E_k/W(k)}$ with respect to f_k). The module with connection $\mathcal{E}_{X_K}$ on X_K is the first de Rham cohomology of E_K relative to X_K with the Gauss-Manin connection. The filtration has only one step and $\mathrm{Fil}^1\mathcal{E}_{X_K} \subset \mathcal{E}_{X_K}$ is $f_*(\Omega^1_{E_K/X_K}$. Then, in this case $\mathbb{L}$ and $\mathcal{E}$ are associated and the conjecture holds. See [AI2].

3 The complex case. The strategy in the p-adic setting

Assume that X is a complex analytic projective variety. Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X. In this consider on the one hand a locally constant sheaf $\mathbb{L}$ of $\mathbb{C}$-vector spaces and a locally free coherent sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules endowed with an integrable connection ∇. Then, $\mathbb{L}$ and $(\mathcal{E},\nabla)$ are associated if $\mathbb{L} = \mathcal{E}^{\nabla=0}$. If this is the case the de Rham complex provides a resolution for $\mathbb{L}$:

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}\otimes_{\mathcal{O}_X}\Omega^1_{X/\mathbb{C}} \longrightarrow \cdots .$$

This provides a canonical comparison isomorphism $\mathrm{H}^i(X,\mathbb{L}) \cong \mathrm{H}^i_{\mathrm{dR}}(X,\mathcal{E})$. Here, we strongly use the analytic topology to prove that the displayed long sequence is exact and this approach certainly will not work for schemes (or even for rigid analytic p-adic spaces). As a refinement of this argument, more algebraic in flavor, we consider the universal covering space $v\colon \widetilde{X} \to X$. Let $x \in X$ be a point and let $\pi_1(X,x)$ be the fundamental group. Then, $v^{-1}(\mathbb{L})$ becomes a constant sheaf on $\widetilde{X}$ and defines (and is defined) by a representation of $\pi_1(X,x)$. Write $v^*(\mathcal{E}) := v^{-1}(\mathcal{E}) \otimes_{v^{-1}(\mathcal{O}_X)} \mathcal{O}_{\widetilde{X}}$. It is a $\mathcal{O}_{\widetilde{X}}$-module endowed with an integrable connection. Then, $\mathbb{L}$ and $\mathcal{E}$ are associated if we have an isomorphism $v^{-1}(\mathbb{L}) \otimes_{\mathbb{C}} \mathcal{O}_{\widetilde{X}} \cong v^{-1}(\mathcal{E}) \otimes_{v^{-1}(\mathcal{O}_X)} \mathcal{O}_{\widetilde{X}}$ of $\mathcal{O}_{\widetilde{X}}$-modules, compatibly with $\pi_1(X,x)$-action and connections. We can consider the complex

$$0 \longrightarrow v^{-1}(\mathbb{L}) \longrightarrow v^*(\mathcal{E}) \longrightarrow v^*(\mathcal{E}) \otimes_{\mathcal{O}_{\widetilde{X}}} \Omega^1_{\widetilde{X}/\mathbb{C}} \longrightarrow \cdots .$$

It is $\pi_1(X,x)$-equivariant and exact. There is an obvious relation between the two complexes. The former is locally given by the $\pi_1(X,x)$-invariants sections of v_* of the latter. This allows to recover the comparison isomorphism over X from the comparison isomorphism $\mathrm{H}^i(\widetilde{X},v^{-1}(\mathbb{L})) \cong \mathrm{H}^i_{\mathrm{dR}}(\widetilde{X},v^*(\mathcal{E}))$ of $\pi_1(X,x)$-modules. Indeed, we have compatible spectral sequences

$$\mathrm{H}^i\big(\pi_1(X,x), \mathrm{H}^j(\widetilde{X},v^{-1}(\mathbb{L}))\big) \Longrightarrow \mathrm{H}^{i+j}(X,\mathbb{L})$$

and

$$\mathrm{H}^i\big(\pi_1(X,x), \mathrm{H}^j_{\mathrm{dR}}(\widetilde{X},v^{-1}(\mathcal{E}))\big) \Longrightarrow \mathrm{H}^{i+j}_{\mathrm{dR}}(X,\mathcal{E})$$

inducing the isomorphism $\mathrm{H}^i(X,\mathbb{L}) \cong \mathrm{H}^i_{\mathrm{dR}}(X,\mathcal{E})$.

We will use the complex case as a guide for the proof in the p-adic case:

Step 1: Construct the analogue of the universal covering space $v\colon \widetilde{X} \to X$.

Step 2: Define the notion of "being associated" on $\widetilde{X}$ and construct the analogue of the exact sequence $0 \longrightarrow v^{-1}(\mathbb{L}) \longrightarrow v^*(\mathcal{E}) \longrightarrow v^*(\mathcal{E}) \otimes_{\mathcal{O}_{\widetilde{X}}} \Omega^1_{\widetilde{X}/\mathbb{C}} \longrightarrow \cdots$. Here, we need to find a replacement for the sheaf of holomorphic functions $\mathcal{O}_{\widetilde{X}}$ introducing Fontaine's sheaves of periods.

Step 3: Deduce the comparison isomorphism for associated objects.

4 The p-adic case; Faltings's topology

We let $X \to \mathrm{Spec}(\mathcal{O}_K)$ be as before. Of course we do not expect to construct a universal covering space. Following Faltings we will construct the sheaves on such space. After all we are interested not in the space itself but in comparing sheaves on it. We give the following:

Definition 4.1. Let $E_{X_{\overline{K}}}$ be the category defined as follows

i) the objects consist of pairs $\big(U, f\colon W \longrightarrow U_{\overline{K}}\big)$ such that U is an open subscheme of X and f is a finite étale morphism with $U_{\overline{K}}$ equal to the base change of U to $\overline{K}$. We will usually denote by (U, W) this object to shorten notations;

ii) a morphism $(U', W') \longrightarrow (U, W)$ in $E_{X_{\overline{K}}}$ consists of a pair (α, β), where $U' \longrightarrow U$ is the inclusion X and $\beta\colon W' \longrightarrow W$ is a morphism commuting with $\alpha \otimes_{\mathcal{O}_K} \mathrm{Id}_M$.

Let us remark that the pair $(X, X_{\overline{K}})$ is a final object in E_{X_M}. Moreover, fibre products exist setting $(U_1, W_1) \times_{(U,W)} (U_2, W_2) := \big(U_1 \cap U_2, W_1 \times_W W_2\big)$.

We say that a family $\{(U_i, W_i) \longrightarrow (U, W)\}_{i\in I}$ is a covering family if $\{U_i \longrightarrow U\}_{i\in I}$ is a covering of X and $\{W_i \longrightarrow W\}_{i\in I}$ is a covering of W i. e., every $\overline{K}$-valued point of W is in the image of a $\overline{K}$-valued point of one of the W_i's.

One verifies that the covering families satisfy the axioms of a pre-topology [SGAIV, Def II 1.3]:

(PT0)&(PT1) Given a morphism $(U', W') \to (U, W)$ and a covering family $\{(U_i, W_i) \longrightarrow (U, W)\}_{i\in I}$ the base change $\{(U_i, W_i) \times_{(U,W)} (U', W')\}_{i\in I}$ is a covering family of (U', W');

(PT2) Given a covering family $\{(U_i, W_i) \longrightarrow (U, W)\}_{i\in I}$ and for every $i \in I$ a covering family a covering family $\{(U_{ij}, W_{ij}) \longrightarrow (U_i, W_i)\}_{j\in J_i}$ the composite family $\{(U_{ij}, W_{ij}) \longrightarrow (U, W)\}_{i\in I, j\in J_i}$ is a covering family;

(PT3) the identity $(U, W) \to (U, W)$ is a covering family.

We let $\mathfrak{X}$ be the category $E_{X_{\overline{K}}}$ with the given pretopology. It should be thought of as the open subsets of the universal cover of X. It turns out that properties (PT0)–(PT3) suffice to define a category of sheaves of abelian groups $\mathrm{Sh}(\mathfrak{X})$ on $\mathfrak{X}$. A presheaf of abelian groups is a controvariant functor $F\colon E^0_{X_{\overline{K}}} \to \mathrm{AbGr}$. It is a sheaf if, for every object (U, W) and every covering family $\{(U_i, W_i) \longrightarrow (U, W)\}_{i\in I}$ the sequence

$$0 \longrightarrow F(U, W) \longrightarrow \prod_{i\in I} F(U_i, W_i) \longrightarrow \prod_{i,j\in I} F\left((U_i, W_i) \times_{(U,W)} (U_j, W_j)\right)$$

is exact. Given a presheaf F one can use the covering families to define the associated sheaf.

Remark 4.2. There are variants of the definition above which will be used in the sequel. Let $\widehat{X}$ be the formal scheme defined by completing X along the special fiber X_k.

(α) One can consider pairs $(\mathcal{U}, \mathcal{W})$ where $\mathcal{U} \subset \widehat{X}$ is Zariski open and $\mathcal{W} \to \mathcal{U}_{\overline{K}}$ is finite étale (meaning that there is a finite extension L such that $\mathcal{W}$ is defined by a finite and étale morphism over $\mathcal{U}_L$ as rigid analytic spaces).

(β) One can consider pairs $\big(U, W\big)$ where $U \to X$ is étale and $W \to U_{\overline{K}}$ is finite étale.

(γ) One can consider pairs $\big(\mathcal{U}, \mathcal{W}\big)$ where $\mathcal{U} \to \widehat{X}$ is p-adically formally étale and $W \to U_{\overline{K}}$ is finite étale.

In case (α) one defined the pretopology $\widehat{\mathfrak{X}}$ as in the algebrai setting. In cases (β) and (γ) Faltings defined in [F3, p. 214]] the pre-topology as before: a family $\{(U_i, W_i) \longrightarrow (U, W)\}_{i\in I}$

is a covering family if $\{U_i \longrightarrow U\}_{i\in I}$ is a covering of X (or $\widehat{X}$) and $\{W_i \longrightarrow W\}_{i\in I}$ is a covering of W. As pointed out by A. Abbes this gives the wrong pre-topology for which the period sheaves behave badly. The right definition, which produces a pre-topology which is coarser than Faltings', is that one takes the pre-topology generated by coverings of the following form: (a) $\{U_i \longrightarrow U\}_{i\in I}$ is a covering of X (or $\widehat{X}$) and $W_i = W \times_U U_i$; (b) $U_i = U$ and $\{W_i \longrightarrow W\}_{i\in I}$ is a covering of W. We refer to [AI2] for details.

We next define a morphism

$$v_*\colon \mathrm{Sh}(\mathfrak{X}) \longrightarrow \mathrm{Sh}(X)$$

analogous to the push–forward functor associated to the morphism $v\colon \widetilde{X} \longrightarrow X$ in the complex case. Let $v^{-1}\colon X^{\mathrm{Zar}} \to \mathfrak{X}$, where X^{Zar} is the Zariski topology on X, be the functor associating to an open $U \subset X$ the pair $(U, U_{\overline{K}})$. This induces a functor v_* from the category of presheaves on $\mathfrak{X}$ to the category of presheaves on X^{Zar} sending a presheaf F to $v_*(F)(U) := F\big(v^{-1}(U)\big)$. One verifies that v sends covering families to covering families and sends intersection of two open subsets to the fibre products in $\mathfrak{X}$. In particular, v_* sends sheaves on $\mathfrak{X}$ to sheaves on X. By construction it is left exact. One can verify that v_* admits a left adjoint v^* which is exact so that v_* sends injective objects to injective objects and we can derive the functor v_*. The rest of this section is devoted to the explicit computation of the functors $\mathrm{R}^i v_*$.

Localization functors: Fix an algebraic closure Ω of the fraction field of $X_{\overline{K}}$. Consider a Zariski open subset $U \subset X$ and let $G_U := \pi_1(U_{\overline{K}}, \Omega)$ be the fundamental group of $U_{\overline{K}}$ with base point defined by Ω. Let $U_{\overline{K}}^{\mathrm{fet}}$ be the category of finite and étale morphisms $W \to U_{\overline{K}}$. It is endowed with a pretopology: for every object W the coverings are morphisms $\{W_i \to W\}$ such that the images of the W_i's cover the whole of W. By Grothendieck's formulation of Galois theory, the category sheaves of abelian groups on $U_{\overline{K}}^{\mathrm{fet}}$ is equivalent to the category $G_U - \mathrm{AbGr}$ of $\mathbb{Z}$-modules with discrete action of G_U. We have a fully faithful morphism $\rho_U^{-1}\colon U_{\overline{K}}^{\mathrm{fet}} \longrightarrow \mathfrak{X}$ sending $W \mapsto (U, W)$. It sends covering families to covering families so that it induces a morphism

$$\rho_{U,*}\colon \mathrm{Sh}(\mathfrak{X}) \longrightarrow \mathrm{Sh}\big(U_{\overline{K}}^{\mathrm{fet}}\big) \equiv G_U - \mathrm{AbGr}$$

which is left exact, admits an exact left adjoint ρ_U^* so that it can be derived. Given a sheaf F we define $\rho_{U,*}(F)$ to be the *localization of F at U*. This construction remains valid also for the variants of the topology given in 4.2. One needs to take care of the choice of a base point, though. We refer to [AI2] for details.

Example 4.3. *Assume that $U = Spec(R_U)$ is affine. Let $\overline{R}_{\mathcal{U}} \subset \Omega$ be the union of all $R_U \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}$-subalgebras S of Ω such that S is normal and $R_U \otimes_{\mathcal{O}_K} \overline{K} \subset S\big[p^{-1}\big]$ is finite and étale. Given a sheaf F on $\mathfrak{X}$ write $F(\overline{R}_U)$ for the direct limit $\lim_S F\left(U, Spec\big(S[p^{-1}]\big)\right)$. It is naturally endowed with a continuous action of G_U and we have $\rho_{U,*}(F) = F\big(\overline{R}_U\big)$ (as G_U-modules).*

In particular, given a sheaf of abelian groups F on $\mathfrak{X}$ we get a spectral sequence a map

$$\mathrm{H}^n\big(G_U, \rho_{U,*}(F)\big) \to \mathrm{R}^n v_*\big(F\big)(U).$$

Define $\mathcal{H}^n_{\mathrm{Gal},X}(F)$ to be the sheaf on X associated to the presheaf $U \mapsto \mathrm{H}^n\big(U^o_{\overline{K},\mathrm{fet}}, \rho_{U,*}(F)\big)$.

Theorem 4.4. *The morphism $\mathcal{H}^n_{\mathrm{Gal},X}(F) \to \mathrm{R}^n v_*\big(F\big)$ is an isomorphism of sheaves on X.*

To prove this it suffices to show that the given morphism induces an isomorphism at level of "stalks". We define a *geometric point* of $\mathfrak{X}$ to be a pair (x,y) where (a) x is a geometric point of X;(b) y is a geometric point of $\mathcal{O}_{X,x}\otimes_{\mathcal{O}_K}\overline{K}$.

Write $G_{(x,y)}$ to be $\pi_1\left(\mathrm{Spec}\big(\mathcal{O}_{X,x}\otimes_{\mathcal{O}_K}\overline{K}\big),y\right)$. Let $J_{(x,y)}$ be set of pairs $\big(U,(W,y')\big)$ where U is an open neighborhood of x and $W\to U_{\overline{K}}$ is a finite and étale morphism with a point y' over y. One verifies that it is a directed set. Given sheaf F on $\mathfrak{X}$ define the stalk $F_{(x,y)}$ as the direct limit $\lim F(U,W)$ taken over all pairs $\big(U,(W,y')\big)\in J_{(x,y)}$. If F is a sheaf of abelian groups, then $F_{(x,y)}$ is a discrete module with continuous action of $G_{(x,y)}$. Since $\mathcal{H}^n_{\mathrm{Gal},X}(F)_x\cong \mathrm{H}^i\big(G_{(x,y)},F_{(x,y)}\big)$ to prove the theorem it suffices to verify that

Lemma 4.5. *(1) A sequence of sheaves of abelian groups is exact if and only if for every geometric point (x,y) the induced sequence of stalks is exact.*

(2) For every geometric point (x,y) and every sheaf F of abelian groups on $\mathfrak{X}$ we have $\mathrm{R}^i_*(F)_x\cong \mathrm{H}^i\big(G_{(x,y)},F_{(x,y)}\big)$.

As a consequence of this theorem we can prove the relation between Faltings cohomology and étale cohomology. Consider the forgetful functor $u^{-1}\colon\mathfrak{X}\to X^{\mathrm{et}}_{\overline{K}}$ sending $(U,W)\mapsto W$. It induces a morphism $u_*\colon \mathrm{Sh}(X^{\mathrm{et}}_{\overline{K}})\longrightarrow \mathrm{Sh}(\mathfrak{X})$. Then, Faltings proves in [F1] that if $\mathbb{L}$ is a locally constant étale sheaf on $X_{\overline{K}}$ we have $\mathcal{H}^n_{\mathrm{Gal},X}\big(u_*(\mathbb{L})\big)\to \mathrm{R}^n(v_*\circ u_*)\big(\mathbb{L}\big)$. This and 4.4 imply that the natural morphism $\mathrm{R}^n v_*\left(u_*\big(F\big)\right)\longrightarrow \mathrm{R}^n(v_*\circ u_*)\big(F\big)$ is an isomorphism. This implies

Proposition 4.6. *We have $\mathrm{H}^i\big(X^{\mathrm{et}}_{\overline{K}},\mathbb{L}\big)\cong \mathrm{H}^i\big(\mathfrak{X},u_*(\mathbb{L})\big)$.*

5 Sheaves of periods

In this section we will introduce the sheaf theoretic analogue of the ring $\mathcal{O}_{\overline{K}}$, $\mathbb{C}_K$ and B_{cris} of p-adic Hodge theory. We start with the analogue of $\mathcal{O}_{\overline{K}}$. First of all notice that given a sheaf $\mathcal{M}$ on X we get a sheaf on $\mathfrak{X}$, which we denote again by $\mathcal{M}$, via the formula $\mathcal{M}(U,W):=\mathcal{M}(U)$. In particular, we can view $\mathcal{O}_X$ and $\Omega^i_{X/\mathcal{O}_K}$ as sheaves on $\mathfrak{X}$.

Definition 5.1. We define the presheaf on $\mathfrak{X}$, denoted $\mathcal{O}_{\mathfrak{X}}$ by

$$\mathcal{O}_{\mathfrak{X}}(U,W):=\text{ the normalization of }\Gamma(U,\mathcal{O}_U)\text{ in }\Gamma(W,\mathcal{O}_W).$$

By construction there is a natural injective ring homomorphism $\mathcal{O}_X\otimes_{\mathcal{O}_K}\mathcal{O}_{\overline{K}}\longrightarrow\mathcal{O}_{\mathfrak{X}}$. We have

Proposition 5.2. *The presheaf $\mathcal{O}_{\mathfrak{X}}$ is a sheaf. Moreover, if $U=Spec(R_{\mathcal{U}})$ is an open affine subset of X then $\mathcal{O}_{\mathfrak{X}}(\overline{R}_{\mathcal{U}})=\overline{R}_{\mathcal{U}}$.*

Before sketching the proof of the proposition we remark that this remains true also for the variants 4.2. For variant (α) we denote by $\mathcal{O}_{\widehat{\mathfrak{X}}}$ the associated sheaf. If one uses the original definition of Faltings the proposition is wrong in cases (β) and (γ). We refer to [AI2] for a counterexample.

Proof. The second statement is clear. For the first, we may assume that W is irreducible and we may consider only the α's such that $W_\alpha\neq\emptyset$. Let $\{(U_\alpha,W_\alpha)\longrightarrow(U,W)\}_\alpha$ be a covering family. We set $U_{\alpha\beta}:=U_\alpha\times_U U_\beta$ and $W_{\alpha\beta}:=W_\alpha\times_W W_\beta$. We have the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathfrak{X}}(U,W) & \stackrel{f}{\longrightarrow} & \prod_\alpha \mathcal{O}_{\mathfrak{X}}(U_\alpha,W_\alpha) & \stackrel{g}{\longrightarrow} & \prod_{(\alpha),(\beta)}\mathcal{O}_{\mathfrak{X}_M}(U_{\alpha\beta},W_{\alpha\beta}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma(W,\mathcal{O}_W) & \longrightarrow & \prod_\alpha\Gamma(W_\alpha,\mathcal{O}_{W_\alpha}) & \longrightarrow & \prod_{\alpha,\beta}\Gamma(W_{\alpha\beta},\mathcal{O}_{\alpha\beta ij})
\end{array}$$

Note that for every α the map $\{W_\alpha \longrightarrow W \times_U U_\alpha\}_i$ is surjective because $W \times_U U_\alpha$ is irreducible and $W_\alpha \neq \emptyset$. Since the $\{U_\alpha \longrightarrow U\}_\alpha$ is a covering of X it follows that $\{W_\alpha \longrightarrow W\}_\alpha$ is a covering in $X_K^{\rm et}$. In particular, the bottom row of the above diagram is exact. Moreover the vertical maps are all inclusions therefore f is injective, i.e. $\mathcal{O}_{\mathfrak{X}}$ is a separated presheaf. Let $x \in \mathrm{Ker}(g)$. Then $x \in \Gamma(W, \mathcal{O}_W) \cap \prod_\alpha \mathcal{O}_{\mathfrak{X}_M}(U_\alpha, W_\alpha)$. We are left to prove that x is integral over $\Gamma(U, \mathcal{O}_U)$. Without loss of generality we may assume that $U_\alpha = \mathrm{Spec}(A_\alpha)$ is affine for every α. Let us denote by x_α the image of x in $\Gamma\bigl(W \times_U U_\alpha, \mathcal{O}_{W\times_U U_\alpha}\bigr)$. Because $W_\alpha \longrightarrow W \times_U U_\alpha$ is surjective, the image x_α of x_α in $\Gamma(W_\alpha, \mathcal{O}_{W_\alpha})$ is in fact in $\mathcal{O}_{\mathfrak{X}}(U_\alpha, W_\alpha)$, hence integral over A_α, it follows that x_α is integral over A_α. Let $P_\alpha(X) \in A_\alpha[X]$ be the (monic) characteristic polynomial of x_α over A_α. Then $P_\alpha(X)|_{U_{\alpha\beta}} = P_\beta(X)|_{U_{\alpha\beta}}$ for all α, β, therefore there is a monic polynomial $P(X) \in \Gamma(U, \mathcal{O}_U)$ such that $P(X)|_{U_\alpha} = P_\alpha(X)$. As $P(x)|_{U_\alpha} = P_\alpha(x_\alpha) = 0$ for every α it follows that $P(x) = 0$, i.e. that x is integral over $\Gamma(U, \mathcal{O}_U)$. $\square$

We come next to the sheaf theoretic analogue of $\mathbb{C}_K$. This coincides with $\widehat{\mathcal{O}}_{\overline{K}}[p^{-1}]$; here $\widehat{\mathcal{O}}_{\overline{K}}$ is the p-adic completion of $\mathcal{O}_{\overline{K}}$ and it is a topological ring and we can view $\widehat{\mathcal{O}}_{\overline{K}}[p^{-1}]$ as the limit of the inductive system $(\widehat{\mathcal{O}}_{\overline{K}})$ with transition maps given by multiplication by p. It is crucial that our definition captures the topology. For example, we would like that the localization of our continuous sheaf at an affine R_U gives the G_U-module $\widehat{\overline{R}}_U$ endowed *not* with the discrete topology but with the p-adic topology. To take this into account we use the category of inverse systems.

Categories of inverse systems. We review some of the results of [J]. Let $\mathcal{A}$ be an abelian category. Denote by $\mathcal{A}^{\mathbb{N}}$ the category of inverse systems indexed by the set of natural numbers. Objects are inverse systems $\{A_n\}_n := \ldots \to A_{n+1} \to A_n \ldots A_2 \to A_1$, where the A_i's are objects of $\mathcal{A}$ and the arrows denote morphisms in $\mathcal{A}$. The morphisms in $\mathcal{A}^{\mathbb{N}}$ are commutative diagrams

$$\begin{array}{ccccccccc} \ldots & \to & A_{n+1} & \to & A_n & \ldots & A_2 & \to & A_1 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \ldots & \to & B_{n+1} & \to & B_n & \ldots & B_2 & \to & B_1, \end{array}$$

where the vertical arrows are morphisms in $\mathcal{A}$. Then, $\mathcal{A}^{\mathbb{N}}$ is an abelian category with kernels and cokernels taken componentwise and if $\mathcal{A}$ has enough injectives, then $\mathcal{A}^{\mathbb{N}}$ also has enough injectives.

Let $h\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. It induces a left exact functor $h^{\mathbb{N}}\colon \mathcal{A}^{\mathbb{N}} \to \mathcal{B}^{\mathbb{N}}$ which, by abuse of notation and if no confusion is possible, we denote again by h. If $\mathcal{A}$ has enough injectives, then also $\mathcal{A}^{\mathbb{N}}$ does. One can derive the functor $h^{\mathbb{N}}$ and $\mathrm{R}^i\bigl(h^{\mathbb{N}}\bigr) = \bigl(\mathrm{R}^i h\bigr)^{\mathbb{N}}$.

If inverse limits over $\mathbb{N}$ exist in $\mathcal{B}$, define the left exact functor $\varprojlim h\colon \mathcal{A}^{\mathbb{N}} \to \mathcal{B}$ as the composite of $h^{\mathbb{N}}$ and the inverse limit functor $\varprojlim\colon \mathcal{B}^{\mathbb{N}} \to \mathcal{B}$. Assume that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. For every $A = \{A_n\}_n \in \mathcal{A}^{\mathbb{N}}$ one then has a spectral sequence

$$\varprojlim{}^{(p)} \mathrm{R}^q h(A_n) \Longrightarrow \mathrm{R}^{p+q}\bigl(\varprojlim h(A)\bigr),$$

where $\varprojlim^{(p)}$ is the p–th derived functor of $\varprojlim$ in $\mathcal{B}$. If in $\mathcal{B}$ infinite products exist and are exact functors, then $\varprojlim^{(p)} = 0$ for $p \geq 2$ and the above spectral sequence reduces to the simpler exact sequence

$$0 \longrightarrow \varprojlim{}^{(1)} \mathrm{R}^{i-1} h(A_n) \longrightarrow \mathrm{R}^i\bigl(\varprojlim h\bigr)(A) \to \varprojlim \mathrm{R}^i h(A_n) \longrightarrow 0.$$

Generalities on inductive systems. Let $\mathcal{A}$ be an abelian category. We denote by $\mathrm{Ind}(\mathcal{A})$, called the category of inductive systems of objects of $\mathcal{A}$, the following category. The objects are $(A_i,\gamma_i)_{i\in\mathbb{Z}}$ with A_i object of $\mathcal{A}$ and $\gamma_i\colon A_i\to A_{i+1}$ morphism in $\mathcal{A}$ for every $i\in\mathbb{Z}$. Given an integer $N\in\mathbb{Z}$ a morphism $f\colon \underline{A}:=(A_i,\gamma_i)_{i\in\mathbb{Z}}\longrightarrow \underline{B}:=(B_j,\delta_j)_{j\in\mathbb{Z}}$ of degree N is a system of morphisms $f_i\colon A_i\to B_{i+N}$ for $i\in\mathbb{Z}$ such that $\delta_{i+N}\circ f_i=f_{i+1}\circ\gamma_i$. Since $\mathcal{A}$ is an additive category, the set of morphisms of degree N form an abelian group with the zero map, the sum of two functions and the inverse of a function defined componentwise. Given a morphism $f=(f_i)_{i\in\mathbb{Z}}\colon \underline{A}\longrightarrow\underline{B}$ of degree N we get a morphism of degree $N+1$ given by $(\delta_{i+N}\circ f_i)_{i\in\mathbb{Z}}$. This defines a group homomorphism from the morphisms $\mathrm{Hom}^N(\underline{A},\underline{B})$ of degree N to the morphisms $\mathrm{Hom}^{N+1}(\underline{A},\underline{B})$ of degree $N+1$. We define the group of morphisms $f\colon (A_i,\gamma_i)_{i\in\mathbb{Z}}\longrightarrow (B_j,\delta_j)_{j\in\mathbb{Z}}$ in $\mathrm{Ind}(\mathcal{A})$ to be the inductive limit $\lim_{N\in\mathbb{Z}}\mathrm{Hom}^N(\underline{A},\underline{B})$ with respect to the transition maps just defined.

One verifies that the category $\mathrm{Ind}(\mathcal{A})$ is an abelian category. Let $\mathcal{B}$ be an abelian category in which direct limits of inductive systems indexed by $\mathbb{Z}$ exist. Consider the induced functor

$$\varinjlim\colon \mathrm{Ind}(\mathcal{B})\longrightarrow\mathcal{B}.$$

Suppose we are given δ-functors $T^n\colon\mathcal{B}\to\mathcal{A}$ with $n\in\mathbb{N}$. Define

$$\varinjlim T^n\colon \mathrm{Ind}(\mathcal{A})\longrightarrow\mathcal{B}$$

as the composite of the functor $\mathrm{Ind}(\mathcal{A})\to\mathrm{Ind}(\mathcal{B})$, given by $(A_i)_{in\mathbb{Z}}\mapsto (T^n(A_i))_{i\in\mathbb{Z}}$, and of the functor $\varinjlim$. Then, if $\varinjlim$ is left exact in $\mathcal{B}$, the functors $\varinjlim T^n$, for varying $n\in\mathbb{N}$, define a δ-functor as well.

Given $F=(F_n)\in\mathrm{Sh}(\mathfrak{X})^{\mathbb{N}}$ define $\mathrm{H}^0(\mathfrak{X},F):=\lim\limits_{\infty\leftarrow n}\mathrm{H}^0(\mathfrak{X},F_n)$. Define $\mathrm{H}^i(\mathfrak{X},F)$ as the i-th derived functor of this. If $G=(G_m)_{m\in\mathbb{N}}\in\mathrm{Ind}\left(\mathrm{Sh}(\mathfrak{X})^{\mathbb{N}}\right)$ with $G_m\in\mathrm{Sh}(\mathfrak{X})^{\mathbb{N}}$ define $\mathrm{H}^i(\mathfrak{X},G)=\lim\limits_{m\to\infty}\mathrm{H}^i(\mathfrak{X},G_m)$.

This bit of abstract non-sense allows us to define inductive limits of topological sheaves $\mathrm{Ind}\left(\mathrm{Sh}(\mathfrak{X})^{\mathbb{N}}\right)$ on $\mathfrak{X}$ where Fontaine's sheaves of periods naturally live. For example, the analogue of $\widehat{\mathcal{O}}_{\overline{K}}$ is the inverse system $\widehat{\mathcal{O}}_{\mathfrak{X}}:=(\mathcal{O}_{\mathfrak{X}}/p^n\mathcal{O}_{\mathfrak{X}})_n$ where the transition maps are the natural reduction maps. Similarly, the analogue of $\mathbb{C}_K$ is the inductive limit $\widehat{\mathcal{O}}_{\mathfrak{X}}[p^{-1}]:=(\widehat{\mathcal{O}}_{\mathfrak{X}})_n$ where the transition maps are given by multiplication by p. We next consider the analogue of B_{cris}. Since we'd like to use the technology above we will first show how to construct B_{cris} as a direct limit of an inverse system of rings. We will then generalize this construction to sheaves.

Review of a construction of B_{cris}. Put $W_n:=W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$ the Witt vectors of length n with values in $\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ and write φ for Frobenius on W_n. We have a ring homomorphism $\theta_n\colon W_n:\longrightarrow \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}$ given by $(s_0,\ldots,s_{n-1})\mapsto\sum_{i=0}^{n-1}p^i\tilde{s}_i^{p^{n-i}}$ where $\tilde{s}_i\in\mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}$ is a lift of s_i for every i. Choose a compatible sequence of roots $(p^{1/p^{n-1}})_{n\geq 1}$ in $\mathcal{O}_{\overline{K}}$ i. e., $(p^{1/p^n})^p=p^{1/p^{n-1}}$. Denote by $\widetilde{p}_n:=\left[p^{1/p^n}\right]\in W_n$ the Teichmüller lift of $p^{1/p^n}\in\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$. Write $\xi_n:=\widetilde{p}_n-p\in W_n$; it is a generator of $\mathrm{Ker}(\theta_n)$. Let $u_{n+1}\colon W_{n+1}\longrightarrow W_n$ be the composite of the natural projection composed with Frobenius φ. Then, $\theta_n\circ u_{n+1}\equiv u_n$ modulo p^n and $u_{n+1}(\xi_{n+1})=\xi_n$. Put A_{cris} as the inverse limit of the rings $W_n\left[\frac{\xi_n^m}{m!}\right]_{m\in\mathbb{N}}$ with respect to the transition morphisms defined by the u_n's. One can prove that $A_{\rm cris}/p^nA_{\rm cris}\cong W_n\left[\frac{\xi_n^m}{m!}\right]_{m\in\mathbb{N}}$. This admits a filtration where $\mathrm{Fil}^r A_{\rm cris}/p^nA_{\rm cris}\cong W_n\left[\frac{\xi_n^m}{m!}\right]_{m\geq r}$. Furthermore, $A_{\rm cris}$ is endowed with (1) an action of G_K, via its action on $\mathcal{O}_{\overline{K}}$, which is continuous for the p-adic topology; (2) a filtration defined by $\mathrm{Fil}^r A_{\rm cris}:=\lim \mathrm{Fil}^r A_{\rm cris}/p^nA_{\rm cris}$; (3) a Frobenius φ defined by Frobenius on the W_n's.

Let $\{\zeta_n\}_{n\in\mathbb{N}}$ be a compatible system of primitive p^n–th roots of unity so that $\zeta_{n+1}^p=\zeta_n$. It defines an element $[\zeta_{n+1}]\in W_n$. Note that $u_{n+1}\big([\zeta_{n+2}]\big)=[\zeta_{n+1}]$ and $[\zeta_{n+1}]-1\in\mathrm{Ker}(\theta_n)$. Let $\varepsilon=\lim[\zeta_{n+1}]\in A_{\rm cris}^+$ be the induced element and write

$$t:=\log(\varepsilon)=\sum_{n=1}^{\infty}\big(\varepsilon-1\big)^n/n.$$

Is an element of $\mathrm{Fil}^1A_{\rm cris}$ and $B_{\rm cris}=A_{\rm cris}\left[t^{-1}\right]$. It inherits a G_K-action, a filtration and a Frobenius from those defined on $A_{\rm cris}$. Note that $t^{[p]}:=t^p/p!\in A_{\rm cris}$ so that $p!t^{[p]}=t^p$ and p is invertible in $B_{\rm cris}$. More precisely, for every $m\in\mathbb{N}$ write $A_{\rm cris}(m)=A_{\rm cris}\cdot t^{-m}$. Since G_K acts on t via the cyclotomic character i. e., for $g\in G_K$ we have $g(t)=\chi(g)t$, then $A_{\rm cris}(m)$ is G_K-stable and as a Galois module it is a Tate twist of $A_{\rm cris}$. Put $\mathrm{Fil}^rA_{\rm cris}(m):=\mathrm{Fil}^{r+m}A_{\rm cris}\cdot t^{-m}$. Eventually, since $\varphi(t)=pt$ then Frobenius on $B_{\rm cris}$ sends $A_{\rm cris}(m)\longrightarrow A_{\rm cris}(pm)$. We note that $B_{\rm cris}=\lim A_{\rm cris}(m)$ compatibly with G_K-action, a filtration and a Frobenius.

The sheaf $\mathbb{B}_{\rm cris}^{\nabla}$. We switch to Faltings' site. Consider the sheaves $\mathbb{W}_n:=W_n\big(\mathcal{O}_{\mathfrak{X}}/p\mathcal{O}_{\mathfrak{X}}\big)$. We have a Frobenius φ on $\mathbb{W}_n$ and on $\mathbb{W}_{n,X}$. As in the classical case we have a homomorphism of sheaves $\theta_n\colon\mathbb{W}_n\longrightarrow\mathcal{O}_{\mathfrak{X}}/p^n\mathcal{O}_{\mathfrak{X}}$. We further have sheaf homomorphisms $u_{n+1}\colon\mathbb{W}_{n+1}\longrightarrow\mathbb{W}_n$, defined be the composite of the natural projection composed with Frobenius. Fix an object (U,W) of $\mathfrak{X}$. Write $S=\mathcal{O}_{\mathfrak{X}}(U,W)$.

Lemma 5.3. *The element ξ_n generates the kernel of $\theta_n\colon W_n(S/pS)\longrightarrow S/p^nS$.*

Proof. Let us first remark that $\theta_n(\xi_n)=(p^{1/p^n})^{p^n}-p=0$, therefore $\xi_n\in\mathrm{Ker}(\theta_n)$. one needs to show that if $x\in\mathrm{Ker}(\theta_n)$ then $x\in\xi_nW_n(S/pS)$. One proceeds by induction on n. For $n=1$, then θ_1 is Frobenius on S/pS whose kernel is generated by $p^{1/p}$ since S is normal. For the inductive step we refer to [AI2] □

In particular, we conclude that the kernel of the map of sheaves θ_n is generated by ξ_n. One defines $\mathbb{A}_{\rm cris}^{\nabla}$ as the inverse system $\mathbb{A}_{\rm cris}/p^n\mathbb{A}_{\rm cris}$ for varying $n\in\mathbb{N}$ where $\mathbb{A}_{\rm cris}^{\nabla}/p^n\mathbb{A}_{\rm cris}^{\nabla}$ is the sheaf $\mathbb{W}_n\otimes_{W_n}\big(A_{\rm cris}/p^nA_{\rm cris}\big)$. The transition morphisms are defined by u_n. Then, $\mathbb{A}_{\rm cris}^{\nabla}$ is endowed with a Frobenius morphism, a filtration defined by $\mathrm{Fil}^r\left(\mathbb{A}_{\rm cris}/p^n\mathbb{A}_{\rm cris}\right):=\mathbb{W}_n\otimes_{W_n}\mathrm{Fil}^r\big(A_{\rm cris}/p^nA_{\rm cris}\big)$ and a G_K-action. One defines similarly the inverse system $\mathbb{A}_{\rm cris}^{\nabla}(m)$ as the inverse system $\mathbb{A}_{\rm cris}^{\nabla}(m)/p^n\mathbb{A}_{\rm cris}^{\nabla}(m):=\mathbb{W}_n\otimes_{W_n}\big(A_{\rm cris}(m)/p^nA_{\rm cris}(m)\big)$ for $m\in\mathbb{N}$ with induced filtration and G_K-action and we let $\mathbb{B}_{\rm cris}^{\nabla}\in\mathrm{Ind}\big(\mathrm{Sh}(\mathfrak{X})^{\mathbb{N}}\big)$ be the induced inductive system of inverse systems of sheaves.

The sheaf $\mathbb{B}_{\rm cris}$. Consider $\mathbb{W}_{n,X}:=W_n(\mathcal{O}_{\mathfrak{X}}/p\mathcal{O}_{\mathfrak{X}})\otimes_{\mathcal{O}_K}\mathcal{O}_X$. Using our assumption that X admits a global lift of Frobenius, we get a Frobenius on $\mathbb{W}_{n,X}$. Extending θ_n and u_{n+1} to $\mathcal{O}_X$-linearly morphisms we get compatible morphisms $\theta_{n,X}\colon\mathbb{W}_{n,X}\longrightarrow\mathcal{O}_{\mathfrak{X}}/p^n\mathcal{O}_{\mathfrak{X}}$ and $u_{n+1,X}\colon\mathbb{W}_{n+1,X}\longrightarrow\mathbb{W}_{n,X}$. We analyze the kernel of $\theta_{n,X}$. Let $(U,W)\in\mathfrak{X}$ with $U=\mathrm{Spec}(R_U)$ affine and put $S:=\overline{\mathcal{O}}_{\mathfrak{X}}(U,W)$. Assume that U admits an étale morphism to $\mathrm{Spec}\big(\mathcal{O}_K[T_1^{\pm1},\ldots,T_d^{\pm1}]\big)$. in this case we say that U is *small*. Assume furthermore that S contains p^{n+1}-th roots $T_i^{1/p^{n+1}}$ for all $1\leq i\leq d$ of the variable T_i. Denote by $\widetilde{T}_{i,n}:=[T_i^{1/p^{n+1}}]\in W_n\big(S/pS\big)$ and $X_{i,n}:==1\otimes T_i-\widetilde{T}_{i,n}\otimes 1\in$

$W_n\big(S/pS\big) \otimes_{\mathcal{O}_K} R_U$. Since the kernel of the ring homomorphism $R_U/p^nR_U \otimes R_U/p^nR_U \to R_U/p^nR_U$ defined by $x\otimes y \to xy$ is the ideal $I = (T_1 \otimes 1 - 1 \otimes T_1, \ldots, T_d \otimes 1 - 1 \otimes T_d)$, we conclude that the kernel of the map $\theta_{n,S} \colon W_n\big(S/pS\big) \otimes_{\mathcal{O}_K} R_U \to S/p^nS$ is the ideal generated by $(\xi_n, X_{1,n}, \ldots, X_{d,n})$. The derivation $d \colon R_U \longrightarrow \Omega^1_{R_U/\mathcal{O}_K} \cong \oplus_{i=1}^d R_U dT_i$ extends to a $W_n\big(S/pS\big) \otimes_{W_n} \frac{A_{\text{cris}}}{p^nA_{\text{cris}}}$–linear connection ∇_S:

$$W_n\big(S/pS\big)\otimes_{\mathcal{O}_K} R_U \left[\frac{\xi_n^m}{m!}, \frac{X_{1,n}^m}{m!}, \ldots, \frac{X_{d,n}^m}{m!}\right] \longrightarrow W_n\big(S/pS\big)\otimes_{\mathcal{O}_K} R_U \left[\frac{\xi_n^m}{m!}, \frac{X_{1,n}^m}{m!}, \ldots, \frac{X_{d,n}^m}{m!}\right] \otimes_{R_U} \Omega^1_{R_U/\mathcal{O}_K}$$

sending $\frac{X_{1,n}^{m_1}}{m_1!} \cdots \frac{X_{d,n}^{m_d}}{m_d!} \mapsto \sum_{i=1}^d \frac{X_{1,n}^{m_1}}{m_1!} \cdots \frac{X_{1,n}^{m_i-1}}{(m_i-1)!} \cdots \frac{X_{d,n}^{m_d}}{m_d!} \cdot dT_i$. One can prove that in fact

$$W_n\big(S/pS\big) \otimes_{\mathcal{O}_K} R_U \left[\frac{\xi_n^m}{m!}, \frac{X_{1,n}^m}{m!}, \ldots, \frac{X_{d,n}^m}{m!}\right] \cong W_n\big(S/pS\big) \otimes_{W_n} \big(A_{\text{cris}}/p^nA_{\text{cris}}\big) \left[\frac{X_{1,n}^m}{m!}, \ldots, \frac{X_{d,n}^m}{m!}\right].$$

It follows from this that de Rham complex

$$0 \longrightarrow W_n\big(S/pS\big) \otimes_{W_n} \frac{A_{\text{cris}}}{p^nA_{\text{cris}}} \longrightarrow W_n\big(S/pS\big) \otimes_{\mathcal{O}_K} R_U \left[\frac{\xi_n^m}{m!}, \frac{X_{1,n}^m}{m!}, \ldots, \frac{X_{d,n}^m}{m!}\right] \otimes_{R_U} \Omega^\bullet_{R_U/\mathcal{O}_K} \longrightarrow 0$$

is exact. Note that $W_n\big(S/pS\big) \otimes_{W_n} \big(A_{\text{cris}}/p^nA_{\text{cris}}\big) \left[\frac{X_{1,n}^m}{m!}, \ldots, \frac{X_{d,n}^m}{m!}\right]$ admits a filtration where Fil^r is generated by the elements $\frac{\xi_n^{m_0}}{m_0!} \frac{X_{1,n}^{m_1}}{m_1!} \cdots \frac{X_{d,n}^{m_d}}{m_d!}$ with $m_0 + m_1 + \cdots m_d \geq r$. Such filtration satisfies Griffith's tranversality with respect to the connection ∇_S. We also notice that the action of the Galois group G_U on S extends to a an action on $W_n\big(S/pS\big) \otimes_{\mathcal{O}_K} R_U \left[\frac{\xi_n^m}{m!}, \frac{X_{1,n}^m}{m!}, \ldots, \frac{X_{d,n}^m}{m!}\right]$. Indeed, let $c_i \colon G_U \longrightarrow \mathbb{Z}_p$ be defined by $\sigma\big(T_i^{1/p^{n+1}}\big) = \zeta_{n+1}^{c_i(\sigma)} T_i^{1/p^{n+1}}$. Then,

$$\sigma\big(X_i^m/m!\big) := \sum_{s=0}^m \frac{X_i^{m-s}}{(m-s)!} \frac{(1 - [\zeta_{n+1}]^{c_i(\sigma)})^s}{s!} \big(\widetilde{T}_{i,n}^s \otimes 1\big).$$

Such action preserves the filtration and commutes with the connection ∇_S. Since the pairs (U, W) with the properties above define a basis for the given pre-topology on $\mathfrak{X}$, one can sheafify this construction to get a sheaf $\mathbb{A}_{\text{cris}}/p^n\mathbb{A}_{\text{cris}}$ of $\mathcal{O}_X \otimes A_{\text{cris}}/p^nA_{\text{cris}}$-modules, endowed with an integrable connection ∇, a decreasing filtration $\text{Fil}^\bullet\mathbb{A}_{\text{cris}}/p^n\mathbb{A}_{\text{cris}}$ which satisfies Griffiths' transversality, a Frobenius φ. Moreover, $\{\mathbb{A}_{\text{cris}}/p^n\mathbb{A}_{\text{cris}}\}_n$ define an inverse system $\mathbb{A}_{\text{cris}} \in \text{Sh}(\mathfrak{X})^{\mathbb{N}}$ of $\mathcal{O}_X\widehat{\otimes}A_{\text{cris}}$-modules, endowed with a connection $\nabla \colon \mathbb{A}_{\text{cris}} \longrightarrow \mathbb{A}_{\text{cris}} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathcal{O}_K}$, filtration $\text{Fil}^\bullet\mathbb{A}_{\text{cris}} := \big\{\text{Fil}^\bullet\big(\mathbb{A}_{\text{cris}}/p^n\mathbb{A}_{\text{cris}}\big)\big\}_n$ and a Frobenius φ. Moreover,

Proposition 5.4. *Consider the complex*

$$\mathbb{A}_{\text{cris}} \xrightarrow{\nabla^1} \mathbb{A}_{\text{cris}} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathcal{O}_K} \xrightarrow{\nabla^2} \mathbb{A}_{\text{cris}} \otimes_{\mathcal{O}_X} \Omega^2_{X/\mathcal{O}} \longrightarrow \cdots \xrightarrow{\nabla^d} \mathbb{A}_{\text{cris}} \otimes_{\mathcal{O}_X} \Omega^d_{X/\mathcal{O}_K} \longrightarrow 0$$

i. it is exact;

ii. the natural inclusion $\mathbb{A}_{\text{cris}}^\nabla \subset \mathbb{A}_{\text{cris}}$ identifies $Ker(\nabla^1)$ with $\mathbb{A}_{\text{cris}}^\nabla$;

iii. (Griffith's transversality) *we have $\nabla\left(\text{Fil}^r\big(\mathbb{A}_{\text{cris}}\big)\right) \subset \text{Fil}^{r-1}\left(\mathbb{A}_{\text{cris}}\right) \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathcal{O}_K}$ for every r;*

iv. for every $r \in \mathbb{N}$ the sequence $0 \longrightarrow \text{Fil}^r\mathbb{A}_{\text{cris}}^\nabla \longrightarrow \text{Fil}^r\mathbb{A}_{\text{cris}} \xrightarrow{\nabla^1} \text{Fil}^{r-1}\mathbb{A}_{\text{cris}} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathcal{O}_K} \xrightarrow{\nabla^2}$ $\text{Fil}^{r-2}\mathbb{A}_{\text{cris}} \otimes_{\mathcal{O}_X} \Omega^2_{X/\mathcal{O}_K} \xrightarrow{\nabla^3} \cdots$, with the convention that $\text{Fil}^s\mathbb{A}_{\text{cris,M}} = \mathbb{A}_{\text{cris}}$ for $s < 0$, is exact;

v. the connection $\nabla\colon \mathbb{A}_{\mathrm{cris,M}} \longrightarrow \mathbb{A}_{\mathrm{cris,M}} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathcal{O}_K}$ *is quasi–nilpotent;*

vi. Frobenius φ *on* $\mathbb{A}_{\mathrm{cris}}$ *is horizontal with respect to* ∇ *i. e.,* $\nabla \circ \varphi = \big(\varphi \otimes dF\big) \circ \nabla$.

Example 5.5. *We consider the case that* $n = 1$. *Take the ring* $S/pS \otimes_{\mathcal{O}_K} R_U \left[\frac{\xi_1^m}{m!}, \frac{X_{1,1}^m}{m!}, \ldots, \frac{X_{d,1}^m}{m!}\right]$. *We claim that it is isomorphic to*

$$S/pS[\delta_0, \delta_1, \ldots X_{i,0}, X_{i,1}, \ldots]_{1\le i\le d}/(\delta_m^p, X_{i,m}^p)_{1\le i\le d, m\ge 0}$$

where $\delta_m = \gamma^{m+1}\xi_1$ *and* $X_{i,j} = \gamma^{j+1}(X_i)$ *and* $\gamma\colon z \mapsto \frac{z^p}{p}$. *In particular, it is a free* S/pS*-module. This allows to compute the localization of* $\mathbb{A}_{\mathrm{cris}}$.

One can also define $\mathbb{A}_{\mathrm{cris}}(m)$" = "$\mathbb{A}_{\mathrm{cris}} \cdot t^{-m}$ for every $m \in \mathbb{N}$ with its filtration and connection. Taking inductive limits of the $\mathbb{A}_{\mathrm{cris}}(m)$ as in the definition of $\mathbb{B}^{\nabla}_{\mathrm{cris}}$ one constructs the object $\mathbb{B}_{\mathrm{cris}} \in \mathrm{Ind}\big(\mathrm{Sh}(\mathfrak{X})^{\mathbb{N}}\big)$, with a filtration, connection and Frobenius so that the analogue of 5.4 holds. One can also prove that an analogue of the fundamental exact sequence holds:

Lemma 5.6. *We have the following exact sequence in* $\mathrm{Ind}\Big(\mathrm{Sh}(\widehat{\mathfrak{X}})^{\mathbb{N}}\Big)$

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow \mathrm{Fil}^0\big(\mathbb{B}^{\nabla}_{\mathrm{cris}}\big) \xrightarrow{\varphi-1} \mathbb{B}^{\nabla}_{\mathrm{cris}} \longrightarrow 0.$$

Proof. We prove the weaker statement that the sequence of inverse systems in $\mathrm{Sh}(\widehat{\mathfrak{X}})^{\mathbb{N}}$

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{A}^{\nabla}_{\mathrm{cris}} \xrightarrow{\varphi-1} \mathbb{A}^{\nabla}_{\mathrm{cris}} \longrightarrow 0.$$

One reduces to prove that

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{A}^{\nabla}_{\mathrm{cris}}/p\mathbb{A}^{\nabla}_{\mathrm{cris}} \xrightarrow{\varphi-1} \mathbb{A}^{\nabla}_{\mathrm{cris}}/p\mathbb{A}^{\nabla}_{\mathrm{cris}} \longrightarrow 0$$

is exact and from this that

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathcal{O}_{\widehat{\mathfrak{X}}}/p\mathcal{O}_{\widehat{\mathfrak{X}}} \xrightarrow{\varphi-1} \mathcal{O}_{\widehat{\mathfrak{X}}}/p\mathcal{O}_{\widehat{\mathfrak{X}}} \longrightarrow 0.$$

This can be proven on stalks and, hence, it suffices to show that for $\mathcal{U} \subset \widehat{X}$ affine the map

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}} \xrightarrow{\varphi-1} \overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}} \longrightarrow 0$$

is exact. By Artin-Schreier theory, the kernel of $\varphi - 1$ is $\mathrm{H}^0\big(\mathrm{Spec}(\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}}), \mathbb{Z}/p\mathbb{Z}\big)$ which is $\mathbb{Z}/p\mathbb{Z}$ since $\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}}$ is connected (here we use that $R_{\mathcal{U}}$ is p-adically complete and separated so that its finite normal extensions are connected if and only if they are connected modulo p). The cokernel of $\varphi - 1$ coincides with $\mathrm{H}^1\big(\mathrm{Spec}(\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}}), \mathbb{Z}/p\mathbb{Z}\big)$ which is alos zero since every $\mathbb{Z}/p\mathbb{Z}$-torsor over $\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}}$ can be lifted to $\overline{R}_{\mathcal{U}}$ and, thus, is trivial by definition of $\overline{R}_{\mathcal{U}}$ (here again we use that $R_{\mathcal{U}}$ is p-adically complete and separated so that every finite extension is henselian with respect to the ideal p). □

6 The computation of $\mathrm{R}^i v_* \mathbb{B}_{\mathrm{cris}}$

Consider the functor $v_*\colon \mathrm{Ind}\big(\mathrm{Sh}(\mathfrak{X})^{\mathbb{N}}\big) \longrightarrow \mathrm{Sh}(X)$ given by $\big\{\big(F_{n,m}\big)_n\big\}_m \mapsto \big\{v_*\big(F_{n,m}\big)_n\big\}_m$. Then,

Theorem 6.1. *We have* $\mathrm{R}^i v_* \mathbb{B}_{\mathrm{cris}} = 0$ *for* $i \geq 1$ *and it is equal to* $\mathcal{O}_X \widehat{\otimes}_{\mathcal{O}_K} B_{\mathrm{cris}}$ *if* $i = 0$. *Similarly,* $\mathrm{R}^i v_* \mathrm{Fil}^r \mathbb{B}_{\mathrm{cris}} = 0$ *for* $i \geq 1$ *and it is equal to* $\mathcal{O}_X \widehat{\otimes}_{\mathcal{O}_K} \mathrm{Fil}^r B_{\mathrm{cris}}$ *if* $i = 0$.

Here, $\mathcal{O}_X \widehat{\otimes}_{\mathcal{O}_K} B_{\mathrm{cris}}$ stands for the inductive limit $\mathcal{O}_X \widehat{\otimes}_{\mathcal{O}_K} A_{\mathrm{cris}}[t^{-1}]$ where $\mathcal{O}_X \widehat{\otimes}_{\mathcal{O}_K} A_{\mathrm{cris}}$ stands for the inverse system of sheaves $\mathcal{O}_X \otimes_{\mathcal{O}_K} (A_{\mathrm{cris}}/p^n A_{\mathrm{cris}})$. Similarly for $\mathcal{O}_X \widehat{\otimes}_{\mathcal{O}_K} \mathrm{Fil}^r B_{\mathrm{cris}}$. In this section we will only sketch the proof that $\mathrm{R}^i v_* \mathbb{B}_{\mathrm{cris}} = 0$ for $i \geq 1$. For the other statements we refer to [AI2] also considering the variants in 4.2.

Recall that an open affine subset $U = \mathrm{Spec}(R_U)$ of X is called small if it admits an étale morphism to $\mathrm{Spec}\big(\mathcal{O}_K[T_1^{\pm 1}, \ldots, T_d^{\pm 1}]\big)$. In this case we write $R_{U,\infty} := \cup_n R_U \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}\Big[T_1^{1/p^n}, \ldots, T_d^{1/p^n}\Big]$. We will freely use the following result proven in [F1, Thm. I.2.4(ii)] as a consequence of his theory of *almost étale extensions*. Given normal extensions $R_{U,\infty} \subset S \subset T$ (contained in Ω) such that $S[p^{-1}] \subset T[p^{-1}]$ is finite, étale and Galois with group $G_{T/S}$ we have that $\mathrm{H}^i\big(G_{T/S}, T\big)$ is annihilated by any element of the maximal ideal of $\mathcal{O}_{\overline{K}}$ for every $i \geq 1$. As an application we prove the following:

Lemma 6.2. *The presheaf* $\overline{\mathcal{O}}_{\mathfrak{X}}/p\mathcal{O}_{\mathfrak{X}}$ *is separated i. e., if* $(\mathcal{U}', \mathcal{W}') \to (\mathcal{U}, \mathcal{W})$ *is a covering, the natural map* $\mathcal{O}_{\mathfrak{X}}(\mathcal{U}, \mathcal{W})/p\mathcal{O}_{\mathfrak{X}}(\mathcal{U}, \mathcal{W}) \longrightarrow \mathcal{O}_{\mathfrak{X}}(\mathcal{U}', \mathcal{W}')/p^n\mathcal{O}_{\mathfrak{X}}(\mathcal{U}', \mathcal{W}')$ *is injective. In particular, if* $U = Spec(R_U)$ *is a small affine open subscheme of* X*, the map*

$$\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}} = \mathcal{O}_{\mathfrak{X}}(\overline{R}_{\mathcal{U}})/p\mathcal{O}_{\mathfrak{X}}(\overline{R}_{\mathcal{U}}) \longrightarrow \big(\mathcal{O}_{\mathfrak{X}}/p\mathcal{O}_{\mathfrak{X}}\big)(\overline{R}_{\mathcal{U}})$$

is injective. Furthermore, its cokernel is annihilated by the maximal ideal of $\mathcal{O}_{\overline{K}}$.

Proof. We first prove the first statement. We may assume that U and U' are affine. We may write $\overline{\mathcal{O}}_{\mathfrak{X}}(U, W) = \cup_i S_i$ (resp. $\mathcal{O}_{\mathfrak{X}}(U', W') = \cup_j S'_j$) as the union of normal and finite $R_{\mathcal{U}}$–algebras (resp. $R_{\mathcal{U}'}$–algebras) of Ω, étale after inverting p such that for every i there exists j_i so that S_i is contained in S'_{j_i} and the map $\mathrm{Spec}(S'_{j_i}) \to \mathrm{Spec}(S_i)$ is surjective on prime ideals containing p. Let $x \in S_i \cap p^n S'_{j_i}$. Let $\mathcal{P} \subset S_i$ be a prime ideal over p and let $\mathcal{P}' \subset S'_{j_i}$ be a height one prime ideal over it. Then, $x \in S_{i,\mathcal{P}} \cap p^n S'_{j_i,\mathcal{P}'}$. Hence, $x \in p^n S_{i,\mathcal{P}}$. Thus, x lies in the intersection of all height one prime ideals of S_i so that $x \in S_i$. We conclude that the map $S_i/p^n S_i \to S'_{j_i}/p^n S'_{j_i}$ is injective. The claimed injectivity follows.

We now pass to the second statement. It follows from the first statement that the value of the sheaf $\mathcal{O}_{\mathfrak{X}}/p\mathcal{O}_{\mathfrak{X}}$ at (U, W) is given by the direct limit, over all coverings (U', W') of (U, W) with U' affine, of the elements b in $\mathcal{O}_{\mathfrak{X}}(U', W')/p\mathcal{O}_{\mathfrak{X}}(U', W')$ such that the image of b in the ring $\mathcal{O}_{\mathfrak{X}}(U'', W'')/p\mathcal{O}_{\mathfrak{X}}(U'', W'')$ is 0 where (U'', W'') is the fiber product of (U', W') with itself over (U, W). Hence,

$$\big(\mathcal{O}_{\mathfrak{X}}/p\mathcal{O}_{\mathfrak{X}}\big)\big(\overline{R}_U\big) = \lim_{S,T} \mathrm{Ker}_{S,T,n}$$

where the notation is as follows. The direct limit is taken over all normal $R_{U,\infty}$–subalgebras S of $\overline{R}_U$, finite and étale after inverting p over $R_{U,\infty}[1/p]$, all affine covers $U' \to U$ and all normal extensions $R_{U',\infty} \otimes_{R_U} S \to T$, finite, étale and Galois after inverting p. Eventually, we put $U'' := \mathrm{Spec}(R_{U''})$ to be the fiber product of U' with itself over U i. e., $R_{U''} := R_{U'} \otimes_{R_U} R_{U'}$. We let

$$\mathrm{Ker}_{S,T,n} := \mathrm{Ker}\left(T/pT \rightrightarrows \widetilde{T}_S/p\widetilde{T}_S\right),$$

where $\widetilde{T}_S$ is the normalization of the base change to $R_{U''}$ of $T\otimes_{(R_{U',\infty}\otimes_{R_{U,\infty}}S)}T$.

Study of Ker$_{S,T,n}$. For every S and T as above, write $G_{S,T}$ for the Galois group of $T\otimes_{\mathcal{O}_K}K$ over $S\otimes_{R_{U,\infty}}R_{U',\infty}\otimes_{\mathcal{O}_K}K$. Then, $\widetilde{T}_S$ is simply the product $\prod_{g\in G_{S,T}}T\otimes_{R_{U'}}R_{U''}$ where we view $R_{U''}$ as $R_{U'}$–algebra choosing the left action. Hence, we have

$$\mathrm{Ker}_{S,T,n}=\mathrm{Ker}\left(T/pT\rightrightarrows\prod_{g\in G_{S,T}}\frac{T\otimes_{R_{U'}}R_{U''}}{pT\otimes_{R_{U'}}R_{U''}}\right),$$

where the two maps in the display are $a\mapsto(a,\cdots,a)$ and $a\mapsto\big(g(a)\big)_{g\in G_{S,T}}$.

Study of Coker$(S/pS\longrightarrow \mathit{Ker}_{S,T,n})$. For the rest of this proof we make the following notations: if B is a normal $R_{U,\infty}$-algebra we denote by $B':=B\otimes_{R_{U,\infty}}R_{U',\infty}=B\otimes_{R_U}R_{U'}$, also $B'':=B\otimes_{R_{U',\infty}}R_{U'',\infty}=B\otimes_{R_{U'}}R_{U''}$. Note that B' and B'' are normal. We then get a commutative diagram

$$\begin{array}{ccccccc}0&\to&S/pS&\longrightarrow&S'/pS'&\rightrightarrows&S''/pS''\\ &&\downarrow&&\downarrow\alpha&&\downarrow\beta\\ 0&\to&\mathrm{Ker}_{S,T,n}&\longrightarrow&T/pT&\rightrightarrows&\widetilde{T}_S/p\widetilde{T}_S=\prod_{g\in G_{S,T}}\big(T''/pT''\big).\end{array}$$

The top row is exact and the bottom row is exact by construction. Since $S'\subset T$ and $S''\subset T''$ are finite extensions of normal rings, the maps α and β are injective. Define Z as $Z:=\mathrm{Coker}\big(S'/pS'\to(T/pT)^{G_{S,T}}\big)\subset\mathrm{Coker}(\alpha)$ and Y as $\mathrm{Coker}(S/pS\longrightarrow\mathrm{Ker}_{S,T,n})$. Since $\mathrm{Ker}_{S,T,n}$ is $G_{S,T}$-invariant, the image of Y in $\mathrm{Coker}(\alpha)$ is contained in Z. Since α and β are injective, the map $Y\to Z$ is injective. Consider the exact sequence

$$0\longrightarrow S'/pS'=T^{G_{S,T}}/pT^{G_{S,T}}\longrightarrow\big(T/pT\big)^{G_{S,T}}\longrightarrow\mathrm{H}^1\big(G_{S,T},T\big).$$

Then, $Y\subset Z\subset\mathrm{H}^1\big(G_{S,T},T\big)$. Since $R_{U',\infty}\to T$ is almost étale, the group $\mathrm{H}^1\big(G_{S,T},T\big)$ is annihilated by any element of the maximal ideal of $\mathcal{O}_{\overline{K}}$ thanks to Faltings' results recalled above. This implies the last claim. □

From this one can deduce the following:

Corollary 6.3. *Let $U=Spec(R_U)$ be a small open affine subset of X. For every $n\in\mathbb{N}$ we have an injective map*

$$W_n\big(\overline{R}_U/p\overline{R}_U\big)\otimes_{W_n}\big(A_{\mathrm{cris}}/p^nA_{\mathrm{cris}}\big)\left[\frac{X_{1,n}^m}{m!},\ldots,\frac{X_{d,n}^m}{m!}\right]\longrightarrow\big(\mathbb{A}_{\mathrm{cris}}/p^n\mathbb{A}_{\mathrm{cris}}\big)(\overline{R}_{\mathcal{U}})$$

with cokernel annihilated by the elements $[\zeta_m]-1\in W_n$ for every $m\in\mathbb{N}$.

We refer to [AI2] for details. Since we now know how to "almost" compute the localizations of $\mathbb{A}_{\mathrm{cris}}/p^n\mathbb{A}_{\mathrm{cris}}$ we can proceed to the computation of $\mathrm{R}^iv_*\left(\mathbb{A}_{\mathrm{cris}}/p^n\mathbb{A}_{\mathrm{cris}}\right)$ using 4.4. Our main theorem, stating that $\mathrm{R}^iv_*(\mathbb{B}_{\mathrm{cris}})=0$ for $i\geq1$ will then amount to prove, see [AB], that

$$\mathrm{H}^i\left(G_U,W_n\big(\overline{R}_U/p\overline{R}_U\big)\otimes_{W_n}\big(A_{\mathrm{cris}}/p^nA_{\mathrm{cris}}\big)\left[\frac{X_{1,n}^m}{m!},\ldots,\frac{X_{d,n}^m}{m!}\right]\right)$$

is annihilated by a fixed power of t, independent of n and U, for every $i\geq1$. These computations, and similar ones for the H^0 and for the cohomology of the filtration, are the content of [AB]. We simply sketch some of the ideas involved.

Consider the extension $R_U \otimes_{\mathcal{O}_K} \overline{K} \subset R_{U,\infty}[p^{-1}]$. It is Galois with group Γ_U isomorphic to $\mathbb{Z}_p^d$. Due to Faltings' almost étale theory, it suffices to prove that

$$\mathrm{H}^i\left(\Gamma_U, W_n\big(R_{U,\infty}/pR_{U,\infty}\big) \otimes_{W_n} \big(A_{\mathrm{cris}}/p^n A_{\mathrm{cris}}\big)\left[\frac{X_{1,n}^m}{m!}, \ldots, \frac{X_{d,n}^m}{m!}\right]\right)$$

is annihilated by a fixed power of t for every $i \geq 1$. Note that there exists a unique homomorphism of $\mathcal{O}_K$-algebras $R_U \longrightarrow W_n\big(R_{U,\infty}/pR_{U,\infty}\big)$ sending $T_i \mapsto \widetilde{T}_{i,n}$. Such map is not G_U-equivariant! Write $\widetilde{R}_{U,n}$ for its image. One can prove that

$$W_n\big(R_{U,\infty}/pR_{U,\infty}\big) \otimes_{W_n} \big(A_{\mathrm{cris}}/p^n A_{\mathrm{cris}}\big) \cong \widetilde{R}_{U,n} \otimes_{W_n(k)} \big(A_{\mathrm{cris}}/p^n A_{\mathrm{cris}}\big)\left[\widetilde{T}_{1,n}^{1/p^m}, \ldots, \widetilde{T}_{d,n}^{1/p^m}\right]_{m \in \mathbb{N}}.$$

Set $A_n = \widetilde{R}_{U,n} \otimes_{W_n(k)} \big(A_{\mathrm{cris}}/p^n A_{\mathrm{cris}}\big)$ and let B_n be the A_n-submodule generated by the elements $\widetilde{T}_{i,n}^{1/p^m}$ for some $m \geq 1$. Then,

$$A_n\left[\frac{X_{1,n}^m}{m!}, \ldots, \frac{X_{d,n}^m}{m!}\right] \cong R_U/p^n R_U \otimes_{W_n(k)} \big(A_{\mathrm{cris}}/p^n A_{\mathrm{cris}}\big)\left[\frac{X_{1,n}^m}{m!}, \ldots, \frac{X_{d,n}^m}{m!}\right]$$

so that, in particular, it is stable under the action of Γ_U and similarly $B_n\left[\frac{X_{1,n}^m}{m!}, \ldots, \frac{X_{d,n}^m}{m!}\right]$ is stable under the action of Γ_U. The problem is reduced to prove:

Lemma 6.4. *(1) The group* $\mathrm{H}^i\left(\Gamma_U, B_n\left[\frac{X_{1,n}^m}{m!}, \ldots, \frac{X_{d,n}^m}{m!}\right]\right)$ *is annihilated by a fixed power of* t *for every* $i \geq 1$.

(2) The group $\mathrm{H}^i\left(\Gamma_U, R_U/p^n R_U \otimes_{W_n(k)} \big(A_{\mathrm{cris}}/p^n A_{\mathrm{cris}}\big)\left[\frac{X_{1,n}^m}{m!}, \ldots, \frac{X_{d,n}^m}{m!}\right]\right)$ *is annihilated by a fixed power of* t *for every* $i \geq 1$.

Proof. We limit ourself to the case that $d = 1$. A Koszul complex argument reduces the general case to this case and we refer to [AB] for details. We remark that in this case Γ_U is the free $\mathbb{Z}_p$-module generated by the element γ which acts on T^{1/p^n} via multiplication by ζ_n. Then, H^1 is simply the cokernel of the operator $\gamma - 1$. We write T for T_1, $\widetilde{T}$ for $\widetilde{T}_{1,n}$ and X for $X_{1,n}$. Since

$$\gamma(X_1) = T - \varepsilon\widetilde{T}_{=}(1-\varepsilon)T + \varepsilon X$$

and hence

$$\begin{aligned}(\gamma - 1)\big(X^{[m]}\big) &= \big((1-\varepsilon)T + \varepsilon X\big)^{[m]} - X^{[m]} \\ &= (\varepsilon^m - 1)X^{[m]} + \sum_{j=1}^{m} (1-\varepsilon)^{[j]} T^j [\varepsilon]^{m-j} X^{[m-j]} \\ &= (1-\varepsilon)\Big(\mu_m X^{[m]} + T\varepsilon^{m-1} X^{[m-1]} + \sum_{j=2}^{m} \beta_j T^j \varepsilon^{m-j} X^{[m-j]}\Big)\end{aligned}$$

where $y^{[m]} := y^m/m!$, $\mu_m = -\frac{\varepsilon^m - 1}{[\varepsilon]-1}$ and $\beta_j = \frac{\varepsilon^j - 1}{\varepsilon - 1}$ which is an element in $\mathrm{Fil}^1 A_{\mathrm{cris}}$ for $j \geq 2$.

(1) Take $b = \sum_{m=0}^{N} b_m X^{[m]}$ with $b_m \in B_n$ for every m. Suppose $N > 0$. Since the cokernel of $\gamma - 1$ on X_n is annihilated by $\varepsilon^{1/p} - 1$, there exists $a_N \in X$ such that $(\gamma - 1)(a_N) = \big(1 - [\varepsilon]^{\frac{1}{p}}\big) b_N$. Then,

$$(\gamma - 1)\big(a_N X^{[N]}\big) = \gamma(a_N)(\gamma - 1)\big(X^{[N]}\big) + (\gamma - 1)(a_N) X^{[N]}$$

and hence

$$\big(1-\varepsilon^{\frac{1}{p}}\big)b-(\gamma-1)\big(a_N X^{[N]}\big)=-\gamma(a_N)(\gamma-1)\big(X^{[N]}\big)+\big(1-\varepsilon^{\frac{1}{p}}\big)\sum_{m=0}^{N-1} b_m X^{[m]}$$
$$\in \big(1-\varepsilon^{\frac{1}{p}}\big)\sum_{m=0}^{N-1} B_n X^{[m]}.$$

Proceeding by descending induction on N we conclude that $\big(1-\varepsilon^{\frac{1}{p}}\big)X_n\left[\frac{X_{1,n}^m}{m!},\ldots,\frac{X_{d,n}^m}{m!}\right]$ is contained in the image of $1-\gamma$. this proves (1).

(2) The matrix of $\gamma-1$ on the module $\sum_{i=0}^N R_U/p^nR_U\otimes_{W_n(k)}\big(A_{\rm cris}/p^nA_{\rm cris}\big)X^{[i]}$ with respect to $1,X,\ldots,X^{[N]}$ is $(1-\varepsilon)G_{n,N}$ with

$$G_n^{(N)}=\begin{pmatrix}
0 & T_i & T^2\beta_2 & T^3\beta_3 & \cdots & \cdots & T^{N-1}\beta_{N-1} & T^N\beta_N\\
0 & \mu_1 & T\varepsilon & T^2\beta_2\varepsilon & \ddots & \ddots & T^{N-2}\beta_{N-2}\varepsilon & T^{N-1}\beta_{N-1}\varepsilon\\
\vdots & \ddots & \mu_2 & T\varepsilon^2 & \ddots & & & T^{N-2}\beta_{N-2}\varepsilon^2\\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & & & \ddots & \ddots & \ddots & T^2\beta_2\varepsilon^{N-3} & T^3\beta_3\varepsilon^{N-3}\\
\vdots & & & & \ddots & \mu_{N-2} & T\varepsilon^{N-2} & T^2\beta_2\varepsilon^{N-2}\\
\vdots & & & & & \ddots & \mu_{N-1} & T\varepsilon^{N-1}\\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \mu_N
\end{pmatrix}$$

Let $\widetilde{G}_{n,N}$ be the matrix obtained erasing the first column and the last row. Then, $\widetilde{G}_{n,N}=\widetilde{U}_{n,N}+\widetilde{N}_{n,N}$ with

$$\widetilde{U}_{n,N}=\begin{pmatrix}
T & 0 & \cdots & \cdots & \cdots & 0\\
\mu_1 & T\varepsilon & \ddots & & & \vdots\\
0 & \mu_2 & T\varepsilon^2 & \ddots & & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & & \ddots & \mu_{N-2} & T\varepsilon^{N-2} & 0\\
0 & \cdots & \cdots & 0 & \mu_{N-1} & T\varepsilon^{N-1}
\end{pmatrix}$$

is invertible and $\widetilde{N}_{n,N}$ has nilpotent coefficients and, hence, it is nilpotent. Hence, also $\widetilde{U}_{n,N}^{-1}\widetilde{N}_{n,N}$ is nilpotent and $\widetilde{G}_{n,N}=\widetilde{U}_{n,N}\big(\mathrm{I}_N+\widetilde{U}_{n,N}^{-1}\widetilde{N}_{n,N}\big)$ is invertible. This implies that the cokernel of $\gamma-1$ on $R_U/p^nR_U\otimes_{W_n(k)}\big(A_{\rm cris}/p^nA_{\rm cris}\big)\left[\frac{X_{1,n}^m}{m!},\ldots,\frac{X_{d,n}^m}{m!}\right]$ is annihilated by $\varepsilon-1$. □

7 The proof of the comparison isomorphism

Definition 7.1. We say that a $\mathbb{Q}_p$-adic étale sheaf $\mathbb{L}=(\mathbb{L}_n)$ and a filtered-F-isocrystal $\mathcal{E}$ on X_K are *associated* if we have an isomorphism $\mathcal{E}\otimes_{\mathcal{O}_X}\mathbb{B}_{\rm cris}\cong\mathbb{L}\otimes_{\mathbb{Z}_p}\mathbb{B}_{\rm cris}$ in $\mathrm{Ind}big\big(\mathrm{Sh}(\mathfrak{X})^{\mathbb{N}}\big)$ (compatibly with all extra structures i.e., Frobenius, Filtrations, connections and $\mathbb{A}_{\rm cris}$-module structures).

It follows from [Bri] that $\mathcal{E}^\vee=\mathrm{Hom}\big(\mathcal{E},(\mathcal{O}_X,d)\big)$ and $\mathbb{L}^\vee=\big(\mathrm{Hom}\big(\mathbb{L}_n,\mathbb{Z}/p^n\mathbb{Z}\big)\big)_n$ are associated since $\mathcal{E}$ and $\mathbb{L}$ are.

Due to 5.4 we have an exact sequence

$$0 \longrightarrow \mathbb{L}\otimes_{\mathbb{Z}_p}\mathbb{B}_{\mathrm{cris}}^{\nabla} \longrightarrow \mathcal{E}\otimes_{\mathcal{O}_X}\mathbb{B}_{\mathrm{cris}}\otimes_{\mathcal{O}_X}\Omega^{\bullet}_{X/\mathcal{O}} \longrightarrow 0.$$

In particular,

$$\mathrm{H}^i\big(\mathfrak{X},\mathbb{L}\otimes_{\mathbb{Z}_p}\mathbb{B}_{\mathrm{cris}}^{\nabla}\big)\cong \mathrm{H}^i\big(\mathfrak{X},\mathcal{E}\otimes_{\mathcal{O}_X}\mathbb{B}_{\mathrm{cris}}\otimes_{\mathcal{O}_X}\Omega^{\bullet}_{X/\mathcal{O}}\big).$$

The isomorphism is compatible with filtration, G_K-action and Frobenius. Due to 6.1 the latter coincides with $\mathrm{H}^i\big(X,\mathcal{E}\otimes_{\mathcal{O}_X}\Omega^{\bullet}_{X/\mathcal{O}_K}\widehat{\otimes}_{\mathcal{O}_K}B_{\mathrm{cris}}\big)$ i. e., $\mathrm{H}^i_{\mathrm{dR}}\big(X,\mathcal{E}\big)\otimes_{\mathcal{O}_K}B_{\mathrm{cris}}$. The isomorphism is compatible with Frobenius and G_K-action and it can be proven to be compatible with filtrations as well (we refer to [AI2] for the non-trivial proof of this fact).

The same isomorphisms hold if we work with $\widehat{\mathfrak{X}}$ instead of $\mathfrak{X}$. In particular, $\mathrm{H}^i\big(\mathfrak{X},\mathbb{L}\otimes_{\mathbb{Z}_p}\mathbb{B}_{\mathrm{cris}}^{\nabla}\big)\cong \mathrm{H}^i\big(\widehat{\mathfrak{X}},\mathbb{L}\otimes_{\mathbb{Z}_p}\mathbb{B}_{\mathrm{cris}}^{\nabla}\big)$. Put $V_i:=\mathrm{H}^i\big(\mathfrak{X},\mathbb{L}\big)\cong \mathrm{H}^1(X_{\overline{K}}^{\mathrm{et}},\mathbb{L})$ and $D_i:=\mathrm{H}^i\big(X,\mathrm{H}^i\big(X,\mathcal{E}\otimes_{\mathcal{O}_X}\Omega^{\bullet}_{X/\mathcal{O}_K}\big)$. One can prove that $V_i\cong \mathrm{H}^i\big(\widehat{\mathfrak{X}},\mathbb{L}\big)$; this is a GAGA type theorem and passes via the variants (β) and (γ) of 4.2; see [AI1]. It then follows from 5.6 that we have long exact sequences

$$\begin{array}{ccccccccc}
\cdots\longrightarrow & V_i & \xrightarrow{\alpha_i} & \mathrm{Fil}^0\big(D_i\otimes_K B_{\mathrm{cris}}\big) & \xrightarrow{1-\varphi} & D_i\otimes_K B_{\mathrm{cris}} & \xrightarrow{\epsilon_i} & V_{i+1} \\
 & \downarrow\beta_i & & \downarrow\gamma_i & & \| & & \downarrow\beta_{i+1} \\
\cdots\longrightarrow & \big(D_i\otimes_K B_{\mathrm{cris}}\big)^{\varphi=1} & \xrightarrow{\omega_i} & D_i\otimes_K B_{\mathrm{cris}} & \xrightarrow{1-\varphi} & D_i\otimes_K B_{\mathrm{cris}} & \longrightarrow & \big(D_{i+1}\otimes_K B_{\mathrm{cris}}\big)^{\varphi=1}
\end{array}$$

We recall a criterion from [CF] for a filtered-F-module D over K to be admissibile i e., to be associated to a crystalline representation of G_K. Let

$$\delta(D)\colon (D\otimes_K B_{\mathrm{cris}})^{\varphi=1}\longrightarrow \frac{D\otimes_K B_{\mathrm{cris}}}{\mathrm{Fil}^0\big(D\otimes_K B_{\mathrm{cris}}\big)}$$

be the natural map. Put $V_{\mathrm{cris}}(D):=\mathrm{Ker}(\delta_D)$

Proposition 7.2 ([CF]). *The filtered φ-module D over K is admissible if and only if* (a) *$V_{\mathrm{cris}}(D)$ is a finite dimensional $\mathbb{Q}_p$-vector space and* (b) *$\delta(D)$ is surjective.*

Moreover, if D is admissible then $V:=V_{\mathrm{cris}}(D)$ is a crystalline representation of G_M and $D_{\mathrm{cris}}(V)\cong D$.

We apply this to the above exact sequence. Consider the part of the above diagram in degrees 0 and $2d$.

$$\begin{array}{ccccccccc}
\cdots\xrightarrow{\epsilon_{2d-1}} & V_{2d} & \xrightarrow{\alpha_{2d}} & \mathrm{Fil}^0(D_{2d}\otimes_{M_0}B_{\mathrm{cris}}) & \xrightarrow{1-\varphi} & D_{2d}\otimes_{M_0}B_{\mathrm{cris}} & \longrightarrow & 0 \\
 & \downarrow\beta_{2d} & & \downarrow\gamma_{2d} & & \| & & \\
\cdots\longrightarrow & \big(D_{2d}\otimes_{M_0}B_{\mathrm{cris}}\big)^{\varphi=1} & \xrightarrow{\omega_{2d}} & D_{2d}\otimes_{M_0}B_{\mathrm{cris}} & \xrightarrow{1-\varphi} & D_{2d}\otimes_{M_0}B_{\mathrm{cris}} & \longrightarrow & 0
\end{array}$$

Note that $\delta(D_{2d})$ is the composite of

$$\big(D_{2d}\otimes_{M_0}B_{\mathrm{cris}}\big)^{\varphi=1}\xrightarrow{\omega_{2d}} D_{2d}\otimes_{M_0}B_{\mathrm{cris}}\longrightarrow \mathrm{Coker}(\gamma_{2d})$$

and also that $\mathrm{Ker}(\delta(D_{2d}))=\mathrm{Ker}\big((1-\varphi)\colon \mathrm{Fil}^0(D_{2d}\otimes_{M_0}B_{\mathrm{cris}})\longrightarrow D_{2d}\otimes_{M_0}B_{\mathrm{cris}}\big)$. It follows that α_{2d} induces a surjective $\mathbb{Q}_p$-linear map $V_{2d}\longrightarrow \mathrm{Ker}(\delta(D_{2d}))$ and that $\delta(D_{2d})$ is surjective. We deduce from 7.2 that D_{2d} is admissible and that we have a $\mathbb{Q}_p$-linear, surjective homomorphism $V_{2d}\longrightarrow V_{\mathrm{cris}}(D_{2d})$ which is G_K-equivariant.

Let D_i^* be $\mathrm{H}^i\big(X,\mathrm{H}^i\big(X,\mathcal{E}^\vee\otimes_{\mathcal{O}_X}\Omega^{\bullet}_{X/\mathcal{O}_K}\big)$ and V_i^* be $\mathrm{H}^i\big(\mathfrak{X},\mathbb{L}\big)\cong \mathrm{H}^1(X_{\overline{K}}^{\mathrm{et}},\mathbb{L}^\vee)$. Then, $D_0^*\cong D_{2d}$ as filtered F-modules (up to shifting the filtration and twisting Frobenius accordingly to Poincaré duality for filtered F-modules). In particular, D_0^* is admissible . Similarly V_0^* is the dual of V_0 (up to Tate twist according to Poincaré duality in étale cohomology). We also have the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_0^* & \xrightarrow{\alpha_0^*} & \mathrm{Fil}^0(D_0^*\otimes_{M_0} B_{\mathrm{cris}}) & \xrightarrow{1-\varphi} & D_0^*\otimes_{M_0} B_{\mathrm{cris}} & \xrightarrow{\epsilon_0^*} & \cdots \\ & & \downarrow \beta_0^* & & \downarrow \gamma_0^* & & || & & \\ 0 & \longrightarrow & \big(D_0^*\otimes_{M_0} B_{\mathrm{cris}}\big)^{\varphi=1} & \xrightarrow{\omega_0^*} & D_0^*\otimes_{M_0} B_{\mathrm{cris}} & \xrightarrow{1-\varphi} & D_0^*\otimes_{M_0} B_{\mathrm{cris}} & \longrightarrow & \cdots \end{array}$$

It follows that $V_0^* \cong \mathrm{Ker}(\delta(D_0^*)) = V_{\mathrm{cris}}(D_0^*)$. Hence, $\dim_{\mathbb{Q}_p}(V_{2d}) = \dim_{\mathbb{Q}_p}(V_0^*) = \dim_K(D_0^*) = \dim_K(D_{2d}) = \dim_{\mathbb{Q}_p}(V_{\mathrm{cris}}(D_{2d}))$ and therefore $V_{2d} \cong V_{\mathrm{cris}}(D_{2d})$. This proves our statement for $i=0$ and $i=2d$.

Let us remark at the same time that as α_{2d} is injective, $\epsilon_{2d-1}=0$. An easy diagram chase shows that $\epsilon_0^*=0$ and therefore we can continue with $i=1$ along exactly the same lines as for $i=0$. By induction the Theorem follows.

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