WEIL AND GROTHENDIECK APPROACHES TO ADELIC POINTS

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1. Introduction

In [We, Ch. 1], Weil defines a process of "adelization" of algebraic varieties over global fields. There is an alternative procedure, due to Grothendieck, using adelic points. One aim of this (largely) expository note is to prove that for schemes of finite type over global fields (i.e., without affineness hypotheses), and also for separated algebraic spaces of finite type over such fields, Weil's adelization process naturally coincides (as a set) with the set of adelic points in the sense of Grothendieck (and that in the affine case the topologies defined by these two viewpoints coincide; Grothendieck's approach doesn't provide a topology beyond the affine case). The other aim is to prove in general that topologies obtained by Weil's method satisfy good functorial properties, including expected behavior with respect to finite flat Weil restriction of scalars. The affine case suffices for most applications, but the non-affine case is useful (e.g., adelic points of G/P for connected reductive groups G and parabolic subgroups P). We also discuss topologizing $X(k)$ for possibly non-separated algebraic spaces X over locally compact fields k; motivation for this is given in Example 5.5.

Although everything we prove (except perhaps for the case of algebraic spaces) is "well known" folklore, and [Oes, I, §3] provides an excellent summary in the affine case, some aspects are not so easy to extract from the available literature. Moreover, (i) some references that discuss the matter in the non-affine case have errors in the description of the topology on adelic points, and (ii) much of what we prove is needed in my paper [Con], or in arithmetic arguments in [CGP]. In effect, these notes can be viewed as an expanded version of [Oes, I, §3], and I hope they will provide a useful general reference on the topic of adelic points of algebro-geometric objects (varieties, schemes, algebraic spaces) over global fields.

In §2 we carry out Grothendieck's method in the affine case over any topological ring R, characterizing the topology on sets of R-points by means of several axioms. The generalization to arbitrary schemes of finite type via a method of Weil is developed in §3. We explore properties of these topologies in §4, especially for adelic points and behavior with respect to Weil restriction of scalars. Finally, in §5 everything is generalized to the case of algebraic spaces.

Notation. We write $\mathbf{A}_F$ to denote the adele ring of a global field F, and likewise $\mathbf{A}_F^n$ denotes Euclidean n-space over $\mathbf{A}_F$. There is no risk of confusion with the common use of such notation to denote affine n-space over $\operatorname{Spec} F$ since we avoid ever using this latter meaning for the notation.

2. Preliminary functorial considerations

Let F be a global field and S a finite non-empty set of places of F, with S always understood to contain the set of archimedean places of F. We let $\mathbf{A}_{F,S} \subseteq \mathbf{A}_F$ denote the open subring of adeles that are integral at all places away from S, so the topological ring $\mathbf{A}_F$ is the direct limit of the open subrings $\mathbf{A}_{F,S}$ over increasing S. For a separated finite type F-scheme X, we would like to endow the set $X(\mathbf{A}_F)$ with a natural structure of Hausdorff locally compact topological space in a manner that is functorial in $\mathbf{A}_F$ and compatible with the formation of fiber products (for topological spaces and F-schemes); in §5 we will address the case of algebraic spaces.

For affine X the coordinate ring $\Gamma(X, \mathscr{O}_X)$ is F-isomorphic to $F[t_1, \ldots, t_n]/I$, so as a set $X(\mathbf{A}_F)$ is identified with the closed subset of the adelic Euclidean space $\mathbf{A}_F^n$ where the functions $f : \mathbf{A}_F^n \to \mathbf{A}_F$ for

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$f \in I$ all vanish. This zero set has a locally compact subspace topology. To see that this topology transferred to $X(\mathbf{A}_F)$ is independent of the choice of presentation of $\Gamma(X, \mathscr{O}_X)$, it is more elegant to uniquely characterize this construction by means of functorial properties, as the proof of the following result shows:

Proposition 2.1. *Let R be a topological ring. There is a unique way to topologize $X(R)$ for affine finite type R-schemes X in a manner that is functorial in X, compatible with the formation of fiber products, carries closed immersions to topological embeddings, and for $X = \operatorname{Spec} R[t]$ gives $X(R) = R$ its usual topology. Explicitly, if A is the coordinate ring of X then $X(R)$ has the weakest topology relative to which all maps $X(R) \to R$ induced by elements of A are continuous, or equivalently the natural injection of $X(R) = \operatorname{Hom}_{R\text{-alg}}(A, R)$ into $\operatorname{Hom}_{\mathrm{Set}}(A, R) = R^A$ endowed with the product topology is a homeomorphism onto its image.*

If R is Hausdorff, then $X(R)$ is Hausdorff and closed immersions $X \hookrightarrow X'$ induce closed embeddings $X(R) \to X'(R)$. If in addition R is locally compact, then $X(R)$ is locally compact.

The Hausdorff property is necessary to require if we want closed immersions to go over to closed embeddings. Indeed, by considering the origin in the affine line we see that such a topological property forces the identity point in R to be closed, and compatibility with products makes $X(R)$ a topological group when X is an R-group scheme, so this forces R to be Hausdorff since (viewing $R = \mathbf{G}_{\mathrm{a}}(R)$ as an additive topological group) a topological group whose identity point is closed must be Hausdorff (because in any category admitting fiber products, the diagonal morphism for a group object is a base change of the identity section). Viewing the topology on $X(R)$ as a subspace topology from R^A is reminiscent of how Milnor topologizes manifolds in [Mil].

Proof. To see uniqueness, we pick a closed immersion $i : X \hookrightarrow \operatorname{Spec} R[t_1, \dots, t_n]$. By forming the induced map on R-points and using compatibility with products (view affine n-space as product of n copies of the affine line), as well as the assumption on closed immersions, the induced set map $X(R) \hookrightarrow R^n$ is a topological embedding into R^n endowed with its usual topology. This proves the uniqueness, and that $X(R)$ has to be Hausdorff when R is Hausdorff. Likewise, we see that $X(R)$ is closed in R^n in the Hausdorff case, so when R is also locally compact then so is $X(R)$.

There remains the issue of existence. Pick an R-algebra isomorphism

$$A := \Gamma(X, \mathscr{O}_X) \simeq R[t_1, \dots, t_n]/I \tag{2.1.1}$$

for an ideal I, and identify $X(R)$ with the subset of R^n on which the elements of I (viewed as functions $R^n \to R$) all vanish. We wish to endow $X(R)$ with the subspace topology, and the main issue is to check that this construction is independent of the choice of (2.1.1) and enjoys all of the desired properties. We claim that the topology defined using (2.1.1) is the same as the subspace topology defined by the canonical injection $X(R) \to R^A$ (so the definition of this topology is independent of the choice of (2.1.1)). Let $a_1, \dots, a_n \in A$ correspond to $t_1 \bmod I, \dots, t_n \bmod I$ via (2.1.1), so the injection $X(R) \to R^n$ is the composition of the natural injection $X(R) \to R^A$ and the map $R^A \to R^n$ defined by $(a_1, \dots, a_n) \in A^n$. Hence, every open set in $X(R)$ is induced by an open set in R^A because $R^A \to R^n$ is continuous. Since every element of A is an R-polynomial in $a_1, \dots, a_n$ and R is a topological ring (so polynomial functions $R^n \to R$ over R are continuous), it follows that the map $X(R) \to R^A$ is also continuous. Thus, indeed $X(R)$ has been given the subspace topology from R^A, so the topology on $X(R)$ is clearly well-defined and functorial in X.

Consider a closed immersion $i : X \hookrightarrow X'$ corresponding to a surjective R-algebra map between coordinate rings $h : A' \twoheadrightarrow A$. The natural map $j : R^A \to R^{A'}$ defined by $(r_a) \mapsto (r_{h(a')})$ is visibly a topological embedding; it topologically identifies R^A with the subset of $R^{A'}$ cut out by a collection of equalities among components, so j is a closed embedding when R is Hausdorff. We have $X'(R) \cap j(R^A) = j(X(R))$ because a set-theoretic map $A \to R$ is an R-algebra map if and only if its composition with the surjection $h : A' \to A$ is an R-algebra map. Hence, $i : X(R) \to X'(R)$ is an embedding of topological spaces, and is a closed embedding when R is Hausdorff. By forming products of closed immersions into affine spaces, we see that $(X \times_{\operatorname{Spec} R} X')(R) \to X(R) \times X'(R)$ is a topological isomorphism via reduction to the trivial special case when X and X' are affine spaces.

Finally, to see that $(X \times_Y Z)(R) \to X(R) \times_{Y(R)} Z(R)$ is a topological isomorphism (for given maps $X \to Y$ and $Z \to Y$ between affine R-schemes), consider the isomorphism

$$X \times_Y Z \simeq (X \times_R Z) \times_{Y \times_R Y} Y$$

and its topological counterpart. Since we have already checked compatibility with absolute products (over the final object in the category), the separatedness of Y over R reduces us to the case in which one of the structure maps of the scheme fiber product is a closed immersion. But we have already seen that closed immersions are carried into topological embeddings, so we are done. ■

Example 2.2. If $R \to R'$ is a continuous map of topological rings (e.g., the inclusion of F into $\mathbf{A}_F$ or of $\mathscr{O}_{F,S}$ into $\mathbf{A}_{F,S}$, with the subring having the discrete topology in both cases), then for any affine finite type R-scheme X with base change X' over R', the natural map $X(R) \to X(R') = X'(R')$ is continuous, and when $R \to R'$ is a topological embedding then so is $X(R) \to X(R')$. Moreover, if R' is closed (resp. open) in R then $X(R) \to X(R')$ is a closed (resp. open) embedding. These claims are immediate from the construction of the topologies by means of closed immersions of X into an affine space over R (and the base change on this to give a closed immersion of X' into an affine space over R'). The same argument shows that if R is discrete in R' then $X(R)$ is discrete in $X(R')$.

Example 2.3. Since F is discrete in $\mathbf{A}_F$, so F^n is discrete in $\mathbf{A}_F^n$, it follows that for any affine finite type F-scheme X, $X(F) \to X(\mathbf{A}_F)$ is a topological embedding onto a discrete subset. Similarly, if X is affine of finite type over $\mathscr{O}_{F,S}$, then $X(\mathscr{O}_{F,S})$ is a discrete subset of $X(\mathbf{A}_{F,S})$. If X is affine of finite type over a discrete valuation ring R with fraction field L then $X(R)$ is open and closed in $X(L) = X_L(L)$.

Example 2.4. Let $R \to R'$ be a module-finite ring extension that makes R' locally free as an R-module. Assume that R and R' are endowed with topological ring structures such that R' has the quotient topology from one (equivalently, any) presentation as a quotient of a finite free R-module. In particular, R has the subspace topology from R' because R' is projective as an R-module (so the inclusion $R \to R'$ admits an R-linear splitting). The main examples of interest are a finite extension of complete discrete valuation rings, local fields, or adele rings of global fields. For an affine R'-scheme X' of finite type, consider the Weil restriction $\mathscr{X} = \mathrm{Res}_{R'/R}(X')$ that is an affine R-scheme of finite type [BLR, §7.6]. (In [CGP, App. A.5] there is given a detailed discussion of properties of Weil restriction, supplementing [BLR, §7.6].) There is a canonical bijection of sets $X'(R') = \mathscr{X}(R)$, and by viewing X' and $\mathscr{X}$ as an R'-scheme and R-scheme respectively we get topologies on both sides of this equality.

We claim that these two topologies agree. Using a closed immersion of X' into an affine space over R' reduces us to the case when X' is such an affine space, because Weil restriction carries closed immersions to closed immersions in the affine case. Choose a finite free R-module P and an R-linear surjection from the dual $P^\vee$ onto the dual module ${R'}^\vee = \mathrm{Hom}_R(R', R)$. The dual map $R' \to P$ is a direct summand, so for any R-algebra A the natural map $R' \otimes_R A \to P \otimes_R A$ is injective and functorially defined by a system of R-linear equations in A. For $M = R^{\oplus n}$ with a suitable $n \geq 0$ we have $X' = \mathrm{Spec}(\mathrm{Sym}_{R'}(M'))$ with $M' = R' \otimes_R M$, so $\mathscr{X}$ is naturally a closed subscheme of $\mathrm{Spec}(\mathrm{Sym}_R(M \otimes_R P^\vee))$. The set $X'(R') = \mathrm{Hom}_{R'}(M', R') = \mathrm{Hom}_R(M, R')$ is endowed with its natural topology as a finite free R'-module, and via the inclusion $R' \hookrightarrow P$ the set $\mathscr{X}(R)$ is $\mathrm{Hom}_R(M, R') = M^\vee \otimes_R R'$ with the subspace topology from $M^\vee \otimes_R P$. Thus, the agreement of topologies comes down to R' inheriting its given topology as a subspace of P. But R' is a direct summand of P, so the subspace topology on R' coincides with the quotient topology via a surjection from P. By hypothesis, such a quotient topology is the given topology on R'.

3. Elimination of affineness hypotheses

When attempting to generalize Proposition 2.1 beyond the affine case, an immediate problem is that if U is an open affine in an affine X of finite type over R, then $U(R) \to X(R)$ need not be an open embedding; it may even fail to be a topological embedding. For example, if X is the affine line over R and U is the complement of the origin, then $U(R) \hookrightarrow X(R)$ is the map $R^\times \to R$ where R has its usual topology but $R^\times$ has a structure of topological group coming from the affine model $U = \mathbf{G}_m \simeq \mathrm{Spec}\, R[x,y]/(xy-1)$ inside the plane (i.e., $r, r' \in R^\times$ are close when r is near r' in R and r^{-1} is near ${r'}^{-1}$ in R). The example of adele rings shows that the unit group of a topological ring need not be a topological group with respect to the induced topology from the ring. Since the topology on $R^\times = \mathbf{G}_m(R)$ is a topological group structure, we see that in such examples the inclusion $R^\times \to R$ cannot be a topological embedding.

More generally, if $X = \operatorname{Spec} A$ and $U = \operatorname{Spec} A_f$ with $f \in A$, then the subset $U(R) \subseteq X(R)$ is the locus where the continuous map $f : X(R) \to R$ is *unit*-valued – the preimage of the subset $R^\times$ – and this preimage might not be open. Such openness in general (for a fixed R) is equivalent to the set of non-units in R being closed, but this fails for adele rings (in which one can find sequences of non-units that converge to 1). Regardless of whether or not $R^\times$ is open in R, since $A_f = A[T]/(fT-1)$ we see that $U(R) \to X(R)$ is a topological embedding onto its image if and only if $1/f : U(R) \to R$ is continuous when $U(R)$ is given the subspace topology from $X(R)$. Taking X to be the affine line and U to be the multiplicative group, such an embedding property for general affine finite type R-schemes would force $R^\times$ to be a topological group with its subspace topology from R (which is false for many R).

We conclude that the failure of openness of $R^\times$ in R or the failure of $R^\times$ to be a topological group with its subspace topology from R are the only obstacles to basic open affine immersions inducing open embeddings on spaces of R-points. Hence, it is natural to try to globalize the topology on $X(R)$ beyond the affine case by gluing along Zariski-opens in X when $R^\times$ is open in R with continuous inversion. In order for the gluing to work, we also need to ensure that if $\{U_i\}$ is an affine open covering of an affine X of finite type over R then $X(R)$ is covered by the subsets $U_i(R)$. This works for local R:

Proposition 3.1. *Let R be a local topological ring such that $R^\times$ is open in R and has continuous inversion. There is a unique way to topologize $X(R)$ for arbitrary locally finite type R-schemes X subject to the requirements of functoriality, carrying closed* (*resp. open*) *immersions of schemes into embeddings* (*resp. open embeddings*) *of topological spaces, compatibility with fiber products, and giving $X(R) = R$ its usual topology when X is the affine line over R.*

This agrees with the earlier construction for affine X, and if R is Hausdorff then $X(R)$ is Hausdorff when X is separated over R. If R is locally compact and Hausdorff, then $X(R)$ is locally compact.

Proof. The key to the proof is to show that if $U \to X$ is an arbitrary open immersion between affine R-schemes of finite type then $U(R) \to X(R)$ is an open immersion relative to the topology already defined in the affine case. Once this is proved, the rest is immediate by gluing arguments, so we explain just this assertion concerning open immersions between affine schemes.

Consider the special case that U is a basic affine open in X, say $U = \operatorname{Spec} A_f$ and $X = \operatorname{Spec} A$ for some $f \in A$. Clearly $U(R)$ is the preimage of the open $R^\times \subset R$ under the map $X(R) \to R$ associated to f. To see that this equips $U(R)$ with a subspace topology coinciding with its intrinsic topology (using that U is affine of finite type over R), the fiber square

$$\begin{array}{ccc} U & \xrightarrow{f} & \mathbf{G}_m \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & \mathbf{A}^1_R \end{array}$$

reduces the problem to the special case $X = \mathbf{A}^1_R$ and $U = \mathbf{G}_m$. In this case $U(R)$ acquires the topology of the hyperbola $xy = 1$ in R^2, and this is homemorphic to $R^\times$ with its subspace topology due to the hypothesis that inversion on $R^\times$ is continuous.

To reduce the general case to the special case just treated, one uses that R is local (and that U is covered by basic affine opens of X, each of which is necessarily a basic affine open in U). The main point is that if $\{U_i\}$ is an open cover of X, then $X(R) = \cup U_i(R)$ because a map $\operatorname{Spec} R \to X$ that carries the closed point into U_i must land entirely inside U_i since the only open subscheme of $\operatorname{Spec} R$ that contains the closed point is the entire space. (The equality $X(R) = \cup U_i(R)$ fails for non-local R in general.) ■

Remark 3.2. If X is a locally finite type scheme over a local field k (such as $\mathbf{C}$ or $\mathbf{Q}_p$), then $X(k)$ is a locally compact topological space via Proposition 3.1. The same goes for $X(\mathscr{O})$ with a compact discrete valuation ring $\mathscr{O}$ and a locally finite type $\mathscr{O}$-scheme X.

Remark 3.3. If Z is a closed subscheme in X and U is its open complement then the disjoint subsets $Z(R)$ and $U(R)$ in $X(R)$ may not cover $X(R)$, even if X is affine. The problem is that "Zariski open" corresponds to a unit condition on R-points whereas "Zariski closed" corresponds to a nilpotence condition on R-points. Thus, if R contains elements that are neither nilpotent nor units then $X(R)$ may fail to be the union of $U(R)$ and $Z(R)$. More geometrically, if we consider maps $\operatorname{Spec} R \to X$ then the image might hit both Z and U (a simple example being the affine R-line X, its origin Z, and complement $U = X - Z$, for which $Z(R)$ and $U(R)$ are both non-empty and do not cover $X(R) = R$ whenever $\operatorname{Spec} R$ is not a point). For local artinian R this does not happen, which is why the construction of a topology on $X(R)$ is especially straightforward when R is a field.

In view of the above discussion, it is a remarkable fact that when $R = \mathbf{A}_F$ is the adele ring of a global field, one can (following a method due to Weil) naturally topologize $X(R)$ for arbitrary finite type F-schemes X. It is not true in such generality that immersions of schemes are carried into topological embeddings, but the topology is functorial and compatible with fiber products, it gives closed embeddings when applied to closed immersions, and it recovers the earlier construction in the affine case. We now present a Grothendieck-style development of Weil's construction.

The key to Weil's construction in the affine case is that if X is a finite type affine F-scheme (for a global field F) then by chasing denominators in a finite presentation of the coordinate ring of X we can find a finite set S of places of F (non-empty and containing the archimedean places, as always) and a finite-type algebra over $\mathscr{O}_{F,S}$ whose generic fiber is the coordinate ring of X. Geometrically, this amounts to giving an affine finite type $\mathscr{O}_{F,S}$-scheme X_S whose generic fiber is X. As will be recorded below, Grothendieck's technique of limits of schemes [EGA, IV_3, §8–§11] shows that an analogous result holds for all finite type F-schemes (not just the affine ones): every finite type F-scheme X is the generic fiber of a finite type $\mathscr{O}_{F,S}$-scheme X_S for some S. We can transfer many properties of X to X_S by increasing S, as we now explain.

We first mention a useful concept: a scheme X over a ring R is *finitely presented* if it is covered by finitely many open affines U_i, each of the form $U_i \simeq \operatorname{Spec}(R[t_1, \dots, t_{n_i}]/(f_{1,i}, \dots, f_{m_i,i}))$ with quasi-compact overlaps $U_i \cap U_{i'}$ (this latter condition being automatic in the separated case, for which an overlap of two affine opens is affine). Finite presentation coincides with finite type when R is noetherian, but the adele ring $\mathbf{A}_F$ is not noetherian. Loosely speaking, finite presentation over R means being "described by a finite amount of information" in R.

Since $F = \varinjlim \mathscr{O}_{F,S}$ and $\mathbf{A}_F = \varinjlim \mathbf{A}_{F,S}$ (limits taken over increasing S), the following link between finite presentation and direct limits is an essential step in Weil's construction (especially beyond the affine case).

Theorem 3.4. *Let $\{A_i\}$ be a directed system of rings, $A = \varinjlim A_i$. Let X be a finitely presented A-scheme.*

(1) *There exists some i_0 and a finitely presented A_{i_0}-scheme X_{i_0} whose base change over A is isomorphic to X. Moreover, if X_{i_0} and Y_{i_0} are two finitely presented A_{i_0}-schemes for some i_0, and we write X_i and Y_i to denote their base changes over A_i for all $i \geq i_0$ (and likewise for X and Y over A), then the natural map of sets*

$$\varinjlim \operatorname{Hom}_{A_i}(X_i, Y_i) \to \operatorname{Hom}_A(X, Y)$$

is bijective.

(2) *A map $f_{i_0} : X_{i_0} \to Y_{i_0}$ acquires property $\mathbf{P}$ upon base change to some A_i if and only if the induced map $f : X \to Y$ over A has property $\mathbf{P}$, where $\mathbf{P}$ is any of the following properties: closed immersion, separated, proper, smooth, affine, flat, open immersion, finite, fibers non-empty and geometrically connected of pure dimension d.*

(3) *Any "descent" X_{i_0} over A_{i_0} of a finitely presented A-scheme X is essentially unique up to essentially unique isomorphism in the following sense: for finitely presented A_{i_0}-schemes X_{i_0} and X'_{i_0} whose base changes over A are identified with X, there exists some $i \geq i_0$ and an isomorphism $h_i : X_i \simeq X'_i$ compatible with the common identification with X upon base change to A, and if h_i and H_i are two such isomorphisms then for some $i' \geq i$ the induced isomorphisms $h_{i'}$ and $H_{i'}$ are equal.*

Proof. Apart from (2), this is [EGA, IV_3, §8.8, §8.9]. To handle the list of properties $\mathbf{P}$ is a lengthy task that is exhaustively developed in [EGA, IV_3, §8.10–§11], where many more properties are also considered (but we only need the ones mentioned above); a good place to begin is [EGA, IV_3, 8.10.5]. ∎

Remark 3.5. In practice, the two examples of $\{A_i\}$ of most interest to us will be $\{\mathbf{A}_{F,S}\}$ (with limit $\mathbf{A}_F$) and $\{\mathscr{O}_{F,S}\}$ (with limit F). Due to the example $\{\mathscr{O}_{F,S}\}$, in which X_S is visualized as fibered over the curve $\operatorname{Spec}\mathscr{O}_{F,S}$ with X as the generic fiber, in general we sometimes call X_{i_0} a "spreading out" of X.

We now apply Theorem 3.4(1) to a finite type F-scheme X: pick a finite set S of places such that there is a finite type $\mathscr{O}_{F,S}$-scheme X_S with generic fiber X. For any finite set S' of places of F containing S, we define $X_{S'}$ over $\mathscr{O}_{F,S'}$ to be the base change of X_S. Note that for any morphism of $\mathscr{O}_{F,S'}$-schemes $\operatorname{Spec}\mathbf{A}_{F,S'} \to X_{S'}$ for some S', if S'' is a finite set of places of F containing S' then we get an induced map of $\mathscr{O}_{F,S''}$-schemes $\operatorname{Spec}\mathbf{A}_{F,S''} \to X_{S''}$ by base change since $\mathbf{A}_{F,S''} = \mathscr{O}_{F,S''} \otimes_{\mathscr{O}_{F,S'}} \mathbf{A}_{F,S'}$. Likewise, by passing to generic fibers we get an F-scheme map $\operatorname{Spec}\mathbf{A}_F \to X$. Putting this together, we get a natural map of sets

$$\varinjlim X_{S'}(\mathbf{A}_{F,S'}) = \varinjlim X_S(\mathbf{A}_{F,S'}) \to X_S(\mathbf{A}_F) = X(\mathbf{A}_F) \tag{3.5.1}$$

that is readily checked to equal the limit of the base change maps. In this limit process we only consider S' containing S, and increasing S at the outset has no impact. Theorem 3.4(1) makes precise the sense in which the direct limit on the left side of (3.5.1) is intrinsic to X. By Theorem 3.4(3), the left side of (3.5.1) is naturally a (set-valued) functor of the F-scheme X.

We can do better: the left side of (3.5.1) is naturally a topological space in a manner that respects functoriality in X, and (3.5.1) is bijective. Before explaining this, we note that the left side of (3.5.1) is what Weil defines to be the *adelization* of a finite type F-scheme X. It is by means of this bijection that we shall trasport a topological structure to the right side of (3.5.1) for general X, recovering the topological construction for affine X in §2.

Bijectivity of (3.5.1) is obvious for affine X, because if $F[t_1, \dots, t_n]/(f_1, \dots, f_m) \to \mathbf{A}_F$ is a map of F-algebras then for some finite set S of places of F, the t_j's all land in $\mathbf{A}_{F,S}$ and the f_j's all have coefficients in $\mathscr{O}_{F,S}$. To establish bijectivity without assuming X to be affine, the key point is that since $\mathbf{A}_F = \varinjlim \mathbf{A}_{F,S'}$ and X_S is of finite type over the noetherian ring $\mathscr{O}_{F,S}$, we can rewrite (3.5.1) as the natural map

$$\varinjlim \operatorname{Hom}_{\mathbf{A}_{F,S'}}(\operatorname{Spec}\mathbf{A}_{F,S'}, (X_S)_{\mathbf{A}_{F,S'}}) \to \operatorname{Hom}_{\mathbf{A}_F}(\operatorname{Spec}\mathbf{A}_F, X_{\mathbf{A}_F}),$$

and this is a bijection by Theorem 3.4(1) (applied to $\mathbf{A}_F = \varinjlim \mathbf{A}_{F,S'}$).

Before we establish some topological properties of (3.5.1), we need some notation. For an $\mathscr{O}_{F,S}$-scheme X_S and a place v of F not in S (i.e., v is a maximal ideal of $\mathscr{O}_{F,S}$), we will write $X_{S,v}$ to denote the base change of X_S over the completion $\mathscr{O}_v$ at v. For any v, we write X_v to denote the base change of X_S (or $X_{S,v}$) over the fraction field F_v of $\mathscr{O}_v$.

Theorem 3.6. *Let X_S be a finite type $\mathscr{O}_{F,S}$-scheme. Using the projections from $\mathbf{A}_{F,S}$ to F_v for $v \in S$ and to $\mathscr{O}_v$ for $v \notin S$, the natural map of sets*

$$X_S(\mathbf{A}_{F,S}) \to \prod_{v \in S} X_v(F_v) \times \prod_{v \notin S} X_{S,v}(\mathscr{O}_v) \tag{3.6.1}$$

is a bijection. When X is affine and we give both sides their natural topologies, using the product topology on the right side, this is a homeomorphism.

In general, if we use the bijection (3.6.1) to define a topology on $X_S(\mathbf{A}_{F,S})$, then for any finite sets of places $S' \subseteq S''$ containing S and the corresponding base changes $X_{S'}$ and $X_{S''}$ of X_S over $\mathscr{O}_{F,S'}$ and $\mathscr{O}_{F,S''}$ respectively, the natural map $X_{S'}(\mathbf{A}_{F,S'}) \to X_{S''}(\mathbf{A}_{F,S''})$ is an open continuous map of topological spaces and it is injective when X_S is separated over $\mathscr{O}_{F,S}$.

In this theorem, we are using Remark 3.2 to give the $X_v(F_v)$'s and $X_{S,v}(\mathscr{O}_v)$'s their natural topologies.

Proof. The bijectivity aspect amounts to the claim that a morphism of $\mathscr{O}_{F,S}$-schemes $\operatorname{Spec}\mathbf{A}_{F,S} \to X_S$ is uniquely determined by its restriction to the open subschemes $\operatorname{Spec} F_v$ ($v \in S$) and $\operatorname{Spec}\mathscr{O}_v$ ($v \notin S$), and that it may be constructed from such arbitrary given data. Note that the quasi-compact $\operatorname{Spec} \mathbf{A}_{F,S}$ is not the disjoint union of these infinitely many pairwise disjoint non-empty affine open subschemes.

The bijectivity assertion has nothing to do with adele rings, and is a special case of the following more general fact. Let $\{R_i\}$ be a collection of C-algebras for a ring C, and let $R = \prod R_i$ denote the product. Note that $\{\operatorname{Spec}(R_i)\}$ is a collection of disjoint open subschemes of the quasi-compact scheme $\operatorname{Spec}(R)$ (so this is not a cover of $\operatorname{Spec}(R)$ if infinitely many of the R_i are nonzero). Let X be an arbitrary C-scheme, and consider the natural map of sets

$$X(R) \to \prod X(R_i) \tag{3.6.2}$$

where $X(R)$ denotes the set of R-valued points of X over C, and similarly for each $X(R_i)$. We claim that this map is injective when X is quasi-separated (i.e., quasi-compact opens in X have quasi-compact overlap, such as locally noetherian or separated X) and is surjective when X is quasi-compact and the R_i's are all local. (This is [Oes, Ch. I, Lemme 3.2], except that the quasi-separatedness hypothesis is missing from the statement but is used in the proof.) By taking $C = \mathscr{O}_{F,S}$, $\{R_i\}$ to be $\{F_v\}_{v \in S} \cup \{\mathscr{O}_v\}_{v \notin S}$, and X to be a scheme of finite type over $\mathscr{O}_{F,S}$, we would then get the asserted bijectivity of (3.6.1).

To prove the injectivity of (3.6.2) when X is quasi-separated, consider $f, g \in X(R)$ that induce the same R_i-points for all i. To prove that $f = g$, it is necessary and sufficient that the product map

$$(f, g) : \operatorname{Spec} R \to X \times_C X$$

factors through the diagonal morphism $\Delta_{X/C}$. Consider the cartesian diagram

$$\begin{array}{ccc} V & \longrightarrow & \operatorname{Spec}(R) \\ \downarrow & & \downarrow{\scriptstyle (f,g)} \\ X & \xrightarrow[\Delta_{X/C}]{} & X \times_C X \end{array}$$

whose bottom side is an immersion (as for any diagonal morphism of schemes). We shall prove that the top side is an isomorphism, which will provide the desired factorization. The immersion $\Delta_{X/C} : X \hookrightarrow X \times_C X$ is a quasi-compact since X is quasi-separated, so V is a quasi-compact subscheme of $\operatorname{Spec}(R)$. Letting $U \subseteq \operatorname{Spec}(R)$ denote the open subscheme that is the union of the disjoint open subschemes $\operatorname{Spec}(R_i) \subseteq \operatorname{Spec}(R)$, by hypothesis $(f, g)|_U$ factors through $\Delta_{X/C}$ and so $U \subseteq V$ as subschemes of $\operatorname{Spec}(R)$. Thus, it suffices to prove that the only quasi-compact (locally closed) subscheme $V \subseteq \operatorname{Spec}(R)$ which contains U is $\operatorname{Spec}(R)$. (This is an assertion entirely about R; we have eliminated X. Note also that when there are infinitely many nonzero R_i's it is essential to assume that V is quasi-compact, as otherwise we could take $V = U$ to get a counterexample.)

By quasi-compactness of the locally closed V in the affine scheme $\operatorname{Spec}(R)$, there is a quasi-compact open subscheme $W \subseteq \operatorname{Spec}(R)$ in which V lies as a closed subscheme. Since $U \subseteq V \subseteq W$, if we first treat the case of quasi-compact open subschemes containing U then we will have $W = \operatorname{Spec}(R)$, which is to say that V is closed in $\operatorname{Spec}(R)$. Hence, it suffices to treat two cases: V is open and V is closed. First suppose V is open. In this case, by quasi-compactness of V the closed complement $\operatorname{Spec}(R) - V$ is the zero locus of a *finitely generated* ideal $I \subseteq R$. The containment $U \subseteq V$ of open subschemes of $\operatorname{Spec}(R)$ is the set-theoretic property that $U = \coprod \operatorname{Spec}(R_i)$ is disjoint from the zero locus of I, or in other words the image of I under each projection $R \to R_i$ is the unit ideal. We are therefore reduced to proving that a finitely generated ideal I in R is the unit ideal if it induces the unit ideal in each R_i. (The finiteness hypothesis on I is crucial; it is easy to construct ideals in $\mathbf{A}_{F,S}$ that are not finitely generated but generate the unit ideal in each standard factor ring: consider the ideal generated by elements that have a uniformizer component in all but finitely many places.) Let $a_1, \ldots, a_n \in R = \prod R_i$ be generators of I. By hypothesis, for each i the elements $a_{1,i}, \ldots, a_{n,i} \in R_i$ generate 1, say $\sum_j r_{j,i} a_{j,i} = 1$ with $r_{j,i} \in R_i$. Hence, for $r_j = (r_{j,i}) \in R$ we have $\sum r_j a_j = 1$ in R, so $I = (1)$.

This settles the case when V is open in $\operatorname{Spec}(R)$, and now consider the case when V is closed. In this case we run through a similar argument with the (perhaps not finitely generated) ideal of R whose zero locus is

V: the algebraic problem is to show that if I is an ideal in R that projects to 0 in each R_i then $I = 0$. But this is trivial, and so completes the proof that (3.6.2) is injective when X is quasi-separated.

(Our trivial argument in the closed case shows that U is scheme-theoretically dense in $\mathrm{Spec}(R)$, but beware that it need not be topologically dense and so it is essential that the containment $U \subseteq V$ is taken in the scheme-theoretic sense rather than in the weaker topological sense. This is illustrated by the following example which was brought to my attention by Moret-Bailly. Take $C = k$ to be a field and $R_n = k[t]/(t^{n+1})$ for $n \geq 0$, and consider the closed subscheme $V = \mathrm{Spec}(R/(r))$ of $\mathrm{Spec}(R)$ defined by killing the "diagonal" element $r = (t, t, \dots)$. This V does contain U topologically since it clearly contains every point of U, but it does not contain U scheme-theoretically since $\mathrm{Spec}(R_n)$ is not contained in V for any $n \geq 1$. Moreover, the underlying space of V is not all of $\mathrm{Spec}(R)$ since r is not nilpotent in R.)

Now we prove that (3.6.2) is surjective when X is quasi-compact and each R_i is local. Assume we are given C-maps $x_i : \mathrm{Spec}\, R_i \to X$ for all i. We claim that there exists $x \in X(R)$ inducing the given local data. Let $\{U_1, \dots, U_n\}$ be a finite affine open covering of X. Since each R_i is a local ring, the image of x_i lands in some U_j (chase the closed point). Pick one such $j(i)$ for each i, and let V_j be the set of i's for which $j(i) = j$ (i.e., those i for which we have selected U_j as an open affine through which x_i factors). We have a natural finite product decomposition $R = \prod_j R_{V_j}$, where R_{V_j} is the subproduct of the product ring R corresponding to local factors for indices $i \in V_j$. Since the Spec functor carries finite products into disjoint unions, we may focus on each R_{V_j} separately. In other words, we may replace X with U_j so as to reduce to the case that X is affine. Now the claim is that if $\phi_i : \mathrm{Spec}\, R_i \to \mathrm{Spec}\, B$ are maps of affine schemes over some affine base $\mathrm{Spec}\, C$, then there exists a map of C-schemes $\phi : \mathrm{Spec}(\prod R_i) \to \mathrm{Spec}\, B$ inducing each ϕ_i. By restating in terms of ring maps, this is obvious.

Now that (3.6.1) is proved to be a bijection, we may use the product topology on its target to endow $X_S(\mathbf{A}_{F,S})$ with a topology. For affine X_S, this recovers the topology constructed earlier: by using a finite presentation of the coordinate ring of X_S as an $\mathscr{O}_{F,S}$-algebra, and recalling how the topology on points of affine schemes (of finite type) was defined by means of embeddings into affine spaces, the problem comes down to the trivial claim that the product topology on $\mathbf{A}^n_{F,S}$ agrees with the product topology on

$$\prod_{v \in S} F_v^n \times \prod_{v \notin S} \mathscr{O}_v^n.$$

Finally, we have to check that if $S' \subseteq S''$ is an inclusion of finite sets of places of F containing S, then the map $X_{S'}(\mathbf{A}_{F,S'}) \to X_{S''}(\mathbf{A}_{F,S''})$ is an open continuous map of topological spaces, and is injective when X_S is separated. Via (3.6.1), this map is (topologically) the product of three maps: the identity maps on $\prod_{v \in S'} X_v(F_v)$ and on $\prod_{v \notin S''} X_{S,v}(\mathscr{O}_v)$, and the base change map

$$\prod_{v \in S'' - S'} X_{S,v}(\mathscr{O}_v) \to \prod_{v \in S'' - S'} X_v(F_v).$$

Thus, we are reduced to show that for $v \notin S$, the natural map $X_{S,v}(\mathscr{O}_v) \to X_v(F_v)$ is continuous and open, and injective when X_S is separated. The injectivity for separated X_S follows from the valuative criterion for separatedness, so we just have to check continuity and openness.

In general, for a finite type scheme X over a complete discrete valuation ring $\mathscr{O}$ with fraction field K given its natural topology, we claim that $X(\mathscr{O}) \to X_K(K)$ is a continuous open map. If U is an open subscheme of X, then by Proposition 3.1, $U(\mathscr{O})$ is open in $X(\mathscr{O})$. Since $X(\mathscr{O})$ is the union of the $U_i(\mathscr{O})$'s for $\{U_i\}$ an open covering of X, our problem is of local nature on X. Hence, we may assume X is affine. By picking a closed immersion of X into an affine space over $\mathscr{O}$, the fact that $\mathscr{O}^n$ is open in K^n then provides what we need. ∎

Using Theorem 3.6 to topologize $X_S(\mathbf{A}_{F,S})$ for finite type $\mathscr{O}_{F,S}$-schemes X_S, it is immediate from the construction that this topology is functorial in X_S, has a countable base of opens, carries fiber products into fiber products, and carries closed immersions into closed embeddings (use Proposition 3.1 and the fact that an arbitrary product of closed embeddings is a closed embedding). For open immersions $U_S \hookrightarrow X_S$ it is not true in general that $U_S(\mathbf{A}_{F,S}) \to X_S(\mathbf{A}_{F,S})$ is an open embedding, though it is a topological embedding. Indeed, an arbitrary product of open embeddings is a topological embedding but usually does not have open image. This is the reason that the construction of the topology on $X_S(\mathbf{A}_{F,S})$ in the non-affine case has to be done globally via the product decomposition in (3.6.1), without trying to glue topologies coming from open affines in X_S.

Corollary 3.7. *Let X_S be a finite type $\mathscr{O}_{F,S}$-scheme. The topological space $X_S(\mathbf{A}_{F,S})$ is locally compact, and is Hausdorff when X_S is separated.*

Proof. Since our topology construction commutes with products and carries closed immersions to closed embeddings, it is clear that if X_S is separated then $X_S(\mathbf{A}_{F,S})$ is Hausdorff. As for local compactness, we want the infinite product space $X_S(\mathbf{A}_{F,S})$ to be locally compact. Since the factor spaces $X_v(F_v)$ are locally compact for $v \in S$, we just have to check that $X_{S,v}(\mathscr{O}_v)$ is compact for $v \notin S$. More generally, for any compact discrete valuation ring R and any finite type R-scheme X, we claim $X(R)$ is compact. Proposition 3.1 shows that for a finite open affine covering $\{U_i\}$ of X the spaces $\{U_i(R)\}$ form a finite open covering of $X(R)$, so the problem comes down to the affine case, which in turn is reduced to the trivial case of affine space (R^n is compact since R is compact). ∎

4. Topological properties

Let X be a finite type F-scheme. We use Theorems 3.4 and 3.6 along with the bijection (3.5.1) to give $X(\mathbf{A}_F)$ a topological structure that is functorial in X and coincides with the topology in Proposition 2.1 when X is affine. To make sense of this, we need to briefly recall how one topologizes direct limits. If $\{T_\alpha\}$ is a directed system of topological spaces, with direct limit set T as sets, we declare $U \subseteq T$ to be open if and only if the preimage of U in each T_α is open. This is readily checked to be a direct limit in the topological category. In general such abstract topologies are hard to handle. However, the case when transition maps are open involves no subtlety: if $T_\alpha \to T_{\alpha'}$ is an open continuous map for all $\alpha' \geq \alpha$, then T is the directed union of the images U_α of the T_α's, and by giving each U_α the quotient topology from T_α it is clear that the topology on T is characterized by declaring the topological spaces U_α to be open subspaces.

The functor $X \rightsquigarrow X(\mathbf{A}_F)$ does not generally carry open immersions over to topological embeddings, but closed immersions do go over to closed embeddings of topological spaces (due to openness of the transition maps in the above topological direct limits). Since the behavior of quotient topologies with respect to fiber products (or even absolute products) is subtle in general, the topology on $X(\mathbf{A}_F)$ is probably rather hard to work with unless we impose a hypothesis on X to ensure injectivity and openness of the transition maps in the limit of $X_{S'}(\mathbf{A}_{F,S'})$'s. We see from the final part of Theorem 3.6, as well as Theorem 3.4(1), that assuming X is *separated* over F ensures the injectivity. Thus, if X is F-separated then (3.5.1) expresses $X(\mathbf{A}_F)$ as a direct limit of locally compact Hausdorff spaces with transition maps that are open embeddings. In this way, we see that $X(\mathbf{A}_F)$ is locally compact and Hausdorff (with a countable base of opens) when X is F-separated, and moreover that this topology is compatible with fiber products for general X.

The preceding defines, for finite type separated F-schemes X, a functorial locally compact Hausdorff topology on $X(\mathbf{A}_F)$ with a countable base of opens, and this topology is compatible with fiber products and carries closed immersions between such F-schemes into closed embeddings of topological spaces. Moreover, if X is the generic fiber of a separated finite type $\mathscr{O}_{F,S}$-scheme X_S, then $X_S(\mathbf{A}_{F,S})$ is naturally an open subset of $X(\mathbf{A}_F)$. As a special case, when X is a group scheme of finite type over F (automatically separated), the set $X(\mathbf{A}_F)$ is naturally a locally compact Hausdorff topological group.

Example 4.1. It is a common mistake to expect that if $\{U_i\}$ is an open affine cover of X then $\{U_i(\mathbf{A}_F)\}$ covers $X(\mathbf{A}_F)$ set-theoretically. This is false even if X is affine, because the image of a morphism $\operatorname{Spec} \mathbf{A}_F \to X$ need not be contained in any of the U_i's. Moreover, the set $\cup U_i(\mathbf{A}_F)$ inside $\prod_v X(F_v)$ is not independent of $\{U_i\}$ in general, and in particular it is not intrinsic to X.

Example 4.2. Let $F \to F'$ be a finite extension of global fields, and X' a quasi-projective F'-scheme. Let $\mathscr{X}$ denote the Weil restriction $\mathrm{Res}_{F'/F}(X')$, which exists and is separated and finite type over F [BLR, pp. 194–196]. (The same reference applies with $F \to F'$ replaced by any finite locally free ring map, such as a finite extension of Dedekind domains. In the generality of finite locally free ring maps, the Weil restriction operation preserves quasi-projectivity, although this is not obvious from the construction; see

[CGP, Prop. A.5.8].) Since naturally $\mathscr{X}(\mathbf{A}_F) = X'(\mathbf{A}_{F'})$ as sets, we are led to ask if this is an equality as topological spaces. Here is an affirmative proof.

In the affine case the equality of topologies follows from Example 2.4 (applied to the base changes of X' and $\mathscr{X}$ over $R' = \mathbf{A}_{F'}$ and $R = \mathbf{A}_F$ respectively). In the general case, fix a finite set S_0 of places of F such that X' extends to a quasi-projective $\mathscr{O}_{F',S'_0}$-scheme $X'_{S'_0}$, where S'_0 is the preimage of S_0 in F'. Thus, $\mathrm{Res}_{\mathscr{O}_{F',S'_0}/\mathscr{O}_{F,S_0}}(X'_{S'_0})$ exists as a finite type and separated $\mathscr{O}_{F,S_0}$-scheme $\mathscr{X}_0$, and $\mathscr{X}_0(\mathbf{A}_{F,S}) = X'_{S'_0}(\mathbf{A}_{F',S'})$ as sets for any finite set S of places of F containing S_0 and for its preimage S' in F'. By the definition of the topology on the adelic points (as a direct limit with open transition maps), the problem of topological equality is reduced to checking that the equality of sets $\mathscr{X}_0(\mathbf{A}_{F,S_0}) = X'_{S'_0}(\mathbf{A}_{F,S'_0})$ (for general S_0) is a homeomorphism. These topologies are defined as product topologies, and so the problem reduces to checking that for each place $v \in S_0$ the equality of sets $\prod_{v'|v} X'(F'_{v'}) = \mathrm{Res}_{F'/F}(X')(F_v)$ is a homeomorphism and that for each place v of F not in S_0 the equality of sets

$$\prod_{v'|v} X'_{S'_0}(\mathscr{O}_{F',v'}) = \mathrm{Res}_{\mathscr{O}_{F',S'_0}/\mathscr{O}_{F,S_0}}(X'_{S'_0})(\mathscr{O}_{F,v})$$

is a homeomorphism. This second homeomorphism claim is a formal consequence of the first one (applied with S_0 increased to contain v), so we can focus on the case of field-valued points with any place v.

Defining $F'_v = F' \otimes_F F_v \simeq \prod_{v'|v} F'_{v'}$ and

$$X'_v = F'_v \otimes_{F'} X' = \coprod_{v'|v} X'_{v'},$$

we have

$$\mathrm{Res}_{F'/F}(X')_{F_v} = \mathrm{Res}_{F'_v/F_v}(X'_v) = \prod_{v'|v} \mathrm{Res}_{F'_{v'}/F_v}(X'_{v'}).$$

Thus, the problem reduces to one over local fields: if k'/k is a finite extension of fields complete with respect to compatible nontrivial absolute values and if Y' is a quasi-projective k'-scheme of finite type, then we claim that the identification of sets $\mathrm{Res}_{k'/k}(Y')(k) = Y'(k')$ is a homeomorphism. Since any finite subset of Y' lies in an open affine, the construction of these Weil restrictions in terms of affine opens reduces us the case when Y' is affine. We can then apply Example 2.4 with the ring extension k'/k. This concludes the proof that Weil restriction for quasi-projective schemes is compatible with the topology on adelic points.

Though Example 2.2 shows that $X(F)$ is a discrete closed set in $X(\mathbf{A}_F)$ for finite type affine F-schemes X (as F is discrete and closed in $\mathbf{A}_F$), globalizing to the non-affine case usually destroys such properties. The following example shows that for separated X, it can happen that the Hausdorff space $X(\mathbf{A}_F)$ is compact and $X(F)$ is a dense proper subset, so $X(F)$ is neither closed nor discrete in $X(\mathbf{A}_F)$ in such cases. (Density is used to deduce non-discreteness from non-closedness; in general a non-closed subset of a compact Hausdorff space can have the discrete topology as its subspace topology, such as $\{e^{2\pi i/n}\}_{n\geq 1}$ inside S^1.)

Example 4.3. Choose $n > 0$. Since $\mathbf{P}^n(\mathscr{O}_v) = \mathbf{P}^n(F_v)$ for all $v \nmid \infty$, the bijection in Theorem 3.6 yields a bijection

$$\mathbf{P}^n(\mathbf{A}_F) = \mathbf{P}^n(F_\infty) \times \prod_{v\nmid\infty} \mathbf{P}^n(\mathscr{O}_v) = \mathbf{P}^n(F_\infty) \times \prod_{v\nmid\infty} \mathbf{P}^n(F_v) = \prod_{v} \mathbf{P}^n(F_v)$$

with the infinite product defining the topology (so it is compact Hausdorff). In the special case $n = 1$, when $\mathbf{A}_F$ is identified with the set of $\mathbf{A}_F$-points of the standard affine line in $\mathbf{P}^1_F$ its resulting subspace topology is induced by the product topology on $\prod_v F_v$ (so it is *not* locally compact).

For any finite non-empty set S of places of F, let $F_S = \prod_{v\in S} F_v$. By weak approximation in the affine space of matrices Mat_{n+1} over F, $\mathrm{GL}_{n+1}(F)$ is dense in $\mathrm{GL}_{n+1}(F_S)$. Thus, $\mathrm{PGL}_{n+1}(F)$ is dense in $\mathrm{PGL}_{n+1}(F_S)$, so any point in $\mathbf{P}^n(F_S)$ can be moved by a suitable projective change of coordinates over F so that its projection into each $\mathbf{P}^n(F_v)$ ($v \in S$) is not in the standard hyperplane at infinity. It then follows from weak approximation in affine n-space that $\mathbf{P}^n(F)$ is dense in $\mathbf{P}^n(F_S)$. Varying S, $\mathbf{P}^n(F)$ is dense in $\mathbf{P}^n(\mathbf{A}_F)$.

Proposition 4.4. *Let $X \to Y$ be a proper map between separated F-schemes of finite type. The induced map $X(\mathbf{A}_F) \to Y(\mathbf{A}_F)$ between locally compact Hausdorff spaces is topologically proper.*

In particular, if X is proper over F then $X(\mathbf{A}_F)$ is compact, and if moreover X_S is a finite type $\mathscr{O}_{F,S}$-scheme with generic fiber X then $X(\mathbf{A}_F) = X_{S'}(\mathbf{A}_{F,S'})$ for every sufficiently large finite set of places S' of F that contains S.

Proof. By increasing S if necessary, by Theorem 3.4(2) we can assume that $X \to Y$ arises from a proper map $X_S \to Y_S$ between separated finite type $\mathscr{O}_{F,S}$-schemes. Since $X(\mathbf{A}_F)$ has an open covering given by the $X_S(\mathbf{A}_{F,S'})$ for S' containing S, the assertions for F-proper X are immediate from the general properness assertion for $X(\mathbf{A}_F) \to Y(\mathbf{A}_F)$. Thus, we focus on this latter assertion.

For any $v \notin S$, the valuative criterion for properness ensures that under the map $X_v(F_v) \to Y_v(F_v)$ the preimage of $Y_{S,v}(\mathscr{O}_v)$ is $X_{S,v}(\mathscr{O}_v)$. Hence, for any S' containing S, the preimage of $Y_S(\mathbf{A}_{F,S'})$ under $X(\mathbf{A}_F) \to Y(\mathbf{A}_F)$ is $X_S(\mathbf{A}_{F,S'})$. Upon renaming S' as S, it suffices to prove that $X_S(\mathbf{A}_{F,S}) \to Y_S(\mathbf{A}_{F,S})$ is proper. Since $Y_S(\mathbf{A}_{F,S})$ is a topological product of the spaces $Y_v(F_v)$ for $v \in S$ and the compact spaces $Y_{S,v}(\mathscr{O}_v) \simeq Y_v(F_v)$ for $v \notin S$, and similarly for X_S, we are reduced to proving that if $f : X \to Y$ is a proper map between separated schemes of finite type over a locally compact field K, then the map $X(K) \to Y(K)$ between locally compact Hausdorff spaces is proper.

We will say that a proper map of schemes is *projective* if it factors, Zariski-locally over the base, as a closed immersion into a projective space over the base. The properness assertion on K-points is clear when $f : X \to Y$ is projective in this sense. In general, we shall argue by induction on $\dim X$ (allowing any Y), the case of dimension 0 being clear (for all Y). We may assume that X is reduced and irreducible, so by Chow's Lemma there is a surjective projective birational K-map $f : X' \to X$ with X' a reduced and irreducible scheme such that X' is also projective over Y. Choose a proper closed subset $Z \subseteq X$ such that f is an isomorphism over $X - Z$. Clearly $X(K) = Z(K) \cup f(X'(K))$, and $Z(K)$ is $Y(K)$-proper since $\dim Z < \dim X$. Also, $X'(K)$ is $Y(K)$-proper and $X(K)$-proper since X' is projective over Y and X, so the maps $Z(K) \coprod X'(K) \to Y(K)$ and $Z(K) \coprod X'(K) \twoheadrightarrow X(K)$ are proper. Hence, the map $X(K) \to Y(K)$ between Hausdorff spaces is proper. ■

The final topic we address in this section is openness properties for the map on adelic points induced by a smooth (e.g., étale) map of schemes. This is inspired by the fact that if $X' \to X$ is a smooth K-morphism between arbitrary algebraic K-schemes for a field K complete with respect to a nontrivial absolute value then the induced map $X'(K) \to X(K)$ is open. Let us first briefly review the reason for such openness on K-points.

By working Zariski-locally, any smooth map factors as as étale map to an affine space [EGA, IV_4, 17.11.4]. This reduces us to the case of an étale map, and by the local structure theorem for such maps [EGA, IV_4,18.4.6(ii)] we may work Zariski-locally to get to the case when $X = \operatorname{Spec} B$ and X' is Zariski-open in $\operatorname{Spec}((B[u]/(h))_{h'})$ for a monic $h \in B[u]$ with positive degree. It therefore suffices to consider the case $X' = \operatorname{Spec}((B[u]/(h))_{h'})$. By expressing B as a quotient of a polynomial ring over K and lifting h to a monic polynomial over such a polynomial ring, we may suppose that X is an affine space over K.

The setup is now a consequence of "continuity of (simple) roots" over K. That is, if $g = \sum c_j u^j \in K[t]$ is a monic polynomial of degree $n > 0$ and if $u_0 \in K$ is a simple root of g then we claim that for any $\varepsilon > 0$ there exists $\delta > 0$ such that every degree-n monic polynomial $G = \sum C_j u^j \in K[u]$ satisfying $|C_j - c_j| < \delta$ for all $j < n$ has a unique root $u_0' \in K$ satisfying $|u_0' - u_0| < \varepsilon$ and it is a simple root. This is very classical in the archimedean case, and in the non-archimedean case it is a key ingredient in the proof of Krasner's Lemma; see [BGR, 3.4, p. 146] (with $t = 1$ there) for a proof.

The analogous openness result for adelic points requires additional hypotheses. For example, the Zariski-open immersion of the multiplicative group into the affine line over F induces the natural inclusion $\mathbf{A}_F^\times \to \mathbf{A}_F$ which is not even a topological embedding and does not have open image. Even if we restrict ourselves to surjective étale maps there are counterexamples: the nth-power map $\mathbf{G}_m \to \mathbf{G}_m$ for $n > 1$ not divisible by $\mathrm{char}(F)$ is a finite étale map that induces the nth-power map $\mathbf{A}_F^\times \to \mathbf{A}_F^\times$ whose image is not open. The defect of these examples is that they have fibers which are either empty or geometrically disconnected. This is bypassed by the hypotheses in the next result.

Theorem 4.5. *Let $f : X' \to X$ be a smooth surjective F-map between separated F-schemes of finite type. If the fibers are geometrically connected then the induced map $X'(\mathbf{A}_F) \to X(\mathbf{A}_F)$ is open.*

This result is stated and proved in [Oes, Ch. I, 3.6] in the affine case (and our proof is simply a more general version of the argument to avoid affineness hypotheses).

Proof. By Theorem 3.4(2) we may and do choose a finite set S of places of F so that f is the map on generic fibers induced by a smooth surjective $\mathscr{O}_{F,S}$-map $f_S : X'_S \to X_S$ with geometrically connected fibers, where X'_S and X_S are separated $\mathscr{O}_{F,S}$-schemes of finite type. By varying through finite $T \supseteq S$ (promptly renamed as S), it suffices to prove that the induced map $X'_S(\mathbf{A}_{F,S}) \to X_S(\mathbf{A}_{F,S})$ is open. This is a map of product spaces, and more specifically is the product of the induced maps $X'_v(F_v) \to X_v(F_v)$ for $v \in S$ and $X'_{S,v}(\mathscr{O}_v) \to X_{S,v}(\mathscr{O}_v)$ for $v \notin S$. These latter maps on $\mathscr{O}_v$-points are induced by the corresponding maps on F_v-points, so (by definition of the topology on a product space) we are reduced to checking two facts: (i) the smooth F_v-map $f_v : X'_v \to X_v$ induces an open map on F_v-points for all v, and (ii) for all but finitely many $v \notin S$, the map $X'_{S,v}(\mathscr{O}_v) \to X_{S,v}(\mathscr{O}_v)$ is surjective.

The openness of the map on F_v-points for all v is a special case of the more general fact, explained in the discussion immediately preceding Theorem 4.5, that if K is any field complete with respect to a nontrivial absolute value and $f : X' \to X$ is a smooth map between K-schemes locally of finite type then the induced map $X'(K) \to X(K)$ is open.

Returning to our setup over $\mathscr{O}_{F,S}$, it remains to show that f_S induces a surjective map on $\mathscr{O}_v$-points for all but finitely many $v \notin S$. Letting k_v denote the finite residue field at v, it suffices to prove surjectivity of the map on k_v-points for all but finitely many such v. Indeed, granting such surjectivity for a particular $v \notin S$, if $x : \operatorname{Spec} \mathscr{O}_v \to X_{S,v}$ is a section then the pullback of the smooth $\mathscr{O}_v$-map $f_{S,v}$ along x is a *smooth* $\mathscr{O}_v$-scheme that (by hypothesis) has a rational point in its special fiber. Since $\mathscr{O}_v$ is henselian, such a rational point in the special fiber lifts to an $\mathscr{O}_v$-point [EGA, IV_4, 18.5.17], and this lies in $X'_{S,v}(\mathscr{O}_v)$ over x as desired. The surjectivity on k_v-points for all but finitely many $v \notin S$ is an assertion in algebraic geometry for separated schemes of finite type over $\mathscr{O}_{F,S}$ and has nothing to do with adelic points. To prove it we may pass to connected components of X and increase S by a finite amount so that the smooth and geometrically connected (and non-empty) fibers of f_S have a common dimension d.

We now appeal to the following relative version of the Lang-Weil estimate for smooth geometrically connected varieties over a finite field, allowing for families over finite fields with varying characteristics:

Lemma 4.6. *Let $f : Y \to B$ be a smooth separated surjective map between finite type $\mathbf{Z}$-schemes such that the fibers are geometrically connected of dimension d. There is a constant $C > 0$ such that for all closed points $b \in B$,*

$$|\#Y_b(k(b)) - q_b^d| \leq C q_b^{d-1/2}, \tag{4.6.1}$$

where $q_b = \#k(b)$.

Proof. This is [Del1, Cor. 3.3.3] applied to the constant sheaf $\mathbf{Q}_\ell$ on Y (which is pure of weight 0), but for the convenience of the reader we say a bit about what underlies the proof. By stratifying B, we can assume it is a $\mathbf{Z}[1/\ell]$-scheme for a prime ℓ. Consider each ℓ-adic sheaf $\mathrm{R}^i f_!(\mathbf{Q}_\ell)$ on B. It is constructible, vanishes for $i > 2d$, and has fiber at a geometric point $\overline{b}$ over a point $b \in B$ naturally identified with $\mathrm{H}^i_c(Y_{\overline{b}}, \mathbf{Q}_\ell)$. Also, for $i = 2d$ this sheaf is $\mathbf{Q}_\ell(-d)$ since f is smooth with geometrically connected non-empty fibers of dimension d.

The Grothendieck-Lefschetz trace formula implies

$$\#Y_b(k(b)) = \sum_{i=0}^{2d} (-1)^i \mathrm{Tr}(\phi_b | \mathrm{H}^i_c(Y_{\overline{b}}, \mathbf{Q}_\ell))$$

for each closed point $b \in B$, where ϕ_b is the geometric Frobenius element in $\mathrm{Gal}(k(\overline{b})/k(b))$. The contribution for $i = 2d$ is q_b^d, and by Deligne's generalization of the Riemann Hypothesis [Del1, Thm. 3.3.1], the eigenvalues of ϕ_b on $\mathrm{H}^i_c(Y_{\overline{b}}, \mathbf{Q}_\ell)$ are q_b-Weil numbers of weight at most i (i.e., algebraic numbers whose complex embeddings all have a common absolute value $q_b^{w/2}$ for some $w \leq i$). In particular, the ith trace term in the above formula is an algebraic number all of whose complex embeddings have absolute value at most $n_i q_b^{i/2}$, where n_i is an upper bound on the fibral ranks of the constructible sheaf $\mathrm{R}^i f_!(\mathbf{Q}_\ell)$. Allowing i to vary from 0 to $2d-1$, we obtain (4.6.1). ■

We apply the lemma to f_S to conclude that for any closed point $x \in X_S$ with associated residue field $k(x)$ of size q_x there is an estimate $|\#f_S^{-1}(x)(k(x)) - q_x^d| \le C q_x^{d-1/2}$ for a constant $C > 0$ that is independent of x. Hence, if q_x is sufficiently large then the fiber $f_S^{-1}(x)$ must have a $k(x)$-rational point. This applies in particular to any k_v-point of X_S when $\#k_v$ is sufficiently large, and so applies to all but finitely many $v \notin S$. ■

5. Algebraic spaces

We now show how Weil's topological method works for adelic points of separated algebraic spaces of finite type over a global field F (and we also consider the non-separated case over local fields). In this section, we assume the reader is familiar with the basic properties of algebraic spaces, as developed in [Kn]. We will work with quasi-separated algebraic spaces (as is the case throughout [Kn]), which is weaker than the separatedness that we shall need to obtain the main topological results in the adelic setting.

The first step is to verify that Theorem 3.4 is valid with finitely presented algebraic spaces in place of finitely presented schemes. This is proved by an étale descent argument to upgrade from schemes to algebraic spaces, and is explained in (the proof of) [Ols, Prop. 2.2] apart from the property of having fibers non-empty and geometrically connected of pure dimension d. So now we address this latter fibral property.

By using étale scheme covers, the condition that fibers are non-empty of pure dimension d can be reduced to the settled scheme case. For the property of geometric connectedness of fibers, we need to do more work. Exactly as in approximation arguments for schemes, it suffices to prove:

Lemma 5.1. *If $X_0 \to \mathrm{Spec}(B_0)$ is a finitely presented algebraic space over a ring B_0 and $n \in \mathbf{Z}$ is an integer then the locus in $\mathrm{Spec}(B_0)$ where the geometric fiber has n connected components is constructible.*

Proof. By applying the descent of finitely presented algebraic spaces through the limit process (using an expression for B_0 as a direct limit of noetherian subrings), it suffices to treat the case when B_0 is noetherian. Noetherian induction reduces the problem to showing that if B_0 is a domain then the number of connected components of the geometric generic fiber coincides with the number of connected components on the geometric fibers over some dense open in the base.

Since we have "spreading out" for algebraic spaces as well as the other properties in Theorem 3.4(2) (especially the properties of being a closed immersion or open immersion), we can conclude by arguing exactly as in the case of schemes [EGA, IV_3, 9.7.7] (using dense open schemes in quasi-compact quasi-separated algebraic spaces, and reducing certain steps in the argument back to the scheme case by using étale scheme covers; e.g., reducedness can be verified using an étale scheme cover, and to carry over [EGA, IV_3, 9.5.3] to algebraic spaces we use that an open subset of a scheme of finite type over a field is dense if and only if the same holds after pullback to an étale cover). ■

We also require the analogue of Theorem 3.6 for algebraic spaces, but we first focus on the set-theoretic aspect:

Proposition 5.2. *Let X_S be a separated algebraic space of finite type over $\mathscr{O}_{F,S}$. The map* (3.6.1) *is bijective.*

Proof. The proof of injectivity goes exactly as in the scheme case, due to the separatedness hypothesis (to circumvent the fact that the diagonal of a general algebraic space does not factor as a closed immersion followed by an open immersion). For surjectivity, we can focus on the factor ring $\prod_{v \notin S} \mathscr{O}_v$ of $\mathbf{A}_{F,S}$ away from S.

Choose a collection of points $x_v \in X_S(\mathscr{O}_v)$ for all $v \notin S$. We seek to construct $x \in X_S(\prod_{v \notin S} \mathscr{O}_v)$ recovering x_v for all $v \notin S$; there is at most one such x, and to prove that such an x exists we will use the settled scheme case and étale descent.

Let $\pi : U_S \to X_S$ be an étale cover by an affine scheme, so this map is separated (as U_S is separated). Its pullback along x_v is an étale cover of $\mathrm{Spec}\, \mathscr{O}_v$, and the special fibers of these maps have degree bounded independently of v since the fibers of π have bounded degree (as for any quasi-compact étale map to a quasi-separated quasi-compact algebraic space). Let N be a uniform upper bound on such fiber degrees, and for each $v \notin S$ let $\mathscr{O}_v \to \mathscr{O}'_v$ be an unramified extension of degree $d = N!$. Thus, the restriction $x'_v \in X_S(\mathscr{O}'_v)$ of x_v lifts to some $u'_v \in U_S(\mathscr{O}'_v)$. By the settled scheme case, there is a unique $u' \in U_S(\prod \mathscr{O}'_v)$ recovering u'_v for every $v \notin S$.

Let $R = \prod_{v \notin S} \mathscr{O}_v$ and $R' = \prod_{v \notin S} \mathscr{O}'_v$, so $R \to R'$ is a finite étale cover of degree d (express each $\mathscr{O}'_v$ in the form $\mathscr{O}_v[t]/(f_v)$ for a monic polynomial $f_v \in \mathscr{O}_v[t]$ with degree d and irreducible reduction, so $R' = R[t]/(f)$ for $f = (f_v) \in R[t] \subset \prod \mathscr{O}_v[t]$). Moreover, this is a $\mathbf{Z}/(d)$-torsor by choosing an identification of $\mathbf{Z}/(d)$ with the cyclic Galois groups for the factors rings. We have constructed a point $x' := \pi \circ u' \in X_S(R')$ which recovers the $\mathscr{O}'_v$-point x'_v for each $v \notin S$, and it suffices to descend x' to an R-point of X_S (since such a descent necessarily recovers x_v for each $v \notin S$, due to the injectivity of $X_S(\mathscr{O}_v) \to X_S(\mathscr{O}'_v)$). Since the functor X_S is an étale sheaf, it suffices to show that x' is $\mathbf{Z}/(d)$-invariant. By the settled injectivity, it suffices to check such invariance on the separate factors. Since x'_v descends to x_v for all $v \notin S$ by construction, we are done. ■

To bring in topologies, we need to address the local case. The role of completeness will be clarified by working with henselian valued fields: a *valued field* is a field k equipped with a nontrivial absolute value, and it is *henselian* if this absolute value uniquely extends to every algebraic extension. A characterization of the henselian property is that k is separably algebraically closed in $\widehat{k}$. (The complete case is all we will actually need, so the reader may skip ahead to Proposition 5.4 and restrict attention to complete ground fields.) By [Ber, 2.4.3], in the non-archimedean case k is henselian if and only if its valuation ring is henselian.

In general if k'/k is a finite separable extension field of a valued field k then the nonzero finite reduced $\widehat{k}$-algebra $k' \otimes_k \widehat{k}$ is the direct product of the completions of k' at the finitely many valuations extending the one on k. Thus, if k is henselian then $k' \otimes_k \widehat{k}$ is a field of degree $[k' : k]$ over $\widehat{k}$, so the archimedean henselian fields are precisely the algebraically closed subfields of $\mathbf{C}$ and the real closed subfields of $\mathbf{R}$ (equipped with the induced valuation). If k is henselian then the functor $k' \rightsquigarrow k' \otimes_k \widehat{k}$ is an equivalence between the category of finite étale k-algebras and the category of finite étale $\widehat{k}$-algebras: this is obvious in the archimedean case, and is [Ber, 2.4.1] in the non-archimedean case.

Lemma 5.3. *Let k be a henselian valued field. For any étale map $Y' \to Y$ between locally finite type k-schemes, the natural map $Y'(k) \to Y(k)$ is a local homeomorphism.*

Proof. We may work Zariski-locally on both Y and Y'. By the Zariski-local structure theorem for étale morphisms [EGA, IV_4, 18.4.6(ii)], we may assume $Y = \operatorname{Spec} B$ is affine and $Y' = \operatorname{Spec}((B[x]/(h))_{h'})$ for a monic $h \in B[x]$ with positive degree, say degree n. Compatibility with base change allows us to reduce to the universal case when Y is affine n-space over k and h is the universal monic polynomial of degree n. The assertion now takes on a concrete form: it is exactly "continuity of simple roots" as discussed just after the proof of Proposition 4.4, except that we are relaxing completeness to the henselian condition.

Since $Y'(\widehat{k}) \to Y(\widehat{k})$ is a local homeomorphism (by the known complete case) and the inclusions $Y'(k) \to Y'(\widehat{k})$ and $Y(k) \to Y(\widehat{k})$ are topological embeddings, it suffices to prove that under the map $Y'(\widehat{k}) \to Y(\widehat{k})$, the fiber over any $y \in Y(k)$ consists entirely of k-rational points. This problem concerns the k-scheme $Y'_y = \operatorname{Spec}((k[x]/(h))_{h'})$ for monic $h \in k[x]$ with degree $n > 0$: we claim that all simple zeros of h in $\widehat{k}$ lie in k. Equivalently, we claim that all $\widehat{k}$-points of a finite étale k-algebra E are k-points. This says that the natural map

$$\operatorname{Hom}_k(E, k) \to \operatorname{Hom}_k(E, \widehat{k}) = \operatorname{Hom}_{\widehat{k}}(\widehat{k} \otimes_k E, \widehat{k})$$

is bijective, which is a special case of the *functorial* equivalence between finite étale k-algebras and finite étale $\widehat{k}$-algebras for henselian valued fields k. ■

Proposition 5.4. *Let k be a henselian valued field, and X a (quasi-separated) algebraic space locally of finite type over k. There is a unique way to topologize $X(k)$ so that the following properties hold: it is functorial, compatible with fiber products and the case of schemes, open (resp. closed) immersions in X are carried to open (resp. closed) embeddings in $X(k)$, and étale maps are carried to local homeomorphisms.*

If X is separated then the topology on $X(k)$ is Hausdorff, and it is totally disconnected (*resp. locally compact*) *when k is non-archimedean* (*resp. locally compact*).

If k is complete and X is smooth then $X(k)$ admits a unique functorial k-analytic manifold structure which agrees with the scheme case and carries étale maps to k-analytic local isomorphisms.

I am grateful to A.J. deJong and L. Moret-Bailly for independently suggesting the method of proof below; it is much simpler than my original method (which required completeness and separatedness throughout, and more importantly rested on the main theorem from [CT], entailing a long detour through Berkovich spaces).

Proof. The uniqueness holds due to the requirement on étale maps and the fact that for every $x \in X(k)$ there exists an étale map $U \to X$ from a scheme U admitting a point $u \in U(k)$ such that $u \mapsto x$ [Kn, II, Thm. 6.4]. (This ensures, using a large disjoint union, that there is an étale scheme cover $U \to X$ such that $U(k) \to X(k)$ is surjective.) For separated X the Hausdorff property of $X(k)$ is a formal consequence of the desired compatibility with closed immersions and fiber products, and the assertions concerning local compactness and total disconnectedness are also clear via the scheme case when X is separated.

To prove existence with the asserted properties, consider the étale maps $f : U \to X$ from finite type k-schemes U. As we vary through such maps, the images $f(U(k)) \subseteq X(k)$ cover $X(k)$. We claim that the *strongest* topology on $X(k)$ making the maps $U(k) \to X(k)$ continuous (i.e., a subset of $X(k)$ is open when its preimage in each such $U(k)$ is open) does the job.

If $f : U \to X$ and $f' : U' \to X$ are two such étale maps, consider the induced maps $\phi : U(k) \to X(k)$ and $\phi' : U'(k) \to X(k)$. For an open set $V' \subseteq U'(k)$, $\phi^{-1}(\phi'(V)) = p_1(p_2^{-1}(V'))$ where p_i is the ith projection on $(U \times_X U')(k) = U(k) \times_{X(k)} U'(k)$. Equip $(U \times_X U')(k)$ with its natural topology using that $U \times_X U'$ is a scheme. Then the p_i are local homeomorphisms, due to Lemma 5.3 and the projections $U \times_X U' \rightrightarrows U, U'$ being étale maps of schemes, so $\phi^{-1}(\phi'(V'))$ is open in $U(k)$. Thus, if a subset of $X(k)$ is the image of an open set in some $U'(k)$ (such as being contained in $\phi'(U'(k))$ with open preimage in $U'(k)$) then it has open preimage in any other $U(k)$. In particular, any open set in $U(k)$ has image in $X(k)$ whose preimage in $U(k)$ is open (by taking $U' = U$).

It follows that if we declare a subset of $X(k)$ to be *open* when it has open preimage in every $U(k)$ (i.e., we consider the strongest topology making all maps $U(k) \to X(k)$ continuous) then in fact all maps $U(k) \to X(k)$ arising from schemes U étale over X are continuous and open. In particular, since there is always an étale map $U \to X$ from a scheme U such that the continuous open map $U(k) \to X(k)$ is surjective, it follows that the topology on $X(k)$ is functorial in X.

To prove that the topology is compatible with fiber products, consider a pair of k-maps $X', X'' \rightrightarrows X$ and compatible k-maps $U', U'' \rightrightarrows U$ among schemes étale over these algebraic spaces. Then $U' \times_U U'' \to X' \times_X X''$ is another such map, and the composite map

$$(U' \times_U U'')(k) = U'(k) \times_{U(k)} U''(k) \stackrel{h}{\to} X'(k) \times_{X(k)} X''(k) = (X' \times_X X'')(k)$$

as well as the middle map h are continuous and open. Thus, since the left map is a homeomorphism, it follows that the right equality is continuous and open on the image of h when we use the fiber product topology on $X'(k) \times_{X(k)} X''(k)$. Varying these étale schemes, it follows that the identification $X'(k) \times_{X(k)} X''(k) = (X' \times_X X'')(k)$ is a continuous open bijection, hence a homeomorphism.

To complete the proof of existence, it remains to verify that if $f : X' \to X$ is an open immersion (resp. closed immersion, resp. étale) then $X'(k) \to X(k)$ is an open embedding (resp. closed embedding, resp. local homeomorphism). Assume f is an open (resp. closed) immersion, and let $U \to X$ be an étale scheme cover such that $U(k) \to X(k)$ is surjective. The pullback $U' := U \times_X X'$ is an open (resp. closed) subscheme in U and $U'(k) = U(k) \times_{X(k)} X'(k)$ topologically due to the established compatibility with fiber products. Since $U' \to U$ is an open (resp. closed) immersion, $U'(k) \to U(k)$ is an open embedding (resp. closed embedding). Thus, for any subset $T \subseteq X'(k)$ that is open (resp. closed), its image in $X(k)$ has pullback in $U(k)$ that is equal to the image under $U'(k) \hookrightarrow U(k)$ of the preimage of T in $U'(k)$. This implies that $f(T)$ is open (resp. closed) in $X(k)$ since $U(k) \to X(k)$ is topologically a quotient mapping.

Now consider the local homeomorphism property for $X'(k) \to X(k)$ when $f : X' \to X$ is étale. Choose a separated étale scheme cover $U \to X$ such that $U(k) \to X(k)$ is surjective, and a separated étale scheme cover $U' \twoheadrightarrow X' \times_X U$ such that $U'(k) \to (X' \times_X U)(k)$ is surjective. Using such covers, by Lemma 5.3 the local homeomorphism property for $X'(k) \to X(k)$ is reduced to the special case of $U(k) \to X(k)$ for an étale map $U \to X$ from a separated scheme. Since the diagonal $U \to U \times_X U$ is an open and closed immersion of schemes (as U is *separated* and $U \to X$ is étale), likewise the natural map $U(k) \to (U \times_X U)(k) = U(k) \times_{X(k)} U(k)$ is an open and closed embedding (when using the fiber product topology on the target). Thus, for every $u \in U(k)$ there is an open neighborhood in $U(k)$ on which $U(k) \to X(k)$ is injective, so the continuous open map $U(k) \to X(k)$ is a local homeomorphism.

Finally, we address the k-analytic manifold structure when X is smooth and k is complete. We wish to use the structure on each $U(k)$ transported via the local homeomorphism $U(k) \to X(k)$ for étale maps $U \to X$ from schemes U. To verify that this defines a k-analytic structure, we have to check the k-analyticity of the transition maps, which amounts to the observation that for any two étale maps $U, U' \rightrightarrows X$ from schemes, the maps $p_1, p_2 : (U \times_X U')(k) = U(k) \times_{X(k)} U'(k) \rightrightarrows U(k), U'(k)$ are local k-analytic isomorphisms (by the known scheme case, ultimately resting on the k-analytic inverse function theorem and the Zariski-local description of étale maps). This k-analytic structure is easily proved to be functorial and to carry étale maps of algebraic spaces over to local k-analytic isomorphisms. ■

Example 5.5. Let G be a unipotent algebraic group over a henselian valued field k of characteristic 0 (such as a p-adic field; i.e., a finite extension of $\mathbf{Q}_p$) and V a reduced k-scheme of finite type equipped with a G-action (e.g., the coadjoint representation $\mathrm{Lie}(G)^*$, as in the orbit method). For $d \geq 0$ let $V_d \subset V$ denote the reduced locally closed subscheme of points whose G-orbit has dimension d. (This is locally closed due to applying semicontinuity of fiber dimension to the action map $G \times V \to V$.)

The universal action map $G \times V_d \to V_d \times V_d$ defined by $(g, v) \mapsto (g.v, v)$ is flat over the diagonal (since the stabilizer scheme in the V-group $G \times V$ is $\exp_{(G \times V)/V}(\ker B)$ for the differentiated vector bundle map $B : \mathfrak{g} \times V \to \mathrm{Tan}_{V/k}$ over V, and $\ker B$ is a subbundle over V_d due to B having constant rank over V_d). Thus, by [Del2, Prop. 3.11] there is a finitely presented algebraic space X over k and a faithfully flat map $V_d \to X$ that identifies X with the fppf sheaf quotient of V_d by its G-action, so we denote X as V_d/G. Generally V_d/G is highly non-separated. The topological space $(V_d/G)(k)$ is locally Hausdorff and locally compact (and locally totally disconnected). For p-adic k and the coadjoint representation $V = \mathrm{Lie}(G)^*$ there is interest in using sheaf theory on $(V_d/G)(k)$ to study the smooth representation theory of $G(k)$ over $\mathbf{C}$.

Corollary 5.6. *Let $f : X \to Y$ be a proper map between (quasi-separated) algebraic spaces locally of finite type over a local field k (possibly archimedean). The map $X(k) \to Y(k)$ is topologically proper.*

Proof. We can choose an étale scheme cover $Y' \to Y$ such that the local homeomorphism $Y'(k) \to Y(k)$ is surjective. It suffices to prove properness of $X(k) \times_{Y(k)} Y'(k) \to Y'(k)$, so we can applying base change along $Y' \to Y$ to reduce to the case that Y is a scheme. By using Chow's Lemma for algebraic spaces [Kn, IV, 3.1], the method of proof of Proposition 4.4 reduces the problem to the easy case when X is a projective space over Y. ■

Corollary 5.7. *Let X be a (quasi-separated) algebraic space locally of finite type over the valuation ring R of a field k equipped with a nontrivial non-archimedean absolute value, and assume that R is henselian. The subset $X(R)$ in $X(k)$ is open and closed, and if k is locally compact and X is of finite type over R then $X(R)$ is quasi-compact.*

Proof. By construction, the topology on $X(k)$ is obtained from that on the spaces $U(k)$ for schemes U étale over X_k. In particular, for any scheme U étale over X the open set $U(R)$ in $U_k(k)$ has open image in $X(k)$. Since R is henselian, any R-point of X is in the image of $U(R)$ for some étale map $U \to X$ (by taking U such that there is a rational point in the fiber of $U \to X$ over the closed point of the chosen R-point of X, and using that R is henselian). This proves that $X(R)$ is open in $X(k)$. Using a huge disjoint union, we can construct an étale scheme cover $U \to X$ such that $U(R) \to X(R)$ and $U(k) \to X(k)$ are surjective. The full preimage of $X(k) - X(R)$ in $U(k)$ is $U(k) - U(R)$, which is open in $U(k)$, so since $U(k) \to X(k)$ is a continuous surjective open map it follows that $X(k) - X(R)$ is open in $X(k)$. Thus, $X(R)$ is also closed in $X(k)$.

Now assume that k is locally compact and X is of finite type over R. To build the étale scheme $U \to X$ such that $U(R) \to X(R)$ is surjective, we just have to lift the rational points in the special fiber of $X \to \operatorname{Spec} R$. But the residue field is a finite field and X is of finite type, so by using a finite stratification of X by schemes we see that there are only finitely many rational points in the special fiber. Thus, U can be constructed as finite type over R, so $U(R)$ is quasi-compact and therefore $X(R)$ is quasi-compact. ■

As an application of Corollary 5.7, we can carry over verbatim the proof of Theorem 3.6 to show that for a separated algebraic space X_S of finite type over $\mathscr{O}_{F,S}$, the product topology on $X_S(\mathbf{A}_{F,S})$ via Proposition 5.2 and Proposition 5.4 is locally compact Hausdorff and induces an open embedding

$$X_{S'}(\mathbf{A}_{F,S'}) \to X_{S''}(\mathbf{A}_{F,S''})$$

where $X_{S'}$ and $X_{S''}$ are as in Theorem 3.6.

Since Theorem 3.4 is valid for algebraic spaces, the natural map

$$\varinjlim X_S(\mathbf{A}_{F,S'}) \to X_S(\mathbf{A}_F)$$

is bijective for any separated algebraic space X_S of finite type over $\mathscr{O}_{F,S}$ (where S' varies through the finite sets of places containing S). Thus, exactly as in the scheme case, we can functorially topologize $X(\mathbf{A}_F)$ for any separated algebraic space X of finite type over F (recovering our earlier topological constructions when X is a separated F-scheme of finite type). Exactly as in the scheme case, this is locally compact, Hausdorff, has a countable base of opens, and is compatible with fiber products and closed immersions. Proposition 4.4 carries over with the same proofs (using Corollary 5.6). For general interest, we record the latter:

Proposition 5.8. *Let $f : X \to Y$ be a proper map between separated algebraic spaces locally of finite type over a global field F. The map $X(\mathbf{A}_F) \to Y(\mathbf{A}_F)$ is topologically proper.*

The openness result for a smooth surjective F-morphism (as in Theorem 4.5) lies somewhat deeper:

Theorem 5.9. *Let $f : X' \to X$ be a smooth surjective map between separated algebraic spaces of finite type over a global field F. Assume that the fibers of f are geometrically connected. Then the map $X'(\mathbf{A}_F) \to X(\mathbf{A}_F)$ is open.*

Proof. The argument for the scheme case carries over except for the step of checking surjectivity at the level of rational points over the finite residue fields at all but finitely many places. For this we just need Lemma 4.6 to be valid for algebraic spaces of finite presentation over $\mathbf{Z}$. The basic formalism of étale cohomology works for noetherian algebraic spaces with essentially the same proofs because of: the finite stratification in locally closed schemes for noetherian algebraic spaces, formal GAGA for noetherian algebraic spaces [Kn, V, §6], Nagata's compactification theorem for algebraic spaces (recently proved, e.g. in [CLO]), and the fact that separated algebraic space curves over a field are schemes [Kn, V, 4.9ff].

The Grothendieck-Lefschetz trace formula also carries over, since excision for cohomology with proper supports allows us to use a stratification in schemes to reduce to the known case of schemes. Thus, we just need that Deligne's Riemann Hypothesis [Del1, Thm. 3.3.1] holds for separated algebraic spaces of finite type over a finite field. Once again we can use the excision sequence and a stratification in schemes to reduce to the known scheme case. ■

Finally, we address how the topology on $X(k)$ for an algebraic space X over a field k as in Proposition 5.4 interacts with Weil restriction through finite extensions k'/k, and then deduce a corresponding global result for adelic points. We first record how Weil restriction behaves for algebraic spaces:

Lemma 5.10. *Let $R \to R'$ be a finite locally free ring extension, and X' a (quasi-separated) algebraic space of finite type over R'. The Weil restriction $X := \operatorname{Res}_{R'/R}(X')$ as a functor on R-schemes is a (quasi-separated) algebraic space of finite type over R. If X' is separated (resp. of finite presentation) over R' then the same holds for X over R.*

See [Ols, Thm. 1.5] for more general results on Weil restriction for algebraic spaces.

Proof. Let $U' \to X'$ be an étale cover by an affine scheme, so $\mathrm{Res}_{R'/R}(U')$ is an affine scheme of finite type over R (and of finite presentation when X' is of finite presentation over R'). Since any finite algebra over a strictly henselian local ring is a finite product of such rings [EGA, IV_4,18.8.10], the induced étale map $\mathrm{Res}_{R'/R}(U') \to \mathrm{Res}_{R'/R}(X')$ of étale sheaves on the category of R-schemes is surjective. Moreover, the fiber square of this map is the functor $\mathrm{Res}_{R'/R}(U' \times_{X'} U')$. The fiber product $U' \times_{X'} U'$ is quasi-compact, separated, and étale over U' under either projection because the same holds for the étale map $U' \to X'$ (since U' is separated and X' is quasi-separated). But any quasi-compact étale map is quasi-finite, so by Zariski's Main Theorem [EGA, IV_3, 8.12.6] such maps $U' \times_{X'} U' \rightrightarrows U'$ are quasi-affine when separated. (See [EGA, II, 5.1.9] for the equivalence of the two natural meanings of "quasi-affine" for finite type schemes over a ring.) Hence, the finite type R'-scheme $U' \times_{X'} U'$ is quasi-affine, so it is also quasi-projective over R'. It follows that $\mathrm{Res}_{R'/R}(U' \times_{X'} U')$ is represented by an R-scheme of finite type (even quasi-projective, by [CGP, A.5.8]).

The projections $\mathrm{Res}_{R'/R}(U' \times_{X'} U') \rightrightarrows \mathrm{Res}_{R'/R}(U')$ are étale since the maps $U' \times_{X'} U' \rightrightarrows U'$ are étale, and the diagonal

$$\delta : \mathrm{Res}_{R'/R}(U' \times_{X'} U') \to \mathrm{Res}_{R'/R}(U') \times_{\mathrm{Spec}(R)} \mathrm{Res}_{R'/R}(U') = \mathrm{Res}_{R'/R}(U' \times_{\mathrm{Spec}(R')} U')$$

is the Weil restriction of $U' \times_{X'} U' \to U' \times_{\mathrm{Spec}(R')} U'$, so δ is a closed immersion when X' is separated.

We conclude that $\mathrm{Res}_{R'/R}(X')$ is an étale sheaf quotient of an affine scheme equipped with a representable étale equivalence relation having a quasi-compact diagonal δ that is a closed immersion when X' is separated. The category of (quasi-separated) algebraic spaces is stable under the formation of quotients by étale equivalence relations having quasi-compact diagonal [LMB, Prop. 1.3], so $\mathrm{Res}_{R'/R}(X')$ is an algebraic space and it is separated when X' is separated. It is finitely presented over R when X' is finitely presented over R' since in such cases by construction $\mathrm{Res}_{R'/R}(X')$ admits a finitely presented étale cover by an affine scheme of finite presentation over R. ■

Proposition 5.11. *Let k'/k be an extension of henselian valued fields, and X a (quasi-separated) algebraic space locally of finite type over k.*

1. *If $[k' : k]$ is finite then for any (quasi-separated) algebraic space Y' of finite type over k', the identification of sets $\mathrm{Res}_{k'/k}(Y')(k) = Y'(k')$ is a homeomorphism.*
2. *The natural map $X(k) \to X(\widehat{k}) = X_{\widehat{k}}(\widehat{k})$ is a topological embedding.*
3. *Assume X is covered by separated Zariski-open subsets. The natural map $X(k) \to X(k') = X_{k'}(k')$ is a topological embedding, and it is a closed embedding when k is closed in k'.*

We will not use (3) (whose proof rests on [CT] when $[k' : k]$ is infinite).

Proof. First consider (1). For $y' \in Y'(k')$, choose an étale map $U' \to Y'$ from an affine scheme U' such that there exists $u' \in U'(k')$ over y'. Then $U := \mathrm{Res}_{k'/k}(U')$ is an affine scheme of finite type over k and the induced map $U \to Y$ is étale (by the functorial criterion, or the construction of Y). Moreover, this latter map carries the point $u \in U(k) = U'(k')$ corresponding to u' over to the point $y \in Y(k) = Y'(k')$ corresponding to y'. In the commutative square

$$\begin{array}{ccc} U(k) & = & U'(k') \\ \downarrow & & \downarrow \\ Y(k) & = & Y'(k') \end{array}$$

the vertical maps are local homeomorphisms onto their images, and the top horizontal map is a homeomorphism due to the known case of affine schemes of finite type. Thus, the bijective bottom horizontal map is a homeomorphism between open neighborhoods of y and y'. Since y' was arbitrary, we are done with (1).

For (2), let $f : U \to X$ be an étale cover by a separated scheme such that $U(k) \to X(k)$ is surjective. In the commutative diagram

$$\begin{array}{ccc} U(k) & \longrightarrow & U(\widehat{k}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ X(k) & \longrightarrow & X(\widehat{k}) \end{array}$$

the vertical maps are local homeomorphisms (with the left side a quotient map), the top map is a topological embedding (since U is a scheme), and the bottom map is injective. It follows that the bottom map is continuous. To prove that it is a topological embedding, let $V \subseteq U(k)$ be an open set which is the preimage of its image in $X(k)$. We can choose an open set $V' \subseteq U(\widehat{k})$ which meets $U(k)$ in exactly V. The image $f(V') \subseteq X(\widehat{k})$ is an open set, and obviously $f(V) \subseteq X(k) \cap f(V')$. But the reverse inclusion also holds. Indeed, if $x \in X(k)$ has the form $f(v')$ for some $v' \in V' \subseteq U(\widehat{k})$ then necessarily $v' \in U(k)$ since the étale k-scheme U_x has all $\widehat{k}$-points necessarily k-rational (as k is henselian). This forces $v' \in V' \cap U(k) = V$, so $x \in f(V)$ as required and (2) is proved.

It follows from (2) that in general the property of $X(k) \to X(k')$ being a topological embedding is reduced to the analogous assertion using the completions of k and k'. If k is closed in k' then the resulting equality $k' \cap \widehat{k} = k$ in $\widehat{k'}$ forces $X(k) = X(k') \cap X(\widehat{k})$ inside $X(\widehat{k'})$, so in such cases $X(k)$ is closed in $X(k')$ when $X(\widehat{k})$ is closed in $X(\widehat{k'})$. Thus, to prove (3) we may and do now work with complete ground fields. (If $[k' : k] < \infty$ then $[\widehat{k'} : \widehat{k}] \leq [k' : k] < \infty$.) We also may and do assume X is separated, since the problem is Zariski-local on X.

First we consider the finite-degree case of (3) (with complete fields), as this admits a simpler proof than the general case. By working Zariski-locally on X we may assume it is of finite type over k, so $\mathrm{Res}_{k'/k}(X_{k'})$ is an algebraic space over k. Consider the diagram

$$X(k) \to X(k') \simeq X_{k'}(k') \simeq \mathrm{Res}_{k'/k}(X_{k'})(k)$$

in which the first bijection defines the topology on $X(k')$ and the second bijection is a homeomorphism (by (1)). The composite map is induced on k-points by the canonical map of k-schemes $j : X \to \mathrm{Res}_{k'/k}(X_{k'})$, so to settle the case when $[k' : k]$ is finite it suffices to prove that j is a closed immersion. It is equivalent to say that the base change $j_{k'} : X_{k'} \to \mathrm{Res}_{k'/k}(X_{k'})_{k'}$ is a closed immersion. This is a section to an instance of the canonical k'-map

$$\pi : \mathrm{Res}_{k'/k}(Y')_{k'} \to Y'$$

defined by $Y'(k' \otimes_k A') \to Y'(A')$ for k'-algebras A' and (quasi-separated) algebraic spaces Y' of finite type over k', so it suffices to note that π is separated when Y' is separated. (If $\Delta_{Y'/k'}$ is a quasi-compact immersion, so the same holds for $\Delta_{\mathrm{Res}_{k'/k}(Y')/k} = \mathrm{Res}_{k'/k}(\Delta_{Y'/k'})$, then any section to π is quasi-compact. Hence, even without completeness, $X(k) \to X(k')$ is a topological embedding whenever $[k' : k]$ is finite and $\Delta_{X/k}$ is a quasi-compact immersion.)

To handle the cases when $[k' : k]$ is not assumed to be finite (so we may and do assume k is non-archimedean, as otherwise we are in the settled finite-degree case), we will appeal to a more difficult (but ultimately equivalent) construction of the topology in the non-archimedean complete case, resting on the main theorem in [CT]. That theorem provides a functorial theory of analytification X^{an} (in the sense of rigid-analytic spaces) for *separated* algebraic spaces X locally of finite type over k, compatible with fiber products, open and closed immersions, étale maps, the scheme case, and extension of the ground field. Moreover, by [CT, Ex. 2.3.2] it satisfies the expected functorial property $X(k) = X^{\mathrm{an}}(k)$ as sets. Thus, by using an admissible affinoid open covering of X^{an}, this provides another way to topologize $X(k)$ compatibly with all of the properties required for the uniqueness in Proposition 5.4 (since rigid-analytic étale maps are local isomorphisms near rational points). Hence, we recover the topology in Proposition 5.4. Since the formation of X^{an} respects extension of the ground field, the injection $X(k) \to X(k')$ is topologically identified with the natural injection $X^{\mathrm{an}}(k) \to (X^{\mathrm{an}})_{k'}(k')$ that is seen to be a closed embedding by working with the constituents of an admissible affinoid open covering of X^{an}. ■

Corollary 5.12. *Let $f : X \to Y$ be a finite map between separated algebraic spaces locally of finite type over a henselian valued field k. If k is algebraically closed in $\widehat{k}$ then $X(k) \to Y(k)$ is topologically proper.*

The hypothesis that k is algebraically closed in $\widehat{k}$ holds if $\mathrm{char}(k) = 0$ or k is non-archimedean with an excellent valuation ring.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} X(k) & \longrightarrow & X(\widehat{k}) \\ \downarrow & & \downarrow \\ Y(k) & \longrightarrow & Y(\widehat{k}) \end{array}$$

in which the horizontal maps are topological embeddings (Proposition 5.11(2)). It follows that the diagram is topologically cartesian since it is set-theoretically cartesian (due to the hypothesis that k is algebraically closed in $\widehat{k}$). Hence, it suffices to consider the case when k is complete. We may also work locally on $Y(k)$, and for any $y \in Y(k)$ there is an étale map $U \to Y$ from a scheme U containing $u \in U(k)$ mapping to y. Then $U(k) \to Y(k)$ is a local homeomorphism near u, so we may pass to $X \times_Y U \to U$ in place of $X \to Y$ to reduce to the case when Y is a scheme.

By working Zariski-locally on Y we can then assume that $Y = \mathrm{Spec}(A)$ is affine and the Y-finite X is a closed subscheme of $\mathrm{Spec}(A[t_1, \dots, t_n]/(h_1, \dots, h_n))$ for some monic $h_j \in A[t_j]$ with positive degree. This reduces the problem to the special case $X = \mathrm{Spec}(A[t]/(h))$ for a monic $h \in A[t]$ with positive degree. Since a topologically closed map between Hausdorff spaces is proper when its fibers are finite, it suffices to prove closedness of the map on k-points. Such closedness follows from the version of "continuity of roots" (without simplicity requirements) given in [BGR, 3.4.1/2]. ■

Example 5.13. We now show if the hypothesis on X in Proposition 5.11(3) (which is always satisfied in the scheme case) is weakened to the condition that the quasi-compact $\Delta_{X/k}$ is an immersion, then the closed embedding property for $X(k) \to X(k')$ can fail even when k is complete with respect to a nontrivial discrete valuation and k'/k is finite separable. As the proof of Proposition 5.11 suggests, the place to look for such X is among those algebraic spaces which fail to admit an analytification in the sense of [CT].

Let k'/k be a separable quadratic extension of fields, and assume k is complete with respect to a nontrivial non-archimedean absolute value. Let X be the algebraic space obtained from the affine line L over k by "replacing" the origin with $\mathrm{Spec}(k')$. In concrete terms, this is the quotient of the affine k'-line L' by the free action of the affine étale L-group G obtained from $(\mathbf{Z}/2\mathbf{Z})_L$ by deleting the non-identity point over the origin of L. The smooth irreducible algebraic space $X = L'/G$ is a lower-dimensional version of the 2-dimensional non-analytifiable example in [CT, Ex. 3.1.1], and as in that example the diagonal $\Delta_{X/k}$ is easily checked to be a quasi-compact immersion (even affine).

By construction there is a natural étale map $X \to L$ that is an isomorphism over $L - \{0\}$ and has fiber $\mathrm{Spec}\, k'$ over 0. Thus, $X(k) \to L(k) = k$ misses 0 and hence is a homeomorphism onto $k^\times$. The construction of X makes sense using any quadratic étale algebra (i.e., we allow $k \times k$, and uniquely identify its k-automorphism group with $\mathbf{Z}/2\mathbf{Z}$). In that sense, the formation of X commutes with any extension of the ground field. Thus, $X_{k'}$ is the affine k'-line with a doubled origin, so $X(k') = X_{k'}(k')$ is the non-Hausdorff space built from k' by doubling the origin. The map $X(k) \to X(k')$ is identified with the inclusion of $k^\times$ into the k'-line with doubled origin. This has non-closed image.

Here is the analogue of Example 4.2 for algebraic spaces:

Proposition 5.14. *Let F'/F be a finite extension of global fields, and X' a separated algebraic space of finite type over F'. For the separated algebraic space $X = \mathrm{Res}_{F'/F}(X')$ of finite type over F, the bijection of sets $X(\mathbf{A}_F) = X'(\mathbf{A}_{F'})$ is a homeomorphism.*

Proof. By carrying over the same argument as in the scheme case, we reduce the problem to the case of local fields. This case is settled by Proposition 5.11(1). ■

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