The original mathematica notebook is attached to this pdf file.

In Acrobat, use "View/Show/Navigation Panes/Attachments" to access the file.

■ Setting up the Tits model

In[1]:=
```
(* Get[NotebookDirectory[]<>"IntegerSmithNormalForm.m"]; *)
Format[a[i_, j_]] := Subscript[a, i, j];
Format[u[i_, j_]] := Subscript[u, i, j];
Format[v[i_, j_]] := Subscript[v, i, j];
Format[w[i_, j_]] := Subscript[w, i, j];
Format[d[i_]] := Subscript[d, i];
Format[s[i_]] := Subscript[s, i];
Format[z[i_]] := Subscript[z, i];
U = Table[u[i, j], {i, 3}, {j, 3}];
V = Table[v[i, j], {i, 3}, {j, 3}];
W = Table[w[i, j], {i, 3}, {j, 3}];
MatrixForm /@ {U, V, W}
(* cubic polynomial δ in the Tits model *)
delta = Det@U + Det@V + Det@W - Sum[(U.V.W)[[i, i]], {i, 3}] // ExpandAll
(* variables in the Tits models *)
vars = Flatten[{U, V, W}]
subzero = Map[# → 0 &, vars];
sube = subzero;
sube[[1, 2]] = 1;
sube[[5, 2]] = 1;
sube[[9, 2]] = 1;
(* sube: the substitution (specilizaiton) that will produce the element "e" *)
MatrixForm /@ ({U, V, W} /. sube)
```

Out[11]= $\left\{\begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{pmatrix}, \begin{pmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{pmatrix}, \begin{pmatrix} w_{1,1} & w_{1,2} & w_{1,3} \\ w_{2,1} & w_{2,2} & w_{2,3} \\ w_{3,1} & w_{3,2} & w_{3,3} \end{pmatrix}\right\}$

Out[12]= $-u_{1,3} u_{2,2} u_{3,1} + u_{1,2} u_{2,3} u_{3,1} + u_{1,3} u_{2,1} u_{3,2} - u_{1,1} u_{2,3} u_{3,2} - u_{1,2} u_{2,1} u_{3,3} + u_{1,1} u_{2,2} u_{3,3} - v_{1,3} v_{2,2} v_{3,1} + v_{1,2} v_{2,3} v_{3,1} + v_{1,3} v_{2,1} v_{3,2} - v_{1,1} v_{2,3} v_{3,2} - v_{1,2} v_{2,1} v_{3,3} + v_{1,1} v_{2,2} v_{3,3} - u_{1,1} v_{1,1} w_{1,1} - u_{1,2} v_{2,1} w_{1,1} - u_{1,3} v_{3,1} w_{1,1} - u_{2,1} v_{1,1} w_{1,2} - u_{2,2} v_{2,1} w_{1,2} - u_{2,3} v_{3,1} w_{1,2} - u_{3,1} v_{1,1} w_{1,3} - u_{3,2} v_{2,1} w_{1,3} - u_{3,3} v_{3,1} w_{1,3} - u_{1,1} v_{1,2} w_{2,1} - u_{1,2} v_{2,2} w_{2,1} - u_{1,3} v_{3,2} w_{2,1} - u_{2,1} v_{1,2} w_{2,2} - u_{2,2} v_{2,2} w_{2,2} - u_{2,3} v_{3,2} w_{2,2} - u_{3,1} v_{1,2} w_{2,3} - u_{3,2} v_{2,2} w_{2,3} - u_{3,3} v_{3,2} w_{2,3} - u_{1,1} v_{1,3} w_{3,1} - u_{1,2} v_{2,3} w_{3,1} - u_{1,3} v_{3,3} w_{3,1} - w_{1,3} w_{2,2} w_{3,1} + w_{1,2} w_{2,3} w_{3,1} - u_{2,1} v_{1,3} w_{3,2} - u_{2,2} v_{2,3} w_{3,2} - u_{2,3} v_{3,3} w_{3,2} + w_{1,3} w_{2,1} w_{3,2} - w_{1,1} w_{2,3} w_{3,2} - u_{3,1} v_{1,3} w_{3,3} - u_{3,2} v_{2,3} w_{3,3} - u_{3,3} v_{3,3} w_{3,3} - w_{1,2} w_{2,1} w_{3,3} + w_{1,1} w_{2,2} w_{3,3}$

Out[13]= $\{u_{1,1}, u_{1,2}, u_{1,3}, u_{2,1}, u_{2,2}, u_{2,3}, u_{3,1}, u_{3,2}, u_{3,3}, v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}, v_{2,3}, v_{3,1}, v_{3,2}, v_{3,3}, w_{1,1}, w_{1,2}, w_{1,3}, w_{2,1}, w_{2,2}, w_{2,3}, w_{3,1}, w_{3,2}, w_{3,3}\}$

Out[19]= $\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}$

■ The Lie algebra of E6

■ Computing the defining equations

In[20]:=
```
(* renaming the 27 variables to z[1],...,z[27] *)
subz = Table[vars[[i]] → z[i], {i, Length@vars}]
deltaz = delta /. subz
AAA[{c_Integer, za_, zb_, zc_}] :=
  (c / 7) (za /. {z[i_] :→ Sum[a[i, j] z[j], {j, 27}]}) zb zc + (c / 7) (zb /. {z[i_] :→ Sum[a[i, j] z[j], {j, 27}]}) za zc + (c / 7) (zc /. {z[i_] :→ Sum[a[i, j] z[j], {j, 27}]}) za zb;
(* lieeqnpoly is such that
  lieeqnpoly ϵ = δ((1+ϵA).z)-δ(z)
   it is a sum of (linear forms in a_ij).(cubic monomials in z_i) *)
lieeqnpoly = Plus @@ (Map[AAA, Map[List @@ (7 #) &, List @@ deltaz]] // ExpandAll);
Short@lieeqnpoly
(* Producing the list of all cubic monomials in z_i *)
cubicmonomials = (7 Sum[z[i], {i, 1, 27}] ^3) // Expand;
cubicmonomials = (List @@ cubicmonomials) /. c_Integer t_ :→ t;
lieeqn = Map[Coefficient[lieeqnpoly, #] &, cubicmonomials];
(* lieeqn is a list of linear forms in a_ij, the quotient by these (of the space of all linear forms in a_ij) is the cotangent space of E6 at the identity *)
lieeqn = Union[lieeqn];
Short@lieeqn
Length@lieeqn
```

Out[20]=

$$\{u_{1,1} \to z_1,\ u_{1,2} \to z_2,\ u_{1,3} \to z_3,\ u_{2,1} \to z_4,\ u_{2,2} \to z_5,\ u_{2,3} \to z_6,\ u_{3,1} \to z_7,\ u_{3,2} \to z_8,\ u_{3,3} \to z_9,\ v_{1,1} \to z_{10},\ v_{1,2} \to z_{11},\ v_{1,3} \to z_{12},\ v_{2,1} \to z_{13},\ v_{2,2} \to z_{14},\ v_{2,3} \to z_{15},\ v_{3,1} \to z_{16},\ v_{3,2} \to z_{17},\ v_{3,3} \to z_{18},\ w_{1,1} \to z_{19},\ w_{1,2} \to z_{20},\ w_{1,3} \to z_{21},\ w_{2,1} \to z_{22},\ w_{2,2} \to z_{23},\ w_{2,3} \to z_{24},\ w_{3,1} \to z_{25},\ w_{3,2} \to z_{26},\ w_{3,3} \to z_{27}\}$$

Out[21]=

$$-z_3 z_5 z_7 + z_2 z_6 z_7 + z_3 z_4 z_8 - z_1 z_6 z_8 - z_2 z_4 z_9 + z_1 z_5 z_9 - z_{12} z_{14} z_{16} + z_{11} z_{15} z_{16} + z_{12} z_{13} z_{17} - z_{10} z_{15} z_{17} - z_{11} z_{13} z_{18} + z_{10} z_{14} z_{18} - z_1 z_{10} z_{19} - z_2 z_{13} z_{19} - z_3 z_{16} z_{19} - z_4 z_{10} z_{20} - z_5 z_{13} z_{20} - z_6 z_{16} z_{20} - z_7 z_{10} z_{21} - z_8 z_{13} z_{21} - z_9 z_{16} z_{21} - z_1 z_{11} z_{22} - z_2 z_{14} z_{22} - z_3 z_{17} z_{22} - z_4 z_{11} z_{23} - z_5 z_{14} z_{23} - z_6 z_{17} z_{23} - z_7 z_{11} z_{24} - z_8 z_{14} z_{24} - z_9 z_{17} z_{24} - z_1 z_{12} z_{25} - z_2 z_{15} z_{25} - z_3 z_{18} z_{25} - z_{21} z_{23} z_{25} + z_{20} z_{24} z_{25} - z_4 z_{12} z_{26} - z_5 z_{15} z_{26} - z_6 z_{18} z_{26} + z_{21} z_{22} z_{26} - z_{19} z_{24} z_{26} - z_7 z_{12} z_{27} - z_8 z_{15} z_{27} - z_9 z_{18} z_{27} - z_{20} z_{22} z_{27} + z_{19} z_{23} z_{27}$$

Out[24]//Short=

$$-a_{9,1} z_1 z_2 z_4 - a_{9,2} z_2^2 z_4 + a_{8,1} z_1 z_3 z_4 + a_{8,2} z_2 z_3 z_4 - a_{9,3} z_2 z_3 z_4 + \ll 6547 \gg + a_{23,27} z_{19} z_{27}^2 - a_{22,27} z_{20} z_{27}^2 - a_{20,27} z_{22} z_{27}^2 + a_{19,27} z_{23} z_{27}^2$$

Out[29]//Short=

$$\{0,\ -a_{1,5},\ a_{1,5},\ -a_{1,6},\ a_{1,6},\ \ll 1657 \gg,\ -a_{8,8} - a_{15,15} - a_{27,27},\ -a_{9,9} - a_{18,18} - a_{27,27},\ -a_{20,20} - a_{22,22} - a_{27,27},\ a_{19,19} + a_{23,23} + a_{27,27}\}$$

Out[30]= 1666

■ Simplifying the defining equations

In[31]:=
```
(* The list of linear forms can be quickly simplified over Q using Gaussian elimination by mathematica.
  But we need to do this over Z.  This can be done by the method of computing elementary divisors (Smith normal form).
  However, I choose to do it by hand, taking advantage of the special shapes of the equations.  *)

(* First, isolate those linear forms of the form ±a_ij.
    We find 540 such a_ij's.  These 540 variables can be eliminated from the 27*27 = 729 variables. *)
zerovars = Union@Select[lieeqn, Head[#] === a &]~Join~(-Select[lieeqn, Head[#] === Times &]);
Short@zerovars
Length@%
lieeqn2 = Union[lieeqn /. (sol1 = Map[# → 0 &, zerovars])];
Short@lieeqn2
Length@lieeqn2
(* After eliminating those 540 variables, most of the remaining linear forms are of the shape ±a_ij±a_pq.
   We identify all such and perform obvious eliminations. *)
lieeqn3 = Union[lieeqn2 /. (sol2 = Solve[Select[lieeqn2, Length@# === 2 &] == 0][[1]])]
Length@%
```

Out[32]//Short=

$$\{a_{1,5},\ a_{1,6},\ a_{1,8},\ a_{1,9},\ a_{1,10},\ a_{1,11},\ a_{1,12},\ a_{1,19},\ a_{1,22},\ a_{1,25},\ \ll 520 \gg,\ a_{27,7},\ a_{27,8},\ a_{27,9},\ a_{27,12},\ a_{27,15},\ a_{27,18},\ a_{27,19},\ a_{27,20},\ a_{27,22},\ a_{27,23}\}$$

Out[33]= 540

Out[35]//Short=

$$\{0,\ a_{1,4}-a_{2,5},\ -a_{1,7}+a_{2,8},\ -a_{1,13}-a_{2,10},\ \ll 1118 \gg,\ -a_{8,8}-a_{15,15}-a_{27,27},\ -a_{9,9}-a_{18,18}-a_{27,27},\ -a_{20,20}-a_{22,22}-a_{27,27},\ a_{19,19}+a_{23,23}+a_{27,27}\}$$

Out[36]= 1126

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[37]=

$$\{0,\ -a_{3,3}-a_{5,5}-a_{7,7},\ a_{2,2}+a_{6,6}+a_{7,7},\ -a_{1,1}-a_{6,6}-a_{8,8},\ a_{3,3}+a_{4,4}+a_{8,8},\ -a_{2,2}-a_{4,4}-a_{9,9},\ a_{1,1}+a_{5,5}+a_{9,9},\ -a_{12,12}-a_{14,14}-a_{16,16},\ a_{11,11}+a_{15,15}+a_{16,16},\ -a_{10,10}-a_{15,15}-a_{17,17},\ a_{12,12}+a_{13,13}+a_{17,17},\ -a_{11,11}-a_{13,13}-a_{18,18},\ a_{10,10}+a_{14,14}+a_{18,18},\ -a_{1,1}-a_{10,10}-a_{19,19},\ -a_{2,2}-a_{13,13}-a_{19,19},\ -a_{3,3}-a_{16,16}-a_{19,19},\ -a_{4,4}-a_{10,10}-a_{20,20},\ -a_{5,5}-a_{13,13}-a_{20,20},\ -a_{6,6}-a_{16,16}-a_{20,20},\ -a_{7,7}-a_{10,10}-a_{21,21},\ -a_{8,8}-a_{13,13}-a_{21,21},\ -a_{9,9}-a_{16,16}-a_{21,21},\ -a_{1,1}-a_{11,11}-a_{22,22},\ -a_{2,2}-a_{14,14}-a_{22,22},\ -a_{3,3}-a_{17,17}-a_{22,22},\ -a_{4,4}-a_{11,11}-a_{23,23},\ -a_{5,5}-a_{14,14}-a_{23,23},\ -a_{6,6}-a_{17,17}-a_{23,23},\ -a_{7,7}-a_{11,11}-a_{24,24},\ -a_{8,8}-a_{14,14}-a_{24,24},\ -a_{9,9}-a_{17,17}-a_{24,24},\ -a_{1,1}-a_{12,12}-a_{25,25},\ -a_{2,2}-a_{15,15}-a_{25,25},\ -a_{3,3}-a_{18,18}-a_{25,25},\ -a_{21,21}-a_{23,23}-a_{25,25},\ a_{20,20}+a_{24,24}+a_{25,25},\ -a_{4,4}-a_{12,12}-a_{26,26},\ -a_{5,5}-a_{15,15}-a_{26,26},\ -a_{6,6}-a_{18,18}-a_{26,26},\ -a_{19,19}-a_{24,24}-a_{26,26},\ a_{21,21}+a_{22,22}+a_{26,26},\ -a_{7,7}-a_{12,12}-a_{27,27},\ -a_{8,8}-a_{15,15}-a_{27,27},\ -a_{9,9}-a_{18,18}-a_{27,27},\ -a_{20,20}-a_{22,22}-a_{27,27},\ a_{19,19}+a_{23,23}+a_{27,27}\}$$

Out[38]= 46

■ Simplifying the "torus" part of the equations

In[39]:=
```
(* After the first two rounds of eliminations, only the "diagonal variables" a_ii's remain.
  They actually give the linear forms defining a maximal torus.
  We rename a_ii to d_i and simplify the equations over Z by hand *)
lieeqn3 /. {a[i_, i_] :> d[i]}
% /. (sol31 = {d[27] → -d[7] - d[12], d[26] → -d[4] - d[12], d[25] → -d[1] - d[12], d[24] → -d[7] - d[11], d[23] → -d[4] - d[11], d[22] → -d[1] - d[11],
      d[21] → -d[7] - d[10], d[20] → -d[4] - d[10], d[19] → -d[1] - d[10], d[18] → -d[11] - d[13], d[17] → -d[12] - d[13], d[16] → -d[12] - d[14]}) // Expand // Sort
% /. (sol32 = {d[15] → d[1] - d[2] + d[12], d[14] → d[1] - d[2] + d[11], d[13] → d[1] - d[2] + d[10]}) // Expand // Sort
% /. (sol33 = {d[12] → -d[1] - d[4] - d[7] - d[10] - d[11], d[9] → -d[2] - d[4], d[8] → -d[3] - d[4]}) // Expand // Sort
% /. (sol34 = {d[7] → d[1] - d[2] - d[3] - d[4], d[6] → -d[1] + d[3] + d[4], d[5] → -d[1] + d[2] + d[4]}) // Expand // Sort
```

Out[39]=

$$\{0,\ -d_3-d_5-d_7,\ d_2+d_6+d_7,\ -d_1-d_6-d_8,\ d_3+d_4+d_8,\ -d_2-d_4-d_9,\ d_1+d_5+d_9,\ -d_{12}-d_{14}-d_{16},\ d_{11}+d_{15}+d_{16},\ -d_{10}-d_{15}-d_{17},\ d_{12}+d_{13}+d_{17},\ -d_{11}-d_{13}-d_{18},\ d_{10}+d_{14}+d_{18},\ -d_1-d_{10}-d_{19},\ -d_2-d_{13}-d_{19},\ -d_3-d_{16}-d_{19},\ -d_4-d_{10}-d_{20},\ -d_5-d_{13}-d_{20},\ -d_6-d_{16}-d_{20},\ -d_7-d_{10}-d_{21},\ -d_8-d_{13}-d_{21},\ -d_9-d_{16}-d_{21},\ -d_1-d_{11}-d_{22},\ -d_2-d_{14}-d_{22},\ -d_3-d_{17}-d_{22},\ -d_4-d_{11}-d_{23},\ -d_5-d_{14}-d_{23},\ -d_6-d_{17}-d_{23},\ -d_7-d_{11}-d_{24},\ -d_8-d_{14}-d_{24},\ -d_9-d_{17}-d_{24},\ -d_1-d_{12}-d_{25},\ -d_2-d_{15}-d_{25},\ -d_3-d_{18}-d_{25},\ -d_{21}-d_{23}-d_{25},\ d_{20}+d_{24}+d_{25},\ -d_4-d_{12}-d_{26},\ -d_5-d_{15}-d_{26},\ -d_6-d_{18}-d_{26},\ -d_{19}-d_{24}-d_{26},\ d_{21}+d_{22}+d_{26},\ -d_7-d_{12}-d_{27},\ -d_8-d_{15}-d_{27},\ -d_9-d_{18}-d_{27},\ -d_{20}-d_{22}-d_{27},\ d_{19}+d_{23}+d_{27}\}$$

Out[40]=

$$\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -d_3-d_5-d_7,\ d_2+d_6+d_7,\ -d_1-d_6-d_8,\ d_3+d_4+d_8,\ -d_2-d_4-d_9,\ d_1+d_5+d_9,\ -d_1-d_4-d_7-d_{10}-d_{11}-d_{12},\ -d_1-d_4-d_7-d_{10}-d_{11}-d_{12},\ -d_1-d_4-d_7-d_{10}-d_{11}-d_{12},\ d_1+d_4+d_7+d_{10}+d_{11}+d_{12},\ d_1+d_4+d_7+d_{10}+d_{11}+d_{12},\ d_1+d_4+d_7+d_{10}+d_{11}+d_{12},\ d_1-d_2+d_{10}-d_{13},\ d_4-d_5+d_{10}-d_{13},\ d_7-d_8+d_{10}-d_{13},\ d_1-d_3+d_{11}+d_{12}+d_{13},\ d_1-d_3+d_{11}+d_{12}+d_{13},\ d_4-d_6+d_{11}+d_{12}+d_{13},\ d_4-d_6+d_{11}+d_{12}+d_{13},\ d_7-d_9+d_{11}+d_{12}+d_{13},\ d_7-d_9+d_{11}+d_{12}+d_{13},\ d_1-d_2+d_{11}-d_{14},\ d_4-d_5+d_{11}-d_{14},\ d_7-d_8+d_{11}-d_{14},\ d_1-d_3+d_{10}+d_{12}+d_{14},\ d_4-d_6+d_{10}+d_{12}+d_{14},\ d_7-d_9+d_{10}+d_{12}+d_{14},\ d_{10}-d_{11}-d_{13}+d_{14},\ d_1-d_2+d_{12}-d_{15},\ d_4-d_5+d_{12}-d_{15},\ d_7-d_8+d_{12}-d_{15},\ -d_{10}+d_{12}+d_{13}-d_{15},\ d_{11}-d_{12}-d_{14}+d_{15}\}$$

Out[41]=

$$\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -d_1+d_2+d_4-d_5,\ -d_1+d_2+d_4-d_5,\ -d_1+d_2+d_4-d_5,\ -d_3-d_5-d_7,\ d_2+d_6+d_7,\ -d_1-d_6-d_8,\ -d_1+d_2+d_7-d_8,\ -d_1+d_2+d_7-d_8,\ -d_1+d_2+d_7-d_8,\ d_3+d_4+d_8,\ -d_2-d_4-d_9,\ d_1+d_5+d_9,\ -d_1-d_4-d_7-d_{10}-d_{11}-d_{12},\ -d_1-d_4-d_7-d_{10}-d_{11}-d_{12},\ -d_1-d_4-d_7-d_{10}-d_{11}-d_{12},\ 2\,d_1-d_2-d_3+d_{10}+d_{11}+d_{12},\ 2\,d_1-d_2-d_3+d_{10}+d_{11}+d_{12},\ 2\,d_1-d_2-d_3+d_{10}+d_{11}+d_{12},\ d_1-d_2+d_4-d_6+d_{10}+d_{11}+d_{12},\ d_1-d_2+d_4-d_6+d_{10}+d_{11}+d_{12},\ d_1-d_2+d_4-d_6+d_{10}+d_{11}+d_{12},\ d_1+d_4+d_7+d_{10}+d_{11}+d_{12},\ d_1+d_4+d_7+d_{10}+d_{11}+d_{12},\ d_1+d_4+d_7+d_{10}+d_{11}+d_{12},\ d_1-d_2+d_7-d_9+d_{10}+d_{11}+d_{12},\ d_1-d_2+d_7-d_9+d_{10}+d_{11}+d_{12},\ d_1-d_2+d_7-d_9+d_{10}+d_{11}+d_{12}\}$$

Out[42]=

$$\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -d_1+d_2+d_4-d_5,\ -d_1+d_2+d_4-d_5,\ -d_1+d_2+d_4-d_5,\ d_1-d_2-d_4+d_5,\ -d_1+d_3+d_4-d_6,\ d_1-d_2-d_3-d_4-d_7,\ d_1-d_2-d_3-d_4-d_7,\ d_1-d_2-d_3-d_4-d_7,\ -d_3-d_5-d_7,\ -d_2-d_6-d_7,\ -d_2-d_6-d_7,\ -d_2-d_6-d_7,\ -d_1+d_2+d_3+d_4+d_7,\ -d_1+d_2+d_3+d_4+d_7,\ -d_1+d_2+d_3+d_4+d_7,\ d_2+d_6+d_7\}$$

Out[43]=

$$\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}$$

In[44]:=
```
(* reorganize the solution and verify it *)
sol3 = sol31;
sol3 = (Join[sol3 /. #, #] &)@sol32;
sol3 = (Join[sol3 /. #, #] &)@sol33;
sol3 = Sort@((Join[sol3 /. #, #] &)@sol34 /. d[i_] :> a[i, i])
lieeqn3 /. sol3
```

Out[47]= $\{a_{5,5} \to -a_{1,1}+a_{2,2}+a_{4,4},\ a_{6,6} \to -a_{1,1}+a_{3,3}+a_{4,4},\ a_{7,7} \to a_{1,1}-a_{2,2}-a_{3,3}-a_{4,4},\ a_{8,8} \to -a_{3,3}-a_{4,4},\ a_{9,9} \to -a_{2,2}-a_{4,4},\ a_{12,12} \to -2\,a_{1,1}+a_{2,2}+a_{3,3}-a_{10,10}-a_{11,11},\ a_{13,13} \to a_{1,1}-a_{2,2}+a_{10,10},\ a_{14,14} \to a_{1,1}-a_{2,2}+a_{11,11},\ a_{15,15} \to -a_{1,1}+a_{3,3}-a_{10,10}-a_{11,11},\ a_{16,16} \to a_{1,1}-a_{3,3}+a_{10,10},\ a_{17,17} \to a_{1,1}-a_{3,3}+a_{11,11},\ a_{18,18} \to -a_{1,1}+a_{2,2}-a_{10,10}-a_{11,11},\ a_{19,19} \to -a_{1,1}-a_{10,10},\ a_{20,20} \to -a_{4,4}-a_{10,10},\ a_{21,21} \to -a_{1,1}+a_{2,2}+a_{3,3}+a_{4,4}-a_{10,10},\ a_{22,22} \to -a_{1,1}-a_{11,11},\ a_{23,23} \to -a_{4,4}-a_{11,11},\ a_{24,24} \to -a_{1,1}+a_{2,2}+a_{3,3}+a_{4,4}-a_{11,11},\ a_{25,25} \to a_{1,1}-a_{2,2}-a_{3,3}+a_{10,10}+a_{11,11},\ a_{26,26} \to 2\,a_{1,1}-a_{2,2}-a_{3,3}-a_{4,4}+a_{10,10}+a_{11,11},\ a_{27,27} \to a_{1,1}+a_{4,4}+a_{10,10}+a_{11,11}\}$

Out[48]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

The final result

In[49]:=
```
E6LieAlgebra = Table[a[i, j], {i, 27}, {j, 27}] /. Join[sol1, sol2, sol3];
(* list of free variables *)
E6vars = Map[a[#[[1]], #[[2]]] &, Select[Flatten[Table[{i, j}, {i, 27}, {j, 27}], 1], E6LieAlgebra[[#[[1]], #[[2]]]] === a[#[[1]], #[[2]]] &]]
(* number of free variables *)
Length@%
tmp = E6LieAlgebra;
Do[tmp[[i, i]] = "*", {i, 27}];
(* the full Lie algebra as a matrix, without the diagonal part *)
tmp // MatrixForm
(* the diagonal part *)
Table[E6LieAlgebra[[i, i]], {i, 27}]
```

Out[50]= $\{a_{1,1}, a_{2,2}, a_{3,3}, a_{4,4}, a_{10,10}, a_{11,11}, a_{12,15}, a_{12,18}, a_{15,12}, a_{15,18}, a_{18,12}, a_{18,15}, a_{20,1}, a_{20,2}, a_{20,3}, a_{21,1}, a_{21,2}, a_{21,3}, a_{21,4}, a_{21,5}, a_{21,6}, a_{21,24}, a_{21,27}, a_{22,10}, a_{22,13}, a_{22,16}, a_{23,1}, a_{23,2}, a_{23,3}, a_{23,10}, a_{23,13}, a_{23,16}, a_{24,1}, a_{24,2}, a_{24,3}, a_{24,4}, a_{24,5}, a_{24,6}, a_{24,10}, a_{24,13}, a_{24,16}, a_{24,21}, a_{24,27}, a_{25,10}, a_{25,11}, a_{25,13}, a_{25,14}, a_{25,16}, a_{25,17}, a_{25,26}, a_{25,27}, a_{26,1}, a_{26,2}, a_{26,3}, a_{26,10}, a_{26,11}, a_{26,13}, a_{26,14}, a_{26,16}, a_{26,17}, a_{26,25}, a_{26,27}, a_{27,1}, a_{27,2}, a_{27,3}, a_{27,4}, a_{27,5}, a_{27,6}, a_{27,10}, a_{27,11}, a_{27,13}, a_{27,14}, a_{27,16}, a_{27,17}, a_{27,21}, a_{27,24}, a_{27,25}, a_{27,26}\}$

Out[51]= 78

Out[54]//MatrixForm=

*	$-a_{15,12}$	$-a_{18,12}$	$-a_{26,25}$	0	0	$-a_{27,25}$	0	0	0	0	0	$-a_{21,6}$	$-a_{24,6}$	$-a_{27,6}$	$a_{21,5}$	$a_{24,5}$	$a_{27,5}$	0	$-a_{27,11}$	$a_{26,11}$	0	$a_{27,10}$	$-a_{26,10}$	0	$-a_{24,10}$	$a_{23,10}$
$-a_{12,15}$	*	$-a_{18,15}$	0	$-a_{26,25}$	0	0	$-a_{27,25}$	0	$a_{21,6}$	$a_{24,6}$	$a_{27,6}$	0	0	0	$-a_{21,4}$	$-a_{24,4}$	$-a_{27,4}$	0	$-a_{27,14}$	$a_{26,14}$	0	$a_{27,13}$	$-a_{26,13}$	0	$-a_{24,13}$	$a_{23,13}$
$-a_{12,18}$	$-a_{15,18}$	*	0	0	$-a_{26,25}$	0	0	$-a_{27,25}$	$-a_{21,5}$	$-a_{24,5}$	$-a_{27,5}$	$a_{21,4}$	$a_{24,4}$	$a_{27,4}$	0	0	0	0	$-a_{27,17}$	$a_{26,17}$	0	$a_{27,16}$	$-a_{26,16}$	0	$-a_{24,16}$	$a_{23,16}$
$-a_{25,26}$	0	0	*	$-a_{15,12}$	$-a_{18,12}$	$-a_{27,26}$	0	0	0	0	0	$a_{21,3}$	$a_{24,3}$	$a_{27,3}$	$-a_{21,2}$	$-a_{24,2}$	$-a_{27,2}$	$a_{27,11}$	0	$-a_{25,11}$	$-a_{27,10}$	0	$a_{25,10}$	$a_{24,10}$	0	$-a_{22,10}$
0	$-a_{25,26}$	0	$-a_{12,15}$	*	$-a_{18,15}$	0	$-a_{27,26}$	0	$-a_{21,3}$	$-a_{24,3}$	$-a_{27,3}$	0	0	0	$a_{21,1}$	$a_{24,1}$	$a_{27,1}$	$a_{27,14}$	0	$-a_{25,14}$	$-a_{27,13}$	0	$a_{25,13}$	$a_{24,13}$	0	$-a_{22,13}$
0	0	$-a_{25,26}$	$-a_{12,18}$	$-a_{15,18}$	*	0	0	$-a_{27,26}$	$a_{21,2}$	$a_{24,2}$	$a_{27,2}$	$-a_{21,1}$	$-a_{24,1}$	$-a_{27,1}$	0	0	0	$a_{27,17}$	0	$-a_{25,17}$	$-a_{27,16}$	0	$a_{25,16}$	$a_{24,16}$	0	$-a_{22,16}$
$-a_{25,27}$	0	0	$-a_{26,27}$	0	0	*	$-a_{15,12}$	$-a_{18,12}$	0	0	0	$-a_{20,3}$	$-a_{23,3}$	$-a_{26,3}$	$a_{20,2}$	$a_{23,2}$	$a_{26,2}$	$-a_{26,11}$	$a_{25,11}$	0	$a_{26,10}$	$-a_{25,10}$	0	$-a_{23,10}$	$a_{22,10}$	0
0	$-a_{25,27}$	0	0	$-a_{26,27}$	0	$-a_{12,15}$	*	$-a_{18,15}$	$a_{20,3}$	$a_{23,3}$	$a_{26,3}$	0	0	0	$-a_{20,1}$	$-a_{23,1}$	$-a_{26,1}$	$-a_{26,14}$	$a_{25,14}$	0	$a_{26,13}$	$-a_{25,13}$	0	$-a_{23,13}$	$a_{22,13}$	0
0	0	$-a_{25,27}$	0	0	$-a_{26,27}$	$-a_{12,18}$	$-a_{15,18}$	*	$-a_{20,2}$	$-a_{23,2}$	$-a_{26,2}$	$a_{20,1}$	$a_{23,1}$	$a_{26,1}$	0	0	0	$-a_{26,17}$	$a_{25,17}$	0	$a_{26,16}$	$-a_{25,16}$	0	$-a_{23,16}$	$a_{22,16}$	0
0	$-a_{25,17}$	$a_{25,14}$	0	$-a_{26,17}$	$a_{26,14}$	0	$-a_{27,17}$	$a_{27,14}$	*	$-a_{24,21}$	$-a_{27,21}$	$a_{12,15}$	0	0	$a_{12,18}$	0	0	0	0	0	$-a_{27,4}$	$a_{27,1}$	$-a_{26,1}$	$a_{24,4}$	$-a_{24,1}$	$a_{23,1}$
0	$a_{25,16}$	$-a_{25,13}$	0	$a_{26,16}$	$-a_{26,13}$	0	$a_{27,16}$	$-a_{27,13}$	$-a_{21,24}$	*	$-a_{27,24}$	0	$a_{12,15}$	0	0	$a_{12,18}$	0	$a_{27,4}$	$-a_{27,1}$	$a_{26,1}$	0	0	0	$-a_{21,4}$	$a_{21,1}$	$-a_{20,1}$
0	$-a_{22,16}$	$a_{22,13}$	0	$-a_{23,16}$	$a_{23,13}$	0	$-a_{24,16}$	$a_{24,13}$	$-a_{21,27}$	$-a_{24,27}$	*	0	0	$a_{12,15}$	0	0	$a_{12,18}$	$-a_{24,4}$	$a_{24,1}$	$-a_{23,1}$	$a_{21,4}$	$-a_{21,1}$	$a_{20,1}$	0	0	0
$a_{25,17}$	0	$-a_{25,11}$	$a_{26,17}$	0	$-a_{26,11}$	$a_{27,17}$	0	$-a_{27,11}$	$a_{15,12}$	0	0	*	$-a_{24,21}$	$-a_{27,21}$	$a_{15,18}$	0	0	0	0	0	$-a_{27,5}$	$a_{27,2}$	$-a_{26,2}$	$a_{24,5}$	$-a_{24,2}$	$a_{23,2}$
$-a_{25,16}$	0	$a_{25,10}$	$-a_{26,16}$	0	$a_{26,10}$	$-a_{27,16}$	0	$a_{27,10}$	0	$a_{15,12}$	0	$-a_{21,24}$	*	$-a_{27,24}$	0	$a_{15,18}$	0	$a_{27,5}$	$-a_{27,2}$	$a_{26,2}$	0	0	0	$-a_{21,5}$	$a_{21,2}$	$-a_{20,2}$
$a_{22,16}$	0	$-a_{22,10}$	$a_{23,16}$	0	$-a_{23,10}$	$a_{24,16}$	0	$-a_{24,10}$	0	0	$a_{15,12}$	$-a_{21,27}$	$-a_{24,27}$	*	0	0	$a_{15,18}$	$-a_{24,5}$	$a_{24,2}$	$-a_{23,2}$	$a_{21,5}$	$-a_{21,2}$	$a_{20,2}$	0	0	0
$-a_{25,14}$	$a_{25,11}$	0	$-a_{26,14}$	$a_{26,11}$	0	$-a_{27,14}$	$a_{27,11}$	0	$a_{18,12}$	0	0	$a_{18,15}$	0	0	*	$-a_{24,21}$	$-a_{27,21}$	0	0	0	$-a_{27,6}$	$a_{27,3}$	$-a_{26,3}$	$a_{24,6}$	$-a_{24,3}$	$a_{23,3}$
$a_{25,13}$	$-a_{25,10}$	0	$a_{26,13}$	$-a_{26,10}$	0	$a_{27,13}$	$-a_{27,10}$	0	0	$a_{18,12}$	0	0	$a_{18,15}$	0	$-a_{21,24}$	*	$-a_{27,24}$	$a_{27,6}$	$-a_{27,3}$	$a_{26,3}$	0	0	0	$-a_{21,6}$	$a_{21,3}$	$-a_{20,3}$
$-a_{22,13}$	$a_{22,10}$	0	$-a_{23,13}$	$a_{23,10}$	0	$-a_{24,13}$	$a_{24,10}$	0	0	0	$a_{18,12}$	0	0	$a_{18,15}$	$-a_{21,27}$	$-a_{24,27}$	*	$-a_{24,6}$	$a_{24,3}$	$-a_{23,3}$	$a_{21,6}$	$-a_{21,3}$	$a_{20,3}$	0	0	0
0	0	0	$-a_{20,1}$	$-a_{20,2}$	$-a_{20,3}$	$-a_{21,1}$	$-a_{21,2}$	$-a_{21,3}$	0	$-a_{22,10}$	$-a_{25,10}$	0	$-a_{22,13}$	$-a_{25,13}$	0	$-a_{22,16}$	$-a_{25,16}$	*	$a_{25,26}$	$a_{25,27}$	$a_{21,24}$	0	0	$a_{21,27}$	0	0
$a_{20,1}$	$a_{20,2}$	$a_{20,3}$	0	0	0	$-a_{21,4}$	$-a_{21,5}$	$-a_{21,6}$	0	$-a_{23,10}$	$-a_{26,10}$	0	$-a_{23,13}$	$-a_{26,13}$	0	$-a_{23,16}$	$-a_{26,16}$	$a_{26,25}$	*	$a_{26,27}$	0	$a_{21,24}$	0	0	$a_{21,27}$	0
$a_{21,1}$	$a_{21,2}$	$a_{21,3}$	$a_{21,4}$	$a_{21,5}$	$a_{21,6}$	0	0	0	0	$-a_{24,10}$	$-a_{27,10}$	0	$-a_{24,13}$	$-a_{27,13}$	0	$-a_{24,16}$	$-a_{27,16}$	$a_{27,25}$	$a_{27,26}$	*	0	0	$a_{21,24}$	0	0	$a_{21,27}$
0	0	0	$-a_{23,1}$	$-a_{23,2}$	$-a_{23,3}$	$-a_{24,1}$	$-a_{24,2}$	$-a_{24,3}$	$a_{22,10}$	0	$-a_{25,11}$	$a_{22,13}$	0	$-a_{25,14}$	$a_{22,16}$	0	$-a_{25,17}$	$a_{24,21}$	0	0	*	$a_{25,26}$	$a_{25,27}$	$a_{24,27}$	0	0
$a_{23,1}$	$a_{23,2}$	$a_{23,3}$	0	0	0	$-a_{24,4}$	$-a_{24,5}$	$-a_{24,6}$	$a_{23,10}$	0	$-a_{26,11}$	$a_{23,13}$	0	$-a_{26,14}$	$a_{23,16}$	0	$-a_{26,17}$	0	$a_{24,21}$	0	$a_{26,25}$	*	$a_{26,27}$	0	$a_{24,27}$	0
$a_{24,1}$	$a_{24,2}$	$a_{24,3}$	$a_{24,4}$	$a_{24,5}$	$a_{24,6}$	0	0	0	$a_{24,10}$	0	$-a_{27,11}$	$a_{24,13}$	0	$-a_{27,14}$	$a_{24,16}$	0	$-a_{27,17}$	0	0	$a_{24,21}$	$a_{27,25}$	$a_{27,26}$	*	0	0	$a_{24,27}$
0	0	0	$-a_{26,1}$	$-a_{26,2}$	$-a_{26,3}$	$-a_{27,1}$	$-a_{27,2}$	$-a_{27,3}$	$a_{25,10}$	$a_{25,11}$	0	$a_{25,13}$	$a_{25,14}$	0	$a_{25,16}$	$a_{25,17}$	0	$a_{27,21}$	0	0	$a_{27,24}$	0	0	*	$a_{25,26}$	$a_{25,27}$
$a_{26,1}$	$a_{26,2}$	$a_{26,3}$	0	0	0	$-a_{27,4}$	$-a_{27,5}$	$-a_{27,6}$	$a_{26,10}$	$a_{26,11}$	0	$a_{26,13}$	$a_{26,14}$	0	$a_{26,16}$	$a_{26,17}$	0	0	$a_{27,21}$	0	0	$a_{27,24}$	0	$a_{26,25}$	*	$a_{26,27}$
$a_{27,1}$	$a_{27,2}$	$a_{27,3}$	$a_{27,4}$	$a_{27,5}$	$a_{27,6}$	0	0	0	$a_{27,10}$	$a_{27,11}$	0	$a_{27,13}$	$a_{27,14}$	0	$a_{27,16}$	$a_{27,17}$	0	0	0	$a_{27,21}$	0	0	$a_{27,24}$	$a_{27,25}$	$a_{27,26}$	*

Out[55]= $\{a_{1,1}, a_{2,2}, a_{3,3}, a_{4,4}, -a_{1,1}+a_{2,2}+a_{4,4}, -a_{1,1}+a_{3,3}+a_{4,4}, a_{1,1}-a_{2,2}-a_{3,3}-a_{4,4}, -a_{3,3}-a_{4,4}, -a_{2,2}-a_{4,4}, a_{10,10}, a_{11,11}, -2\,a_{1,1}+a_{2,2}+a_{3,3}-a_{10,10}-a_{11,11}, a_{1,1}-a_{2,2}+a_{10,10}, a_{1,1}-a_{2,2}+a_{11,11}, -a_{1,1}+a_{3,3}-a_{10,10}-a_{11,11}, a_{1,1}-a_{3,3}+a_{10,10}, a_{1,1}-a_{3,3}+a_{11,11}, -a_{1,1}+a_{2,2}-a_{10,10}-a_{11,11}, -a_{1,1}-a_{10,10}, -a_{4,4}-a_{10,10}, -a_{1,1}+a_{2,2}+a_{3,3}+a_{4,4}-a_{10,10}, -a_{1,1}-a_{11,11}, -a_{4,4}-a_{11,11}, -a_{1,1}+a_{2,2}+a_{3,3}+a_{4,4}-a_{11,11}, a_{1,1}-a_{2,2}-a_{3,3}+a_{10,10}+a_{11,11}, 2\,a_{1,1}-a_{2,2}-a_{3,3}-a_{4,4}+a_{10,10}+a_{11,11}, a_{1,1}+a_{4,4}+a_{10,10}+a_{11,11}\}$

Standard maximal torus of E6

In[56]:=
```
diag = Table[E6LieAlgebra[[i, i]], {i, 27}]
(* a_ii, i=1,2,3,4,10,11; six variables *)
ss[1] = (diag /. a[1, 1] → 1) /. a[i_, j_] :→ 0
ss[2] = (diag /. a[2, 2] → 1) /. a[i_, j_] :→ 0
ss[3] = (diag /. a[3, 3] → 1) /. a[i_, j_] :→ 0
ss[4] = (diag /. a[4, 4] → 1) /. a[i_, j_] :→ 0
ss[5] = (diag /. a[10, 10] → 1) /. a[i_, j_] :→ 0
ss[6] = (diag /. a[11, 11] → 1) /. a[i_, j_] :→ 0
E6Torus = Table[Product[s[i] ^ (ss[i][[j]]), {i, 6}], {j, 27}]
(* Verifying the preservation of the cubic form *)
(deltaz /. {z[i_] :> E6Torus[[i]] z[i]}) - deltaz
```

Out[56]= $\{a_{1,1}, a_{2,2}, a_{3,3}, a_{4,4}, -a_{1,1}+a_{2,2}+a_{4,4}, -a_{1,1}+a_{3,3}+a_{4,4}, a_{1,1}-a_{2,2}-a_{3,3}-a_{4,4}, -a_{3,3}-a_{4,4}, -a_{2,2}-a_{4,4}, a_{10,10}, a_{11,11}, -2\,a_{1,1}+a_{2,2}+a_{3,3}-a_{10,10}-a_{11,11}, a_{1,1}-a_{2,2}+a_{10,10}, a_{1,1}-a_{2,2}+a_{11,11}, -a_{1,1}+a_{3,3}-a_{10,10}-a_{11,11}, a_{1,1}-a_{3,3}+a_{10,10}, a_{1,1}-a_{3,3}+a_{11,11}, -a_{1,1}+a_{2,2}-a_{10,10}-a_{11,11}, -a_{1,1}-a_{10,10}, -a_{4,4}-a_{10,10}, -a_{1,1}+a_{2,2}+a_{3,3}+a_{4,4}-a_{10,10}, -a_{1,1}-a_{11,11}, -a_{4,4}-a_{11,11}, -a_{1,1}+a_{2,2}+a_{3,3}+a_{4,4}-a_{11,11}, a_{1,1}-a_{2,2}-a_{3,3}+a_{10,10}+a_{11,11}, 2\,a_{1,1}-a_{2,2}-a_{3,3}-a_{4,4}+a_{10,10}+a_{11,11}, a_{1,1}+a_{4,4}+a_{10,10}+a_{11,11}\}$

Out[57]= {1, 0, 0, 0, -1, -1, 1, 0, 0, 0, 0, -2, 1, 1, -1, 1, 1, -1, -1, 0, -1, -1, 0, -1, 1, 2, 1}

Out[58]= {0, 1, 0, 0, 1, 0, -1, 0, -1, 0, 0, 1, -1, -1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, -1, -1, 0}

Out[59]= {0, 0, 1, 0, 0, 1, -1, -1, 0, 0, 0, 1, 0, 0, 1, -1, -1, 0, 0, 0, 1, 0, 0, 1, -1, -1, 0}

Out[60]= {0, 0, 0, 1, 1, 1, -1, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 0, -1, 1, 0, -1, 1}

Out[61]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -1, 1, 0, -1, 1, 0, -1, -1, -1, -1, 0, 0, 0, 1, 1, 1}

Out[62]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 1, -1, 0, 1, -1, 0, 0, 0, -1, -1, -1, 1, 1, 1}

Out[63]= $\left\{s_1, s_2, s_3, s_4, \frac{s_2 s_4}{s_1}, \frac{s_3 s_4}{s_1}, \frac{s_1}{s_2 s_3 s_4}, \frac{1}{s_3 s_4}, \frac{1}{s_2 s_4}, s_5, s_6, \frac{s_2 s_3}{s_1^2 s_5 s_6}, \frac{s_1 s_5}{s_2}, \frac{s_1 s_6}{s_2}, \frac{s_3}{s_1 s_5 s_6}, \frac{s_1 s_5}{s_3}, \frac{s_1 s_6}{s_3}, \frac{s_2}{s_1 s_5 s_6}, \frac{1}{s_1 s_5}, \frac{1}{s_4 s_5}, \frac{s_2 s_3 s_4}{s_1 s_5}, \frac{1}{s_1 s_6}, \frac{1}{s_4 s_6}, \frac{s_2 s_3 s_4}{s_1 s_6}, \frac{s_1 s_5 s_6}{s_2 s_3}, \frac{s_1^2 s_5 s_6}{s_2 s_3 s_4}, s_1 s_4 s_5 s_6\right\}$

Out[64]= 0

▪ Roots of E6

▪ Roots

In[65]:=
```
rootspaces = E6LieAlgebra /. {a[i_, i_] :> 0};
rootvars = Complement[E6vars, Table[a[i, i], {i, 27}]];
Length@rootvars

(* The eigenvalues/eigenvectors are found by the following shortcut.
  I observed that the eigenvectors seem all obtained by specializing the Lie algebra matrix
 by setting one variable to 1, the rest to 0.
 The computation below confirms that indeed each such specialization gives an eienvector.
  By counting we know that we have all the eigenvectors/eigenvalues *)
rootvecs = Table[0, {Length@rootvars}];
roots = Table[rootv = (rootspaces /. rootvars[[k]] → 1) /. a[i_, j_] :> 0;
   rootvecs[[k]] = rootv;
   rootv1 = rootv;
   Do[rootv1[[i, j]] = rootv[[i, j]] E6Torus[[i]] / E6Torus[[j]], {i, 27}, {j, 27}];
   i = 1; j = 1; While[rootv[[i, j]] === 0, j++; If[j > 27, i++; j = 1]];
   eigenvalue = rootv1[[i, j]] / rootv[[i, j]];
   If[Union @@ (rootv1 - eigenvalue rootv) =!= {0}, Print["When k=", k, ", the vector is not an eigenvector!"]];
   eigenvalue, {k, 1, Length@rootvars}];
roots = roots /. {Times → Plus, s[i_]^j_ :> j s[i]};
roots = roots /. {s[i_] :> IdentityMatrix[6][[i]]}
```

Out[67]= 72

Out[71]=
```
{{-1, 1, 0, 0, 0, 0}, {-1, 0, 1, 0, 0, 0}, {1, -1, 0, 0, 0, 0}, {0, -1, 1, 0, 0, 0}, {1, 0, -1, 0, 0, 0}, {0, 1, -1, 0, 0, 0}, {-1, 0, 0, -1, -1, 0}, {0, -1, 0, -1, -1, 0}, {0, 0, -1, -1, -1, 0},
 {-2, 1, 1, 1, -1, 0}, {-1, 0, 1, 1, -1, 0}, {-1, 1, 0, 1, -1, 0}, {-1, 1, 1, 0, -1, 0}, {0, 0, 1, 0, -1, 0}, {0, 1, 0, 0, -1, 0}, {0, 0, 0, 0, -1, 1}, {-2, 1, 1, 0, -2, -1}, {-1, 0, 0, 0, -1, -1},
 {-2, 1, 0, 0, -1, -1}, {-2, 0, 1, 0, -1, -1}, {-1, 0, 0, -1, 0, -1}, {0, -1, 0, -1, 0, -1}, {0, 0, -1, -1, 0, -1}, {0, 0, 0, -1, -1, -1}, {-1, 1, 0, -1, -1, -1}, {-1, 0, 1, -1, -1, -1},
 {-2, 1, 1, 1, 0, -1}, {-1, 0, 1, 1, 0, -1}, {-1, 1, 0, 1, 0, -1}, {-1, 1, 1, 0, 0, -1}, {0, 0, 1, 0, 0, -1}, {0, 1, 0, 0, 0, -1}, {-1, 1, 1, 1, -1, -1}, {-2, 2, 1, 1, -1, -1}, {-2, 1, 2, 1, -1, -1},
 {0, 0, 0, 0, 1, -1}, {-2, 1, 1, 0, -1, -2}, {1, -1, -1, 0, 0, 1}, {1, -1, -1, 0, 1, 0}, {0, 0, -1, 0, 0, 1}, {0, 0, -1, 0, 1, 0}, {0, -1, 0, 0, 0, 1}, {0, -1, 0, 0, 1, 0}, {-1, 0, 0, 1, 0, 0},
 {0, -1, -1, -1, 0, 0}, {1, -1, -1, -1, 1, 1}, {2, -2, -1, -1, 1, 1}, {2, -1, -2, -1, 1, 1}, {2, -1, -1, -1, 0, 1}, {2, -1, -1, -1, 1, 0}, {1, 0, -1, -1, 0, 1}, {1, 0, -1, -1, 1, 0}, {1, -1, 0, -1, 0, 1},
 {1, -1, 0, -1, 1, 0}, {1, 0, 0, -1, 0, 0}, {1, -1, -1, -2, 0, 0}, {0, 0, 0, 1, 1, 1}, {1, -1, 0, 1, 1, 1}, {1, 0, -1, 1, 1, 1}, {1, 0, 0, 0, 1, 1}, {2, -1, 0, 0, 1, 1}, {2, 0, -1, 0, 1, 1},
 {1, 0, 0, 1, 0, 1}, {1, 0, 0, 1, 1, 0}, {0, 1, 0, 1, 0, 1}, {0, 1, 0, 1, 1, 0}, {0, 0, 1, 1, 0, 1}, {0, 0, 1, 1, 1, 0}, {2, -1, -1, 0, 2, 1}, {2, -1, -1, 0, 1, 2}, {0, 1, 1, 1, 0, 0}, {-1, 1, 1, 2, 0, 0}}
```

▪ Coroots

In[72]:=
```
(* t1,t2,t3,t4,t5,t6 form a basis of the Lie (standard maximal torus of E6), as a submodule of Lie (standard maximal torus of GL(27)) *)
t1 = Map[(D[#, t] /. t → 1) &, (E6Torus /. s[1] → t) /. s[i_] :> 1];
t2 = Map[(D[#, t] /. t → 1) &, (E6Torus /. s[2] → t) /. s[i_] :> 1];
t3 = Map[(D[#, t] /. t → 1) &, (E6Torus /. s[3] → t) /. s[i_] :> 1];
t4 = Map[(D[#, t] /. t → 1) &, (E6Torus /. s[4] → t) /. s[i_] :> 1];
t5 = Map[(D[#, t] /. t → 1) &, (E6Torus /. s[5] → t) /. s[i_] :> 1];
t6 = Map[(D[#, t] /. t → 1) &, (E6Torus /. s[6] → t) /. s[i_] :> 1];
coroots = Table[0, {Length@roots}];
negroots = Table[Position[roots, -roots[[i]]][[1, 1]], {i, Length@roots}];
Table[
 X1 = rootvecs[[i]];
 X2 = rootvecs[[negroots[[i]]]];
 br = X1.X2 - X2.X1;
 tmp = br;
 Do[tmp[[i, i]] = 0, {i, 27}];
 If[Union @@ tmp =!= {0}, Print["the Lie bracket is not a a diagonal matrix!"]];
 sol = Solve[Table[br[[j, j]], {j, 27}] == c1 t1 + c2 t2 + c3 t3 + c4 t4 + c5 t5 + c6 t6, {c1, c2, c3, c4, c5, c6}][[1]];
 pairing = ({c1, c2, c3, c4, c5, c6} /. sol).(roots[[i]]);
 coroots[[i]] = (2 / pairing) {c1, c2, c3, c4, c5, c6} /. sol;
 pairing, {i, Length@roots}]
(* That the output from the above are all 2 or -2, confirms [X_a,X_{-a}]=±(coroot of a)_*(generator of Lie G_m/Z) for every a. *)
Union[Abs /@ %]
```

Out[80]=
```
{2, 2, 2, 2, 2, 2, -2, -2, -2, 2, 2, 2, -2, -2, -2, 2, 2, -2, -2, -2, 2, 2, 2, 2, 2, 2, -2, -2, -2, 2, 2, 2,
 -2, -2, -2, 2, 2, 2, -2, 2, -2, 2, -2, 2, 2, -2, -2, -2, -2, 2, -2, 2, -2, 2, 2, 2, 2, 2, 2, -2, -2, -2, 2, -2, 2, -2, 2, -2, 2, 2, 2, 2}
```

Out[81]= {2}

Simple roots and Cartan matrix

In[82]:=
```
(* Make sure that (1,3,11,13,19,41) lies in no root hyperplane *)
And @@ Map[#.{1, 3, 11, 13, 19, 41} =!= 0 &, roots]
posroots = Select[roots, #.{1, 3, 11, 13, 19, 41} > 0 &];
Length@%
simpleroots = Complement[posroots, Union @@ Table[posroots[[i]] + posroots[[j]], {i, 36}, {j, i + 1, 36}]]
(* Positive roots as linear combinations of the simple roots *)
Sort[posroots.Inverse[simpleroots], Plus @@ #1 > Plus @@ #2 &]
Position[roots, #][[1, 1]] & /@ simpleroots
simplepos = {71, 49, 39, 1, 11, 54};
(* Reorder the simple roots so that the resulting Cartan matrix is identical to the one in Bourbaki *)
Table[roots[[simplepos[[i]]]].coroots[[simplepos[[j]]]], {i, 6}, {j, 6}] // MatrixForm
Det@%
```

Out[82]= True

Out[84]= 36

Out[85]= {{-1, 0, 1, 1, -1, 0}, {-1, 1, 0, 0, 0, 0}, {0, 1, 1, 1, 0, 0}, {1, -1, -1, 0, 1, 0}, {1, -1, 0, -1, 1, 0}, {2, -1, -1, -1, 0, 1}}

Out[86]= {{2, 3, 1, 2, 1, 2}, {2, 3, 1, 2, 1, 1}, {2, 2, 1, 2, 1, 1}, {1, 2, 1, 2, 1, 1}, {2, 2, 1, 1, 1, 1}, {1, 2, 1, 1, 1, 1}, {1, 2, 1, 2, 0, 1}, {2, 2, 0, 1, 1, 1}, {1, 2, 1, 1, 0, 1}, {1, 1, 1, 1, 1, 1}, {1, 2, 0, 1, 1, 1}, {1, 1, 1, 1, 1, 0}, {1, 1, 1, 1, 0, 1}, {1, 1, 0, 1, 1, 1}, {1, 2, 0, 1, 0, 1}, {1, 1, 1, 1, 0, 0}, {0, 1, 1, 1, 0, 1}, {1, 1, 0, 0, 1, 1}, {1, 1, 0, 1, 1, 0}, {1, 1, 0, 1, 0, 1}, {0, 1, 1, 1, 0, 0}, {0, 1, 0, 1, 0, 1}, {1, 1, 0, 1, 0, 0}, {1, 1, 0, 0, 0, 1}, {1, 1, 0, 0, 1, 0}, {0, 0, 1, 1, 0, 0}, {0, 1, 0, 0, 0, 1}, {0, 1, 0, 1, 0, 0}, {1, 1, 0, 0, 0, 0}, {1, 0, 0, 0, 1, 0}, {0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 1}, {0, 0, 0, 1, 0, 0}, {1, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0}}

Out[87]= {11, 1, 71, 39, 54, 49}

Out[89]//MatrixForm=

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Out[90]= 3

The Lie algebra of F4

Computing and simplifying the defining equations

In[91]:=
```
E0 = Table[0, {27}];
E0[[1]] = E0[[5]] = E0[[9]] = 1;
F4lieeqn = Transpose[E6LieAlgebra.Transpose[{E0}]][[1]]
```

Out[93]= $\{a_{1,1}, -a_{12,15}-a_{26,25}, -a_{12,18}-a_{27,25}, -a_{15,12}-a_{25,26}, -a_{1,1}+a_{2,2}+a_{4,4}, -a_{15,18}-a_{27,26}, -a_{18,12}-a_{25,27}, -a_{18,15}-a_{26,27}, -a_{2,2}-a_{4,4}, -a_{26,17}+a_{27,14}, a_{26,16}-a_{27,13}, -a_{23,16}+a_{24,13}, a_{25,17}-a_{27,11}, -a_{25,16}+a_{27,10}, a_{22,16}-a_{24,10}, -a_{25,14}+a_{26,11}, a_{25,13}-a_{26,10}, -a_{22,13}+a_{23,10}, -a_{20,2}-a_{21,3}, a_{20,1}-a_{21,6}, a_{21,1}+a_{21,5}, -a_{23,2}-a_{24,3}, a_{23,1}-a_{24,6}, a_{24,1}+a_{24,5}, -a_{26,2}-a_{27,3}, a_{26,1}-a_{27,6}, a_{27,1}+a_{27,5}\}$

In[94]:=
```
F4zerovars = Union@Select[F4lieeqn, Head[#] === a &] ~Join~ (-Select[F4lieeqn, Head[#] === Times &])
F4lieeqn2 = Union[F4lieeqn /. (F4sol1 = Map[# → 0 &, F4zerovars])];
F4lieeqn3 = Union[F4lieeqn2 /. (F4sol2 = Solve[Select[F4lieeqn2, Length@# === 2 &] == 0][[1]])]
```

Out[94]= $\{a_{1,1}\}$

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[96]= {0}

- **The final result**

In[97]:=
```
F4LieAlgebra = E6LieAlgebra /. Join[F4sol1, F4sol2];
(* list of free variables *)
F4vars = Map[a[#[[1]], #[[2]]] &, Select[Flatten[Table[{i, j}, {i, 27}, {j, 27}], 1], F4LieAlgebra[[#[[1]], #[[2]]]] === a[#[[1]], #[[2]]] &]]
(* number of free variables *)
Length@%
tmp = F4LieAlgebra;
Do[tmp[[i, i]] = "*", {i, 27}];
(* the full Lie algebra as a matrix, without the diagonal part *)
tmp // MatrixForm
(* the diagonal part *)
Table[F4LieAlgebra[[i, i]], {i, 27}]
```

Out[98]= $\{a_{3,3}, a_{4,4}, a_{10,10}, a_{11,11}, a_{20,3}, a_{21,2}, a_{21,3}, a_{21,4}, a_{21,5}, a_{21,6}, a_{21,24}, a_{21,27}, a_{22,10}, a_{23,3}, a_{23,10}, a_{23,13}, a_{24,2}, a_{24,3}, a_{24,4}, a_{24,5}, a_{24,6}, a_{24,10}, a_{24,13}, a_{24,16}, a_{24,21}, a_{24,27}, a_{25,10}, a_{25,11}, a_{25,26}, a_{25,27}, a_{26,3}, a_{26,10}, a_{26,11}, a_{26,13}, a_{26,14}, a_{26,25}, a_{26,27}, a_{27,2}, a_{27,3}, a_{27,4}, a_{27,5}, a_{27,6}, a_{27,10}, a_{27,11}, a_{27,13}, a_{27,14}, a_{27,16}, a_{27,17}, a_{27,21}, a_{27,24}, a_{27,25}, a_{27,26}\}$

Out[99]= 52

Out[102]//MatrixForm=

*	$a_{25,26}$	$a_{25,27}$	$-a_{26,25}$	0	0	$-a_{27,25}$	0	0	0	0	0	$-a_{21,6}$	$-a_{24,6}$	$-a_{27,6}$	$a_{21,5}$	$a_{24,5}$	$a_{27,5}$	0	$-a_{27,11}$	$a_{26,11}$	0	$a_{27,10}$	$-a_{26,10}$	0	$-a_{24,10}$	$a_{23,10}$
$a_{26,25}$	*	$a_{26,27}$	0	$-a_{26,25}$	0	0	$-a_{27,25}$	0	$a_{21,6}$	$a_{24,6}$	$a_{27,6}$	0	0	0	$-a_{21,4}$	$-a_{24,4}$	$-a_{27,4}$	0	$-a_{27,14}$	$a_{26,14}$	0	$a_{27,13}$	$-a_{26,13}$	0	$-a_{24,13}$	$a_{23,13}$
$a_{27,25}$	$a_{27,26}$	*	0	0	$-a_{26,25}$	0	0	$-a_{27,25}$	$-a_{21,5}$	$-a_{24,5}$	$-a_{27,5}$	$a_{21,4}$	$a_{24,4}$	$a_{27,4}$	0	0	0	0	$-a_{27,17}$	$a_{27,14}$	0	$a_{27,16}$	$-a_{27,13}$	0	$-a_{24,16}$	$a_{24,13}$
$-a_{25,26}$	0	0	*	$a_{25,26}$	$a_{25,27}$	$-a_{27,26}$	0	0	0	0	0	$a_{21,3}$	$a_{24,3}$	$a_{27,3}$	$-a_{21,2}$	$-a_{24,2}$	$-a_{27,2}$	$a_{27,11}$	0	$-a_{25,11}$	$-a_{27,10}$	0	$a_{25,10}$	$a_{24,10}$	0	$-a_{22,10}$
0	$-a_{25,26}$	0	$a_{26,25}$	*	$a_{26,27}$	0	$-a_{27,26}$	0	$-a_{21,3}$	$-a_{24,3}$	$-a_{27,3}$	0	0	0	$-a_{21,5}$	$-a_{24,5}$	$-a_{27,5}$	$a_{27,14}$	0	$-a_{26,11}$	$-a_{27,13}$	0	$a_{26,10}$	$a_{24,13}$	0	$-a_{23,10}$
0	0	$-a_{25,26}$	$a_{27,25}$	$a_{27,26}$	*	0	0	$-a_{27,26}$	$a_{21,2}$	$a_{24,2}$	$a_{27,2}$	$a_{21,5}$	$a_{24,5}$	$a_{27,5}$	0	0	0	$a_{27,17}$	0	$-a_{27,11}$	$-a_{27,16}$	0	$a_{27,10}$	$a_{24,16}$	0	$-a_{24,10}$
$-a_{25,27}$	0	0	$-a_{26,27}$	0	0	*	$a_{25,26}$	$a_{25,27}$	0	0	0	$-a_{20,3}$	$-a_{23,3}$	$-a_{26,3}$	$-a_{21,3}$	$-a_{24,3}$	$-a_{27,3}$	$-a_{26,11}$	$a_{25,11}$	0	$a_{26,10}$	$-a_{25,10}$	0	$-a_{23,10}$	$a_{22,10}$	0
0	$-a_{25,27}$	0	0	$-a_{26,27}$	0	$a_{26,25}$	*	$a_{26,27}$	$a_{20,3}$	$a_{23,3}$	$a_{26,3}$	0	0	0	$-a_{21,6}$	$-a_{24,6}$	$-a_{27,6}$	$-a_{26,14}$	$a_{26,11}$	0	$a_{26,13}$	$-a_{26,10}$	0	$-a_{23,13}$	$a_{23,10}$	0
0	0	$-a_{25,27}$	0	0	$-a_{26,27}$	$a_{27,25}$	$a_{27,26}$	*	$a_{21,3}$	$a_{24,3}$	$a_{27,3}$	$a_{21,6}$	$a_{24,6}$	$a_{27,6}$	0	0	0	$-a_{27,14}$	$a_{27,11}$	0	$a_{27,13}$	$-a_{27,10}$	0	$-a_{24,13}$	$a_{24,10}$	0
0	$-a_{27,11}$	$a_{26,11}$	0	$-a_{27,14}$	$a_{26,14}$	0	$-a_{27,17}$	$a_{27,14}$	*	$-a_{24,21}$	$-a_{27,21}$	$-a_{26,25}$	0	0	$-a_{27,25}$	0	0	0	0	0	$-a_{27,4}$	$-a_{27,5}$	$-a_{27,6}$	$a_{24,4}$	$a_{24,5}$	$a_{24,6}$
0	$a_{27,10}$	$-a_{26,10}$	0	$a_{27,13}$	$-a_{26,13}$	0	$a_{27,16}$	$-a_{27,13}$	$-a_{21,24}$	*	$-a_{27,24}$	0	$-a_{26,25}$	0	0	$-a_{27,25}$	0	$a_{27,4}$	$a_{27,5}$	$a_{27,6}$	0	0	0	$-a_{21,4}$	$-a_{21,5}$	$-a_{21,6}$
0	$-a_{24,10}$	$a_{23,10}$	0	$-a_{24,13}$	$a_{23,13}$	0	$-a_{24,16}$	$a_{24,13}$	$-a_{21,27}$	$-a_{24,27}$	*	0	0	$-a_{26,25}$	0	0	$-a_{27,25}$	$-a_{24,4}$	$-a_{24,5}$	$-a_{24,6}$	$a_{21,4}$	$a_{21,5}$	$a_{21,6}$	0	0	0
$a_{27,11}$	0	$-a_{25,11}$	$a_{27,14}$	0	$-a_{26,11}$	$a_{27,17}$	0	$-a_{27,11}$	$-a_{25,26}$	0	0	*	$-a_{24,21}$	$-a_{27,21}$	$-a_{27,26}$	0	0	0	0	0	$-a_{27,5}$	$a_{27,2}$	$a_{27,3}$	$a_{24,5}$	$-a_{24,2}$	$-a_{24,3}$
$-a_{27,10}$	0	$a_{25,10}$	$-a_{27,13}$	0	$a_{26,10}$	$-a_{27,16}$	0	$a_{27,10}$	0	$-a_{25,26}$	0	$-a_{21,24}$	*	$-a_{27,24}$	0	$-a_{27,26}$	0	$a_{27,5}$	$-a_{27,2}$	$-a_{27,3}$	0	0	0	$-a_{21,5}$	$a_{21,2}$	$a_{21,3}$
$a_{24,10}$	0	$-a_{22,10}$	$a_{24,13}$	0	$-a_{23,10}$	$a_{24,16}$	0	$-a_{24,10}$	0	0	$-a_{25,26}$	$-a_{21,27}$	$-a_{24,27}$	*	0	0	$-a_{27,26}$	$-a_{24,5}$	$a_{24,2}$	$a_{24,3}$	$a_{21,5}$	$-a_{21,2}$	$-a_{21,3}$	0	0	0
$-a_{26,11}$	$a_{25,11}$	0	$-a_{26,14}$	$a_{26,11}$	0	$-a_{27,14}$	$a_{27,11}$	0	$-a_{25,27}$	0	0	$-a_{26,27}$	0	0	*	$-a_{24,21}$	$-a_{27,21}$	0	0	0	$-a_{27,6}$	$a_{27,3}$	$-a_{26,3}$	$a_{24,6}$	$-a_{24,3}$	$a_{23,3}$
$a_{26,10}$	$-a_{25,10}$	0	$a_{26,13}$	$-a_{26,10}$	0	$a_{27,13}$	$-a_{27,10}$	0	0	$-a_{25,27}$	0	0	$-a_{26,27}$	0	$-a_{21,24}$	*	$-a_{27,24}$	$a_{27,6}$	$-a_{27,3}$	$a_{26,3}$	0	0	0	$-a_{21,6}$	$a_{21,3}$	$-a_{20,3}$
$-a_{23,10}$	$a_{22,10}$	0	$-a_{23,13}$	$a_{23,10}$	0	$-a_{24,13}$	$a_{24,10}$	0	0	0	$-a_{25,27}$	0	0	$-a_{26,27}$	$-a_{21,27}$	$-a_{24,27}$	*	$-a_{24,6}$	$a_{24,3}$	$-a_{23,3}$	$a_{21,6}$	$-a_{21,3}$	$a_{20,3}$	0	0	0
0	0	0	$-a_{21,6}$	$a_{21,3}$	$-a_{20,3}$	$a_{21,5}$	$-a_{21,2}$	$-a_{21,3}$	0	$-a_{22,10}$	$-a_{25,10}$	0	$-a_{23,10}$	$-a_{26,10}$	0	$-a_{24,10}$	$-a_{27,10}$	*	$a_{25,26}$	$a_{25,27}$	$a_{21,24}$	0	0	$a_{21,27}$	0	0
$a_{21,6}$	$-a_{21,3}$	$a_{20,3}$	0	0	0	$-a_{21,4}$	$-a_{21,5}$	$-a_{21,6}$	0	$-a_{23,10}$	$-a_{26,10}$	0	$-a_{23,13}$	$-a_{26,13}$	0	$-a_{24,13}$	$-a_{27,13}$	$a_{26,25}$	*	$a_{26,27}$	0	$a_{21,24}$	0	0	$a_{21,27}$	0
$-a_{21,5}$	$a_{21,2}$	$a_{21,3}$	$a_{21,4}$	$a_{21,5}$	$a_{21,6}$	0	0	0	0	$-a_{24,10}$	$-a_{27,10}$	0	$-a_{24,13}$	$-a_{27,13}$	0	$-a_{24,16}$	$-a_{27,16}$	$a_{27,25}$	$a_{27,26}$	*	0	0	$a_{21,24}$	0	0	$a_{21,27}$
0	0	0	$-a_{24,6}$	$a_{24,3}$	$-a_{23,3}$	$a_{24,5}$	$-a_{24,2}$	$-a_{24,3}$	$a_{22,10}$	0	$-a_{25,11}$	$a_{23,10}$	0	$-a_{26,11}$	$a_{24,10}$	0	$-a_{27,11}$	$a_{24,21}$	0	0	*	$a_{25,26}$	$a_{25,27}$	$a_{24,27}$	0	0
$a_{24,6}$	$-a_{24,3}$	$a_{23,3}$	0	0	0	$-a_{24,4}$	$-a_{24,5}$	$-a_{24,6}$	$a_{23,10}$	0	$-a_{26,11}$	$a_{23,13}$	0	$-a_{26,14}$	$a_{24,13}$	0	$-a_{27,14}$	0	$a_{24,21}$	0	$a_{26,25}$	*	$a_{26,27}$	0	$a_{24,27}$	0
$-a_{24,5}$	$a_{24,2}$	$a_{24,3}$	$a_{24,4}$	$a_{24,5}$	$a_{24,6}$	0	0	0	$a_{24,10}$	0	$-a_{27,11}$	$a_{24,13}$	0	$-a_{27,14}$	$a_{24,16}$	0	$-a_{27,17}$	0	0	$a_{24,21}$	$a_{27,25}$	$a_{27,26}$	*	0	0	$a_{24,27}$
0	0	0	$-a_{27,6}$	$a_{27,3}$	$-a_{26,3}$	$a_{27,5}$	$-a_{27,2}$	$-a_{27,3}$	$a_{25,10}$	$a_{25,11}$	0	$a_{26,10}$	$a_{26,11}$	0	$a_{27,10}$	$a_{27,11}$	0	$a_{27,21}$	0	0	$a_{27,24}$	0	0	*	$a_{25,26}$	$a_{25,27}$
$a_{27,6}$	$-a_{27,3}$	$a_{26,3}$	0	0	0	$-a_{27,4}$	$-a_{27,5}$	$-a_{27,6}$	$a_{26,10}$	$a_{26,11}$	0	$a_{26,13}$	$a_{26,14}$	0	$a_{27,13}$	$a_{27,14}$	0	0	$a_{27,21}$	0	0	$a_{27,24}$	0	$a_{26,25}$	*	$a_{26,27}$
$-a_{27,5}$	$a_{27,2}$	$a_{27,3}$	$a_{27,4}$	$a_{27,5}$	$a_{27,6}$	0	0	0	$a_{27,10}$	$a_{27,11}$	0	$a_{27,13}$	$a_{27,14}$	0	$a_{27,16}$	$a_{27,17}$	0	0	0	$a_{27,21}$	0	0	$a_{27,24}$	$a_{27,25}$	$a_{27,26}$	*

Out[103]= $\{0, -a_{4,4}, a_{3,3}, a_{4,4}, 0, a_{3,3}+a_{4,4}, -a_{3,3}, -a_{3,3}-a_{4,4}, 0, a_{10,10}, a_{11,11}, a_{3,3}-a_{4,4}-a_{10,10}-a_{11,11}, a_{4,4}+a_{10,10}, a_{4,4}+a_{11,11}, a_{3,3}-a_{10,10}-a_{11,11}, -a_{3,3}+a_{10,10}, -a_{3,3}+a_{11,11}, -a_{4,4}-a_{10,10}-a_{11,11}, -a_{10,10}, -a_{4,4}-a_{10,10}, a_{3,3}-a_{10,10}, -a_{11,11}, -a_{4,4}-a_{11,11}, a_{3,3}-a_{11,11}, -a_{3,3}+a_{4,4}+a_{10,10}+a_{11,11}, -a_{3,3}+a_{10,10}+a_{11,11}, a_{4,4}+a_{10,10}+a_{11,11}\}$

▪ Standard maximal torus of F4

In[104]:=
```
diag = Table[F4LieAlgebra[[i, i]], {i, 27}]
(* a_ii, i=3,4,10,11; six variables *)
ss[1] = (diag /. a[3, 3] → 1) /. a[i_, j_] ⧴ 0
ss[2] = (diag /. a[4, 4] → 1) /. a[i_, j_] ⧴ 0
ss[3] = (diag /. a[10, 10] → 1) /. a[i_, j_] ⧴ 0
ss[4] = (diag /. a[11, 11] → 1) /. a[i_, j_] ⧴ 0
F4Torus = Table[Product[s[i]^(ss[i][[j]]), {i, 4}], {j, 27}]
(* Verifying the preservation of the cubic form *)
(deltaz /. {z[i_] ⧴ F4Torus[[i]] z[i]}) - deltaz
(* Verifying the preservation of E0 *)
Table[F4Torus[[i]] E0[[i]], {i, 27}] - E0
```

Out[104]= $\{0, -a_{4,4}, a_{3,3}, a_{4,4}, 0, a_{3,3}+a_{4,4}, -a_{3,3}, -a_{3,3}-a_{4,4}, 0, a_{10,10}, a_{11,11}, a_{3,3}-a_{4,4}-a_{10,10}-a_{11,11}, a_{4,4}+a_{10,10}, a_{4,4}+a_{11,11}, a_{3,3}-a_{10,10}-a_{11,11}, -a_{3,3}+a_{10,10}, -a_{3,3}+a_{11,11}, -a_{4,4}-a_{10,10}-a_{11,11}, -a_{10,10}, -a_{4,4}-a_{10,10}, a_{3,3}-a_{10,10}, -a_{11,11}, -a_{4,4}-a_{11,11}, a_{3,3}-a_{11,11}, -a_{3,3}+a_{4,4}+a_{10,10}+a_{11,11}, -a_{3,3}+a_{10,10}+a_{11,11}, a_{4,4}+a_{10,10}+a_{11,11}\}$

Out[105]= {0, 0, 1, 0, 0, 1, -1, -1, 0, 0, 0, 1, 0, 0, 1, -1, -1, 0, 0, 0, 1, 0, 0, 1, -1, -1, 0}

Out[106]= {0, -1, 0, 1, 0, 1, 0, -1, 0, 0, 0, -1, 1, 1, 0, 0, 0, -1, 0, -1, 0, 0, -1, 0, 1, 0, 1}

Out[107]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -1, 1, 0, -1, 1, 0, -1, -1, -1, -1, 0, 0, 0, 1, 1, 1}

Out[108]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 1, -1, 0, 1, -1, 0, 0, 0, -1, -1, -1, 1, 1, 1}

Out[109]= $\left\{1, \frac{1}{s_2}, s_1, s_2, 1, s_1 s_2, \frac{1}{s_1}, \frac{1}{s_1 s_2}, 1, s_3, s_4, \frac{s_1}{s_2 s_3 s_4}, s_2 s_3, s_2 s_4, \frac{s_1}{s_3 s_4}, \frac{s_3}{s_1}, \frac{s_4}{s_1}, \frac{1}{s_2 s_3 s_4}, \frac{1}{s_3}, \frac{1}{s_2 s_3}, \frac{s_1}{s_3}, \frac{1}{s_4}, \frac{1}{s_2 s_4}, \frac{s_1}{s_4}, \frac{s_2 s_3 s_4}{s_1}, \frac{s_3 s_4}{s_1}, s_2 s_3 s_4\right\}$

Out[110]= 0

Out[111]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

▪ Roots of F4

▪ Roots

In[112]:=
```
rootspaces = F4LieAlgebra /. {a[i_, i_] ⧴ 0};
rootvars = Complement[F4vars, Table[a[i, i], {i, 27}]];
Length@rootvars

(* The eigenvalues/eigenvectors are found by the following shortcut.
  I observed that the eigenvectors seem all obtained by specializing the Lie algebra matrix
 by setting one variable to 1, the rest to 0.
 The computation below confirms that indeed each such specialization gives an eienvector.
  By counting we know that we have all the eigenvectors/eigenvalues *)
rootvecs = Table[0, {Length@rootvars}];
roots = Table[rootv = (rootspaces /. rootvars[[k]] → 1) /. a[i_, j_] ⧴ 0;
   rootvecs[[k]] = rootv;
   rootv1 = rootv;
   Do[rootv1[[i, j]] = rootv[[i, j]] F4Torus[[i]] / F4Torus[[j]], {i, 27}, {j, 27}];
   i = 1; j = 1; While[rootv[[i, j]] === 0, j++; If[j > 27, i++; j = 1]];
   eigenvalue = rootv1[[i, j]] / rootv[[i, j]];
   If[Union @@ (rootv1 - eigenvalue rootv) =!= {0}, Print["When k=", k, ", the vector is not an eigenvector!"]];
   eigenvalue, {k, 1, Length@rootvars}];
roots = roots /. {Times → Plus, s[i_]^j_ ⧴ j s[i]};
roots = roots /. {s[i_] ⧴ IdentityMatrix[4][[i]]}
```

Out[114]= 48

Out[118]= {{-1, -1, -1, 0}, {1, 1, -1, 0}, {0, 0, -1, 0}, {1, -1, -1, 0}, {1, 0, -1, 0}, {0, -1, -1, 0}, {0, 0, -1, 1}, {1, -1, -2, -1}, {0, 0, -1, -1}, {-1, -1, 0, -1}, {0, -1, -1, -1}, {0, -2, -1, -1}, {1, 1, 0, -1}, {0, 0, 0, -1}, {1, -1, 0, -1}, {1, 0, 0, -1}, {0, -1, 0, -1}, {1, 0, -1, -1}, {1, -1, -1, -1}, {2, 0, -1, -1}, {0, 0, 1, -1}, {1, -1, -1, -2}, {-1, 1, 0, 1}, {-1, 1, 1, 0}, {0, 1, 0, 0}, {-1, 0, 0, 0}, {-2, 0, 1, 1}, {-1, 0, 0, 1}, {-1, 0, 1, 0}, {-1, -1, 0, 1}, {-1, -1, 1, 0}, {0, -1, 0, 0}, {-1, -1, 0, 0}, {0, 2, 1, 1}, {-1, 1, 1, 1}, {0, 0, 1, 1}, {0, 1, 1, 1}, {-1, 0, 1, 1}, {0, 1, 0, 1}, {0, 1, 1, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}, {1, 1, 0, 1}, {1, 1, 1, 0}, {-1, 1, 2, 1}, {-1, 1, 1, 2}, {1, 0, 0, 0}, {1, 1, 0, 0}}

■ Coroots

```
In[119]:= (* t1,t2,t3,t4 form a basis of the Lie (standard maximal torus of E6), as a submodule of Lie (standard maximal torus of GL(27)) *)
t1 = Map[(D[#, t] /. t → 1) &, (F4Torus /. s[1] → t) /. s[i_] :→ 1];
t2 = Map[(D[#, t] /. t → 1) &, (F4Torus /. s[2] → t) /. s[i_] :→ 1];
t3 = Map[(D[#, t] /. t → 1) &, (F4Torus /. s[3] → t) /. s[i_] :→ 1];
t4 = Map[(D[#, t] /. t → 1) &, (F4Torus /. s[4] → t) /. s[i_] :→ 1];
coroots = Table[0, {Length@roots}];
negroots = Table[Position[roots, -roots[[i]]][[1, 1]], {i, Length@roots}];
Table[
 X1 = rootvecs[[i]];
 X2 = rootvecs[[negroots[[i]]]];
 br = X1.X2 - X2.X1;
 tmp = br;
 Do[tmp[[i, i]] = 0, {i, 27}];
 If[Union @@ tmp =!= {0}, Print["the Lie bracket is not a a diagonal matrix!"]];
 sol = Solve[Table[br[[j, j]], {j, 27}] == c1 t1 + c2 t2 + c3 t3 + c4 t4, {c1, c2, c3, c4}][[1]];
 pairing = ({c1, c2, c3, c4} /. sol).(roots[[i]]);
 coroots[[i]] = (2 / pairing) {c1, c2, c3, c4} /. sol;
 pairing, {i, Length@roots}]
(* That the output from the above are all 2 or -2, confirms [X_a,X_{-a}]=±(coroot of a)_*(generator of Lie G_m/Z) for every a. *)
Union[Abs /@ %]

Out[125]= {-2, 2, 2, -2, -2, -2, 2, 2, -2, 2, -2, 2, -2, -2, 2, 2, 2, -2, 2, -2, 2, 2, 2, -2, 2, 2, -2, 2, -2, -2, 2, 2, 2, 2, 2, -2, -2, -2, 2, -2, -2, 2, 2, -2, 2, 2, 2, 2}

Out[126]= {2}
```

■ Simple roots and Cartan matrix

```
In[127]:= (* Make sure that (1,3,19,41) lies in no root hyperplane *)
And @@ Map[#.{1, 3, 19, 41} =!= 0 &, roots]
posroots = Select[roots, #.{1, 3, 19, 41} > 0 &];
Length@%
simpleroots = Complement[posroots, Union @@ Table[posroots[[i]] + posroots[[j]], {i, 24}, {j, i + 1, 24}]]
(* Positive roots as linear combinations of the simple roots *)
Sort[posroots.Inverse[simpleroots], Plus @@ #1 > Plus @@ #2 &]
Position[roots, #][[1, 1]] & /@ simpleroots
simplepos = {7, 31, 25, 47};
(* Reordering the simple roots so that the resulting Cartan matrix is identical to the one in Bourbaki *)
Table[roots[[simplepos[[i]]]].coroots[[simplepos[[j]]]], {i, 4}, {j, 4}] // MatrixForm
Det@%

Out[127]= True

Out[129]= 24

Out[130]= {{-1, -1, 1, 0}, {0, 0, -1, 1}, {0, 1, 0, 0}, {1, 0, 0, 0}}

Out[131]= {{3, 2, 4, 2}, {3, 1, 4, 2}, {2, 1, 4, 2}, {2, 1, 3, 2}, {2, 1, 2, 2}, {2, 1, 3, 1}, {1, 1, 2, 2}, {2, 1, 2, 1}, {1, 0, 2, 2}, {1, 1, 2, 1}, {2, 1, 2, 0}, {1, 1, 1, 1},
 {1, 0, 2, 1}, {1, 1, 2, 0}, {1, 0, 1, 1}, {1, 1, 1, 0}, {1, 0, 2, 0}, {0, 0, 1, 1}, {1, 1, 0, 0}, {1, 0, 1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, 0, 1, 0}, {0, 1, 0, 0}}

Out[132]= {31, 7, 25, 47}

Out[134]//MatrixForm=
 2  -1   0   0
-1   2  -2   0
 0  -1   2  -1
 0   0  -1   2

Out[135]= 1
```

Verifying Proposition 6.6

- **1st computation**

In[136]:=
```
(* We compute the differential of delta as a linear combination of ∂/∂t, for all t in the list of variables in vars *)
DdeltaatE0 = (D[delta, #] /. sube) & /@ vars
(* the indivisibility of this vector is the first computation needed. *)
```

Out[136]= {1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

- **2nd computation: surjectivity of the orbit map on the tangent spaces**

In[137]:=
```
E6LieAlgebra.E0
%.DdeltaatE0
Solve[%% == {-v[5] - v[9]}~Join~Table[v[i], {i, 2, 27}]]
(* that the solution of this set of linear equations is visibly over Z, confirms the surjectivity. *)
```

Out[137]= $\{a_{1,1},\ -a_{12,15}-a_{26,25},\ -a_{12,18}-a_{27,25},\ -a_{15,12}-a_{25,26},\ -a_{1,1}+a_{2,2}+a_{4,4},\ -a_{15,18}-a_{27,26},\ -a_{18,12}-a_{25,27},\ -a_{18,15}-a_{26,27},\ -a_{2,2}-a_{4,4},\ -a_{26,17}+a_{27,14},\ a_{26,16}-a_{27,13},\ -a_{23,16}+a_{24,13},\ a_{25,17}-a_{27,11},$
$-a_{25,16}+a_{27,10},\ a_{22,16}-a_{24,10},\ -a_{25,14}+a_{26,11},\ a_{25,13}-a_{26,10},\ -a_{22,13}+a_{23,10},\ -a_{20,2}-a_{21,3},\ a_{20,1}-a_{21,6},\ a_{21,1}+a_{21,5},\ -a_{23,2}-a_{24,3},\ a_{23,1}-a_{24,6},\ a_{24,1}+a_{24,5},\ -a_{26,2}-a_{27,3},\ a_{26,1}-a_{27,6},\ a_{27,1}+a_{27,5}\}$

Out[138]= 0

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[139]= $\{\{a_{12,15}\to -a_{26,25}-v[2],\ a_{12,18}\to -a_{27,25}-v[3],\ a_{15,12}\to -a_{25,26}-v[4],\ a_{1,1}\to -v[5]-v[9],\ a_{2,2}\to -a_{4,4}-v[9],\ a_{15,18}\to -a_{27,26}-v[6],\ a_{18,12}\to -a_{25,27}-v[7],\ a_{18,15}\to -a_{26,27}-v[8],\ a_{26,17}\to a_{27,14}-v[10],$
$a_{26,16}\to a_{27,13}+v[11],\ a_{23,16}\to a_{24,13}-v[12],\ a_{25,17}\to a_{27,11}+v[13],\ a_{25,16}\to a_{27,10}-v[14],\ a_{22,16}\to a_{24,10}+v[15],\ a_{25,14}\to a_{26,11}-v[16],\ a_{25,13}\to a_{26,10}+v[17],\ a_{22,13}\to a_{23,10}-v[18],$
$a_{20,2}\to -a_{21,3}-v[19],\ a_{20,1}\to a_{21,6}+v[20],\ a_{21,1}\to -a_{21,5}+v[21],\ a_{23,2}\to -a_{24,3}-v[22],\ a_{23,1}\to a_{24,6}+v[23],\ a_{24,1}\to -a_{24,5}+v[24],\ a_{26,2}\to -a_{27,3}-v[25],\ a_{26,1}\to a_{27,6}+v[26],\ a_{27,1}\to -a_{27,5}+v[27]\}\}$

- **3rd computation: See the section on coroots of F4, where the requested computation is verified for all roots**

Verifying Lemma C.3

- **The required verification was done in during the computation of the Lie algebra of E6. The cotangent space is the quotient of the free module on the first list below of 99 linear forms by the submodule generated by the 21 linear forms obtained as follows: for each of the 21 expressions LHS → RHS in the second list below, LHS - RHS is one of the generators.
One sees immediately that the quotient is free.
Further, the cotangent space of GL(27) at the identity element is the free module on $\{a_{ij} : 1 \leq i, j \leq 27\}$,
of which our free module of rank 99 is a quotient in an obvious way.**

In[140]:=
```
Union[Table[a[i, i], {i, 27}], E6vars]
sol3
{Length@%%, Length@%, Length@%% - Length@%}
```

Out[140]= $\{a_{1,1},\ a_{2,2},\ a_{3,3},\ a_{4,4},\ a_{5,5},\ a_{6,6},\ a_{7,7},\ a_{8,8},\ a_{9,9},\ a_{10,10},\ a_{11,11},\ a_{12,12},\ a_{12,15},\ a_{12,18},\ a_{13,13},\ a_{14,14},\ a_{15,12},\ a_{15,15},\ a_{15,18},\ a_{16,16},\ a_{17,17},\ a_{18,12},\ a_{18,15},\ a_{18,18},$
$a_{19,19},\ a_{20,1},\ a_{20,2},\ a_{20,3},\ a_{20,20},\ a_{21,1},\ a_{21,2},\ a_{21,3},\ a_{21,4},\ a_{21,5},\ a_{21,6},\ a_{21,21},\ a_{21,24},\ a_{21,27},\ a_{22,10},\ a_{22,13},\ a_{22,16},\ a_{22,22},\ a_{23,1},\ a_{23,2},\ a_{23,3},\ a_{23,10},\ a_{23,13},\ a_{23,16},\ a_{23,23},$
$a_{24,1},\ a_{24,2},\ a_{24,3},\ a_{24,4},\ a_{24,5},\ a_{24,6},\ a_{24,10},\ a_{24,13},\ a_{24,16},\ a_{24,21},\ a_{24,24},\ a_{24,27},\ a_{25,10},\ a_{25,11},\ a_{25,13},\ a_{25,14},\ a_{25,16},\ a_{25,17},\ a_{25,25},\ a_{25,26},\ a_{25,27},\ a_{26,1},\ a_{26,2},\ a_{26,3},\ a_{26,10},$
$a_{26,11},\ a_{26,13},\ a_{26,14},\ a_{26,16},\ a_{26,17},\ a_{26,25},\ a_{26,26},\ a_{26,27},\ a_{27,1},\ a_{27,2},\ a_{27,3},\ a_{27,4},\ a_{27,5},\ a_{27,6},\ a_{27,10},\ a_{27,11},\ a_{27,13},\ a_{27,14},\ a_{27,16},\ a_{27,17},\ a_{27,21},\ a_{27,24},\ a_{27,25},\ a_{27,26},\ a_{27,27}\}$

Out[141]= $\{a_{5,5}\to -a_{1,1}+a_{2,2}+a_{4,4},\ a_{6,6}\to -a_{1,1}+a_{3,3}+a_{4,4},\ a_{7,7}\to a_{1,1}-a_{2,2}-a_{3,3}-a_{4,4},\ a_{8,8}\to -a_{3,3}-a_{4,4},\ a_{9,9}\to -a_{2,2}-a_{4,4},\ a_{12,12}\to -2\,a_{1,1}+a_{2,2}+a_{3,3}-a_{10,10}-a_{11,11},\ a_{13,13}\to a_{1,1}-a_{2,2}+a_{10,10},\ a_{14,14}\to a_{1,1}-a_{2,2}+a_{11,11},$
$a_{15,15}\to -a_{1,1}+a_{3,3}-a_{10,10}-a_{11,11},\ a_{16,16}\to a_{1,1}-a_{3,3}+a_{10,10},\ a_{17,17}\to a_{1,1}-a_{3,3}+a_{11,11},\ a_{18,18}\to -a_{1,1}+a_{2,2}-a_{10,10}-a_{11,11},\ a_{19,19}\to -a_{1,1}-a_{10,10},\ a_{20,20}\to -a_{4,4}-a_{10,10},\ a_{21,21}\to -a_{1,1}+a_{2,2}+a_{3,3}+a_{4,4}-a_{10,10},$
$a_{22,22}\to -a_{1,1}-a_{11,11},\ a_{23,23}\to -a_{4,4}-a_{11,11},\ a_{24,24}\to -a_{1,1}+a_{2,2}+a_{3,3}+a_{4,4}-a_{11,11},\ a_{25,25}\to a_{1,1}-a_{2,2}-a_{3,3}+a_{10,10}+a_{11,11},\ a_{26,26}\to 2\,a_{1,1}-a_{2,2}-a_{3,3}-a_{4,4}+a_{10,10}+a_{11,11},\ a_{27,27}\to a_{1,1}+a_{4,4}+a_{10,10}+a_{11,11}\}$

Out[142]= {99, 21, 78}

Verifying Lemma C.4

- **See the section on coroots of E6, where the requested computation is verified for all roots**