Calculating deformation rings

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1 Introduction

We are interested in computing local deformation rings away from \( p \). That is, if \( L \) is a finite extension of \( \mathbb{Q}_\ell \) and \( V \) is a 2-dimensional representation of \( G_L \) over \( F \), where \( F \) is a finite extension of \( \mathbb{F}_p \) for \( \ell \neq p \), we wish to study the deformation rings \( R_V^\square \) and \( R_V^{\psi,\square} \). Here \( \psi : G_L \to \mathcal{O}^\times \) is a continuous unramified character, \( \mathcal{O} \) is the ring of integers of a finite extension \( E \) of \( \mathbb{Q}_p \) which has residue field \( F \), and \( R_V^{\psi,\square} \) is the quotient of \( R_V^\square \) corresponding to deformations with determinant \( \psi \chi \), where \( \chi : G_L \to \mathbb{Z}_p^\times \) is the cyclotomic character.

Note that \( R_V^{\psi,\square} \) exists: There is a natural determinant map from the universal 2-dimensional (framed) representation to the universal 1-dimensional (framed) representation, and we take the fiber over the closed point corresponding to \( \chi \psi \).

We define the following two deformation problems:

- \( D_V^{ur, \psi,\square} \) is the deformation functor which spits out unramified framed deformations with determinant \( \psi \chi \)

- \( L_V^{\chi,\square} \) is the deformation functor which spits out pairs \( (V_A, L_A) \) of framed deformations with determinant \( \chi \), together with a \( G_L \)-stable \( A \)-line with \( G_L \) acting via \( \chi \) on \( L_A \). That is, \( L_A \) is a projective rank 1 \( A \)-module such that \( V_A/L_A \) is a projective \( A \)-module with trivial \( G_L \)-action.

Most of this talk will be about the structure of the ring representing the second functor.
2 Lies I will tell, and auxiliary categories of rings

The minor lie I will tell is that I will entirely suppress the language of categories fibered in groupoids, and pretend we are working with functors. This will allow me to avoid 2-categorical language. But to make what I say literally true, one has to handle non-trivial isomorphisms of deformations via the language of groupoids.

The more major lie I will tell is that after I finish this section, I will try to avoid talking about the various categories of algebras that are involved.

The basic set-up is representing certain deformations of a fixed residual representation (in characteristic $p$). The deformations are a priori to finite local artinian rings with fixed residue field. But we want to be able to take generic fibers of our representing objects in a sensible way, so we need techniques for passing to characteristic 0 points.

To do this, we need a variety of confusing auxiliary categories of algebras. To demonstrate, let $E/Q_p$ be a finite extension with residue field containing $F$, let $O \subset O_E$ be a discrete valuation ring finite over $W(F)$, and let $D$ be a deformation functor on the category $\mathfrak{A}_O$ of finite local artinian $O$-algebras with residue field $O/m_O$, and let $E/Q_p$ be a finite extension with residue field containing $F$. We will be interested in the category $\mathfrak{A}_E$ of finite local $W(F)[1/p]$-algebras with residue field $E$. We also introduce the following categories:

$\hat{\mathfrak{A}}_O$: $\hat{\mathfrak{A}}_O$ is the category of complete local noetherian $O$-algebras with residue field $O/m_O$.

$\hat{\mathfrak{A}}_{O,(O_E)}$: $\hat{\mathfrak{A}}_{O,(O_E)}$ is the category of $O$-algebras $A$ in $\hat{\mathfrak{A}}_O$ equipped with a map of $O$-algebras $A \to O_E$.

Int$_B$: Given $B \in \mathfrak{A}_E$, Int$_B$ is the category of finite $O_E$-subalgebras $A \subset B$ such that $A \otimes_{O_E} E = B$.

Note that Int$_B$ is a subcategory of $\hat{\mathfrak{A}}_{O,(O_E)}$ ($A$ obviously has a map to $E$, and by finiteness or the same sort of arguments as in Brian’s talk, it actually lands in $O_E$), and there is a natural functor $\hat{\mathfrak{A}}_{O,(O_E)} \to \hat{\mathfrak{A}}_O$. 


Also note that we can canonically extend $D$ to a groupoid on $\mathcal{M}_\mathcal{O}$, by setting $D(\varprojlim R/m_R^{n+1}) = \varprojlim D(R/m_R^{n+1})$.

Now fix some $\xi \in D(\mathcal{O}_E)$, which makes sense by the preceding comment. We define a groupoid $D(\xi)$ on $\mathcal{M}_{\mathcal{O}_E}$ by setting $D(\xi)$ to be the fiber over $\xi$. More precisely, $D(\xi)(A)$ consists of objects of $D(A)$ together with morphisms (in $D$) covering the given map $A \to \mathcal{O}_E$.

Finally, we can extend $D(\xi)$ to $\mathcal{M}_E$. We note that $B \in \mathcal{M}_E$ can be exhausted by objects in $\text{Int}_B$, so we set $D(\xi)(B) = \varprojlim_{A \in \text{Int}_B} D(\xi)(A)$.

Now Kisin proves two crucial lemmas about these groupoids (which he calls a lemma and a proposition). The first tells us how to get universal deformation rings for the groupoids on $\mathcal{M}_E$, and the second relates those groupoids to the ones we would naively expect (for some deformation problems we already care about):

**Lemma 2.1.** If $D$ is pro-represented by a complete local $\mathcal{O}$-algebra $R$, then $D(\xi)$ is pro-represented (on $\mathcal{M}_E$) by the complete local $\mathcal{O}[1/p]$-algebra $\hat{R}_\xi$ obtained by completing $R \otimes \mathcal{O} E$ along the kernel $I_\xi$ of the map $R \otimes \mathcal{O} E \to E$ induced by $\xi$.

**Lemma 2.2.** Fix a residual representation $V$ over $F$, and carry out the above program for $D_V$ and $D^\Box_V$. Then there are natural isomorphisms of groupoids $D_{V,\xi} \sim D_V$ and $D_{V,\xi}^\Box \sim D_V^\Box$.

### 3 Main result

The main result we will prove is the following:

**Theorem 3.1.** Let $V$ be any 2-dimensional representation of $G_L$ (over $F$). Fix a continuous unramified character $\psi : G_L \to \mathcal{O}^\times$ and consider $R^\psi\Box_V$, the quotient of $R^\Box_V$ corresponding to deformations of $V$ with determinant $\psi$. Then $\text{Spec} \, R^\psi\Box_V[1/p]$ is 3-dimensional, and it is the scheme-theoretic union of formally smooth components.

There are several claims implicit in this theorem, namely the existence, smoothness, connectedness, and dimension of $R^{ur,\psi\Box}_V$ and $R^{\gamma,\gamma\Box}_V$, as well as the connectedness of $R^\psi\Box_V$. We assume these for the moment and go on with the proof.
Proof. Let $E'/E$ be a finite extension, let $x : R^{\psi, \square}_V[1/p] \to E'$ be a point of $\text{Spec} R^{\psi, \square}_V[1/p]$ with residue field $E'$ (so that it is actually an $E'$-point), and let $V_x$ be the induced representation with coefficients in $E'$. We know (from Brian’s talk on characteristic 0 points of deformation rings) that the completion of $R^{\psi, \square}_V[1/p]$ at the maximal ideal $m_x = \ker x$ represents deformations of $V_x$. The tangent space at $x$ is $H^1(G_L, \text{ad}^0 V_x)$. Obstructions to deforming representations live in $H^2$ groups, so $R^{\psi, \square}_V[1/p]$ at $x$ will be formally smooth at any point $x$ where $H^2(G_L, \text{ad}^0 V_x)$ vanishes.

Given any framed deformation problem $D^{\square}$ (with coefficients in some unspecified field $H$), there is a natural morphism $D^{\square} \to D$ to the unframed problem given by “forgetting the basis”. This morphism is formally smooth in the sense that artinian points of $D$ can be lifted.

Furthermore, the fibers of the morphism of tangent spaces $D^{\square}(H[\varepsilon]) \to D(H[\varepsilon])$ are principal homogeneous spaces under $\text{ad} / \text{ad}^{G_L}$. Specifically, given a residual representation $V_H$ and a choice of (unframed) deformation $V_H[\varepsilon], \ker(\text{GL}_2(H[\varepsilon]) \to \text{GL}_2(H)) = 1 + \varepsilon M_2(H[\varepsilon]) \cong \text{End}_H V_H$ acts (via conjugation) on the fiber over $V_H[\varepsilon]$. Then it is easy to check that $1 + \varepsilon M$ acts trivially on the fiber if and only if $M$ is in $\text{ad}^0 V_H$.

Counting dimensions,

$$\dim_F \text{D}^{\square}(\text{F}[\varepsilon]) = \dim_F \text{D}(\text{F}[\varepsilon]) + \dim_F \text{ad} - \dim_F H^0(G_L, \text{ad}) \quad (3.1)$$

Using this formula, we see that the tangent space to $\text{Spec} R^{\psi, \square}_V[1/p]$ at $x$ has $E'$-dimension

$$\dim_{E'} H^1(G_L, \text{ad}^0 V_x) + \dim_{E'} \text{ad} V_x - \dim_{E'} H^0(G_L, \text{ad} V_x)$$

$$= \dim_{E'} H^1(G_L, \text{ad}^0 V_x) + \dim_{E'} \text{ad} V_x - (\dim_{E'} H^0(G_L, \text{ad}^0 V_x) - 1)$$

$$= - (\dim_{E'} H^2(G_L, \text{ad}^0 V_x) - \dim_{E'} H^1(G_L, \text{ad}^0 V_x) + \dim_{E'} H^0(G_L, \text{ad}^0 V_x))$$

$$+ \dim_{E'} H^2(G_L, \text{ad}^0 V_x) + \dim_{E'} \text{ad} V_x - 1$$

$$= \dim_{E'} H^2(G_L, \text{ad}^0 V_x) + 3$$

the last step following by the Euler characteristic formula for $p$-adic coefficients. Thus, if $H^2(G_L, \text{ad}^0 V_x) = 0$, $x$ will be a formally smooth point of $\text{Spec} R^{\psi, \square}_V[1/p]$ with a 3-dimensional tangent space.

Now suppose $H^2(G_L, \text{ad}^0 V_x) \neq 0$. By the $p$-adic version of Tate local duality,

$$\dim_{E'} H^2(G_L, \text{ad}^0 V_x) = \dim_{E'} H^0(G_L, (\text{ad}^0 V_x)^*)$$

which is $\dim_{E'} H^0(G_L, \text{ad}^0 V_x(1))$
(because \( \text{ad}^0 \) is self-dual). Now we have the split exact sequence of \( GL \)-modules

\[
0 \to \text{ad}^0 V_x(1) \to \text{ad} V_x(1) \to E'(1) \to 0
\]

which gives us an exact sequence in cohomology:

\[
0 \to H^0(GL, \text{ad}^0 V_x(1)) \to H^0(GL, \text{ad} V_x(1)) \to H^0(GL, E'(1))
\]

But \( H^0(GL, E'(1)) = 0 \) so

\[
H^0(GL, \text{ad}^0 V_x(1)) = H^0(GL, \text{ad} V_x(1)) = H^0(GL, \text{Hom}(V_x, V_x(1)))
\]

In particular, if \( H^2(GL, \text{ad}^0 V_x) \neq 0 \), there is a non-zero homomorphism (of \( GL \)-modules) \( V_x \to V_x(1) \). It has 1-dimensional \((GL\text{-stable})\) image and kernel, so there is some character \( \gamma \) such that \( 0 \to \gamma \to V_x \to \gamma(1) \to 0 \) is exact. But such extensions are classified by \( H^1(GL, E'(-1)) \), which is 0: the Euler characteristic formula says that

\[
\dim_{E'} H^0(GL, E'(-1)) - \dim_{E'} H^1(GL, E'(-1)) + \dim_{E'} H^2(GL, E'(-1)) = 0,
\]

but \( H^0(GL, E'(-1)) \) is clearly zero, and \( H^2(GL, E'(-1)) \) is dual to \( H^0(GL, E'(2)) \), which is zero, so \( H^1(GL, E'(-1)) \) is zero as well. So this extension splits.

We have shown that if \( H^2(GL, \text{ad}^0 V_x) \neq 0 \), then \( V_x = \gamma \oplus \gamma \chi \) for some character \( \gamma : GL \to E'^{\times} \). If \( \gamma \) is unramified, then this implies that \( x \) is in the image of both \( R_{V^\gamma \chi}^{ur, \gamma, \square} \) and \( R_{V^\gamma \chi}^{ur, \gamma, \square} \).

So the only singular points of \( \text{Spec} R_{V^\square}^{\psi}[1/p] \) lie in the intersection of two formally smooth components.

\[\square\]

The definition of formal smoothness requires us to be able to lift through any square-zero thickening, but we only looked at what happens at artinian points of \( \text{Spec} R_{V^\square}^{\psi}[1/p] \); the commutative algebra necessary to justify this is discussed in Brian’s notes on \( \ell = p \).

### 4 Unramified deformations

We’ve seen previously that for the unframed case, the tangent space at \( x \) for unramified deformations with fixed determinant is \( H^1(GL/I_L, (\text{ad}^0 V_x)^I_L) \),
and the obstruction space should be $H^2(G_L/I_L, (\text{ad}^0 V_x)^{I_z}) = 0$. We have the exact sequence

$$0 \to (\text{ad}^0 V_x)^{G_L} \to (\text{ad}^0 V_x)^{I_L} \xrightarrow{\text{Frob} - \text{id}} (\text{ad}^0 V_x)^{I_L} \to (\text{ad}^0 V_x)^{I_L} / (\text{Frob} - \text{id})(\text{ad}^0 V_x)^{I_L} \to 0$$

This implies that $\dim E' H^0(G_L, \text{ad}^0 V_x) = \dim E' H^1(G_L/I_L, (\text{ad}^0 V_x)^{I_L})$. And since the tangent space for the framed case has dimension $\dim_{\mathcal{E}'} H^1(G_L/I_L, (\text{ad}^0 V_x)^{I_L}) + \dim_{\mathcal{E}'} \dim_{\mathcal{E}'} H^0(G_L, \text{ad}^0 V_x)$ by the discussion in the previous section, this implies that the tangent space of $\mathcal{R}^r_{ur, \psi, \square}$ has dimension $\dim_{\mathcal{E}'} \text{ad}^0 V_x = 3$.

So granting existence, $\mathcal{R}^r_{ur, \psi, \square}$ is formally smooth and 3-dimensional.

5 $R^\chi_{\gamma, \gamma, \square}$

We begin this section with a general lemma.

**Lemma 5.1.** Let $\mathcal{O}$ be a local $W(k)$-algebra with residue field $k$, with $K$ the fraction field of $W(k)$, and let $X$ be a proper residually reduced $\mathcal{O}$-scheme. Then the components of the fiber of $X$ over the closed point of $\mathcal{O}$ are in bijection with the components of $X[1/p]$.

**Proof.** Consider a connected component of $X[1/p] = X \otimes_{W(k)} K$ and let $e$ be the idempotent which is 1 on this component and 0 on the others. Then if $w$ is a uniformizer of $W(k)$, there is some $n$ such that $w^n e$ extends to a global section of $X$. But $(w^n e)^2 = w^n (w^n e)$, so if $n > 0$, as a function on the special fiber $X \otimes_{\mathcal{O}} k$, $w^n e$ is nilpotent. This contradicts our reducedness hypothesis, so $n = 0$ and $e$ is already a global section of $X$.

So we know that the components of $X \otimes_{W(k)} K$ are in bijection with the components of $X$ itself. But if $X^\wedge$ is the completion of $X$ along its special fiber, the components of the special fiber $X \otimes_{\mathcal{O}} k$ are in bijection with the components of $X^\wedge$ (because they have the same underlying topological space), and formal GAGA implies that the components of $X^\wedge$ are in bijection with the components of $X$ ($X$ is proper over $\mathcal{O}$, so we can apply formal GAGA to see that the global idempotent functions on $X$ and $X^\wedge$ are in bijection). \qed
5.1 Representability

Proposition 5.2. The morphism $|L^\chi_V| \to |D^\square_V|$ is represented by a projective morphism $\Theta_V : \mathcal{L}^\chi_V \to R^\chi_V$.

Proof. Given an $A$-point of $R^\chi_V$, the $A$-points of $\mathcal{L}^\chi_V$ should be certain line bundles on $\text{Spec} A$, so we will cut $\mathcal{L}^\chi_V$ out of $P_1^{R^\chi_V}$.

Consider $\mathbb{P}$, the projectivization of the universal rank 2 $R^\chi_V$-module. That is, if $V_R$ is the universal rank 2 $R^\chi_V$-module (equipped with a representation of $G_L$), then $\mathbb{P} := \text{Proj} \text{Sym} V_R \cong \text{Proj} R^\chi_V[x_0, x_1]$.

If $A$ is an an $R^\chi_V$-algebra with residue field $F$, a morphism $\text{Spec} A \to \mathbb{P}$ (over $R^\chi_V$) is the same as a surjection (of sheaves) $A^2 \to \mathcal{L} \to 0$.

Given a morphism $f : \text{Spec} A \to \mathbb{P}$, there is a natural $G_L$-action on the quotient $\mathcal{L}$ if and only if $g^*f = f$ for all $g \in G_L$. The $g^*$-fixed locus of $\mathbb{P}$ is $H_g$ defined by the Cartesian square

$$
\begin{array}{ccc}
H_g & \longrightarrow & \mathbb{P} \\
\downarrow & & \downarrow \text{(id,} g^*) \\
\mathbb{P} & \Delta \rightarrow & \mathbb{P} \times_{R^\chi_V} \mathbb{P}
\end{array}
$$

Since $\mathbb{P}$ is separated, $H_g$ is a closed subscheme of $\mathbb{P}$. Thus, the intersection $H := \bigcap_{g \in G} H_g$ is a closed subscheme of $\mathbb{P}$ parametrizing $G_L$-equivariant quotients $A^2 \to \mathcal{L} \to 0$.

Now if $A$ is a complete local $W(F)$-algebra, there is a natural map from $H$ to the universal deformation of the residually trivial 1-dimensional representation, given (in the language of the functor of points) by sending $A^2 \to \mathcal{L} \to 0$ to $\mathcal{L}$. Then we can take the fiber over the (closed) point corresponding to the trivial representation to get a closed subscheme of $\mathbb{P}$ representing $L^\chi_V$ on $\mathfrak{M}_{W(F)}$.

Now take limits to get representability of $L^\chi_V$ on $\mathfrak{Aug}_{W(F)}$.

5.2 Smoothness and connectedness

Next we want to study smoothness and connectedness.
Proposition 5.3. \( \mathcal{L}_V^\square \) is formally smooth over \( W(F) \). Furthermore, the \( W(F)[1/p] \)-scheme \( \mathcal{L}_V^\square \otimes_{W(F)} W(F)[1/p] \) is connected.

Proof. First, we will show that for any finite group \( M \) of \( p \)-power order, the natural map \( H^1(G_L, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} M \to H^1(G_L, \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} M) \) is an isomorphism. It suffices to consider the case \( M = \mathbb{Z}/p^n\mathbb{Z} \). In that case, we have the exact sequence

\[
0 \to \mathbb{Z}_p(1) \xrightarrow{p^n} \mathbb{Z}_p(1) \to M \to 0
\]

Then the long exact sequence in group cohomology shows that

\[
0 \to H^1(G_L, \mathbb{Z}_p(1))/p^n H^1(G_L, \mathbb{Z}_p(1)) \to H^1(G_L, M) \to H^2(G_L, \mathbb{Z}_p(1))[p^n]
\]

is exact. The middle arrow is the natural map we started with, so we wish to show that \( H^2(G_L, \mathbb{Z}_p(1))[p^n] \) is 0. But by Tate local duality (as in Simon’s talk), \( H^2(G_L, \mathbb{Z}_p(1)) \) is Pontryagin dual to \( \mathbb{Q}_p/\mathbb{Z}_p \), so has no \( p^n \)-torsion.

Thus, for any artinian algebra \( A \), the composition

\[
\text{Ext}^1_{\mathbb{Z}_p[G_L]}(\mathbb{Z}_p, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} A \to H^1(G_L, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} A \to H^1(G_L, \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} A) \to \text{Ext}^1_{\mathbb{Z}_p[G_L]}(A, A(1))
\]

is an isomorphism.

To prove smoothness, it suffices to show that for any surjection of artinian rings \( A \to A' \), the map \( |L_V^\square|(A) \to |L_V^\square|(A') \) is a surjection. Now consider a pair \((V_A, L_A)\) in \( |L_V^\square|(A') \). It corresponds to an element of \( \text{Ext}^1_{\mathbb{Z}_p[G_L]}(A, A(1)) \), so by the isomorphism we just proved, it corresponds to an element of \( \text{Ext}^1_{\mathbb{Z}_p[G_L]}(\mathbb{Z}_p, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} A' \). But such an element clearly lifts to an element of \( \text{Ext}^1_{\mathbb{Z}_p[G_L]}(\mathbb{Z}_p, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} A \), which is to say, an element of \( |L_V^\square|(A) \).

Now we wish to prove connectedness after inverting \( p \), and for this we use the lemma on connected components. Specifically, since \( \mathcal{L}_V^\square \) is smooth, its special fiber \( \mathcal{L}_V^\square \otimes_{W(F)} F \) is reduced, so to show \( \mathcal{L}_V^\square[1/p] \) is connected, it suffices to show that the special fiber \( \mathcal{L} \otimes_{R_V} F \) is connected.

But the special fiber is simply the fiber over the residual representation. If \( F \cong F(1) \) and the representation is split (i.e., the residual representation is trivial), any line in \( F^2 \) is \( G_L \)-stable with \( G_L \)-acting by \( \chi = \text{id} \), so the fiber is a full \( P^1 \). Otherwise, there is at most one \( G_L \)-line with \( G_L \) acting via \( \chi \), and this is true for any \( A \)-point of the fiber, so it is either empty or it consists of a single reduced point. So the special fiber is connected. \( \square \)
The next proposition will show that $\mathcal{L}_V^{\square}[1/p] \to \text{Spec } R_V^{\square}[1/p]$ is a monomorphism. More precisely, it shows that this morphism is injective on artinian points, but, as before, Brian’s notes on $\ell = p$ explain why this is sufficient to let us conclude that it is actually a monomorphism.

**Proposition 5.4.** Let $E/\mathbb{Q}_p$ be a finite extension, and let $\xi$ refer to both an $\mathcal{O}_E$-valued point of $R_V^{\square}$ and an $\mathcal{O}_E$-valued point in the fiber of $\mathcal{L}_V^{\square}$ above it. Then the morphism of groupoids (functors) on $\mathfrak{M}_E L_V^{\square} \to D_V^{\square}$ is fully faithful. If the representation over $E V_\xi$ corresponding to $\xi$ is indecomposable, then this is an equivalence.

**Proof.** Let $B$ be an object of $\mathfrak{M}_E$, and let $V_B$ be an object of $D_V^{\square}(B)$. To prove the first assertion, we need to show that $V_B$ admits at most one $G_L$-stable $B$-line $L_B \subset V_B$ such that $G_L$ acts trivially on $V_B/L_B$. But $\text{Hom}_{B[G_L]}(B(1), V_B/L_B) = \{0\}$ because the $G_L$-action on the target is trivial, so $\text{Hom}_{B[G_L]}(B(1), V_B) = \text{Hom}_{B[G_L]}(B(1), L_B)$ and $L_B$ is unique.

Now suppose $V_\xi$ is indecomposable; we wish to show that $V_B$ actually does admit a suitable $B$-line. We will do this by showing that $V_B$ is isomorphic to the trivial deformation $V_\xi \otimes_E B$. Note that by Tate local duality

$$\dim_E H^1(G_L, \text{ad}^0 V_\xi) = \dim_E H^0(G_L, \text{ad}^0 V_\xi) + \dim_E H^0(G_L, \text{ad}^0 V_\xi(1)) = 0$$

the last equality following from indecomposability of $V_\xi$. The result then follows by induction on the length of $B$, since this calculation holds for any indecomposable extension of $A(1)$ by $A$. 

But since we have a proper monomorphism of schemes $\mathcal{L}_V^{\square}[1/p] \to \text{Spec } R_V^{\square}[1/p]$, it is a closed immersion.

Now we can prove the following proposition and corollary.

**Proposition 5.5.** Let $\text{Spec } R_V^{\square,1}$ be the scheme-theoretic image of the morphism $\mathcal{L}_V^{\square} \to \text{Spec } R_V^{\square}$. Then

1. $R_V^{\square,1}$ is a domain of dimension 4 and $R_V^{\square,1}$ is formally smooth over $W(F)$.

2. If $E/\mathbb{Q}_p$ is a finite extension, then a morphism $\xi : R_V^{\square} \to E$ factors through $R_V^{\square,1}$ if and only if the corresponding two-dimensional representation $V_\xi$ is an extension of $E$ by $E(1)$.
Proof. Since \( R_{V,\square}^{\chi,1} \) is smooth and connected, it is a domain. We will find its dimension via a tangent space calculation. Suppose \( V_\xi \) is indecomposable (which we may assume, since most points on \( R_{V,\square}^{\chi,1} \) are indecomposable). Then the dimension of \( R_{V,\square}^{\chi,1}[1/p] \) is
\[
\dim_E |D_{V_\xi}|(E[\varepsilon]) = \dim_E |D_{V_\xi}|(E[\varepsilon]) + 4 - \dim_E (\text{ad} V_\xi)^{G_L} \\
= \dim_E H^1(G_L, \text{ad}^0 V_\xi) + 3 = 3
\]
So \( R_{V,\square}^{\chi,1} \) itself is 4-dimensional, and we have proven the first part.

The second part follows from the definition of \( L_{V,\square}^{\chi,1} \) and \( R_{V,\square}^{\chi,1} \).

Corollary 5.6. Let \( \mathcal{O} \) be the ring of integers in a finite extension of \( W(F)[1/p] \), and \( \gamma : G_L \to \mathcal{O}^\times \) a continuous unramified character. Write \( R_{V,\mathcal{O}}^\square = R_{V,\mathcal{O}}^\square \otimes_{W(F)} \mathcal{O} \). Then there exists a quotient \( R_{V,\mathcal{O}}^{\chi,\gamma,\square} \) such that

- \( R_{V,\mathcal{O}}^{\chi,\gamma,\square} \) is a domain of dimension 4 and \( R_{V,\mathcal{O}}^{\chi,\gamma,\square}[1/p] \) is formally smooth over \( \mathcal{O} \).

- If \( E/\mathcal{O}[1/p] \) is a finite extension, then a map \( \xi : R_{V,\mathcal{O}}^\square \to E \) factors through \( R_{V,\mathcal{O}}^{\chi,\gamma,\square} \) if and only if \( V_\xi \) is an extension of \( \gamma \) by \( \gamma(1) \).

Proof. This basically follows because universal deformation rings behave reasonably well with respect to twisting by fixed characters, at least once the question makes sense.

More precisely, we may replace \( F \) by the residue field of \( \mathcal{O} \) (corresponding to tensoring \( R_{V,\square}^\square \) with \( \mathcal{O} \)). Then twisting by \( \gamma^{-1} \) induces an isomorphism \( R_{V,\mathcal{O}}^{\chi,\gamma,\square} \to R_{V,\mathcal{O}}^{\chi,\gamma^{-1},\mathcal{O}} \) (because twisting the residual representation by \( \gamma^{-1} \) doesn’t change this deformation problem (except to multiply the determinant by \( \gamma^2 \)), and the quotient \( R_{V,\mathcal{O}}^{\chi,\gamma,\square} \) corresponds to \( R_{V,\mathcal{O}}^{\chi,1,\square} \otimes_{W(F)} \mathcal{O} \) under this isomorphism. \( \square \)