MODULAR FORMS AND AUTOMORPHIC REPRESENTATIONS

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Contents

1. Some notations.

Let $\mathfrak{H} = \{z \in \mathbf{C}; \Im(z) > 0\}$ be the Poincaré upper-half plane.

Let k and N be two integers, and, as usual, $\Gamma_0(N)$ be the subgroup of $\mathrm{SL}(2, \mathbf{Z})$ of matrices whose lower left entries are divisible by N. It acts on $\mathfrak{H}$ by fractional linear transformations: $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \cdot z = \frac{az+b}{cz+d}$.

Let χ be a Dirichlet character modulo q: it defines a character on $\Gamma_0(N)$, by evaluating χ at the upper left entry. It will be convenient to define $\chi(n) = 0$ if the integer n is not coprime with N.

If X is a finite set, $|X|$ denotes its cardinality; we reserve the letters p, ℓ for prime numbers, and n, m for integers.

The letters K, E, k (resp. K_λ) denote fields (resp. the completion of K with respect to the valuation associated to λ), and $\mathcal{O}_K, \mathcal{O}_\lambda$ stand for the rings of integers of K, K_λ in the relevant situations.

The set of adeles of $\mathbf{Q}$ is denoted $\mathbb{A}_\mathbf{Q}$, and for a finite set of primes S containing ∞, one denotes $\mathbb{A}_{\mathbf{Q},S} = \prod_{v \in S} \mathbf{Q}_v \times \prod_{v \notin S} \mathbf{Z}_v$. The finite adeles are denoted $\mathbb{A}_f$.

For a complex number z, the notation $\mathrm{e}(z)$ stands for $\exp(2\pi i z)$.

The notation $f(x, A) \ll_A g(x)$ means that for any A, there exists a real number $C(A)$ such that for any x, $|f(x, A)| \leq C(A) \cdot |g(x)|$; if one adds "as

$x \to \infty$", it means that the last inequality holds for $x \geq x(A)$ for some real number $x(A)$. In the same spirit, the notation $f(x) = o_{x\to x_0}(g(x))$ (resp. $f(x) = O_{x\to x_0}(g(x))$) means that the quotient $f(x)/g(x)$ is defined in a (pointed) neighbourhood of x_0, and that $|f(x)/g(x)|$ tends to zero (resp. stays bounded) when x tends to x_0.

Some spaces of funtions: let $\mathfrak{X}$ be a locally compact Hausdorff space.

- $\mathcal{C}_c(\mathfrak{X})$ is the space of continuous compactly supported complex valued funcions.
- $\mathcal{C}_c^\infty(\mathfrak{X})$ denotes the subspace of smooth functions in the latter (when $\mathfrak{X}$ is a manifold, this means "locally constant" if the manifold is totally disconnected).

2. Modular forms

2.1. For any holomorphic function f defined on $\mathfrak{H}$ and $\gamma \in \Gamma_0(N)$, we define:

$$f_{|\gamma}(z) = \chi(\gamma)^{-1}(cz+d)^{-k}f(\gamma\cdot z)$$

Consider the following properties:

(M1): For any $\gamma \in \Gamma_0(N)$, $f_{|\gamma} = f$. This implies, by Fourier analysis, that

for any $\sigma \in \mathrm{SL}(2,\mathbf{Z})$, there exists a positive integer $h(\sigma)$ (with $h(\mathrm{I}) = 1$) such that one has an absolutely convergent decomposition:

$$f_{|\gamma}(z) = \sum_{n\in\mathbf{Z}} c_n(f,\sigma)\mathrm{e}(nz/h(\sigma))$$

The holomorphy at $i\infty$ is then expressed by:

(M2): For any $\sigma \in \mathrm{SL}(2,\mathbf{Z})$, $c_n(f,\sigma) = 0$ for all negative n.

"Cuspidality" is:

(M2′): For any $\sigma \in \mathrm{SL}(2,\mathbf{Z})$, $c_n(f,\sigma) = 0$ for all $n \geq 0$.

2.2. The space of modular forms of weight k, level q and nebentypus χ is the set of holomorphic functions satisfying (M1) and (M2) above; the subspace of modular forms satisfying (M2′) as well is called the space of cusp forms, noted $\mathcal{S}_k(N,\chi)$. The latter is finite dimensional (as is the first), and equipped with the Petersson inner product, invariant under the group action (it is a quotient of a Haar measure on $\mathfrak{H} = \mathrm{SO}_2(\mathbf{R})\backslash\mathrm{SL}_2(\mathbf{R})$):

$$\langle f,g\rangle = \int_{\Gamma_0(N)\backslash\mathfrak{H}} f(x+iy)\overline{g(x+iy)}y^k\frac{dxdy}{y^2}$$

Note right now that by taking $\gamma = -\mathrm{I}$, (M1) gives $f(z) = (-1)^k\chi(-1)f(z)$, so if χ and k don't have the same parity, the space of modular forms is $\{0\}$; we shall exclude this case.

2.3. **Hecke operators.** On the space of modular forms of weight k and level q, one has the so-called Hecke operators, defined as follows. Let $n \geq 1$ be an integer, and let $\Delta_0(N) = \{\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{M}_2(\mathbf{Z}) \ : \ \det(\gamma) > 0, N|c, (a,N) = 1\}$. For $\alpha \in \Delta_0(N)$, one defines first:

$$T_\alpha(f)(z) = \det(\alpha)^{k-1}\chi(\alpha)^{-1}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$$

To define the level n Hecke operator, one considers the set $\{\alpha \in \Delta_0(N) \ : \ \det(\alpha) = n\}$ on which $\Gamma_0(N)$ acts on the left. One proves that one can write it as a finite disjoint union $\sqcup_j \Gamma_0(N)\alpha_j$, and one defines:

$$T_n(f)(z) = \sum_j (T_{\alpha_j} f)(z)$$

More explicily, one has:

$$\{\alpha \in \Delta_0(N) \ : \ \det(\alpha) = n\} = \bigcup_{\substack{ad=n \\ a>0 \\ (a,q)=1}} \bigcup_{0 \leq b \leq d-1} \Gamma_0(N) \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

from which one deduces ($\chi(a) = 0$ if a and N are not coprime):

$$T_n(f)(z) := n^{k-1} \sum_{\substack{ad=n \\ a>0 \\ 0 \leq b \leq d-1}} \chi(a) d^{-k} f\left(\frac{az+b}{d}\right)$$

With this definition, one sees easily that the T_n's preserve the modularity and cupsidality. One can then give the action of the T_p, for p prime, on the Fourier expansion of a modular form (but the modularity is hardly seen from this expression):

- If $(p,N) = 1$, $T_p(f)(z) = \sum_n c_{pn}(f)\mathrm{e}(nz) + \chi(p)p^{k-1}\sum_n c_n(f)\mathrm{e}(pnz)$ – p is called a good prime.
- If $p|N$, $T_p(f)(z) = \sum_n c_{pn}(f)\mathrm{e}(nz)$ – p is a bad prime.

The Hecke operators preserve the space of cusp forms; the Hecke operators at good primes all commute, and are normal with respect to the Petersson inner product. These important facts are explained in Miyake [M], as are the multiplicativity relations. In particular, if f is an eigenfunction for all the Hecke operators at good primes, with eigenvalues $\{a_p(f)\}$, one has $c_p(f) = c_f(1)a_p(f)$ at good p. To diagonalize further the Hecke operators, and get a good definition of L-series, it is necessary to introduce

2.4. **Newforms and oldforms.** Suppose χ defines a Dirichlet character modulo N', for $N'|N$. For any cusp form g in $\mathcal{S}_k(N',\chi)$, one checks easily that $z \mapsto g(dz)$ defines an element of $\mathcal{S}_k(N,\chi)$, for any $d|(N/N')$. Let

$$\mathcal{S}_k^{\mathrm{old}}(N,\chi) = \bigcup_{\substack{\chi \text{ factors through } N'|N \\ d|(N/N')}} \{z \mapsto g(dz) \ : \ g \in \mathcal{S}_k(N',\chi)\}$$

be the space of oldforms, and let

$$\mathcal{S}_k^{\mathrm{new}}(N,\chi) = \mathcal{S}_k^{\mathrm{old}}(N,\chi)^{\perp}$$

be the space of newforms (it may be zero!). Then it can be shown that the whole Hecke algebra (i.e. including bad primes) can be diagonalized on the space of newforms. The primitive Hecke eigenforms (those with $c_1(f) = 1$) have pairwise distinct systems of eigenvalues outside a finite number of primes ("multiplicity one", well explained in the adelic setting by Casselman [C], cf. Gelbart [G] as well). Their L-series have an Euler product, absolutely convergent if $\Re(s) > 1 + k/2$:

$$L(s,f) := \sum_n \frac{a_n(f)}{n^s} = \prod_p L(s,f_p)$$

with

$$\begin{aligned} L(s,f_p) &= \Big(1 - a_p(f)p^{-s} + \chi(p)p^{k-1-2s}\Big)^{-1} \\ &= \Big(1 - \alpha_1(p,f)p^{-s}\Big)^{-1}\Big(1 - \alpha_2(p,f)p^{-s}\Big)^{-1} \end{aligned}$$

at a good prime p, and

$$L(s,f_p) = \Big(1 - a_p(f)p^{-s}\Big)^{-1}$$

at a bad prime, along with an analytic continuation (easy to see with the Mellin transform), functional equation – cf. Bump [Bu], Miyake [M], Iwaniec [I], etc.

When one proves a theorem, one can often reduce it to the case of newforms, thanks to this decomposition.

2.5. **Ramanujan conjecture.** Let f be a primitive newform. The Ramanujan conjecture is the following inequality:

$$|a_p(f)| \leq 2p^{\frac{k-1}{2}}$$

for good p, which is equivalent to $|\alpha_i(p,f)| = p^{\frac{k-1}{2}}$. It has been a theorem for 35 years now, proven by Deligne for weight greater than two. In the case of bad p one can compute the possibilities for $a_p(f)$ rather explicitly (see [M]).

2.6. **Rationality properties.** Let $f \in \mathcal{S}_k(N,\chi)$ be a eigenform for all the Hecke operators at good primes, with Hecke eigenvalues $\{a_f(p)\}_{p\nmid N}$. Then:

$$\mathbf{Q}(f) := \mathbf{Q}(a_f(p), \chi(p) : p \nmid N)$$

is a finite extension of $\mathbf{Q}$, and all the Hecke eigenvalues are integers in this extension. If the nebentypus is trivial, then this extension in totally real. Serre explains all of this in terms of arithmetic geometry in his Durham lectures.

2.7. An interesting problem is the evaluation of the dimension of the space of cusp forms, when one or more of the parameters (k, N) vary. For instance, using Eichler-Selberg trace formula one can prove that (see Knightly-Li [KL] theorem 29.5):

$$\dim(\mathcal{S}_k(N,\chi)) = \frac{k-1}{12}\psi(N) + O\Big(N^{1/2}\tau(N)\Big) \tag{1}$$

uniform in $k \geq 2$ and N, where $\psi(N) = q\prod_{p|N}(1+p^{-1})$ and $\tau(N)$ is the number of divisors of N.

Similarly, one can bound the dimension of the space of newforms using the Petersson trace formula (Iwaniec-Luo-Sarnak [ILS]), and one has a uniform estimate for N squarefree, $k \geq 2$:

$$\dim(\mathcal{S}_k^{\mathrm{new}}(N,\chi)) = \frac{k-1}{12}\varphi(N) + O\Big((kN)^{2/3}\Big) \tag{2}$$

with $\varphi(N) = q\prod_{p|N}(1-p^{-1})$ the Euler phi function.

3. Representation theory

If G is a locally compact group, V a complex topological vector space ($\mathbf{C}$ is endowed either with the discrete or the euclidean topology), a *representation* of G in V is a group homomorphism $\rho : G \to \mathrm{Aut}(V)$, such that the mapping $(g, v) \in G \times V \mapsto \rho(g)v \in V$ is continuous. One says: "(ρ, V) is a representation", or simply "let ρ be a representation". But this notion is not sufficient in applications: if G is an algebraic group (over $\mathbf{Q}$ say), then $G(\mathbf{R})$ has a natural structure of a Lie group in which case the notion of $(\mathfrak{g}, K)$-module is important. On $G(\mathbf{Q}_p)$ for $p \geq 2$, one is led to consider also "smooth" representations. On $G(\mathbb{A}_{\mathbf{Q}})$, the notion of "automorphic representation" has at least three interpretations. The point of this section is to provide some background and references on this topic. We chose to minimize the amount of references, but all that follows can be found in any serious book on the subject.

3.1. Let $\mathfrak{H}$ be a Hilbert space. A *unitary* representation is a representation $(\rho, \mathfrak{H})$ such that $\rho(g)$ is unitary for any g in G. Examples:

(1) Given $f \in L^2(G)$ (dg here is a right Haar measure), put

$$R(g)(f)(h) := f(hg)$$

Then $(L^2(G), R)$ is a unitary representation of G, called the right regular representation. Indeed, let $f_1, f_2 \in L^2(G)$, $x, y \in G$. As $\|R(x)f_1 - R(y)f_2\|_2 \leq \|f_1 - f_2\|_2 + \|R(xy^{-1})f_2 - f_2\|_2$, it suffices to prove that for any $f \in L^2(G)$:

$$\lim_{x\to e} \|R(x)f - f\|_2 = 0$$

By using exactly the same argument, and approximating f with compactly supported continuous φ, it suffices to do it for φ instead of f, and this is trivial.

One can extend this idea to the general situation, and prove using the uniform boundedness theorem that if for any v in a fixed dense subset of $\mathfrak{H}$ and $w \in \mathfrak{H}$, the mapping $g \mapsto \langle \rho(g)v, w\rangle$ is continuous then ρ is a representation: see [Wal] lemma 1.1.3, [War] proposition 4.2.2.1, [Ro] chapter 13.

(2) With exactly the same proof, the right action of G on $L^2(H\backslash G)$ (H is a closed subgroup of G, both of which are unimodular say) provides a representation.

(3) If G is compact, and $(\rho, \mathfrak{H})$ a representation, one can show that one can put an inner product on $\mathfrak{H}$, without changing the topology, so that ρ becomes unitary: see lemma 1.4.8 of [Wal] (the idea is to average over G the inner product of course).

One says that $(\rho, \mathfrak{H})$ is *irreducible* if $\mathfrak{H}$ has no closed nontrivial G-invariant proper subspaces. When a representation is not irreducible, it may (or may not) be a Hilbert sum of irreducible subrepresentations. Two unitary representations $(\rho, \mathfrak{H}), (\rho', \mathfrak{H}')$ are *equivalent* if there exist a G-equivariant linear homeomorphism between $\mathfrak{H}$ and $\mathfrak{H}'$: it can be shown that such an isomorphism can be chosen to be an isometry (cf. [Bo2], 5.2). Note that if $(\rho, \mathfrak{H})$ is an irreducible unitary representation of G, then $\mathrm{span}(\rho(g)v \ : \ g \in G)$ is dense in $\mathfrak{H}$. This implies that the Hilbert dimension of $\mathfrak{H}$ is less than $\mathrm{card}(G)$, and therefore the *set* of unitary irreducible representations of G up to equivalence (or isomorphism) is a well defined object: it is denoted $\widehat{G}$.

Theorem 3.1 (Schur's lemma). *If $(\rho, \mathfrak{H})$ is irreducible, then $\mathrm{Hom}_G(\mathfrak{H}, \mathfrak{H}) = \mathbf{C}\mathrm{Id}_{\mathfrak{H}}$. Furthermore, if $(\rho', \mathfrak{H}')$ is another (not necessarily irreducible) unitary representation, then any nonzero element of $\mathrm{Hom}_G(\mathfrak{H}, \mathfrak{H}')$ is a positive real scalar multiple of an isometry.*

Reference: [Wal] section 1.2, [KL] proposition 10.14.

Application: Let Z denote the center of G, and let $(\pi, \mathfrak{H})$ denote an irreducible unitary representation of G. Then Schur's lemma implies that the action of Z on $\mathfrak{H}$ is by a unitary character; i.e. there exists a continuous character $\omega_\pi : Z \to \mathbf{S}^1$ such that $\pi(z)x = \omega_\pi(z)x$ for any $x \in \mathfrak{H}$: this is called the *central character* of π.

Remark: Let $(\mathfrak{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space. A convenient way to check that a unitary representation $(\rho, \mathfrak{H})$ is irreducible is to prove that any G-invariant continuous inner product $\langle\cdot,\cdot\rangle_2$ on $\mathfrak{H}$ is a multiple of $\langle\cdot,\cdot\rangle$. Indeed, if ρ contains a nonzero invariant closed proper subspace $\mathfrak{H}_0$ then under the decomposition $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_0^\perp$ we can change the inner product on $\mathfrak{H}_0^\perp$ by positive scalars while leaving the one on $\mathfrak{H}_0$ unchanged and this preserves the G-invariance property. But such a modification inner product on $\mathfrak{H}$ is clearly not a scalar multiple of the given one, so no such $\mathfrak{H}_0$ exists.

Conversely, if $(\rho, \mathfrak{H})$ is irreducible, and $\langle\cdot,\cdot\rangle_2$ is a G-invariant inner product on $\mathfrak{H}$, let $\mathfrak{H}'$ be the Hausdorff completion of $(\mathfrak{H}, \langle\cdot,\cdot\rangle_2)$: the natural embedding $(\mathfrak{H}, \langle\cdot,\cdot\rangle) \to (\mathfrak{H}', \langle\cdot,\cdot\rangle_2)$ is continuous, G-equivariant, with dense image. By Schur's lemma, it is a scalar multiple of an isometry: this proves our contention.

Lemma 3.1. *Let $(\rho, \mathfrak{H})$ be a unitary representation of G, and let $\mathfrak{H}'$ be a closed G-invariant subspace. If $(\rho, \mathfrak{H})$ is a Hilbert sum of irreducible representations, then so are $(\rho, \mathfrak{H}')$ and $(\rho, \mathfrak{H}/\mathfrak{H}')$.*

Proof: Using duality and/or orthogonal complements, it suffices to treat $\mathfrak{H}/\mathfrak{H}'$. Let's write:

$$\mathfrak{H} = \widehat{\bigoplus_{i \in \mathrm{I}}} \mathfrak{H}_i$$

where $\mathfrak{H}_i$ is an irreducible closed G-subspace of $\mathfrak{H}$ (the set of index I is at most countable if $\mathfrak{H}$ is separable, which will be the case in all our applications). We can also assume $\mathfrak{H}/\mathfrak{H}' \neq 0$.

The projection p onto $\mathfrak{H}/\mathfrak{H}'$ is G-equivariant, so $\mathfrak{H}/\mathfrak{H}'$ is spanned (in the Hilbert sense) by the $p(\mathfrak{H}_i)$ $(i \in I)$. In particular, some $p(\mathfrak{H}_i)$ is nonzero. But this projection is a closed G-invariant subspace of $\mathfrak{H}/\mathfrak{H}'$, so the set $\mathscr{X}$ of collections of pairwise orthogonal closed G-invariant irreducible subspaces of $\mathfrak{H}/\mathfrak{H}'$ is non-empty. By Zorn's Lemma there is a maximal element in $\mathscr{X}$, and the corresponding Hilbert direct sum is a closed G-invariant subspace W of $\mathfrak{H}/\mathfrak{H}'$. We just have to rule out the possibility that it is a proper subspace. If so, then clearly its orthogonal complement (in $\mathfrak{H}/\mathfrak{H}'$) contains no closed irreducible G-invariant subspace, so by replacing $\mathfrak{H}'$ with the preimage in $\mathfrak{H}$ corresponding to W we arrive at the case when the nonzero $\mathfrak{H}/\mathfrak{H}'$ contains no irreducible G-invariant closed subspaces. It has already been seen that such a situation cannot occur. QED

3.2. In some common situations, unitary representations are Hilbert sums of irreducibles representations: this is the content of the next theorems.

Theorem 3.2. *Let G be a compact group. Then any unitary representation is a Hilbert sum of irreducible representations. Furthermore any irreducible representation is finite dimensional.*

References: [Wal] prop. 1.4.1 and 1.4.2; [Ro] chapter 5 or the excellent [BR] chapter 7 for instance.

Remark: Let $(\rho, \mathfrak{H})$ be a unitary representation of G, and K be a compact subgroup. One can therefore write:

$$\mathfrak{H} = \widehat{\bigoplus_{i \in \mathrm{I}}} \mathfrak{H}_i$$

where each $\mathfrak{H}_i$ is a K-irreducible closed subspace of $\mathfrak{H}$. This decomposition is not unique (think of the trivial representation, for which any Hilbert basis provides such a decomposition), and two $\mathfrak{H}_i$'s may be unitarily K-equivalent. One usually rewrites the decomposition as follows: for each (isomorphism class of) irreducible representation π of K , let I_π be the set of $i \in \mathrm{I}$ for which $(\rho_{|_K}, \mathfrak{H}_i)$ is equivalent to π. The cardinal number $m_\pi = \mathrm{card}(\mathrm{I}_\pi)$ is the *multiplicity* of π in $\rho_{|_K}$: by Schur lemma, this cardinal number is independent of the decomposition we started with. One writes $\mathfrak{H}(\pi) = \bigoplus_{i \in \mathrm{I}_\pi} \mathfrak{H}_i = m_\pi \rho$, and the above Hilbert sum is written:

$$\mathfrak{H} = \widehat{\bigoplus_{\pi \in \widehat{K}}} \mathfrak{H}(\pi) = \widehat{\bigoplus_{\pi \in \widehat{K}}} m_\pi \pi.$$

One says that ρ is K-*admissible* if m_π is a finite cardinal for each $\pi \in \widehat{K}$. We'll see later on that any irreducible unitary representation of a connected reductive group is admissible (for K a maximal compact subgroup in the archimedean case, and maximal compact open subgroup in the non-archimedean case).

3.3. The next examples require the use of the integration in topological vector spaces. A thorough treatment can be found in Bourbaki, Integration, chap VI, §1,2 and chap VII, §2 for the application on representations; [War] section 4.1.1; [Ro] section 6 for some comments. Let $(\pi, \mathfrak{H})$ be a unitary representation of a locally compact group G (so $\mathfrak{H}$ is a Hilbert space, though to integrate continuous vector-valued functions it suffices to assume that $\mathfrak{H}$ is locally convex and quasi-complete). Let $f \in \mathcal{C}_c(G)$, $v, w \in \mathfrak{H}$, one can consider the absolutely converging integral:

$$l_v(w) := \int_G f(g) \langle \pi(g)v, w \rangle \, dg$$

The mapping $w \mapsto l_v(w)$ is continuous and linear, therefore by Riesz' representation theorem it defines an element of $\mathfrak{H}$ denoted

$$\pi(f)v := \int_G f(g)\pi(g)v dg.$$

It is clearly linear in f and v, continuous as $\|\pi(f)v\| \leq \|f\|_1 \|v\|$ and can be extended by density to $L^1(G)$ (actually even to the space of compactly supported complex measures, cf. Bourbaki): in particular, one checks easily that $f \in L^1(G) \mapsto \pi(f) \in \mathrm{End}(\mathfrak{H})$ is a continuous homomorphism of Banach algebras.

Remark: It is sometimes convenient to consider a continuous function f whose support is contained in a compact subgroup K of G. If K is negligible in G, then $\pi(f)$ as defined above is zero. However, the same arguments shows that the integral $\int_K f(k)\pi(k)vdk$ is absolutely convergent: by an abuse of notations, we will denote this integral $\pi(f)v$. As soon as the Haar measures are suitably normalized, this defines the same operator in the case K is also open, so we hope this won't cause any confusion.

Theorem 3.3. *Let $(\pi, \mathfrak{H})$ be a unitary representation of G. Assume the existence of a delta-sequence $(f_n)_{n\in\mathbf{N}}$ in $\mathcal{C}_c(G)$, i.e. satisfying:*

$$\mathrm{supp}(f_{n+1}) \subset \mathrm{supp}(f_n), \quad \bigcap_{n\geq 1} \mathrm{supp}(f_n) = \{e\}$$

$$\forall n \in \mathbf{N},\ \forall g \in G,\ f_n(g) = f_n(g^{-1}), f_n \geq 0, \int_G f_n = 1$$

such that the operator $\pi(f_n)$ is compact for all n. Then $(\pi, \mathfrak{H})$ is a Hilbert sum of irreducible representations, each occuring with finite multiplicities.

References: [Wal] proposition 1.4.1, [L] I §3. Note that the invariance of the f_n under $g \mapsto g^{-1}$ insures that $\pi(f_n)$ is self-adjoint: the proof uses the spectral decomposition of such operators.

Remark: This theorem is fundamental in the theory of automorphic forms: the most common proofs that the space of cusp forms splits as a sum of irreducible representations is based on it – though Jacquet-Langlands seem to have a purely algebraic proof of this fact.

Remark: Let G be a locally compact group. One says that G (actually its stellar algebra: see [Dix], 13.9) is *liminal* if for any $(\pi, \mathfrak{H})$ irreducible unitary representation of G, and any $f \in \mathcal{C}_c(G)$, $\pi(f)$ is compact. We'll see later that all reductive groups over locally compact fields are liminal, and to what extent this plays a role in the tensor product theorem.

Theorem 3.4. *Let G be a locally compact group, K a compact subgroup of G, and $(\pi, \mathfrak{H})$ a unitary representation of G. Assume that π is K-admissible. Then there exists a delta-sequence satisfying the condition of the previous theorem, and therefore $(\pi, \mathfrak{H})$ splits as a Hilbert direct sum of irreducible representations.*

Proof: (cf. [Bo2], 5.9 corollaire) If $\rho \in \widehat{K}$ occurs in π, denote its character χ_ρ: by hypothesis $\pi(\chi_\rho)$ is compact. As any central function f of K is a uniform limit of linear combinations of characters (cf. [Ro], 7.1, proposition), so $\pi(f)$ is compact as well (the subspace of compact operators is closed in $\mathrm{End}(\mathfrak{H})$ for the topology of uniform convergence on bounded sets). To conclude, one uses a delta-sequence made of central functions (by averaging over K of course), and one applies the previous theorem. qed

4. The case of reductive groups

In this section, let G be a reductive algebraic group over a local field F (say $F = \mathbf{R}$ or $\mathbf{Q}_p$ for some prime p). One denotes $\mathfrak{g}$ its Lie algebra. Let K be a compact subgroup of $G(F)$ such that:

- if F is archimedean, K is a maximal compact subgroup of $G(F)$ (e.g. $K = \mathbf{O}_2(\mathbf{R})$ if $G = \mathbf{GL}_2, F = \mathbf{R}$)
- if F is non-archimedean, K is open (e.g. $K = \mathbf{GL}_2(\mathbf{Z}_p)$ if $G = \mathbf{GL}_2, F = \mathbf{Q}_p$)

Theorem 4.1. *Let $(\pi, \mathfrak{H})$ be an irreducible unitary representation of $G(F)$. Then $(\pi, \mathfrak{H})$ is K-admissible.*

References: for F archimedean, cf. [Wal] theorem 3.4.10, [Bo2] théorème 5.27. In the non-archimedean case, it is quoted by Cartier in [Cor] and it is discussed in the unpublished notes of Garrett [Ga1]. As we are mainly interested in the case of $G = \mathbf{GL}(2)$, refer to [Su] theorem 5.1. for the real case, and to [BH] in the p-adic case, where the smooth representations are completely classified – so that one is left to observe the admissibility.

Corollary 4.1.1. *Let $(\pi, \mathfrak{H})$ be an irreducible unitary representation of $G(F)$, and $f \in \mathcal{C}_c(G)$. Then $\pi(f)$ is compact.*

References: Théorème 5.27 in [Bo2] for the real case. In the p-adic case, $\mathcal{C}_c^\infty(G)$ is dense in $\mathcal{C}_c(G)$ (for its natural inductive limit topology, stronger than the uniform convergence): as the subspace of compact operators is closed in $\mathrm{End}(\mathfrak{H})$, it suffices to prove the claim for $f \in \mathcal{C}_c^\infty(G)$. But for such an f, it is immediate that one can find a compact open subgroup K_f of G such that $f(kgk^{-1}) = f(g)$ for any $g \in G$, $k \in K_f$, in which case one concludes as in the proof of theorem 3.4.

Remark: This proves that the (stellar algebra of) reductive groups are liminal, as claimed above.

Before we state the next corollary, which will be useful in our discussion of the tensor product theorem, let's recall that given two Hilbert spaces $\mathfrak{H}_1, \mathfrak{H}_2$, the bilinear map induced by

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle := \langle x_1, y_1 \rangle_1 \langle x_2, y_2 \rangle_2$$

provides $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ with a non-degenerate inner product, whose completion is denoted $\mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2$: cf [Bour-EVT], chap V, §3, No 1 and 2. Let G_1, G_2 be two locally compact groups, and let $(\pi_i, \mathfrak{H}_i)$ $(i = 1, 2)$ be two unitary representations. Then $\pi_1 \widehat{\otimes} \pi_2$ denotes the unitary representation of $G_1 \times G_2$ on $\mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2$ deduced from the representation on the pre-Hilbert space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, itself induced by:

$$(\pi_1 \otimes \pi_2)(g_1, g_2)(x_1 \otimes x_2) = \pi_1(g_1)x_1 \otimes \pi_2(g_2)x_2.$$

Let's briefly justify this is a unitary representation: first, for any $g_1 \in G_1, g_2 \in G_2$, the operator $\pi_1(g_1) \otimes \pi_2(g_2)$ is unitary ([Bour-EVT], V, §4, No 1, proposition 3 and the paragraph following proposition 2); as for the continuity, given that $G = G_1 \times G_2$ acts by unitary operators, it suffices to prove that the mappings $g \in G \mapsto \pi(g)v$ are continuous for v in a total subset of $\mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2$ (cf [War] section 4.1.1 page 219): if we take this total subset to be $\{x_1 \otimes x_2 \ : \ x_1 \in \mathfrak{H}_1, x_2 \in \mathfrak{H}_2\}$, our contention is clear.

It is easy to see that $\pi_1 \widehat{\otimes} \pi_2$ is irreducible if π_1, π_2 are. Indeed, let Q be a $G_1 \times G_2$-invariant continuous inner product on $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. Fix two nonzero vectors $x_2, y_2 \in \mathfrak{H}_2$: then the inner product on $\mathfrak{H}_1$ defined by $(x_1, y_1) \mapsto Q(x_1 \otimes x_2, y_1 \otimes y_2)$ is continuous and G_1-invariant, so is equal to $\langle \cdot, \cdot \rangle_1$ up to some constant by irreducibility of π_1. One determines the constant the same way, by varying x_2, y_2. Conversely:

Corollary 4.1.2. *Let G_1, G_2 be two reductive groups over two local fields (maybe distinct). Then any irreducible representation π of $G = G_1 \times G_2$ is equivalent to a representation of the shape $\pi_1 \widehat{\otimes} \pi_2$, with π_1, π_2 irreducible.*

Reference: [Dix] proposition 13.1.8, where it is proven that if at least one of G_1, G_2 are of type 1, then the conclusion holds (see also [GGP] appendix to chapter 2 and [Ro], section 20). It can be shown (cf. [Dix], theorem 5.5.2 and 13.9.4) that a group is of type 1 as soon as its stellar algebra is liminal, which is the case here, by the corollary 4.1.1.

Remark: The previous corollary was stated only in the case of reductive groups: it is of course true in the generality of type 1 groups, as the references justify it.

4.1. **Smooth vectors and $(\mathfrak{g}, K)$-modules.** References: [Bu] chapter 2, [Wal1], and [Wal] chapter 3. As we mentionned earlier, there are also more algebraic counterparts of representation theory. In the case of archimedean Lie groups, $(\mathfrak{g}, K)$-modules play an important role. Let G be an archimedean reductive Lie group, $\mathfrak{g}$ its complex Lie algebra, K a maximal compact subgroup.

One can attach canonically to $\mathfrak{g}$ an associative unitary algebra $\mathcal{U}(\mathfrak{g})$ called the (complexified) universal enveloping algebra, which gives rise to differential operators acting on $\mathcal{C}_c^\infty(G)$. We will denote $\mathfrak{z}$ the center of $\mathcal{U}(\mathfrak{g})$ (if G is of inner type, this is also the set of elements z in $\mathcal{U}(\mathfrak{g})$ such that $\mathrm{Ad}(g)z = z$ for any g in G, cf [Wal] 3.4.1: this is the case for $\mathbf{GL}_n(\mathbf{R})$), which is finitely generated, and generalizes the Laplace-Beltrami operator: cf [Wal] section 0.4.

A $(\mathfrak{g}, K)$-*module* is a complex vector space V (without topology), together with

- a structure of a K-module, continuous in the following sense: if $v \in V$, then there exists a finite dimensional subspace W_v such that $Kv \subset W_v$ and the mapping $K \to \mathrm{Aut}(W_v)$ is continuous (therefore analytic),
- a structure of a $\mathfrak{g}$-module,

such that:

(1) $k \cdot X \cdot v = (\mathrm{Ad}(k)X) \cdot k \cdot v$ for $k \in K, X \in \mathfrak{g}, v \in V$,
(2) $\frac{d}{dt}(\exp(tX)v)_{|t=0} = Xv$ for $v \in V$ and X in the Lie algebra $\mathfrak{k}$ of K.

In these conditions, one can prove that V is a semisimple K-module (cf. [Wal] lemma 3.3.3). The $(\mathfrak{g}, K)$-module V is *admissible* if the ρ-isotypic subspace $V(\rho)$ is finite-dimensional for any $\rho \in \widehat{K}$.

Fundamental example: Let $(\pi, \mathfrak{H})$ be a unitary representation of G. Let $\mathfrak{H}^\infty$ be the subspace of smooth vectors (i.e. the vectors $v \in \mathfrak{H}$ such that $g \in G \mapsto \pi(g)v$ is smooth). The real Lie algebra $\mathfrak{g}_\mathbf{R}$ acts on $\mathfrak{H}^\infty$ by $d\pi(X)v =$

$\frac{d}{dt}(\exp(tX)v)_{|t=0}$ [†], hence an action of $\mathfrak{g}$. Gårding's theorem states that $\mathfrak{H}^\infty$ is dense in $\mathfrak{H}$ (exercise: use a δ-sequence of smooth functions to prove it, cf [Wal] section 1.6). Write $\mathfrak{H} = \widehat{\bigoplus}_{\rho\in\widehat{K}}\mathfrak{H}(\rho)$: one can prove that $\mathfrak{H}^\infty \cap \mathfrak{H}(\rho)$ is dense in $\mathfrak{H}(\rho)$ for any $\rho \in \widehat{K}$. Define $\mathfrak{H}_K := \bigoplus_{\rho\in\widehat{K}}(\mathfrak{H}(\rho)\cap\mathfrak{H}^\infty)$. Then $\mathfrak{H}_K$ is stable under the action of $K, \mathfrak{g}$, satisfies the aforementioned compatibilities and is called the $(\mathfrak{g}, K)$-module *associated* to the unitary representation $(\pi, \mathfrak{H})$. By construction, $\mathfrak{H}_K$ is dense in $\mathfrak{H}$. One can prove that $\mathfrak{H}_K$ is irreducible (=contains no algebraic submodule) if and only if the representation π is (topologically) irreducible, thanks to this density: cf. [Wal], theorem 3.4.11 – this uses the admissibility of π.

Remark: Note that if $(\pi, \mathfrak{H})$ is admissible, as $\mathfrak{H}^\infty \cap \mathfrak{H}(\rho)$ is dense in $\mathfrak{H}(\rho)$, it must be *equal* to it. This implies that given an irreducible (necessarily admissible) unitary representation of a reductive group G, its associated $(\mathfrak{g}, K)$-module is actually $\bigoplus_{\rho\in\widehat{K}}\mathfrak{H}(\rho)$, and that the K-finite vectors of $\mathfrak{H}$ are smooth.

Remark: A $(\mathfrak{g}, K)$-module does not afford a representation of G. However, one can define an "extension" of G, called the *Hecke algebra* and denoted $\mathcal{H}_G$, such that $(\mathfrak{g}, K)$-modules correspond naturally to $\mathcal{H}_G$-modules: see [Bu] proposition 3.4.4.

Remark: There is a version of the Schur lemma for irreducible $(\mathfrak{g}, K)$-modules: cf [Wal] lemma 3.3.2. One can say a bit more in the case of an irreducible unitary representation of $G(\mathbf{R})$ for reductive G: the center of the universal algebra $\mathfrak{z}$ acts on $\mathfrak{H}^\infty$ by a character (here this means an homomorphism of $\mathbf{C}$-algebras $\chi : \mathfrak{z} \to \mathbf{C}$), this is the content of lemma 1.6.5 of [Wal].

Remark: The complex conjugation on $\mathfrak{g}$ extends to an conjugate-linear anti-automorphism on $\mathcal{U}(\mathfrak{g})$ (cf [Wal] 1.6.5) denoted $x \mapsto x^\star$. The proof of lemma 1.6.5 (ibid.) implies that if $x \in \mathcal{U}(\mathfrak{g})$, then for any $v, w \in \mathfrak{H}$, with $(\pi, \mathfrak{H})$ unitary representation of G, $\langle d\pi(x)v, w\rangle = \langle v, d\pi(x^\star)w\rangle$. In particular if $x = x^\star$, then $d\pi(x)$ is self-adjoint. This applies to the Laplace-Beltrami operator Δ of $\mathbf{SL}_2(\mathbf{R})$ acting for example on $L^2(\mathbf{SL}_2(\mathbf{Z})\backslash\mathbf{SL}_2(\mathbf{R}))$, giving a representation-theoritic proof of such self-adjointness in this case, usually proved by Green's identity, cf [Bu] section 2.1.

Remark: About K and $\mathfrak{z}$-finiteness, useful in the context of automorphic forms. If $(\pi, \mathfrak{H})$ is a unitary representation, a vector v is K*-finite* if $\pi(K)v$ is finite dimensional: this makes sense for any vector in the representation. If v is a smooth vector, then v is $\mathfrak{z}$-finite if $d\pi(\mathfrak{z})v$ is finite dimensional. However, it is technically important to define it for non-smooth vectors as well: this is

[†] The limit in consideration is with respect to the norm of $\mathfrak{H}$: when $\mathfrak{H}$ is a space of functions, the derivative can also taken with respect to the pointwise convergence, which may not be coherent with the latter. For instance, the smooth vectors in $L^2(\mathbf{R})$ is not $\mathcal{C}^\infty(\mathbf{R})$!

way distributions play an important role in the theory of automorphic forms, often implicitly. In this context, a vector $v \in \mathfrak{H}$ defines a (vector-valued) distribution $T_v : \mathcal{C}_c^\infty(G) \to \mathfrak{H}$ by:

$$T_v(\varphi) = \int_G \varphi(g)\pi(g)v dg.$$

One says that v is $\mathfrak{z}$-finite (as a distribution) if $\mathrm{span}\{zT_v : z \in \mathfrak{z}\}$ is a finite-dimensional subspace of $\mathfrak{H}$-valued distributions, where xT_v is the distribution defined for $x \in \mathcal{U}(\mathfrak{g})$ by:

$$(xT_v)(\varphi) := \int_G (\varphi * \check{x})(g)\pi(g)dg.$$

(Here and below, for $x \in \mathcal{U}(\mathfrak{g})$ the notation $\varphi * \check{x}$ denotes the action of x on $\mathcal{C}_c^\infty(G)$ arising from the action of $\mathfrak{g}$ via differential operators.) Note that if v is smooth then $xT_v = T_{d\pi(x)v}$ and that in the case where $(\pi, \mathfrak{H})$ is the right regular representation of $L^2(G)$, $f \in L^2(G)$ is $\mathfrak{z}$-finite in the above sense if and only if the real valued representations $\varphi \mapsto \int_G (\varphi * \check{z}) g f(g) dg$ span, when z varies in $\mathfrak{z}$, a finite dimensional subspace of (real valued) distributions (by using the right regular representation on $\mathcal{C}_c^\infty(G)$).

4.2. **Smooth representations of non-archimedean groups.** References [BH] chapter 1, [Bu] chapter 4 for a thorough discussion of this topic. This is the p-adic counterpart of the preceeding paragraph. Let G be a totally disconnected locally compact group, K an open compact subgroup. A *smooth representation* of G is a vector space V together with an group homomorphism $\pi : G \to \mathrm{Aut}(V)$ such that

(1) any $v \in V$ is smooth, i.e. the subgroup $\{g \in G \ : \ \pi(g)v = v\}$ is compact and open in G.

In this situation, the restriction of the representation π to K is semisimple (cf. [BH] lemma 2.2). It is said to be *admissible* if furthermore the space of K-fixed vectors V^K is finite dimensional: this implies that one can write

$$V = \bigoplus_{\rho \in \widehat{K}} V(\rho)$$

where each $V(\rho)$ is finite-dimensional.

Fundamental example: Let $(\pi, \mathfrak{H})$ be a unitary representation of G; denote by $\mathfrak{H}^\infty$ the subspace of smooth vectors in $\mathfrak{H}$, which is stable under G. Then the corestriction of π to $\mathfrak{H}^\infty$ is a smooth representation of G.

Note that $\mathfrak{H}^\infty$ is dense in $\mathfrak{H}$: indeed, if $v \in V$, then $\pi(f)v \in \mathfrak{H}^\infty$ for any $f \in \mathcal{C}_c^\infty(G)$. Let $\mathfrak{C}$ be the filter generated by open and compact neighbourhoods of the identity and let $f_\mathfrak{c}$ be the characteristic function of $\mathfrak{c} \in \mathfrak{C}$: then $\pi(f_\mathfrak{c})v \to_\mathfrak{C} v$, as claimed. This density implies that given an admissible unitary representation $(\pi, \mathfrak{H})$, then $(\pi, \mathfrak{H})$ is irreducible if and only if $(\pi, \mathfrak{H}^\infty)$ is algebraically irreducible.

Remark: Smooth representations of G are in one-to-one correspondence with smooth representations of the Hecke algebra of G: cf [BH] section 1.

Case of $\mathbf{GL}_2$: spherical representations. Here F denotes a non-archimedean field, and $\mathcal{O}_F$ its integers with maximal ideal $\mathfrak{p}_F$, ϖ a uniformizer and q_F the cardinality of the residue field. As we mentioned earlier, the smooth irreducible representations of $\mathbf{GL}_2(F)$ are classified (up to equivalence), and fall into three families: principal series, special representations and supercuspidals (see [BH]) . We will need later a few facts on unramified representations (as defined in the following result):

Theorem 4.2. *Let (π, V) be a smooth irreducible representation of $\mathbf{GL}_2(F)$ which is unramified in the sense that it contains nonzero spherical vectors; i.e.,*

$$V^{\mathbf{GL}_2(\mathcal{O}_F)} := \{v \in V \ : \ \pi(k)v = v \text{ for all } k \in \mathbf{GL}_2(\mathcal{O}_F)\} \neq \{0\}$$

Then (π, V) is equivalent to an unramified principal series representation, and furthermore the space $V^{\mathbf{GL}_2(\mathcal{O}_F)}$ is one-dimensional.

This means that $\pi \cong \pi(\chi_1, \chi_2)$ for some unramified quasi-characters of $F^\times$. The one-dimensionality result comes from the fact that the spherical Hecke algebra is commutative (cf [Bu] theorem 4.6.2)

A natural question is, given a unitary irreducible representation π of $\mathbf{GL}_2(F)$, how to determine the characters χ_1, χ_2 from π: this leads to the introduction of Hecke operators in this local setting.

First of all, an unramified quasi-character χ of F can be written $\chi(x) = |x|^t$, for some $t \in \mathbf{C}$ (uniquely determined modulo $2i\pi \log(q_F)^{-1}\mathbf{Z}$). As it is known that $\pi \cong \pi(\chi_1, \chi_2)$ and $\pi \cong \pi(\chi_2, \chi_1)$ are unitarily equivalent, it suffices to determine the set $\{q_F^{t_1}, q_F^{t_2}\}$ of complex numbers (here the t_i's are actually imaginary, as the quasi-characters χ_1, χ_2 are unitary). By looking at the central characters, one has for any $x \in F^\times$:

$$\omega_\pi(x) = \chi_1(x)\chi_2(x)$$

so this gives a condition on $q_F^{t_1} q_F^{t_2}$.

To determine completely our set, it suffices to get a condition on the sum $q_F^{t_1} + q_F^{t_2}$. Let φ_0 denote the characteristic function of the (compact and open) subset $\mathbf{GL}_2(\mathcal{O}_F)\left[\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right]\mathbf{GL}_2(\mathcal{O}_F)$; then for any vector v, $\pi(\varphi_0)v$ is spherical (if non-zero). Thus, in our setting, if one denotes v_0 a non-zero spherical vector in $\pi(\chi_1, \chi_2)$, $\pi(\varphi_0)v$ and v are colinear, more precisely – and this will end our discussion:

$$\pi(\varphi_0)v = q_F^{1/2}(q_F^{t_1} + q_F^{t_2})v.$$

For a proof, see for example [Bu] proposition 4.6.6: the standard notation for the operator $\pi(\varphi_0)$ is $T(\mathfrak{p}_F)$ or $T_{\mathfrak{p}_F}$ – we'll see later that the Hecke operators introduced in section 2.3 "correspond" to these $T(\mathfrak{p}_F)$'s once the adeles are introduced.

5. The adelization of a modular form and adelic Hecke operators

In this section, we use the same notation as in section 2.

If G is an algebraic group over $\mathbf{Q}$ (we'll use $G = \mathbf{GL}_2$ only), we'll denote $g = (g_v)_{v\leq\infty}$ an element of $G(\mathbb{A}_\mathbf{Q})$. Given a place v of $\mathbf{Q}$, and an element $g_v \in G(\mathbf{Q}_v)$, we'll denote also g_v the element of $G(\mathbb{A}_\mathbf{Q})$ whose component at v is g_v, and at $w \neq v$ is 1. We'll denote sometimes g_f the "finite" part of g (i.e. $(g_f)_\infty = 1$ and $(g_f)_p = g_p$ for p prime). For $G = \mathbf{GL}_2$, we recall that a maximal compact subgroup of $G(\mathbb{A}_\mathbf{Q})$ is $K = \prod_{v\leq\infty} K_v$, with $K_\infty = \mathbf{O}_2(\mathbf{R})$ and $K_p = \mathbf{GL}_2(\mathbf{Z}_p)$ for p prime; the center of $G(\mathbb{A}_\mathbf{Q})$ will always be denoted $Z(\mathbb{A}_\mathbf{Q})$. If $N \in \mathbf{N}$, we'll denote $K_0(N)$ the subgroup of K_f made of matrices whose lower left entry is in $N\widehat{\mathbf{Z}}$ (so the component at infinity is 1).

5.1. From a classical modular form to an automorphic form on $\mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$.

References: [KL] section 12.2, [G] §3, [Bu] section 3.6, [BCSGKK] section 7.

Let f be a modular form of weight k, nebentypus χ and level N. The Dirichlet character χ is associated with a finite order idele class character of $\mathbb{A}_\mathbf{Q}^\times/\mathbf{Q}^\times$ denoted ω_χ called the adelization of χ: see [KL] section 12.1 for its construction. The strong apprimation theorem states that:

$$\mathbf{GL}_2(\mathbb{A}_\mathbf{Q}) = \mathbf{GL}_2(\mathbf{Q})\mathbf{GL}_2^+(\mathbf{R})K_0(N).$$

This means that any element g in $\mathbf{GL}_2(\mathbb{A})$ can be (non-uniquely) written $g = \gamma h_\infty k$, with $\gamma \in \mathbf{GL}_2(\mathbf{Q}), h_\infty \in \mathbf{GL}_2^+(\mathbf{R}), k \in K_0(N)$ (in other words the continuous map $\mathbf{GL}_2(\mathbf{Q})\times\mathbf{GL}_2^+(\mathbf{R})\times K_0(N) \to \mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$ is surjective). See [KL] section 6.3 for an elementary proof in this setting.

One can prove as a consequence that $\mathrm{vol}(Z(\mathbb{A}_\mathbf{Q})\mathbf{GL}_2(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A}_\mathbf{Q})) < \infty$: cf [KL] section 7.11. We will still denote ω_χ the character on $K_0(N)$ defined by the evaluation of ω_χ at the lower right entry.

Definition 5.1. *Let f be a modular form of weight k, nebentypus χ and level N. The* adelization *of f is the function $\varphi_f : \mathbf{GL}_2(\mathbb{A}_\mathbf{Q}) \to \mathbf{C}$ defined by:*

$$\varphi_f(g) = j(h_\infty, i)^{-k} f(h_\infty \cdot i)\omega_\chi(k)$$

where:

1. *$h_\infty \in \mathbf{GL}_2^+(\mathbf{R}), k \in K_0(N)$ are chosen so that $g = \gamma h_\infty k$ for some $\gamma \in \mathbf{GL}_2(\mathbf{Q})$,*
2. *for any $z \in \mathbf{C} - \mathbf{R}$, $j(h_\infty, z) = \det(h_\infty)^{-1/2}(cz + d)$, if one write $h_\infty = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$*

Remarks:

- This is well defined (i.e. the number $\varphi_f(g)$ does not depend on the choices of γ, h_∞, k) because of the modularity of f: see section 12.2 of [KL].
- The function φ_f is continuous. Indeed, its restriction to the (open) subset $\gamma\mathbf{GL}_2^+(\mathbf{R})K_0(N)$ (for any $\gamma \in \mathbf{GL}_2(\mathbf{Q})$) is continuous by definition, so by the "gluing lemma" φ_f is continuous on $\mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$.
- For any $\gamma \in \mathbf{GL}_2(\mathbf{Q}), g \in \mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$, $\varphi_f(\gamma g) = \varphi_f(g)$.
- For a fixed finite adelic point g_f, $g_\infty \mapsto \varphi_f(g_\infty, g_f)$ is smooth.
- For a fixed $g_\infty \in \mathbf{GL}_2^+(\mathbf{R})$, $g_f \mapsto \varphi_f(g_\infty, g_f)$ is locally constant on the finite adeles.

The last three points are obvious.

As the title of this section indicates, the function φ_f is actually an automorphic form on $\mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$: we list below the properties statisfied by φ_f to inherit such a name: the proofs are to be found in [KL] or [G].

(1) ($\mathbf{GL}_2(\mathbf{Q})$-left invariance) For any $\gamma \in \mathbf{GL}_2(\mathbf{Q}), g \in \mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$, one has: $\varphi_f(\gamma g) = \varphi_f(g)$.

(2) (K-finiteness) For $k_\infty = \left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right] \in \mathbf{SO}_2(\mathbf{R}), k_f \in K_0(N), g \in \mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$, $\varphi_f(gk_\infty k_f) = \omega(k_f)\exp(2\pi ik\theta)\varphi_f(g)$, where $k \in \mathbf{Z}$ is the weight of f. In the adelic setting, the condition of *K-finiteness* on φ_f means that the subspace $\mathrm{span}(R(g)\varphi_f \ : \ g \in K)$ is finite-dimensional. The link with the classical setting is that all finite-dimensional continuous representations of the circle group $\mathbf{SO}_2(\mathbf{R}) = \mathbf{R}/(2\pi\mathbf{Z})$ are direct sums of 1-dimensional representations with the character $\theta \mapsto \exp(ik\theta)$ for various $k \in \mathbf{Z}$.

(3) ($\mathfrak{z}$-finiteness) One has the differential equation: $\Delta\varphi_f = \frac{k}{2}\left(1 - \frac{k}{2}\right)\varphi_f$ (where the Casimir operator Δ acts on the infinite component). This implies that φ is Δ-finite: this, and the next item, implies that φ is $\mathfrak{z}$-finite, as the center of the universal algebra is generated by Δ and I). In other words, the subspace $\mathrm{span}(\varphi_f * \check{z} \ : \ z \in \mathfrak{z})$ is finite dimensional.

(4) (Action of the center) For any $z \in Z(\mathbb{A}_\mathbf{Q}), g \in \mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$, $\varphi_f(zg) = \omega_\chi(z)\varphi_f(g)$.

(5) (Growth condition) For any norm $\|\cdot\|$ on $\mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$, there exists a real number $A > 0$ such that: $\varphi_f(g) \ll \|g\|^A$. In other words, φ_f is moderate growth. This point is not obvious: see [Bo1] section 5, Borel-Jacquet in [Cor] and [Wal] for norms on Lie groups. It is simpler to prove that if f is a cusp form, then φ_f is actually bounded: this is because of the basic fact that f is cuspidal if and only if the mapping $g_\infty \in \mathbf{GL}_2^+(\mathbf{R}) \mapsto j(g_\infty, i)^{-k} f(g_\infty \cdot i)$ is bounded: see [KL] proposition 12.2.

(6) (Cuspidality) If f is a cusp form, then φ_f is cuspidal, in the sense that for any $g \in \mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$:

$$\int_{\mathbf{Q}\backslash\mathbb{A}_\mathbf{Q}} \varphi_f\left(\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] g\right) dx = 0.$$

See [KL] proposition 12.3 for a proof (if the form is unramified, then [G] proves it as well, but for general levels, one has to use all the cusps).

We tried to list the properties so that they are easily modified to define an autmorphic form on a general reductive group G over $\mathbf{Q}$ (even over a number field); the cuspidality condition is more difficult to handle, as one has to write the vanishing condition on the unipotent radical of any parabolic $\mathbf{Q}$-subgroup of G.

It is important to notice that since φ_f is bounded for f cuspidal, $|\varphi_f|$ is square integrable on $Z(\mathbb{A})\mathbf{GL}_2(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$.

Remark: By using the strong approximation theorem, one can characterize the image of $\mathcal{S}_k(N,\chi)$ under this construction (which is clearly linear in f): refer to [KL] section 12.4.

5.2. **From a classical cuspidal modular form to a unitary automorphic representations of $\mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$.** We keep the same notations, and refer to [G] for more details that we won't cover (chapter 5 is especially relevant).

Definition 5.2. *Let f be a cuspidal modular form of weight k, nebentypus χ and level N. The unitary automorphic representation attached to f is the restriction of the right regular representation of $\mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$ on the closed subspace $\mathfrak{H}_f$ of $L^2_0(Z(\mathbb{A}_\mathbf{Q})\mathbf{GL}_2(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A}),\omega_\chi)$ defined by:*

$$\mathfrak{H}_f := \overline{\mathrm{span}(R(g)\varphi_f \ : \ g \in \mathbf{GL}_2(\mathbb{A}_\mathbf{Q}))}$$

This unitary representation is denoted π_f.

Remark: see the appendix for a definition of the space of cuspidal functions $L^2_0(Z(\mathbb{A}_\mathbf{Q})\mathbf{GL}_2(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A}),\omega_\chi)$.

Remark: We chose to work with unitary automorphic representations; if instead one wishes to work with the more algebraic theory of (admissible) automorphic representations, one can attach to a modular form f the $\mathcal{H}_{\mathbf{GL}_2}$-submodule $\mathcal{H}_{\mathbf{GL}_2}\varphi_f$ of the space of automorphic forms on $\mathbf{GL}_2$ with central character ω_χ: here $\mathcal{H}_{\mathbf{GL}_2}$ denotes the adelic Hecke algebra, which is a restricted tensor product of the local Hecke algebras – cf [Bu] section 3.4.

The main result is the following:

Theorem 5.1. *Let f be a cuspidal modular form of weight k, nebentypus χ and level N. Assume that there exists a finite set of primes S such that f is a Hecke eigenform for the T_p, $p \notin S$. Then the unitary representation π_f is irreducible.*

We will sketch the proof below – references: [Bu] section 3.6, and [G] section 5.B. We will need the tensor product theorem and multiplicty one to achieve that.

5.3. **Hecke operators.** Let f be a cuspidal modular form of weight k, nebentypus χ and level N, and φ_f its adelization. We denote ω the adelization of the Dirichlet character χ. Let p be a prime not dividing q (for simplicity).

Let H_p be the compact open subset of $\mathbf{GL}_2(\mathbf{Q}_p)$ defined by

$$H_p = \mathbf{GL}_2(\mathbf{Z}_p)\left[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right]\mathbf{GL}_2(\mathbf{Z}_p)$$

For $\varphi \in L^2_0(Z(\mathbb{A})\mathbf{GL}_2(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}}), \omega)$, we define:

$$\mathbb{T}_p(\varphi) = \int_{H_p} f(gk_p)dk_p.$$

By using the disjoint union decomposition:

$$H_p = \bigcup_{b=0}^{p-1} \left[\begin{smallmatrix} p & b \\ 0 & 1 \end{smallmatrix}\right]\mathbf{GL}_2(\mathbf{Z}_p) \cup \left[\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right]\mathbf{GL}_2(\mathbf{Z}_p)$$

one proves easily that (cf [G] lemma 3.7):

$$\mathbb{T}_p(\varphi_f) = \varphi_{p^{1-k/2}T_p(f)}.$$

On the other hand, by using the disjoint union decomposition:

$$H_p = \bigcup_{b=0}^{p-1} \mathbf{GL}_2(\mathbf{Z}_p)\left[\begin{smallmatrix} 1 & b \\ 0 & p \end{smallmatrix}\right] \cup \mathbf{GL}_2(\mathbf{Z}_p)\left[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right]$$

one sees that if φ is $\mathbf{GL}_2(\mathbf{Z}_p)$-invariant on the right then

$$\mathbb{T}_p(\varphi) = (p+1)\int_{\mathbf{GL}_2(\mathbf{Z}_p)} \varphi\left(gk_p\left[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right]\right)dk_p.$$

The last integral can be modified in order to adelize the ramified Hecke operators. Reference: [M2] and [We] chapter VI.

Remark: In [KL] section 13, it is explained how to use f to construct a smooth function ψ on $\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})$ as a product $\psi = \prod_v \psi_v$, where ψ_∞ is integrable modulo the center for weights ≥ 3 and the finite components are smooth and compactly supported modulo the center: this is technically important in order to use the (relative) trace formula.

6. The tensor product theorem

In this section, we collect some facts leading to the statement and proof of the tensor product theorem. Again, our choice is to deal with unitary representations, for two reasons:

- Automorphic representations are not representations of $\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})$, but of the Hecke algebra, which is difficult to define (see [Bu] section 3.4). It is simpler to define unitary representations in this context.

- Sooner or later in the theory, one really *needs* properties of Hilbert spaces, compact operators, trace class operators. The author is not sure to what extend a completely algebraic theory can do the job.

6.1. **A construction.** Let G denote the algebraic group $\mathbf{GL}_2$; all the notations introduced in the preceding section remain in force. For each place $v \leq \infty$ of $\mathbf{Q}$, let $(\pi_v, \mathfrak{H}_v)$ be a unitary representation of $G(\mathbf{Q}_v)$. Denote $\langle \cdot, \cdot \rangle_v$ the inner product of $\mathfrak{H}_v$. Let's assume that there exists a finite set of finite primes S_0 containing ∞ such that for any $v \notin S_0$, the space of K_v-fixed vectors is one-dimensional. For each place $v \notin S_0$, we choose a *unitary* vector in $\mathfrak{H}_v^{K_v}$, which we will denote ξ_v^0. We will construct a unitary representation π of $\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})$ which is usually denoted

$$\pi = \widehat{\bigotimes_{v \leq \infty}} \pi_v$$

but one has to keep in mind that it might a priori depend on the choice of $\boldsymbol{\xi}^0 = (\xi_v^0)_{v \notin S_0}$ (and so $\widehat{\bigotimes_{v \leq \infty}}^{\xi} \pi_v$ would be a better notation).

Step 1: construction of the Hilbert space on which $\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})$ will act: reference [Gui].

For each finite set S of primes containing S_0, one denotes by $\mathfrak{H}_S$ the prehilbert space

$$\mathfrak{H}_S = \bigotimes_{v \in S} \mathfrak{H}_v$$

By general properties of the tensor product of modules, one does not have to choose any order on the set of places. For two such sets S, T with $S \subset T$, there is a unique mapping $j_{S,T} : \mathfrak{H}_S \to \mathfrak{H}_T$ defined for each family $(x_v \in \prod_{v \in S} \mathfrak{H}_v)$ by:

$$j_{S,T}(\bigotimes_{v \in S} x_v) = \bigotimes_{v \in S} x_v \otimes \bigotimes_{v \in T-S} \xi_v^0.$$

It is obvious that these mappings are injective Put on $\mathfrak{H}_S$ the (positive definite) inner product $\langle \cdot, \cdot \rangle_S$ induced by

$$\langle \bigotimes_{v \in S} x_v, \bigotimes_{v \in S} y_v \rangle_S := \prod_{v \in S} \langle x_v, y_v \rangle_v$$

See section 4 for some facts on these tensor products. One sees immediately that the embeddings $j_{S,T}$ are isometric for these inner products. Denote $\mathfrak{H}^{\mathrm{alg}}$ the inductive limit of the system $(\mathfrak{H}_S, j_{S,T})$ (the directed set is the set of finite sets of primes, ordered by inclusion), and $j_S : \mathfrak{H}_S \to \mathfrak{H}^{\mathrm{alg}}$ the canonical embedding.

For $x, y \in \mathfrak{H}^{\mathrm{alg}}$, there exist a finite set of places S, and elements x_S, y_S of $\mathfrak{H}_S$ such that $x = j_S(x_S), y = j_S(y_S)$ and we define an inner product on $\mathfrak{H}^{\mathrm{alg}}$ by:

$$\langle x, y \rangle = \langle x_S, y_S \rangle_S$$

This is well-defined, and this makes each j_S an isometry. Finally, denote by $\mathfrak{H}$ the completion of $\mathfrak{H}^{\mathrm{alg}}$ for this inner product: this space is denoted in [Gui] as $\widehat{\bigotimes_{v\leq\infty}}^{\xi}\mathfrak{H}_v$.

Remark: In [Gui], it is proved that *canonically* there is no dependence on the vectors $\boldsymbol{\xi}^0$: this is Proposition 1.3 loc. cit., which can be applied as the space of K_v-fixed vectors is one-dimensional (for another choice $\boldsymbol{\xi}^1$, one has $\xi_v^1 = \alpha_v \xi_v^0$ for a unique complex α_v of modulus one, so the hypothesis is trivially satisfied). Of course, the choice of the finite set S_0 is unimportant, by general properties of inductive limits.

Remark: If $x = \bigotimes_{v\in S} x_v$ is a vector in $\mathfrak{H}_S$, one often denotes its image in $\mathfrak{H}$ using the notation $\bigotimes_{v\in S} x_v \otimes \bigotimes_{v\notin S} \xi_v^0$: this is an abuse of language, as the latter makes sense only in the algebraic infinite tensor product $\bigotimes_{v\leq\infty}\mathfrak{H}_v$ as defined in Bourbaki, Algèbre, chap II, §3, No 9, but this is common.

Step 2: construction of the representation.

For each finite set S of primes containing S_0, $\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q},S})$ acts on $\mathfrak{H}_S$ via the unitary representation $\bigotimes_{v\in S}\pi_v$: unitarity and continuity of this representation has been checked in section 4. One sees at once that one gets an inductive system of unitary representations (of course by using the K_v-invariance of the ξ_v^0's), and so one gets an algebraic representation π^{alg} of $\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})$ on $\mathfrak{H}^{\mathrm{alg}}$ by unitary operators. One can extend by uniform continuity each operator $\pi^{\mathrm{alg}}(g)$, for $g \in \mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})$, to a unitary operator on the completion $\mathfrak{H}$, which we denote π: π affords an algebraic representation of $\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})$ by unitary operators, so we need only justify the continuity of this action.

It suffices to prove that the mappings $g \in \mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}}) \mapsto \pi(g)x \in \mathfrak{H}$ are continuous for each x in a total subset of $\mathfrak{H}$ (see section 4), so it is sufficient to check this continuity for x of the shape $x = j_S(x_S)$ for some finite set S of primes containing S_0. Because of the topology on the adeles, it suffices to prove the continuity of $g \in \mathbf{GL}_2(\mathbb{A}_{\mathbf{Q},T}) \mapsto \pi(g)x \in \mathfrak{H}$ for T a fixed finite set of primes with $S \subset T$, but in this case one has:

$$\pi(g)x = j_T\left(\bigotimes_{v\in S}\pi_v(g_v)x_v \otimes \bigotimes_{v\in T-S}\pi_v(g_v)\xi_v^0\right)$$

and the continuity is clear.

We have therefore constructed from the data the unitary tensor product representation π, denoted in the litterature

$$\pi = \widehat{\bigotimes_{v\leq\infty}}\pi_v.$$

A slightly better notation would be $\widehat{\bigotimes_{v\le\infty}}^{\boldsymbol{\xi}^0} \pi_v$ a priori, but because of the remark we made above, if one changes the family $\boldsymbol{\xi}^0$, one *canonically* gets a unitarily equivalent representation.
Remark: If furthermore all the local representations π_v are irreducible, then so is their unitary tensor product: this is a simple adaptation of an argument given in section 4. See [G], §4.C.
Remark: Let π be a unitary representation of $\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})$ constructed as above from irreducible unitary representations π_v of $\mathbf{GL}_2(\mathbf{Q}_v)$. Then the restriction of π to $\mathbf{GL}_2(\mathbf{Q}_v)$ splits as a Hilbert direct sum of irreducible representations, all equivalent to π_v: this means that (the equivalence class of) π_v is uniquely determined by π.

Remark: If an irreducible unitary representation π of $\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})$ is equivalent to a unitary tensor product as above, then π is admissible. Indeed, any $\rho \in \widehat{K}$ is equivalent to a unitary tensor product representation $\widehat{\bigotimes_v}\rho_v$, where ρ_v is an irreducible representation of K_v for each place v, almost all of which are trivial of dimension 1 ([Bu], lemma 3.3.1): each of these local representations appear with finite multiplicities in their respective space, which proves the claim.

Remark: If one is interested in the algebraic theory of the automorphic representations, one has to modify slightly the above construction to a more algebraic one: this is explained in Bump (ibid.).

6.2. **The tensor product theorem.** In this section, one is interested in a converse statement of the previous construction. We take $G = \mathbf{GL}_2$, but this would work mutatis mutandis for a reductive group over a number field, as these are liminal.

Theorem 6.1 (The tensor product theorem). *Let π be an irreducible unitary representation of $G(\mathbb{A}_{\mathbf{Q}})$. Then there exist a finite set S_0 of primes containing ∞, an irreducible unitary representation π_v of $G(\mathbf{Q}_v)$ for each place v such that π_v is spherical for $v \notin S_0$, and a unitary K_v-fixed vector ξ_v^0 for each $v \notin S_0$, so that π is equivalent to $\widehat{\bigotimes_{v\le\infty}}^{\boldsymbol{\xi}^0} \pi_v$ for $\boldsymbol{\xi}^0 = (\xi_v^0)_{v\notin S_0}$.*

Remark: see [Bu] section 3.4 for a statement and proof of the algebraic counterpart, as well as Cogdell in [CKM] lecture 3 for a statement without proof of the various versions of the tensor product theorem.

References: Depending on the strength of the statement, there are more or less difficult proofs of this result.

- Godement in [Go] §3.2 assumes furthermore that π is admissible. Under this assumption, he considers the restriction of π to $G(\mathbf{Q}_v)$, which is also admissible, and therefore splits as a Hilbert direct sum of irreducible representations by theorem 3.4. As π is irreducible, the Schur lemma insures that any continuous operator in the space of π commuting with $\pi_{|_{G(\mathbf{Q}_v)}}$ and with the operators commuting with $\pi_{|_{G(\mathbf{Q}_v)}}$ are scalars, so that $\pi_{|_{G(\mathbf{Q}_v)}}$ is a factor representation.

As G is of type 1, this implies that $\pi_{|_{G(\mathbf{Q}_v)}}$ is isotypical, i.e. isomorphic to a Hilbert direct sum of equivalent representations (cf [Dix] or [Ro] section 20). Therefore, one has at one's disposal a family of irreducible unitary representations π_v of $G(\mathbf{Q}_v)$ for each place v, and the rest of the proof is a tedious construction allowing to "glue" together the local pieces. Note that along the way one chooses unitary K_v-fixed vectors, getting for each choice a factorization into a unitary tensor product – hence another justification in this context of the "independence" in the choice of $\boldsymbol{\xi}^0$.

- In [GGP] chapter 3 §3.3, there is a proof which does not make use of any admissibility condition. As a consequence, this proves that any irreducible representation of $G(\mathbb{A}_{\mathbf{Q}})$ is admissible, as explained in the previous subsection. Without a doubt, one could adapt Godement's arguments in order not to assume that π is admissible, as this is used only to find the local pieces π_v: [GGP] get these another way, yet the rest of the proofs are pretty close.
- If one is only interested in unitary cuspidal representations, one can prove first the admissibility of these, and use Godement's argument, or even the algebraic tensor product theorem on the space of K-finite vectors: in the last case, one gets a factorization into a restricted tensor product of smooth representations, which are unitarizable because π is.

To prove the admissibility of an irreducible unitary cuspidal automorphic representation (π, V_π), that is (a unitary representation equivalent to a) $G(\mathbb{A}_{\mathbf{Q}})$-invariant irreducible closed subspace of the space $L^2_0(Z(\mathbb{A}_{\mathbf{Q}})\mathbf{GL}_2(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A}), \omega)$ for some unitary character ω of the idele class group, one can proceed as follows: let $\rho = \widehat{\bigotimes_v}\rho_v \in \widehat{K}$, where ρ_v is an irreducible representation of K_v for each place v. As K_f is totally disconnected, and ρ is finite dimensional, there exists an open compact normal subgroup K_1 of K_f such that the restriction of ρ to K_1 is trivial.

One wants to prove that $V_\pi(\rho)$ is finite dimensional. The latter is contained in the space of K_1-fixed vectors in V_π which we denote $V_\pi^{K_1}$. Note right now that $V_\pi^{K_1}$ is stable under the restriction of the right regular representation of $\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})$ to $\mathbf{GL}_2(\mathbf{R})$, so that it suffices to prove that $V_\pi^{K_1}(\rho_\infty)$ is finite dimensional.

Consider $\varphi = \varphi_\infty\varphi_f \in \mathcal{C}_c^\infty(G(\mathbb{A}_{\mathbf{Q}}))$, where:

- the function (defined on $\mathbf{GL}_2(\mathbb{A}_f)$) φ_f is the characteristic function of K_1,

– the archimedean component $\varphi_\infty \in \mathcal{C}_c^\infty(\mathbf{GL}_2(\mathbf{R}))$ is arbitrary. The mapping $R(\varphi)$ defined for $f \in L_0^2(Z(\mathbb{A})\mathbf{GL}_2(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A}_\mathbf{Q}), \omega)$ by:

$$R(\varphi)(f)(x) = \int_{\mathbf{GL}_2(\mathbb{A}_\mathbf{Q})} \varphi(y)f(xy)dy \ \text{ for any } x \in \mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$$

is a compact operator on $L_0^2(Z(\mathbb{A})\mathbf{GL}_2(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A}_\mathbf{Q}), \omega)$, and so is its restriction to V_π. But for $f \in V_\pi^{K_1}, x \in \mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$, one has:

$$(R(\varphi)f)(x) = \mathrm{vol}(K_1)\int_{\mathbf{GL}_2(\mathbf{R})} \varphi_\infty(y_\infty)f(xy_\infty)dy_\infty$$

which is also a compact operator. As a consequence, theorem 3.3 implies that $V_\pi^{K_1}$ splits as a Hilbert direct sum of irreducible representations $(V_i)_{i\in\mathrm{I}}$ of $\mathbf{GL}_2(\mathbf{R})$, each occuring with finite multiplicities.

We also have:

$$V_\pi^{K_1}(\rho_\infty) = \widehat{\bigoplus_{i\in\mathrm{I}}} V_i(\rho_\infty)$$

so it suffices to prove that only finitely many i are such that $V_i(\rho_\infty) \neq \{0\}$.

One the other hand, $\mathfrak{z}$ acts by characters on the smooth vectors of V_i^∞ for each $i \in \mathrm{I}$: in particular, there exists a complex number λ such that $V := \ker(\Delta - \lambda\mathrm{Id}) \neq \{0\}$ (here we take the kernel in V_π^∞), where Δ denotes the Casimir element of $\mathbf{GL}_2(\mathbf{R})$. This subspace V is obviously stable under the action of $\mathbf{GL}_2(\mathbb{A}_f)$ and of $\mathbf{GL}_2(\mathbf{R})$: therefore it must be dense in V_π. This implies that Δ acts by λ on *each* V_i^∞ (by using the self-adjointness of Δ). By the classification of irreducible unitary representations of $\mathbf{GL}_2(\mathbf{R})$, there are only finitely many equivalence classes of representations of $\mathbf{GL}_2(\mathbf{R})$ containing ρ_∞, with central character ω_∞, such that Δ acts by λ on the smooth vectors: this ends the proof.

Remark: This argument works in generality for reductive groups (use [Bo2] théorème 5.29), but instead of using the Casimir element, one can argue as follows – this affects the last paragraph of the previous proof: $\mathfrak{z}$ acts by characters on the smooth vectors of V_i^∞ for each $i \in \mathrm{I}$: let χ be one of them. Let V be the space of smooth vectors son which $\mathfrak{z}$ acts through χ: V is dense in V_π for the same reasons. This implies that for *any* $v \in V_\pi$, $d\pi(z) = \chi(z)v$ as a distribution, for any $z \in \mathfrak{z}$ (by taking a sequence in V tending to v), and so that on *any* smooth vector of V_π, $\mathfrak{z}$ acts through χ. This implies that the infinitesimal character of each V_i is χ, and again, there are only finitely many irreducible unitary representations containing ρ_∞, with central character ω_∞ and infinitesimal character χ.

7. Proof of theorem 5.1

Let f be a cuspidal modular form of weight k, nebentypus χ and level N. Assume that there exists a finite set of primes S_f such that f is a Hecke eigenform for the T_p, $p \notin S_f$: we may and will assume that S_f contains the divisors of q. We wish to prove that that the unitary representation $(\mathfrak{H}_f, \pi_f)$ we attached to f in section 5 is irreducible.

To simplify the proof, we'll make use of the strong multiplicity one theorem, which asserts that given two irreducible unitary cuspidal representations $\pi \cong \widehat{\bigotimes_{v\leq\infty}} \pi_v, \pi' \cong \widehat{\bigotimes_{v\leq\infty}} \pi'_v$ of $\mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$ are equivalent if and only if there exists a finite set of primes S (containing or not ∞) such that $\pi_v \cong \pi'_v$ for each $v \notin S$. This theorem holds actually for irreducible automorphic representations, and can be proven using Whittaker models (cf [Bu] section 3.5, [Go] §3.5 and [G] §6) or the Rankin-Selberg L-function (as in [CKM] theorem 9.3): in any case, the proof makes use of the algebraic theory of automorphic forms, in the sense that Whittaker models are smooth models, not unitary representations.

As $L^2_0(Z(\mathbb{A})\mathbf{GL}_2(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A}_\mathbf{Q}), \omega_\chi)$ is $\mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$-invariant, $\mathfrak{H}_f$ is a subspace of it. Also, as $L^2_0(Z(\mathbb{A})\mathbf{GL}_2(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A}_\mathbf{Q}), \omega_\chi)$ splits as a Hilbert sums of irreducibles, then so does $(\mathfrak{H}_f, \pi_f)$ by Lemma 3.1. We may therefore write:

$$\mathfrak{H}_f = \widehat{\bigoplus_{i\in \mathrm{I}}} \mathfrak{H}_i$$

where each $\mathfrak{H}_i$ is a closed subspace of $\mathfrak{H}_f$ stable and irreducible under $\mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$. We have to prove that $\mathrm{card}(\mathrm{I}) = 1$.

To do so, let's denote π_i the representation of $\mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$ on $\mathfrak{H}_i$. By the tensor product theorem, for each i we can write (with alleged notations)

$$\pi_i \cong \widehat{\bigotimes_{v\leq\infty}} \pi_{i,v}.$$

To prove the theorem, due to the multiplicity one theorem, it is sufficent to prove that $\pi_{i,p} \cong \pi_{j,p}$ for any $p \notin S_f$ and each $i, j \in \mathrm{I}$.

As the adelization of f, φ_f, is in $\mathfrak{H}_f$, we can write, in a unique way – with convergence in L^2:

$$\varphi_f = \sum_{i\in\mathrm{I}} \varphi_i$$

with $\varphi_i \in \mathfrak{H}_i - \{0\}$. If K' denotes the product of the $\mathbf{GL}_2(\mathbf{Z}_p)$'s for $p \notin S_f$, then φ_f is K' right invariant, and we can assume that so are the φ_i's (if not, one writes $\varphi_f(g) = \frac{1}{\mathrm{vol}(K')}\int_{K'} \psi(k)\varphi_f(gk)dk = \sum_i \int_{K'} \psi(k)\varphi_i(gk)dk$ with $\psi =$ the characteristic function of K', and the job is done, or one simply projects on the K'-invariant vectors).

Let $i \in \mathrm{I}$. By the tensor product theorem, one can write $\mathfrak{H}_i = \widehat{\bigotimes_{v \leq \infty}} \mathfrak{H}_{i,v}$, and so there exists a set J such that:

$$\varphi_i = \sum_{j \in J} x_j$$

where x_j is of the form $x_j = j_{S_j}(x_{S_j})$ with $x_{S_j} \in \mathfrak{H}_{i,S_j}$ – see section 6 for the notations. As we did above, one can assume that each of the x_j's are K'-invariant, i.e. that $S_j \subset S_f$. All this proves that the π_i's are unramified outside S_f.

Let $p \notin S_f$ be a prime. Obviously, all the π_i's have the same central character (namely ω): so do all the $\pi_{i,p}$ (for various i's). To prove the theorem, it is sufficient to prove that the $\pi_{i,p}$ share the same eigenvalue under the Hecke operator T_p we introduced in section 4.2. We defined in section 5.3 the adelization of the classical Hecke operator $\mathbb{T}_p$. By hypothesis, we have:

$$\mathbb{T}_p \varphi_f = \lambda_f(p) \varphi_f$$

and so, by continuity for each $i \in \mathrm{I}$:

$$\mathbb{T}_p \varphi_i = \lambda_f(p) \varphi_i. \tag{3}$$

As each x_j is a "pure tensor", say $x_j = \otimes_v x_{j,v}$ (for $p \notin S_f$, $x_{j,p} = \xi^0_{j,p}$, the K_p-fixed vector), one has :

$$\mathbb{T}_p x_j = \bigotimes_{v \neq p} x_{j,v} \otimes (T_p(x_{j,p})).$$

The vector $x_{j,p}$ is K_p-invariant, so by denoting $c_{i,p}$ the Hecke eigenvalue of $\pi_{i,p}$:

$$T_p(x_{j,p}) = c_{i,p} x_{j,p}.$$

By comparing with (3), one gets $\lambda_f(p) = c_{i,p}$ for any $i \in I$, so we get exactly what we wanted. QED

8. Appendix

8.1. **Appendix 1.** Let G be a locally compact unimodular group, H a closed unimodular subgroup of G, Z the center of G (or more generally a closed subgroup of the center) and $\omega : Z \to \mathbb{S}^1$ a character. We want to first define in this appendix what is meant in the literature by $L^2(ZH\backslash G, \omega)$, as it was mentioned in section 5.2. So let $L^2(ZH\backslash G, \omega)$ ($L^2(\omega)$ is a useful abbreviation if no confusion arises) be the space of classes of functions (the equivalence is equality almost everywhere on $H\backslash G$) $f : H\backslash G \to \mathbf{C}$ such that:

1. f is Borel-measureable on $H\backslash G$,
2. $|f|$ is Borel-measureable on $ZH\backslash G$,
3. for any $z \in Z$, $f(zx) = \omega(z) f(x)$ for almost all $x \in H\backslash G$,
4. $\int_{ZH\backslash G} |f|^2 < \infty$

We claim that:

(1) $L^2(ZH\backslash G,\omega)$ is a Hilbert space for the inner product $\langle\cdot,\cdot\rangle$ defined by:

$$\langle f,g\rangle=\int_{ZH\backslash G} f(x)\overline{g(x)}dx$$

(2) The space of bounded continuous functions $\mathcal{C}_b(ZH\backslash G,\omega)$ satisfying:

$$f(zx)=\omega(z)f(x)$$

for all $x\in H\backslash G$, is dense in $L^2(ZH\backslash G,\omega)$

Let (f_n) be a Cauchy sequence in $L^2(ZH\backslash G,\omega)$. One can find a subsequence φ_n such that:

$$\|\varphi_{n+1}-\varphi_n\|\leq 2^{-n}$$

This implies that the series $\sum_{n\in\mathbf{N}}|\varphi_{n+1}-\varphi_n|$ converges almost everywhere on $ZH\backslash G$, and thus that $\sum_{n\in\mathbf{N}}(\varphi_{n+1}-\varphi_n)$ is absolutely convergent almost everywhere on $H\backslash G$: this proves the first claim.

To prove the second claim, note first that the subspace $L^2_c(ZH\backslash G,\omega)$ of functions with compact support in $ZH\backslash G$ is dense in $L^2(ZH\backslash G,\omega)$: for instance, if K is a (large) compact subset of $ZH\backslash G$, $f\times \mathrm{Char}_{K'}$ will do the job (K' is the inverse image of K under the projection $H\backslash G\to ZH\backslash G$). Then, let f be in $L^2_c(ZH\backslash G,\omega)$, and let φ_n a continuous δ-sequence in G. Consider the function:

$$f_n(x)=\int_G f(xg)\varphi_n(g)dg$$

It is well-defined: denoting by K_n a compact of G containing the support of φ_n, one has:

$$\begin{aligned}\int_{ZH\backslash G}|f_n(x)|^2dx &\leq \int_{ZH\backslash G}\left(\int_G|\varphi_n(g)|^2dg\int_G|f(xg)|^2\mathrm{Char}_{K_n}(g)dg\right)dx\\ &\leq \mathrm{vol}(K_n)\times\int_G|\varphi_n(g)|^2dg\times\int_{ZH\backslash G}|f(x)|^2dx\end{aligned}$$

The function f_n is continuous, as this is easily seen after a legal change of variable and a use of Lebesgue dominated convergence theorem, and its support is compact (because it is the convolution of two such functions). This ends our contention.

8.2. **On cuspidal functions.** We refer to [L] and [Bo1] (especially chap 8) for a rigorous definition of this space, in the classical setting. In the litterature, given a unitary Grossencharakter $\omega:\mathbb{A}_{\mathbf{Q}}^{\times}/\mathbf{Q}^{\times}\to\mathbf{C}^{\times}$, the space of square-integrable cusp forms is often "defined" by:

$$L^2_0(Z(\mathbb{A}_{\mathbf{Q}})\mathbf{GL}_2(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}}),\omega) =$$

$$\Big\{f : \mathbf{GL}_2(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}}) \to \mathbf{C} \ : \ \text{ for all } z \in Z(\mathbb{A}_Q), f(zg) = \omega(z)f(g)$$

$$\text{for almost all } g, \int_{Z(\mathbb{A}_{\mathbf{Q}})\mathbf{GL}_2(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})} |f|^2 < \infty \text{ and}$$

$$\int_{\mathbf{Q}\backslash\mathbb{A}_{\mathbf{Q}}} f\left(\left[\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right]g\right) dx = 0 \text{ for almost all } g\Big\}$$

The problem is that the last condition is not a closed one, a priori. One option is to define this space as the closure in $L^2(\omega)$ –which we defined in the previous paragraph – of the space of bounded continuous satisfying the above (well defined) conditions: this is what Lang does.

Another possibility is to consider, for a compactly supported function φ on $U(\mathbb{A}_{\mathbf{Q}})\backslash\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})$ (here U denotes the usual unipotent subgroup of $\mathbf{GL}_2$), the linear form:

$$f \in L^2(\omega) \to \Lambda_\varphi(f) = \int_{U(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})} f(g)\varphi(g)dg$$

This mapping is well-defined, and continuous: indeed, the support of φ, viewed as a function on $\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})$, is contained in a set of the shape $U(\mathbf{Q})\Omega$, where Ω is a compact subset of $\mathbf{GL}_2(\mathbb{A})$ – this is because $\mathbf{Q}$ is cocompact inside $\mathbb{A}_{\mathbf{Q}}$. So one has:

$$|\Lambda_\varphi(f)| \leq \|\varphi\|_\infty \int_\Omega |f(x)|dx$$

To finish, one covers Ω with finitely many (say m) relatively compact open sets U_i of $\mathbf{GL}_2(\mathbb{A})$ such that the projection $\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}}) \to \mathbf{GL}_2(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})$ induces on each U_i a homeomorphism onto its image – which is possible by the discreteness of $\mathbf{GL}_2(\mathbf{Q})$ in $\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})$:

$$\int_\Omega |f(x)|dx \leq m \int_{Z(\mathbb{A}_{\mathbf{Q}})\mathbf{GL}_2(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})} |f(x)|dx \times \sup_i(\mathrm{vol})(U_i)$$

this proves the continuity, because $\mathrm{vol}(Z(\mathbb{A}_{\mathbf{Q}})\mathbf{GL}_2(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})) < \infty$.

Finally, to see the link with the cuspidal condition, one notes that:

$$\Lambda_\varphi(f) = \int_{U(\mathbb{A}_{\mathbf{Q}})\backslash\mathbf{GL}_2(\mathbb{A})} W_f(g)\varphi(g)dg$$

where

$$W_f(g) := \int_{\mathbf{Q}\backslash\mathbb{A}_{\mathbf{Q}}} f\left(\left[\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right]g\right) dx,$$

and the space of cusp forms can be identified with the intersection of the kernels of all the Λ_φ, when φ varies among such functions (it is a posteriori easy to see this, by using convolution with a δ-sequence).

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