1. Z-Finiteness of Hecke Algebras

Let $S_k$ denote the complex vector space $S_k(\Gamma_1(N))$ of cusp forms of weight $k \geq 2$ on $\Gamma_1(N)$. Let $T$ be the $Z$-subalgebra of $\text{End}_{\mathbb{C}}(S_k)$ generated by Hecke operators $T_d$ for every prime $p$ and diamond operators $(d)$ for every $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. In this section our aim is to prove that $T$ is a finite free $Z$-module. As it is clear that $T$ is torsion-free, it is enough to show that $T$ is a finitely generated $Z$-module. We show this in Theorem 1.6.

We begin with some general constructions for any congruence subgroup $\Gamma$ of $\text{SL}_2(\mathbb{Z})$. Let $\{e, e\}'$ be a $\mathbb{C}$-basis for $C^2$. The group $\Gamma$ acts on $C^2$ via the embedding $\text{SL}_2(\mathbb{Z}) \hookrightarrow \text{SL}_2(\mathbb{C})$ with respect to the basis $\{e, e\}'$: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $c_1 e + c_2 e' \in C^2$,

$$\gamma \cdot (c_1 e + c_2 e') = (ac_1 + bc_2)e + (cc_1 + dc_2)e'.$$

This action induces an action on $V_k := \text{Sym}^{k-2}(C^2)$.

Fix any $z_0$ in the upper half-plane $\mathfrak{h}$. Let $f$ be any element of the $\mathbb{C}$-vector space $M_k(\Gamma)$ of modular forms of weight $k$ on $\Gamma$. We define the function $I_f : \Gamma \rightarrow V_k$ by

$$I_f(\gamma) = \int_{\gamma z_0} (ze + e')^{k-2} f(z) dz$$

for every $\gamma \in \Gamma$.

**Proposition 1.1.** The function $I_f$ in (1.1) is a $1$-cocycle and its class in $H^1(\Gamma, V_k)$ is independent of $z_0$.

**Proof.** First, we show that $I_f$ is in $Z^1(\Gamma, V_k)$. Let $\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma_2$ be elements of $\Gamma$. Since $f|_{k\gamma_1} = f$, we have

$$\gamma_1 \cdot I_f(\gamma_2) = \int_{\gamma_2 z_0} ((az + b)e + (cz + d)e')^{k-2} f(z) dz,$$

or

$$\gamma_1 \cdot I_f(\gamma_2) = \int_{\gamma z_0} (\gamma_1(z)e + e')^{k-2} f(\gamma_1 z) dz = \int_{\gamma z_0} (\gamma_1(z)e + e')^{k-2} f(\gamma z) dz = \int_{\gamma z_0} (ze + e')^{k-2} f(z) dz.$$

It follows that

$$\gamma_1 \cdot I_f(\gamma_2) + I_f(\gamma_1) = \int_{\gamma z_0} (ze + e')^{k-2} f(z) dz + \int_{\gamma z_0} (ze + e')^{k-2} f(z) dz = I_f(\gamma_1 \gamma_2),$$

as desired.

Now we show that $I_f$ modulo $B^1(\Gamma, V_k)$ is independent of $z_0$. Choose $z_1 \in \mathfrak{h}$. For any $\gamma \in \Gamma$ the difference $\int_{\gamma z_0} (ze + e')^{k-2} f(z) dz - \int_{\gamma z_1} (ze + e')^{k-2} f(z) dz$ is equal to

$$\int_{\gamma z_1} (ze + e')^{k-2} f(z) dz - \int_{\gamma z_1} (ze + e')^{k-2} f(z) dz.$$

The calculations in (1.2) with $\gamma z_0$ replaced by $\gamma z_1$ show that $\int_{\gamma z_1} (ze + e')^{k-2} f(z) dz = \gamma \cdot \int_{\gamma z_1} (ze + e')^{k-2} f(z) dz$. Hence, we see that the difference is a $1$-coboundary.
By Proposition 1.1 we can define the $C$-linear map
\begin{equation}
(1.3) \quad j : M_k(\Gamma) \rightarrow H^1(\Gamma, V_k)
\end{equation}
by $j(f) = I_f$, where $I_f$ is given in (1.1).

**Proposition 1.2.** Choose $z_0 \in \mathfrak{h}$. The restriction
\begin{equation*}
(1.4) \quad j : S_k(\Gamma) \rightarrow H^1(\Gamma, V_k)
\end{equation*}
\begin{equation*}
f \mapsto \gamma \mapsto \int_{z_0}^{\gamma z_0} (ze + e')^{k-2} f dz,
\end{equation*}
of (1.3) is injective.

**Proof.** For any $h \in S_k(\Gamma)$ consider the holomorphic map
\begin{equation*}
(ze + e')^{k-2} h(z) : \mathfrak{h} \rightarrow V_k.
\end{equation*}
Since $\mathfrak{h}$ is simply connected, we can choose a holomorphic function $G_h : \mathfrak{h} \rightarrow V_k$ so that
\begin{equation*}
dG_h = (ze + e')^{k-2} h(z) dz.
\end{equation*}
For any $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ we see that
\begin{equation*}
d(G_h \sigma) = G'_h(\sigma(z)) d\sigma(z),
\end{equation*}
\begin{equation*}
= \left( \left( \frac{az + b}{cz + d} \right) e + e' \right)^{k-2} h(\sigma(z)) \frac{dz}{(cz + d)^2},
\end{equation*}
\begin{equation*}
= (az + b) e + (cz + d) e')^{k-2} (h|_{k\sigma})(z) dz,
\end{equation*}
where $(h|_{k\sigma})(z) = (cz + d)^{-k} h(\sigma(z))$. Therefore, for every $\sigma \in \text{SL}_2(\mathbb{Z})$ we have
\begin{equation}
(1.5) \quad G_h \sigma = \sigma \cdot G_{h|k\sigma} + v_{\sigma}
\end{equation}
for our fixed choice of antiderivative $G_{h|k\sigma}$ of $(ze + e')^{k-2} (h|_{k\sigma})$ and some $v_{\sigma} \in V_k$.

Let $\text{SL}_2(\mathbb{Z})$ act on the holomorphic maps $G : \mathfrak{h} \rightarrow V_k$ as follows:
\begin{equation*}
(\sigma \ast G)(z) = \sigma \cdot (G(\sigma^{-1}(z))).
\end{equation*}

For each member $\hat{h}$ of $\text{SL}_2(\mathbb{Z})$-orbit of $h$ (under $\sigma \mapsto h|_{k\sigma}$) we choose an antiderivative $G_{\hat{h}}$ as above, so by (1.4) for every $\sigma \in \text{SL}_2(\mathbb{Z})$ we have
\begin{equation}
(1.6) \quad \sigma \ast G_{\hat{h}} = G_{h|_{k\sigma^{-1}}} + c_{\sigma}
\end{equation}
for some $c_{\sigma} \in V_k$.

Consider $f \in S_k(\Gamma)$ in the kernel of $j$; that is, the 1-cocycle
\begin{equation*}
\gamma \mapsto \int_{z_0}^{\gamma z_0} (ze + e')^{k-2} f(z) dz = G_f(\gamma z_0) - G_f(z_0)
\end{equation*}
is a 1-coboundary. Then, for every $\gamma \in \Gamma$ we have
\begin{equation}
(1.7) \quad G_f(\gamma z_0) - G_f(z_0) = \gamma \cdot v - v
\end{equation}
for some $v \in V_k$. Our aim is to show that $f = 0$.

For $\gamma \in \Gamma$ the equation (1.5) becomes
\begin{equation}
(1.8) \quad \gamma \ast G_f = G_f + c_{\gamma}
\end{equation}
for some $c_{\gamma} \in V_k$. We evaluate this equation at $\gamma z_0$ and obtain that $c_{\gamma} = (\gamma \ast G_f)(z_0) - G_f(\gamma z_0)$. By using equation (1.6) we see that $c_{\gamma} = \gamma \cdot (G_f(\gamma^{-1} z_0) - v) - (G_f(z_0) - v)$. We may replace $G_f$ with $G_f - (G_f(z_0) - v_{\gamma})$, so (1.7) becomes
\begin{equation}
(1.9) \quad \gamma \ast G_f = G_f
\end{equation}
for all $\gamma \in \Gamma$.

Recall that for the upper half-plane $\mathfrak{h}$, we topologize $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$ using $\text{SL}_2(\mathbb{Z})$-translates of bounded vertical strips
\begin{equation*}
\{ z \in \mathfrak{h} \mid \text{Im}(z) > c, \quad a < \text{Re}(z) < b \}
\end{equation*}
for $a, b \in \mathbb{R}$ and $c > 0$. Now we prove the following claim.

Claim 1: As we approach any fixed cusp in $\mathfrak{h}^*$, the function $G_f$ remains bounded in $V_k$.

Proof of Claim 1: Let $s \in \mathfrak{h}^*$ be any cusp and choose $\sigma \in \text{SL}_2(\mathbb{Z})$ such that $\sigma(s) = \infty$. To prove the claim, it is enough to prove that $\sigma \ast G_f$ is bounded as we approach $\infty$ in $\mathfrak{h}$. By (1.5), this is just an antiderivative of $f|_k \sigma^{-1}$. Thus, it suffices to prove that each coefficient function of $(ze + e')^{k-2}(f|_k \sigma^{-1})(z)$ has bounded antiderivative as $\text{Im}(z) \to \infty$ in any bounded vertical strip $\{z \in \mathfrak{h} | \text{Re}(z) < a\}$ where $a \in \mathbb{R}^+$. Since $f \in S_k(\Gamma)$, we have $(f|_k \sigma^{-1})(z) \in S_k(\sigma \Gamma \sigma^{-1})$. Let $\tilde{f}(z) := (f|_k \sigma^{-1})(z)$. Since $\tilde{f}$ is a cusp form for $\sigma \Gamma \sigma^{-1}$, for any $a > 0$ there exists $c \in \mathbb{R}^+$ such that

$$|\tilde{f}(z)| \ll e^{-c \text{Im}(z)} \quad \text{as } \text{Im}(z) \to \infty$$

uniformly for $|\text{Re}(z)| < a$. Thus, for any $x \in [-a, a]$ and $y_0 \geq M > 0$ the coefficients of $G_f(x + iy) - G_f(x + iy_0)$ are linear combinations of terms $\int_{y_0}^{y} y'' \tilde{f}(x + iy) dy$ with uniformly bounded coefficients. This integral is bounded above by $|P_r(Y)|e^{-cY} + |P_s(y_0)|e^{-cy_0}$, where $P_r$ is a fixed polynomial of degree $r$, and as $Y \to \infty$ this tends to $|P_r(y_0)|e^{-cy_0}$ uniformly in $|x| \leq a$. This shows that each coefficient function of $(ze + e')^{k-2}(f(z))$ has bounded antiderivative as $\text{Im}(z) \to \infty$ in the mentioned vertical strips. Hence, Claim 1 follows.

Using the $\text{SL}_2(\mathbb{Z})$-invariant bilinear pairing $B : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ defined by the determinant, we obtain the induced bilinear pairing

$$B_k : V_k \times V_k \to \mathbb{C},$$

which is also $\text{SL}_2(\mathbb{Z})$-invariant. For $\omega_f = (ze + e')^{k-2} f \, dz$, consider the 2-form

$$B_k(\omega_f, \bar{\omega}_f) = (k - 2)! |f|^2 \text{det}(ze + e', \bar{ze} + e') \, dz \wedge d\bar{z},$$

where $z = x + iy$. Since $f$ is a cusp form, $B_k(\omega_f, \bar{\omega}_f)$ has finite integral over a fundamental domain $F$ of $\Gamma$. Before computing this integral, we compute $B_k(\omega_f, \bar{\omega}_f)$ in another way.

Since $\omega_f = dG_f = gdz$ for $g = (ze + e')^{k-2}f,$

$$B_k(\omega_f, \bar{\omega}_f) = B_k(g, \bar{g}) \, dz \wedge d\bar{z}.$$ 

But $g$ is holomorphic, so $\frac{\partial g}{\partial \bar{z}} = 0$ and hence

$$B_k(g, \bar{g}) = \frac{\partial B_k(G_f, \bar{g})}{\partial z}.$$ 

Thus, we see that

$$B_k(\omega_f, \bar{\omega}_f) = \frac{\partial B_k(G_f, \bar{g})}{\partial z} \, dz \wedge d\bar{z} = d(B_k(G_f, \bar{g}) dz).$$

By using this equality and Stoke’s Theorem we obtain

$$\int_F B_k(\omega_f, \bar{\omega}_f) = \int_{\partial F} B_k(G_f, \bar{dG_f}).$$

Now, we want to compute $\int_{\partial F} B_k(G_f, \bar{dG_f})$. To do this, for each cusp $c$ we choose $\gamma \in \text{SL}_2(\mathbb{Z})$ such that $\gamma(c) = \infty$. We define the “loop” $R_{c, h}$ around $c$ in $F$ to be $\gamma^{-1}(L)$ where $L$ is the horizontal segment joining the two edges at a common “height” $h$ emanating from $\infty$ in $\gamma(F)$. Define the “closed disc” $D_{c, h} = \gamma^{-1}(U_L)$ where $U_L$ is the closed vertical strip above $L$ including $\infty$. Then, this integral is equal to

$$\lim_{h \to \infty} \left( \int_{\partial(F - U_L, D_{c, h})} B_k(G_f, \bar{dG_f}) + \sum_{c \in \{\text{cusps of } F\}} \int_{R_{c, h}} B_k(G_f, \bar{dG_f}) \right).$$

To calculate the first integral in (1.11) we prove the following claim.

Claim 2: For any $\gamma \in \Gamma$, the pullback $\gamma^*(B_k(G_f, \bar{dG_f}))$ is equal to $B_k(G_f, \bar{dG_f})$. 

Proof of Claim 2: Let $\gamma \in \Gamma$. Since $B_k$ is $SL_2(\mathbb{Z})$-invariant, we have
$$\gamma^*(B_k(G_f, d\overline{G_f})) = B_k(G_f \gamma, d(\overline{G_f} \gamma)).$$
Since $\gamma \in \Gamma$, by (1.8) we see that $G_f = \gamma^{-1} * G_f$. With this equality we obtain $G_f \gamma = \gamma^{-1} \cdot G_f$. Thus, the above equality gives us
$$\gamma^*(B_k(G_f, d\overline{G_f})) = B_k(\gamma^{-1} \cdot G_f, d(\gamma^{-1} \cdot \overline{G_f})),
= B_k(\gamma^{-1} \cdot G_f, \gamma^{-1} \cdot d(\overline{G_f})),
= B_k(G_f, d\overline{G_f}).$$
The last equality holds because $B_k$ is $SL_2(\mathbb{Z})$-invariant. Hence, Claim 2 follows.

By Claim 2, the integrals on edges $L_1$ and $L_2$ of $F$ such that $L_1 = \gamma L_2$ for some $\gamma \in \Gamma$ cancel. That gives us
\begin{equation}
(1.12) \int_{\partial(F - \cup_c D_{c,h})} B_k(G_f, d\overline{G_f}) = 0
\end{equation}
for any $h$. Now, consider any cusp $c$ of $F$ and the loop $R_{c,h}$ around it. We want to compute
$$\lim_{h \to \infty} \int_{R_{c,h}} B_k(G_f, d\overline{G_f}).$$
Choose $\sigma \in SL_2(\mathbb{Z})$ such that $\sigma(\infty) = c$. We have
$$\int_{R_{c,h}} B_k(G_f, d\overline{G_f}) = \int_{\sigma^{-1}(R_{c,h})} \sigma^*(B_k(G_f, d\overline{G_f})),
= \int_{\sigma^{-1}(R_{c,h})} B_k(G_f \sigma, d\overline{G_f} \sigma),$$
the last equality holds because $B_k$ is $SL_2(\mathbb{Z})$-invariant. The loop $\sigma^{-1}(R_{c,h})$ is a loop $R_{\infty,h}$ around $\infty$ at height $h$. By equation (1.4), the function $G_f \sigma$ is just $\sigma \cdot G_{f|k} \sigma$ up to translation by a constant in $V_k$. Thus, as $B_k$ is $SL_2(\mathbb{Z})$-invariant, instead of computing the limit with integral (1.13), we may compute it with $\int_{R_{\infty,h}} B_k(G_{f|k} \sigma, d\overline{G_{f|k} \sigma})$ with any choice of antiderivative $G_{f|k} \sigma$. We do this by calculating the integrals of the $\{e, e'\}$-coefficients of the integrand.

By Claim 1, any antiderivative $G_{f|k} \sigma$ is bounded in $V_k$ as we approach $\infty$ in a bounded vertical strip, and $d\overline{G_{f|k} \sigma}$ has an explicit formula in terms of the cusp form $\tilde{f}_{|k} \sigma$. Thus, for any $a > 0$ there exists $b > 0$ such that
$$|\tilde{f}_{|k}(z)| \ll e^{-b \text{Im}(z)} \quad \text{as Im}(z) \to \infty$$
uniformly for $|\text{Re}(z)| < a$, so $\lim_{h \to \infty} \int_{R_{\infty,h}} B_k(G_f, d\overline{G_f}) = 0$. As a result, for each cusp $c$ of $F$ and the loop $R_{c,h}$ around it $\lim_{h \to \infty} \int_{R_{c,h}} B_k(G_f, d\overline{G_f}) = 0$. Hence,
\begin{equation}
(1.14) \lim_{h \to \infty} \sum_{c \in \{\text{cusps of } F\}} \int_{R_{c,h}} B_k(G_f, d\overline{G_f}) = 0.
\end{equation}
By (1.12) and (1.14), we see that the integral (1.10) becomes
$$\int_{F} B_k(\omega_f, \overline{\omega_f}) = 0.$$
Now, we want to construct operators acting on $H^1(\Gamma, V_k)$ compatible via $j$ with the Hecke operators acting on $S_k$ and preserving the $\mathbb{Z}$-structure on $H^1(\Gamma, V_k)$. To do this we view Hecke operators acting on $S_k$ as double cosets $\Gamma \alpha \Gamma$ where $\alpha$ is an element of

\[ \Delta = \{ \beta \in M_2(\mathbb{Z}) \mid \det(\beta) > 0, \beta \equiv \begin{pmatrix} 1 & \ast \\ 0 & \ast \end{pmatrix} \mod N \}. \]

It suffices to construct some $T_\alpha$ acting on $H^1(\Gamma, V_k)$ for every $\alpha \in \Delta$ such that

(i) the map $j$ in (1.15) carries $[\Gamma \alpha \Gamma]$-action on the left to $T_\alpha$-action on the right,
(ii) $T_\alpha$ preserves the $\mathbb{Z}$-structure on $H^1(\Gamma, V_k)$ coming from the one on $V_k$.

The following three lemmas give such $T_\alpha$.

**Lemma 1.3.** Choose $\alpha \in \Delta$ and coset representatives $\{\alpha_i\}$ for the left multiplication action of $\Gamma$ in $\Gamma \alpha \Gamma$, so that $\Gamma \alpha \Gamma = \bigsqcup_{i=1}^n \Gamma \alpha_i \Gamma$. For every $i$ and $\gamma \in \Gamma$, define $j[i]$ uniquely via $\alpha_i \gamma = j[i] \alpha_{j[i]}$. There is a well-defined operator

\[ T_\alpha : H^1(\Gamma, V_k) \rightarrow H^1(\Gamma, V_k). \]

\[ c \mapsto \sum_{i=1}^n (\det \alpha)^{k-1} \alpha_i^{-1} \cdot c(\gamma_i), \]

which does not depend on the coset representatives.

Let $\Gamma_\alpha := \alpha^{-1} \Gamma \alpha \cap \Gamma$. Using the natural finite-index inclusion $\iota_1 : \Gamma_\alpha \hookrightarrow \Gamma$ and the finite-index inclusion $\iota_2 : \Gamma_\alpha \hookrightarrow \Gamma$ defined by $\iota_2(\beta) = \alpha \beta \alpha^{-1}$, the resulting composite map of the restriction and corestriction maps

\[ H^1(\Gamma, V_k) \xrightarrow{\text{Res along } \iota_2} H^1(\Gamma_\alpha, V_k) \xrightarrow{\text{Cor along } \iota_1} H^1(\Gamma, V_k) \]

is the operation $T_\alpha$.

**Proof.** We first show that if we use another choice of coset representatives $\{\alpha'_i\}$ for $\Gamma$ in $\Gamma \alpha \Gamma$, then the operator $T_\alpha$ on 1-cocycles (valued in 1-cochains) changes by 1-coboundaries. Consider

\[ \alpha'_i = \tilde{\gamma}_i \alpha_i \]

where $\tilde{\gamma}_i \in \Gamma$ for every $i$. Since we have $\alpha_i \gamma = \gamma_i \alpha_{j[i]}$ for every $i$ and $\gamma \in \Gamma$, with the new choice of coset representatives we obtain $\tilde{\gamma}_i^{-1} \alpha'_i \gamma = \gamma_i \tilde{\gamma}_j[i] \alpha'_{j[i]}$. Writing $\gamma'_i := \tilde{\gamma}_i \gamma_i \tilde{\gamma}_j[i]$, we get

\[ \alpha'_i \gamma = \gamma'_i \alpha'_{j[i]} \]

for every $i$ and $\gamma \in \Gamma$. With the new choice of coset representatives $\{\alpha'_i\}$, for $c \in Z^1(\Gamma, V_k)$ and $\gamma \in \Gamma$ we have the equalities
\[
\sum_{i=1}^{n} (\det \alpha)^{k-1} \alpha_{i}^{\cdot 1} \cdot c(\gamma_{i}) = \sum_{i=1}^{n} (\det \alpha)^{k-1} \alpha_{i}^{-1} \cdot c(\tilde{\gamma}_{i}),
\]
\[
= \sum_{i=1}^{n} (\det \alpha)^{k-1} \alpha_{i}^{-1} \cdot c(\tilde{\gamma}_{i}) + \sum_{i=1}^{n} (\det \alpha)^{k-1} \alpha_{i}^{-1} \cdot c(\gamma_{j[i]}),
\]
\[
= \sum_{i=1}^{n} (\det \alpha)^{k-1} \alpha_{i}^{-1} \cdot c(\tilde{\gamma}_{i}) + \sum_{i=1}^{n} (\det \alpha)^{k-1} \alpha_{i}^{-1} \gamma_{j[i]} \cdot c(\tilde{\gamma}_{j[i]}),
\]
\[
+ \sum_{i=1}^{n} (\det \alpha)^{k-1} \alpha_{i}^{-1} \cdot c(\gamma_{i}),
\]
\[
= - \sum_{i=1}^{n} (\det \alpha)^{k-1} \alpha_{i}^{-1} \cdot c(\tilde{\gamma}_{i}) + \sum_{i=1}^{n} (\det \alpha)^{k-1} \alpha_{j[i]}^{-1} \cdot c(\tilde{\gamma}_{j[i]}),
\]
\[
+ \sum_{i=1}^{n} (\det \alpha)^{k-1} \alpha_{i}^{-1} \cdot c(\gamma_{i}),
\]
\[
= \sum_{i=1}^{n} (\det \alpha)^{k-1} \alpha_{i}^{-1} \cdot c(\gamma_{i}) + (\gamma \cdot v_{0} - v_{0}),
\]

where \(v_{0} = \sum_{i=1}^{n} (\det \alpha)^{k-1} \alpha_{i}^{-1} \cdot c(\tilde{\gamma}_{i})\). Hence, we have shown that the operator \(T_{\alpha}\) on 1-cocycles does not depend on the chosen coset representatives if we view its values modulo \(B^{1}(\Gamma, V_{k})\). Now, we want to show that it is a well-defined operator.

We choose coset representatives \(\{\alpha_{i}\} \) for \(\Gamma \backslash \Gamma\alpha\) so that \(\Gamma = \bigsqcup \Gamma_{\alpha}(\alpha^{-1}\alpha_{i})\). We can do this by [1, Lemma 5.1.2]. Since we have \(\alpha_{i} \gamma = \gamma_{i} \alpha_{j[i]}\) for every \(\gamma \in \Gamma\), we see that \((\alpha^{-1}\alpha_{i})\gamma = (\alpha^{-1}\gamma_{i}\alpha)\alpha^{-1}\alpha_{j[i]}\). Since \(\alpha^{-1}\alpha_{i} \in \Gamma\) for every \(i\), we have \((\alpha^{-1}\alpha_{i})\gamma(\alpha^{-1}\alpha_{j[i]}^{-1})^{-1} \in \Gamma\). Thus, it follows from [2, p. 45] that

\[
\text{Cor} : H^{1}(\Gamma, V_{k}) \rightarrow H^{1}(\Gamma_{\alpha}, V_{k}),
\]
\[
c \mapsto (\gamma \mapsto \sum_{i=1}^{n} (\alpha^{-1}\alpha_{i})^{-1} \cdot c(\gamma^{-1}\alpha_{j[i]}^{-1} \cdot (\alpha^{-1}\gamma_{i}\alpha)) = \sum_{i=1}^{n} \alpha_{i}^{-1} \alpha \cdot c(\alpha_{j[i]}^{-1} \gamma_{i}\alpha))
\]

where \(\alpha_{i} \gamma = \gamma_{i} \alpha_{j[i]}\). To compute the restriction map along \(\iota_{2}\), observe that the isomorphism

\[
V_{k} \rightarrow V_{k},
\]
\[
v \mapsto \alpha \cdot v
\]

is equivariant for the \(\Gamma_{\alpha}\)-action on the left-side and \(\Gamma\)-action on the right-side via the embedding \(\iota_{2}\). Thus, the restriction map is computed as follows

\[
\text{Res} : H^{1}(\Gamma_{\alpha}, V_{k}) \rightarrow H^{1}(\Gamma, V_{k})
\]
\[
c \mapsto (\gamma \mapsto \alpha^{-1} \cdot c(\alpha \gamma \alpha^{-1})).
\]

As a result, we see that the composite map \(\text{Cor} \circ \text{Res}\) is the desired map. Hence, \(T_{\alpha}\) is a well-defined action \(H^{1}(\Gamma, V_{k})\).

\[\rule{1cm}{0.1mm}\]

**Lemma 1.4.** The \(T_{\alpha}\)-action on \(H^{1}(\Gamma, V_{k})\) is induced by scalar extension of the analogous operation on \(H^{1}(\Gamma, \text{Sym}^{k-2}(\mathbb{Z}^{2}))\).

**Proof.** Since \(k \geq 2\), we have \((\det \alpha)^{k-1} \alpha_{i}^{-1} = (\det \alpha)^{k-2}((\det \alpha) \alpha_{i}^{-1})\), with \((\det \alpha) \alpha_{i}^{-1}\) a matrix having \(\mathbb{Z}\) entries. The result then follows from the cocycle formula for \(\Gamma_{\alpha}\).

\[\rule{1cm}{0.1mm}\]
Lemma 1.5. Consider the action of $T_\alpha$ on $H^1(\Gamma, V_k)$ that we defined in Lemma 1.3. The injective map $j$ in (1.15) carries the $[\Gamma_0 \Gamma]$-action on $S_k$ over to the $T_\alpha$-action on $H^1(\Gamma, V_k)$ for every $\alpha$ in $\Delta$ as in (1.16).

Proof. Choose $\alpha \in \Delta$ and coset representatives $\{a_i\}$ for $\Gamma \backslash \Gamma \alpha \Gamma$, so $\Gamma \alpha \Gamma = \bigsqcup_{i=1}^n \Gamma a_i$. For $f \in S_k$ we have $f|_{k[\Gamma_0 \Gamma]} = \sum_{i=1}^n f|_{k \alpha_i}$. Now for each $i$ and $\gamma \in \Gamma$, we compute $I_{f|_{k \alpha_i}}(\gamma)$ via (1.1):

$$I_{f|_{k \alpha_i}}(\gamma) = \int_{z_0}^{\gamma z_0} (ze + \epsilon)^{k-2}(f|_{k \alpha_i})dz,$$

$$= \alpha_i^{-1} \int_{z_0}^{\gamma z_0} \alpha_i \cdot (ze + \epsilon)^{k-2}(f|_{k \alpha_i})dz,$$

$$= \alpha_i^{-1} \cdot (\det \alpha_i)^{k-1} \int_{\alpha_i z_0}^{\gamma \alpha_i z_0} (ze + \epsilon)^{k-2}f dz.$$

The last equality follows by the calculations that are similar to the ones that we did in (1.2). Since for $\gamma \in \Gamma$ right multiplication by $\gamma$ permutes $\Gamma \alpha _i$, for every $i$ and $\gamma \in \Gamma$ there exists a unique $j[i]$ and $\gamma_i \in \Gamma$ such that $\alpha \gamma = \gamma_i \alpha_j[i]$. By using this equality we compute

$$I_{f|_{k \Gamma_0 \Gamma}}(\gamma) = (\det \alpha)^{-k+1} \sum_{i=1}^n \alpha_i^{-1} \int_{\alpha_i z_0}^{\gamma \alpha_i z_0} (ze + \epsilon)^{k-2}f dz,$$

$$= (\det \alpha)^{-k+1} \sum_{i=1}^n \alpha_i^{-1} \cdot \left( \int_{z_0}^{\gamma \alpha_i z_0} (ze + \epsilon)f dz - \int_{z_0}^{\alpha_i z_0} (ze + \epsilon)^{k-2}f dz \right),$$

$$= (\det \alpha)^{-k+1} \sum_{i=1}^n \alpha_i^{-1} \cdot \left( \int_{\gamma_i z_0}^{\alpha_i z_0} (ze + \epsilon)f dz + \int_{\gamma_i z_0}^{\gamma z_0} (ze + \epsilon)^{k-2}f dz \right)$$

$$- \int_{z_0}^{\alpha_i z_0} (ze + \epsilon)^{k-2}f dz,$$

$$= (\det \alpha)^{-k+1} \sum_{i=1}^n \alpha_i^{-1} \cdot (\gamma_i \int_{z_0}^{\alpha_i z_0} (ze + \epsilon)f dz + \int_{z_0}^{\gamma_i z_0} (ze + \epsilon)^{k-2}f dz)$$

$$- \int_{z_0}^{\alpha_i z_0} (ze + \epsilon)^{k-2}f dz,$$

by similar calculations done in (1.2),

$$= (\det \alpha)^{-k+1} \left( \sum_{i=1}^n \gamma \alpha_i^{-1} \int_{z_0}^{\gamma z_0} (ze + \epsilon)f dz + \sum_{i=1}^n \alpha_i^{-1} \int_{z_0}^{\gamma_i z_0} (ze + \epsilon)^{k-2}f dz \right)$$

$$- \sum_{i=1}^n \alpha_i^{-1} \int_{z_0}^{\alpha_i z_0} (ze + \epsilon)f dz,$$

since $\alpha_i^{-1} \gamma_i = \gamma_j^{-1}$.

$$= (\det \alpha)^{-k+1} \left( \sum_{i=1}^n \alpha_i^{-1} \int_{z_0}^{\gamma_i z_0} (ze + \epsilon)f dz + (\gamma \cdot v_1 - v_1) \right),$$

where $v_1 = \sum_{i=1}^n \alpha_i^{-1} \int_{z_0}^{\alpha_i z_0} (ze + \epsilon)^{k-2}f dz$. Therefore, we see that for every $\alpha \in \Delta$ and $f \in S_k$ we have the quality $j(f|_{k[\Gamma_0 \Gamma]}) = T_\alpha(j(f))$ in $H^1(\Gamma, V_k)$. Hence, the lemma follows.

Theorem 1.6. Let $T$ be the $Z$-subalgebra of $\text{End}_C(S_k)$ generated by Hecke operators $T_p$ for every prime $p$ and diamond operators $(d)$ for every $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. Then $T$ is finitely generated as a $Z$-module.

Proof. By Proposition 1.2, we have $C$-linear injection

$$j : S_k \rightarrow H^1(\Gamma, V_k)$$

for $\Gamma = \Gamma_1(N)$. By Lemma 1.3, for every $\alpha \in \Delta$ (see (1.16)) we have a well-defined action $T_\alpha$ on $H^1(\Gamma, V_k)$. By Lemma 1.5, the action $T_\alpha$ on $H^1(\Gamma, V_k)$ is compatible with the action of $[\Gamma_0 \Gamma]$ on $S_k$. 


Let $T'$ be the $\mathbb{Z}$-subalgebra of $\text{End}_C(H^1(\Gamma, V_k))$ generated by the $T_\alpha$ for every $\alpha \in \Delta$. Then, by Lemma 1.4, the $\mathbb{Z}$-algebra $T'$ is in the image of the $\mathbb{Z}$-subalgebra of $\text{End}_\mathbb{Z}(H^1(\Gamma, \text{Sym}^{k-2}(\mathbb{Z}^2)))$. Since $H^1(\Gamma, \text{Sym}^{k-2}(\mathbb{Z}^2))$ is a finitely generated $\mathbb{Z}$-module, $T'$ is also a finitely generated $\mathbb{Z}$-module. By construction, the $T'$-action on $H^1(\Gamma, V_k)$ preserves $S_k$, so we get a restriction map

$$\nu : T' \longrightarrow \text{End}_C(S_k)$$

defined by $\nu(T) = T|_{S_k}$ for every $T \in T'$. The image of $\nu$ in $\text{End}_C(S_k)$ is $T$. Therefore, since $T'$ is finitely generated $\mathbb{Z}$-module, $T$ is finitely generated $\mathbb{Z}$-module.

\section{Some Commutative Algebra}

In this section we again assume that $\Gamma = \Gamma_1(N)$. Remember that we denote the $\mathbb{C}$-vector space $S_k(\Gamma_1(N))$ of cusp forms of weight $k$ on $\Gamma$ by $S_k$. Let $S_k(\Gamma, \mathbb{Q})$ be the space of cusp forms with in $S_k$ with Fourier coefficients in $\mathbb{Q}$. By [4, Thm. 3.52], we know that $S_k$ has a $\mathbb{C}$ basis that comes from $S_k(\Gamma, \mathbb{Q})$ and so we have a surjection

$$S_k(\Gamma, \mathbb{Q}) \otimes \mathbb{Q} \mathbb{C} \longrightarrow S_k.$$  

Actually, this basis also spans the $\mathbb{Q}$-vector space $S_k(\Gamma, \mathbb{Q})$ and so this surjection is in fact an isomorphism. This “justifies” the following two definitions.

\textbf{Definition 2.1.} For any field $F$ with characteristic 0,

$$S_k(\Gamma, F) := S_k(\Gamma, \mathbb{Q}) \otimes \mathbb{Q} F.$$  

Remember that $T$ is the $\mathbb{Z}$-subalgebra of $\text{End}_C(S_k)$ generated by Hecke operators $T_p$ for every prime $p$ and diamond operators $\langle d \rangle$ for every $d \in (\mathbb{Z}/N\mathbb{Z})^\times$.  

\textbf{Definition 2.2.} For any domain $R$ with characteristic 0, we define

$$T_R := T \otimes_{\mathbb{Z}} R$$

acting on $S_k(\Gamma, \text{Frac}(R))$.

\textbf{Remark 2.3.} By Theorem 1.6 we know that $T_R$ is a finite free $R$-module.

Let $\ell$ be a prime number. Fix an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_\ell$. Let $K$ be a finite extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$. Let $\mathcal{O}$ be its ring of integers and $\lambda$ be its maximal ideal. Consider the finite flat $\mathcal{O}$-algebra $T_\mathcal{O}$.

\textbf{Proposition 2.4.} The minimal prime ideals of $T_\mathcal{O}$ are those lying over the prime ideal $(0)$ of $\mathcal{O}$.

\textbf{Proof.} Let $P$ be a minimal prime ideal of $T_\mathcal{O}$. Since $T_\mathcal{O}$ is a flat $\mathcal{O}$-algebra, the going down theorem holds between $T_\mathcal{O}$ and $\mathcal{O}$ (see [3, Thm. 9.5]). Therefore, $P \cap \mathcal{O} = (0)$. Now, suppose that $P'$ is a prime ideal of $T_\mathcal{O}$ such that $P' \subset P$ and $P' \cap \mathcal{O} = (0)$. As $T_\mathcal{O}$ is an integral extension of $\mathcal{O}$, there are no strict inclusions between prime ideals lying over $(0)$. Thus, $P' = P$. Hence, the proposition follows. $\blacksquare$

The $K$-algebra $T_K$ is Artinian. Hence, it has only a finite number of prime ideals, all of which are maximal. By Proposition 2.4, the natural map

$$T_\mathcal{O} \leftrightarrow T_\mathcal{O} \otimes_{\mathcal{O}} K \cong T_K$$

induces a bijection

$$(2.1) \quad \{\text{minimal prime ideals of } T_\mathcal{O}\} \leftrightarrow \{\text{prime ideals of } T_K\}.$$  

Moreover, since $\mathcal{O}$ is complete, $T_\mathcal{O}$ is $\lambda$-adically complete and by [3, Thm. 8.15] there is an isomorphism

$$T_\mathcal{O} \cong \prod_m T_m.$$  

The product is taken over the finite set of maximal ideals $m$ of $T_\mathcal{O}$ and $T_m$ denotes the localization of $T_\mathcal{O}$ at $m$. Each $T_m$ is a complete local $\mathcal{O}$-algebra which is finite free as an
\( \mathcal{O} \)-module. With this isomorphism we see that every prime ideal of \( T_{\mathcal{O}} \) is contained in the unique maximal ideal of \( T_{\mathcal{O}} \). Hence, we have a surjection

\[
(2.2) \quad \{ \text{minimal prime ideals of } T_{\mathcal{O}} \} \twoheadrightarrow \{ \text{maximal ideals of } T_{\mathcal{O}} \}.
\]

Let \( G_K \) be the absolute Galois group of \( K \). Suppose \( f = \sum a_n q^n \) is a normalized eigenform in \( S_k(\Gamma, \overline{K}) \). Then \( T \to (Tq^{-}\text{eigenvalue of } f) \) defines a ring map \( T \rightarrow \overline{K} \) and so induces a \( K \)-algebra homomorphism \( \Theta_f : T_K \rightarrow \overline{K} \). The image is the finite extension of \( K \) generated by the \( a_n \) and the kernel is a maximal ideal of \( T_K \) which depends only on the \( G_K \)-conjugacy class of \( f \). Thus, we have the map

\[
(2.3) \quad \varphi : \left\{ \text{normalized eigenforms in } S_k(\Gamma, \overline{K}) \text{ modulo } G_K-\text{conjugacy} \right\} \rightarrow \{ \text{maximal ideals of } T_K \}
\]

defined by \( \varphi(f) = \ker(\Theta_f) \).

**Proposition 2.5.** The map \( \varphi \) in (2.3) is a bijection.

**Proof.** For any maximal ideal \( m \) of \( T_K \), all \( K \)-algebra embeddings \( T_K/m \rightarrow \overline{K} \) are obtained from a single one by composing with an element of \( G_K \). Thus, we can make the identification

\[
\{ \text{maximal ideals of } T_K \} = \text{Hom}_{K-\text{alg}}(T_K, \overline{K})/(G_K-\text{action}).
\]

Thus, to prove the proposition it is enough to show that the \( G_K \)-equivariant map

\[
\psi : \{ \text{normalized eigenforms in } S_k(\Gamma, \overline{K}) \} \rightarrow \text{Hom}_{K-\text{alg}}(T_K, \overline{K})
\]

defined by \( \psi(f)(T) = (Tq^{-}\text{eigenvalue of } f) \) is bijective. To do this, consider the \( \overline{K} \)-linear map

\[
(2.4) \quad \delta : S_k(\Gamma, \overline{K}) \rightarrow \text{Hom}_{K-\text{vp}}(T_K, \overline{K})
\]

\[
f \mapsto (\alpha_f : T \mapsto a_1(Tf)).
\]

If we can show that \( \delta \) is an isomorphism of \( \overline{K} \)-vector spaces, then we claim we are done. Because in (2.4) we claim that \( f \in S_k(\Gamma, \overline{K}) \) is a normalized eigenform if and only if \( \alpha_f \) is a ring homomorphism. To see this, suppose \( f \in S_k(\Gamma, \overline{K}) \) is a normalized eigenform, so there exists a \( K \)-algebra homomorphism \( \Theta_f : T_K \rightarrow \overline{K} \) defined by \( T_\cdot = \Theta_f(T)f \) for every \( T \in T_K \). Clearly \( \delta(f) = \alpha_f \) where

\[
\alpha_f(T) = a_1(Tf) = a_1(\Theta_f(T)f) = \Theta_f(T)a_1(f) = \Theta_f(T)
\]

for every \( T \in T_K \). Thus, \( \alpha_f \) is a \( K \)-algebra homomorphism. Conversely, consider any \( K \)-algebra homomorphism \( \alpha : T_K \rightarrow \overline{K} \), so \( \alpha(T) = a_1(Tf) \) for some unique \( f \in S_k(\Gamma, \overline{K}) \). Let \( \lambda_n = \alpha(T_n) \) for every \( T_n \in T_K \). Then we have

\[
a_1(TT_nf) = \alpha(TT_n) = \alpha(T)\alpha(T_n) = \lambda_n a_1(Tf) = a_1(T\lambda_nf)
\]

for every \( T \in T_K \) and \( n \geq 1 \). Taking \( T = T_m \) for every \( m \geq 1 \) gives \( T_nf = \lambda_nf \) for every \( n \geq 1 \), proving that \( f \) is an eigenform. Moreover, as \( \alpha \) is a \( K \)-algebra map, \( 1 = \alpha(\text{id}) = a_1(f) \). Hence, \( f \) is a normalized eigenform in \( S_k(\Gamma, \overline{K}) \).

Now, we will show that \( \delta \) is an isomorphism of \( \overline{K} \)-vector spaces. For injectivity, suppose \( \delta(f) = \alpha_f \) is the zero map, so \( a_1(Tf) = 0 \) for every \( T \in T_K \). In particular, \( a_n(f) = a_1(T_nf) = 0 \) for every \( n \geq 1 \), which implies that \( f = 0 \). To prove surjectivity of \( \delta \), it is enough to show that

\[
(2.5) \quad \dim_{\overline{K}}\text{Hom}_{K-\text{vp}}(T_K, \overline{K}) \leq \dim_{\overline{K}}S_k(\Gamma, \overline{K}).
\]

Since \( \text{Hom}_{K-\text{vp}}(T_K, \overline{K}) \cong \text{Hom}_{\overline{K}}(T_K, \overline{K}) \), we can work with \( \text{Hom}_{\overline{K}}(T_K, \overline{K}) \). Actually, with this identification, studying the map \( \delta \) is the same as studying the \( \overline{K} \)-bilinear mapping

\[
S_k(\Gamma, \overline{K}) \times T_K \rightarrow \overline{K}
\]

\[
(f, T) \mapsto a_1(Tf)
\]
between finite-dimensional $\overline{K}$-vector spaces. Thus, to prove (2.5), it is enough to show that the map
\[
\epsilon : T_{\overline{K}} \rightarrow \operatorname{Hom}_K(S_k(\Gamma, \overline{K}), \overline{K})
\]
\[T \rightarrow (f \rightarrow a_1(Tf))\]
is injective. Suppose $\epsilon(T)$ vanishes for some $T$. Thus, for every $f \in S_k(\Gamma, \overline{K})$ and for every integer $n \geq 1$ we have $a_1(T_nTf) = a_1(TT_nf) = 0$. Therefore, $Tf = 0$ for every $f \in S_k(\Gamma, \overline{K})$. Since $T_{\overline{K}}$ acts faithfully on $S_k(\Gamma, K)$, we get $T = 0$, proving that the map $\epsilon$ is injective. Hence, the proposition follows.

Combining the bijections (2.1) and (2.3) and the surjection (2.2), we have the following diagram.

\[
\begin{array}{ccc}
\{\text{minimal prime ideals of } T_O\} & \rightarrow & \{\text{maximal ideals of } T_O\} \\
\downarrow & & \downarrow \\
\{\text{prime ideals of } T_K\} & \rightarrow & E = \{\text{normalized eigenforms in } S_k(\Gamma, \overline{K}) \text{ modulo } G_K-\text{conjugacy}\}
\end{array}
\]

Let $\mathfrak{m}$ be any maximal ideal of $T_O$, so $\mathfrak{m}$ is the kernel of a map $\Phi : T_O \rightarrow \overline{F}_\ell$. We want to attach a residual representation $\overline{\rho}_\mathfrak{m}$ over $\overline{F}_\ell$ to $\mathfrak{m}$ using the diagram (2.6). Let $\{f_1, \ldots, f_r\}$ be a set of representatives of all normalized eigenforms in $E$ such that in the diagram (2.6) their corresponding minimal prime ideals $\mathfrak{p}_{f_i}$ in $T_O$ are inside the maximal ideal $\mathfrak{m}$. For each $i$, let $\mathfrak{p}_{f_i}'$ be the corresponding prime ideal in $T_K$, so $\mathfrak{p}_{f_i}' \subset T_O = \mathfrak{p}_{f_i}$. Thus, for each $i$, we have a map
\[
\Theta_{f_i} : T_O \rightarrow \overline{O}
\]
\[T_n \rightarrow a_n(f_i)\]
with kernel $\mathfrak{p}_{f_i}$. Since each $\mathfrak{p}_{f_i} \subset \mathfrak{m}$, the map $\Phi : T_O \rightarrow \overline{F}_\ell$ factors through $\operatorname{Im} \Theta_{f_i}$ for each $i$ as follows,

\[
\begin{array}{ccc}
\operatorname{Im} \Theta_{f_1} & \rightarrow & \overline{F}_\ell \\
\uparrow & \cdots & \downarrow \\
T_O & \rightarrow & \overline{F}_\ell.
\end{array}
\]

For each $i$, the quotient $T_K/\mathfrak{p}_{f_i}'$ is a finite extension $K_{f_i}$ of $K$. Let $O_{f_i}$ be its ring of integers and $k_{f_i}$ be its residue field. Each map $\operatorname{Im} \Theta_{f_i} \rightarrow \overline{F}_\ell$ lifts to $O_{f_i}$, lifting the embedding of the residue field of $\operatorname{Im} \Theta_{f_i}$ to an embedding of $k_{f_i}$ into $\overline{F}_\ell$. The above commutative diagram tells us that for every integer $n \geq 1$, we have
\[
a_n(f_1) = \ldots = a_n(f_r)
\]
in $\overline{F}_\ell$. Consider the semisimplified residual representation $\overline{\rho}_{f_i}$ associated to each $f_i$; it is defined over $k_{f_i}$. For every prime $p$ such that $p \not| N\ell$ we have
\[
\operatorname{tr}(\overline{\rho}_{f_i}(\text{Frob}_p)) = \ldots = \operatorname{tr}(\overline{\rho}_{f_r}(\text{Frob}_p))
\]
over $\mathbb{F}_ℓ$. We obtain a similar result for the determinants of $\tilde{\rho}_f(Frob_p)$’s when we compare the characters $\chi_f$ associated to $f_i$’s. Therefore, we obtain

$$\tilde{\rho}_{f_1} \cong \ldots \cong \tilde{\rho}_{f_1}$$

over $\mathbb{F}_ℓ$. We let $\tilde{\rho}_m$ denote this common residual representation.

3. The Main Theorem

In this section we prove the following theorem.

**Theorem 3.1.** Let $K$ be a finite extension of $\mathbb{Q}_ℓ$ such that its ring of integers $\mathcal{O}$ is big enough to contain all Hecke eigenvalues at level $N$. Let $\lambda$ be its maximal ideal, $k$ its residue field and $\mathfrak{m}$ a maximal ideal of $\mathcal{T}_\mathcal{O}$. Consider the associated residual representation

$$\tilde{\rho}_m : G_\mathbb{Q} \rightarrow GL_2(k)$$

over $k$. Assume $\tilde{\rho}_m$ is absolutely irreducible. Then there exists a unique deformation

$$\rho_m : G_\mathbb{Q} \rightarrow GL_2((\mathcal{T}_m)_\text{red})$$

such that

1. $\rho_m$ is unramified at every prime $p$ such that $p \nmid N\ell$,
2. For every prime $p$ such that $p \nmid N\ell$, the characteristic polynomial of $\rho_m(Frob_p)$ is $x^2 - T_p x + p^{k-1}(p)$.

Before proving this theorem, consider the following theorem which was proved by Akshay in his talk. The corollary of this theorem will be the main ingredient while proving Theorem 3.1.

**Theorem 3.2.** Let $R$ be a complete local Noetherian ring and let $\rho : G_\mathbb{Q} \rightarrow GL_2(R)$ be a residually absolutely irreducible representation. If $S$ is a complete local Noetherian subring of $R$ which contains all the traces of $\rho$, then the Galois representation $\rho$ is conjugate to a representation $G_\mathbb{Q} \rightarrow GL_2(S)$.

**Corollary 3.3.** Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_ℓ$, with maximal ideal $\lambda$ and residue field $k$. Let $\Sigma$ be a finite set of places of $\mathbb{Q}$ containing $\ell$. Let $\rho : G_\mathbb{Q} \rightarrow GL_2(R)$ be the universal deformation unramified outside $\Sigma$ for an absolutely irreducible representation $\tilde{\rho} : G_\mathbb{Q} \rightarrow GL_2(k)$ unramified outside $\Sigma$, taken on the category of complete local Noetherian $\mathcal{O}$-algebras with residue field $k$. The traces $\text{tr}(\rho(Frob_p))$ for all but finitely many primes $p \not\in \Sigma$ generate a dense $\mathcal{O}$-subalgebra of $R$.

**Proof.** Let $M_R$ be the maximal ideal of $R$. By successive approximation, it is enough to show that such $\text{tr}(\rho(Frob_p))$ generate $R/(\lambda, M_R^2)$ as $k$-algebras. Let $R_1 := R/(\lambda, M_R^2)$. The ring $R_1$ is the universal deformation ring for $\tilde{\rho}$ for $k$-algebras with residue field $k$ such that the square of the maximal ideal is zero. Let $S$ be a $k$-subalgebra of $R_1$ generated by $\text{tr}(\rho(Frob_p))$ for almost all primes $p \not\in \Sigma$. Being a subring of $R_1$, the square of the maximal ideal of $S$ is also zero. If we can show that $R_1 = S$, then we’re done.

By Theorem 3.2 we have the following commutative diagram (up to conjugation) which lifts $\tilde{\rho}$

$$\begin{array}{c}
G_\mathbb{Q} \rightarrow GL_2(S) \\
\rho_1 \downarrow \downarrow \downarrow \\
GL_2(R_1)
\end{array}$$

Also, since $R_1$ is the universal deformation ring of $\tilde{\rho}$ we have the following commutative diagram (up to conjugation) which lifts $\tilde{\rho}$

$$\begin{array}{c}
G_\mathbb{Q} \rightarrow GL_2(R_1) \\
\rho_1 \downarrow \downarrow \downarrow \\
GL_2(S)
\end{array}$$
As a result we have the following composition of maps

$$R_1 \longrightarrow S \hookrightarrow R_1$$

which carries \( \rho_1 \) to itself and hence is the identity map. Thus, \( S = R_1 \).

**Proof of Theorem 3.1.** Let \( f \) be a normalized eigenform in \( S_k(\Gamma, \overline{K}) \) such that the corresponding minimal prime ideal \( \mathfrak{p}_f \) in \( T_0 \) is contained in \( \mathfrak{m} \) (see diagram (2.6)). By Deligne, we have a Galois representation \( \rho_f \) over \( \mathcal{O} \) associated to \( f \) whose residual reduction is \( \overline{\rho}_m \):

$$\xymatrix{ G\mathbb{Q} \ar[r]^{\rho_f} & \text{GL}_2(\mathcal{O}) \ar[d] \ar[r] & \text{GL}_2(k) \ar[d] \ar[r] & \text{GL}_2(\mathcal{O}) \ar[d] \ar[r] & \text{GL}_2(\mathbb{Q}) \ar[d] }$$

Let \( (R, \rho : G \longrightarrow \text{GL}_2(R)) \) be the universal deformation of \( \overline{\rho}_m \) unramified outside \( N\ell \). Then \( \rho_f \) corresponds to an \( \mathcal{O} \)-algebra map \( R \longrightarrow \mathcal{O} \), so the diagram

$$\xymatrix{ G\mathbb{Q} \ar[r]^{\rho} & \text{GL}_2(R) \ar[d] \ar[r] & \text{GL}_2(\mathcal{O}) \ar[d] \ar[r] & \text{GL}_2(\mathbb{Q}) \ar[d] }$$

commutes up to conjugation by \( 1 + M_2(\lambda) \) in \( \text{GL}_2(\mathcal{O}) \). By Corollary 3.3, we see that the set of \( \text{tr}(\rho(\text{Frob}_q)) \) for every prime \( q \nmid N\ell \) generates a dense \( \mathcal{O} \)-subalgebra in \( R \).

Consider the map

$$\eta : R \longrightarrow \prod_{\mathfrak{p}_f \subset \mathfrak{m}} \mathcal{O}$$

$$\text{tr}(\rho(\text{Frob}_q)) \mapsto \prod_{\mathfrak{p}_f} a_q(f)$$

where the product is taken over minimal primes \( \mathfrak{p}_f \) contained in \( \mathfrak{m} \), with \( f \) the corresponding normalized eigenform in \( S_k(\Gamma, \overline{K}) \). Consider the embedding

$$\left( T_m \right)_{\text{red}} \hookrightarrow \prod_{\mathfrak{p}_f \subset \mathfrak{m}} T_\mathfrak{O}/\mathfrak{p}_f$$

$$T_q \hookrightarrow \prod_{\mathfrak{p}_f} T_q (\text{mod } \mathfrak{p}_f).$$

With the identification

$$\prod_{\mathfrak{p}_f \subset \mathfrak{m}} \mathcal{O} = \prod_{\mathfrak{p}_f \subset \mathfrak{m}} T_\mathfrak{O}/\mathfrak{p}_f$$

$$\prod_{\mathfrak{p}_f} a_q(f) \mapsto \prod_{\mathfrak{p}_f} T_q (\text{mod } \mathfrak{p}_f),$$

we see that all \( \text{tr}(\rho(\text{Frob}_q)) \) for \( q \nmid N\ell \) land in the closed subalgebra \( (T_m)_{\text{red}} \). Since they generate dense algebra in \( R \), the ring \( R \) also lands in there under \( \eta \), say inducing \( h : R \longrightarrow (T_m)_{\text{red}} \). Thus, we get

$$\rho_m : G\mathbb{Q} \overset{\rho}{\longrightarrow} \text{GL}_2(R) \overset{h}{\longrightarrow} \text{GL}_2((T_m)_{\text{red}}).$$

This gives existence and also uniqueness since any other \( \rho'_m \) would give another map \( h' : R \longrightarrow (T_m)_{\text{red}} \) and compatibility with traces of representations then forces \( \text{tr}(\rho(\text{Frob}_q)) \mapsto T_q \). Thus, \( h \) and \( h' \) coincide on a dense set, hence \( h = h' \). By checking in each \( T_\mathfrak{O}/\mathfrak{p}_f = \mathcal{O} \), we see that \( \rho_m(\text{Frob}_q) \) has the expected characteristic polynomial for every \( q \nmid N\ell \).
4. Reduced Hecke Algebras

In this section, let $K$ be a finite extension of $\mathbb{Q}_\ell$ and $\mathcal{O}$ its ring of integers. For any ring $A$, let $T_A$ be the $A$-subalgebra of $T_A$ generated by the Hecke operators $T_p$ for $p \nmid N\ell$ and diamond operators $\langle d \rangle$ for every $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. Fix a maximal ideal $\mathfrak{m}$ of $T_{\mathcal{O}}$. We have a map $T_{\mathcal{O}} \to \overline{\mathbb{F}}_\ell$ with kernel $\mathfrak{m}$. Since $T_{\mathcal{O}}$ is an integral extension of $T_{\mathcal{O}}$ and $\overline{\mathbb{F}}_\ell$ is algebraically closed, this map can be extended to $T_{\mathcal{O}}$. Let $\mathfrak{m}'$ be the kernel of this extended map, so it is a maximal ideal of $T_{\mathcal{O}}$. Consider common (up to isomorphism) residual representation $\bar{\rho}_f$ for all normalized eigenforms $f$ whose corresponding minimal primes $p_f$ (see (2.6)) are contained in $\mathfrak{m}'$. Call it $\bar{\rho}_m$. In this section we prove the following theorem.

**Theorem 4.1.** If the Serre conductor $\mathcal{N}(\bar{\rho}_m)$ is equal to $N$ then the $\mathcal{O}$-algebra $(T_{\mathcal{O}})_{\mathfrak{m}}$ is reduced.

**Proof.** Since the Serre conductor $\mathcal{N}(\bar{\rho}_m)$ is equal to $N$, the minimal possible level of a normalized eigenform $f$ such that $\bar{\rho}_f \simeq \bar{\rho}_m$ over $\overline{\mathbb{F}}_\ell$ is $N$. Thus, such $f$ are newforms. To prove the theorem, we will show that $(T_{\mathcal{O}})_{\mathfrak{m}} \otimes_{\mathcal{O}} K$, which contains $(T_{\mathcal{O}})_{\mathfrak{m}}$, is reduced. We have the equality

$$(T_{\mathcal{O}})_{\mathfrak{m}} \otimes_{\mathcal{O}} K = \prod_{p \mathfrak{K}} (T_K)_{p \mathfrak{K}}$$

where the product is taken over all prime ideals $p \mathfrak{K}$ of the Artinian ring $T_K$ such that $p \mathfrak{K} \cap T_{\mathcal{O}} \subseteq \mathfrak{m}$ and $(T_K)_{p \mathfrak{K}}$ denotes the localization of $T_K$ at $p \mathfrak{K}$. Thus, each $p \mathfrak{K}$ in the product corresponds to a newform. To prove the theorem it is therefore enough to show that $(T_K)_p$ is a field when $p$ corresponds to a newform.

Assume the prime ideal $p$ of $T_K$ corresponds to a newform $f \in S_k(\Gamma, K)$ of level $N$. We can increase $K$ to a finite extension. Thus, without loss of generality we can assume that $K$ is big enough to contain the Hecke eigenvalues of all normalized eigenforms at level $N$. Since $S_k(\Gamma, K)$ is faithful $T_K$-module, the localization $(S_k(\Gamma, K))_p$ at $p$ is faithful $(T_K)_p$-module. If we can prove that $(S_k(\Gamma, K))_p$ is one dimensional as a vector space over $K$ then we are done, because this would force $(T_K)_p$ to be equal to $K$.

We have

$$S_k(\Gamma, K) = K f \oplus \left( \bigoplus_g S_g(\Gamma, K) \right)$$

where the direct sum is taken over all newforms $g$ of level $N_g$ and $S_g(\Gamma, K)$ is spanned by $g(vz)$ for the divisors $v$ of $N/N_g$. By multiplicity one, for every $g$ which is different from $f$, there exists a prime $q \nmid N\ell$ such that

$$a_q(g(vz)) = a_q(g(z)) \neq a_q(f(z))$$

for every $v|(N/N_g)$. We know that $(T_q - a_q(f)) \in p$ and it acts on $g(vz)$ as

$$(T_q - a_q(f))g(vz) = T_q(g(vz)) - a_q(f)g(vz) = (a_q(g) - a_q(f))g(vz).$$

By the above argument, $(a_q(g) - a_q(f)) \in K^\times$. But $(T_K)_p$ is Artin local, so its maximal ideal is nilpotent. This forces $\left( \bigoplus_{g \neq f} S_g(\Gamma, K) \right)_p = 0$. As a result, $(S_k(\Gamma, K))_p = Kf$ and the theorem follows.$\blacksquare$

**References**


