2009-10 SEMINAR ON MODULARITY-LIFTING (383-N, THURSDAYS, 1:30PM–3:30PM)

The aim of the seminar is to cover the "modern" version of the Taylor-Wiles method for proving modularity of p-adic Galois representations that are residually modular. We focus on the "simplest" natural class of examples, those which are ordinary at p. The original version of the method made extensive use of subtle integral models of modular curves and elaborate constructions with cohomology and commutative algebra. Thanks to subsequent generalizations and improvements, mainly due to Kisin and Taylor, the intervention of algebraic geometry has been basically eliminated in the global setting (but in the refined study of local deformation problems at p, Kisin's recent methods use advanced techniques from EGA). This is essential for applying the Taylor-Wiles method beyond the classical case (GL_2 over $\mathbf{Q}$), and even within the classical case it yields tremendous simplifications. By focusing on the ordinary case we can avoid p-adic Hodge theory and get a clear understanding of how the method works without extensive technical digressions.

There are three different topics to be covered: Galois cohomology and deformation theory, automorphic representations and their associated Galois representations (especially the Jacquet-Langlands correspondence, which is the key to removing modular curves from the original method), and finally the Taylor-Wiles method. In the schedule below, we leave a lot of open slots at the end so that when (inevitably) people run overtime it will not screw up the later lectures.

Oct. 2 [Brian] Galois representations associated to classical modular forms: local properties, existence, and application to FLT via Faltings' theorem and to analytic properties of L-functions for varieties. Mod p variant, residual modularity.

Oct. 9 [Akshay] How p-adic representations come from geometry (abelian varieties, étale cohomology), information they encode. In the abstract: good primes, conductor, stable lattice, reduction. Serre's conjecture. What is an "$R = \mathbf{T}$" theorem, and why do we care?

Oct. 16 [Mok] Notion of Galois deformation for $\overline{\rho} : \Gamma \to \mathrm{GL}_n(\mathbf{F})$ for suitable profinite Γ and finite $\mathbf{F}$: universal deformation ring $R_{\overline{\rho}}^{\mathrm{univ}}$ and framed deformation ring $R_{\overline{\rho}}^{\square}$ (allow trivial $\overline{\rho}$!). Work out 1-dimensional case via CFT. Construct and describe $R_{\overline{\rho}}^{\square}$ for trivial $\overline{\rho}$ and $\Gamma = G_F$ for $[F : \mathbf{Q}_\ell] < \infty$ ($\ell \neq p$). Interpret $\mathbf{F}[\epsilon]$-points via H^1 and Z^1, discuss change of dvr coefficients. Formal smoothness of $R_{\overline{\rho}}^{\mathrm{univ}} \to R_{\overline{\rho}}^{\square}$.

Oct. 23 [Brian] Structure of $R[1/p]$, interpretation of its MaxSpec and identification of such completions $R[1/p]_{\wp}^{\wedge}$ as characteristic-0 deformation ring. Relation of its tangent space with p-adic coefficient cohomology.

Oct. 30 [Brandon] State Schlessinger, discuss "deformation conditions", mention $R_{\overline{\rho}}^{\chi}$ (deformation with fixed determinant χ) and its framed analogue. For GL_2: define ordinary deformation problem for $\ell = p$, explain why $R_{\overline{\rho}}^{\mathrm{ord}}$ exists via Schlessinger in the "residually distinguished" case. Describe framed analogue explicitly for $\overline{\rho}$ trivial (using structure of tame inertia).

Nov. 6 [Samit] Relate $\mathrm{H}^1(\Gamma, \mathrm{ad}(\overline{\rho}))$ to extension classes for GL_2, variants for "fixed determinant" condition (GL_n), as well as "ord" (GL_2) and "unramified" (GL_n) for local Galois groups Γ. Discuss general formula (abstract Γ) for minimal number of generators and relations of $R_{\overline{\rho}}^{\mathrm{univ}}$ in terms of Γ-cohomology in low degrees. Given finitely many maps $\Gamma_v \to \Gamma$, discuss map $\widehat{\otimes}_v R_v^{\square} \to R^{\square}$: minimal number of algebra generators, relative dimension.

Nov. 13 [Rebecca] Specialize to $\Gamma = G_{F,S}$ for number field F. Review Tate local duality and local Euler characteristic, Poitou-Tate exact sequence, and global Euler characteristic. Highlights from proofs. Use these to simplify earlier formulas for Galois deformation rings (generators, relations). Effect of adding auxiliary unramified primes v to S with $q_v \equiv 1 \bmod p$. Wiles product formula for Selmer group and its dual (with dual conditions).

Nov. 20 [Burcu] Classical Hecke rings: $\mathbf{Z}$-finiteness, maximal and minimal primes in relation to modular eigenforms and congruences, reducedness and newforms. Interpret $\mathbf{T}_{\mathfrak{m}}$ in residually absolutely irreducible case, construct $\rho_{\mathfrak{m}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{T}_{\mathfrak{m}})$ via traces.

Dec. 4 [for falling behind schedule]

Holiday Break

Jan. 8 [Marty] Smoothness and admissibility for complex $G(F)$-representations, F local non-archimedean field ($G = \mathrm{GL}_n$, etc.). Induction and compact induction: basic properties, principal series. Hecke algebra and representations. The three basic examples of irreducible smooth admissible representations for $G = \mathrm{GL}_2$. Rough statement of local Langlands for GL_2, relate unramifiedness to unramified principal series, and relate Frobenius eigenvalues to a "Hecke polynomial".

Jan. 15: No seminar, due to AMS meeting in San Francisco.

Jan. 22 [Denis] Passing from classical modular forms to automorphic forms on and representations of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ (including definition of latter). Interpretation of level, cuspidality, newforms, Hecke operators (at "good" primes) on automorphic side. "Canonical" decomposition $\pi = \otimes' \pi_v$ for suitable $\mathrm{GL}_n(\mathbf{A}_F^{\infty})$-representations (any global field F and $n \geqslant 1$).

Jan. 29 [Andrew] For totally real F and quaternion division algebra D over F ramified at infinity, discuss automorphic forms on and representations of $D^{\times}$ (cusp forms, level, functions on finite set). Algebraic Hecke characters with "coefficients": complex, p-adic, artinian, etc.

Feb. 5 [Mike] Central simple algebras over general fields and over local/global fields.

Feb. 12 [Akshay] Jacquet-Langlands correspondence and local Langlands correspondence,

Feb. 19 [Alex] Notions of automorphic Galois representation in GL_2 and $D^{\times}$ over totally real field. Cyclic base change and applications: sufficient to check automorphy after solvable base change. Local-to-global for solvability via Grunwald–Wang.

Feb. 26 [Andrew] Summary of the automorphic side, why Hecke algebras on $D^{\times}$ are useful, and how this all fits into the big picture (and is better than the original approach via modular curves).

March 5 [Mok] Hecke algebra on $D^{\times}$ (over totally real field F) with coefficients. Congruences, and construction of $\rho_{\mathfrak{m}} : G_F \to \mathrm{GL}_2(\mathbf{T}_{\mathfrak{m}})$ with suitable local properties (for appropriate D). Construct $R_{\overline{\rho}}^{\mathrm{univ}} \to \mathbf{T}_{\mathfrak{m}}$. Framed variant.

March 12 [Andrew] Overview of Taylor-Wiles method.

Spring Break

April 1 [Simon] Review of what we learned and forgot about Galois deformation theory.

April 8 [Rebecca] Structure of (framed) deformation ring away from p for the "Steinberg" condition. (ref: 2.6 in Kisin's "Moduli of finite flat group schemes" paper)

April 15 [Brian] Structure of ordinary crystalline (framed) deformation ring, Kisin's "normalization" of it (ref: 2.4 in Kisin's "2-adic..." paper)

April 22 [Andrew] "Framed" modular deformation to Hecke algebra with T_p's; verify ord/crys.

April 29 [Akshay] Level raising/lowering away from p (Ihara's Lemma, Skinner-Wiles argument)

May 6 [Mike] Existence of Taylor-Wiles primes. (ref: DDT, also look at Kisin's papers for context)

May 13 [Samit] Congruences at TW primes, verify only gives oldforms. (ref: notes of Andrew)

May 20 [Brandon] Patching argument. (ref: Kisin's papers, notes on Hida's webpage)

May 27 [TBD] Application to a conjecture of some French or Japanese mathematician.