2010-11 SEMINAR ON FALTINGS' PROOF OF MORDELL CONJECTURE (383-N, WEDNESDAYS, 1PM–3PM)

The aim of the seminar is to cover Faltings' proof of the Mordell Conjecture for higher genus curves over number fields K, in the course of which he also proved two other spectacular results: Tate's isogeny conjecture for abelian varieties over K (previously known only for elliptic curves, by work of Serre and Parshin) and Shafarevich's conjecture on finiteness for the set of isomorphism classes of abelian varieties of a fixed dimension g over K with good reduction outside of a fixed finite set S of places of K (previously known only for elliptic curves, by using Siegel's finiteness theorem for S-integral points of affine curves, and in dimension 2 by Parshin).

Faltings' proof did not use Diophantine approximation or Siegel's theorem. His methods are a synthesis of deep techniques in arithmetic geometry: finite flat group schemes (especially work of Raynaud [R]), p-divisible groups (especially work of Tate [T1]), abelian varieties, Néron models, semistable reduction for curves, and various moduli spaces. We won't need any "Arakelov theory".

We'll begin with an overview of Faltings' method (including how the rather different-looking conjectures of Mordell, Tate, and Shafarevich are intimately related), a survey of basic facts about abelian varieties (building on experience with elliptic curves), and Tate's ground-breaking proof of his isogeny conjecture for abelian varieties over finite fields. This proof of Tate inspires some of the key ideas in Faltings' method. Then we will move on to the two core topics needed to understand the proof: finite flat group schemes and p-divisible groups in the fall (with many examples culled from the study of curves and abelian varieties), and Néron models and heights and the moduli space of curves in the winter. Then in the spring we will bring it all together to give Faltings' proof, following the original paper of Faltings (which won't be so bad, given our preparations).

In the schedule below, we leave some open slots at the end so that when (inevitably) people run overtime it will not be a big deal to push ahead the schedule.

Sept. 22 [Akshay]: Overview of Faltings' proof, "properties of heights imply everything".

Sept. 29 [Brian]: Overview of abelian varieties, including Jacobians and duality.

Oct. 6 [Sam]: Proof of Tate conjecture for abelian varieties over finite fields. (Ref: Tate's original paper, or appendix I to Mumford's book "Abelian Varieties".)

Oct. 13 [Samit] Finite flat (commutative) group schemes: definitions, lots of examples (from abelian varieties, α_{p^n}, μ_N, finite étale, base change), relation with (unramified) Galois modules, method of schematic closure, proof of Nagell-Lutz using finite flat group schemes. Ref: [T2], [Sch], [Sh]

Oct. 20 [Akshay] "finite flat group schemes give properties of heights." Elliptic curves only! Ref: [S2]

Oct. 27 [Simon] Cartier duality, quotients for finite flat (commutative) group schemes, exact sequences (and how to use them), examples: connected-étale sequence over field and complete local noetherian ring, interaction with Cartier duality and base change, torsion in abelian varieties. Ref: [T2], [Sch], [Sh]

Nov. 3 [Rebecca]: Raynaud's classification of finite flat group schemes: motivation, examples from elliptic curves (via formal group), F-vector groups, statement of main results, deduction of Theorem 4.1.1. Ref: [R]

Nov. 10 [Brian]: Raynaud's classification of finite flat group schemes: highlights from proofs. Ref: [R]

Nov. 17 [Mike]: p-divisible groups: examples from abelian varieties, connected-étale sequence, relation with formal groups (and special case of abelian varieties), notion of dimension, discriminant at finite level. Ref: Ch. 2 of [T1]

Dec. 1 [Brandon]: statements of Tate's p-adic Hodge decomposition for p-divisible groups, discussion of basic formalism of Hodge–Tate representations, special case of dimension 1, implications for "full faithfulness" of generic fiber, Raynaud's variant. Ref: Section 1 of [R], Ch. 4 of [T1]

Holiday Break

Jan. 5 [Sam]: Definitions of smooth, étale maps, Néron models. Examples: good reduction (abelian scheme), Néron-Ogg-Shafarevich criterion (include statement of Grothendieck's p-adic version with p-divisible groups), Tamagawa numbers, elliptic curves (esp. split multiplicative reduction). Ref: [N], [S1]

Jan. 12 [Christian]: Overview of fundamental moduli spaces: Hilbert schemes and Picard schemes. Precise definitions, intuition, idea of construction, some properties (e.g., properness and smoothness aspects), some examples (e.g., Mumford examples, Jacobians and generalized Jacobians in families). Ref: [FGA]

Jan. 19 [Brian]: Semistable reduction for curves and abelian varieties, Lemma 1 in §2 of Faltings' paper, Grothendieck's criterion for semistable reduction, Raynaud's theorem for Jacobians. Ref: [L], [DM], [N], [SGA7].

Jan. 26 [Ravi] $\overline{\mathcal{M}}_{g,n}$ over $\mathbf{Z}$ (crucial to be over $\mathbf{Z}$), Pic^0 for semistable curves over any (noetherian) base.

Feb. 2 [Akshay]: Definition of the Faltings height. Definition of Weil heights. Precise statement of main property: *Faltings height is close to a Weil height.* Explicit proof for elliptic curves (ref: [S2]). Give a precise version of BSD Conjecture, since we will have covered the fudge factors in detail!

Feb. 9 [Brian]: Semiabelian schemes, Raynaud-Gruson stuff, and Gabber's lemma. Ref: [D, p32–34]

Feb. 23 [Akshay]: Logarithmic singularities. Ref: [D, p34]

March 2 [TBD] (If time permits!): *Sketch*[1] of the proof *assuming* existence of the semiabelian scheme over a "compactification" of $\mathcal{A}_g$. Ref: [D, p. 32] or [C, §5].

March 9 [Burcu]: Review of finite flat group schemes, discussion of Fontaine's proof that there are no abelian varieties over $\mathbf{Q}$ with everywhere good reduction (as good warm-up to what comes next). Ref: [Sch]

Spring Break

March 30 [TBD]: Proof of Tate Conjecture (bypassing the small error in Faltings' argument; see appendix to Faltings' paper).

April 6 [TBD]: Finiteness theorems for the number of isogeny classes, finiteness of isogeny classes (using Raynaud's results).

April 13 [TBD]: Deduction of Mordell, discussion of effectivity aspects of Faltings' proof.

April 20 [TBD]: Mordell over finitely generated fields of characteristic 0 (by induction on transcendence degree, beginning from settled number field case).

April 27 [TBD]: Room for overflow from hereon out.

May 4 [TBD]:

May 11 [TBD]:

May 18 [TBD]:

May 25 [TBD]:

June 1 [TBD]:

[1]Chai's article is quite complicated.

References

[N] Bosch, Lütkebohmert, Raynaud. *N*éron models.
[C] Chai. *Siegel moduli schemes and their compactifications.*
[D] Deligne. *Conjectures de Tate et Shafarevich.*
[DM] Deligne–Mumford.
[FGA] *FGA Explained* (chapters on Hilbert and Picard functors).
[SGA7] Grothendieck, SGA 7, App. to Exp. I.
[L] Q. Liu, *Algebraic geometry and arithmetic curves.*
[R] Raynaud. *Schémas en groupes de type* $(p, \dots, p)$.
[Sch] Schoof. *Introduction to finite group schemes.* `www.cems.uvm.edu/~voight/notes/274-Schoof.pdf`
[Sh] Shatz. *Group schemes, formal groups and p-divisible groups.*
[S1] Silverman. *Advanced topics in the arithmetic of elliptic curves.*
[S2] Silverman. *Heights and elliptic curves*
[T1] Tate. *p-divisible groups.*
[T2] Tate. *Finite flat group schemes*