Math 396. The Topologists’ Sine Curve

We want to present the classic example of a space which is connected but not path-connected. Define

\[ S = \{(x, y) \in \mathbb{R}^2 \mid y = \sin(1/x)\} \cup \{(0) \times [-1, 1]\} \subseteq \mathbb{R}^2, \]

so \( S \) is the union of the graph of \( y = \sin(1/x) \) over \( x > 0 \), along with the interval \([-1, 1]\) in the \( y \)-axis. Geometrically, the graph of \( y = \sin(1/x) \) is a wiggly path that oscillates more and more frequently (between the lines \( y = \pm 1 \)) as we get near the \( y \)-axis (more precisely, over the tiny interval \( 1/(2\pi(n+1)) \leq x \leq 1/(2\pi n) \) the function \( \sin(1/x) \) goes through an entire wave).

We’ll write \( S_+ \) and \( S_0 \) for these two parts of \( S \) (i.e., \( S_+ \) is the graph of \( y = \sin(1/x) \) over \( x > 0 \) and \( S_0 = \{(0) \times [-1, 1]\} \). It is clear that \( S_+ \) is path-connected (and hence connected), as is the graph of any continuous function (we use \( t \mapsto (t, \sin(1/t)) \) to define a path from \([a, b]\) to join up \((a, \sin(1/a))\) and \((b, \sin(1/b))\) for any \( 0 < a \leq b \), and then reparameterize the source variable to make our domain \([0, 1]\)). We will show that \( S \) is connected but is not path-connected. Intuitively, a path from \( S_+ \) that tries to get onto the \( y \)-axis part of \( S \) cannot get there in finite time, due to the crazy wigging of \( S_+ \). Of course, we have to convert this idea into precise mathematics.

1. Connectedness of \( S \)

We begin with a lemma which shows how to recover \( S \) from \( S_+ \). This will enable us to show that \( S \) is connected.

**Lemma 1.1.** The closure of \( S_+ \) in \( \mathbb{R}^2 \) is equal to \( S \).

The point of the lemma is that we’ll show the closure of a connected subset of a topological space is always connected, so the connectedness of \( S_+ \) and this lemma then implies the connectedness of \( S \). The fact that \( S \) turns out to not be path-connected then shows that forming closure can destroy the property of path connectedness for subsets of a topological space (even a metric space).

**Proof.** To show that \( S \) lies in the closure of \( S_+ \), we have to express each \( p \in S \) as a limit of a sequence of points in \( S_+ \). If \( p \in S_+ \) we use the constant sequence \( \{p, p, \ldots\} \). If \( p = (0, y) \) with \( |y| \leq 1 \), we argue as follows. Certainly \( y = \sin(\theta) \) for some \( \theta \in [-\pi, \pi] \), whence \( y = \sin(\theta + 2n\pi) \) for all positive integers \( n \). Thus, for \( x_n = 1/(\theta + 2n\pi) \), we have \( \sin(1/x_n) = y \) for all \( n \). Since \( x_n \to 0 \) as \( n \to \infty \), we have \((x_n, \sin(1/x_n)) = (0, y) \to (0, y) \). Geometrically, this is the infinite sequence of points where the horizontal line through \( y \) cuts the graph of \( \sin(1/x) \).

Now that we have shown that the set \( S \) containing \( S_+ \) lies inside the closure of \( S_+ \), to show that it is the closure of \( S_+ \) we just have to show that \( S \) is closed (as the closure of \( S_+ \) in \( \mathbb{R}^2 \) is the unique minimal closed subset of \( \mathbb{R}^2 \) which contains \( S_+ \)). Let \( \{(x_n, y_n)\} \) be a sequence in \( S \) with limit \((x, y) \in \mathbb{R}^2 \). We must prove \((x, y) \in S \). Since \( x = \lim x_n \) and \( y = \lim y_n \), we know that \( x \geq 0 \) and \( |y| = \lim |y_n| \leq 1 \). If \( x = 0 \), then clearly \((x, y) = (0, y) \in S \) since \( |y| \leq 1 \). If \( x > 0 \), then upon dropping the first few terms of the sequence we can assume \( x_n > 0 \) for all \( n \). Then \((x_n, y_n) \in S \) must lie on \( S_+ \), so \( y_n = \sin(1/x_n) \). Since the function \( t \mapsto \sin(1/t) \) on \((0, \infty)\) is continuous, from the condition \( x_n \to x \) we conclude

\[ y = \lim y_n = \lim \sin(1/x_n) = \sin(1/x). \]

Thus, \((x, y) \in S_+ \subseteq S \) once again.

Thanks to the lemma, the connectedness of \( S \) is an immediate consequence of the following general fact (applied to the topological space \( \mathbb{R}^2 \) and the connected subset \( S_+ \)):

**Theorem 1.2.** Let \( X \) be a topological space and \( Y \) a connected subset. Then the closure \( \overline{Y} \) of \( Y \) in \( X \) is connected.
Proof. Without loss of generality, \( Y \neq \emptyset \). Suppose that \( \{U, V\} \) is a separation of \( Y \). That is, \( U \) and \( V \) are disjoint opens of \( Y \) with union equal to \( Y \). We want one of them to be empty. The intersections \( U' = U \cap Y \) and \( V' = V \cap Y \) give a separation of \( Y \) (why?), so by connectedness of \( Y \) we have that one of \( U' \) or \( V' \) is empty and the other is equal to \( Y \). Without loss of generality, we may suppose \( U' = Y \) and \( V' = \emptyset \).

Since \( U \) is closed in \( Y \), it has the form \( U = Y \cap Z \) for some closed subset \( Z \) in \( X \). But \( Y = U' \subseteq U \subseteq Z \), so by closedness of \( Z \) it follows that \( Y \subseteq Z \). Then \( U = Y \cap Z = Y \), and by disjointness \( V \) must then be empty. Hence, \( Y \) indeed has no non-trivial separations, so it is connected.

\[ \square \]

2. \( S \) is not path-connected

Now that we have proven \( S \) to be connected, we prove it is not path-connected. More specifically, we will show that there is no continuous function \( f : [0, 1] \rightarrow S \) with \( f(0) \in S_+ \) and \( f(1) \in S_0 = \{0\} \times [-1, 1] \). Assuming such an \( f \) exists, we will deduce a contradiction. Thanks to path-connectedness of \( S_0 \), we can extend our path to suppose \( f(1) = (0, 1) \). Choose \( \varepsilon = 1/2 > 0 \). By continuity, for some small \( \delta > 0 \) we have \( \|f(t) - (0, 1)\| < 1/2 \) whenever \( 1 - \delta \leq t \leq 1 \). If you draw the picture, you’ll see that the graph of \( \sin(1/x) \) keeps popping out of the disc around \((0, 1)\) of radius \( 1/2 \), and that will contradict the existence of a continuous path \( f \).

To be precise, consider the image \( f([1 - \delta, 1]) \), which must be connected since \( f \) is continuous and \([1 - \delta, 1]\) is connected. Let \( f(1 - \delta) = (x_0, y_0) \). Consider the composite of \( f : [1 - \delta, 1] \rightarrow \mathbb{R}^2 \) and projection to the \( x \)-axis. Both such maps are continuous, hence so is their composite, so the image of the composite map is a connected subset of \( \mathbb{R} \) which contains 0 (the \( x \)-coordinate of \( f(1) \)) and \( x_0 \) (the \( x \)-coordinate of \( f(1 - \delta) \)). But since connected subsets of \( \mathbb{R} \) must be intervals, it follows that the set of \( x \)-coordinates of points in \( f([1 - \delta, 1]) \) includes the entire interval \([0, x_0] \). Thus, for all \( x_1 \in (0, x_0] \) there exists \( t \in [1 - \delta, 1] \) such that \( f(t) = (x_1, \sin(1/x_1)) \).

In particular, if \( x_1 = 1/(2n\pi - \pi/2) \) for large \( n \) then \( 0 < x_1 < x_0 \) yet \( \sin(1/x_1) = \sin(-\pi/2) = -1 \).

Thus, the point \((1/(2n\pi - \pi/2), -1)\) has the form \( f(t) \) for some \( t \in [1 - \delta, 1] \), and hence this point lies within a distance of 1/2 from the point \((0, 1)\). But that’s a contradiction, since the distance from \((1/(2n\pi - \pi/2), -1)\) to \((0, 1)\) clearly at least 2 (as is the distance between any point on the line \( y = 1 \) and any other point on the line \( y = -1 \)).