Math 396. Paracompactness and local compactness

1. Motivation

There is an extraordinarily useful weaking of compactness that is satisfied by virtually all "nice" topological spaces that arise in geometry and analysis. To get started, consider $\mathbf{R}^n$ with its usual topology. This is not compact, but if $\{U_i\}$ is an open cover then we can "refine" it in a manner that still retains finiteness properties locally on the space. More specifically, for each $x \in \mathbf{R}^n$ we can pick an open ball $B_{r_x}(x)$ contained in some $U_{i(x)}$ with $r_x < 1$. For each integer $N > 0$ finitely many of the balls $B_{r_x}(x)$ cover the compact set $\overline{B}_N(0) - B_{N-1}(0)$, say $B_{r_{x_1}}(x_1), \dots, B_{r_{x_m}}(x_m)$, so we may write $\{V_{j,N}\}$ to denote these finitely many opens (with j running through a finite range that depends on N and the U_i's).

As we vary j and N, the $V_{j,N}$'s certainly cover the whole space $\mathbf{R}^n$ (even the origin), and this covering "refines" $\{U_i\}$ in the sense that every $V_{j,N}$ lies in some U_i (though many U_i's could fail to contain any $V_{j,N}$'s), and one more property holds: the collection $\{V_{j,N}\}$ is *locally finite* in the sense that any point $x \in \mathbf{R}^n$ has a neighborhood meeting only finitely many $V_{j,N}$'s. Indeed, since $V_{j,N}$ is a ball of radius at most 1 and it touches $\overline{B}_N(0) - B_{N-1}(0)$, by elementary considerations with the triangle inequality we see that a bounded region of $\mathbf{R}^n$ meets only finitely many $V_{j,N}$'s. Thus, we have refined $\{U_i\}$ to an open covering that is "locally finite"; this is weaker than compactness, but often adequate for many purposes. It is this feature of $\mathbf{R}^n$ that we seek to generalize.

2. Definitions

Let X be a topological space.

Definition 2.1. The space X is *locally compact* if each $x \in X$ admits a compact neighborhood N.

If X is locally compact and Hausdorff, then all compact sets in X are closed and hence if N is a compact neighborhood of x then N contains the closure the open $\mathrm{int}(N)$ around x. Hence, in such cases every point $x \in X$ lies in an open whose closure is compact. Much more can be said about the local structure of locally compact Hausdorff spaces, though it requires some serious theorems in topology (such as Urysohn's lemma) which, while covered in basic topology books, are too much of a digression for us and are not necessary for our purposes. We record one interesting aspect of locally compact spaces:

Lemma 2.2. *If X is a locally compact Hausdorff space that is second countable, then it admits a countable base of opens $\{U_n\}$ with compact closure.*

Proof. Let $\{V_n\}$ be a countable base of opens. For each $x \in X$ there exists an open U_x around x with compact closure, yet some $V_{n(x)}$ contains x and is contained in U_x. The closure of $V_{n(x)}$ is a closed subset of the compact $\overline{U}_x$, and so $\overline{V}_{n(x)}$ is also compact. Thus, the V_n's with compact closure are a countable base of opens with compact closure. ■

Definition 2.3. An open covering $\{U_i\}$ of X *refines* an open covering $\{V_j\}$ of X if each U_i is contained in some (perhaps many) V_j.

A simple example of a refinement is a subcover, but refinements allow much greate flexibility: none of the U_i's needs to be a V_j. For example, the covering of a metric space by open balls of radius 1 is refined by the covering by open balls of radius 1/2. We are interested in special refinements:

Definition 2.4. An open covering $\{U_i\}$ of X is *locally finite* if every $x \in X$ admits a neighborhood N such that $N \cap U_i$ is empty for all but finitely many i.

For example, the covering of $\mathbf{R}$ by open intervals $(n-1, n+1)$ for $n \in \mathbf{Z}$ is locally finite, whereas the covering of $(-1, 1)$ by intervals $(-1/n, 1/n)$ (for $n \geq 1$) barely fails to be locally finite: there is a problem at the origin (but nowhere else).

Definition 2.5. A topological space X is *paracompact* if every open coverings admits a locally finite refinement. (It is traditional to also require paracompact spaces to be Hausdorff, as paracompactness is never used away from the Hausdorff setting, in contrast with compactness – though many mathematicians implicity require compact spaces to be Hausdorff too and they reserve a separate word (quasi-compact) for compactness without the assumption of the Hausdorff condition.)

Obviously any compact space is paracompact (as every open cover admits a finite subcover, let alone a locally finite refinement). Also, an arbitrary disjoint union $\coprod X_i$ of paracompact spaces (given the topology wherein an open set is one that meets each X_i is an open subset) is again paracompact. Note that it is *not* the case that open covers of a paracompact space admit locally finite *subcovers*, but rather just locally finite refinements. Indeed, we saw at the outset that $\mathbf{R}^n$ is paracompact, but even in the real line there exist open covers with no locally finite subcover: let $U_n = (-\infty, n)$ for $n \geq 1$. All U_n's contain $(-\infty, 0)$, and any subcollection of U_{n_i}'s that covers $\mathbf{R}$ has to be infinite since each U_n is "bounded on the right". Thus, no subcover can be locally finite near a negative number.

In general, paracompactness is a slightly tricky property: there are counterexamples that show that an open subset of a paracompact Hausdorff space need not be paracompact. Thus, to prove that an open subset of $\mathbf{R}^n$ is paracompact we will have to use special features of $\mathbf{R}^n$. However, just as closed subsets of compact sets are compact, closed subsets of paracompact spaces are paracompact; the argument is virtually the same as in the compact case (extend covers by using the complement of the closed set), so we leave the details to the reader. It is a non-trivial theorem in topology that *any* metric space is paracompact! This can be found in any introductory topology book, but we will not need it. Our interest in paracompact spaces is due to:

Theorem 2.6. *Any second countable Hausdorff space X that is locally compact is paracompact.*

Proof. Let $\{V_n\}$ be a countable base of opens in X. Let $\{U_i\}$ be an open cover of X for which we seek a locally finite refinement. Each $x \in X$ lies in some U_i and so there exists a $V_{n(x)}$ containing x with $V_{n(x)} \subseteq U_i$. The $V_{n(x)}$'s therefore consistute a refinement of $\{U_i\}$ that is countable. Since the property of one open covering refining another is transitive, we therefore lose no generality by seeking locally finite refinements of countable covers. We can do better: by Lemma 2.2, we can assume that all $\overline{V}_n$ are compact. Hence, we can restrict our attention to countable covers by opens U_n for which $\overline{U}_n$ is compact. Since closure commutes with finite unions, by replacing U_n with $\cup_{j \leq n} U_j$ we preserve the compactness condition (as a finite union of compact subsets is compact) and so we can assume that $\{U_n\}$ is an increasing collection of opens with compact closure (with $n \geq 0$). Since $\overline{U}_n$ is compact yet is covered by the open U_i's, for sufficiently large N we have $\overline{U}_n \subseteq U_N$. If we recursively replace U_{n+1} with such a U_N for each n, then we can arrange that $\overline{U}_n \subseteq U_{n+1}$ for each n. Let $K_0 = \overline{U}_0$ and for $n \geq 1$ let $K_n = \overline{U}_n - U_{n-1} = \overline{U}_n \cap (X - U_{n-1})$, so K_n is compact for every n (as it is closed in the compact $\overline{U}_n$) but for any fixed N we see that U_N is disjoint from K_n for all $n > N$.

Now we have a situation similar to the concentric shells in our earlier proof of paracompactness of $\mathbf{R}^n$, and so we can carry over the argument from Euclidean spaces as follows. We seek a locally finite refinement of $\{U_n\}$. For $n \geq 2$ the open set $W_n = U_{n+1} - \overline{U}_{n-2}$ contains K_n, so for each $x \in K_n$ there exists some $V_m \subseteq W_n$ around x. There are finitely many such V_m's that actually cover the compact K_n, and the collection of V_m's that arise in this way as we vary $n \geq 2$ is clearly a locally finite collection of opens in X whose union contains $X - \overline{U}_0$. Throwing in finitely many

V_m's contained in U_1 that cover the compact $\overline{U}_0$ thereby gives an open cover of X that refines $\{U_i\}$ and is locally finite. ■

In the definition of a topological manifold, we imposed the condition of second countability (in addition to the Hausdorff condition) on top of the condition of being a topological premanifold. Since the connected components of a topological premanifold are open, second countability is equivalent to two conditions: countability of the set of connected components and second countability of each connected component (here we use countability of a countable union of countable sets). Requiring that there be at most countably many connected components is a very mild condition in practice (one never encounters natural examples that violate it), and our motivation for insisting upon it is purely technical: in a later important theorem (Frobenius Integrability Theorem) it will be necessary to require this property (essentially to rule out the possibility of an uncountable disjoint union of points with the discrete topology). More specifically, the result that is guaranteed (as will be seen later) by having at most countably many connected components is that a bijective C^∞-map between C^∞-manifolds whose "derivative" is injective must have C^∞ (and especially, continuous!) inverse. Aside from the countability aspect of the set of connected components, the following corollary shows that the second countability in the definition of a manifold could have been replaced with other conditions, particularly paracompactness, and we would have obtained an equivalent concept:

Corollary 2.7. *Let X be a Hausdorff topological premanifold. The following properties of X are equivalent: its connected components are countable unions of compact sets, its connected components are second countable, and it is paracompact.*

Proof. If $\{U, V\}$ is a separation of X and X is paracompact then it is clear that both U and V are paracompact. Hence, since the connected components of X are open, X is paracompact if and only if its connected components are paracompact. We may therefore restrict our attention to connected X. For such X, we claim that it is equivalent to require that X be a countable union of compact sets, that X be second countable, and that X be paracompact. By the preceding theorem, if X is second countable then it is paracompact. By the self-contained first Lemma in Appendix A of the text, since X is connected, Hausdorff, and locally compact, if it is paracompact then it is a countable union of compacts. Hence, to complete the cycle of implications it remains to check that if X is a countable union of compacts then it is second countable. Let $\{K_n\}$ be a countable collection of compacts that cover X, so if $\{U_i\}$ is a covering of X by open sets homeomorphic to an open set in a Euclidean space we may find finitely many U_i's that cover each K_n. As there are only countably many K_n's, in this way we find countably many U_i's that cover X. Since each U_i is certainly second countable (being open in a Euclidean space), a countable base of opens for X is given by the union of countable bases of opens for each of the U_i's. Hence, X is second countable. ■

We conclude by commenting on the metrizability of topological manifolds. It follows from Exercise 1 in Chapter 3 that such spaces are automatically metrizable, but we will not need this result; nonetheless, it shows *a priori* that the framework of the course text (where manifolds are assumed to be metrizable, even though such a hypothesis is never really needed in the foundations and the choice of a metric is completely artificial) is equivalent to the framework we use (aside from the fact that we require manifolds to have at most countably many connected components, for technical reasons to be explained later).