This will be the first part of several lectures discussing how to translate the classical theory of modular forms into the study of appropriate subspaces of the Hilbert space of $L^2$-functions on certain adelic coset spaces.

1. Correspondence between automorphic forms and $L^2$-functions on adelic coset spaces

For any connected reductive group $G$ over a global field $k$ and non-empty finite set $S$ of places of $k$ containing the archimedean places, an $S$-arithmetic subgroup of $G(k)$ (or more accurately: of $G$) is a subgroup commensurable with $G(\mathcal{O}_k,S)$ for a flat affine $\mathcal{O}_k,S$-group scheme $G$ of finite type with generic fiber $G$ (e.g., if $G = \text{SL}_n$ over $\mathcal{O}_k$ then we can take $G$ to be $\text{SL}_n$ over $\mathcal{O}_{k,S}$). Such a $G$ always exists: the schematic closure of $G$ in the $\mathcal{O}_k,S$-group $G_\mathbb{A}$ relative to a closed $k$-subgroup inclusion of $G$ into the $k$-group $G_\mathbb{A}$. (This schematic closure method gives rise to all $G$’s since any flat affine group scheme of finite type over a Dedekind domain is a closed subgroup scheme of some $GL_n$, by adapting the well-known analogue over fields, using that any finitely generated torsion-free module over a Dedekind domain is projective and hence a direct summand of a finite free module.) The notion of $S$-arithmeticity is independent of the choice of $G$ and has reasonable functorial properties in the $k$-group $G$; see [6, Ch. 1, 3.1.1(iv), 3.1.3(a)]. In the special case $k = \mathbb{Q}$ and $S = \{\infty\}$ one usually says “arithmetic” rather than “$S$-arithmetic”.

Let $\Gamma$ be an arithmetic subgroup of $\text{SL}_2(\mathbb{Q})$ (e.g., any congruence subgroup of $\text{SL}_2(\mathbb{Z})$). Consider a holomorphic function $f : \mathbb{H} \to \mathbb{C}$, where $\mathbb{H}$ is the upper half-plane $\{a + bi \in \mathbb{C} \mid b > 0\}$ (with respect to a fixed choice of $i = \sqrt{-1}$) on which $\text{SL}_2(\mathbb{R})$ acts transitively via linear fractional transformations. Recall that $\text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R}) \simeq \mathbb{H}$ as real-analytic manifolds via $g \mapsto g(i)$.

**Definition 1.1.** For $g \in \text{GL}_2(\mathbb{R})$ (the identity component of $\text{GL}_2(\mathbb{R})$, or equivalently the subgroup of $\text{GL}_2(\mathbb{R})$ with positive determinant), define

\[ (f|k)g (z) = f(gz) \left( \frac{\sqrt{\det g}}{cz+d} \right)^k. \]

Using this definition, we have:

**Definition 1.2.** The holomorphic function $f$ is a $\Gamma$-automorphic form of weight $k$ if $f|k\gamma = f$ for all $\gamma \in \Gamma$ and $f$ is moreover “holomorphic at the cusps”.

Let’s review from an “algebraic group” viewpoint what it means that $f$ is holomorphic at the cusps. The action of $\text{SL}_2(\mathbb{Q})$ on $\mathbb{H}$ via linear fractional transformations extends to a transitive action on $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ that encodes the transitive conjugation action of $\text{SL}_2(\mathbb{Q})$ on the set of Borel $\mathbb{Q}$-subgroups of $\text{SL}_2$ via the $\mathbb{Q}$-isomorphism $\mathbb{P}^1 \simeq \text{SL}_2/B_\infty$ (inverse to $g \mapsto g(\infty)$) onto
the “variety of Borel subgroups” of $\text{SL}_2$, with $B_\infty$ the upper-triangular Borel subgroup (which is the $\text{SL}_2$-stabilizer of $\infty$). The action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Z})$ is transitive, so the action of $\Gamma$ has finitely many orbits.

If we consider the quotient of $H$ by the action of $\Gamma$, it is non-compact but for a suitable “horocycle topology” on $H \cup \mathbb{P}^1(\mathbb{Q})$ we get a compactification by adding in the finitely many $\Gamma$-orbits on $\mathbb{P}^1(\mathbb{Q})$, which we call cusps of $\Gamma$. The complex structure on $\Gamma \backslash H$ extends to this compactification in a standard (and even unique) manner, as is explained in many introductory books on classical modular forms. The stabilizer in $\text{SL}_2(\mathbb{R})$ of $s \in \mathbb{P}^1(\mathbb{Q})$ is $B_s(\mathbb{R})$ for the Borel $\mathbb{Q}$-subgroup $B_s$ corresponding to $s$, so the $\Gamma$-stabilizer $\Gamma_s := \Gamma \cap B_s(\mathbb{Q})$ of $s$ is an arithmetic subgroup of $B_s$ (arithmeticty interacts well with passage to algebraic subgroups). Letting $U_s$ denote the unipotent radical of $B_s$, we have:

**Lemma 1.3.** If $\Gamma_0$ is an arithmetic subgroup of $B_s$ then $\Gamma_0 \cap U_s(\mathbb{Q})$ is infinite cyclic. Moreover, $\Gamma_0 \cap U_s(\mathbb{Q})$ has index at most $2$ in $\Gamma_0$, with index $2$ if and only if $\Gamma_0$ contains an element $\gamma$ whose eigenvalues are equal to $-1$.

The conjugation action of $\Gamma_0$ on $U_s(\mathbb{Q})$ is trivial, and $\Gamma_0$ is infinite cyclic except exactly when $-1 \in \Gamma_0$, in which case $\Gamma_0 = (-1) \times (\Gamma_0 \cap U_s(\mathbb{Q}))$.

**Proof.** The intersection $\Gamma_0 \cap U_s(\mathbb{Q})$ is an arithmetic subgroup of $U_s \simeq G_m$, so it is commensurable with $G_m(\mathbb{Z}) = \mathbb{Z}$. A subgroup of $G_m(\mathbb{Q}) = \mathbb{Q}$ commensurable with $\mathbb{Z}$ is clearly infinite cyclic.

Since $G_m(\mathbb{Z}) = \mathbb{Z}^\times = \{ \pm 1 \}$ is the entire torsion subgroup of $G_m(\mathbb{Q}) = \mathbb{Q}^\times$, the only arithmetic subgroups of $G_m$ over $\mathbb{Q}$ are the trivial group and $\{ \pm 1 \}$. In particular, the trivial subgroup of $G_m(\mathbb{Q})$ is arithmetic, so by [6] 3.1.3(a) applied to the quotient map $B_s \to B_s/U_s \simeq G_m$ it follows that any arithmetic subgroup of $B_s(\mathbb{Q})$ contains its intersection with $U_s(\mathbb{Q})$ with finite index. Thus, any element $\gamma$ of $\Gamma_0 \subset B_s(\mathbb{Q})$ has some positive power belonging to $U_s(\mathbb{Q})$. Since $B_s$ is conjugate to $B_\infty \subset \text{SL}_2$, it follows that the eigenvalues of $\gamma$ are inverse roots of unity in $\mathbb{Q}^\times$ and hence are either both equal to $1$ or both equal to $-1$, with the former happening if and only if $\gamma$ is unipotent, which is equivalent to the condition that $\gamma \in U_s(\mathbb{Q})$.

When there exist non-unipotent elements of $\Gamma_0$, if $\gamma, \gamma'$ are two such elements, then $\gamma'\gamma^{-1}$ is unipotent (as we see by hand upon computing with $B_\infty$, for example), so $\Gamma_0 \cap U_s(\mathbb{Q})$ has index $1$ or $2$ in $\Gamma_0$, with index $2$ precisely when $\Gamma_0$ contains an element $\gamma$ whose eigenvalues are $-1$.

The conjugation action of $B_s$ on the commutative $U_s \simeq G_m$ factors through a character of $B_s/U_s \simeq G_m$ that is a square in the character lattice (as is well-known for the root system of $\text{SL}_2$ and anyway is seen explicitly for $s = \infty$). That must kill the $2$-torsion elements in $G_m(\mathbb{Q}) = \mathbb{Q}^\times$ and so kills the image of $\Gamma_0$ in $(B_s/U_s)(\mathbb{Q})$. Thus, the $\Gamma_0$-conjugation on $U_s(Q)$ is trivial. Since $\Gamma_0 \cap U_s(\mathbb{Q})$ is infinite cyclic, the only cases when infinite cyclicity might fail is when there exists a non-unipotent $\gamma \in \Gamma_0$. Assuming such a $\gamma$ exists, we have just seen that $\gamma$ commutes with the infinite cyclic $\Gamma_0 \cap U_s(\mathbb{Q})$, so $\Gamma_0$ is commutative and generated by $\gamma$ and $\Gamma_0 \cap U_s(\mathbb{Q})$ with $\gamma^2 \in \Gamma_0 \cap U_s(\mathbb{Q})$. The known structure of finitely generated abelian groups then implies that $\Gamma_0$ fails to be infinite cyclic if and only if $\Gamma_0$ contains such a $\gamma$ with order $2$. Any such $\gamma$ must be semisimple and so is geometrically diagonalizable, yet both eigenvalues are $-1$ and so it is geometrically conjugate to the central element $-1$. Thus, necessarily $\gamma = -1$, so we are done.

For each $s \in \mathbb{P}^1(\mathbb{Q})$ there exists $g_s \in \text{SL}_2(\mathbb{Q})$ such that $g_s \cdot \infty = s$. We can even arrange that $g_s \in \text{SL}_2(\mathbb{Z})$ since $\mathbb{P}^1(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Z})$. The holomorphic function $f_{k|g_s}$ is invariant under the group $g_s^{-1}\Gamma_s g_s \subset B_\infty(\mathbb{Q})$ that is as described in Lemma [1.3] in particular, this group lies inside
\( \{\pm 1\} \times U_\infty(\mathbb{Q}) \). If we take \( g_s \) to come from \( \text{SL}_2(\mathbb{Z}) \) then all choices of \( g_s \) are related through \( B_\infty(\mathbb{Z}) \) whose conjugation action on \( U_\infty \cong \Gamma_\mathbb{A} \) is through multiplication against squares in \( \mathbb{Z}^\times = \{\pm 1\} \) (and whose conjugation action on \(-1\) is obviously trivial), so under this condition on \( g_s \) we can say that \( g_s^{-1}\Gamma g_s \) is independent of the choice of \( g_s \). But without such an integrality requirement then varying the choice of \( g_s \) has the effect of scaling by \( (\mathbb{Q}^\times)^2 \) on \( U_\infty \).

By Lemma 1.3, \( g_s^{-1}\Gamma g_s \) meets \( U_\infty(\mathbb{Q}) \) with index at most 2 and the standard \( \mathbb{Q} \)-isomorphism \( U_\infty \cong \Gamma_\mathbb{A} \) defined by \( (\frac{1}{b} 1) \mapsto x \), identifies the intersection of \( g_s^{-1}\Gamma g_s \) and \( U_\infty(\mathbb{Q}) \) with the infinite cyclic group generated by a unique \( h_s \in \mathbb{Q}_{>0} \).

The function \( f|_k g_s \) is invariant under weight-\( k \) slashing against \( g_s^{-1}\Gamma g_s \) since \( f \) is \( \Gamma \)-automorphic of weight \( k \), but weight-\( k \) slashing against \( (\frac{1}{b} 1) \in U_\infty(\mathbb{R}) \) has nothing to do with \( k \): it is just composition with additive translation by \( x \) on \( \mathbb{H} \). Hence, this holomorphic function is invariant under \( z \mapsto z + h_s \), so it descends to a holomorphic function on the open punctured unit disc \( \Delta^* \) via the quotient map

\[
q_{h_s} = e^{2\pi i z/h_s} : \mathbb{H} \rightarrow \mathbb{H}/(z \sim z + h_s) = \Delta^*.
\]

As such, we have a Fourier–Laurent expansion \( f|_k g_s = \sum_{n \in \mathbb{Z}} a_{n,s}(f) q_{h_s}^n \), and “holomorphicity at \( s \)” means that this Fourier expansion has no negative-degree terms; i.e., \( a_{n,s}(f) = 0 \) for all \( n < 0 \).

**Remark 1.4.** Although \( f|_k g_s \) is unaffected by replacing \( g_s \) with \( \gamma g_s \) for \( \gamma \in \Gamma \) (and \( \gamma g_s \) is a valid choice for \( g_{\gamma_s} \), provided we don’t require \( g_s \) to belong to \( \text{SL}_2(\mathbb{Z}) \), except of course when \( \Gamma \subset \text{SL}_2(\mathbb{Z}) \)), the full range of choices of \( g_s \) for a given \( s \) is given by right multiplication against anything in \( B_\infty(\mathbb{Q}) \). Making such a change in \( g_s \) for a given \( s \) changes \( f|_k g_s \) by weight-\( k \) slashing against an element of \( B_\infty(\mathbb{Q}) \) and also changes \( h_s \) through multiplication against a nonzero rational square. Thus, in general \( h_s \) is not intrinsic to the orbit \( \Gamma \)’s but we see that the property “holomorphicity at \( s \)” is independent of the choice of \( g_s \) and consequently is intrinsic to \( \Gamma \)’s (since \( \gamma g_s \) is a valid choice of \( g_{\gamma_s} \)).

When \( \Gamma \subset \text{SL}_2(\mathbb{Z}) \) and we require \( g_s \in \text{SL}_2(\mathbb{Z}) \) (so \( \gamma g_s \) remains a valid choice for \( g_{\gamma_s} \)) then the triviality of \( (\mathbb{Z}^\times)^2 \) makes \( h_s \) intrinsic to \( \Gamma \)’s and the intervention of \( U_\infty(\mathbb{Z}) \) descends to rotation of \( \Delta^* \) via multiplication against \( e^{2\pi i m/h_s} \) for \( m \in \mathbb{Z} \). Thus, the Fourier expansion of \( f|_k g_s \) is intrinsic to the orbit \( \Gamma \)’s up to precisely the operation of multiplying the \( n \)th coefficient by \( \zeta^n \) for all \( n \in \mathbb{Z} \) with \( \zeta \) an \( h_s \)th root of unity. Continuing to assume \( \Gamma \subset \text{SL}_2(\mathbb{Z}) \), if moreover the period \( h_s \) of the cusp is equal to 1 (classically \( s \) is called a regular cusp in this case, and all cusps of \( \Gamma_0(N) \) and \( \Gamma_1(N) \) are regular except for some specific small \( N \) then there is no ambiguity at all and hence the “\( q \)-expansion at \( s \)” is really intrinsic to the cusp (i.e., independent of \( g_s \in \text{SL}_2(\mathbb{Z}) \) and of the representative \( s \in \mathbb{P}^1(\mathbb{Q}) \) for a given \( \Gamma \)-orbit).

**Definition 1.5.** If \( f \) is a \( \Gamma \)-automorphic form of weight \( k \), we say that it is a cusp form if its Fourier expansion at each cusp has vanishing constant term.

One can also express holomorphicity at the cusps as a “moderate growth” condition, which we discuss later. It is this condition which translates more easily to the automorphic setting. In any case, we are most interested in cusp forms; for such forms the “moderate growth” condition will also follow from a representation-theoretic formulation of cuspidality that makes sense in a much wider setting.

**Definition 1.6.** We define \( M_k(\Gamma) \) to be the space of \( \Gamma \)-automorphic forms of weight \( k \), and \( S_k(\Gamma) \) to be the space of \( \Gamma \)-cusp forms of weight \( k \).
For a Dirichlet character \( \psi \mod N \), \( S_k(N, \psi) \) denotes the space of \( \Gamma_1(N) \)-cusp forms of weight \( k \) such that if \( g \in \Gamma_0(N) \), then \( f|_k g = \psi([g]) f \), where \([g]\) is the class of \( g \) in \( \Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times \).

In order to identify modular forms with certain \( L^2 \) functions on adelic coset spaces for certain congruence subgroups \( \Gamma \subset \text{SL}_2(\mathbb{Z}) \), the idea is to first define a map sending \( \Gamma \)-cusp forms to \( L^2 \) functions on \( \Gamma \backslash \text{SL}_2(\mathbb{R}) \), and then use strong approximation for \( \text{SL}_2 \) to identify \( \Gamma \backslash \text{SL}_2(\mathbb{R}) \) with \( K_\Gamma \backslash \text{GL}_2(\mathbb{A})/\text{GL}_2(\mathbb{Q}) \), where \( \mathbb{A} = \mathbb{A}_Q \) is the adele ring of \( \mathbb{Q} \) and \( K_\Gamma \) is a specific compact open subgroup of \( \text{GL}_2(\mathbb{Z}) \). One important part of this identification will be to identify the images of \( S_k(\Gamma) \) and \( S_k(N, \psi) \) on the adelic side.

### 1.1 The case of \( S_k(\Gamma) \)

We define a map \( M_k(\Gamma) \to C^\infty(\Gamma \backslash \text{SL}_2(\mathbb{R})) \) by sending \( f \) to \( \phi_f : g \mapsto (f|_k g)(i) \). This is injective because it is easy to verify that

\[
f(x + iy) = \phi_f \left( \frac{y^{1/2} xy^{-1/2}}{y^{-1/2}} \right) y^{-k/2}.
\]

**Proposition 1.7.** For \( f \in S_k(\Gamma) \) we have \( \phi_f \in L^2(\Gamma \backslash \text{SL}_2(\mathbb{R})) \) when \( \Gamma \backslash \text{SL}_2(\mathbb{R}) \) is equipped with the \( \text{SL}_2(\mathbb{R}) \)-invariant measure arising from a choice of Haar measure on the unimodular group \( \text{SL}_2(\mathbb{R}) \). If \( f \in M_k(\Gamma) \) and \( f \not\in S_k(\Gamma) \) then \( \phi_f \not\in L^p(\Gamma \backslash \text{SL}_2(\mathbb{R})) \) for all \( p \geq 1 \) when \( k \geq 2 \).

**Proof.** Define \( F(z) = |f(z)|(|\text{Im} \, z|^{k/2}) \). For \( g \in \text{SL}_2(\mathbb{R}) \), we have:

\[
|\phi_f(g)| = \left| f \left( \frac{ai + b}{ci + d} \right) \frac{1}{(ci + d)^k} \right|
= \left| f \left( \frac{ai + b}{ci + d} \right) \left( \frac{1}{|ci + d|^2} \right)^{k/2} \right|
= F(g(i)).
\]

In particular, \( |\phi_f| \) is right-invariant by the \( \text{SL}_2(\mathbb{R}) \)-stabilizer \( K := \text{SO}_2(\mathbb{R}) \) of \( i \) which is moreover a (maximal) compact subgroup of \( \text{SL}_2(\mathbb{R}) \), so the \( L^2 \) property can be checked using the induced function on the quotient space

\[
\Gamma \backslash \text{SL}_2(\mathbb{R})/K = \Gamma \backslash \mathbb{H}
\]

with its induced \( \text{SL}_2(\mathbb{R}) \)-invariant measure arising from the Haar measure, namely a positive constant multiple of \( (dx \, dy)/y^2 \).

Since \( \text{SL}_2(\mathbb{R}) \) acts transitively on \( \mathbb{H} \) and \( \phi_f \) is left \( \Gamma \)-invariant, it follows that \( F \) is also left \( \Gamma \)-invariant. Hence, \( F \) defines a continuous function on the quotient space \( \Gamma \backslash \mathbb{H} \) that is compactified with finitely many cusps and has finite volume for its \( \text{SL}_2(\mathbb{R}) \)-invariant measure \( (dx \, dy)/y^2 \). The \( L^2 \)-property for \( \phi_f \) on \( \Gamma \backslash \text{SL}_2(\mathbb{R}) \) is therefore the same as that for \( F \) on \( \Gamma \backslash \mathbb{H} \) near each of the finitely many cusps relative to the measure \( (dx \, dy)/y^2 \).

For any \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}) \) the identity

\[
(cz + d)^2 \text{Im} \left( \frac{az + b}{cz + d} \right) = (\text{Im} \, z) \left( \frac{cz + d}{cz + d} \right)
\]
implies

\[ F(gz) = |f(gz)|(\text{Im } gz)^{k/2} \]
\[ = |(f|kg)(z)|(cz + d)^k|(|\text{Im } g z|^{k/2}) \]
\[ = |(f|kg)(z)|(\text{Im } z)^{k/2}. \]

By using the horocycle topology on \( \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) \), to analyze the behavior of \( F \) near a cusp on \( \Gamma \backslash \mathbb{H} \) is the same as to analyze \( F(g_s z) \) with \( z \) far up in a vertical strip of bounded width for \( g_s \in \text{SL}_2(\mathbb{Q}) \) as above (note that typically \( g_s \notin \Gamma! \)). But this is exactly \( |(f|kg_s)(z)|(\text{Im } z)^{k/2} \), so if \( f \) is not “cuspidal at \( s \)” then \( |(f|kg_s)(z)| \) approaches a nonzero constant value as \( \text{Im } z \to \infty \) so \( \phi_f \) fails to be \( L^p \) for any \( p \geq 1 \) since \( y^{k/2}/y^2 \notin L^p((c, \infty)) \) for any \( c > 0 \) when \( k \geq 2 \). (This calculation that near each cusp on \( \Gamma \backslash \text{SL}_2(\mathbb{R}) \) we have \( \phi_f = O(y^{k/2}) \) as \( y \to \infty \) will turn into a “moderate growth” condition in the adelic theory.)

Now suppose \( f \in S_k(\Gamma) \). The exponential decay of the Fourier expansion of \( f|kg_s \) swamps out the growth of \( (\text{Im } z)^{k/2} \) as this imaginary part gets large: by cuspidality the Fourier expansion is bounded in absolute value by a constant multiple of \( |e^{2\pi i(x+iy)}| = e^{-2\pi y} \), and \( e^{-2\pi y y^{k/2}} \) tends to 0 as \( y \to \infty \). Thus, \( F \) extends to a continuous function on the compactification of \( \Gamma \backslash \mathbb{H} \) by assigning it value 0 at each cusp, so \( \phi_f \) is a bounded function on the space \( \Gamma \backslash \text{SL}_2(\mathbb{R}) \) that has finite volume. Hence, \( \phi_f \) belongs to \( L^\infty \subset L^2 \).

We conclude that the injective map \( M_k(\Gamma) \to C^\infty(\Gamma \backslash \text{SL}_2(\mathbb{R})) \) defined by \( f \mapsto \phi_f \) has image that meets \( L^2(\Gamma \backslash \text{SL}_2(\mathbb{R})) \) in exactly \( S_k(\Gamma) \). Let’s see how the properties of \( f \in M_k(\Gamma) \) translate into properties of \( \phi_f \).

- Let \( s \in \mathbb{P}^1(\mathbb{Q}) \) represent a cusp of \( \Gamma \), and let \( U_s \) be the unipotent radical of the corresponding Borel Q-subgroup of \( \text{SL}_2 \), so by Lemma 1.3 we know that \( \Gamma \cap U_s(\mathbb{R}) \) is a subgroup of the \( s \)-stabilizer \( \Gamma_s \) with index at most 2 and is an infinite cyclic subgroup of \( U_s(\mathbb{R}) \simeq \mathbb{R} \). We claim that \( f \) is “cuspidal at \( s \)” (i.e., its Fourier expansion at \( s \) has vanishing constant term, a property intrinsic to the cusp rather than depending on the representative \( s \)) if and only if \( \int_{(\Gamma \cap U_s(\mathbb{R})) \backslash U_s(\mathbb{R})} \phi_f(ug) \, du = 0 \) for all \( g \in \text{SL}_2(\mathbb{R}) \) (using any Haar measure \( du \) on \( U_s(\mathbb{R}) \), the choice of which clearly doesn’t matter).

Once this is shown, it is indeed true that an element \( f \in M_k(\Gamma) \) is cuspidal if and only if for the unipotent radical \( U \) of every Borel Q-subgroup of \( \text{SL}_2 \) we have

\[ \int_{U \cap U(\mathbb{R})} \phi_f(ug) \, du = 0 \]

with \( U_\Gamma := \Gamma \cap U(\mathbb{Q}) \) an arithmetic subgroup (so infinite cyclic in \( U(\mathbb{R}) \)). This vanishing condition for a given \( U \) only depends on the \( \Gamma \)-conjugacy class of \( U \) (since \( \phi_f \) is left \( \Gamma \)-invariant and we vary through all \( g \in \text{SL}_2(\mathbb{R}) \)), and its validity for all \( U \) will be called cuspidality for \( \phi_f \) (to be appropriately adapted to the \( L^2 \) adelic theory).

To verify the asserted reformulation of cuspidality at \( s \), we will calculate with the Fourier expansion at \( s \). Note that \( U_s = g_s \cdot U_\infty \cdot g_s^{-1} \) for a choice of \( g_s \in \text{SL}_2(\mathbb{Q}) \) satisfying \( g_s(\infty) = s \). Write \( g_s^{-1} g = (a \ b \ c \ d) \) so for any function \( F : \mathbb{H} \to \mathbb{C} \) we have \( (F|k(g_s^{-1} g))(w) = (cw + d)^{-k} F((g_s^{-1} g) \cdot w) \). Let \( z = (g_s^{-1} g) \cdot i \) and \( \alpha = (ci + d)^{-k} \), so for any function
\( F : \mathbf{H} \to \mathbf{C} \) we have \( (F|_k(g^{-1}_s g))(i) = \alpha F(z) \). Finally, define \( f_s = f|_k g_s \). Now, we may calculate the integral via the Fourier expansion at \( s \) as:

\[
\int_{(\Gamma \cap U_s(\mathbf{R})) \setminus U_s(\mathbf{R})} \phi_f(ug) \ du = \int_{((g^{-1}_s \Gamma_s g_s) \cap U_\infty(\mathbf{R})) \setminus U_\infty(\mathbf{R})} \phi_f(g_s u g_s^{-1} g) \ du \\
= \int_{((g^{-1}_s \Gamma_s g_s) \cap U_\infty(\mathbf{R})) \setminus U_\infty(\mathbf{R})} ((f_s)|_k u)(g^{-1}_s g) \ (i) \ du \\
= \alpha \int_{((g^{-1}_s \Gamma_s g_s) \cap U_\infty(\mathbf{R})) \setminus U_\infty(\mathbf{R})} ((f_s)|_k u)(z) \ du \\
= \alpha \int_{h_s \mathbf{Z} \setminus \mathbf{R}} f_s(z + x) \ dx \\
= \alpha \int_0^{h_s} f_s(z + x) \ dx \\
= \alpha \int_0^{h_s} \sum_{n \geq 0} a_{n,s}(f)(e^{2\pi (z+x)/h_s})^n \ dx \\
= \alpha \cdot h_s \cdot a_{0,s}(f)
\]

Note that \( \alpha h_s \in \mathbf{C}^\times \), so this equals 0 if and only if \( a_{0,s}(f) = 0 \), which amounts to the element \( f \in M_k(\Gamma) \) being cuspidal at \( s \).

Let \( K = \text{SO}_2(\mathbf{R}) \) be the \( \text{SL}_2(\mathbf{R}) \)-stabilizer of \( i \in \mathbf{H} \); this is a maximal compact subgroup of \( \text{SL}_2(\mathbf{R}) \). Defining

\[
r(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},
\]

it is easy to check from the definition of \( \phi_f \) in terms of \( f \) that \( \phi_f(g \cdot r(\theta)) = \phi_f(g)e^{-ik\theta} \); this is an important refinement of the fact seen in the proof of Proposition 1.7 that \( |\phi_f| \) is right \( K \)-invariant. For \( f \in S_k(\Gamma) \), this says that \( \phi_f \) is an eigenfunction for \( K \) under the right regular representation \( R \) of \( \text{SL}_2(\mathbf{R}) \) on \( L^2(\Gamma \setminus \text{SL}_2(\mathbf{R})) \) defined by \( (R(g) \cdot \phi)(h) = \phi(hg) \) (which makes sense on the \( L^2 \)-space since \( \text{SL}_2(\mathbf{R}) \) is unimodular), with eigencharacter \( r(\theta) \mapsto e^{-ik\theta} \) encoding the weight \( k \) of \( f \).

The property of \( f \) that we still need to capture is that it is a holomorphic function on \( \mathbf{H} \). There is a unique (up to scaling) right-invariant Riemannian metric on \( \text{SL}_2(\mathbf{R}) \). To any Riemannian manifold \( M \) there is a canonically associated Laplace–Beltrami operator \( \Delta_p \) on the space \( \Omega^p(M) \) of smooth global \( p \)-forms on \( M \) (see \([9, \S 6.1]\) for this definition), so for \( p = 0 \) and \( M = \text{SL}_2(\mathbf{R}) \) we get an associated Laplace–Beltrami operator \( \Delta \) on \( C^\infty(\text{SL}_2(\mathbf{R})) \) that commutes with \( R \) by invariance for the metric under the right \( \text{SL}_2(\mathbf{R}) \)-action.

By the PBW theorem giving the structure of the universal enveloping algebra of a finite-dimensional Lie algebra, the space of right-invariant differential operators on \( C^\infty(\text{SL}_2(\mathbf{R})) \) is identified with \( U(\mathfrak{sl}_2(\mathbf{R})) \). Hence, \( \Delta \) arises from a unique degree-2 element of the center \( \mathfrak{z} \) of \( U(\mathfrak{sl}_2(\mathbf{R})) \). For any finite-dimensional semisimple Lie algebra \( \mathfrak{g} \) over a field \( F \) of characteristic 0 the symmetric bilinear Killing form \( \kappa : \mathfrak{g} \times \mathfrak{g} \to F \) defined by \( \kappa(X, Y) = \text{Tr}(\text{ad}(X) \circ \text{ad}(Y)) \) is
non-degenerate and thereby gives rise to a canonical nonzero central element $C \in \mathfrak{z}$ in degree 2 called the *Casimir element*; this is defined in [2 Ch. I, §3.7].

For $\mathfrak{g} = \mathfrak{sl}_2(F)$ it is a classical fact that $\mathfrak{z}$ is a polynomial ring generated by a nonzero element in degree 2, so $\Delta$ must be an $\mathbb{R}^\times$-multiple of (the effect of) $C$. Scaling the invariant metric by $a > 0$ scales the Hodge-star by $1/a^2$; hence, we can scale the metric so that $\Delta = \pm 2C$ for a unique sign. In Appendix A we will see that the sign comes out to be negative; i.e., we can arrange that $\Delta = -2C$. The right-invariance of $\Delta$ by design and the invariance of $C$ under all automorphisms of a Lie algebra (such as the adjoint representation!) together imply the left-invariance of $\Delta$, so it descends to an operator on $C^\infty(\Gamma \setminus \text{SL}_2(\mathbb{R}))$.

If $g \in \text{SL}_2(\mathbb{R})$, we can use the NAK decomposition to write $g = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \cdot r(\theta)$. Then we can write down $\Delta$ in terms of these coordinates:

$$\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - y \frac{\partial^2}{\partial x \partial \theta}.$$

**Proposition 1.8.** If $f \in S_k(\Gamma)$, then $\Delta \phi_f = - \frac{k}{2} \left( \frac{k}{2} - 1 \right) \phi_f$

*Proof.*

$$\phi_f(g) = \phi_f\left( \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \cdot e^{-ik\theta} y^{k/2} f(x + iy) e^{-ik\theta} \right)$$

This gives us that

$$\Delta \phi_f = (-y^{k/2+2} e^{-ik\theta}) \frac{\partial^2 f}{\partial x^2} + (-y^{k/2+2} e^{-ik\theta}) \frac{\partial^2 f}{\partial y^2} + (-y^{k/2+1} e^{-ik\theta} \cdot k) \frac{\partial f}{\partial y}$$

$$= - \frac{k}{2} \left( \frac{k}{2} - 1 \right) y^{k/2} f(x + iy) e^{-ik\theta} + (-y^{k/2+1} e^{-ik\theta} \cdot (-ik)) \frac{\partial f}{\partial x}$$

The last equality follows from the Cauchy-Riemann equations characterizing holomorphicity of $f$ (recall that $\frac{\partial}{\partial x} = \frac{1}{2}(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y})$).

**Proposition 1.9.** The image of $S_k(\Gamma)$ in $L^2(\Gamma \setminus \text{SL}_2(\mathbb{R}))$ is exactly characterized as the set of all $\phi$ such that:

- $R(\theta)(\phi) = e^{-ik\theta} \phi$; i.e., $\phi$ is an eigenfunction for $K$ under the right regular representation, with “weight” $k$ under the standard parameterization $r: \mathbb{R} / 2\pi \mathbb{Z} \simeq K$,

- $\phi$ is an eigenfunction for $\Delta$ in the distributional sense, with eigenvalue $-(k/2)(k/2 - 1)$ (so $\phi$ is smooth by elliptic regularity, as the Laplacian for any Riemannian metric tensor is elliptic \[9, 6.34–6.36\]; it suffices to check smoothness after pullback to $\text{SL}_2(\mathbb{R})$),

---

1In Appendix A (the discussion preceding Lemma A.1) we give a reference for this computation; in particular, the given expression indeed describes the effect of $-2C$ in these coordinates.
it satisfies the “cuspidality condition”

\[ \int_{(\Gamma \cap U(R)) \setminus U(R)} \phi(ug) \, du = 0 \]

for all \( g \in \text{SL}_2(R) \) and all unipotent radicals \( U \) of Borel \( Q \)-subgroups of \( \text{SL}_2 \).

For \( \phi \) satisfying the conditions of Proposition 1.9 so \( \phi \) is smooth, the formula for \( f \) in terms of \( \phi_f \) motivates making the definition

\[
f(x + iy) := \phi \left( \left( \frac{y^{1/2} x y^{-1/2}}{y^{1/2}} \right) \right) y^{-k/2}
\]

for \( x \in \mathbb{R} \) and \( y > 0 \). This is smooth function on \( H \). It’s straightforward to check that \( f|_k \gamma = f \) for \( \gamma \in \Gamma \) precisely because of the left \( \Gamma \)-invariance of \( \phi \), and that if \( f \) is holomorphic then its zeroth Fourier coefficient vanishes at each cusp of \( \Gamma \) precisely because of the cuspidality condition.

Thus, \( f \in S_k(\Gamma) \) as soon as we know \( f \) is holomorphic with \( f|_k g_s \) bounded near \( s \) for each \( s \in P^1(Q) \) (with \( g_s \in \text{SL}_2(Q) \) carrying \( \infty \) to \( s \), the choice of which we have seen doesn’t matter). This boundedness property for all \( s \) can be deduced from a “moderate growth” condition on \( \phi \) that is a consequence of the cuspidality condition (see Remark 1.12), but the proof of holomorphicity (in effect, that \( f \) satisfies the Cauchy-Riemann equations \( \partial f / \partial z = 0 \)) lies rather deeper: it requires some input from representation theory. Moreover, this deduction of holomorphicity for \( f \) built from such \( \phi \) seems to be omitted from many standard references discussing the representation-theoretic aspects of classical modular forms (such references generally only prove the much easier converse direction), so we give a proof in Appendix A using the representation theory of \( \text{sl}_2(\mathbb{R}) \).

Proposition 1.9 motivates the following definition:

**Definition 1.10.** A \( \Gamma \)-cusp form on \( G = \text{SL}_2(\mathbb{R}) \) is a smooth function \( \phi : G \to \mathbb{C} \) such that:

1. \( \phi(\gamma \cdot g) = \phi(g) \) for all \( \gamma \in \Gamma \)
2. \( \phi \) is “\( K \)-finite” for a maximal compact subgroup \( K \) of \( G \); i.e., the span of \( \{ \kappa \cdot \phi \}_{\kappa \in K} \) is finite-dimensional (this property is independent of the choice of \( K \) since all such \( K \)'s are related to each other through \( G \)-conjugation);
3. \( \phi \) is an eigenfunction of \( \Delta \) (which forces \( \phi \) to be real-analytic, by real-analyticity of the elliptic operator \( \Delta \); see [1] Appendix, Ch. 4, Part II] for an elegant proof of preservation of real-analyticity in the elliptic regularity theorem);\(^2\)
4. \( \phi \) is cuspidal (i.e., \( \int_{(\Gamma \cap U(R)) \setminus U(R)} \phi(ug) \, du = 0 \) for all \( g \in G \) and unipotent radicals \( U \) of Borel \( Q \)-subgroups of \( \text{SL}_2 \)).

Proposition 1.9 says that if \( \phi \) is an eigenfunction for \( K \) with eigencharacter \( r(\theta) \mapsto e^{-ik\theta} \) and the eigenvalue of \( \Delta \) on \( \phi \) is \( -(k/2)(k/2 - 1) \) then we get exactly the functions \( \phi_f \) for \( f \in S_k(\Gamma) \). However, there are lots of cusp forms for \( \Gamma \) that are eigenfunctions for \( K \) with an eigencharacter that

\(^2\)To generalize beyond \( \text{SL}_2(\mathbb{R}) \), one needs to replace the eigenfunction condition for \( \Delta \) with one for the entire center of the universal enveloping algebra acting naturally through differential operators on smooth functions; in the \( \text{SL}_2 \)-case, this center is generated over \( \mathbb{R} \) by \( \Delta \).
has nothing to do with its Laplacian eigenvalue, so these don’t arise from $S_k(\Gamma)$. Such additional $\phi$ arise from non-holomorphic Maass forms. An example of a non-cuspidal Maass form (although Maass cusp forms also exist) is the Eisenstein series:

$$E\left(z, \frac{1}{2} + \frac{it}{2}\right) = \sum_{(c,d)=1 \atop c,d \in \mathbb{Z}} \frac{(\text{Im } z)^{1/2+it/2}}{|cz+d|^{1+it}}$$

Note that $\Delta E = \frac{1+t^2}{4} E$. Although we are focused on cusp forms, we take this opportunity to define more general automorphic forms:

**Definition 1.11.** A $\Gamma$-automorphic form on $G = \text{SL}_2(\mathbb{R})$ is a smooth function $\phi : G \to \mathbb{C}$ satisfying (1), (2), and (3) of Definition 1.10 plus

1. $\phi$ is of “moderate growth” at the cusps of $\Gamma$: there is an $A > 0$ such that for each cusp $s$ of $\Gamma$ and $g_s \in G$ such that $g_s(\infty) = s$,

$$\left| \phi\left(g_s\left(\begin{smallmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{smallmatrix}\right)\right) \right| \ll y^A.$$  

2. As a function on $\Gamma \backslash G$, $\phi$ is square-integrable.

**Remark 1.12.** First, Proposition 1.7 shows that $\phi_f$ is even bounded, hence of moderate growth. In general, the cuspidality condition for all cusps implies the moderate growth condition at all cusps. Morally, having a negative Fourier coefficient at a cusp $s$ prevents us from being square-integrable near that cusp, but see [7, Cor. 3.4.3] for a rigorous argument (which has the benefit of being more generalizable).

Second, as we’ve phrased it, it’s not clear how to generalize to other reductive groups $G$ the statement itself of the moderate growth condition. We note (without proving, although it follows from the Iwasawa decomposition) that it is equivalent to the following suggestive reformulation. The group $\text{SL}_2(\mathbb{R})$, as a closed subspace of $\text{Mat}_2(\mathbb{R})$, inherits the sup norm $\|\cdot\|$; i.e. $\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\| = \max(|a|, |b|, |c|, |d|)$. Then $\phi$ as above is of moderate growth at each cusp if and only if there exists an $A' > 0$ such that $\phi(g) \ll \|g\|^{A'}$. See [4] for more details and the generalization to reductive $G$.

### 1.2 The case of $S_k(N, \psi)$

What about $S_k(N, \psi)$? Fix a Dirichlet character $\psi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. We can consider $\psi$ as a unitary character on the idèles in the usual way, via the identification $A^\times/Q^\times \mathbb{R}^\times = \hat{Z}^\times$.

Let $G = \text{GL}_2/Q$, and for finite places $v$ define $K_v = \text{GL}_2(\mathcal{O}_v)$, a maximal compact subgroup of $\text{GL}_2(Q_v)$. Also, let $Z$ denote the scheme-theoretic center of $G$ (i.e. the diagonal scalar matrices). For all $v | N$, we set

$$K_v^N = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_v \mid c \equiv 0 \mod N \}$$

By strong approximation for $\text{SL}_2$ we can see that:

$$\text{SL}_2(A) = \text{SL}_2(Q) \cdot \text{SL}_2(R) \cdot \prod_{v|N} \text{SL}_2(\mathcal{O}_v)^N \cdot \prod_{v|N\infty} \text{SL}_2(\mathcal{O}_v),$$

9
where $\text{SL}_2(\mathcal{O}_v)^N$ consists of elements of $\text{SL}_2(\mathcal{O}_v)$ which are upper triangular modulo $N$. Then observing that the left side is the kernel of $G(\mathbb{A})$ under the determinant map, the right side is the kernel of the determinant map on

$$G(\mathbb{Q}) \cdot \text{GL}_2^+(\mathbb{R}) \cdot \prod_{v|N} K_v^N \cdot \prod_{v|N\infty} K_v$$

and both $G(\mathbb{A})$ and the product above yield the same image (namely, all ideles) under the determinant map. Hence,

$$G(\mathbb{A}) = G(\mathbb{Q}) \cdot \text{GL}_2^+(\mathbb{R}) \cdot \prod_{v|N} K_v^N \cdot \prod_{v|N\infty} K_v.$$

Hence, any $g \in G(\mathbb{A})$ may be written as $g = \gamma \cdot g_\infty \cdot \kappa_0$ with $\gamma \in G(\mathbb{Q}), g_\infty \in \text{GL}_2^+(\mathbb{R}), \kappa_0 \in \prod_{v|N} K_v^N \cdot \prod_{v|N\infty} K_v =: K_0^N$. We define a map $S_k(N, \psi) \rightarrow L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ by setting $\phi_f(g) = (f|_{k\cdot g_\infty})(i)\psi(\kappa_0)$. The notation $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ (used here and throughout) is misleading. The elements thereof are not square-integrable on $G(\mathbb{Q})\backslash G(\mathbb{A})$; rather, they are required by definition to admit a unitary (idélic) central character (i.e. defined on $Z(\mathbb{A})$ and trivial on $Z(\mathbb{Q})$) and to have their resulting absolute value function on $(Z(\mathbb{A}) \cdot G(\mathbb{Q}))\backslash G(\mathbb{A})$ be square-integrable (for the measure arising from the Haar measure on the unimodular $G(\mathbb{A})$).

Similarly to the previous case, we have:

**Proposition 1.13.** For $f \in S_k(N, \psi)$, we have the following properties:

- $\phi_f$ is well-defined\(^3\) and an eigenfunction for the right regular action of $Z(\mathbb{A})$ with unitary central character $\psi$; i.e., $\phi_f(gz) = \psi(z)\phi_f(g)$ for any $z \in Z(\mathbb{A})$ and $g \in G(\mathbb{A})$ (so $|\phi_f|$ descends to a continuous function on $Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$),

- $\phi_f$ belongs to $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ (which makes sense by the previous condition)\(^4\),

- $\phi_f : G(\mathbb{A}) \rightarrow \mathbb{C}$ is smooth (i.e., it is locally constant in the finite-adelic part for fixed archimedean component and it is smooth in the archimedean component for fixed finite-adelic part),

- $\phi_f$ is a “weight $k$” eigenfunction for the right regular action of $K$ (i.e., $\phi_f(g \cdot r(\theta)) = e^{-ik\theta}\phi_f(g)$);

- $\phi_f$ is cuspidal in the sense that $\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \phi_f(ug) du = 0$ for all $g \in G(\mathbb{A})$ and all unipotent radicals $U$ of Borel $\mathbb{Q}$-subgroups of $\text{GL}_2$ (this integral makes sense since $U(\mathbb{Q})\backslash U(\mathbb{A}) = \mathbb{Q} \backslash \mathbb{A}$ is compact, and since we allow variation across all $g$ it follows from the left $G(\mathbb{Q})$-invariance of $\phi_f$ that it is enough to check this vanishing property for one $U$ because the collection of such $U$ is a single $G(\mathbb{Q})$-conjugacy class);

$$\Delta \cdot |\phi_f|_{\text{GL}_2^+(\mathbb{R})} = -\frac{k}{2} \left( \frac{k}{2} - 1 \right) \phi_f|_{\text{GL}_2^+(\mathbb{R})}$$

\(^3\)This follows from $G(\mathbb{Q}) \cap \left( \text{GL}_2^+(\mathbb{R}) \cdot \prod_{v|N} K_v^N \cdot \prod_{v|N\infty} K_v \right) = \Gamma_0(N)$ and the action of $\Gamma_0(N)$ on $S_k(N, \psi)$.

\(^4\)This follows from “Theorem F” in the lectures on reduction theory applied to $\text{GL}_2$ and Proposition 1.7. In particular, volume-finiteness of $Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$ follows from volume-finiteness of $[\text{GL}_2]$. 10
• $\phi_f(g \cdot \kappa_0) = \phi_f(g) \cdot \psi(\kappa_0)$ for any $\kappa_0 \in K_0^N$.

Furthermore, any element $\phi \in L^2(G(\mathbf{Q}) \backslash G(\mathbf{A}))$ which satisfies the above properties is of the form $\phi_f$ for a uniquely-determined $f \in S_k(N, \psi)$.

Again the key point is to verify holomorphicity; one applies the argument from Appendix A mutatis mutandi.

**Remark 1.14.** To give a definition of cuspidality that is more intrinsic to $L^2(G(\mathbf{Q}) \backslash G(\mathbf{A}))$, it is better to speak in terms of “almost every $g$” rather than consider each $g$ in isolation (since it doesn’t make sense to integrate an $L^p$-function along a measure-0 subset); there are subtleties in doing so, discussed in the Appendix of [8].

**Definition 1.15.** A cuspidal automorphic form on $G = \text{GL}_2$ is a smooth function $\phi : G(\mathbf{A}) \to \mathbf{C}$ such that:

1a) $\phi(\gamma \cdot g) = \phi(g)$ for all $\gamma \in G(\mathbf{Q})$ and all $g \in G$;

1b) There exists a unitary character $\psi$ on $Z(\mathbf{Q}) \backslash Z(\mathbf{A}) = \mathbf{Q}^\times \backslash \mathbf{A}^\times$ such that $\phi(gz) = \phi(g) \cdot \psi(z)$ for all $z \in Z(\mathbf{A})$ and $g \in G(\mathbf{A})$;

2a) $\phi$ is invariant under a compact open subgroup of $G(\mathbf{A})$;

2b) $\phi$ is $K_{\infty}$-finite;

3) $\phi$ is $3$-finite, where 3 is the center of the universal enveloping algebra of Lie$(G_{\infty})$ (it doesn’t matter in this definition if one takes 3 over $\mathbf{R}$ or over $\mathbf{C}$, but in practice one always takes it over $\mathbf{C}$ for convenience beyond the setting of split reductive groups such as $\text{SL}_2$);

4) $\phi$ is cuspidal.

**Remark 1.16.** As in the classical setting, one can define the space of automorphic forms on $G(\mathbf{A})$ that are not necessarily cuspidal. As before, one must then separately insist on an adelic moderate growth condition similar to that of Remark 1.12. We record this here briefly; see [4] §4 for more. Also, see [5] for relevant sanity-checks about the adelic topology (for example one can use the results therein to verify that the moderate growth condition doesn’t depend on the choice of norm we make below).

Because $G$ is not Zariski-closed in $\text{Mat}_2$, we cannot simply use norms of entries of $g \in G(\mathbf{A})$ to define an adelic norm. Instead, if $a, b, c, d$ are the entries of $g$, we can let

$$\|g\| = \sup_v \left( \max \|a_v\|_v, \|b\|_v, \|c\|_v, \|d\|_v, \|ad - bc\|_v^{-1} \right).$$

With this definition, we say $\phi : G(\mathbf{A}) \to \mathbf{C}$ is of moderate growth if there exists an $A > 0$ such that $|\phi(g)| \ll \|g\|^4$ for all $g \in G(\mathbf{A})$. This may look ad hoc; in general, the idea is to control the size using a finite generating set of the coordinate ring of the algebraic group (the choice of which turns out not to matter).

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5This means that it is smooth at the archimedean place $\infty$ and locally constant at the finite-adelic part.
2 \textit{L}-functions

Recall that in his proof of the analytic continuation and functional equation for the zeta function, Riemann defines:

\[ Z(2s) := \pi^{-s}\Gamma(s)\zeta(2s) = \int_0^\infty \frac{\theta(t) - 1}{2} t^s \, dt \]

where \( \theta(t) \) is the theta function \( \theta(t) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} \). The Poisson summation formula implies that \( \theta(s) = s^{1/2}\theta(s) \), and this in turn implies the functional equation \( Z(s) = Z(1-s) \), or in other words:

\[ \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{1-s/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \]

as meromorphic functions with poles at 0 and 1. Also, we have the Euler product formula:

\[ \zeta(s) = \prod_p (1 - p^{-s})^{-1} \]

for \( s \) with \( \Re(s) > 1 \).

\textit{Tate’s thesis} provides an adelic interpretation of these facts. He shows that:

\[ Z(s) = \int_{\hat{A}^\times} f(a)|a|^s d^\times a \]

(1)

where \( |\cdot| \) is the idelic norm and \( f(a) = \prod_p f_p(a) \cdot \left(e^{-\pi t^2}\right)_\infty \), with \( f_p \) the indicator function of \( \mathbb{Z}_p \) on \( \mathbb{Q}_p \). Then, we can manipulate \([1]\) when \( \Re(s) > 1 \) as:

\[ Z(s) = \left(\prod_p \int_{\mathbb{Z}_p} |a|^s d^\times a\right) \int_0^\infty e^{-\pi t^2} t^s \, dt = \left(\prod_p (1 - p^{-s})^{-1}\right) \pi^{s/2}\Gamma(s/2) \]

Then, we get the functional equation for \( Z(s) \) by adelic Poisson summation.

We can also do this with a Dirichlet character \( \chi: (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times \). In the classical version, we define:

\[ L_{\text{fin}}(s, \chi) = \sum_{n=1}^\infty \frac{\chi(n)}{n^s} \]

Note that this definition only “sees” primes \( p \mid N \), i.e. that \( \chi(m) = 0 \) whenever \( (m, N) \neq 1 \).

For the adelic version of this, we pull back \( \chi \) from \( (\mathbb{Z}/N\mathbb{Z})^\times \) via the quotient map \( \mathbb{Q}^\times/\mathbb{A}^\times \cong \mathbb{R}^\times \times \hat{\mathbb{Z}}^\times \to (\hat{\mathbb{Z}}/N\hat{\mathbb{Z}})^\times \) to an idèle character \( \psi: \mathbb{Q}^\times/\mathbb{A}^\times \to \mathbb{C}^\times \). Then we define the adelic \( L \)-function:

\[ L(s, \psi) = \int_{\mathbb{A}^\times} f(a)\psi(a)|a|^s d^\times a \]

\( L(s, \psi) \) will be the product of \( L_{\text{fin}}(s, \chi) \) with local factors at \( p \mid N \) and \( \infty \). Here, we have to choose \( f \) more carefully than before, according to the conductor of \( \chi \). Then, applying adelic Poisson summation, we get a functional equation for this \( L \)-function. We can get a functional equation for \( L_{\text{fin}}(s, \chi) \) from this by using local functional equations for the extra local factors and dividing through by these factors.
We want to repeat this story for modular forms. In the classical version, we can define $L$-functions of modular forms via the Fourier expansion. Let $f = \sum_{n \geq 1} a_n q^n \in S_k^{new}(N, \chi)$. We assume that $f$ is a normalized eigenform, meaning that it is a simultaneous eigenfunction for the Hecke operators and that $a_1 = 1$ (in which case the eigenvalue for the Hecke operator $T_p$ is $a_p$). The subset $S_k^{new}(N, \chi)$ is the orthogonal complement (with respect to the Petersson inner product) to the space of “oldforms” defined by $f(z) = g(Mz)$ where $g$ is a cusp form with level dividing $N/M$ with $M > 1$.

Then we define:

$$L_{\text{fin}}(s, f) := \sum_{n \geq 1} \frac{a_n}{n^s} = \prod_{p\mid N} \left(1 - a_p p^{-s} + \chi(p)p^{\kappa-1+2s}\right) \prod_{p\not\mid N} \left(1 - a_p p^{-s}\right)^{-1}$$

Here, the second equality holds for $\Re(s)$ sufficiently large.

A slightly better version is:

$$L(s, f) := \int_0^\infty f(iy) y^{s} \, d^{\times}y = \int_0^\infty \left(\sum_{n \geq 1} a_n e^{-2\pi ny}\right) y^{s-1} \, dy = (2\pi)^{-s} \Gamma(s) \cdot L_{\text{fin}}(s, f)$$

Observe that $\phi_f \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)_\infty = f(iy) \cdot y^{k/2}$. Then we have:

$$L(s + k/2, f) = \int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \phi_f \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)_\infty |\psi^{-1}(y_0)| |y_0|^s \, d^{\times}y$$

where $y = y_Q \cdot y_\infty \cdot y_0$ with $y_Q \in \mathbb{Q}^\times$, $y_\infty \in \mathbb{R}_{>0}$, $y_0 \in \mathbb{Z}^\times$.

Now, this gives an $L$-function for $f$ which is written in terms of adelic information, but it is unsatisfying in a number of ways: it does not clearly generalize to settings other than $\text{GL}_2(\mathbb{Q})$, and it does not obviously lead to a functional equation or product formula.

The idea is that we can find a “canonical function” which depends on a representation associated to $\phi_f$ (e.g. a matrix coefficient) and take an appropriate adelic version of its “Mellin transform.” Depending on how we phrase this, we might need to also include a well-chosen Bruhat–Schwartz function to ensure convergence (e.g. $f(a)$ in (1)), which is compactly-supported at the finite places and Schwartz at the infinite place).

Recall that $\mathbb{A}$ is its own Pontryagin dual. This comes from compatible versions of self-duality at each place: at the real place, this is the familiar statement that $\mathbb{R} \simeq \hat{\mathbb{R}}$, normalized such that $\xi \mapsto (\chi_\xi : x \mapsto e^{-2\pi i \xi x})$. At each place $p < \infty$, we have an isomorphism $\mathbb{Q}_p \simeq \hat{\mathbb{Q}}_p$ having the normalization $\xi \mapsto (\chi_\xi : x \mapsto e^{2\pi i \xi x})$.

Note these isomorphisms respect the natural actions of $\mathbb{Q}_v^\times$ (including $v = \infty$) on each side (therefore, we have the same for the global self-duality statement). Moreover, under the induced isomorphism $\mathbb{A} \simeq \hat{\mathbb{A}}$ (denoted $a \mapsto \chi_a$), we see that $\chi_a$ vanishes on $\mathbb{Q}$ if and only if $a \in \mathbb{Q} \subset \mathbb{A}$. In particular, the Pontryagin dual $(\mathbb{A}/\mathbb{Q})$ is isomorphic to $\mathbb{Q}$, and this respects the multiplication action of $\mathbb{Q}_v^\times$ on each side. We call $\lambda_1 \in (\mathbb{A}/\mathbb{Q})$ the character corresponding to $1 \in \mathbb{Q}$. Concretely,

\footnote{This turns out not to be necessary for \(\mathbb{A}\).}

\footnote{This is well-defined as a character on $\mathbb{Q}$ and continuous for the $p$-adic topology, so it extends to $\mathbb{Q}_p$.}
\( \lambda_1(a_v) = e^{-2\pi i a_v} \prod_{v \in \infty} e^{2\pi i a_v} \). Alternatively, one can use strong approximation for \( A \) to (make sense of) and write \( \lambda_1(a_v) = e^{-2\pi i a_v} \prod_{v \in \infty} \phi_v(a_v) \).

We set \( W_\phi(g) := \phi_1(g) = \int_A \phi\left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \lambda_1(x) \, dx \). Then Fourier inversion gives us a new version of (2) to get a canonical definition of the \( L \)-function. In particular, we have

\[
L(s + k/2, f) = \int_{\mathbf{Q} \setminus \mathbf{A}} \sum_{\xi \in \mathbf{Q}^\times} W_\phi\left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) \psi^{-1}(y_0) |y|^s \, d^\times y
\]

(3)

\[
= \int_{\mathbf{A}^\times} W_\phi\left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) \psi^{-1}(y_0) |y|^s \, d^\times y.
\]

(4)

The upshot, as we will see after the talk on Kirillov models, is that the Whittaker functional \( W_\phi \) arises locally. Moreover, we can characterize it uniquely using the local representation at each place. This leads to a canonical definition of the completed \( L \)-function with a built-in product formula. We finish by listing some tasks that still remain (some of these will be addressed in “Adelization of Modular Forms, Part II”; others will be major themes of the seminar at large):

1. Use Whittaker models to define the global \( L \)-function purely from the perspective of representation theory, and deduce its product formula and functional equation (depending on a version of the \( |k \mu_N \) operator), plus functional equations for the local factors.

2. Explain the action of the Hecke operators \( T_p \) in terms of corresponding Hecke operators acting on automorphic forms.

3. Identify the space \( S_{\text{new}}^k(N, \psi) \) of newforms inside \( L^2_{\text{cusp}}(G(\mathbf{Q}) \setminus G(\mathbf{A}), \psi) \) using an adelic version of the Petersson inner product.

4. Understand how the representation \( \pi_f \) generated by \( \phi_f \) behaves (especially locally):
   
   (i) Show that \( \pi_f \) is a completed tensor product of local representations at each place.
   
   (ii) Show that \( \pi_f \) is irreducible if \( f \) is a cuspidal eigenform.
   
   (iii) Prove representation-theoretic multiplicity-one statements and the Converse Theorem. For example, we would like to deduce the following fact from representation-theoretic knowledge: “if \( f_1, f_2 \) are normalized Hecke eigenforms in the new space for which the eigenvalues \( a_p \) are equal for all but finitely many \( p \) then \( f_1 = f_2 \)”

A Proof of Proposition 1.9

Recall that \( \phi : \Gamma \setminus \SL_2(\mathbf{R}) \to \mathbf{C} \) is a smooth function with \( \SO_2(\mathbf{R}) \)-weight \( k \in \mathbf{Z} \) such that \( \Delta \phi = -(k/2)(k/2-1)\phi \). Our aim is to show that \( f(x+iy) := \phi\left( \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} \right) \) is holomorphic if \( \phi \) is cuspidal.

The key point is going to be that the \( K = \SO_2(\mathbf{R}) \)-weight \( k \) and the \( \Delta \) eigenvalue \( -\frac{k}{2}\left( \frac{k}{2} - 1 \right) \) of \( \phi \) are compatible. This allows us to show that \( \phi \) is the lowest-weight vector in an irreducible \((\mathfrak{g}, K)\)-module (a copy of the so-called \( D^+_k \)). In particular, the lowering operator in \( \mathfrak{g} \) kills \( \phi \); a
straightforward calculation shows that $L\phi$ vanishes exactly when $\frac{\partial f}{\partial \theta}$ does; see Lemma A.3. Our proof will be phrased, however, in the language of differential operators.

Lemma A.1 encodes the compatibility of the $K$-weight and the $\Delta$-eigenvalue of $\phi$ in terms of the action of the raising and lowering operators. Lemma A.3 shows the connection between $\phi$ being annihilated by the lowering operator and $f$ being holomorphic. The key step is Lemma A.2 whose representation-theoretic interpretation is that the $(g, K)$-module generated by $\phi$ is unitary, allowing us to deduce that the lowering operator indeed kills $\phi$. For more representation-theoretic context, including the theory of $(g, K)$-modules and the definition of $D^+_k$, see Ch. 2 of [3].

The right-regular action of $\text{SL}_2(\mathbb{R})$ on $C^\infty(\Gamma \backslash \text{SL}_2(\mathbb{R}))$ induces an action of the Lie algebra $g = \mathfrak{sl}_2(\mathbb{R})$ (and even its universal enveloping algebra $U(g)$). This was implicit during the discussion of the Casimir operator in §1. Moreover, since $C^\infty(\Gamma \backslash \text{SL}_2(\mathbb{R}))$ is a complex vector space, we can complexify this action to one of $g_C = g \otimes_{\mathbb{R}} \mathbb{C} \simeq \text{Lie}(\text{SL}_2(\mathbb{C}))$ (as well as of $U(g_C)$). One can consider $U(g)$ as the algebra of left-$\text{SL}_2(\mathbb{R})$-invariant differential operators, so we would like to write down these differential operators in the $(x, y, \theta)$ coordinates we have been using on $\text{SL}_2(\mathbb{R})$. Throughout this discussion, we’ll use the same notation for elements of $g_C$ and their effect in the representation on $C^\infty(\Gamma \backslash \text{SL}_2(\mathbb{R}))$.

We have the usual basis for $g$

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfying the familiar commutation relations:

$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$

Similarly, a basis for $g_C$ is given by the elements

$$R := \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad L := \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad H = -i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that $R$ and $L$ are complex conjugates, we have “equivalent” commutation relations:


and $H = ir^t(0)$ for the parameterization $r(\theta) = (\cos \theta, -\sin \theta, -\cos \theta)$ that we have been using for $K = \text{SO}_2(\mathbb{R}) \subset \text{SL}_2(\mathbb{R})$.

Using the commutation relations, it is a classical exercise to check that the Killing form $\mathfrak{sl}_2$ in characteristic 0 is $(X, Y) \mapsto 4 \cdot \text{tr}(XY)$ and from this that the Casimir operator on $g = \mathfrak{sl}_2(\mathbb{R})$ and $g_C$ is given by

$$\frac{1}{8}(H^2 + 2RL + 2LR) = C = \frac{1}{8}(h^2 + 2ef + 2fe).$$

\[\text{Recall that for us, } \phi \text{ having “weight } k\text{” means that } \phi(\text{gr}(\theta)) = \phi(g)e^{-ik\theta}, \text{ where we use the usual “counterclockwise” parametrization } r(\theta) \text{ of } \text{SO}_2(\mathbb{R}). \text{ Both of these are the opposite conventions to those in } [3], \text{ where the clockwise parametrization and the opposite definition of weights are used. The net effect is that an eigenfunction } \phi \text{ has the same weight } k \text{ in our setting as it does in } [3].\]
Returning to the action on \( C^\infty(\Gamma \backslash \text{SL}_2(\mathbb{R})) \), one can check (via differentiation)\(^9\) that the \( \mathfrak{sl}_2 \)-triple \( \{ R, L, H \} \) giving a basis of \( g_C \) acts via the following operators:

\[
R := e^{-2i\theta} \left( i y \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \frac{1}{2i} \frac{\partial}{\partial \theta} \right)
\]

\[
L := e^{2i\theta} \left( -iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{1}{2i} \frac{\partial}{\partial \theta} \right)
\]

\[
H := i \frac{\partial}{\partial \theta}.
\]

One can then compute by hand\(^10\) that

\[
- y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - y \frac{\partial^2}{\partial x \partial \theta} = -\frac{1}{4} (H^2 + 2RL + 2LR)
\]

\[
= -2C
\]

\[
= -RL + \frac{1}{4} \frac{\partial^2}{\partial \theta^2} - \frac{1}{2i} \frac{\partial}{\partial \theta}.
\]

We claim that the left side equals the Laplacian \( \Delta \) for a suitable invariant metric (so that Laplacian is equal to \(-2C\), as asserted in §1). This is a direct calculation using the metric tensor on \( \text{SL}_2(\mathbb{R}) \) given by\(^11\)

\[
ds^2 = \text{???}
\]

that is seen to be right-invariant.

Before proceeding with the proof of Proposition 1.9, we make one more observation: with these operators in place, the weight-\( k \) eigenfunction relation \( R(r(\theta)) : \phi \mapsto e^{-ik\theta} \phi \) for \( \phi \) under the \( \text{SO}_2(\mathbb{R}) \)-action implies at the level of the action of \( \mathfrak{sl}_2(\mathbb{R}) \) via differential operators that \( r'(0) : \phi \mapsto -ik\phi \). Since \( H = ir'(0) \) in \( \mathfrak{sl}_2(\mathbb{C}) \), we conclude that \( \phi \) is an eigenfunction for \( H \) with eigenvalue \( k \); i.e. \( H\phi = k\phi \).

Lemma A.1. Let \( \phi : \Gamma \backslash \text{SL}_2(\mathbb{R}) \rightarrow \mathbb{C} \) be a smooth function satisfying the first two conditions of Proposition 1.9. That is, \( \phi \) is an eigenfunction for \( \text{SO}_2(\mathbb{R}) \) with weight \( k \) and an eigenfunction for \( \Delta \) with eigenvalue \( -(k/2)(k/2 - 1) \). Then \( RL\phi = 0 \).

Proof. Using the Iwasawa (i.e., NAK) decomposition, the function

\[
\tilde{\phi}(x, y, \theta) := e^{ik\theta} \phi(x, y, \theta)
\]

is independent of \( \theta \) due to \( \phi \) being an \( \text{SO}_2(\mathbb{R}) \)-eigenfunction of weight \( k \). Therefore,

\[
\left( \frac{1}{4} \frac{\partial^2}{\partial \theta^2} - \frac{1}{2i} \frac{\partial}{\partial \theta} \right) \phi = \left( \frac{1}{4} \frac{\partial^2}{\partial \theta^2} - \frac{1}{2i} \frac{\partial}{\partial \theta} \right) (e^{-ik\theta} \tilde{\phi}) = -\frac{k^2}{4} \phi + \frac{k}{2} \phi = -(k/2)(k/2 - 1) \phi,
\]

yet

\[
RL = (1/4)\partial^2_\theta - (1/2i)\partial_\theta - \Delta,
\]

so the \( \Delta \)-eigenfunction hypothesis on \( \phi \) implies that \( RL(\phi) = 0 \) as desired. \( \Box \)

\(^9\)see [3, Prop. 2.2.5] for this computation, keeping in mind that the opposite parametrization of \( \text{SO}_2(\mathbb{R}) \) is used there.\(^10\)again, see [3, Prop. 2.2.5].\(^11\)Need to say what this right-invariant metric is, to justify that this really is a Laplace-Beltrami operator and not just some hocus-pocus called a “Laplacian” by hand. Is there a reference computing such a right-invariant metric?
Lemma A.2. Let $\phi : \Gamma \backslash \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ be a smooth eigenfunction for $\mathrm{SO}_2(\mathbb{R})$ of weight $k$ such that $RL\phi = 0$. If $\phi$ is cuspidal then $L\phi = 0$.

Granting this lemma for a moment, we connect the lowering operator to holomorphicity of our candidate modular form $f$:

Lemma A.3. Let $\phi : \Gamma \backslash \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ be a smooth eigenfunction of $\mathrm{SO}_2(\mathbb{R})$-weight $k$. Define $f(x + iy) := \phi \left( \frac{y^{1/2}}{x-1/2} \right) y^{-k/2}$. Then

$$2 \frac{\partial f}{\partial z} = iy^{-k/2-1}e^{i(k-2)\theta} L\phi.$$ 

In particular, $L\phi = 0$ if and only if $f$ is holomorphic.

Proof. We can rewrite $f = y^{-k/2}e^{ik\theta} \phi(x, y, \theta)$ since $\phi$ is a weight-$k$ eigenfunction for $K = \mathrm{SO}_2(\mathbb{R})$. Then

$$2 \frac{\partial f}{\partial z} = \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f = y^{-k/2}e^{ik\theta} \left( \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} \right) - \frac{i k}{2} y^{-k/2-1} e^{i k \theta} \phi$$

$$= iy^{-k/2-1}e^{i k \theta} \left( -i y \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} \right) - \frac{i k}{2} y f$$

$$= iy^{-k/2-1}e^{i k \theta} \left( - \frac{1}{2i} \frac{\partial \phi}{\partial \theta} \right) - \frac{i k}{2y} f + iy^{-k/2-1}e^{i(k-2)\theta} L\phi,$$

where the final equality uses the description of $L$ as a first-order differential operator. By “differentiating” the condition that $\phi$ is a weight-$k$ eigenfunction for $K$, we have $\partial \phi / \partial \theta = -ik\phi$. Plugging this into the first term on the right side at the end, we obtain that

$$2 \frac{\partial f}{\partial z} = iy^{-k/2-1}e^{i k \theta} \left( \frac{k}{2} \phi \right) - \frac{i k}{2y} f + iy^{-k/2-1}e^{i(k-2)\theta} L\phi$$

$$= \frac{i k}{2y} f - \frac{i k}{2y} f + iy^{-k/2-1}e^{i(k-2)\theta} L\phi$$

$$= iy^{-k/2-1}e^{i(k-2)\theta} L\phi.$$ 

It remains to prove Lemma A.2

Proof. Using the $\mathrm{NAK}$ decomposition for $\mathrm{SL}_2(\mathbb{R})$ and the identification of $\mathrm{SL}_2(\mathbb{R})/K$ with $H$, the Haar measure on $\mathrm{SL}_2(\mathbb{R})$ in such “$(x, y, \theta)$” coordinates is $(y^{-2}dx dy)d\theta$. We will show that $\int_{\Gamma \backslash \mathrm{SL}_2(\mathbb{R})} |L\phi|^2 \frac{dxdy}{y^2} d\theta$ vanishes.

The proof strategy is to show that $R$ and $L$ roughly behave as adjoints with respect to the inner product for certain $K$-eigenfunctions in $L^2_{\text{cusp}}(\Gamma \backslash \mathrm{SL}_2(\mathbb{R}))$ (depending on the $K$-weight of those functions). More specifically, we will find that our integrand is exact, and more specifically is equal to $d\omega$ for a 2-form $\omega$ on $\Gamma \backslash \mathrm{SL}_2(\mathbb{R})$ such that $\omega(x, y, \theta)$ decays rapidly as $(x, y)$ approaches any cusp of $\Gamma$ (ultimately due to cuspidality of $\phi$). A careful limiting application of Stokes’ Theorem to
compact “cutoffs” of the $K$-bundle $\Gamma' \backslash \mathrm{SL}_2(\mathbb{R}) \to \Gamma' \backslash \mathbb{H}$ near the cusps will then give the result, where $\Gamma'$ is a suitable finite-index subgroup of $\Gamma$.

By the Leibniz rule for $R$, together with the fact that $R$ and $L$ are complex conjugates, we have

$$|L\phi|^2 = (L\phi)\overline{(L\phi)} = R\left(L\phi(\bar{\phi})\right) - (RL\phi)\overline{\left(\bar{\phi}\right)} = R\left(L\phi(\bar{\phi})\right).$$

We claim this function multiplied against the oriented volume form $(y^{-2}dx\,dy)d\theta$ is exact. Indeed, consider the 2-form

$$\omega = -e^{-2i\theta}(L\phi)\overline{\phi}\left(y^{-1}\frac{dzd\theta}{y} + i\frac{dx\,dy}{2y^2}\right).$$

Note that

$$\frac{\partial\phi}{\partial\theta} = ik\phi \quad \text{and} \quad \frac{\partial}{\partial\theta}\left(L\phi\right) = -iHL\phi = -i(LH - 2L)\phi = -i(k - 2)(L\phi)$$

In particular, this implies that

$$\frac{\partial}{\partial\theta}\left((L\phi)\overline{\phi}\right) = 2i(L\phi)\overline{\phi},$$

so

$$\frac{\partial}{\partial\theta}\left(e^{-2i\theta}(L\phi)\overline{\phi}\right) = 0.$$

Hence, we have

$$d\omega = d\left(-e^{-2i\theta}(L\phi)\overline{\phi}\left(y^{-1}xd\theta - iy^{-1}y\,dy\,d\theta + \frac{i}{2}y^{-2}xd\,dy\right)\right)$$

$$= e^{-2i\theta}\frac{\partial}{\partial y}\left(y^{-1}(L\phi)\overline{\phi}\right)dx\,dy\,d\theta + ie^{-2i\theta}y^{-1}\frac{\partial}{\partial x}\left(L\phi\overline{\phi}\right)dx\,dy\,d\theta$$

$$- \frac{i}{2y^2}\frac{\partial}{\partial \theta}\left(e^{-2i\theta}(L\phi)\overline{\phi}\right)dx\,dy\,d\theta.$$ 

The final term being subtracted vanishes, so

$$d\omega = \left(e^{-2i\theta}(-y^{-2})(L\phi)\overline{\phi} + e^{-2i\theta}y^{-1}\frac{\partial}{\partial y}\left((L\phi)\overline{\phi}\right) + ie^{-2i\theta}y^{-1}\frac{\partial}{\partial x}\left(L\phi\overline{\phi}\right)\right)dx\,dy\,d\theta$$

$$= \left(-e^{-2i\theta}(L\phi)\overline{\phi}\right)\frac{dx\,dy\,d\theta}{y^2} + \left(e^{-2i\theta}y\frac{\partial}{\partial y}\left((L\phi)\overline{\phi}\right) + e^{-2i\theta}iy\frac{\partial}{\partial x}\left((L\phi)\overline{\phi}\right)\right)\frac{dx\,dy\,d\theta}{y^2}$$

$$= \left(-e^{-2i\theta}(L\phi)\overline{\phi}\right)\frac{dx\,dy\,d\theta}{y^2} + \left(R\left((L\phi)\overline{\phi}\right) + e^{-2i\theta}\frac{1}{2i}\frac{\partial}{\partial \theta}\left((L\phi)\overline{\phi}\right)\right)\frac{dx\,dy\,d\theta}{y^2},$$

the final equality by the determination of $R$ as a first-order differential operator. Rearranging the order of summation, we get

$$d\omega = \left(R\left((L\phi)\overline{\phi}\right) + \left(-e^{-2i\theta} + e^{-2i\theta}\frac{1}{2i}\frac{\partial}{\partial \theta}\right)\left((L\phi)\overline{\phi}\right)\right)\frac{dx\,dy\,d\theta}{y^2}$$

$$= \left(R\left((L\phi)\overline{\phi}\right) + \left(-e^{-2i\theta} + e^{-2i\theta}\right)\left((L\phi)\overline{\phi}\right)\right)\frac{dx\,dy\,d\theta}{y^2}$$

$$= R\left(L\phi(\overline{\phi})\right)\frac{dx\,dy\,d\theta}{y^2}.$$
(the second equality due to the identity \( \partial_y((L\phi)\bar{\phi}) = 2i(L\phi)\bar{\phi} \).

We conclude that
\[
\int_{\Gamma \setminus \text{SL}_2(\mathbb{R})} |L\phi|^2 \frac{dxdy}{y^2} d\theta = \int_{\Gamma \setminus \text{SL}_2(\mathbb{R})} d\omega
\]
as oriented integrals, provided that the right side is at least \textit{finite}. In particular, it is harmless to replace \( d\omega \) with the corresponding density \( |d\omega| \) (so all convergence issues involve manifestly non-negative quantities).

The \textit{NAK}-decomposition realizes \( \text{SL}_2(\mathbb{R}) \) as an \( S^1 \)-bundle over \( H \). As long as \( H \) acts freely on \( R \), the coset space \( \Gamma \setminus \text{SL}_2(\mathbb{R}) \) is also an \( S^1 \)-bundle over \( \Gamma \setminus H \). For convenience, we will denote the structure map in that case as \( \pi : \Gamma \setminus \text{SL}_2(\mathbb{R}) \to Y(\Gamma) \). Suppose \( \Gamma' \subset \Gamma \) is a finite-index subgroup. Then \( L\phi = 0 \) if and only if \( L\phi' = 0 \), where \( \phi' \) is the pullback of \( \phi \) to the finite cover \( \Gamma' \setminus \text{SL}_2(\mathbb{R}) \). Since \( \Gamma \) is arithmetic, it has a finite-index subgroup \( \Gamma' \subset \text{SL}_2(\mathbb{Z}) \) such that \( \Gamma' \cap K = \{1\} \). Then \( \Gamma' \) acts freely on \( H \) and we may use it in place of \( \Gamma \) for the remainder of the argument.

Let \( M \) be the 2-manifold with boundary obtained from \( X(\Gamma) \) by removing an (open) \( \epsilon \)-neighborhood of each cusp. Then \( E := \pi^{-1}(M) \subset \Gamma \setminus \text{SL}_2(\mathbb{R}) \) is a compact oriented Riemannian 3-manifold with boundary (\( \partial E \) is a disjoint union of \( S^1 \)-bundles over circles of radius \( \epsilon \), indexed by the cusps of \( \Gamma \)). Then
\[
\int_{\Gamma \setminus \text{SL}_2(\mathbb{R})} |L\phi|^2 \frac{dxdy}{y^2} d\theta = \int_{\Gamma \setminus \text{SL}_2(\mathbb{R})} |d\omega| = \int_{E} |d\omega| + \int_{(\Gamma \setminus \text{SL}_2(\mathbb{R})) - E} |d\omega| \leq \int_{\partial E} |\omega| + \int_{(\Gamma \setminus \text{SL}_2(\mathbb{R})) - E} |d\omega|,
\]
the final inequality by Stokes’ Theorem for the oriented compact manifold with boundary \( E \). We claim that both of these final integrals decay to 0 as \( \epsilon \to 0 \). Since \( \phi \) is automorphic and \( L \) is \( \text{SL}_2(\mathbb{R}) \)-invariant, the formula defining \( \omega \) shows that the decay (in \( y \)) of the first integral is reduced to the decay of \( (L\phi)\bar{\phi} \) near each cusp. Likewise, our computation of \( d\omega \) reduces the decay of the second integral to the decay (in \( y \)) of \( R((L\phi)\bar{\phi}) \) near each cusp. We explain how to deduce these decay properties from results proved in \cite{7}.

In \cite[Cor. 3.6.1]{7}, it is shown that for each smooth compactly-supported function \( \alpha \in C_\infty^c(\text{SL}_2(\mathbb{R})) \), there is a constant \( c_0(\alpha) > 0 \) such that
\[
|\langle \psi \ast \alpha \rangle(g) \rangle \leq c_0(\alpha)\|\psi\|_2
\]
for any \( \psi \in L^2_{\text{cusp}}(\Gamma \setminus \text{SL}_2(\mathbb{R})) \). But the method of proof is robust as follows: in view of the identity \( \mathcal{L}_X(\psi \ast \alpha) = \psi \ast \mathcal{L}_X(\alpha) \) for any left-invariant differential operator \( \mathcal{L}_X \), one can apply the statement to the functions \( \mathcal{L}_X(\alpha) \in C_\infty^c(\text{SL}_2(\mathbb{R})) \) to conclude that
\[
|\mathcal{L}_X(\psi \ast \alpha)(g)| \leq c_0(\mathcal{L}_X(\alpha))\|\psi\|_2.
\]

\footnote{The \( L \) operator on \( C_\infty^c(\Gamma \setminus \text{SL}_2(\mathbb{R})) \) takes the same form as the \( L \) operator on \( C_\infty^c(\Gamma' \setminus \text{SL}_2(\mathbb{R})) \).}

\footnote{This corresponds to cutting away the “large-\( y \)” portion of a Siegel domain in \( Y(\Gamma) \) for each cusp.}

\footnote{Whoa, how do we know that for smooth functions cuspidal \textit{in the sense defined above} that they belong to \( L^2 \)? Is this implicit in Ngo’s proofs? No circularity?}
By integration over $\Gamma_s \backslash U_s(\mathbb{R})$ (for $s$ a cusp), we get the same estimate\(^{15}\) for the “constant term” at $s$:
\[
|L_X(\psi * \alpha)_B(g)| \leq c_0(L_X(\alpha))\|\psi\|_2,
\]
where $F_B(g) := \int_{\Gamma_s \backslash U_s(\mathbb{R})} F(ug) \, du$.

On the other hand, [7, Prop. 3.5.4] implies (by cuspidality of $\psi$, hence of $\psi * \alpha$\(^{15}\)) that
\[
|(\psi * \alpha)(g_s, g)| \leq cy^{-1} \left( \sum_{i=1}^{3} |L_{X_i}(\psi * \alpha)_B(g_s, g)| \right),
\]
for $g \in \text{SL}_2(\mathbb{R})$ such that $g(i) = x + iy, g_s \in \text{SL}_2(\mathbb{Q})$ satisfying $g_s(\infty) = s$, a basis $\{X_1, X_2, X_3\}$ of $g$, and $c > 0$ a constant depending on the choice of $g_s$ and $\{X_1, X_2, X_3\}$.\(^{17}\) Combining these two inequalities, we see that there is a constant $c_{-1}(\alpha) > 0$ depending only on $\alpha$ such that
\[
|(\psi * \alpha)(g_s, g)| \leq c_{-1}(\alpha)y^{-1}\|\psi\|_2
\]
for any $\psi \in L^2_{\text{cusp}}(\Gamma_s \backslash \text{SL}_2(\mathbb{R}))$ and cusp $s$. By induction\(^{18}\) we thereby find constants $c_{-n}(\alpha) > 0$ for each $n$ such that
\[
|(\psi * \alpha)(g_s, g)| \leq c_{-n}(\alpha)y^{-n}\|\psi\|_2
\]
(5)
for any $\psi \in L^2_{\text{cusp}}(\Gamma_s \backslash \text{SL}_2(\mathbb{R}))$ and cusp $s$.

By [7, Prop. 3.3.5], there is an $\alpha_0 \in C_0(\text{SL}_2(\mathbb{R}))$ such that $\phi = \phi * \alpha_0$. Recalling that the operators $L$ and $R$ are $\text{SL}_2(\mathbb{R})$-invariant, we apply \(^{16}\) with $\psi = \phi$ and $\alpha = \alpha_0$ (or $\alpha$ equal to one of $L\alpha_0, RL\alpha_0, R\alpha_0$, as needed) to see that $|(L\phi)\varphi|$ and $|R((L\phi)\varphi)|$ decay faster than any power of $y^{-n}$ near the cusp $s$. Therefore, both of the integrals $\int_{\partial \epsilon} |\omega|$ and $\int_{(\Gamma_s \backslash \text{SL}_2(\mathbb{R})) - \epsilon} |d\omega|$ decay to 0 as $\epsilon \to 0$, proving that $L\phi = 0$. \qed

\begin{thebibliography}{99}


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\(^{15}\) Why the same constant? Shouldn’t the volume of $\Gamma_s \backslash U_s(\mathbb{R})$ intervene?

\(^{16}\) How do we know that convolution against $\alpha$ preserves cuspidality? Reference?

\(^{17}\) For example, if $g_s = 1$ and we use the basis $\{e, f, h\}$, then $c = 1$ suffices.