Lecture 4: Adelic heights and SL\(_2\), GL\(_2\), PGL\(_2\) over \(k_0 = \mathbb{Q}, \mathbb{F}_q(t)\)

Lecture by Brian Conrad

Stanford Number Theory Learning Seminar

October 25, 2017

Notes by Dan Dore

We’ll follow \(\S 1.8 - \S 1.11\) of [2] for the bulk of today’s lecture. Recall the statement of Theorems C and F of reduction theory:

**Theorem 1** (“Theorem C”). For \(G\) a connected reductive group over a global field \(k\), \([G] = G(\mathbb{A})^1 / G(k)\) is compact if and only if \(\mathcal{D}G\) is \(k\)-anisotropic (“semi-simple \(k\)-rank is 0”).

Last time, we proved the forward implication: if \(\mathcal{D}G\) contains a nontrivial split \(k\)-torus then \([G]\) is non-compact. In particular, this verifies Theorem C for any split \(G\) such that \(GL_2, SL_2, PGL_2\).

**Theorem 2** (“Theorem F”). Assume \(\mathcal{D}G\) is \(k\)-isotropic. Choose a maximal split \(k\)-torus \(S \subset G\) and minimal parabolic \(k\)-subgroup \(P\) with \(S \subseteq P \subseteq G\). Then

\[
G(\mathbb{A})^1 = K \cdot S(c)P(\mathbb{A})^1 G(k)
\]

for some compact subset \(K \subseteq G(\mathbb{A})^1\) and some \(c > 0\), with

\[
S(c) := \{ s \in S(\mathbb{A}) \cap G(\mathbb{A})^1 \mid |\alpha(s)|_k \leq c \text{ for all } \alpha \in \Delta \}
\]

where \(\Delta\) is the basis for the positive system of roots \(\Phi(P, S)\) in the relative root system \(\Phi(G, S)\) of non-trivial \(S\)-weights on \(\text{Lie}(G)\).

In the case \(G = SL_2\) and \(k = \mathbb{Q}\), we saw for suitable \(S\) and \(P\) that the image of \(S(c)\) in \([G]/G(\hat{\mathbb{Z}})\) may be identified with the ray \(\{iy \mid y \geq \frac{1}{c^2}\}\) in the upper half-plane.

**Remark 3.** Last time we saw that once both theorems are proved, we can replace \(P(\mathbb{A})^1\) in this expression with a compact subset \(K' \subseteq P(\mathbb{A})^1\) by applying Theorem C to any Levi factor of \(P\) (i.e., a \(k\)-subgroup mapping isomorphically onto the connected reductive \(P/\mathcal{R}_{u,k}(P)\)).

To prove these theorems, we’ll start off by showing Theorem F when \(G\) is one of \(SL_2, GL_2, PGL_2\) and \(k\) is one of \(\mathbb{Q}, \mathbb{F}_q(t)\) (since Theorem C for split groups has been settled). We will adapt the classical proof describing the fundamental domain for the action of \(SL_2(\mathbb{Z})\) on the upper half-plane into something more group-theoretic (and adelic).

Let \(V\) be a non-zero finite-dimensional \(k\)-vector space. We’ll denote \(V \otimes \mathbb{A}\) as \(V_{\mathbb{A}}\). Given a point of \(V_{\mathbb{A}}\), can we assign this a meaningful “norm”? Consider norms \(\| \cdot \|_v\) on \(V_v = V \otimes_k v\) for each place \(v\); these are required to be compatible with \(\| \cdot \|_v\) on \(k_v\) in the sense that \(\|ax\|_v = |a|_v \|x\|_v\) and we furthermore ask it to satisfy the ultrametric inequality when \(v\) is non-archimedean. We will impose a further compatibility condition: for all but finitely many \(v\), \(\| \cdot \|_v\) is the “sup-norm” with respect to a common choice of \(k\)-basis.
Remark 4. Why don’t we just pick a $k$-basis for $V$ at the start and ask $\| \cdot \|_v$ to be a sup-norm for this basis for all $v$? Our definition has the following advantage: if $g \in \mathrm{GL}(V \otimes A) = \mathrm{GL}(V)(A)$, then $\{\|g(\cdot)\|_v\}$ is another such collection. Indeed, if we choose a $k$-basis for $V$ to define $k^n \simeq V$ then we get a lattice $L_v \simeq O_v^\otimes \subseteq V_v$ for each finite $v$, and this is preserved by $g$ for all but finitely many $v$; i.e., $g_v \in \mathrm{GL}_n(L_v)$ for all but finitely many $v$. Thus $\| \cdot \|_v$ is the sup-norm for all but finitely many $v$, so our notion is preserved by global automorphisms of $V_A := V \otimes A$.

We call $\xi \in V_A$ primitive if $\xi \in \mathrm{GL}(V)(A) \cdot (V - \{0\}) \subseteq V_A$. For such $\xi$, $\|\xi_v\|_v \neq 0$ for all $v$, and $\|\xi_v\|_v = 1$ for almost all $v$, so $\prod_v \|\xi_v\|_v$ is a product of finitely many non-zero terms, so it converges to a positive number. For such $\xi$, we define the adelic height\footnote{This has nothing to do with other notions of height in arithmetic geometry!} $\|\xi\|_\Xi$ as $\prod_v \|\xi_v\|_v$. By the remark, for any $g \in \mathrm{GL}(V_A)$, $\|g(\cdot)\|$ also gives a height on primitive vectors, and the ratio of $\|g(\cdot)\|$ to $\| \cdot \|$ is clearly bounded above and below by positive constants.

Proposition 5 (Properties of Heights).

(i) $\|t\xi\| = |t|_k \cdot \|\xi\|$ for $t \in A^\times$ and $\xi$ primitive.

(ii) For $\| \cdot \|$, $\| \cdot \|$’ two heights, there is some $c, C > 0$ such that

$$c \leq \|\xi\|'/\|\xi\| \leq C$$

for all primitive $\xi$.

(iii) If $\{\xi_n\}$ are primitive and $\xi_n \to 0$ in $V_A$, then $\|\xi_n\| \to 0$.

(iv) (Approximate converse to (iii)) If $\{\xi_n\}$ are primitive with $\|\xi_n\| \to 0$, then there exist $\lambda_n \in k^\times$ such that $\lambda_n \xi_n \to 0$.

The first two properties are easy: (ii) follows from the easy analysis fact that all norms on a finite-dimensional topological vector space over a locally compact field are metrically equivalent (i.e. the analogous inequality as in the statement of (ii) holds), and (i) is immediate from the construction.

By (ii), in order to prove (iii) it is harmless to fix a $k$-basis of $V$ and to let $\| \cdot \|_v$ be the sup norm with respect to this basis for all $v$. Then, it is easy to verify that $\xi_n \to 0$ implies that the sequence $\|\xi_n\|$ is at least bounded. Springer provides a proof of (iii), so we will just highlight some of the important ingredients. The first key fact is this boundedness statement. We also need:

- The part of $A^\times/k^\times$ with $|\cdot|_k$ in a compact subset of $R_{>0}$ is compact.

- $\{\alpha \in k^\times \mid |\alpha|_v \leq 1 \text{ for all } v\} = \mu_\infty(k)$ (i.e. the set of roots of unity in $k$) is finite.

Here’s a shorter proof of (iii): fixing a $k$-basis of $V$ and assuming that $\| \cdot \|_v$ is the sup-norm with respect to this basis for all $v$. Then the condition that $\xi_n \to 0$ in $V_A$ implies that for large enough $n$, $\|\xi_n\|_v \leq 1$ for all $v$, and that there is some finite set of places $v_1, \ldots, v_n$ such that $\|\xi_n\|_{v_i} \to 0$ for each $i$. Then we can see directly that this implies that the product $\|\xi_n\| = \prod_v \|\xi_n\|_v$ goes to 0.

Note that (iv) is as good as we can hope for, since if $\|\xi_n\| \to 0$, then by (i) $\|\lambda_n \xi_n\| = |\lambda_n|_k \|\xi_n\| = \|\xi_n\| \to 0$ as well by the product formula. Certainly multiplying by such elements could destroy the property that $\xi_n \to 0$ in $V_A$. 

To prove (iv), Springer says that we can easily reduce to the one-dimensional case, but now that I am standing here I am a bit puzzled why this should be true. Hmm, I thought it was clear when reading the paper, but now I am a bit perplexed if this is even plausible. Ugh. I’ll have to come back to this later; property (iv) is crucial for later purposes in Springer’s paper, so this had better be correct or else the whole paper will collapse. Let’s grant it for now. In the 1-dimensional case (so we can suppose $V = k$), one can conclude by using the strong approximation for the adele ring $A$ to see that $(A^\times)^1$ gets its usual topology as the subspace topology from $A$.

We will first use adelic heights and their properties in the case that $\dim V = 2$ and $k_0 = \mathbb{Q}, F_q(t)$ to study $GL_2, SL_2, PGL_2$ for these fields. (They also play a crucial role for more general $V$ in the treatment of more general $G$ later.) For now, let’s fix a basis of $V$ and assume that we fix the height when $k_0 = \mathbb{Q}$ to be the sup-norm with respect to this basis for all $v \neq \infty$ and the usual Euclidean length for for $v = \infty$, and fix it to be the sup-norm for all $v$ when $k_0 = F_q(t)$.

We have a (maximal) compact subgroup of $GL(k_v)$ given by

$$K_v = \{ g \in GL(V_v) \mid \| g(\cdot) \|_v = \| \cdot \|_v \} = \begin{cases} GL_2(\mathcal{O}_v), v \neq \infty; \\ O_2(\mathbb{R}), v = \infty, \end{cases}$$

Taking the directly product, we get a compact subgroup $K = \prod_v K_v \subseteq GL(V_\mathbb{A})$.

For $B = (\delta^* \delta^*)$ the upper-triangular Borel subgroup of $GL_2$, we have $GL_2/B \simeq P^1$ by sending $g$ to $g(\infty)$. A mild argument shows that there is a topological isomorphism

$$GL_2(A)/B(A) \simeq (GL_2/B)(A) = P^1(A) = \prod_v P^1(k_v)$$

This uses that for $v \neq \infty$, by the valuative criterion of properness (or just clearing denominators of homogeneous coordinates) we have $P^1(k_v) = P^1(\mathcal{O}_v)$. Hence, as an exercise we get $GL_2(A) = K \cdot B(A)$.

For $c > 0$, we let:

$$B(c) = \{(t_1^* t_2) \in B(A) \mid |t_1/t_2|_k \leq c\}$$

**Proposition 6.** For any $\epsilon_0 > 0$ and $c = (2/\sqrt{3}) + \epsilon_0$ we have:

$$GL_2(A) = K \cdot B(c) \cdot GL_2(k)$$

This implies Theorem F for $GL_2$ by restricting to the “norm-1” parts of $GL_2(A)$ and $B(c)$, i.e. by restricting to $g \in GL_2(A)$ such that $|\det g| \leq 1$ and $b \in B(c)$ such that $|t_1 t_2| \leq 1$. The case of $SL_2$ is treated by a variant of the same method, and the case of $PGL_2$ can be deduce from that of $GL_2$ (details left to the reader).

**Proof.** This proof is inspired by classical arguments with $SL_2(\mathbb{Z})$ acting on the upper half-plane, such as those appearing in [1, Chapter VII]. The strategy will be to take a point in $GL_2(A)$, vary it across the entire orbit of $G(k)$, show that the height is bounded away from 0, and pick an element which is close to the infimum. We shall use the adelic height as above defined with respect to the standard basis of $k^2$. 

3
Choose $g \in \text{GL}_2(A)$. We seek $\gamma \in \text{GL}_2(k)$ with $g \cdot \gamma \in K \cdot B(c)$. For varying $\xi \in k^2 - \{0\}$ (which is acted on transitively by $\text{GL}_2(k)$), we claim that the height $\|g(\xi)\|$ is bounded away from 0. If not, pick some sequence $\{\xi_n\}$ with $\|g(\xi_n)\| \to 0$. By Proposition 5 (iv), we get $\lambda_n \in k^\times$ such that $\lambda_n g(\xi_n) = g(\lambda_n \xi_n) \to 0$. Since $g$ is fixed, this means that $\lambda_n \xi_n \in k^2 - \{0\}$ goes to 0 in $A^2$, which is a contradiction since $k^2$ is discrete in $A^2$ and $\lambda_n \xi_n \neq 0$ for all $n$.

Fix some small $\epsilon > 0$. Pick some $\xi_0$ such that $\|g(\xi_0)\| \leq (1 + \epsilon) \inf_\xi \|g(\xi)\|$. By right multiplication on $g$ by an element $\gamma_0$ of $\text{GL}_2(k)$, we can change the basis to reduce to the case $\xi_0 = e_1$. Thus, we are in the situation where:

$$\text{for all } \xi \in k^2 - \{0\}, \|g(e_1)\| \leq (1 + \epsilon)\|g(\xi)\|$$

(1)

This property is invariant under left multiplication on $g$ by $K = \prod_v K_v$ since for $v \neq \infty$, $\text{GL}_2(\mathcal{O}_v)$ preserves the sup norm $\|\cdot\|$ and $\mathcal{O}_2(\mathbb{R})$ preserves the Euclidean norm. We want to show that the inequality (1) implies that $g \in K \cdot B(c)$. Since $\text{GL}_2(A) = K \cdot B(A)$, we know that $g = mb$ for $m \in K$, $b = \begin{pmatrix} t_1 & t_2 \\ 0 & 1 \end{pmatrix} \in B(A)$ with $t_1, t_2 \in A^\times$, $u \in A$. Without loss of generality, we can replace $g$ with $m^{-1}g = b$. Now, since left multiplication by $K$ preserves the property (1), we know that:

$$\|t_1|_k = \|ge_1\| \leq (1 + \epsilon)\|g(\lambda e_1 + \mu e_2)\| \text{ for all } (\lambda, \mu) \in k^2 - \{0\}$$

Note that $g(\lambda e_1 + \mu e_2) = (\lambda + \mu u)t_1e_1 + \mu t_2e_2$. Fix $\mu = 1$.

We can divide by $|t_2|_k$, and get, defining $\alpha = t_1/t_2$:

$$x := |\alpha|_k \leq (1 + \epsilon)\|\lambda + u\alpha \cdot e_1 + e_2\|$$

for all $\lambda \in k$ and some $u \in A$. We want to show that this implies $|x| \leq \frac{2}{\sqrt{3}} + \epsilon'$ with $\epsilon' \to 0$ as $\epsilon \to 0$.

We’ll handle the two cases $k = \mathbb{Q}, k = \mathbb{F}_q(t)$ separately, starting with the case $k = \mathbb{Q}$.

It is harmless to scale $\alpha$ by $Q^\times$, so we may assume (by strong approximation) that $\alpha \in R_{>0} \times \hat{\mathbb{Z}}^\times$. This means that $|\alpha|_Q = |\alpha|_\infty$. We can choose $\lambda \in \mathbb{Q}$ such that $|\lambda + u|_v \leq 1$ for all $v \neq \infty$, and $|\lambda + u|_\infty \leq \frac{1}{2}$ for $v = \infty$. This uses the fact that the map $[0, 1) \times \hat{\mathbb{Z}} \to A_Q/Q$ induced from the injection $[0, 1) \times \hat{\mathbb{Z}} \hookrightarrow A_Q$ is bijective since the “polar parts” of elements of $Q$, are in $Q$, and only finitely many are non-zero.

Then, since $\|\cdot\|$ is the product of the sup-norms at each finite place with the Euclidean norm at $\infty$, we see that since $|\alpha|_v = 1$, $\|\lambda + u\alpha e_1 + e_2\|_v \leq 1$ for each finite $v$, so we get from [1]:

$$x = |t_1/t_2|_k = |\alpha|_\infty \leq (1 + \epsilon)\|\lambda + u\alpha e_1 + e_2\|_\infty \leq \sqrt{\frac{x^2}{4} + 1}$$

and this implies that $x \leq \frac{2}{\sqrt{3}} + o(\epsilon)$.

When $k = \mathbb{F}_q(t)$, we can do this the same way with $\mathbb{F}_q[t]$ in the role of $\mathbb{Z}$. We can scale $\alpha$ by $\mathbb{F}_q(t)$ so that $|\alpha|_v \leq 1$ for $v \neq \infty$ and pick $\lambda \in \mathbb{F}_q(t)$ such that $|\lambda + \mu|_v \leq 1$ for all $v \neq \infty$ and $|\lambda + \mu|_\infty \leq \frac{1}{q}$ at $\infty$. Then we get that the sup norm $\|\lambda + u\alpha e_1 + e_2\|$ is at most $\|\lambda + u\|_\infty \leq 1$ at each place $v \neq \infty$ and it is at most 1, $\frac{x}{q}$ at $v = \infty$, so we get $x \leq \max 1, \frac{x}{q}$, so we must have $x \leq 1$ (since $x \leq \frac{x}{q}$ unless $x = 0$).
Now, let’s briefly discuss some ideas in the deduction Theorems C and F for general groups $G$ in place of GL$_2$, SL$_2$, PGL$_2$ (which will use these special cases!). We saw in the last lecture how to reduce the general case to the case $k = k_0$ (either $\mathbb{Q}$ or some $\mathbb{F}_q(t)$) by taking Weil restrictions. However, even if we started with GL$_2$ over some general $k$ (which is always finite separable over some $k_0$) we would thereby end up with some more complicated, often non-split group over $k_0$!

Consider some connected reductive $G$ over $k_0$. If $G$ has positive semisimple $k_0$-rank (so the relative root system is non-empty), let $S$ be a maximal $k_0$-split torus. For $\alpha \in \Phi(G, S)$, we have $S_\alpha = (\ker \alpha)^0_{\text{red}} \subseteq S$ is a codimension-1 torus killed by $\alpha : S \to \mathbb{G}_m$. Then $Z_G(S_\alpha)$ is a connected reductive group with semisimple $k_0$-rank 1. When $G$ is split, so is $Z_G(S_\alpha)$.

Via extensive use of the serious structure theory of reductive groups over fields, one can reduce Theorems C and F for $G$ to the corresponding theorems for all of the $Z_G(S_\alpha)$’s (I will try to find time to write up an exposition of how this is done, fleshing out more fully how this is done in Springer’s paper), so we thereby reduce the general case to the case where $G$ has semisimple $k_0$-rank at most 1.

It is a fundamental fact that those $G$ that are split of semisimple rank 1 are precisely the group $H \times T$ for a split torus $T$ and $H = SL_2, GL_2, PGL_2$. Thus, $[G] = [H] \times [T]$, and it is classical that $[T]$ is compact (the adelic synthesis of finiteness of generalized class groups and the $S$-unit theorem). So Theorems C, F for general $k_0$-split $G$ reduce to the known case of $H$ over $k_0$!

This then settles the general split case over $k_0$. By an additional argument, one can use this to settle the split case over any $k$.

Now, we will use the settled split case in general to settle the general case over $k_0$ where $G$ has semisimple $k_0$-rank at most 1. In particular, if the semisimple $k_0$-rank is 0, we need to show that $[G]$ is compact. Pick $k/k_0$ splitting the $k_0$-group $G$, and we get a natural map:

$$[G] \hookrightarrow [R_{k/k_0}(G_k)] = [G_k];$$

this map is a closed embedding as we saw Lecture 2! Since $G_k$ is split, we know that Theorem F is true for $G_k$, so we can get a description $G_k(A_k)^1 = K \cdot T'(c')P(A_k)^1G_k(k)$ where $T'$ is some $k$-split maximal torus of $G_k$, which we can assume contains $S_k$. One can then use the general relationship of $\Phi(G, S)$ and $\Phi(G_k, T')$ applied in this setting with empty $\Phi(G, S)$ to (eventually) see that the way $[G]$ lies inside $[G_k]$ is controlled by the compact part of $T'(c')$, from which the desired compactness of $[G]$ follows. Thus, Theorem C is settled in general.

For the case of semisimple rank 1, we choose a faithful representation $G \hookrightarrow GL(V)$ such that $P$ is the stabilizer of a line. One has to use lots of arguments with adelic heights on $V$ (including property (iv)!!) to harness the explicit description of $[G_k]$ (by Theorem F and Theorem C in the settled split case) to get the desired description of $[G]$ in Theorem F.

As mentioned, later I’ll try to write a more genuine exposition of the omitted details using the structure theory of reductive groups to bootstrap from SL$_2, GL_2, PGL_2$ over $k_0$. And we need to sort out why property (iv) of heights is true.

References
