Spherical representations and the Satake isomorphism

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Topics: Motivation for the study of spherical representations; Satake isomorphism stated for the general case of a connected reductive group (taking Bruhat-Tits theory as a black box); interpretation (spherical principal series, Satake parameter, representations of dual reductive group); Satake made more explicit for the split case (key calculation); idea of proof in the general case (mostly punted to Cartier's Corvallis article [Car79]).

1 Introduction

Let $\mathbb{G}$ be a connected reductive group over a local field F with ring of integers $\mathscr{O}$. Let $G = \mathbb{G}(F)$; this convention (normal font for the group of F-points) will be used for subgroups of $\mathbb{G}$. Let K be a maximal compact (hence open) subgroup of G.

Definition 1.1. A smooth representation (π, V) of G is called *spherical* (or *unramified*) with respect to K if it contains a nonzero K-fixed vector, i.e. if $V^K \neq 0$. The Hecke algebra $\mathscr{H}(G, K)$ is called the *spherical Hecke algebra* with respect to K.

Note that this notion depends on the conjugacy class of K. By definition, every vector in a smooth representation is fixed by *some* sufficiently small compact open subgroup. Spherical representations with respect to K may thus be thought of, in some sense, as the simplest smooth representations.

This notion is motivated by automorphic representations. To explain this, we first review some basic topological features of $\mathbb{G}(\mathbb{A}_k)$ for a connected reductive group $\mathbb{G}$ over a global field k. Recall that $\mathbb{G}(\mathbb{A}_k)$ is topologized exactly in accordance with the canonical manner in which one topologizes $X(R)$ for any topological ring R and affine R-scheme X of finite type (applied to $X = \mathbb{G}_{\mathbb{A}_k}$ over $R = \mathbb{A}_k$), namely by using any closed immersion of X into an affine space over R (the choice of which does not matter). In particular, one certainly does not *define* the topology as a restricted direct product, but we will see soon that it can nonetheless be "computed" in such a manner.

For any finite set S of places of k, let $\mathbb{A}_k^S$ be the restricted direct product of k_v's over all $v \notin S$ and let $k_S = \prod_{v \in S} k_v$. The topological ring isomorphism $\mathbb{A}_k = k_S \times \mathbb{A}_k^S$ induces a topological isomorphism $X(\mathbb{A}_k) = X(k_S) \times X^S(\mathbb{A}_k^S)$ for any affine $\mathbb{A}_k$-scheme X of finite type, where X^S is its pullback over $\mathbb{A}_k^S$, so setting $X = \mathbb{G}_{\mathbb{A}_k}$ yields the identification $\mathbb{G}(\mathbb{A}_k) = \mathbb{G}(k_S) \times \mathbb{G}(\mathbb{A}_k^S)$ as topological groups, with $\mathbb{G}(k_S) = \prod_{v \in S} \mathbb{G}(k_v)$. By choosing S large enough, we can arrange that $\mathbb{G}$ extends to a reductive $\mathscr{O}_{k,S}$-group scheme $\mathscr{G}$ (i.e., a smooth affine $\mathscr{O}_{k,S}$-group with connected reductive fibers), as discussed in Conrad's lecture notes, so $\mathscr{G}(\prod_{v\notin S}\mathscr{O}_{k,v})$ is a compact open subgroup of $\mathbb{G}(\mathbb{A}_k^S)$ since $\prod_{v\notin S}\mathscr{O}_{k,v}$ is a compact open subring of $\mathbb{A}_k^S$. If K is any compact open subgroup of $\mathbb{G}(\mathbb{A}_k^\infty)$ then it meets $\mathscr{G}(\prod_{v\notin S}\mathscr{O}_{k,v}) = \prod_{v\notin S}\mathscr{G}(\mathscr{O}_{k,v})$ in a compact open subgroup. Hence, for sufficiently large S' containing S, $K = K_{S'} \times \prod_{v\notin S'}\mathscr{G}(\mathscr{O}_{k,v})$ for a compact open subgroup $K_{S'} \subset \prod_{v\in S', v\nmid\infty}\mathbb{G}(k_v)$. In other words, for the hyperspecial maximal compact open subgroup $K_v := \mathscr{G}(\mathscr{O}_{k,v}) \subset \mathbb{G}(k_v)$ for $v \notin S'$, we have $\mathbb{G}(\mathbb{A}_k^{S'}) = \prod'_{v\notin S'}\mathbb{G}(k_v)$ where the restricted direct product is taken with respect to the K_v's. This restricted direct product construction is well-posed (i.e., intrinsic to $\mathbb{G}$ and not dependent on S' and $\mathscr{G}$) in the sense that any two reductive $\mathscr{O}_{k,S'}$-models of the k-group $\mathbb{G}$ coincide as $\mathscr{O}_{k,S''}$-models for some S'' containing S'.

To summarize, $\mathbb{G}(\mathbb{A}_k) = \prod'_{K_v}\mathbb{G}_v(k_v)$, where K_v defined for all finite v are the hyperspecial maximal compact subgroups of $\mathbb{G}_v(k_v)$ arising from a reductive $\mathscr{O}_{k,\Sigma}$-model of $\mathbb{G}$ for some non-empty finite set Σ of places of k containing the archimedean places. In particular, we do *not* simply use arbitrary hyperspecial maximal compact open subgroups K_v in $\mathbb{G}(k_v)$ (for almost all v) having no link to each other through a "global" reductive model over some ring of Σ-integers of k. For example, if $\mathbb{G} = \mathrm{GL}_n$ then we *must* use $K_v = \mathrm{GL}_n(\mathscr{O}_{k,v})$ for almost all v (or equivalently, $K_v = \mathrm{GL}(\Lambda_v)$ for almost all v with Λ an $\mathscr{O}_k$-lattice in k^n); we cannot take K_v to be the automorphism group of an $\mathscr{O}_{k,v}$-lattice in k_v^n such that these local lattices do not arise from a common global lattice in k^n at almost all places. (By weak approximation, this latter condition is the same as arising from a common global lattice at *all* non-archimedean places of k).

In this set-up, we have

Theorem 1.2 (Flath's tensor product theorem)**.** *Any irreducible admissible representation π of $\mathbb{G}(\mathbb{A}_k)$ is uniquely a restricted tensor product of irreducible smooth representations π_v of $\mathbb{G}_v(k_v)$, with π_v spherical with respect to K_v for almost all v, where $\{K_v\}$ arise as above from a compact open subgroup of $\mathbb{G}(\mathbb{A}_k^S)$ or equivalently from a reductive $\mathscr{O}_{k,S}$-model of $\mathbb{G}$ for some large S.*

For details, see the short article [Fla79] by Flath.

So we wish to understand spherical representations. Back in the local picture, recall that, for a smooth representation π of G, π^K is a smooth $\mathscr{H}(G,K)$-module. The Satake isomorphism allows us to analyze the structure of $\mathscr{H}(G,K)$ and hence understand spherical representations.

To state the most general form of the Satake isomorphism, for an arbitrary connected reductive group $\mathbb{G}$, we need to be careful about the choice of K. Fix a maximal split torus $\mathbb{S}$ of $\mathbb{G}$, with centralizer $\mathbb{M}$, and let $W = W(\mathbb{G},\mathbb{S}) := N_{\mathbb{G}}(\mathbb{S})(F)/Z_{\mathbb{G}}(\mathbb{S})(F) = N_G(S)/Z_G(S)$ be the relative Weyl group. Then M has a unique maximal compact subgroup ${}^\circ M$, namely ${}^\circ M = (\mathbb{S}^{\mathrm{an}} \cdot \mathscr{D}(\mathbb{M}))(F)$, where $\mathbb{S}^{\mathrm{an}} \subset \mathbb{S}$ is the maximal F-anisotropic subtorus of $\mathbb{S}$, so $M/{}^\circ M$ is commutative; W clearly acts on $M/{}^\circ M$. In this set-up, Bruhat-Tits theory produces what is called a *special* maximal compact subgroup K of G (the properties we need from this definition will be explained later, and it includes the hyperspecial case). For such K, we have

Theorem 1.3 (Satake Isomorphism (general case of connected reductive group))**.** *There exists a canonical isomorphism*

$$\mathscr{H}(G,K) \cong \mathscr{H}(M, {}^\circ M)^W.$$

Now assume that $\mathbb{G}$ has a reductive model $\mathscr{G}$ over $\mathscr{O}$ such that $\mathbb{S}$ extends to a split $\mathscr{O}$-torus in $\mathscr{G}$ (Remark: it is a nontrivial fact that the existence of such a model is equivalent to $\mathbb{G}$ being *unramified*, i.e. quasi-split over F and split over an unramified extension of F). In this case, the claims in the paragraph preceeding the theorem are easily proved, and the *hyperspecial* maximal compact subgroup $K = \mathscr{G}(\mathscr{O})$ turns out to be special. In the case that $\mathbb{G}$ is even split, so that $\mathbb{S} = \mathbb{M} = \mathbb{T}$, an F-split maximal torus, we have ${}^\circ M = T^1 := \mathbb{T}(\mathscr{O})$ (using the unique $\mathscr{O}$-torus structure on the F-torus $\mathbb{T}$), and the Satake isomorphism takes the form

$$\mathscr{H}(G, K) \cong \mathscr{H}(T, T^1)^W.$$

Remark 1.4. One could similarly define a spherical representation for an arbitrary t.d. group and a maximal compact open subgroup (if any exists). However, this is never studied. In this generality, there is little that can be said about the properties of maximal compact subgroups or about the spherical Hecke algebra.

2 References

In addition to asking Akshay and Brian, the following were the main references I used to learn this material and write these notes: lecture notes by Sakellaridis (available at http://math.newark.rutgers.edu/~sakellar/automorphic/) to orient myself, and for the interpretation of the Satake isomorphism; Gross's article [Gro98] for concrete calculations; and Cartier's Corvallis article [Car79] for the general case (still taking Bruhat-Tits as a black box).

3 Interpretation

3.1 Unramified characters

The point (or at least *a* point) of the Satake isomorphism is it tells us that there is a bijection

$$\{\text{irreducible spherical representations of } G\} \longleftrightarrow \{\text{unramified characters of } M\}/W. \tag{1}$$

We need to explain the right side.

First suppose that $\mathbb{G}$ is split, so $\mathbb{S} = \mathbb{M} = \mathbb{T}$. An *unramified character* of T is a character unramified with respect to, i.e. trivial on, the unique maximal compact subgroup $T^1 := \mathbb{T}(\mathscr{O})$. We have a short exact sequence

$$1 \longrightarrow T^1 \longrightarrow T \xrightarrow{\mathrm{ord}_T} X_*(\mathbb{T}) \longrightarrow 1,$$

where ord_T is defined by $\chi \circ \mathrm{ord}_T(t) = \mathrm{ord}_F(\chi(t))$ for $\chi \in X^*(\mathbb{T})$. Thus we have a W-equivariant isomorphism

$$\begin{aligned} \mathscr{H}(T, T^1) &\cong \mathbb{C}[X_*(\mathbb{T})] \\ 1_{\lambda(\pi)T^1} &\leftrightarrow [\lambda], \end{aligned}$$

where the right side is the group algebra. Since W is finite, the $\mathbb{C}$-points of $\operatorname{Spec}\mathbb{C}[X_*(\mathbb{T})]^W$ correspond to W-orbits of the set of $\mathbb{C}$-points of $\operatorname{Spec}\mathbb{C}[X_*(\mathbb{T})]$. Thus characters of $\mathscr{H}(T,T^1)^W$ are identified with $\operatorname{Hom}(X_*(\mathbb{T}),\mathbb{C}^\times)/W = \operatorname{Hom}(T/T^1,\mathbb{C}^\times)/W$, i.e. with unramified characters of T up to the W-action.

In the general case of a connected reductive group $\mathbb{G}$, an unramified character of M is again a character trivial on the unique maximal compact subgroup ${}^\circ M$. To carry out the argument above, one now defines

$$\begin{aligned}\operatorname{ord}_M : M &\to X^*(\mathbb{M})^\vee\\ m &\mapsto (\chi \mapsto \operatorname{ord}_F(\chi(h))).\end{aligned}$$

It is easy to check that the kernel of this map is ${}^\circ M$. It may no longer be surjective, but the image has finite index. Denoting the image Λ, we now have the short exact sequence

$$1 \longrightarrow {}^\circ M \longrightarrow M \xrightarrow{\operatorname{ord}_M} \Lambda \longrightarrow 1,$$

and we again obtain a W-equivariant isomorphism $\mathscr{H}(M,{}^\circ M) \cong \mathbb{C}[\Lambda]$, and characters of $\mathscr{H}(M,{}^\circ M)^W$ are identified with unramified characters of M up to the W-action. For some details of this construction, see Cartier p.134.

3.2 Spherical representations and unramified Weyl-invariant characters

Here is a summary that help us interpret the Satake isomorphism. Given an irreducible spherical representation π of G, π^K is an irreducible $\mathscr{H}(G,K)$-module. One deduces from the Satake isomorphism that $\mathscr{H}(G,K)$ is commutative (or directly for K hyperspecial), so that by Schur's lemma, $\dim \pi^K = 1$ and $\mathscr{H}(G,K)$ acts via a character. View this via the Satake isomorphism as a character of $\mathscr{H}(M,{}^\circ M)^W$, which as above is identified with an unramified character of M up to the W-action.

This process can be reversed. Let χ be an unramified character of M. View it as a character of P, and let

$$I(\chi) = \operatorname{Ind}_P^G(\delta^{1/2}\chi) = \{f : G \to \mathbb{C} \text{ locally constant } : f(pg) = \delta^{1/2}(p)\chi(p)f(g)\}.$$

K special implies in particular the Iwasawa decomposition $G = PK$ and that $M \cap K = {}^\circ M$. Since $\chi|_M$ is trivial on ${}^\circ M = M \cap K$, χ is trivial on $P \cap K$, which together with the Iwasawa decomposition implies $\dim I(\chi)^K = 1$. This $I(\chi)$ is called an *spherical principal series representation.* Since the functor $(\cdot)^K$ is exact, $I(\chi)$ has a unique irreducible spherical subquotient π_χ. Moreover, earlier in this seminar, Iurie showed (see his notes) that $\pi_{w\cdot\chi} \cong \pi_\chi$ for all $w \in W$.

This gives the bijection (1).

3.3 Split case: Satake parameter, representations of the dual group

In the unramified case, there is another interpretation of the right side of the bijection (1). For simplicity, assume $\mathbb{G}$ is split. Let $\widehat{\mathbb{G}}$ be the complex dual reductive group with maximal torus $\widehat{\mathbb{T}}$, so we have a canonical identification $X_*(\mathbb{T}) \cong X^*(\widehat{\mathbb{T}})$. The Satake isomorphism then says

$$\mathscr{H}(G,K) \cong \mathscr{H}(T,T^1)^W \cong \mathbb{C}[X^*(\widehat{\mathbb{T}})]^W.$$

Characters of $\mathscr{H}(G,K)$ are therefore identified with elements of $\widehat{\mathbb{T}}/W$, i.e. semisimple conjugacy classes in $\widehat{\mathbb{G}}$. This is called the *Satake parameter* of the corresponding spherical representation.

Note also that the right side is isomorphic to the algebra of (virtual finite-dimensional complex algebraic) representations of $\widehat{\mathbb{G}}$ under tensor product, by taking the character.

Remark 3.1. For quasi-split $\mathbb{G}$, there is an interpretation of W-orbits on X as *twisted conjugacy classes*, using a more sophisticated replacement for $\widehat{\mathbb{G}}$, the so-called Langlands dual group ${}^L\mathbb{G}$.

4 Definition of the Satake transform

Again let $\mathbb{G}$ be a connected reductive group over a local field F. Fix a maximal F-split torus $\mathbb{S}$, and let $\mathbb{P}$ be a minimal F-parabolic containing $\mathbb{S}$. Let $\mathbb{U}$ be the unipotent radical of $\mathbb{P}$, and let $\mathbb{M}$ be the centralizer of $\mathbb{S}$. Let $W = N_G(S)/Z_G(S)$, the relative Weyl group, and $\Phi = \Phi(\mathbb{G},\mathbb{S})$, the relative root system.

That K is a special maximal compact subgroup implies in particular the following decompositions:

Theorem 4.1 (Iwasawa Decomposition)**.** *$G = PK$.*

Theorem 4.2 (Cartan Decomposition)**.** *G is the disjoint union of $K\operatorname{ord}_M^{-1}(\lambda)K$ for λ running over Λ^-, where Λ^- is a some subset of Λ such that every element of Λ is conjugate under W to a unique element in Λ^-.*

In the general setting of a connected reductive group $\mathbb{G}$, special maximal compact open subgroups and the corresponding Λ^- as above are defined in terms of, and the above decompositions are proved using, the Bruhat-Tits theory of the building associated to $\mathbb{G}$ [BT72, BT84]. See Section 3.3 of Tits' article [Tit79] in the Corvallis proceedings for a summary of these decompositions.

The Satake transform $S : \mathscr{H}(G,K) \to \mathscr{H}(M, {}^\circ M)$ is then defined by

$$(Sf)(m) = \delta(m)^{1/2} \int_U f(mu)du,$$

where du is the Haar measure on U normalized so that $U \cap K$ has unit measure, and δ is the modulus character on P.

In the unramified case, $\Lambda = X_*(\mathbb{S})$, and the corresponding Λ^- may be taken to be the set of dominant cocharacters P^+. The Cartan decomposition simplifies to

Theorem 4.3 (Cartan Decomposition)**.** *$G = \bigsqcup_{\lambda \in P^+} K\lambda(\pi)K$.*

The Satake transform is a map

$$\begin{aligned} \mathscr{H}(G,K) \xrightarrow{S} \mathscr{H}(T,T^1) &\cong \mathbb{C}[X_*(\mathbb{T})] \\ 1_{\lambda(\pi)K} &\leftrightarrow [\lambda]. \end{aligned}$$

5 An explicit calculation in GL_n

(This calculation in fact only assumes $\mathbb{G}$ split.)

Let $\mathbb{G} = \mathrm{GL}_n$, $\mathbb{B}$ the upper triangular Borel, $\mathbb{T}$ the diagonal torus, and $K = \mathrm{GL}_n(\mathscr{O})$. The positive cocharacters with respect to $\mathbb{B}$ are then $\lambda : t \mapsto \mathrm{diag}(\pi^{a_1}, \dots, \pi^{a_n})$ with $a_1 \geq \dots \geq a_n$.

By the Cartan decomposition, the characteristic functions $1_{K\lambda(\pi)K}$, $\lambda \in P^+$, form a $\mathbb{C}$-basis of $\mathscr{H}(G, K)$. Write

$$S1_{K\lambda(\pi)K} = \sum_{\mu \in X_*(\mathbb{T})} a_\lambda(\mu)\mu.$$

To evaluate these coefficients explicitly, write $K\lambda(\pi)K = \bigsqcup x_i K$. By the Iwasawa decomposition, we may take $x_i = t_i u_i$, where $t_i \in T$, $u_i \in U$. Then

$$\begin{aligned}
a_\lambda(\mu) &= (S1_{K\lambda(\pi)K})(\mu(\pi)) \\
&= \delta(\mu(\pi))^{1/2} \int_U 1_{K\lambda(\pi)K}(\mu(\pi)u)du \\
&= q^{-\langle \mu, \rho \rangle} \sum_i \int_{U \cap \mu(\pi)^{-1} x_i K} du \\
&= q^{-\langle \mu, \rho \rangle} \#\{i : t_i \equiv \mu(\pi) \pmod{T^1}\}
\end{aligned}$$

where for the last line we note that there is a contribution of 1 for each i such that $t^{-1}t_i \in T^1$. In particular, $a_\lambda(\lambda) \geq 1$. Since $KIK = IK$, it is also clear that S sends the identity to the identity. It can be proved that $a_\lambda(\mu) = 0$ unless $\mu \leq \lambda$, so that

$$S1_{K\lambda(\pi)K} = a_\lambda(\lambda)\lambda + \sum_{\mu \leq \lambda} a_\lambda(\mu)\mu.$$

Once S is shown to land in the W-invariants, this triangularity shows by a diagonalization argument that S is a bijection.

Here is a very simple example in GL_2. We have $\delta(\begin{smallmatrix} a & \\ & b \end{smallmatrix}) = |a||b|^{-1}$. Let $g \in K$ and $m \geq n$ be integers. If $g \in K(\begin{smallmatrix} \varpi^m & \\ & \varpi^n \end{smallmatrix})K$, then clearly $\mathrm{ord}_F(\det g) = m + n$ and the minimum of the ord_F of the entries of g is n, and it is also easy to show the converse. It follows that

$$K(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix})K = (\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix})K \sqcup \left(\bigsqcup_{a \in \mathscr{O}/(\varpi)} (\begin{smallmatrix} \varpi & a \\ & 1 \end{smallmatrix})K \right),$$

so $S1_{K(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix})K}$ is supported on $(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix})T^1 \sqcup (\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix})T^1$, and

$$\begin{aligned}
(S1_{K(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix})K})(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}) &= |\varpi|^{-1/2} = q^{1/2} \\
(S1_{K(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix})K})(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}) &= |\varpi|^{1/2} q = q^{1/2}.
\end{aligned}$$

Thus

$$S1_{K\left(\begin{smallmatrix}\varpi & \\ & 1\end{smallmatrix}\right)K} = q^{1/2}1_{\left(\begin{smallmatrix}\varpi & \\ & 1\end{smallmatrix}\right)T^1} + q^{1/2}1_{\left(\begin{smallmatrix}1 & \\ & \varpi\end{smallmatrix}\right)T^1},$$

which is indeed W-invariant.

6 Sketch of proof in the general (connected reductive) case

All measures below are Haar measures normalized so that the intersection with K has unit measure. The key proposition is the following.

Proposition 6.1 (Adjunction)**.** *For any $f \in \mathscr{H}(G,K)$ and any unramified character χ of M,*

$$\langle Sf, \chi\rangle = \langle f, \pi_\chi\rangle,$$

where the right side denotes how f acts on the one-dimensional space π_χ^K.

Proof. By linearity, we may assume $f = 1_{KgK}$. Let π_χ have spherical vector v_χ. Recalling that functions in π_χ^K are determined by their value at 1,

$$\begin{aligned}
\langle 1_{KgK}, \pi_\chi\rangle &= \frac{1}{v_\chi(1)}(\pi_\chi(1_{KgK})v_\chi)(1) \\
&= \frac{1}{v_\chi(1)}\int_G 1_{KgK}(x)v_\chi(x)dx \\
&= \frac{1}{v_\chi(1)}\int_M\int_U\int_K 1_{KgK}(muk)v_\chi(muk)dkdudm \\
&= \int_M \chi(m)\delta(m)^{1/2}\int_U\int_K 1_{KgK}(muk)dkdudm \\
&= \int_M \chi(m)\delta(m)^{1/2}\int_U 1_{KgK}(mu)dudm \\
&= \int_M \chi(m)(S1_{KgK})(m)dm \\
&= \langle S1_{KgK}, \chi\rangle.
\end{aligned}$$

The normalization of the Haar measure is used in the third equality from the bottom. □

In fact, this calculation is how one might discover the Satake isomorphism. Also note that this adjunction, once the Satake isomorphism is proved, shows the bijection claimed earlier between irreducible spherical representations of G and unramified characters of M up to the Weyl action.

We prove the Satake isomorphism in three steps.

Step 1: S is an algebra homomorphism.

For any unramified character χ of M,

$$\langle f * g, \pi_\chi\rangle = \langle f, \pi_\chi\rangle\langle g, \pi_\chi\rangle$$

since π_χ^K is an $\mathscr{H}(G,K)$-module. For $F, G \in \mathscr{H}(M, {}^\circ M)$,

$$\begin{aligned}
\langle F * G, \chi\rangle &= \int_M \chi(m)(F*G)(m)dm \\
&= \int_M \chi(m)\int_M F(m_1)G(m_1^{-1}m)dmdm \\
&= \left(\int_M \chi(m_1)F(m_1)dm\right)\left(\int_M \chi(m_2)G(m_2)dm\right) \\
&= \langle F,\chi\rangle\langle G,\chi\rangle.
\end{aligned}$$

Hence by the adjunction, $\langle Sf * Sg, \chi\rangle = \langle f * g, \pi_\chi\rangle = \langle S(f*g), \chi\rangle$ for all χ, which implies $Sf * Sg = S(f*g)$.

This step may also be proved by a direct calculation involving some manipulation of integrals and Haar measures using Iwasawa decomposition (see [Car79, p.147]).

Step 2: The image of S is W-invariant

Earlier in this seminar, Iurie proved (see his notes) that $\pi_{w\cdot\chi} \cong \pi_\chi$ for all $w \in W$. This step now follows from the adjunction.

Cartier has a more direct (but less enlightening, I found) proof, which proceeds as follows; for details and the definition of regular elements in M, see [Car79, p.147]. We need to show that $(Sf)(xmx^{-1}) = (Sf)(m)$ for $m \in M$ and $x \in N_G(S) \cap K$. It suffices to show this for the *regular* elements in M, which are dense in M. This becomes another integral calculation once one rewrites the Satake transform as an orbital integral (twisted by the modular character).

Step 3: S is a bijection

This is similar to the discussion for the split case in the previous section, using a certain partial order on Λ generalizing the partial order on the cocharacters $X_*(\mathbb{S})$ (recall that in the unramified case, $\Lambda = X_*(\mathbb{S})$ and $\Lambda^- = P^+$, the dominant cocharacters). From the Cartan decomposition, $1_{K\,\mathrm{ord}_M^{-1}(\lambda)K}$, $\lambda \in \Lambda^-$, is a $\mathbb{C}$-basis of $\mathscr{H}(G,K)$. Since any element of Λ is conjugate under W to a unique element in Λ^-,

$$\sum_{w\in W} w\lambda, \quad \lambda \in \Lambda^-$$

is a $\mathbb{C}$-basis of $\mathbb{C}[\Lambda]^W$. Choosing $m, m' \in M$ mapping under ord_M to λ, λ', we immediately find that the λ'-coefficient of $S1_{K\,\mathrm{ord}_M^{-1}(\lambda)K}$ under this basis is $\delta(m)^{-1/2}\mu(Km'K \cap UmK)$. As in the split case, this is obviously nonzero for $\lambda' = \lambda$, and to complete the proof it remains to show that $Km'K \cap UmK \neq \emptyset$ implies $\lambda' \leq \lambda$. I won't even attempt to sketch a proof of this triangularity, which also relies on K being special. Cartier states this without proof [Car79, p.147, italicized statement after equation (27)]. A reference in the general case is [HR10].

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