THE JACQUET-LANGLANDS CORRESPONDENCE FOR GL_2

Contents

We follow [1, §8] with some examples and details supplemented.

1. Local statement and examples

Let F be a local field of characteristic not 2. Let D be a quaternion algebra over F (unique up to isomorphism).

1.1. **Matching regular semisimple conjugacy classes.** Let $X(F) = F \times F^\times$, viewed as the space of semisimple conjugacy classes in $\mathrm{GL}_2(F)$ via $\mathrm{GL}_2(F) \to X(F)$ sending $\gamma \mapsto (\mathrm{Tr}(\gamma), \det(\gamma))$. Similarly we have $D^\times \to X(F)$ sending $\gamma' \mapsto (\mathrm{Tr}_{D/F}(\gamma'), \mathrm{Nm}_{D/F}(\gamma'))$, where $\mathrm{Tr}_{D/F}$ and $\mathrm{Nm}_{D/F}$ mean reduced trace and norm.

For regular semisimple $\gamma \in \mathrm{GL}_2(F)$ and $\gamma' \in D^\times$, we write $\gamma \sim \gamma'$ if they have the same image in $X(F)$. This gives a bijection beween elliptic regular semisimple classes in $\mathrm{GL}_2(F)$ and regular semisimple classes in $D^\times$.

Let $\omega : F^\times \to \mathbb{C}^\times$ be a smooth character.

1.2. **Theorem** (Local JL)**.** *There is a unique bijection*

$$\begin{aligned} & \{\textit{irreducible smooth representations of } D^\times \textit{ with central character } \omega\} \\ \leftrightarrow\ & \{\textit{irreducible discrete series representations of } \mathrm{GL}_2(F) \textit{ with central character } \omega\} \end{aligned}$$

such that for $\pi' \leftrightarrow \pi$ and regular semisimple $\gamma' \in D^\times$ and $\gamma \in \mathrm{GL}_2(F)$ with $\gamma \sim \gamma'$, we have

$$\Theta_\pi(\gamma) = -\Theta_{\pi'}(\gamma').$$

Here Θ_π and $\Theta_{\pi'}$ are distribution characters of π and π', which are represented by functions when restricted to regular semisimple loci of $\mathrm{GL}_2(F)$ and $D^\times$.

Compatibility with twists: if $\pi' \leftrightarrow \pi$, then $\pi' \otimes (\chi \circ \mathrm{Nm}) \leftrightarrow \pi \otimes (\chi \circ \det)$ for any smooth character $\chi : F^\times \to \mathbb{C}^\times$.

1.3. $F = \mathbb{R}$**.** Let $D = \mathbb{H}$ be the Hamiltonian quaternions over $\mathbb{R}$. Then $\mathbb{H}^\times = \mathbb{R}^{>0} \cdot \mathrm{SU}(2)$. For a central character $\omega : \mathbb{R}^\times \to \mathbb{C}^\times$, let ω^+ denote its restriction to $\mathbb{R}^{>0}$ and let $\epsilon \in \mathbb{Z}/2\mathbb{Z}$ be such that $\omega(-1) = (-1)^\epsilon$. Then irreducible representations of $\mathbb{H}^\times$ with central character ω are of the form $\omega^+ \boxtimes \mathrm{Sym}^n(\mathbb{C}^2)$ for $n \equiv \epsilon \mod 2$.

Discrete series representations of $\mathrm{GL}_2(\mathbb{R}) = \mathbb{R}^{>0} \cdot \mathrm{SL}_2^\pm(\mathbb{R})$ with central charater ω are of the form $\omega^+ \boxtimes D_n^\pm$ with $n \equiv \epsilon \mod 2$ and $n \geq 2$. Here D_n^+ and D_n^- are the holomorphic and antiholomorphic discrete series representation of $\mathrm{SL}_2(\mathbb{R})$ with $\mathrm{SO}(2)$-weights $\pm n, \pm(n+2), \cdots$ respectively, and $D_n^\pm = D_n^+ \oplus D_n^- = \mathrm{Ind}_{\mathrm{SL}_2(\mathbb{R})}^{\mathrm{SL}_2^\pm(\mathbb{R})} D_n^+$.

The local JL correspondence in this case is $\omega^+ \boxtimes \mathrm{Sym}^n(\mathbb{C}^2) \leftrightarrow \omega^+ \boxtimes D_{n+2}^\pm$. It preserves infinitesimal characters (under an identification of the complexified Lie algebras of $\mathbb{H}^\times$ and $\mathrm{GL}_2(\mathbb{R})$).

Character relation. We have an exact sequence $0 \to \mathrm{Sym}^n(\mathbb{C}^2) \to \mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(\chi) \to D_{n+2}^{\pm}$ for some character χ of $T(\mathbb{R})$. For $\gamma \sim \gamma'$, they have the same trace on $\mathrm{Sym}^n(\mathbb{C}^2)$, hence it suffices to show that $\mathrm{Tr}(\gamma, \mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(\chi)) = 0$. This is because γ is elliptic and does not have any fixed point on $\mathbb{P}^1(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})/B(\mathbb{R})$.

1.4. F **is local non-archimedean.** Let k be the residue field of F. Let ϖ_F be a uniformizer. Discrete series representations of $\mathrm{GL}_2(F)$ are twisted Steinberg representations and supercuspidal representations.

1.4.1. *Trivial vs Steinberg.* The trivial representation of $D^\times$ corresponds to the Steinberg representation St of $\mathrm{GL}_2(F)$.

Character relation. Since $0 \to \mathbb{C} \to \mathrm{Fun}(\mathbb{P}^1(F)) \to \mathrm{St} \to 0$, we need to show that $\mathrm{Tr}(\gamma, \mathrm{Fun}(\mathbb{P}^1(F))) = 0$ for any elliptic element $\gamma \in \mathrm{GL}_2(F)$ (i.e., $\gamma \sim \gamma'$ for some $\gamma' \in D^\times$). This is clear because γ does not have a fixed point on $\mathbb{P}^1(F)$. More generally, 1-dim characters of $D^\times$ (which have to factor through $F^\times$ by the reduced norm) correspond to twisted Steinberg of $\mathrm{GL}_2(F)$.

1.4.2. *Depth zero supercuspidals.* Choose a uniformizer ϖ_D of D, then $\mathcal{O}_D^\times \supset 1 + \varpi_D\mathcal{O}_D \supset 1 + \varpi_D^2\mathcal{O}_D \supset \cdots$. We have $\mathcal{O}_D^\times/(1+\varpi_D\mathcal{O}_D) \cong k'^\times$ where $[k' : k] = 2$. Conjugation by ϖ_D acts by Galois involution on $k'^\times$. Let $\theta : k'^\times \to \mathbb{C}^\times$ be a character that doesn't factor through the norm map to $k^\times$. Then θ extends to a character of $F^\times\mathcal{O}_D^\times$ by requiring $\varpi_F \mapsto c$. Note that $[D^\times : F^\times\mathcal{O}_D^\times] = 2$. Induce $c\theta$ from $F^\times\mathcal{O}_D^\times$ to $D^\times$ we get a two-dim representation $\pi'(\theta, c)$ of $D^\times$.

For $\mathrm{GL}_2(F)$, we consider $\mathrm{GL}_2(k)$ as the quotient of $\mathrm{GL}_2(\mathcal{O}_F)$. Let $T = \mathrm{Res}_{k'/k}\mathbb{G}_m \subset \mathrm{GL}_2$ be a nonsplit torus. Take the same character $\theta : k'^\times = T(k) \to \mathbb{C}^\times$ as above. The Deligne-Lusztig representation $R_{T,\theta}$ is an irreducible representation constructed as $\mathrm{H}_c^1(\widetilde{X}_T, \overline{\mathbb{Q}}_\ell)_\theta$ where $\widetilde{X}_T$ is a $T(k)$-torsor over $\mathbb{P}^1 - \mathbb{P}^1(k)$ with a $\mathrm{GL}_2(k)$-action. We inflate $R_{T,\theta}$ to a representation of $\mathrm{GL}_2(\mathcal{O}_F)$ and extend it to $F^\times\mathrm{GL}_2(\mathcal{O}_F)$ by sending ϖ_F to c. Then compactly induce it to $\mathrm{GL}_2(F)$ we get a supercuspidal representation $\pi(\theta, c)$ of $\mathrm{GL}_2(F)$.

Character relation. Let $\gamma' \in D^\times$ be regular semisimple. If $\mathrm{val}(\gamma')$ is odd then its action on the induced representation takes the form $\pi'(\theta, c)$ is $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ under an obvious basis, hence $\mathrm{Tr}(\gamma', \pi'(\theta, c)) = 0$. If $\mathrm{val}(\gamma') = 2n$ we may write $\gamma' = \varpi_F^n\gamma_0'$ for some $\gamma_0' \in \mathcal{O}_D^\times$ with image $t \in k'^\times$. Then we have $\mathrm{Tr}(\gamma', \pi'(\theta, c)) = c^n(\theta(t) + \theta(\bar{t}))$ where $\bar{t}$ is the Galois conjugate of t (over k).

Now let $\gamma \in \mathrm{GL}_2(F)$ be regular semisimple. The parity of the valuation of the determinant gives a map $e : \mathrm{GL}_2(F)/F^\times\mathrm{GL}_2(\mathcal{O}_F) \to \mathbb{Z}/2\mathbb{Z}$. Then $\pi(\theta, c)$ is a direct sum of two spaces $V_0 \oplus V_1$ consisting of functions supported on $e^{-1}(0)$ and $e^{-1}(1)$ respectively. If $\mathrm{val}(\det(\gamma))$ is odd, it maps V_0 to V_1 and maps V_1 to V_0, therefore $\mathrm{Tr}(\gamma, \pi(\theta, c)) = 0$. If $\mathrm{val}(\det(\gamma)) = 2n$, we may write $\gamma = \varpi_F^n\gamma_0$. Assume the characteristic polynomial of γ_0 is irreducible after reduction mod ϖ_F. In this case γ_0 has a unique fixed point on $\mathrm{GL}_2(F)/F^\times\mathrm{GL}_2(\mathcal{O}_F)$, which we denote by $gF^\times\mathrm{GL}_2(\mathcal{O}_F)$. We have $\mathrm{Tr}(\gamma, \pi(\theta, c)) = c^n\mathrm{Tr}(g^{-1}\gamma_0 g, R_{T,\theta})$. We have $\gamma \sim \gamma' \in D^\times$ where $\gamma' = \varpi_F^n\gamma_0'$ and the image of γ_0' in $k'^\times$ is $t \in k'^\times - k^\times$. Then the image of $g^{-1}\gamma_0 g \in \mathrm{GL}_2(\mathcal{O}_F)$ in $\mathrm{GL}_2(k)$ is conjugate to $t \in k'^\times$. We have $\mathrm{Tr}(t, R_{T,\theta}) = -\theta(t) - \theta(\bar{t})$. The minus sign comes from the fact that $R_{T,\theta}$ is H_c^1 (applying Lefschetz trace formula for the action of t on $\widetilde{X}_T$). Therefore $\mathrm{Tr}(\gamma, \pi(\theta, c)) = -\mathrm{Tr}(\gamma', \pi'(\theta, c))$ for the γ considered above.

2. Global statement and examples

Let F be a global field of characteristic not 2. Let D be a quaternion algebra over F ramified exactly at places S. Let $\omega : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$ be a smooth character.

2.1. **Theorem** (Global JL)**.** *There is a unique injection*

$$\begin{aligned} & \{\textit{irreducible automorphic representations of } \mathbb{A}_D^\times \textit{ of dimension} > 1 \textit{ with central character } \omega\} \\ \hookrightarrow\ & \{\textit{irreducible cuspidal automorphic representations of } \mathrm{GL}_2(\mathbb{A}_F) \textit{ with central character } \omega\} \end{aligned}$$

such that $\pi' \leftrightarrow \pi$ if and only if $\pi'_v \cong \pi_v$ for all $v \notin S$ and $\pi'_v \leftrightarrow \pi_v$ for all $v \in S$ in the sense of Theorem 1.2. The image of this injection consists exactly of those cuspidal automorphic π of $\mathrm{GL}_2(\mathbb{A}_F)$ with π_v in the discrete series for all $v \in S$.

Compatibility with twists: if $\pi' \leftrightarrow \pi$, then $\pi' \otimes (\chi \circ \mathrm{Nm}) \leftrightarrow \pi \otimes (\chi \circ \det)$ for any smooth character $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$.

2.2. $F = \mathbb{Q}$**.** Let $N = p_1 \cdots p_r$ be square-free with r odd. Let D be the definite quaternion algebra over $\mathbb{Q}$ ramified at p_i. Then global JL correspondence gives a bijection between normalized cuspidal Hecke eigen newforms of weight 2, level $\Gamma_0(N)$, and automorphic representations π' of $\mathbb{A}_D^\times$ of dim> 1, with the fixed central character (the norm map $|\cdot| : \mathbb{A}_\mathbb{Q}^\times \to \mathbb{R}^{>0}$) such that π'_∞ is trivial on $\mathrm{SU}(2)$ and π'_p is trivial when restricted to $\mathcal{O}_{D_p}^\times$ at $p|N$, and unramified elsewhere. Counting numbers of newforms for GL_2 and all forms for D, we get

$$\dim S_2^{\mathrm{new}}(\Gamma_0(N)) = g(X_0(N))^{\mathrm{new}} = \#(D^\times \mathbb{A}_\mathbb{Q}^\times \backslash \mathbb{A}_{D,f}^\times / \prod_p \mathcal{O}_{D_p}^\times) - 1 = h(D) - 1.$$

Here $\mathcal{O}_{D_p}$ is a maximal order in D_p, unique if $p|N$, and $h(D)$ is the class number of D: the number of maximal orders in D up to left multiplication by $D^\times$. We subtract by one in the end of the RHS to exclude the unique 1-dim representation with given central character $|\cdot|$.

For $N = p$, we may remove the superscripts "new". In particular, $h(D) = 1$ iff $p = 2, 3, 5, 7$ and 13.

For indefinite D ramified at $p_1, \cdots, p_r$ (r even), we can similarly get a relation between the new part of the genus of $X_0(p_1 \cdots p_r)$ and the genus of a certain Shimura curve.

3. Proofs

3.1. **Matching orbital integrals.** We are in the situation where F is local. Fix Haar measures dg on $\mathrm{GL}_2(F)$ and dg' on $D^\times$. Two functions $\varphi \in C_c^\infty(\mathrm{GL}_2(F), \omega)$ and $\varphi' \in C_c^\infty(D^\times, \omega)$ have matching orbital integrals if

(1) For $\mathrm{GL}_2(F) \ni \gamma \sim \gamma' \in D^\times$ (both regular semisimple), we have
$$O_\gamma(\varphi) = O_{\gamma'}(\varphi')$$
with compatible choice of measures on $T_\gamma \cong T_{\gamma'}$ (centralizers). Note that the definition of orbital integrals uses the quotient measures on $T_\gamma(F)\backslash \mathrm{GL}_2(F)$ and $T_{\gamma'}(F)\backslash D^\times$.

(2) For hyperbolic elements $\gamma \in \mathrm{GL}_2(F)$, $O_\gamma(\varphi) = 0$.

Notation $\varphi \sim \varphi'$.

3.2. **Lemma.** *Given $\varphi' \in C_c^\infty(D^\times, \omega)$, there exists a $\varphi \in C_c^\infty(\mathrm{GL}_2(F), \omega)$ such that $\varphi \sim \varphi'$. Moreover, for such a φ, $O_\gamma(\varphi) = 0$ for $\gamma = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, $b \neq 0$ and*

$$-\varphi(z)/d(\mathrm{St}, dg) = \varphi'(z)\mathrm{vol}(F^\times \backslash D^\times, dg') \tag{3.1}$$

for scalar matrices $z \in F^\times$. Here St is the Steinberg representation for $\mathrm{GL}_2(F)$ and $d(\mathrm{St}, dg)$ its formal degree with respect to the chosen Haar measure dg.

Sketch of proof (when F is non-archimedean): See [4, §6] and [3, Lemma 6.2]. Langlands has characterized functions $(\gamma; T; \mu) \mapsto \Phi(\gamma; T; \mu) \in \mathbb{C}$ where T is a torus in GL_2, $\gamma \in T(F)$ and μ is a Haar measure on $T(F)$ that come from functions $\varphi \in C_c^\infty(\mathrm{GL}_2(F))$ by taking orbital integrals $(\gamma; T; \mu) \mapsto O_\gamma(\varphi)$ (T and μ are implicitly used in the integrals), i.e., which functions $\Phi(\gamma; T; \mu)$ coincide with $O_\gamma(\varphi)$ for all regular semisimple γ for some φ. Consider the map $\mathrm{GL}_2(F) \to X(F) = F \times F^\times$ defined by $g \mapsto (\mathrm{Tr}(g), \det(g))$. The discriminant locus $\Delta \subset X$ is defined by $\mathrm{tr}^2 = 4\det$. Then the functions $\Phi(\gamma; T; \mu)$ arising from orbital integrals, when restricted to the regular semisimple locus of GL_2, should descend to smooth functions with relatively compact support on $X - \Delta$; and they have specific aymptotic behavior when T is given and $T(F) \ni \gamma \to z \in Z(F)$, known as the Shalika germ expansion. Using this characterization one can verify that $\Phi(\gamma; T; \mu) = O_{\gamma'}(\varphi')$ for T nonsplit and zero otherwise satisfies all the conditions for being from an orbital integral.

Formula (3.1) is crucial in the proof of the character relation under local JL correspondence. The proof uses the Shalika germ expansion and the calculation of the germ corresponding to the trivial unipotent class using Euler-Poincaré measures. See [5].

3.3. **Preparation on characters.** Again we are in the local situation, with a fixed central character ω. Recall $X(F) = F \times F^\times$ is the space of semisimple conjugacy classes of $\mathrm{GL}_2(F)$. Let $X^{\mathrm{ell}}(F)$ be the elliptic regular locus of $X(F)$. This is a disjoint union of $(E^\times - F^\times)/\mathrm{Gal}(E/F)$, where E runs over isomorphism classes of quadratic extensions of F. For irreducible smooth representations π' of $D^\times$, the characters $\Theta_{\pi'}$ are smooth functions on $X^{\mathrm{ell}}(F)$. There is a measure on $X^{\mathrm{ell}}(F)$ making $\{\Theta_{\pi'}\}$ (π' has central character ω) an orthonormal basis of $L^2(X^{\mathrm{ell}}(F), \omega)$. The measure on $X^{\mathrm{ell}}(F)$ (see [2, formula before Lemma 15.3]) is chosen according to the Weyl integration formula, making $\langle \Theta_{\pi'}, \varphi' \rangle_{L^2(X^{\mathrm{ell}}(F),\omega)} = \mathrm{Tr}(\varphi'|\pi')$ for $\varphi' \in C_c^\infty(D^\times, \omega)$.

For irreducible smooth representations π of $\mathrm{GL}_2(F)$, their charatcers Θ_π are smooth functions on the regular semisimple locus of $\mathrm{GL}_2(F)$, and in particular on $X^{\mathrm{ell}}(F)$. As π runs over the discrete series with central character ω, Θ_π also form an orthonormal basis of $L^2(X^{\mathrm{ell}}(F), \omega)$. In fact we don't need the completeness of these characters, only orthonormality is needed, for which see [2, Lemma 15.4]. The essential ingredients in the proof are the Weyl integration formula and the vanishing of non-elliptic orbital integrals of matrix coefficients of supercuspidal representations.

3.4. **Comparison of trace formulae.** Back to the global situation. Let F be a global field of characteristic not 2. Let D be a quaternion algebra over F with ramification set S. Fix an isomorphism $D_v := D(F_v) \cong \mathrm{Mat}_2(F_v)$ for $v \notin S$, and use it to identify $D_v^\times$ with $\mathrm{GL}_2(F_v)$. Choose Haar measures on $\prod dg_v$ on $\mathrm{GL}_2(\mathbb{A}_F)$ and $\prod_v dg'_v$ on $\mathbb{A}_D^\times$ such that $dg_v = dg'_v$ for $v \notin S$.

Let $\varphi' = \otimes \varphi'_v$ for $\varphi'_v \in C_v^\infty(D_v^\times, \omega_v)$ (almost all of them are char functions of $\mathrm{GL}_2(\mathcal{O}_{F_v})$), and let $\varphi = \otimes \varphi_v$ where

- $\varphi_v = \varphi'_v$ for $v \notin S$ (we have identified $D_v^\times$ with $\mathrm{GL}_2(F_v)$);
- $\varphi_v \sim \varphi'_v$ for $v \in S$.

Trace formulae for $D^\times$:

$$\mathrm{Tr}(\varphi'|L^2(D^\times \backslash \mathbb{A}_D^\times, \omega)) = \mathrm{vol}(Z'(\mathbb{A}_F)D^\times \backslash \mathbb{A}_D^\times)\varphi'(1) + \sum_{\gamma' \in D^{\times,\mathrm{reg}}/\sim} \mathrm{vol}(T_{\gamma'}(F)\backslash T_{\gamma'}(\mathbb{A}_F))O_{\gamma'}(\varphi'). \tag{3.2}$$

Trace formulae for GL_2:

$$\begin{aligned} &\mathrm{Tr}(\varphi|L^2_{\mathrm{disc}}(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F), \omega)) \\ = \ &\mathrm{vol}(Z(\mathbb{A}_F)\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F))\varphi(1) + \sum_{\gamma \in \mathrm{GL}_2(F)^{S-\mathrm{ell}}/\sim} \mathrm{vol}(T_\gamma(F)\backslash T_\gamma(\mathbb{A}_F))O_\gamma(\varphi). \end{aligned} \tag{3.3}$$

Here $\mathrm{GL}_2(F)^{S-\mathrm{ell}}$ are regular semisimple elements γ that are elliptic in $\mathrm{GL}_2(F_v)$ for all $v \in S$. We have used the fact that when $v \in S$, $O_\gamma(\varphi_v) = 0$ for γ hyperbolic and $v \in S$. Since $\#S \geq 2$, the complementary terms in the trace formula (contribution of the continuous spectrum, hyperbolic and unipotent orbital integrals) all vanish.

Note that $D^{\times,\mathrm{reg}}/\sim$ is in natural bijection with $\mathrm{GL}_2(F)^{S-\mathrm{ell}}/\sim$ because both are parametrized by pairs (E, γ) where E/F is a quadratic extension that is nonsplit over S, and $\gamma \in E^\times - F^\times$ up to Galois involution. Therefore the second terms on the RHS of (3.2) and (3.3) match.

The first terms in the two trace formulae also match up. By (3.1), the equality of the first terms is equivalent to the identity

$$\mathrm{vol}(Z(\mathbb{A}_F)\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)) \prod_{v \in S} d(\mathrm{St}_v) = \mathrm{vol}(Z'(\mathbb{A}_F)D^\times \backslash \mathbb{A}_D^\times) \prod_{v \in S} \mathrm{vol}(F_v^\times \backslash D_v^\times)^{-1}. \tag{3.4}$$

Note that $(-1)^{\#S}$ disappears because $\#S$ is even. Also both sides above are independent of the choice of dg_v at $v \in S$. (3.4) is a nontrivial fact, which can be deduced after knowing that the second terms in both trace formulae are equal. See [2, Lemma 16.1.2].

We conclude that the RHS of (3.2) and (3.3) are equal. Hence the LHS are also equal. We may cancel the terms of 1-dim representations for $D^\times$ and GL_2 which have the same contribution to the LHS of (3.2) and (3.3), and get

$$\mathrm{Tr}(\varphi'|L_0^2(D^\times\backslash\mathbb{A}_D^\times,\omega)) = \mathrm{Tr}(\varphi|L^2_{\mathrm{cusp}}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F),\omega)) \tag{3.5}$$

Here $L_0^2(D^\times\backslash\mathbb{A}_D^\times,\omega)$ is the direct sum of irreducible automorphic representations of $D^\times$ of dim >1 and central char ω. We shall write the two Hilbert spaces in (3.5) simply by L_0^2 and L^2_{cusp}.

3.5. **Independence of charaters.** We separate the sums in (3.5) according to representations of $\mathrm{GL}_2(\mathbb{A}_F^S)$. For an irreducible representation $\sigma^S \cong \otimes'_{w\notin S}\sigma_w$ of $\mathrm{GL}_2(\mathbb{A}_F^S)$, let $L_0^2(\sigma^S)$ (resp. $L^2_{\mathrm{cusp}}(\sigma^S)$) be the subspace of L_0^2 (resp. L^2_{cusp}) consisting of those π' such that $\pi'^S \cong \sigma^S$ (resp. those π such that $\pi^S \cong \sigma^S$). We may rewrite (3.5) as

$$\sum_{\sigma^S}\mathrm{Tr}(\varphi^S|\sigma^S)\left(\sum_{\pi'\subset L_0^2(\sigma^S)}\prod_{v\in S}\mathrm{Tr}(\varphi'_v|\pi'_v) - \sum_{\pi\subset L^2_{\mathrm{cusp}}(\sigma^S)}\prod_{v\in S}\mathrm{Tr}(\varphi_v|\pi_v)\right). \tag{3.6}$$

Since $\varphi^S = \otimes_{w\notin S}\varphi_w$ is arbitrary, by Lemma 3.6 below (or rather its variant with a fixed central character), we conclude that the terms in the bracket above have to be zero, i.e.,

$$\sum_{\pi'\subset L_0^2(\sigma^S)}\prod_{v\in S}\mathrm{Tr}(\varphi'_v|\pi'_v) = \sum_{\pi\subset L^2_{\mathrm{cusp}}(\sigma^S)}\prod_{v\in S}\mathrm{Tr}(\varphi_v|\pi_v) \tag{3.7}$$

for all σ^S. Note on both sides π and π' are counted with multiplicities of their appearance in the automorphic spectra.

3.6. **Lemma** (Linear independence of characters, variant of [2, Lemma 16.1.1])**.** *Let G be a locally compact unimodular group and $\{\pi_i\}_{i\in I}$ be a family of distinct irreducible Hilbert space representations of G. Let $\mathcal{S}_G \subset L^1(G)$ be a dense subspace stable under convolution and taking adjoint $\varphi \mapsto \varphi^*(g) = \overline{\varphi(g^{-1})}$. Let $c_i \in \mathbb{C}$ ($i\in I$) be such that $\sum_i c_i\pi_i(\varphi)$ is a Hilbert-Schmidt operator for any $\varphi\in\mathcal{S}_G$.*

If $\sum_i c_i\mathrm{Tr}(\varphi*\varphi^*|\pi_i) = 0$ for all $\varphi\in\mathcal{S}_G$, then all $c_i = 0$.

Proof. Let $||\cdot||_{HS}$ is the Hilbert-Schmidt norm: for any $\varphi\in\mathcal{S}_G$, $\mathrm{Tr}(\varphi*\varphi^*|\pi_i) = ||\pi_i(\varphi)||^2_{HS}$.

Suppose not all c_i are zero, let $0\in I$ be an index such that $c_0\neq 0$. We may assume $c_0 = 1$. Then $||\pi_0(\varphi)||^2_{HS} = \sum_{i\neq 0}c_i||\pi_i(\varphi)||^2_{HS}$. This implies that $||\pi_0(\varphi)||^2_{HS} \le \sum_{i\neq 0}|c_i|\cdot||\pi_i(\varphi)||^2_{HS}$ for all $\varphi\in\mathcal{S}_G$. Redefining the norms on the spaces of π_i we may reduce to the case $|c_i| = 0$ or 1 for all $i\neq 0$. Let π be the direct sum of all π_i such that $i\neq 0$ and $c_i\neq 0$, we get an inequality

$$||\pi_0(\varphi)||^2_{HS} \le ||\pi(\varphi)||^2_{HS}, \forall\varphi\in\mathcal{S}_G. \tag{3.8}$$

Let V_0 and V the Hilbert spaces on which G acts through π_0 and π.

First we show that for any nonzero $v_0\in V_0$, and any $\epsilon>0$, there is a function $\varphi\in\mathcal{S}_G$ with $||\pi(\varphi)||^2_{HS}\le\epsilon||\pi_0(\varphi)v_0||^2$. Suppose the contrary, then $||\pi(\varphi)||^2_{HS}\ge c||\pi_0(\varphi)v_0||^2$ for some $c>0$, which implies that if φ has zero matrix coefficient in $V\otimes V$ (i.e., $||\pi(\varphi)||^2_{HS} = 0$), then $\pi_0(\varphi)v_0 = 0$. Therefore the map $\mathcal{S}_G\to V_0 : \varphi\mapsto\pi_0(\varphi)v_0$ factors through the matric coefficient map $m:\mathcal{S}_G\to V\otimes V : \varphi\mapsto\sum e_\alpha\otimes\pi(\varphi)e_\alpha$, inducing a nonzero G-map $\overline{\mathrm{Im}(m)}\to V_0$ (the G-mod structure on $V\otimes V$ is given by the action on the second factor). This contradicts the assumption that π_0 is not isomorphic to any irreducible constituent of π.

Next choose $\varphi_0 = h*h^*\in\mathcal{S}_G$ with $\pi_0(\varphi_0)\neq 0$ (hence a self-adjoint HS operator). We may assume the largest eigenvalue of $\pi_0(\varphi_0)$ is 1, with unit eigenvector v_0. Let λ be the largest eigenvalue of $\pi(\varphi_0)$ on V. Applying the remarks of the previous paragraph to v_0 we find that we may choose $\varphi_\epsilon\in\mathcal{S}_G$ such that $||\pi(\varphi_\epsilon)||^2_{HS}\le\epsilon||\pi_0(\varphi_\epsilon)v_0||^2$. Then

$$||\pi(\varphi_\epsilon*\varphi_0)||^2_{HS}\le\lambda^2||\pi(\varphi_\epsilon)||^2_{HS}\le\epsilon\lambda^2||\pi_0(\varphi_\epsilon)v_0||^2 = \epsilon\lambda^2||\pi_0(\varphi_\epsilon*\varphi_0)v_0||^2\le\epsilon\lambda^2||\pi_0(\varphi_\epsilon*\varphi_0)||^2_{HS}.$$

Since ϵ can be arbitrarily small, for $\epsilon<\lambda^{-2}$ and $\varphi = \varphi_\epsilon*\varphi_0$ we have $||\pi(\varphi)||^2_{HS}\le||\pi_0(\varphi)||^2_{HS}$, which contradicts (3.8). This completes the proof. □

3.7. **Proof of Theorem 2.1.** The direction $\pi' \mapsto \pi$. We need to show that for each automorphic $\pi' \subset L_0^2$ there is a unique $\pi \subset L^2_{\mathrm{cusp}}$ such that $\pi^S \cong \pi'^S$. Moreover, π_v is in the discrete series for each $v \in S$. Uniqueness is by strong multiplicity one for GL_2. Existence is proved by contradiction. Let $\sigma^S = \pi^S$. Suppose no such π, then the RHS of (3.7) is zero, hence the LHS is also zero, for *all* choices of $\varphi'_v \in C_c^\infty(D_v^\times, \omega_v)$. But this is impossible because we can choose φ'_v to be matrix coefficients of π'_v to make the product nonzero.

For $v \in S$, π_v has to be in the discrete series: for otherwise $\mathrm{Tr}(\varphi_v|\pi_v)$ is still zero for $\varphi_v \sim \varphi'_v$ and we get contradiction as above. To see why $\mathrm{Tr}(\varphi_v|\pi_v) = 0$ we use the Weyl integration formula to write it as a sum over conjugacy classes of tori $T(F) \subset \mathrm{GL}_2(F)$ with contribution $\int_{T(F)_{\mathrm{reg}}} \Theta_{\pi_v}(t) O_t(\varphi_v)|D_{G/T}(t)|dt$. For elliptic T, $\Theta_{\pi_v}(t) = 0$ because π_v is a principal series representation; for split T, $O_t(\varphi_v) = 0$ by §3.1(2).

We show that the correspondence $\pi' \mapsto \pi$ is surjective, i.e., for any $\pi \subset L^2_{\mathrm{cusp}}$ with π_S in the discrete series, there is some $\pi' \subset L_0^2$ such that $\pi'^S \cong \pi^S$. Taking $\sigma^S = \pi^S$ and using strong multiplicity one for GL_2, (3.7) becomes

$$\sum_{\pi' \subset L_0^2, \pi'^S \cong \pi^S} \prod_{v \in S} \mathrm{Tr}(\varphi'_v|\pi'_v) = \prod_{v \in S} \mathrm{Tr}(\varphi_v|\pi_v). \tag{3.9}$$

For an irreducible representation $\sigma'_S = \otimes \sigma'_v$ of $\mathbb{A}^\times_{D,S}$. Let $\varphi'_v = \Theta_{\sigma'_v}$ and $\varphi_v \sim \varphi'_v$, then $\mathrm{Tr}(\varphi_v|\pi_v) = \langle \Theta_{\sigma'_v}, \Theta_{\pi_v} \rangle$ again by Weyl integration formula. Applying (3.9) to these test functions we get

$$\#\{\pi' \subset L_0^2 | \pi' \cong \pi^S \otimes \sigma'_S\} = \prod_{v \in S} \langle \Theta_{\sigma'_v}, \Theta_{\pi_v} \rangle. \tag{3.10}$$

Recall the discussion in §3.3. Since $\{\Theta_{\sigma'_v}\}$ form a basis for $L^2(X^{\mathrm{ell}}(F_v), \omega_v)$, for each $v \in S$ there is some σ'_v such that $\langle \Theta_{\sigma'_v}, \Theta_{\pi_v} \rangle \neq 0$. This shows that the LHS of (3.9) contains at least one term, and hence π comes from some π' under the map $\pi' \mapsto \pi$.

Finally we show that the correspondence $\pi' \mapsto \pi$ is injective. We use (3.10) again. Since characters are unit vectors in $L^2(X^{\mathrm{ell}}(F_v), \omega_v)$, $|\langle \Theta_{\sigma'_v}, \Theta_{\pi_v} \rangle| \leq 1$. Therefore the LHS of (3.10) is ≤ 1, proving the multiplicity one for $D^\times$. For some choice of σ'_S, both sides of (3.10) are nonzero, hence both must be 1. In particular, $|\langle \Theta_{\sigma'_v}, \Theta_{\pi_v} \rangle| = 1$, and $\Theta_{\sigma'_v} = a_v \Theta_{\pi_v}$ as functions on $X^{\mathrm{ell}}(F_v)$ for some constant $|a_v| = 1$. It also implies that the LHS of (3.9) has only one isomorphism class of π': for otherwise we have some π'' not isomorphic to π', and we may choose $\sigma''_S = \pi''_S$ and repeat the above argument, then the RHS of (3.9) would be a product of $\langle \Theta_{\sigma''_v}, \Theta_{\pi_v} \rangle$, which is zero for some $v \in S$. This together with the multiplicity one for $D^\times$ we just showed implies that the LHS of (3.9) contains exactly one term. Therefore the global JL is proved.

3.8. **Proof of Theorem 1.2.** In the above argument we have shown that for each $v \in S$ and discrete series representation π_v of $\mathrm{GL}_2(F_v)$ realized as a local component of a cuspidal representation, there is a unique irreducible representation π'_v of $D_v^\times$ such that $\Theta_{\pi'_v}(\gamma') = a_v \Theta_{\pi_v}(\gamma)$ (for $\gamma \sim \gamma'$) for some $|a_v| = 1$ independent of γ. Lemma 3.9 below shows that a_v is negative real number, therefore $a_v = -1$. This proves the direction $\pi_v \mapsto \pi'_v$ and the character relation for π_v coming from automorphic representations. But any discrete series representation π_v can be realized as a local component of a cuspidal automorphic representation (for supercuspidals, take its matrix coefficients as test functions; for Steinberg use Euler-Poincaré function, then use trace formula for GL_2).

Conversely, for any irreducible representation π'_v of $D_v^\times$ realized as a local component of an automorphic π' of $\mathbb{A}_D^\times$ of dimension > 1, the argument in §3.7 (especially (3.10)) shows that there exists a cusp automorphic representation π of GL_2 such that $\pi^S \cong \pi'^S$ and π_v satisfies character relation with respect to π'_v for all $v \in S$. This gives the direction $\pi'_v \mapsto \pi_v$ for π'_v coming from automorphic π'. We can similarly show that any π'_v comes from automorphic π' using trace formula. This completes the proof of the local JL correspondence.

3.9. **Lemma.** *Let F be a local field and D/F a quaternion algebra. Let π' be an irreducible representation of $D^\times$ and π be an irreducible discrete series representation of $\mathrm{GL}_2(F)$. Suppose for some $a \in \mathbb{C}$, Θ_π and*

$a\Theta_{\pi'}$ take the same values on elliptic regular semisimple conjugacy classes, then

$$a = -\frac{d(\pi)}{\dim \pi' \cdot d(\mathrm{St})}. \tag{3.11}$$

Note that the ratio of formal degrees is independent of the choice of a Haar measure on $\mathrm{GL}_2(F)$.

Proof. The archimedean case and twisted Steinberg case follow from direct calculation. Let us assume F is non-archimedean and π is supercuspidal.

Recall the formal degree $d(\pi) = d(\pi, dg) > 0$ of a discrete series representation π of G (depending on the choice of a Haar measure dg on G) is defined to make the following hold

$$\int_G \langle \pi(g)u_1, v_1\rangle \overline{\langle \pi(g)u_2, v_2\rangle} dg = d(\pi)^{-1}\langle u_1, u_2\rangle\langle v_1, v_2\rangle.$$

Let V_π be the Hilbert space on which $\mathrm{GL}_2(F)$ acts through π and let $v \in V_\pi$ be a unit vector. Consider the matrix coefficient $\varphi(g) = d(\pi)\langle \pi(g)v, v\rangle$ ($\varphi \in C_c^\infty(\mathrm{GL}_2(F))$). Then for each elliptic torus $T(F) \subset \mathrm{GL}_2(F)$ with a Haar measure dt on it we have $O_t(\varphi) = \mathrm{vol}(F^\times\backslash T(F), dt)^{-1}\Theta_\pi(t)$ for $t \in T(F)_{\mathrm{reg}}$. For hyperbolic $t \in \mathrm{GL}_2(F)$ we have $O_t(\varphi) = 0$ by cuspidality of π. Similarly, for a Haar measure dg' on $D^\times$, the matrix coefficient $\varphi'(g') = d(\pi')\langle \pi'(g')v', v'\rangle$ for π' and it satisfies $O_{t'}(\varphi') = \mathrm{vol}(F^\times\backslash T'(F), dt')^{-1}\Theta_{\pi'}(t')$ for $t' \in T'(F)_{\mathrm{reg}}$. Since $\Theta_\pi(t) = a\Theta_{\pi'}(t')$ when $t \sim t'$, φ and $a\varphi'$ have matching orbital integrals in the sense of §3.1. In particular, $\varphi(1)/d(\mathrm{St}) = -a\mathrm{vol}(F^\times\backslash D^\times)\varphi'(1)$ by Lemma 3.2. Using $\mathrm{vol}(F^\times\backslash D^\times)d(\pi') = \dim \pi'$, we get (3.11). □

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