p-adic L-functions for Dirichlet characters

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1 Notation and conventions

Before we begin, we fix a bit of notation.

We make the following convention: for a fixed prime p, we set $q = p$ if p is odd, and we set $q = 4$ if $p = 2$.

We will always view our Dirichlet characters as primitive; since we can obtain the L-function of an imprimitive Dirichlet character by throwing away Euler factors from the L-function of its associated primitive Dirichlet character, this convention will not affect interpolation questions. It does mean, however, that the product $\chi_1\chi_2$ of two characters is not necessarily the pointwise product.

The most important Dirichlet character will be the Teichmüller character, which we will denote ω. There is a canonical isomorphism $\mathbf{Z}_p^\times \cong (\mathbf{Z}/q\mathbf{Z})^\times \times (1+q\mathbf{Z}_p)$, so for any $a \in \mathbf{Z}_p^\times$, we can write $a = \omega(a)\langle a\rangle$, where $\omega(a) \in (\mathbf{Z}/q\mathbf{Z})^\times$ and $\langle a\rangle$ is a 1-unit. We can view ω either as a Dirichlet character modulo p or as a character on $\mathbf{Z}_p^\times$, via the composition $\mathbf{Z}_p^\times \to (\mathbf{Z}/q\mathbf{Z})^\times \to \mathbf{Z}_p^\times$.

We also fix an embedding $\overline{Q} \hookrightarrow \overline{Q}_p$. Since classical Dirichlet characters take algebraic values, this lets us view our Dirichlet characters as valued simultaneously in $\overline{Q}$ and in $\mathbf{C}_p$.

The Bernoulli numbers are defined by the generating function

$$\sum_{k\geq 0} B_k \frac{t^k}{k!} = \frac{t}{e^t - 1}$$

Given a Dirichlet character χ of conductor f, we define the generalized

Bernoulli numbers $B_{\chi,k}$ via the generating function

$$\sum_{k\geq 0} B_{\chi,k}\frac{t^k}{k!} = \frac{t}{e^{ft}-1}\sum_{a=1}^{f}\chi(a)e^{at}$$

The $B_{\chi,k}$ are algebraic numbers, and they live in $\mathbf{Q}(\chi)$, the extension of $\mathbf{Q}$ defined by adjoining the values of χ. If χ is the trivial character, we recover the ordinary Bernoulli numbers, except at $k = 1$. Then

$$L(\chi, 1-k) = -\frac{B_{\chi,k}}{k}$$

for $k \geq 1$. We also define the adjusted Bernoulli numbers

$$B^*_{\chi,k} := (1 - \frac{\chi(p)}{p^{1-k}})B_{\chi,k}$$

Finally, we define the Bernoulli polynomials by

$$\frac{te^{Xt}}{e^t - 1} = \sum_{n=1}^{\infty} B_n(X)\frac{t^n}{n!}$$

At $X = 0$, we recover the classical Bernoulli numbers. At $X = 1$, we recover the classical Bernoulli numbers, except for B_1.

We record the results of three generating function calculations for later use:

Proposition 1.1. $B_n(X) = \sum_{i=0}^{n} \binom{n}{i} B_i X^{n-i}$.

Proposition 1.2. $N^{k-1}\sum_{a=0}^{N-1} B_k(\frac{X+a}{N}) = B_k(X)$

Proposition 1.3. *Let F be any multiple of f. Then* $B_{\chi,n} = F^{n-1}\sum_{a=1}^{F}\chi(a)B_n\left(\frac{a}{F}\right)$.

2 Kummer congruences

Given a Dirichlet character $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$, we have an L-function $L(\chi, s) := \sum_{n\geq 1}\frac{\chi(n)}{n^s}$ for complex s with $\Re(s) > 1$ (here we say that $\chi(n) = 0$ if $(n, N) \neq 1$), which we can analytically continue to a meromorphic function on all of $\mathbf{C}$. We would like to define a p-adic analogue of this L-function.

That is, we would like to view $L(\chi, s)$ as an analytic function of a p-adic variable s valued in $\mathbf{C}_p$.

There are some immediate problems with this. First of all, there are terms in the sum with arbitrarily large powers of p in the denominator, so the sum above will diverge badly. To have any hope of defining a p-adic analogue of $L(\chi, s)$, we will need to remove the n with $p|n$ and consider $L^*(\chi, s) := (1-\frac{1}{p^s})L(\chi, s)$ instead of $L(\chi, s)$. The series $\sum_{\substack{n\geq 1 \\ p\nmid n}} \frac{\chi(n)}{n^s}$ still does not converge (as each term has p-adic absolute value 1), but at least the absolute values are note blowing up.

The second problem is that even if $n \in \mathbf{Z}_p^\times$, n^s is not a p-adically continuous function of s unless $n \in 1 + p\mathbf{Z}_p$. So we can't do something as naive as evaluating the sum defining $L^*(\chi, s)$ for $s \gg 0$, $s \in \mathbf{Z}$, say that it's clearly continuous, and say that a p-adically continuous function on a dense subset of $\mathbf{Z}_p$ uniquely interpolates to a continuous function on all of $\mathbf{Z}_p$.

Note, however, that if we restrict s to a single residue class modulo $p-1$, then n^s is a p-adically continuous function of s for any fixed $n \in \mathbf{Z}_p^\times$. This suggests that if we can make sense of $L(\chi, s)$ as a p-adic function at all, we should also expect a dependence on residue classes modulo $p-1$.

Instead, we will interpolate special values of the analytic continuation of $L^*(\chi, s)$. Recall that we can evaluate the Riemann ζ-function at negative integers, and

$$\zeta(1-k) = -\frac{B_k}{k}$$

for $k \geq 1$, where B_k is the kth Bernoulli number (and $B_1 = -\frac{1}{2}$); $\zeta(1-k) = 0$ for all odd integers k. More generally, for any Dirichlet character χ,

$$L(\chi, 1-k) = -\frac{B_{\chi,k}}{k}$$

and

$$L^*(\chi, 1-k) = -\frac{B^*_{\chi,k}}{k}$$

We will interpolate these special values.

The classical Kummer congruences state the following p-adic continuity result about the values $-\frac{B_k}{k}$:

Theorem 2.1. *Let m, n be positive even integers with $m \equiv n \pmod{p^{a-1}(p-1)}$ and $n \not\equiv 0 \pmod{p-1}$. Then B_m/m and B_n/n are p-integral and*

$$(1-p^{m-1})\frac{B_m}{m} \equiv (1-p^{n-1})\frac{B_n}{n} \pmod{p^a}$$

This tells us that on $p-2$ of the $p-1$ residue classes for $\mathbf{Z}(p-1)\mathbf{Z}$, we can interpolate the Riemann zeta function to a p-adically continuous function, which gives us $p-2$ distinct "p-adic zeta functions".

Rather amazingly, there is a stronger result refining the Kummer congruences.

Theorem 2.2. *Suppose $\chi \neq 1$ is a power of the Teichmüller character. Then if $m \equiv n \pmod{p^{a-1}}$, we have*

$$(1-\chi\omega^{-m}(p)p^{m-1})\frac{B_{\chi\omega^{-m},m}}{m} \equiv (1-\chi\omega^{-n}(p)p^{n-1})\frac{B_{\chi\omega^{-n},n}}{n} \pmod{p^a}$$

In other words, twisting χ by appropriate powers of the Teichmüller character has eliminated the requirement that m and n be in the same residue class modulo $p-1$, as well as the requirement that they not be divisible by $p-1$.

Thus, granting these refined congruences, we can define a p-adic L-function as follows: For $s \in \mathbf{Z}_p$, choose a sequence of positive integers k_i converging to $1-s$ p-adically. Then we set

$$L_p(\chi, s) = \lim_{i\to\infty}(1-\chi\omega^{-k_i}(p)p^{k_i-1})\frac{B_{\chi\omega^{-k_i},k_i}}{k_i}$$

This defines a continuous function on $\mathbf{Z}_p$ which interpolatates special values of several classical L-functions.

We actually have the following stronger result:

Theorem 2.3. *Let χ be a Dirichlet character of conductor f. Then there is a p-adic meromorphic function $L_p(\chi, s)$ on $\{s \in \mathbf{C}_p \| s| < qp^{-1/(p-1)}\}$ such that*

$$L_p(\chi, s) = -(1-\chi\omega^{-n}(p)p^{n-1})\frac{B_{\chi\omega^{-n},n}}{n} = L^*(\chi\omega^{-n}, 1-n)$$

If χ is not the trivial character, then $L_p(\chi, s)$ is analytic. If $\chi = 1$, then the only pole of $L_p(\chi, s)$ is at $s = 1$, where the residue is $1 - 1/p$.

In fact, we have an explicit formula. If F is any multiple of q and f, then

$$L_p(\chi, s) = \frac{1}{F}\frac{1}{s-1}\sum_{\substack{a=1\\ p\nmid p}}^{F} \chi(a)\langle a\rangle^{1-s} \sum_{j\geq 0} \binom{1-2}{j}(B_j)\left(\frac{F}{a}\right)^j$$

A proof (including a proof of the explicit formula) can be found in [3]. We will give a different proof of analyticity after we explain the source of the twistedness of the interpolation.

3 Weight space

Rather than view p-adic L-functions as analytic (or meromorphic) functions on $\mathbf{Z}_p$, we can view them as functions on the *weight space*.

Definition 3.1. The *weight space* $\mathcal{X}$ is the set of continuous characters $\mathrm{Hom}_{cont}(\mathbf{Z}_p^\times, \mathbf{C}_p^\times)$.

To understand this set of characters, first note that we can rewrite $\mathbf{Z}_p^\times$ as

$$\mathbf{Z}_p^\times = (\mathbf{Z}/q\mathbf{Z})^\times \times (1 + q\mathbf{Z}_p) \cong (\mathbf{Z}/q\mathbf{Z})^\times \times \mathbf{Z}_p$$

while we can rewrite $\mathbf{C}_p^\times$ as

$$\mathbf{C}_p^\times = p^{\mathbf{Q}} \times W \times U_1$$

where W is the group of roots of unity of $\mathbf{C}_p^\times$ of order prime to p, and $U_1 = \{x \in \mathbf{C}_p | |x-1|_p < 1\}$.

Any continuous map $\mathbf{Z}_p^\times \to \mathbf{C}_p^\times$ must send $\mu_{p-1}(\mathbf{Z}_p)$ to W and U_1 to U_1. Therefore, we can specify any character χ by a pair (i, s), where $i \in \mathbf{Z}/(p-1)\mathbf{Z}$ and $s \in U_1$ (s is the image of some fixed topological generator of $1+q\mathbf{Z}_p$). After fixing a topological generator γ of $1+q\mathbf{Z}_p$, we write χ_s for the character sending γ to s.

We can embed $\mathbf{Z}$ in $\mathcal{X}$ by sending k to the character $\psi_k : x \mapsto x^k$. However, this map $\mathbf{Z} \to \mathcal{X}$ is not p-adically continuous, because for a fixed x, x^k is generally not a p-adically continuous function of k. To get a continuous map $\mathbf{Z} \to \mathcal{X}$, we need to kill the $\mathbf{Z}/(p-1)\mathbf{Z}$ part of the map. For example, we could instead send k to $\psi_k\omega^{-k} = \langle\cdot\rangle^k$.

Thus, we get a natural embedding of $D = \{s \in \mathbf{C}_p \| |s|_p < qp^{-1/(p-1)}\}$ into $\mathcal{X}$, by sending $s \mapsto \langle\cdot\rangle^s$.

Given a Dirichlet character χ, we constructed a family of $p-1$ p-adic L-functions $L_p(\chi\omega^i, s)$, so they define a (meromorphic) function $\mathcal{L} : \mathcal{X} \to \mathbf{C}_p$ via $\mathcal{L}(\omega^i\langle\cdot\rangle^s) = L_p(\chi\omega^i, -s)$. But then

$$-\frac{B^*_{\chi,n}}{n} = L^*(\chi, 1-n) = L_p(\chi\omega^n, 1-n) = \mathcal{L}(\omega^n\langle\cdot\rangle^{n-1}) = \mathcal{L}(\psi_n\langle\cdot\rangle^{-1})$$

So we find the special values of the classical L-function by evaluating $\mathcal{L}$ on a translate of the naive embedding of $\mathbf{Z}$ into $\mathcal{X}$.

4 Mazur-Swinnerton-Dyer

Fix a Dirichlet character χ of conductor $p^n M$ with $p \nmid M$ and let $\mathbf{Z}_{p,M} = \mathbf{Z}/M\mathbf{Z} \times \mathbf{Z}_p$ and $\mathbf{Z}_{p,M}^\times = (\mathbf{Z}/M\mathbf{Z})^\times \times \mathbf{Z}_p^\times$, so that χ is a character on $\mathbf{Z}_{p,M}^\times$.

We will construct a measure on $\mathbf{Z}_{p,M}^\times$ and define the L-function by integrating characters in the weight space over $\mathbf{Z}_{p,M}^\times$.

Definition 4.1. A measure μ on $\mathbf{Z}_{p,M}$ assigns to each compact open subset $U \subset \mathbf{Z}_{p,M}$ a number $\mu(U) \in \mathbf{C}_p$ such that the distribution property is satisfied. That is, $\mu(\coprod U_i) = \sum \mu(U_i)$.

Given a measure μ, we can integrate locally constant functions $f = \sum c_i(1)_{U_i}$ (here $\mathbf{1}$ is the characteristic function of U_i) by taking

$$\int_{\mathbf{Z}_{p,M}} f d\mu = \sum c_i \mu(U_i)$$

This sum converges because $\mathbf{Z}_{p,M}$ is compact.

If f is a continuous function, we would like to integrate f by approximating f on $a + p^m M\mathbf{Z}_{p,M}$ by $f(a)$ and setting

$$\int_{\mathbf{Z}_{p,M}} f d\mu := \lim_{m\to\infty} \sum f(a)\mu(a + p^m M\mathbf{Z}_{p,M})$$

This will work so long as μ is *bounded*, that is, there is a constant $C \in \mathbf{R}$ such that $|\mu(U)|_p \leq C$ for all compact open subsets $U \subset \mathbf{Z}_{p,M}$.

As a first attempt, we define a family of measures μ_k on $\mathbf{Z}_{p,M}$ by

$$\mu_k(a+p^mM\mathbf{Z}_{p,M}) = -(p^mM)^{k-1}\frac{B_k(\{\frac{a}{p^mM}\})}{k}$$

where $\{\frac{a}{p^mM}\}$ is the fractional part of $\frac{a}{p^mM}$. Then Proposition 1.2 tells us that this is a finitely additive measure on open sets of $\mathbf{Z}_{p,M}$.

Furthermore, we can write $\mathbf{Z}_{p,M}^\times = \coprod_{a\in(\mathbf{Z}/p^nM)^\times}(a+p^nM\mathbf{Z}_{p,M})$ for any $n \geq 1$. Thus, if p divides the conductor of χ, we can use Proposition 1.3 to compute

$$\begin{aligned}\int_{\mathbf{Z}_{p,M}^\times}\chi d\mu_k &= \sum_{a\in(\mathbf{Z}/p^nM)^\times}\chi(a)\mu(a+p^nM\mathbf{Z}_{p,M}) \\ &= -(p^nM)^{k-1}\sum_{a=1}^{p^nM}\chi(a)\frac{B_k(\frac{a}{p^nM})}{k} \\ &= -\frac{B_{\chi,k}}{k} = -\frac{B_{\chi,k}^*}{k}\end{aligned}$$

If p does not divide the conductor of χ,

$$\begin{aligned}\int_{\mathbf{Z}_{p,M}^\times}\chi d\mu_k &= \sum_{a\in(\mathbf{Z}/pM)^\times}\chi(a)\mu(a+pM\mathbf{Z}_{p,M}) \\ &= -(pM)^{k-1}\left(\sum_{a=1}^{pM}\chi(a)\frac{B_k(\frac{a}{pM})}{k} - \sum_{a=1}^{M}\chi(pa)\frac{B_k(\frac{a}{M})}{k}\right) \\ &= -(1-\chi(p)p^{k-1})\frac{B_{\chi,k}}{k} = -\frac{B_{\chi,k}^*}{k}\end{aligned}$$

To define μ_k, we used $\{0,\ldots,p^mM-1\}$ as distinguished integral representatives of $(\mathbf{Z}/p^mM\mathbf{Z})^\times$. If we had chosen a different set and used it to define a different measure $\tilde{\mu}_k$, we would have

$$\mu_k(a+p^mM\mathbf{Z}_{p,M}) \equiv \tilde{\mu}_k(a+p^mM\mathbf{Z}_{p,M}) \pmod{p^mM}$$

so our choices made very little difference.

However, μ_k is not a bounded measure (because $B_k(\frac{a}{p^mM})$ is p-adically large). We modify it as follows:

Fix a $u \neq 1$, $u \in \mathbf{Z}_{p,M}^\times$, and define

$$\mu_{k,u}(a+p^mM\mathbf{Z}_{p,M}) = \mu_k(a+p^mM\mathbf{Z}_{p,M}) - u^k\mu_k(u^{-1}a+p^mM\mathbf{Z}_{p,M})$$

For each $m \geq 0$, there is a unique $a'_m \in \mathbf{Z}$, $0 \leq a'_m < p^m M$ such that $u^{-1}a \equiv a'_m \pmod{p^m M}$. Then $\mu_{k,u}$ is bounded, because

$$\begin{aligned}\mu_{k,u}(a + p^m M\mathbf{Z}_{p,M}) &= -(p^m M)^{k-1}\frac{1}{k}\left(B_k\left(\frac{a}{p^m M}\right) - u^k B_k\left(\frac{a'_m}{p^m M}\right)\right)\\ &= -\frac{1}{k}\sum_{i=0}^{k}\binom{k}{i}(B_i)(p^m M)^{i-1}\left(a^{k-i} - u^k (a'_m)^{k-i}\right)\end{aligned}$$

Every term in this sum is p-adically bounded as m gets large, except possibly the $i = 0$ term. But

$$a^k - u^k(a'_m)^k \equiv 0 \pmod{p^m M}$$

so even the $i = 0$ term is p-adically bounded for m large.

In fact, rewriting a'_m as $u^{-1}a - q_m(p^m M)$, for some $q_m \in \mathbf{Z}_{p,M}$, we get

$$\begin{aligned}(p^m M)^{-1}(a^k - u^k(a'_m)^k) &= (p^m M)^{-1}\left(a^k - u^k\sum_{j=0}^{k}\binom{k}{j}(u^{-1}a)^{k-j}(-q_m p^m M)^j\right)\\ &= u^k\sum_{j=1}^{k}\binom{k}{j}(u^{-1}a)^{k-j}(-q_m)^j(p^m M)^{j-1}\\ &\equiv -uka^{k-1}q_m \pmod{p^m}\end{aligned}$$

If we reduce our expression for $\mu_{k,u}(a + p^m M\mathbf{Z}_{p,M})$ modulo p^{m-1} for $m \gg 0$ (in case some of the B_k have a p in the denominator), we get

$$-(1 - \chi(p)p^{-(1-k)})\left(-ua^{k-1}q_m + B_1 a^{k-1}(1-u)\right)$$

which is the same as

$$\frac{1 - \chi(p)p^{-(1-k)}}{1 - \chi(p)}a^{k-1}\mu_{1,u}(a + p^m M\mathbf{Z}_{p,M})$$

This shows that for f a continuous function on $\mathbf{Z}_{p,M}$,

$$\int_{\mathbf{Z}_{p,M}} f d\mu_{k,u} = \int_{\mathbf{Z}_{p,M}} x^{k-1}f(x)d\mu_{1,u}$$

Another calculation shows

$$\int_{\mathbf{Z}_{p,M}^\times} \chi d\mu_{k,u} = -(1-\chi(u)u^k)\frac{B_{\chi,k}^*}{k}$$

Now we put everything together and define a function $\mathcal{L}$ on the weight space $\mathcal{X}$ by

$$\mathcal{L}(\varphi) := \frac{1}{1-\chi(u)\varphi(u)\langle u\rangle} \int_{\mathbf{Z}_{p,M}^\times} \chi(x)\omega(x)^{-1}\varphi(x) d\mu_{1,u}$$

This is defined for any $\varphi \in \mathcal{X}$ except $\langle\cdot\rangle^{-1}$. And for $\varphi = \psi_n\langle\cdot\rangle^{-1}$,

$$\mathcal{L}(\varphi) = \frac{1}{1-\chi(u)u^k} \int_{\mathbf{Z}_{p,M}^\times} \chi(x)x^{k-1} d\mu_{1,u}$$

Thus, $\mathcal{L}$ is a meromorphic function on the weight space whose only pole is at $\langle\cdot\rangle^{-1}$ and which interpolates our special values correctly.

References

[1] Neal Koblitz. *p-adic numbers, p-adic analysis, and zeta-functions, 2nd ed.*

[2] Jay Pottharst. Many Twisted Interpolations, Part I.

[3] Lawrence C. Washington. *Cyclotomic Fields.*