Algebraic Hecke Characters

Motivation

This motivation was inspired by the excellent article [Serre-Tate, § 7].
Our goal is to prove the main theorem of complex multiplication. The Galois theoretic formulation is as follows:

Theorem (Main Theorem of Complex Multiplication). *Let (A, α) be a CM abelian variety over a number field $K \subset \overline{\mathbb{Q}}$, and let (L, Φ) be its CM-type.*

(i) The reflex field $E \subset \overline{\mathbb{Q}}$ is contained in K.

(ii) There exists a unique algebraic Hecke character $\epsilon : \mathbb{A}_K \to L^\times$ such that for each ℓ, the continuous homomorphism

$$\phi_\ell : Gal(K^{ab}/K) \to L_\ell^\times$$
$$\phi_\ell(a) = \epsilon(a)\epsilon_{alg}(a_\ell^{-1})$$

is equal to the ℓ-adic representation of $Gal(K^{ab}/K)$ on the ℓ-adic Tate module $V_\ell(A)$.

(iii) The algebraic part of ϵ is

$$\epsilon_{alg} = N_\Phi \circ Nm_{K/E},$$

where $N_\Phi : Res_{E/Q}\mathbb{G}_m \to Res_{L/Q}\mathbb{G}_m$ denotes the reflex norm.

(v) If v is a finite place of K where A has good reduction, say $\overline{A}/\kappa_v$, then for any uniformizer π_v of O_{K_v}, we have $\epsilon(\pi_v) = Fr_{\overline{A},q_v}$, where $\#\kappa_v = q_v$.

Let's meditate on this statement to understand the need for algebraic Hecke characters.

We start with (v). Because A/K has CM by L, we have a diagram

$$\bar{i} : L \hookrightarrow End^0(A) \xrightarrow{red_v} End^0(A_L)$$

where the second map is the reduction at v map. The second map is injective. Indeed, for any m prime to q_v, $A[m]$ is a finite etale group scheme over O_{K_v}; for such group schemes, the reduction map is always isomorphism.

Now $Fr_{\overline{A},q_v}$ commutes with every k_v endomorphism of $\overline{A}$. In particular, it centralizes the image of $\bar{i}$ in $End^0(\overline{A})$. But because $\overline{A}$ is CM, so in particular, $[\bar{i}L : \mathbb{Q}] = 2\dim(\overline{A})$, we must have

$Fr_{\overline{A},q_v} \in im(\bar{i})$. Let $\pi'_v \in L$ denote the unique preimage of $Fr_{\overline{A},q_v}$. All Galois representations ϕ_ℓ on $V_\ell(A)$ are completely encoded in the collection $\{\pi'_v\}$, by the Chebotarev density theorem.

So the representation of Galois on torsion points is completely encapsulated by the (partially defined) homomorphism

$$\epsilon : \mathbb{A}_K^\times \to L^\times$$
$$\epsilon(a) = \prod_v (\pi'_v)^{val_v(a_v)} \ (*)$$

at least for those $a \in I_S$, the group of ideles a with $a_v = 1$ for all archimedean places or places of bad reduction for A. Let $\epsilon_0 = \epsilon|_{K^\times}$.
If ϵ could be extended to a continuous homomorphism on all of $\mathbb{A}_K^\times$, then the formula $(*)$ would hold in an open subgroup N containing I_S. Then by weak approximation, we can express any $a \in \mathbb{A}_K^\times$ as $a = fn$ for some $f \in K$ and $n \in N$. Thus,

$$\epsilon(a) = \epsilon(fn) = \epsilon_0(f) \prod_v (\pi'_v)^{val_v(n_v)}.$$

This would give a well defined continuous homomorphism provided

$$\epsilon_0(a) = \prod_v (\pi'_v)^{val_v(a_v)} \ (*) \text{ for all } a \in K^\times \cap N.$$

But the *Shimura-Taniyama formula* states that the valuation of $N_\Phi \circ Nm_{K/E}$ equals the valuation of the right hand side at every finite place v of good reduction. But why should that imply the equality (*)? We need more a priori information about the homomorphism ϵ_0.

For example, suppose ϵ_0 were the the restriction to $\underline{K}^\times(\mathbb{Q}) \to \underline{L}^\times(\mathbb{Q})$ of an algebraic map $\underline{K}^\times \to \underline{L}^\times$. Then

$$f = \epsilon_0 \cdot (N_\Phi \circ Nm_{K/E})^{-1} : \underline{K}^\times \to \underline{L}^\times$$

would be algebraic, and for all a in a congruence subgroup of $K^\times$, $f(a)$ would be a v-unit for all finite places v. But this would imply that $f(a) = 1$.

In fact, our main goal will be to show that "locally algebraic" abelian ℓ-adic Galois representations are in bijection with algebraic Hecke characters, those continuous homomorphisms $\epsilon_0 : \mathbb{A}_K^\times$ for which $\epsilon|_{K^\times}$ is algebraic. Fortunately, large classes of natural abelian ℓ-adic Galois representations fit this bill. For example, all abelian Hodge-Tate ℓ-adic Galois representations are locally algebraic, and this includes all ℓ-adic representations of abelian varieties with CM by L.

Algebraic Tori

A torus T/k is defined to be a group scheme over k for which $T_{\overline{k}} \cong (\mathbb{G}_m^r)_{\overline{k}}$. But something much better is true: $T_{k_s} \cong (\mathbb{G}_m^r)_{k_s}$. This makes tori succeptible to Galois theory.

Let $X^*(T)$ denote the group of characters $T_{k_s} \to \mathbb{G}_m$. It is a free abelian group because $T_{k_s} \cong \mathbb{G}_m^r$ and $Hom(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$ (over any connected base, in fact).

Because T is defined over $k, X^*(T)$ also admits an action of $\Gamma = Gal(k_s/k)$ by group automorphisms. Since T is actually split over a finite separable extension L/k, the finite index subgroup of Γ fixing L acts trivially on $X^*(T)$. Thus, the action of Γ on $X^*(T)$ is continuous, $X^*(T)$ having the discrete topology.

Let Tori/k denote the category of tori over k with morphisms algebraic group homomorphisms. Let Γ-Lat denote the category of (finite rank) free abelian groups with a continuous action of Γ by group automorphisms with morphisms Γ-equivariant group homomorphisms. Then:

Theorem. *The functor*

$$Tori/k \to \Gamma\text{-}Lat$$

$$T \to X^*(T)$$

is an anti-equivalence of categories. A quasi-inverse is given by

$$\Lambda \mapsto (k_s \otimes_k k[\Lambda])^\Gamma.$$

This equivalence is very useful.

Restriction of Scalars

In what follows, we will constantly refer to "algebraic homomorphisms $K^\times \to L^\times$" and the like. This is defined through restriction of scalars.

Let X be a K-scheme and K/k a separable field extension (or even finite etale, which is also useful.) Then we define its *restriction of scalars* by $R_{K/k}(X)$ using its functor of points:

For any scheme S/k, define

$$R_{\text{K}/k}(X)(S) := X(S_K).$$

It turns out that if X is finite-type, quasiprojective, then $R_{K/k}(X)$ is representable. Intuitively, this is not much different than "viewing a complex manifold as a real manifold."

Example. Let $X = \mathbb{C}[z_1, z_2]/(xy - 1)$. To find the coordinate ring of the restriction of scalars, we formally replace $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$, so each generator is replaced by $[\mathbb{C} : \mathbb{R}]$ generators for each coordinate of X, and each relation becomes $[\mathbb{C} : \mathbb{R}]$ relations:

$$1 = z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1).$$

One can check that $\mathbb{R}[x_1, y_2, x_2, y_2]/(x_1x_2 - y_1y_2 = 1, x_1y_2 + x_2y_1 = 0)$ satisfies the right functorial property to be the restriction of scalars.

Comment. Though we will only use $R_{K/k}$ for affine X, the functorial description allows this construction to globalize.

Note that both

$$R_{K/k}(\cdot) : \text{Sch/K} \to \text{Sch/k}$$

and

$$\cdot \otimes_k K : \text{Sch/k} \to \text{Sch/K}$$

are functors. And by definition, they satisfy the following adjointness property:

$$Mor_k(X, R_{K/k}(Y)) \cong Mor_K(X_K, Y).$$

When Weil originally defined restriction of scalars, he used a different approach:
We can form

$$Z = \prod_{i:K \hookrightarrow k_s} X_i$$

a product of copies of X indexed by the embeddings $i : K \hookrightarrow k_s$. Let the Galois group act on X by permuting the embeddings through automorphisms of k_s. By Galois descent, there is a unique Y/k for which $Y_K = Z$. Weil then defined Y to be the restriction of scalars. From our perspective, we just use the good functorial properties of $R_{K/k}$ to check compatibility with Weil's definition:

$$\begin{aligned}
(R_{K/k}(X))_{k_s} &= R_{K \otimes_k k_s / k_s}(X_{K \otimes_k k_s}) \\
&= R_{\sqcup_{i:K \hookrightarrow k_s} Spec(k_s)}(X_{k_s, i}) \\
&= \prod_{i:K \hookrightarrow k_s} X_i
\end{aligned}$$

with Galois action as described above.

Interaction with Tori

Let $k_s/K/k$ be a tower of fields with K/k finite. Let's form the k-scheme

$$Y = R_{K/k}(\mathbb{G}_m).$$

- By the functorial description of $R_{K/k}$, it is clear that Y is a group scheme.
- By Weil's description of restriction of scalars as a Galois descent, we know that (as group schemes)

  $$Y_{k_s} = \prod_{i:K \hookrightarrow k_s} \mathbb{G}_m.$$

  So Y_{k_s} is a torus.

By the description of Y_{k_s}, it is already more or less clear that $X^*(Y) = Ind_{\Gamma_K}^{\Gamma_k} \mathbb{Z}$. But there is also a completely formal reason, using the adjunction of restriction of scalars and base change.

To be specific, our equivalence of categories transforms $R_{K/k}$ into some functor K from Γ_K-Lat to Γ_k-Lat. Meanwhile, it is clear that base change corresponds to restriction of the Galois representation. Thus, our anti-equivalence of categories transforms to

$$Mor_{\Gamma_k}(F(\Lambda), \Lambda') = Mor_{\Gamma_K}(\Lambda, \Lambda'_{\Gamma_K}).$$

But by Frobenius reciprocity, we already know that $Ind_{\Gamma_K}^{\Gamma_k}$ is adjoint to restriction. So $F(\Lambda) = Ind_{\Gamma_K}^{\Gamma_k}(\Lambda)$ functorially. It follows that induction gives the Galois lattice corresponding to restriction of scalars!

Algebraic Hecke characters

What follows draws heavily from [Conrad-Chai-Oort], especially §2.4. Let K, L be number fields.

Definition. *An algebraic Hecke character* $\mathbb{A}_K^\times \xrightarrow{\epsilon} L^\times$ *is a continuous homomorphism for which* $\epsilon|_{K^\times} : K^\times = \underline{K}^\times(\mathbb{Q}) \to \underline{L}^\times(\mathbb{Q}) = L^\times$ *is the restriction of an algebraic group homomorphism* $\epsilon_0 : \underline{K}^\times \to \underline{L}^\times$.

In the above definition, $\underline{L}^\times$ is given the discrete topology, so $ker(\epsilon)$ is open.

Examples.

(1) Let $K/\mathbb{Q}, w = \#O_K^\times, h =$ class number of O_K. For every fractional ideal J, we have $J^h = (a)$ for a which is unique up to $O_K^\times$. Since $\#O_K^\times = w$, the assignment

$$\epsilon : J \mapsto a^w$$

gives a well-defined homomorphism from the group of all fractional ideals (not the class group, the actual group of fractional ideals) to $K^\times$. Also, $\epsilon(a) = a^{hw}$ for any $a \in K^\times$. So ϵ is an algebraic Hecke character.

(2) By the Main Theorem of CM, to each CM abelian variety over a number field K with CM-type (L, Φ) and reflex field E, we naturally associate an algebraic Hecke character $\epsilon : \mathbb{A}_K^\times \to L^\times$ with algebraic part given by $\epsilon = N_\Phi \circ N_{K/E}$.

Serre Tori

We will express algebraic Hecke characters in terms of morphisms of certain algebraic groups $S_\mathfrak{m}$ to be constructed below.

H^2 classes and group extensions

Recall that the isomorphism class of a (central) extension of abelian groups

$$0 \to Y_1 \to Y_2 \to Y_3 \to 0$$

is classified by a 2-cocycle c. We can think of Y_2 as being the set $Y_1 \times Y_3$ with group action given by

$$(y_1, y_3) + (y_1', y_3') = (y_1 + y_1' + c(y_3, y_3'), y_3 + y_3').$$

If $i : Y_1 \to X_2$ is any map of abelian groups, then we can similarly define a new group X_2 which is $X \times Y_3$ as a set with group structure

$$(x_1, y_3) + (x_1', y_3') = (x_1 + x_1' + i(c(y_3, y_3')), y_3 + y_3')$$

and then $0 \to X_1 \to X_2 \to Y_3 \to 0$ is an exact sequence and

$$\begin{array}{ccccc} 0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 \\ & & {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow \\ 0 & \longrightarrow & X_1 & \longrightarrow & X_2 \end{array}$$

is a pushout diagram, with

$$j : Y_2 \to X_2$$
$$(y_1, y_3) \mapsto (i(y_1), y_3).$$

These considerations work out in exactly the same way if X_1/k happens to be an abelian algebraic group with $i : Y_1 \to X_1(k)$. And then X_2 the above diagram is still a pushout diagram, say in the category of commutative algebraic groups over k.

Serre Tori and a second take on algebraic Hecke characters

Let $U_{\mathfrak{m}} = K_\infty^\times \times \prod_{v|m}(1 + \mathfrak{m}O_{K,v}) \times \prod_{v\nmid\infty,\mathfrak{m}} O_{K,v}^\times$
be a standard compact open congruence subgroup of $\mathbb{A}_K^\times$.
We have an exact sequence

$$1 \to K^\times/(K^\times \cap U_{\mathfrak{m}}) \to \mathbb{A}_K^\times/U_{\mathfrak{m}} \to C_{\mathfrak{m}} = \mathbb{A}_K^\times/(U_{\mathfrak{m}}K^\times) \to 1.$$

$C_{\mathfrak{m}}$ is a finite ideal class group with level structure.

We want to algebraize this exact sequence. We let $T_{\mathfrak{m}}$ be the quotient of $Res_{K/\mathbb{Q}}\mathbb{G}_m$ by the Zariski closure of $K^\times \cap U_{\mathfrak{m}}$. Then there is a natural map

$$i : K^\times/(K^\times \cap U_{\mathfrak{m}}) \to T_{\mathfrak{m}}(\mathbb{Q}).$$

As in the previous section, we can form the pushout exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K^\times/(K^\times \cap U_{\mathfrak{m}}) & \longrightarrow & \mathbb{A}_K^\times/U_{\mathfrak{m}} & \longrightarrow & C_{\mathfrak{m}} = \mathbb{A}_K^\times/(U_{\mathfrak{m}}K^\times) & \longrightarrow & 1 \\ & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_{K,\mathfrak{m}}} & & \downarrow{\scriptstyle =} & & \\ 1 & \longrightarrow & T_{\mathfrak{m}} & \longrightarrow & S_{\mathfrak{m}} & \longrightarrow & C_{\mathfrak{m}} & \longrightarrow & 1 \ (*)_{\mathfrak{m}}. \end{array}$$

These commutative diagrams form an inverse system. For $\mathfrak{m}$ sufficiently divisible, $T_{\mathfrak{m}}$ doesn't change. Indeed, the connected component of $\overline{K^\times \cap U_{\mathfrak{m}}}$ does not change with $\mathfrak{m}$. And the component group must stabilize at some point. Call

$$\mathfrak{S}_K = \lim_{\leftarrow} T_{\mathfrak{m}}$$

the *Serre torus.*
Taking the inverse limit, we get a map of exact sequences of proalgebraic groups

$$1 \to \mathfrak{S}_K \to S_K \to Gal(K^{ab}/K) \to 1 \ (*);$$

this is the pullback of the finite level exact sequence, at sufficiently divisible level $\mathfrak{m}$, via the surjection $Gal(K^{ab}/K) \to C_{\mathfrak{m}}$.
We also have associated exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K^\times & \longrightarrow & \mathbb{A}_K^\times & \longrightarrow & \mathbb{A}_K^\times/K^\times & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i_K} & & \downarrow{\scriptstyle rec_K} & & \\ 1 & \longrightarrow & \mathfrak{S}_K(\mathbb{Q}) & \longrightarrow & S_K(\mathbb{Q}) & \longrightarrow & Gal(K^{ab}/K) & \longrightarrow & 1. \end{array}$$

In this setup, we can give another alternative characterization of algebraic Hecke characters.

Lemma. *(1) Let $\rho : S_K \to \underline{L}^\times$ be a $\mathbb{Q}$-homomorphism. Then the composition*

$$\mathbb{A}_K^\times \xrightarrow{i_K} S_K(\mathbb{Q}) \to \underline{L}^\times(\mathbb{Q}) = L^\times$$

is an algebraic Hecke character. This can also be phrased at finite level:

If $\rho : S_\mathfrak{m} \to \underline{L}^\times$ is a $\mathbb{Q}$-homomorphism, then the composition

$$I_\mathfrak{m} = \mathbb{A}_K^{\times, S(\mathfrak{m})} / \prod_{v \notin S(\mathfrak{m})} O_{K,v}^\times \xrightarrow{i_{K,\mathfrak{m}}} S_\mathfrak{m}(\mathbb{Q}) \xrightarrow{\rho} L^\times$$

is an algebraic Hecke character of conductor $\leq \mathfrak{m}$.

(2) Every algebraic Hecke character (of conductor $\leq \mathfrak{m}$) arises via the above procedure.

Associated ℓ-adic representations

We will construct splittings

$$\phi_\ell : Gal(K^{ab}/K) \to S_K(\mathbb{A}_K)$$

to the exact sequence

$$1 \to \mathfrak{S}_K \to S_K \to Gal(K^{ab}/K) \to 1 \ (*),$$

viewed as an exact sequence of $\mathbb{A}_{K,f}$ groups. The ℓ-adic Galois representations associated to algebraic Hecke characters will factor through these splittings.

The main obstacle, of course, is to modify $\epsilon = \mathbb{A}_K^\times \to \mathbb{A}_K^{\times,S} / \prod_{v \notin S(\mathfrak{m})} O_{K,v}^\times \xrightarrow{i_{K,\mathfrak{m}}} S_\mathfrak{m}(\mathbb{Q}) \subset S_\mathfrak{m}(\mathbb{Q}_\ell)$ so as to kill the global points $K^\times \subset \mathbb{A}_K^\times$.

Let $\pi : T = Res_{K/\mathbb{Q}} \mathbb{G}_m \to T_\mathfrak{m} \to S_\mathfrak{m}$ be quotient projection. Then let

$$\alpha_\ell = \mathbb{A}_K^\times \to (K \otimes_\mathbb{Q} \mathbb{Q}_\ell)^\times = T(\mathbb{Q}_\ell) \xrightarrow{\pi_\ell} S_\mathfrak{m}(\mathbb{Q}_\ell).$$

Claim. *α_ℓ and $i_{K,m}$ coincide on $K^\times$.*

This follows by the commutativity of the diagram

Proof.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K^\times & \longrightarrow & \mathbb{A}_K^\times & \longrightarrow & \mathbb{A}_K^\times/K^\times & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i_K} & & \downarrow{\scriptstyle rec_K} & & \\ 1 & \longrightarrow & \mathfrak{S}_K(\mathbb{Q}) & \longrightarrow & S_K(\mathbb{Q}) & \longrightarrow & Gal(K^{ab}/K) & \longrightarrow & 1. \end{array}$$

□

We thus define

$$\phi_\ell = i_{K,m} \cdot \alpha_\ell^{-1} : \mathbb{A}_K^\times \to S_\mathfrak{m}(\mathbb{Q}_\ell).$$

By the claim, this homomorphism factors through $\mathbb{A}_K^\times/K^\times$. Even better, because the target is totally disconnected, it factors through $\pi_0(\mathbb{A}_K^\times/K^\times) = Gal(K^{ab}/K)$, the latter identification coming through Artin reciprocity.
Abusing notation, we let ϕ_ℓ denote the associated map $Gal(K^{ab}/K) \to S_\mathfrak{m}(\mathbb{Q}_\ell)$.

The compatibility with change in $\mathfrak{m}$ is easy to check. So this gives us our ℓ-adic splittings

$$\phi_\ell : Gal(K^{ab}/K) \to S_K(\mathbb{Q}_\ell)$$

which can be assembled into a single map

$$\phi : Gal(K^{ab}/K) \to S_K(\mathbb{A}_K)$$

if we like.

Lemma 1. *Let $\epsilon : \mathbb{A}_K^\times \to L^\times$ be an algebraic Hecke character corresponding to a $\mathbb{Q}$-rational homomorphism $\rho : S_K \to \underline{L}^\times$. Then $\psi_\ell = \rho \circ \phi_\ell : Gal(K^{ab}/K) \to L_\ell^\times$ is a continuous homomorphism with the following properties.*

(i) There exists a finite set Σ of places of K, containing all places above ℓ and all archimedean places, such that $\psi_\ell(W_v) \subset L^\times$ for the Weil subgroup of the decomposition group at $v, W_v \subset D_v$, for $v \notin \Sigma$. Furthermore,

$$\epsilon(\pi_v) = \psi_\ell(Frob_v)$$

for any local uniformizer π_v at $v \notin \Sigma$ and $Frob_v$ the arithmetic Frobenius.

(ii) The composition

$$\psi_\ell \circ rec_{K,\ell} : K_\ell^\times \to L_\ell^\times$$

coincides with the algebraic $\mathbb{Q}$-homomorphism

$$\epsilon_{alg}^{-1} : K_\ell^\times = \underline{K}^\times(\mathbb{Q}_\ell) \to \underline{L}^\times(\mathbb{Q}_\ell) = L_\ell^\times$$

on an open subgroup of $K_\ell^\times$. Thus, $\psi_\ell \circ rec_{K,\ell}$ is <u>locally algebraic</u>.

Remark. If ρ factors through $\rho_\mathfrak{m} : S_\mathfrak{m} \to \underline{L}^\times$, then

$$\psi_\ell = \rho_\mathfrak{m} \circ (\phi_\mathfrak{m})_\ell.$$

Proof. (i) This is clear from the definitions.

(ii) Suppose ϵ has conductor $\leq \mathfrak{m}$.

– For any $x \in K^\times$,

$$\psi_\ell(rec_{K,\ell}(x)) = \psi_\ell\left(rec_K((x)^{\{v|\ell\}})\right)^{-1},$$

where for any set S of places of K, $(x)^S$ is the idele

$$(x)^S_v := \begin{cases} x_v & \text{if } v \notin S \\ 1 & \text{otherwise.} \end{cases}$$

For any $v \in \Sigma, v \nmid \ell$, the image of inertia I_v in $Gal(K^{ab}/K)$ has finite pro-ℓ part; this is because the image of inertia is topologically isomorphic to $O^\times_{K_v}$ via the local reciprocity map, and this latter group has finite pro-ℓ part. So for x sufficiently close to 1 at each $v \in \Sigma, v \nmid \ell$,

$$\psi_\ell(rec_K((x)^\Sigma)) = \psi_\ell\left(rec_K((x)^{\{v|\ell\}})\right).$$

By (i), we also get that

$$\psi_\ell(rec_K((x)^\Sigma)) = \epsilon((x)^\Sigma).$$

So putting this all together, we see that for $x \in K^\times, x \equiv 1 \; mod \; \mathfrak{m}$ (remembering that $cond(\epsilon)$ divides $\mathfrak{m}$)

$$\psi_\ell(rec_{K,\ell}(x)) = \epsilon((x)^\Sigma)^{-1} = \epsilon_{alg}(x)^{-1}.$$

The above nearness conditions define an open subgroup of 1 in $K_\ell^\times$. Since $K^\times$ is dense, continuity implies the same relationship in that neighborhood.

□

Now, we'll prove a converse statement: roughly, all locally algebraic (semisimple) abelian ℓ-adic Galois representations arise as some ϕ_ℓ described above.

Lemma 1. *Let K and L be number fields. Let ℓ be prime and let Σ be a set of places, including all places of K over ℓ and all archimedean places. Let $\psi_\ell : Gal(K^{ab}/K) \to L_\ell^\times$ be a continuous homomorphism which is unramified outside of Σ and for which $\psi_\ell(W_v) \subset L^\times$. Assume moreover that ψ_ℓ is "locally algebraic". This means that there exists a open subgroup $1 \in U_\ell \subset K_\ell^\times$ and an algebraic $\mathbb{Q}$-homomorphism $\chi_{alg} : \underline{K}^\times \to \underline{L}^\times$ such that $\psi_\ell \circ rec_{K,\ell}|_{U_\ell} = \chi_{alg,\mathbb{Q}_\ell}$.*

Then there exists a unique algebraic Hecke character $\epsilon : \mathbb{A}_K^\times \to L^\times$ with algebraic part χ_{alg}^{-1} such that ϵ induces ϕ_ℓ in the manner described above.

Proof. Our hand is forced by the statement of the theorem; we must define ϵ by

$$\epsilon(x) = \psi_\ell(rec_K(x)))\chi_{alg}(x_\ell)^{-1}.$$

- ϵ is clearly a continuous homomorphism $\mathbb{A}_K^\times \to L_\ell^\times$.
- – By assumption, $\epsilon(x)$ is trivial on U_ℓ.

  - Because L_ℓ is totally disconnected, ϵ is trivial on $(K_\infty^\times)^0$.
  - Because $L_\ell^\times$ is ℓ-adic, ϵ must kill an open subgroup W of $\prod_{v\in\Sigma, v\nmid\ell,\infty} K_v^\times$.
  - Because ψ_ℓ is unramified outside Σ and $rec_K(O_{K,v}^\times) = I_v$, ϵ must kill $\prod_{v\notin\Sigma} O_{K,v}^\times$.

  Let U be the open subgroup $(K_\infty^\times)^0 \times U_\ell \times W \times \prod_{v\notin\Sigma} O_{K,v}^\times$. So ϵ is continuous when $L_\ell^\times$ is given the discrete topology, not just the ℓ-adic topology. So at least it's conceivable that ϵ could factor through $L^\times$.

- ϵ equals χ_{alg}^{-1} when restricted to $K^\times$ because rec_K kills $K^\times$.

- $rec_K(\mathbb{A}_K^{\Sigma,\times}) \subset \langle W_v \rangle$. Since we are assuming $\psi_\ell(W_v) \subset L^\times$, it follows that $\psi_\ell(\mathbb{A}_K^{\Sigma,\times}) \subset L^\times$. Since

$$\mathbb{A}_K^\times = K^\times \cdot U \cdot \mathbb{A}_K^{\Sigma,\times},$$

  it follows that

$$\epsilon(\mathbb{A}_K^\times) \subset L^\times$$

  with $\epsilon|_{K^\times} = \chi_{alg}^{-1}$. Thus, ϵ really is an algebraic Hecke character.

To prove uniqueness, we need to show that if ϵ is a finite order Hecke character with $\rho_\epsilon \circ \psi_\ell = 1$, then $\epsilon = 1$. But this equality implies that $\epsilon(\pi_v) = 1$ for almost all v. As above,

$$\mathbb{A}_K^\times = K^\times \cdot U \cdot \mathbb{A}_K^{\Sigma,\times}.$$

So it follows that ϵ is identically 1.

□

Remark. It's a true and unobvious fact that for abelian, semisimple, p-adic Galois representations, locally algebraic is equivalent to Hodge-Tate. Proving this would be far beyond the scope of these notes and my capabilities. I mention this only because many geometric Galois representations are known to be Hodge-Tate. For example, the Galois representation on the Tate module of any p-divisible group is Hodge-Tate. This, in particular, implies that the action of $D_v, v|p$ on $V_p(A)$ for a CM abelian variety A is a priori locally algebraic.

The Archimedean Avatar

Let $\epsilon : \mathbb{A}_K^\times \to L^\times$ be an algebraic Hecke character. We formed "ℓ-adic avatars" by making a simple modification at ℓ, namely by setting

$$\psi_\ell : \mathbb{A}_K^\times \to L_\ell^\times$$
$$\psi_\ell(a) = \epsilon(a)\epsilon_{alg}(a_\ell^{-1}).$$

This factors through $\mathbb{A}_K^\times / K^\times \cdot (K_\infty^\times)^0 = Gal(K^{ab}/K)$ and so gives rise to Galois representations.

Though there is no Galois interpretation, one could ask what happens when we make a local modification to ϵ at ∞:

$$\psi_\infty : \mathbb{A}_K^\times \to \mathbb{A}_K^\times/K^\times \to (L \otimes_{\mathbb{Q}} \mathbb{R})^\times$$
$$\psi_\infty(a) = \epsilon(a)\epsilon_{alg}(a_\infty)^{-1}.$$

Composing with all of the embeddings $\tau : K \hookrightarrow \mathbb{C}$, we get a collection of Hecke Grossencharaktere:

$$\psi_{\infty,\tau} := \mathbb{A}_K^\times/K^\times \to (L \otimes_{\mathbb{Q}} \mathbb{R})^\times \to \mathbb{C}^\times.$$

Example: CM abelian varieties

The above Grossencharaktere formed from ψ_∞ encode all of the Galois theoretic information of the compatible family ψ_ℓ.

For example, suppose that A/K is a CM abelian variety with CM type (L, Φ). The main theorem of complex multiplication tells us that that there is an algebraic Hecke character ϵ_0 with $\epsilon_{0,alg} = N_\Phi \circ N_{K/E}$ such that the ℓ-adic avatar

$$\phi_\ell(a) = \epsilon_0(a)\epsilon_{0,alg}(a_\ell^{-1}) : Gal(K^{ab}/K) \to L_\ell^\times$$

equals the ℓ-adic representation of $Gal(K^{ab}/K)$ on $V_\ell(A)$. Let's then compute the L-function of A.

Claim [Milne, § 2] 1. $L(A/K, s) = \prod_{\tau:K\to\mathbb{C}} L(\psi_{\infty,\tau}, s)$.

Proof. The local factor at v is $L_v(A/K, T) := det(1 - TFr_v|V_\ell(A)^{I_v})^{-1}$ for any $v \nmid \ell$.

- Because A has potentially good reduction at every place v, we know that $V_\ell(A)^{I_v} = 0$ at primes v of bad reduction. Because each $\phi_{\infty,\tau}$ is unramified if and only if A has good reduction at v, we are reduced to checking the equality of local factors at places v of good reduction.

- The main theorem tells us Fr_v acting on $V_\ell(A)$ has the same characteristic polynomial as $\epsilon_0(\pi_v)$ acting on $L \otimes_{\mathbb{Q}} \mathbb{C}$. So

  $$det(1 - TFr_v|V_\ell(A))^{-1} = \prod_{\tau:L\to\mathbb{C}} (1 - \psi_{\infty,\tau}(\pi_v))^{-1}$$

  at places of good reduction.

It follows that

$$L(A/K, s) = \prod_{\tau:L\to\mathbb{C}} L(\psi_{\infty,\tau}, s).$$

□

References

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