SEMISTABLE REDUCTION FOR ABELIAN VARIETIES

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1. Introduction

The semistable reduction theorem for curves was discussed in Christian's notes. In these notes, we will use that result to prove an analogous theorem for abelian varieties. After some preliminaries on semi-abelian varieties (to convince us that the notion is a robust one), we will review the notion of a semi-abelian scheme (introduced in Christian's lecture), recall the statement of the semistable reduction theorem, and explain why it suffices to prove that result in the special case of Jacobians. A discussion of the proof in that case, indicating the role of results of Artin and Raynaud concerning Picard functors, will be given in the final section of the notes. The bulk of these notes are concerned with proving many useful facts about abelian varieties by using the semistable reduction theorem as a "black box" (another reason we postpone its proof until the end of the notes).

2. Some algebraic group generalities

Let k be a field. We'd like to study general smooth connected commutative k-groups, with an eye towards understanding the structure of the identity component of the special fiber of the Néron model of an abelian variety over the fraction field of a discrete valuation ring with residue field k. As usual, we insist on allowing arbitrary k, without perfectness hypotheses, because we want the theory to be applicable at codimension-1 points on normalizations of some flat moduli spaces over $\mathbf{Z}$ with generic dimension > 0, for which the residue field at generic points in positive characteristic is never perfect.

In this section we discuss some preliminaries concerning exact sequences of group schemes of finite type over a field, and then review some facts concerning tori, unipotent groups, and commutative linear algebraic groups. A robust theory of quotients of smooth group varieties modulo normal smooth closed subgroup varieties emerges from the following result:

Proposition 2.1. *Let $f : G' \to G$ be a homomorphism between finite type group schemes over a field k.*

(1) *The image $f(G')$ in G is closed, and f is a closed immersion if and only if* $\ker f = 1$. *In particular, if f is a monomorphism then it is a closed immersion.*
(2) *If H' is a closed k-subgroup scheme of G' then there exists a faithfully flat quotient map $q : G' \to G'/H'$ onto a finite type k-scheme G'/H'; that is, q is initial among k-morphisms $G' \to X$ that are invariant with respect to the right action of H' on G'. The formation of this quotient commutes with extension on k, it is smooth if G' is smooth, and when G' and H' are smooth this coincides with the notion of quotient in the sense of* [Bo].

(3) *If H' is a closed normal k-subgroup scheme of G' then G'/H' admits a unique k-group structure making q a homomorphism, and* $\ker q = H'$. *If moreover G' is affine then G'/H' is affine.*
(4) *If G and G' are smooth then $f(G')$ is a smooth closed k-subgroup of G and the map $G'/\ker f \to f(G')$ is an isomorphism. In particular, if f is surjective and G and G' are smooth then f is a quotient mapping.*

For part (3), we recall that a monomorphism $H \to G$ between group schemes over a base scheme S is a *normal* subgroup scheme if $H(T)$ is normal in $G(T)$ for every S-scheme T, or equivalently the map $G \times_S H \to G$ defined by $(g, h) \mapsto ghg^{-1}$ factors through H. Hence, in the smooth case over $\operatorname{Spec} k$ with H and G of finite type, it is equivalent to say that $H(\overline{k})$ is a normal subgroup of $G(\overline{k})$.

Proof. For a proof of (1), see [SGA3, $\mathrm{VI_B}$, 1.2, 1.4.2]. For (2), see [SGA3, $\mathrm{VI_A}$, §3] apart from the smoothness of G'/H' when G' is smooth and the consistency with [Bo]. To prove the smoothness when G' is smooth, we may and do increase the ground field to be algebraically closed, so smoothness and regularity coincide. By faithful flatness of $G' \to G$ it follows that every local ring $\mathscr{O}_{G,g}$ admits a faithfully flat map to $\mathscr{O}_{G',g'}$ for $g' \in f^{-1}(g) \neq \emptyset$. But a noetherian ring is regular if it admits a faithfully flat extension ring that is regular [Mat, Thm. 23.7]. Hence, all local rings on G are regular, so G is smooth (as we arranged $k = \overline{k}$). When G' and H' are both smooth, the agreement with the quotient in the sense of [Bo] is due to both notions of quotient serving as an initial object among H'-invariant maps from G' to *smooth* k-schemes.

To prove (3), in case G' and H' are smooth we may appeal to [Bo, II, §5–§6] (thanks to the agreement noted in (2)). The general case is reduced to the case of smooth G' and H' in [SGA3, $\mathrm{VI_B}$, 11.17]. Finally, (4) is explained in [CGP, Ex. A.1.12]. ■

Definition 2.2. A *short exact sequence* of finite type k-group schemes is a diagram

$$1 \to G' \xrightarrow{j} G \xrightarrow{q} G'' \to 1$$

such that $G' = \ker q$ (via j) and $G'' = G/G'$ (via q). In particular, q is initial for homomorphisms from G that annihilate G', and q is faithfully flat.

By Proposition 2.1(2), these properties persist after any extension of the ground field. Since q is faithfully flat of finite type, it follows from descent theory that the associated diagram of group functors (on the category of k-schemes, for which these functors are sheaves for the fppf topology) is short exact as a diagram of group sheaves for the fppf topology.

Example 2.3. For a surjective map $f : G' \twoheadrightarrow G$ between smooth finite type k-groups, and $H = \ker f$, it follows from Proposition 2.1(4) that $1 \to H \to G' \to G \to 1$ is a short exact sequence. Equally important is a kind of converse: if G' is a smooth k-group scheme of finite type and H is a closed normal k-subgroup scheme then the quotient $G = G'/H$ not only makes $1 \to H \to G' \to G \to 1$ a short exact sequence (thanks to Proposition 2.1(3)), but G is necessarily *smooth*. This is part of Proposition 2.1(2).

Example 2.4. If G' is a smooth connected affine k-group and G is an abelian variety, then there are no nontrivial k-homomorphisms from G' to G or from G to G'. Indeed, by Proposition 2.1, any homomorphism $f : G' \to G$ has normal kernel and hence $f(G')$ is a closed smooth *affine* k-subgroup of G. But since G is proper, so is the closed $f(G')$. Hence, $f(G')$ is proper and affine, so it is finite, and thus by connectedness it is a single point. This must be the identity, so f is trivial. In the reverse direction, any map from a proper k-scheme to an affine k-scheme of finite type has finite image, so by connectedness considerations the only homomorphism from G to G' is the trivial one.

Lemma 2.5. *Let G be a connected k-group scheme that is locally of finite type. Then G is finite type (equivalently, quasi-compact) and geometrically connected over k. In particular, a smooth connected k-group is necessarily of finite type.*

Proof. This is a special case of [SGA3, $\mathrm{VI_A}$, Prop. 2.4]. Here is a sketch. First one handles geometric connectedness using arguments not specific to groups (any connected k-scheme with a k-point remains connected after an arbitrary extension of the ground field). Then for the quasi-compactness we may assume $k = \overline{k}$ and replace G with G_{red} so that G is smooth. Since smooth connected k-schemes are irreducible, for any non-empty affine open set U and $g \in G(k)$ the open sets U and gU^{-1} meet. Hence, the image of $U \times U$ under the multiplication map $G \times G \to G$ contains $G(k)$, so all k-points of G lies in a quasi-compact subset of G. But $k = \overline{k}$, so this forces G to be quasi-compact. ∎

It may seem like useless hypergenerality to assume just locally finite type and not finite type, but in our later work with certain Picard varieties we really will just know "locally finite type" at the outset. Of more immediate importance to us is that connectedness implies geometric connectedness for such k-groups, as that permits us to extend the ground field in proofs without ruining connectedness conditions. (The case of most interest is the identity component of the special fiber of a Néron model.)

Theorem 2.6 (Chevalley)**.** *Let G be a smooth connected group scheme over a perfect field k. There exists a unique short exact sequence*

$$1 \to G^{\mathrm{aff}} \to G \to A \to 1$$

with G^{aff} affine and A an abelian variety.

This theorem is the basis for the work of Lang and Rosenlicht using generalized Jacobians to classify finite abelian (ramified) coverings of smooth proper geometrically connected curves over perfect fields. For finite fields, this recovers their geometric approach to class field theory of global function fields.

Proof. For an exposition of the proof in the language of schemes, see [C1]. (The references therein also point to the original proofs.) Briefly, the plan of the proof goes as follows. A Galois descent argument (using uniqueness, as well as the Galois property of $\overline{k}/k$ for the perfect field k) reduces the problem to the case $k = \overline{k}$. One further reduces to the case that G has no nontrivial homomorphism to an abelian variety (in particular, to a Jacobian), and then the aim is to prove that G is affine.

The absence of maps to Jacobians is used to build big projective representations of G on spaces of rational functions on G. By noetherian induction, such a representation is constructed for which the associated homomorphism $f : G \to \mathrm{PGL}_n$ is injective on geometric points and hence has infinitesimal kernel. But $f(G)$ is a smooth *closed* k-subgroup of PGL_n

(Proposition 2.1(1)), so it inherits affineness from PGL_n and hence we get a short exact sequence $1 \to H \to G \to f(G) \to 1$ with infinitesimal (hence k-finite) H. A descent theory argument then shows that the k-finiteness of H implies the finiteness of the "H-torsor" morphism $G \to f(G)$, so G is finite over the affine $f(G)$ and hence is affine. ∎

We respectively call G^{aff} and A the *affine part* and *abelian part* of G. By Example 2.4, these are *functorial* in G; we will use this later. The exact sequence in Chevalley's theorem is called the *Chevalley decomposition* of G.

Remark 2.7. The perfectness hypothesis in Chevalley's theorem cannot be dropped. For counterexamples over any imperfect field, see [CGP, Ex. A.3.8] (and see [CGP, Thm. A.3.9] for an interesting salvage over arbitrary fields of positive characteristic). We will always apply Chevalley's theorem after passing to the case of an algebraically closed ground field.

Recall that a homomorphism $f : G' \to G$ between finite type k-group schemes is an *isogeny* if f is a finite flat surjection, or equivalently (by descent theory) it is faithfully flat and $\ker f$ is finite. When G' and G are smooth, by the Miracle Flatness Theorem [Mat, 23.1] (or generic flatness combined with homogeneity considerations) it is equivalent to say that f is surjective and $\dim G' = \dim G$ (or that f is surjective and $\ker f$ is finite).

Proposition 2.8. *If $f : G' \to G$ is a surjective homomorphism* (*resp. isogeny*) *between smooth connected k-groups and k is perfect then ${G'}^{\mathrm{aff}} \to G^{\mathrm{aff}}$ is surjective* (*resp. an isogeny*) *and likewise for the induced map $G'/{G'}^{\mathrm{aff}} \to G/G^{\mathrm{aff}}$ between the abelian parts.*

Proof. Since $\dim G = \dim G^{\mathrm{aff}} + \dim(G/G^{\mathrm{aff}})$, and similarly for G' in place of G, once the surjective case is settled the isogeny case will be immediate by dimension considerations. The closed k-subgroup $f({G'}^{\mathrm{aff}})$ in G^{aff} is normal in $f(G') = G$, so the smooth connected quotient group $G/f({G'}^{\mathrm{aff}})$ makes sense and is a quotient of the abelian variety $G'/{G'}^{\mathrm{aff}}$, so it is an abelian variety. Thus, its smooth connected closed k-subgroups are abelian varieties, such as $G^{\mathrm{aff}}/f({G'}^{\mathrm{aff}})$. But this latter quotient is affine by Proposition 2.1(3), so it is trivial. This proves surjectivity between affine parts, and surjectivity between abelian parts is obvious. ∎

From the viewpoint of Chevalley's theorem, semi-abelian varieties are the smooth connected commutative k-groups whose affine part over $\overline{k}$ is a power of $\mathbf{G}_m$. Before we take up the study of semi-abelian varieties, we need to record a basic structural fact concerning smooth connected commutative affine groups over a field k (to be applied to the affine part of the Chevalley decomposition over $\overline{k}$). Thus, we now review some basic definitions and results connected to linear algebraic groups. We will say much more than we need, mainly to convey the sense in which tori are much nicer than unipotent groups (thereby explaining why the definition of "semi-abelian variety" is cannot be improved).

Remark 2.9. The phrase "linear algebraic group" means "smooth affine group" over a field; this terminology is justified because any smooth affine group over a field k is k-isomorphic to a closed k-subgroup of some GL_n over k [Bo, 1.10]. Such embeddings are a powerful tool in the theory, allowing concepts like "Jordan decomposition" for geometric points to be carried over functorially from GL_n to any linear algebraic group [Bo, 4.4].

By [Bo, 10.9], over an algebraically closed field every 1-dimensional smooth connected affine group is isomorphic to $\mathbf{G}_m$ or $\mathbf{G}_a$. This leads to the two general classes of smooth connected affine groups which pervade the basic structure theory of linear algebraic groups: tori and unipotent groups. We take up each in turn.

Definition 2.10. A *k-torus* is a smooth affine commutative k-group T such that $T_{\overline{k}} \simeq \mathbf{G}_m^r$ for some $r \geq 0$. If T is k-isomorphic to $\mathbf{G}_m^r$ for some r, then T is *k-split*.

By [Bo, §8] (see especially [Bo, 8.3]), any k-torus T becomes split after a finite Galois extension on k and all smooth connected k-subgroups or quotients of T are tori. In particular, $T_{k_s} \simeq \mathbf{G}_m^r$ and hence the (geometric) *character group*

$$\mathrm{X}(T_{k_s}) := \mathrm{Hom}_{k_s}(T_{k_s}, \mathbf{G}_m) \simeq \mathrm{End}_{k_s}(\mathbf{G}_m)^{\oplus r} = \mathbf{Z}^{\oplus r}$$

is a finite free $\mathbf{Z}$-module whose natural $\mathrm{Gal}(k_s/k)$-action is discrete. (Note that $\mathrm{X}(T_{k_s}) = \mathrm{X}(T_{\overline{k}})$ since the $\overline{k}$-endomorphisms of $\mathbf{G}_m$ are defined over k_s and even over the prime field.) The arguments in [Bo, §8] show that $T \rightsquigarrow \mathrm{X}(T_{k_s})$ is an anti-equivalence between the category of k-tori and the category of $\mathrm{Gal}(k_s/k)$-lattices; this makes tori accessible objects. It is also true that if the outer terms of a short exact sequence of smooth connected k-groups are tori then so is the middle term (without assuming it to be commutative), and that then the outer terms are k-split if and only if the middle is, but we do not need these facts.

Just as tori are related to groups built up from $\mathbf{G}_m$, there is an analogue for $\mathbf{G}_a$ called *unipotence*. The starting point is the observation that $\mathbf{G}_a$ is isomorphic to the closed subgroup $\{(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix})\}$ in GL_2. There are two ways to define unipotence; the definitions look quite different, but are equivalent. As a starting point, we make a definition inspired by [SGA3]:

Definition 2.11. A smooth connected affine k-group U is *unipotent* if $U_{\overline{k}}$ admits a composition series whose successive quotients are isomorphic to $\mathbf{G}_a$.

Remark 2.12. Unipotence can be defined more generally (without smoothness or connectedness), but we do not need it in such generality. It can be shown (via descent theory) that the affineness hypothesis on U in the definition is redundant, but we do not need that either.

Remark 2.13. In [Bo] the definition of unipotence is that all points in $U(\overline{k})$ have unipotent image under some k-subgroup inclusion $U \hookrightarrow \mathrm{GL}_n$ (the choice of which turns out not to matter). This is the traditional definition in the theory of linear algebraic groups. Let's establish the equivalence with the above definition. The well-definedness and functoriality of Jordan decomposition for linear algebraic groups [Bo, 4.4(3),(4)] ensure that our definition of unipotence implies the traditional one (for smooth connected affine k-groups). Conversely, by [Bo, 4.8, 10.6(2)], a smooth connected affine k-group that is unipotent in the traditional sense is unipotent in our sense.

The traditional definition of unipotence makes it clear that unipotence is inherited by smooth connected k-subgroups and quotients. Less evident but true is that any unipotent smooth connected affine k-group is k-isomorphic to a closed k-subgroup of the strictly upper-triangular subgroup in some GL_n over k [Bo, 15.5(ii)] (thereby justifying the terminology "unipotent").

Definition 2.14. A unipotent smooth connected affine k-group U is *split* if it admits a composition series of smooth connected k-subgroups such that the successive quotients are k-isomorphic to $\mathbf{G}_a$.

Here are two striking contrasts with tori. First, any unipotent smooth connected affine k-group U becomes split over a finite purely inseparable extension of k [Bo, 15.5(ii)], and hence U is k-split when k is perfect. Second, the k-split property is inherited by quotients [Bo, 15.4(i)], but is *not* inherited by smooth connected closed subgroups in general when k is imperfect. (The classic counterexample in $\mathbf{G}_a^2$ over an arbitrary imperfect field was given in Samit's talk in the fall.)

Lemma 2.15. *Let T be a k-torus, and U a smooth connected unipotent k-group. There are no nontrivial homomorphisms between T and U (in either direction).*

Proof. We may assume $k = \overline{k}$, and then use composition series to reduce to the case $T = \mathbf{G}_m$ and $U = \mathbf{G}_a$. The density of nontrivial torsion in T with order not divisible by $\mathrm{char}(k)$ and the absence of such torsion in $\mathbf{G}_a$ forces any homomorphism $\mathbf{G}_m \to \mathbf{G}_a$ to vanish. In the opposite direction, a homomorphism $\mathbf{G}_a \to \mathbf{G}_m$ corresponds to a nowhere-vanishing function on $\mathbf{G}_a$. But the only such functions are the constants, so such a homomorphism is trivial. ■

The importance of tori and unipotent groups for our purposes is through their role in the following general structure theorem in the commutative case:

Proposition 2.16. *Let G be a smooth connected commutative affine k-group.*

(1) *There is a unique k-torus T in G that contains all k-tori of G, and $U = G/T$ is unipotent.*
(2) *If k is perfect then there is a unique splitting $G = T \times U$.*
(3) *The formation of T and U is functorial in G, and a surjective homomorphism (resp. isogeny) $G' \to G$ between smooth connected commutative affine k-groups induces surjective homomorphisms (resp. isogenies) $T' \to T$ and $U' \to U$. In particular, G' is a torus if and only if G is a torus, and likewise for unipotence.*

Note that by Lemma 2.15, the formation of T and U automatically commutes with extension of the ground field (since if S is a torus in G_K then its image in $G_K/T_K = (G/T)_K = U_K$ must be trivial, forcing $S \subseteq T_K$).

Proof. Part (1) is [Bo, 10.6(3)], which also gives (2) over $\overline{k}$. For perfect k this product decomposition of $G_{\overline{k}}$ descends to k by Galois descent. (A simpler proof over perfect k is given by [Bo, 4.7].) The functoriality in (3) follows from Lemma 2.15. Since $\dim G = \dim T + \dim U$ and likewise for G', and the rest goes as in the proof of Proposition 2.8 (using T in the role of G^{aff}). ■

Remark 2.17. The perfectness hypothesis in Proposition 2.16(2) cannot be dropped. See [CGP, Ex. 1.1.3] for a counterexample over any imperfect field.

3. Semi-abelian varieties

Recall that Chrisitan define a semi-abelian variety over k to be a smooth connected commutative k-group G such that $G_{\overline{k}}$ is an extension of an abelian variety by a torus; equivalently,

$G^{\rm aff}_{\overline{k}}$ is a torus. By Proposition 2.8 and Proposition 2.16(3), it follows that for an isogeny $G' \to G$ between smooth connected commutative k-groups, G' is semi-abelian if and only if G is. Also, again using Proposition 2.16, a smooth connected commutative k-group G is semi-abelian if and only if $G^{\rm aff}_{\overline{k}}$ (or equivalently, $G_{\overline{k}}$) contains no nontrivial smooth connected unipotent $\overline{k}$-subgroup. This is the main reason that the semi-abelian condition is powerful (since unipotent groups are generally a source of headaches).

Despite the failure of Chevalley's theorem over imperfect fields, we now show that the extension structure on $G_{\overline{k}}$ can always be descended to k in the semiabelian case:

Theorem 3.1. *Let G be a smooth connected k-group, and assume that $G^{\rm aff}_{\overline{k}}$ is a torus. Then G is necessarily commutative and there is a unique short exact sequence of k-groups*

$$1 \to T \to G \to A \to 1$$

for a k-torus T and abelian variety A over k.

This shows that in the definition of "semi-abelian variety" we do not need to assume commutativity of G. The descent of the torus to k will be essential in our later study of abelian varieties with semistable reduction.

Proof. We first prove the commutativity, so we may and do assume $k = \overline{k}$. It now suffices to show that any smooth connected extension E of an abelian variety A by the k-group $T = \mathbf{G}_m^r$ is necessarily commutative. First we show that T is central in E. The centrality of T in E is equivalent to the triviality of the E-action on T via conjugation. Consider the closed subgroups $T[N]$ for $N \geq 1$ not divisible by ${\rm char}(k)$. These are étale, since $[N] : T \to T$ induces multiplication by N on the tangent space at the identity (an isomorphism since ${\rm char}(k) \nmid N$). Thus, the $T[N]$ are finite constant groups (since $k = \overline{k}$), and they are clearly functorial in T (e.g., preserved under any automorphism).

By consideration of each $\mathbf{G}_m$-factor of T, we see that the finite constant subgroups $T[N]$ are collectively Zariski-dense, so to prove the triviality of the E-action on T it suffices to show that E acts trivially on each $T[N]$. But E is connected and $T[N]$ is a finite constant k-group, so the triviality of the action is immediate (as each point of $T[N]$ has open and closed stabilizer in E). Hence, T is central in E, as desired. Combining this centrality with the commutativity of $A = E/T$, the commutator map $E \times E \to E$ factors through a map

$$A \times A = (E/T) \times (E/T) = (E \times E)/(T \times T) \to T$$

carrying $(1, 1)$ to 1. But $A \times A$ is proper and connected, and T is affine, so this map vanishes and thus E is commutative.

Returning to our initial k, now with G known to be commutative, we will descend $G^{\rm aff}_{\overline{k}}$ to a k-torus T in G. Once this is done, G/T must be an abelian variety because $(G/T)_{\overline{k}} = G_{\overline{k}}/G^{\rm aff}_{\overline{k}} = A$ is proper (and properness descends through a ground field extension). The uniqueness of the resulting extension structure on G is a formal consequence of existence, by Example 2.4.

To carry out the descent from $\overline{k}$ to k, if we can solve the descent down to k_s then the uniqueness over k_s (granting existence) implies via Galois descent that we can push the descent down to k. Hence, we may and do rename k_s as k, so now $k = k_s$. We will construct the desired k-torus T in G via closure of suitable finite étale torsion subgroups.

(The technique of construction via closure of étale torsion is very useful; it will occur later in the proof of Grothendieck's inertial criterion for semistable reduction.)

Choose a prime $\ell \neq \mathrm{char}(k)$, so each $G[\ell^n]$ is a finite étale k-group, and hence is constant because $k = k_s$. In other words, $G_{\overline{k}}[\ell^n]$ consists entirely of k-rational points; i.e., $G(\overline{k})[\ell^n] = G(k)[\ell^n]$. Thus, every subgroup of $G(\overline{k})[\ell^n]$ arises from a unique subgroup of $G(k)[\ell^n]$. We therefore get a rising chain of subgroups $S_n \subseteq G(k)[\ell^n]$ corresponding to the ℓ^n-torsion in the torus $G^{\mathrm{aff}}_{\overline{k}}$, so $\cup_n S_n$ is a subgroup of $G(k)$. The Zariski closure in G of any abstract subgroup of $G(k)$ is a smooth closed k-subgroup of G whose *formation commutes with extension of the ground field* (see [Bo, 1.3(b)]). Hence, the identity component T of this closure is a smooth connected closed subgroup of G such that $T_{\overline{k}} \subseteq G^{\mathrm{aff}}_{\overline{k}}$. But $G^{\mathrm{aff}}_{\overline{k}}$ is a torus, so its smooth connected closed subgroups are tori and no proper subtorus can contain all of its ℓ-power torsion (as the dimension of a k-torus can be read off from the order of its ℓ-power torsion subgroups, akin to the case of abelian varieties). Thus, $T_{\overline{k}} = G^{\mathrm{aff}}_{\overline{k}}$. ■

Corollary 3.2. *Let $1 \to G' \to G \to G'' \to 1$ be a short exact sequence of smooth connected groups over a field k. Then G is semi-abelian if and only if G' and G'' are semi-abelian.*

Proof. We may and do assume $k = \overline{k}$. Assume G is semi-abelian. The smooth connected closed subgroup G'^{aff} of the torus G^{aff} must be a torus, so G' is semi-abelian. The $G^{\mathrm{aff}} \to G''^{\mathrm{aff}}$ is surjective by Proposition 2.8, so G''^{aff} inherits the torus property from G^{aff}, and hence G'' is semi-abelian. Conversely, assume G' and G'' are semi-abelian. To prove that G is semi-abelian we have to prove that G^{aff} is a torus, or equivalently (by Proposition 2.16) that a unipotent smooth connected subgroup U in G^{aff} is trivial. Its image in the torus G''^{aff} is trivial, so $U \subseteq \ker(G \to G'') = G'$ and hence U is contained in G'^{aff}. But G'^{aff} is a torus, so $U = 1$. ■

Corollary 3.3. *Let G be a semi-abelian variety over a field k, and H a closed k-subgroup scheme. If the ℓ-torsion scheme $H[\ell]$ vanishes for some prime ℓ then H is finite.*

Proof. We may assume $k = \overline{k}$ and replace H with H^0_{red} so that H is smooth and connected. We will prove $H = 0$. By Corollary 3.2, H is semi-abelian. Its maximal torus T satisfies $T[\ell] \subseteq H[\ell] = 0$ yet $T[\ell]$ has order $\ell^{\dim T}$, so $T = 0$. Thus, H is an abelian variety. But then $0 = H[\ell]$ has order $\ell^{2\dim H}$, so $H = 0$. ■

4. Semistable reduction

Let R be a discrete valuation ring, K its fraction field, and k its residue field.

Proposition 4.1. *Let $f : A \to B$ be a map between abelian varieties over K, and let $\mathscr{A}$ and $\mathscr{B}$ denote the respective Néron models of A and B over R.*

(1) *If f is an isogeny and either of $\mathscr{A}^0_k$ or $\mathscr{B}^0_k$ is semi-abelian then so is the other, and then $f^0_k : \mathscr{A}^0_k \to \mathscr{B}^0_k$ is an isogeny.*
(2) *Conversely, if the reduction f^0_k is an isogeny then f is an isogeny. f is an isogeny.*

Proof. First we prove (1). There exists an isogeny $h : B \to A$ such that $f \circ h = [n]_B$ and $h \circ f = [n]_A$ for some nonzero integer n. Since multiplication by n is an isogeny on any semi-abelian variety over k, once the equivalence of semi-abelian reductions is established it follows that f_k^0 is an isogeny if and only if h_k^0 is. Hence, by symmetry we may replace f with h if necessary so that $\mathscr{A}_k^0$ is semi-abelian. Then $\ker f_k^0$ is contained in the $\ker([n]_A)_k^0 = \ker[n]_{\mathscr{A}_k^0}$, which is finite. Since $\dim \mathscr{A}_k^0 = \dim \mathscr{A}_k = \dim \mathscr{A}_K = \dim A = \dim B = \dim \mathscr{B}_k^0$, f_k^0 is therefore an isogeny. In particular, $\mathscr{B}_k^0$ inherits the semi-abelian property from $\mathscr{A}_k^0$.

Now we prove (2). The isogeny over k implies equality of dimensions, so $\dim A = \dim B$. Hence, to deduce that f is an isogeny it suffices to prove that it is quasi-finite. Consider the "quasi-finite locus" $U \subseteq \mathscr{A}$ consists of $a \in \mathscr{A}$ that are isolated in $f^{-1}(f(a))$. This locus is non-empty since by hypothesis it contains $\mathscr{A}_k^0$. By semicontinuity of fiber dimension for finite type maps between noetherian schemes, U is open. Since $\mathscr{A}$ is R-flat, it follows that U meets the generic fiber $A = \mathscr{A}_K$. Hence, $f : A \to B$ has an isolated point in some fiber, so by homogeneity f has finite fibers. ■

Recall that an abelian variety A over K has *semistable reduction* if $\mathscr{A}_k^0$ is semi-abelian (with $\mathscr{A}$ denoting the Néron model of A over R). We may a similar definition when R is semi-local Dedekind (requiring the identity component of the fiber of $\mathscr{A}$ at each closed point of $\operatorname{Spec} R$ to be semi-abelian). Our eventual goal is to use the semistable reduction theorem for curves to prove:

Theorem 4.2 (Semistable reduction for abelian varieties)**.** *For an abelian variety A over K, there exists a finite separable extension K'/K such that $A_{K'}$ has semistable reduction over the integral closure R' of R in K.*

Note that by separability of K'/K, R' is module-finite over R. When R is complete, or more generally henselian, then R' is again local. However, in general R' is merely semi-local (and Dedekind). We want to apply the theory over number fields, so insisting on the henselian (or completeness) hypothesis is unwise (though it is largely harmless, due to Krasner's Lemma and the compatibility of the formation of Néron models with respect to local base change to the completion or henselization).

We will also prove Grothendieck's inertial criterion for semistable reduction, which generalizes the Néron–Ogg–Shafarevich criterion for good reduction. For now, we show that to prove Theorem 4.2, it suffices to consider the case of Jacobians:

Proposition 4.3. *To prove the semistable reduction theorem over K, it suffices to treat the special case of Jacobians of smooth and geometrically connected proper curves X over K.*

We refer the reader to [Mi2] for a general discussion of the theory of Jacobian varieties.

Proof. Let A be an abelian variety over K. By using a projective embedding of A over K and Bertini-style slicing arguments, it is a classical fact (see [Mi2, Thm. 10.1], which applies since K is infinite) that there exists a smooth and geometrically connected proper curve X over K such that A is a quotient of its Jacobian J_X over K. By Poincaré reducibility over K, we get an isogeny $A \times B \sim J_X$ for an abelian variety B over K. By hypothesis there is a finite separable extension K'/K such that $(J_X)_{K'}$ has semistable reductive over the integral closure R' of R in K'. By Proposition 4.1 applied over each of the finitely many localizations of R' at a maximal ideal, $(A \times B)_{K'} = A_{K'} \times B_{K'}$ has semistable reduction over R'. But Néron models commute with products, and a (smooth) direct factor of a semiabelian variety is semiabelian (as a special case of Corollary 3.2). Hence, $A_{K'}$ has semistable reduction over R'. ■

Consider the task of proving the semistable reduction theorem for the Jacobian of a curve X over K (smooth, proper, and geometrically connected). We will later review the definition of and existence results concerning the Picard variety $\mathrm{Pic}^0_{X/K}$ (which is also explained in [Mi2] over a field); this is the dual of the Jacobian, so the two are isogenous (and in fact isomorphic, via the canonical autoduality of the Jacobian), so our problem is to find a finite separable extension K'/K such that $(\mathrm{Pic}^0_{X/K})_{K'} = \mathrm{Pic}^0_{X_{K'}/K'}$ has semistable reduction over R'. By the semistable reduction theorem for curves (!), there exists a finite separable extension K'/K such that $X_{K'} = \mathscr{X}' \otimes_{R'} K'$ for a projective flat morphism $\mathscr{X}' \to \mathrm{Spec}\, R'$ with $\mathscr{X}'$ regular and geometric closed fibers all semistable. (Note that such a map is necessarily its own Stein factorization, due to normality of R' and integrality and connectedness of the geometric generic fiber, so its closed fibers are automatically geometrically connected; see Lemma 7.1.)

How can we get a handle on the identity components of the special fibers of the "abstract" Néron model over R' of the abelian variety $\mathrm{Pic}^0_{X_{K'}/K'}$ over K'? A key idea, which goes back to work of Raynaud, is to construct the Néron model of $\mathrm{Pic}^0_{X_{K'}/K'}$ in another manner, more closely related to the geometry of regular proper R'-models $\mathscr{X}'$ of $X_{K'}$. In §7 we will discuss the definition of and representability results for the functor $\mathrm{Pic}^0_{\mathscr{X}'/R'}$ as a separated, smooth, finite type R'-group scheme. This lies very deep, for reasons explained in §7.

Once the scheme $\mathrm{Pic}^0_{\mathscr{X}'/R'}$ is in hand, its generic fiber is $\mathrm{Pic}^0_{\mathscr{X}'_{K'}/K'} = \mathrm{Pic}^0_{X_{K'}/K'}$ and its fiber at any closed point $s' \in \mathrm{Spec}\, R'$ is the Picard variety $\mathrm{Pic}^0_{\mathscr{X}'_{s'}/k(s')}$ of the semistable curve $\mathscr{X}'_{s'}$ over $k(s')$. The wonderful fact is that this is visibly intrinsic to $\mathscr{X}'_{s'}$, so we can try to study its properties using the geometry of $\mathscr{X}'_{s'}$. More generally, we can study the Picard variety of any semistable proper (and geometrically connected) curve C over a field F, and it will turn out that $\mathrm{Pic}^0_{C/F}$ is always *semi-abelian* (with abelian and toric parts related to the irreducible components and singularities of C). Hence, $\mathrm{Pic}^0_{\mathscr{X}'/R'}$ will be a *semi-abelian scheme* over R'. That will settle the semistable reduction theorem (conditional on the robust theory of Picard functors as just outlined) provided that the natural map $\mathrm{Pic}^0_{\mathscr{X}'/R'} \to N(\mathrm{Pic}^0_{X_{K'}/K'})$ is an open immersion (then necessarily recovering the identity component on fibers, so the Néron model has semiabelian fibral identity components, as desired). This open immersion property follows from the following useful result (applied over the local rings of R' at its maximal ideals):

Theorem 4.4. *Let $\mathscr{G}$ be a separated R-group scheme of finite type such that its open relative identity component $\mathscr{G}^0$ (i.e., the complement of the union of the non-identity components in the special fiber) is semi-abelian, and assume that its generic fiber A is an abelian variety. The natural map $\mathscr{G} \to N(A)$ is an open immersion.*

In particular, if $\mathscr{A}$ is a semi-abelian R-scheme and its generic fiber A is an abelian variety then the natural map $\mathscr{A} \to N(A)^0$ is an isomorphism.

Before we prove Theorem 4.4, let's put the result in context by discussing its application to base change of Néron models. Consider a local extension $R \to R'$ of discrete valuation rings, inducing an extension $K \to K'$ on fraction fields. For any abelian variety A over K the R'-group $N(A)_{R'}$ is smooth and separated with generic fiber $A_{K'}$, so by the universal property of Néron models there is a unique R'-group map $N(A)_{R'} \to N(A_{K'})$. This is the *base change morphism* for Néron models relative to $R \to R'$. There is an induced map $N(A)^0_{R'} \to N(A_{K'})^0$ between relative identity components. (Note that $(N(A)_{R'})^0 = (N(A)^0)_{R'}$ because of the geometric connectedness in Lemma 2.5, so the notation $N(A)^0_{R'}$ is unambiguous.) There is also an induced map

$$\pi_0(N(A)_{k_s}) = N(A)_{k_s}/N(A)^0_{k_s} \to N(A_{K'})_{k'_s}/N(A_{K'})^0_{k'_s} = \pi_0(N(A_{K'})_{k'_s})$$

between the geometric component groups, and this is visibly injective if the base change morphism between identity components is an isomorphism. (Note that the component groups are unaffected by replacing k_s with $\overline{k}$.)

The base change morphism and its induced effect on relative identity components and geometric component groups give a natural meaning to the questions: does the Néron model commute with base change? How about its identity component or component group? The phenomenon of bad reduction that is potentially good shows that all of these questions have a negative answer in general. However, in the semistable case things work out very nicely for the identity component:

Corollary 4.5. *If A has semistable reduction, then the natural map $N(A)^0_{R'} \to N(A_{K'})^0$ is an isomorphism.*

Proof. Let $\mathscr{A} = N(A)^0$, so $\mathscr{A}_{R'}$ is a semi-abelian R'-scheme with generic fiber $A_{K'}$. By Theorem 4.4 the natural map $\mathscr{A}_{R'} \to N(A_{K'})$ is an isomorphism onto $N(A_{K'})^0$. This is exactly the map we wanted to prove to be an isomorphism. ■

To summarize, the identity component is "preserved" under base change once we are in the semistable case. Hence, under any further base change all that can happen is that the component group of the special fiber may "grow". More specifically, if we extend the place v on K to a place v_s fixed separable closure K_s (or what comes to the same, replace R with its henselization), then as K' varies through the finite extensions of K contained in K_s and we equip K' with the restriction v' of v_s (and let R' be the corresponding discrete valuation ring) then the component groups $\pi_0(N(A_{K'})_{\overline{k}})$ form a directed system with injective transition maps (due to Theorem 4.4, where there are no connectedness hypotheses on the special fiber of $\mathscr{G}$!). Thus, we can ask about the structure of the direct limit.

Example 4.6. Consider an elliptic curve $A = E$ with split multiplicative reduction over a complete discrete valuation ring R with uniformizer π. The relationship between Néron models and minimal regular proper models of elliptic curves implies that the minimal regular proper model $\mathscr{E}$ of E over R is an n-gon of $\mathbf{P}^1_k$'s arranged in a loop, where $n = \operatorname{ord}(q_E) = -\operatorname{ord}(j)$, and the étale local equation on $\mathscr{E}$ at each singularity of the special fiber is given by $xy = \pi$. The R-smooth locus of the proper R-flat $\mathscr{E}$ is the Néron model of E, and its special fiber is $\mathbf{G}_m \times (\mathbf{Z}/n\mathbf{Z})$ as a k-group. In particular, $\pi_0(N(E)_{\overline{k}}) = \mathbf{Z}/n\mathbf{Z}$.

Now consider a finite separable base change K'/K with ramification degree e, so on $\mathscr{E}_{R'}$ the étale local equation at the singularities of the special fiber is $xy = u'\pi'^e$ for some $u' \in R'^\times$ and uniformizer π' of R'. This is non-regular if $e > 1$. A blow-up calculation (which is local for the étale topology!) shows that blowing up $\mathscr{E}_{R'}$ at a singularity in the special fiber creates a new projective line linking the two passing through the chosen singularity, and if $e = 2$ then it is regular whereas if $e > 2$ then at one of the new singularities the local equation is $xy = u'\pi'^{e-2}$ while at the other it is regular. Thus, after $[e/2]$ blow-ups over each singularity on $\mathscr{E}_{R'}$ we reach a regular scheme, and this has special fiber that is a loop of $\mathbf{P}^1_{k'}$'s: we have inserted $e-1$ new lines at each of the n original singularities, so the number of lines now is $n + n(e-1) = ne$. The geometry of a loop of projective lines implies that it is the minimal regular proper model of its generic fiber, so we have reached the Néron model of $E_{K'}$ (which also has split multiplicative reduction).

The map $N(E)_{R'} \to N(E_{K'})$ does *not* extend to a morphism from $\mathscr{E}_{R'}$ to the minimal regular proper model of $E_{K'}$ when $e > 1$ (as otherwise by properness and dominance it would be surjective, contradicting the number of projective lines in the two special fibers). By uniqueness, this map of Néron models must be the inclusion into the complement of the exceptional locus of the blow-up, so the induced map on component groups is the natural map $\mathbf{Z}/n\mathbf{Z} \to \mathbf{Z}/(ne)\mathbf{Z}$ corresponding to multiplication by e; equivalently, it is the natural inclusion $(1/n)\mathbf{Z}/\mathbf{Z} \to (1/ne)\mathbf{Z}/\mathbf{Z}$. Passing to the limit, we get $\mathbf{Q}/\mathbf{Z}$.

In view of the preceding example, it is not surprising that Serre conjectured in general that for A with semistable reduction, the limit of the component groups $\pi_0(N(A_{K'}))$ is $\mathrm{Hom}(\mathrm{X}(T_{\overline{k}}), \mathbf{Q}/\mathbf{Z})$ where T is the maximal torus in the semi-abelian $N(A)^0_k$. (Of course, by the semistable reduction theorem the same is true without assuming semistable reduction, by using the maximal torus in the "common" identity components $N(A_{K'})^0_{\overline{k}}$ for K'/K sufficiently large.) This conjecture was proved by Grothendieck in [SGA7, Exp. IX, 11.9]. By far the hardest part of the proof is to control the p-part of the component group when $\mathrm{char}(k) = p > 0$; for this, Grothendieck uses very sophisticated group scheme constructions.

The moral of the story is that once we reached the semistable case, the component group continues to grow a lot, just as in the case of elliptic curves.

The preceding considerations were all conditional on Theorem 4.4, so we end this section by proving the theorem. This will require a number of new ideas and constructions that will be useful in our further study of properties of semistable reduction, so the effort is worthwhile. Let $N = N(A_K)$. The canonical map $f : \mathscr{G} \to N$ is an isomorphism between K-fibers, and we note that since N is R-flat it is irreducible. In particular, N is connected even though N_k may be disconnected. Also note that N is normal, since it is R-smooth. Thus, both $\mathscr{G}$ and N are integral R-schemes. The essential step is to prove that f_k has finite kernel, so f_k has finite fibers and hence f is *quasi-finite*. (This makes essential use of the semi-abelian hypothesis. For example, the map $[p]_E$ on an elliptic curve with additive reduction in residue characteristic $p > 0$ reduces to the zero map on the special fiber of the Néron model.)

Let's grant the quasi-finiteness of f and see how to conclude that f is an open immersion. What is the general nature of a quasi-finite map? If we begin with a finite map and remove a closed subset then we get a quasi-finite separated map. Remarkably, they essentially always arise this way:

Theorem 4.7 (Zariski's Main Theorem)**.** *If $h : X \to S$ is a quasi-finite and separated map between noetherian schemes, then it factors as an open immersion $j : X \hookrightarrow \overline{X}$ into a finite S-scheme.*

This result is [EGA, IV_3, 8.12.6] (or see [EGA, IV_4, 18.12.13] for Deligne's ultimate generalization without noetherian hypotheses), and it relies on virtually everything that precedes it in EGA IV. In Raynaud's book on henselian rings, he gives a more direct proof. Even in the case that S is the spectrum of a discrete valuation ring, this result is not obvious.

We now apply Zariski's Main Theorem to the map f, arriving at a factorization

$$\begin{array}{ccc} \mathscr{G} & \xrightarrow{j} & \mathscr{Y} \\ & {}_{f}\searrow & \downarrow \\ & & N \end{array}$$

where j is an open immersion and $\mathscr{Y}$ is N-finite. We may replace $\mathscr{Y}$ with the schematic closure of $\mathscr{G}$, so $\mathscr{Y}$ is irreducible and $\mathscr{O}_{\mathscr{Y}}$ is a subsheaf of $j_*(\mathscr{O}_{\mathscr{G}})$. In particular, $\mathscr{Y}$ is also reduced, hence integral, and it is R-flat too (since R is a discrete valuation ring, so flatness is the same as being torsion-free). Since f_K is an isomorphism and j_K is an open immersion into the integral $\mathscr{Y}_K$, it follows that j_K is an isomorphism. Thus, $\mathscr{Y} \to N$ is a finite surjective map between integral schemes and it is an isomorphism on K-fibers. But N is *normal* (even regular), since it is smooth over R, so this map must be an isomorphism. It follows that f is an open immersion! Hence, we just have to prove that f is quasi-finite, or equivalently that the k-homomorphism f_k is quasi-finite.

Remark 4.8. This same argument shows that a birational proper map between normal quasi-projective varieties over a field cannot be quasi-finite if it is not an isomorphism. That is, some fiber must have positive dimension. This was the viewpoint through which Zariski thought about his "Main Theorem". See Mumford's Red Book for an illuminating discussion of various results called "Zariski's Main Theorem" and how they are related.

It remains to prove that the closed subgroup scheme $H := \ker f_k$ in $\mathscr{G}_k$ is finite. Since $\mathscr{G}_k/\mathscr{G}_k^0$ is k-finite, it is equivalent to prove that $H \cap \mathscr{G}_k^0$ is finite. But $\mathscr{G}_k^0$ is semi-abelian, so by Corollary 3.3 it suffices to prove that $H[\ell] = 0$ for some prime ℓ. In other words, we aim to prove that the natural map

$$\mathscr{G}_k[\ell] \to N[\ell]$$

is a closed immersion for some prime ℓ. It is harmless to make a base change to a strict henselization R^{sh}, so now R is henselian and $k = k_s$.

Choose $\ell \neq \mathrm{char}(k)$, so $\mathscr{G}_k[\ell]$ is a finite étale k-scheme and thus is a constant group (as $k = k_s$). Hence, our task (with the choice of ℓ just made) is equivalent to proving that $\mathscr{G}(k)[\ell] \to N[\ell](k)$ is injective. The multiplication maps $\ell : \mathscr{G} \to \mathscr{G}$ and $\ell : N \to N$ between smooth separated R-groups of finite type are étale on fibers (as $\ell \neq \mathrm{char}(k)$), so their kernels $\mathscr{G}[\ell]$ and $N[\ell]$ are quasi-finite *étale* R-groups.

Consider the commutative diagram

$$\begin{array}{ccc} \mathscr{G}[\ell](K) & = & N[\ell](K) \\ \uparrow & & \| \\ \mathscr{G}[\ell](R) & \longrightarrow & N[\ell](R) \\ \downarrow & & \downarrow \\ \mathscr{G}[\ell](k) & \longrightarrow & N[\ell](k) \end{array}$$

in which the top horizontal equality is due to the equality of generic fibers $\mathscr{G}_K = N_K$ (via f_K), the upper vertical maps are injective due to the separatedness of $\mathscr{G}$ and N over R, and the bottom horizonal map is what we wish to be injective. It follows from commutativity that the middle horizontal map is injective, so we'll be done provided that the lower vertical maps are equalities. In other words, we are reduced to proving:

Lemma 4.9. *Let X be a quasi-finite separated étale scheme over a henselian local ring R. The natural map $X(R) \to X(k)$ is bijective.*

To prove this lemma, as a warm-up we consider the case when X is finite étale, and then we reduce the general case to this case. In the R-finite case, we invoke a basic fact from the theory of henselian local rings: a finite R-scheme is a disjoint union of finitely many finite local R-schemes. This is [EGA, IV_4, 18.5.11, 18.5.13], and is probably proved in a more direct way in Raynaud's book on henselian local rings; I don't remember. (For complete local noetherian R this is the familiar fact [Mat, 8.15] that every module-finite algebra over a complete local noetherian ring is a product of finitely many such local algebras. That proof involves lifting the primitive idempotents from the special fiber, which is an instance of Hensel's Lemma in the upstairs algebra relative to the étale equation $t^2 - t = 0$.) Hence, in the finite étale case

$$X = \coprod \operatorname{Spec} R_i$$

for local finite étale algebras $R \to R_i$. Letting k_i denote the corresponding residue field, we have $X_k = \coprod \operatorname{Spec} k_i$. The set $X(R)$ corresponds to those R_i equal to R (since the existence of an R-algebra section $R_i \to R$ to an *étale* R-algebra forces a ring-theoretic splitting $R_i = R \times R'_i$, and hence by locality of R_i we must have $R'_i = 0$).

We conclude (for X finite étale over R) that the map $X(R) \to X(k)$ is bijective provided that $R_i = R$ if and only if $k_i = k$. When R is a complete discrete valuation ring, this is the classical classification of finite unramified extensions in terms of the residue field extension. In general, the theory of henselian local rings ensures that the functor $R' \rightsquigarrow k'$ from local finite étale R-algebras to finite separable extensions of k is an equivalence of categories. In particular, if $k_i = k$ then $R_i = R$.

How can we bootstrap from the finite étale case to the general case? For this we invoke an important consequence of Zariski's Main Theorem:

Theorem 4.10 (Structure theorem for quasi-finite morphisms)**.** *Let X be quasi-finite and separated over a henselian local ring R. There is a unique decomposition $X = X_{\mathrm{f}} \coprod X_\eta$ where X_{f} is R-finite and X_η has empty special fiber.*

The formation of the "finite part" X_{f} is functorial in X and commutes with products, so it is an R-subgroup when X is an R-group.

Before deducing this structure theorem from Zariski's Main Theorem, we make some remarks. The relevance of this structure theorem is that by connectedness of $\operatorname{Spec} R$ we have $X_\eta(R) = \emptyset$, so $X(R) = X_{\mathrm{f}}(R)$. Since X_{f} is R-étale when X is (as X_{f} is an open subscheme of X), the structure theorem does indeed complete the reduction to the settled finite étale case.

The structure theorem for quasi-finite morphisms is a very deep fact, even when R is a discrete valuation ring. Here is an illustration of its meaning in an elementary (perhaps too elementary?) example:

Example 4.11. Let R be a (henselian) discrete valuation ring and $X \subseteq (\mathbf{Z}/n\mathbf{Z})_R$ be the open R-subgroup obtained by removing the nonzero points from the special fiber. Then X_{f} is the zero section and X_η consists of the nonzero K-points on the generic fiber. (This is a weak example, since it works without the henselian condition on R. To see the necessity of a henselian hypothesis, consider the localization of $\mathbf{Z}[\sqrt{2}]$ at one of the two primes over 7. That is a quasi-finite étale algebra over the non-henselian local $\mathbf{Z}_{(7)}$ and it does not decompose as in the structure theorem.)

This weak example also shows that X_η is generally not functorial in X: consider the zero map of X as an R-group.

Finally, we probe the structure theorem for quasi-finite morphisms.

Proof. Granting the existence of the proposed decomposition of X, let's show that the finite part is unique, functorial, and compatible with products (and we'll also see why X_η is not compatible with products). To establish the uniqueness and functoriality of X_{f}, we again use (as we did above) the non-obvious fact (which is elementary for complete local noetherian base rings) that a finite scheme over a henselian local algebra is a disjoint union of finitely many *local* schemes $X_i = \operatorname{Spec} R_i$. Consider an R-morphism $X \to Y$ between quasi-finite separated R-schemes. To prove that X_{f} lands in Y_{f} we may replace X with each connected component X_i of X_{f} to reduce to the case when X is R-finite and *local*. In particular, X is connected, so it lands in either Y_{f} or Y_η. It cannot land in Y_η since then its unique closed point would have nowhere to go (as Y_η has empty special fiber). This proves the uniqueness (taking $Y = X$ with perhaps a different decomposition, and $X \to Y$ to be the identity map), as well as the functoriality.

The compatibility with products is a calculation: if Y is another quasi-finite separated R-scheme then

$$\begin{aligned} X \times Y &= (X_{\mathrm{f}} \coprod X_\eta) \times (Y_{\mathrm{f}} \coprod Y_\eta) \\ &= (X_{\mathrm{f}} \times Y_{\mathrm{f}}) \coprod (X_{\mathrm{f}} \times Y_\eta \coprod X_\eta \times Y_{\mathrm{f}} \coprod X_\eta \times Y_\eta). \end{aligned}$$

By inspection, the three final terms in this disjoint union have empty special fiber whereas $X_{\mathrm{f}} \times Y_{\mathrm{f}}$ is R-finite, so by uniqueness this must be the decomposition of $X \times Y$ according to the structure theorem. In particular, $(X \times Y)_{\mathrm{f}} = X_{\mathrm{f}} \times Y_{\mathrm{f}}$.

Finally we come to the heart of the matter, which is to construct the desired decomposition of X. By Zariski's Main Theorem, X admits an open immersion into a *finite* R-scheme $\overline{X}$.

Since R is henselian, $\overline{X} = \coprod \operatorname{Spec} R_i$ for finite local R-algebras R_i. Thus, $X = \coprod X_i$ for the open subschemes $X_i = X \cap \operatorname{Spec} R_i$ in X. Since R_i is local, for each i either $X_i = \operatorname{Spec} R_i$ or X_i has empty special fiber (and not both!). Take X_{f} to be the disjoint union of the X_i for which $X_i = \operatorname{Spec} R_i$, and take X_η to be the disjoint union of the X_i with empty special fiber. This completes the proof of Theorem 4.4. ■

5. Applications of the semistable reduction theorem

We now grant the semistable reduction theorem (for abelian varieties) as a "black box", much as everyone treated the existence of Néron models for many years, and we will deduce from it many interesting results. So we fix an abelian variety A of dimension $g > 0$ over the fraction field K of a discrete valuation ring R with residue field k.

Our initial discussion will proceed under the assumption that A has semistable reduction; i.e., its Néron model $\mathscr{A}$ has semi-abelian identity component $\mathscr{A}_k^0$ in its special fiber. In particular, we will not actually make logical use of the semistable reduction theorem right away. But it will come up later.

By Theorem 3.1 there is an exact sequence of k-groups

$$0 \to T \to \mathscr{A}_k^0 \to B \to 0$$

with T a torus and B an abelian variety. Define $t = \dim T$ and $a = \dim B$, so $g = a + t$. Define $\Phi = \mathscr{A}_k/\mathscr{A}_k^0$; this is a finite étale k-group.

Since the dual abelian variety A' is K-isogenous to A, it also has semistable reduction (Proposition 4.1) and a choice of K-isogeny $A' \to A$ induces an isogeny $\mathscr{A}'^0_k \to \mathscr{A}_k^0$ and thus isogenies $T' \to T$ and $B' \to B$ between toric and abelian parts (Proposition 2.8 applied to $\overline{k}$-fibers). In particular, the numerical parameters t' and a' for A' coincide with the ones for A: $t' = t$ and $a' = a$.

Remark 5.1. It is natural to wonder if there is a canonical duality relationship between B' and B, as well as between the Galois lattices $\mathrm{X}(T'_{k_s})$ and $\mathrm{X}(T_{k_s})$. The answer is affirmative, and is developed in SGA7, but the justification is very lengthy; it involves a digression into the theory of bi-extensions. We will have no need for these refined facts, so we say nothing more on the matter.

The filtration on $\mathscr{A}_k^0$ via its toric and abelian parts does not lift to $\mathscr{A}$. However, we will now show how to use the structure theorem for quasi-finite morphisms to carry out such lifting at the level of *torsion subgroups* when R is henselian (e.g., complete). This will have striking and important consequences for the structure of the Tate modules of A as Galois modules, vastly generalizing the familiar Galois-module structure of Tate curves (the ur-example of semistable reduction).

Fix an integer $N \geq 1$. (We allow the possibility that $\mathrm{char}(k)|N$, since we want to permit N to be a power of p when K is a p-adic field.) Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T & \longrightarrow & \mathscr{A}_k^0 & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle [N]} & & \downarrow{\scriptstyle [N]} & & \downarrow{\scriptstyle [N]} & & \\
0 & \longrightarrow & T & \longrightarrow & \mathscr{A}_k^0 & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

The rows are exact sequences in the abelian category of sheaves for the fppf topology on the category of k-schemes (or even just k-schemes of finite type), and the left vertical map is an epimorphism (as it is an isogeny, since T is a torus), so we conclude via the snake lemma that the natural left exact sequence of finite commutative k-group schemes

$$0 \to T[N] \to \mathscr{A}_k^0[N] \to B[N] \to 0 \tag{5.1}$$

is a short exact sequence (since it is equivalent to short exactness in the sense of fppf abelian sheaves). This proves:

Lemma 5.2. *For $N \geq 1$, the finite k-group $\mathscr{A}_k^0[N]$ has order N^{t+2a}.*

We now make several observations about the terms in (5.1). The left term $T[N]$ has étale Cartier dual (as we may check over $\overline{k}$, where $T[N]$ becomes a power of μ_N). The middle term is an open subgroup of $\mathscr{A}_k[N]$, and the étale quotient $\mathscr{A}_k[N]/\mathscr{A}_k^0[N]$ is contained in $\Phi[N] \subseteq \Phi$, so its order is bounded *independently of* N. This will be very useful when we pass to limits over increasing N (in the multiplicative sense).

Assume R is henselian, so by the structure theorem for quasi-finite morphisms

$$\mathscr{A}[N] = \mathscr{A}[N]_{\mathrm{f}} \coprod \mathscr{A}[N]_\eta$$

with $\mathscr{A}[N]_{\mathrm{f}}$ finite over R and having special fiber $\mathscr{A}[N]_k = \mathscr{A}_k[N]$. Observe that $\mathscr{A}[N]$ is a *flat* R-group since $[N] : \mathscr{A} \to \mathscr{A}$ is flat due to the fibral flatness criterion and the fact that $[N]_k$ is flat (thanks to its surjectivity on the *semi-abelian* $\mathscr{A}_k^0$). Hence, $\mathscr{A}[N]_{\mathrm{f}}$ is a *finite flat* R-group. The order of this R-group is not a nice function of N, since its special fiber $\mathscr{A}_k[N]$ may be a bit bigger than the group $\mathscr{A}_k^0[N]$ of order N^{t+2a}: it is influenced by $\Phi[N]$ too.

Remark 5.3. If $\mathrm{char}(k) \nmid N$ then the Galois module $\mathscr{A}[N]_{\mathrm{f}}(K_s) \subseteq A[N](K_s)$ is exactly the *unramified* submodule (i.e., the K^{un}-points). To see this we may make scalar extension to R^{sh} (as that is compatible with the formation of the Néron model) so that $k = k_s$ and finite étale R-schemes are constant (i.e., a disjoint union of copies of $\mathrm{Spec}\, R$). Since $\mathscr{A}[N]_{\mathrm{f}}$ is finite étale (by the hypothesis $N \in R^\times$), our problem is to prove that $\mathscr{A}[N]_{\mathrm{f}}(K) = \mathscr{A}[N](K)$. By the *Néron mapping property*, it is equivalent to check that $\mathscr{A}[N]_{\mathrm{f}}(R) = \mathscr{A}[N](R)$. But this is immediate from the very definition of the "finite part" (since $\mathrm{Spec}\, R$ is connected)!

Consider the unique open finite R-subgroups

$$\mathscr{A}[N]_{\mathrm{t}} \subseteq \mathscr{A}[N]_{\mathrm{f}}^0 \subseteq \mathscr{A}[N]_{\mathrm{f}}$$

which are *defined* to be the respective lifts of the open and closed k-subgroups $T[N]$ and $\mathscr{A}_k^0[N]$ inside of $\mathscr{A}_k[N]$. (Here we use the decomposition of a finite R-scheme into its local parts to see that k-subgroups of the special fiber uniquely lift to open and closed R-subgroups.) The finite flat R-group $\mathscr{A}[N]_{\mathrm{t}}$ has étale Cartier dual, since its special fiber $T[N]$ has that property. Despite the notation, $\mathscr{A}[N]_{\mathrm{t}}$ and $\mathscr{A}[N]_{\mathrm{f}}^0$ are not intrinsic to $\mathscr{A}[N]$; they rely on the ambient $\mathscr{A}$. We are going to use these finite flat R-groups with N increasing through powers of a prime ℓ in order to get a handle on the ℓ-adic Tate module of A when $\ell \neq \mathrm{char}(K)$ and the ℓ-divisible group of A when $\ell = \mathrm{char}(K)$.

Clearly the orders of $\mathscr{A}[N]_{\mathrm{t}}$ and $\mathscr{A}[N]_{\mathrm{f}}^0$ are N^t and N^{t+2a} respectively. The following is then almost immediate from the definitions (and the details are left to the reader; keep in mind that $\#\Phi$ is finite, so Remark 5.3 yields an analogous result using the inverse limit of the $\mathscr{A}[\ell^n]^0_{\mathrm{f}}(K_s)$'s):

Lemma 5.4. *For any prime ℓ, the directed systems $\{\mathscr{A}[\ell^n]_{\mathrm{t}}\}$ and $\{\mathscr{A}[\ell^n]^0_{\mathrm{f}}\}$ are ℓ-divisible groups over R of heights t and $t+2a$. Viewing their generic fibers inside of the ℓ-divisible group of A_K, if $\ell \neq \mathrm{char}(K)$ then these yield $\mathrm{Gal}(K_s/K)$-stable saturated $\mathbf{Z}_\ell$-submodules*

$$\mathrm{T}_\ell(A)_{\mathrm{t}} \subseteq \mathrm{T}_\ell(A)_{\mathrm{f}} \subseteq \mathrm{T}_\ell(A).$$

If $\ell \neq \mathrm{char}(k)$ then $\mathrm{T}_\ell(A)_{\mathrm{f}} = \mathrm{T}_\ell(A)^{I_K}$ is the inertial fixed part.

In the saturated filtration of $\mathrm{T}_\ell(A)$ by its "toric part" and "finite part" for $\ell \neq \mathrm{char}(K)$, observe that the toric part has $\mathbf{Z}_\ell$-rank t and the finite part has corank $2g-(2t+a)=t$. This becomes interesting when it is combined with the duality of abelian varieties. For the dual A' of A and a prime $\ell \neq \mathrm{char}(K)$, we also have a toric part $\mathrm{T}_\ell(A')_{\mathrm{t}}$ and a finite part $\mathrm{T}_\ell(A')_{\mathrm{f}}$. Consider the perfect Weil pairing

$$\mathrm{T}_\ell(A) \times \mathrm{T}_\ell(A') \to \mathbf{Z}_\ell(1).$$

The finite part $\mathrm{T}_\ell(A)_{\mathrm{f}}$ is $\mathbf{Z}_\ell$-saturated with corank t in $\mathrm{T}_\ell(A)$, so its annihilator in $\mathrm{T}_\ell(A')$ is a canonical saturated $\mathbf{Z}_\ell$-submodule of rank t. It is only natural to guess that this annihilator must be the toric part $\mathrm{T}_\ell(A')_{\mathrm{t}}$. This is confirmed by the following result of Grothendieck which is used in Faltings' paper:

Theorem 5.5 (Orthogonality theorem, semistable case)**.** *Under the Weil pairing as above, $\mathrm{T}_\ell(A)_{\mathrm{f}}$ and $\mathrm{T}_\ell(A')_{\mathrm{t}}$ are exact annihilators of each other. In particular, $\mathrm{T}_\ell(A)/\mathrm{T}_\ell(A)_{\mathrm{f}}$ is Cartier dual to $\mathrm{T}_\ell(A')_{\mathrm{t}}$ and hence has trivial I_K-action (as $\mathscr{A}'[N]_{\mathrm{t}}$ has étale Cartier dual for all $N \geq 1$).*

Remark 5.6. An interesting consequence of this theorem arises for $\ell \neq \mathrm{char}(K)$ when we describe the action of $\mathrm{Gal}(K_s/K)$ on $\mathrm{T}_\ell(A)$ by choosing a basis whose initial members lie in $\mathrm{T}_\ell(A)_{\mathrm{f}}$. By Remark 5.3 and the final part of the orthogonality theorem, we get the matrix block description of the Galois action as $\left(\begin{smallmatrix}\rho_\ell & * \\ 0 & 1\end{smallmatrix}\right)$ where ρ_ℓ arises from the generic fiber of an ℓ-divisible group over R. In particular, if $\ell \neq \mathrm{char}(k)$ then ρ_ℓ is unramified, so for such ℓ the I_K-action is "unipotent of height ≤ 2": $(\sigma-1)^2=0$ for all $\sigma \in I_K$. This is a vast generalization of the description of Tate modules of Tate curves.

The proof of the orthogonality theorem relies on Tate's theorem on p-divisible groups in case $\ell = \mathrm{char}(k) \neq \mathrm{char}(K)$. The interested reader who accepts Tate's theorem in the equicharacteristic case (which was proved by deJong) can readily adapt the statement and our proof of the orthogonality theorem to the case $\ell = \mathrm{char}(k) = \mathrm{char}(K) > 0$ by using ℓ-divisible groups in place of ℓ-adic Tate modules. In [SGA7, Exp. IX, 2.4, 5.2] Grothendieck does this, and he also gives a version of the orthogonality theorem for $\ell \neq p$ without semistability; we have no need for that and hence do not discuss it. Grothendieck's approach to proving the orthogonality theorem even for $\ell \neq p$ in the semistable case is via his elaborate theory of bi-extensions, which we definitely do not want to get into. So we will give a different proof, inspired by Deligne's Appendix to [SGA7, Exp. I], though still ultimately relying on Tate's theorem in case $\ell = \mathrm{char}(k)$ as Grothendieck did.

Proof. In view of the the $\mathbf{Z}_\ell$-saturatedness and rank properties, it is equivalent to prove the weaker assertion that $\mathrm{T}_\ell(A)_{\mathrm{f}}$ and $\mathrm{T}_\ell(A')_{\mathrm{t}}$ annihilate each other under the ℓ-adic Weil pairing.

To treat the cases $\ell = \mathrm{char}(k)$ and $\ell \neq \mathrm{char}(k)$ in a uniform manner, we will generally express the ideas in terms of ℓ-divisible groups (which work well over K and R, for any ℓ) rather than with ℓ-adic Tate modules (which only work over K, especially when allowing $\ell = \mathrm{char}(k)$). Let $\Gamma = \{\mathscr{A}[\ell^n]^0_{\mathrm{f}}\}$ and $\Gamma' = \{\mathscr{A}'[\ell^n]_{\mathrm{t}}\}$ be the ℓ-divisible groups over R whose generic fibers respectively define $\mathrm{T}_\ell(A)_{\mathrm{f}}$ and $\mathrm{T}_\ell(A')_{\mathrm{t}}$. The $\mathbf{Z}_\ell(1)$-valued pairing between these can be expressed as a homomorphism from one to the Cartier dual of the other. That is, it is encoded in terms of a map of ℓ-divisible groups $f : \Gamma_K \to \mathbf{D}(\Gamma')_K$ over K; here, $\mathbf{D}(\cdot)$ denotes Cartier duality. By Tate's theorem (which applies since $\ell \neq \mathrm{char}(K)$, though it is trivial if $\ell \neq \mathrm{char}(k)$), f arises from a homomorphism $\Gamma \to \mathbf{D}(\Gamma')$ between ℓ-divisible groups over R. We will prove that the *only* homomorphism between these ℓ-divisible groups over R is the zero map, which will complete the proof.

By construction, Γ' lifts the Cartier dual of $T'[\ell^\infty]$ and Γ lifts $\mathscr{A}^0_k[\ell^\infty]$. We saw in Mike's lecture on p-divisible groups that the reduction map $\mathrm{Hom}_R(G', G) \to \mathrm{Hom}_k(G'_k, G_k)$ is injective for any ℓ-divisible groups G and G' over R (this is trivial if $\ell \neq \mathrm{char}(k)$), so we are reduced to proving the vanishing of $\mathrm{Hom}_k(\mathscr{A}^0_k[\ell^\infty], \mathbf{D}(T'[\ell^\infty]))$. At this point we focus on the case that k is finite, which is all that is needed for Faltings' paper. The problem becomes that of proving:

Lemma 5.7. *Let k be a finite field, G a semi-abelian variety over k, and T' a k-torus. For any prime ℓ,* $\mathrm{Hom}_k(G[\ell^\infty], \mathbf{D}(T'[\ell^\infty])) = 0$.

To prove this lemma we shall use a very powerful trick called "weight-chasing", which is rather effective at exploiting geometry over finite fields to deduce results in more general settings. The case of R with arbitrary residue field can still be reduced to Lemma 5.7, at the expense of considering local henselian integrally closed domains R with higher dimension. To be precise (since the theory of Néron models certainly does not work over a higher-dimensional normal base), when k is general we can use direct limit techniques from [EGA, IV_3, §8ff.] to descend the relative identity components of the Néron models to semiabelian schemes over the henselization at a maximal ideal of an integrally closed finite type $\mathbf{Z}$-subalgebra of R. We then use Tate's theorem over such a normal noetherian local domain and the faithfulness of the specialization functor on ℓ-divisible groups (which was proved in Mike's notes over any local noetherian base) to reduce the problem again to Lemma 5.7.

To prove Lemma 5.7, let T be the maximal torus of G and $B = G/T$ the abelian part (over k). We then have an exact sequence of ℓ-divisible groups

$$1 \to T[\ell^\infty] \to G[\ell^\infty] \to B[\ell^\infty] \to 1,$$

so it suffices to prove that any map from the outer terms to $\mathbf{D}(T'[\ell^\infty])$ must vanish.

Making a finite extension on k is harmless, so we may assume that T and T' are k-split. Hence, their ℓ-divisible groups are powers of μ_{ℓ^∞}, so we are reduced to proving the vanishing of $\mathrm{Hom}_k(\mu_{\ell^\infty}, \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ and $\mathrm{Hom}_k(B[\ell^\infty], \mathbf{Q}_\ell/\mathbf{Z}_\ell)$. For the first of these, we separately treat the cases $\ell \neq p := \mathrm{char}(k)$ and $\ell = p$. If $\ell \neq p$ then μ_{ℓ^∞} is étale so we can convert the problem into the language of ℓ-adic Tate modules viewed as Galois modules for the Galois group G_k of k: the assertion is that $\mathrm{Hom}_{\mathbf{Z}_\ell[G_k]}(\mathbf{Z}_\ell(1), \mathbf{Z}_\ell) = 0$. This is obvious, since the

ℓ-adic cyclotomic character of G_k is nontrivial. If instead $\ell = p$ then μ_{ℓ^n} is an infinitesimal k-group for all n, so it is clear that $\mathrm{Hom}_k(\mu_{\ell^\infty}, \mathbf{Q}_\ell/\mathbf{Z}_\ell) = 0$.

Next consider $\mathrm{Hom}_k(B[\ell^\infty], \mathbf{Q}_\ell/\mathbf{Z}_\ell)$; this is where "weights" enter in a substantial way. First assume $\ell \neq p$, so we can convert the problem into the language of ℓ-adic Tate modules: does $\mathrm{Hom}_{\mathbf{Z}_\ell[G_k]}(\mathrm{T}_\ell(B), \mathbf{Z}_\ell)$ vanish? Indeed it does, since the Riemann Hypothesis for abelian varieties over finite fields implies that the Frobenius action on $\mathrm{T}_\ell(B)$ does not admit 1 as an eigenvalue. (In fancier terms, the eigenvalues are Weil q-numbers of weight $1/2$, whereas 1 is a Weil q-number of weight 0; here, $q = \#k$). If $\ell = p$ then we convert the problem into the language of contravariant Dieudonné modules instead, and argue the same way: we have to prove that there are no nonzero $W(k)$-linear maps from $W(k)$ to the Dieudonné module of $B[p^\infty]$ that are compatible with the action of the absolute Frobenius operators. Even better, there are no such maps that are compatible with the $W(k)$-linear q-Frobenius operators ($q = \#k$). This is because the Riemann Hypothesis implies that the $W(k)$-linear q-Frobenius on the Dieudonné module of $B[p^\infty]$ does not have 1 as an eigenvalue (over $W(k)$). ■

Here is Grothendieck's inertial semistable reduction criterion (and we remind the reader that throughout this section we are accepting the semistable reduction theorem for abelian varieties as a black box; we will use it in the proof below):

Theorem 5.8. *Let A be an abelian variety over K, and choose a prime $\ell \neq \mathrm{char}(k)$. Fix a place of K_s extending the one on K arising from R, and let $I_K \subseteq G_K := \mathrm{Gal}(K_s/K)$ denote the resulting inertia subgroup. Then A has semistable reduction if and only if the I_K-action on $\mathrm{V}_\ell(A)$ is unipotent.*

The condition of unipotence of the I_K-action can be defined in either the pointwise sense (each $\sigma \in I_K$ acts as a unipotent automorphism) or the group-theoretic sense (the action of I_K can be strictly upper-triangularized). To prove that these definitions are equivalent, the problem is to show that the pointwise condition implies the group-theoretic one. Under the pointwise condition, the Zariski-closure U of the image of I_K in $\mathrm{GL}(\mathrm{V}_\ell(A))$ has all elements with characteristic polynomial $(X-1)^{2g}$ (as this holds for elements from I_K). Hence, U is a unipotent $\mathbf{Q}_\ell$-group in the sense of Jordan decomposition of its geometric points. In characteristic 0 such groups are always connected, so by [Bo, 15.5(ii)] we can strictly upper triangularize U over $\mathbf{Q}_\ell$.

Proof. The implication "⇒" was deduced above from the orthogonality theorem (where we got the stronger conclusion $(\sigma - 1)^2 = 0$ for all $\sigma \in I_K$). For the converse, we may replace R with R^{sh} so that $I_K = G_K$ and hence the G_K-action is unipotent. By the semistable reduction theorem (!), there is a finite separable extension K'/K inside of K_s such that the Néron model $\mathscr{A}'$ of $A_{K'}$ over the local integral closure R' of R in K' has special fiber with semi-abelian identity component. Thus, by Theorem 3.1 there is a short exact sequence

$$0 \to T' \to \mathscr{A}'^{0}_{k'} \to B' \to 0$$

of group varieties over the residue field k' of R', where T' is a torus and B' is an abelian variety.

The base change morphism $\mathscr{A}^0_{R'} \to \mathscr{A}'^0$ between the relative identity components of the Néron models is mysterious at the outset (since we do not yet know that A has semistable reduction). We will use the trick of descent of torsion points to descend T' (up to isogeny) to a k-torus in $\mathscr{A}^0_k$ such that $\mathscr{A}^0_k/T$ is an abelian variety. To do the descent through the possibly ramified extension K'/K, we have to exploit the unipotence hypothesis for the G_K-action in a clever way. The role of unipotence is to prove:

Lemma 5.9. *The inclusion* $\mathrm{V}_\ell(A)^{G_{K'}} \subseteq \mathrm{V}_\ell(A)^{G_K}$ *is an equality.*

Proof. For any $g \in G_K$, some power g^n lies in $G_{K'}$ with $n > 0$. Thus, it suffices to prove rather generally that if g is a unipotent automorphism of a finite-dimensional vector space V over a field of characteristic 0 then $g - 1$ and $g^n - 1$ have the same kernel.

Writing $g = 1 + N$ with nilpotent N, we have

$$g^n = 1 + nN + N^2(\cdots) = 1 + nN(1 + N(\cdots))$$

where the terms in $(\cdot)$ are linear combinations of powers of N. Thus, $g^n - 1 = nN(1+N(\cdots))$ where $1 + N(\cdots)$ is invertible. Since n acts invertibly (as we are in characteristic 0), we are done. ∎

The lemma implies that $A[\ell^n](K) = A[\ell^n](K')$ for all $n \geq 1$, so by the *Néron mapping property* we have $\mathscr{A}[\ell^n](R) = \mathscr{A}'[\ell^n](R')$ for all $n \geq 1$. Since $\ell \neq \mathrm{char}(k)$ and $k = k_s$, all finite flat commutative group schemes over R' of ℓ-power order are constant. Also, even though $\mathscr{A}^0_k$ is not yet known to be semiabelian, the endomorphism of multiplication by ℓ is étale, so $\ell : \mathscr{A} \to \mathscr{A}$ over R is étale. Hence, all $\mathscr{A}[\ell^n]$'s are quasi-finite étale over the strictly henselian R and thus $\mathscr{A}[\ell^n]_{\mathrm{f}}$ makes sense as a finite *constant* R-subgroup of $\mathscr{A}$ (though we have not defined a notion of "toric part" for it). We can likewise make sense of $\mathscr{A}[\ell^n]^0_{\mathrm{f}}$, though it will not be of any use to us.

It follows that $\mathscr{A}'[\ell^n]_{\mathrm{t}}$ and $\mathscr{A}'[\ell^n]^0_{\mathrm{f}}$ (viewed as finite constant R'-groups) descend into the finite constant R-group $\mathscr{A}[\ell^n]_{\mathrm{f}}$ (as all R-points of $\mathscr{A}[\ell^n]$ factor through the finite part!). Passing to the special fiber, inside of the mysterious $\mathscr{A}_k$ we have produced two constant ℓ-divisible subgroups $\Gamma_{\mathrm{t}} \subseteq \Gamma$ such that the special fiber $\mathscr{A}_{k'} \to \mathscr{A}'_{k'}$ of the base change morphism carries Γ_{t} *isomorphically* onto $T'[\ell^\infty]$ and carries Γ *isomorphically* onto $\mathscr{A}'^0_{k'}[\ell^\infty]$.

Let $T \subset \mathscr{A}^0_k$ denote the identity component of the Zariski closure in $\mathscr{A}_k$ of the points in $\Gamma_{\mathrm{t}}(k) \subset \mathscr{A}_k(k)$. (We do *not* yet know that this is a torus; it is just an abstract smooth connected k-subgroup of $\mathscr{A}^0_k$.) The formation of T commutes with extension on k, so by density reasons over k' the map $\mathscr{A}_{k'} \to \mathscr{A}'_{k'}$ carries $T_{k'}$ into T'. But the image of $T_{k'}$ in T' contains $T'[\ell^\infty]$, so $T_{k'}$ maps onto T'. It follows that $\dim T \geq \dim T'$, with equality if and only if $T_{k'} \twoheadrightarrow T'$ is an isogeny, in which case T must be a torus (by Proposition 2.16 applied over $\overline{k}$).

Consider the constant ℓ-divisible group Γ in $\mathscr{A}_k$. By finiteness of the component group of $\mathscr{A}_k$, Γ is contained in $\mathscr{A}^0_k$. Under the special fiber $f : (\mathscr{A}^0_k)_{k'} \to \mathscr{A}'^0_{k'}$ of the base change morphism, $\Gamma_{k'}$ is carried isomorphically onto $\mathscr{A}'^0_{k'}[\ell^\infty]$, which is Zariski-dense in the *semi-abelian* $\mathscr{A}'^0_{k'}$. Hence, f is surjective. But we have seen that $f(T_{k'}) = T'$, so the induced map $\overline{f} : (\mathscr{A}^0_k/T)_{k'} \to \mathscr{A}'^0_{k'}/T'$ is surjective (with target an abelian variety). Since

$$\dim(\mathscr{A}^0_k/T)_{k'} = \dim \mathscr{A}^0_k - \dim T = g - \dim T \leq g - \dim T' = \dim \mathscr{A}'^0_{k'}/T',$$

it follows that $\overline{f}$ must be an isogeny and $\dim T = \dim T'$. We conclude that T is a torus and $\mathscr{A}_k^0/T$ is an abelian variety. ■

6. Applications of Grothendieck's inertial criterion

We now illustrate the power of the inertial criterion for semistable reduction in Theorem 5.8 (whose proof was conditional on the semistable reduction theorem for abelian varieties, which remains to be proved in the special case of Jacobians). First, we show that semistable reduction enjoys many of the familiar properties of good reduction, by virtually the same proofs. Semistable reduction is insensitive to isogenies: this is rather elementary, as we saw in Proposition 4.1. The situation for exact sequences is deeper, since Néron models have poor behavior with respect to exact sequence of abelian varieties (apart from situations such as mixed characteristic $(0, p)$ with $e < p - 1$, to which Raynaud's work on finite flat group schemes can be applied). But Grothendieck's inertial criterion effortlessly settles such problems:

Proposition 6.1. *If $0 \to A' \to A \to A'' \to 0$ is a short exact sequence of abelian varieties over K then A has semistable reduction if and only if A' and A'' do.*

Proof. Consider the Galois-equivariant exact sequence of V_ℓ's for a prime $\ell \neq \mathrm{char}(k)$. Since unipotence of an automorphism of a finite-dimensional vector space amounts to the characteristic polynomial have 1 as its only root, it can be checked on the successive quotients of a flag stable under the automorphism. ■

Example 6.2. If A is an abelian variety over K and it is K-isogenous to $\prod A_i$ for abelian varieties A_i over K, then A is semistable if and only if all A_i are semistable. Indeed, since semistability is insensitive to isogenies we can assume $A = \prod A_i$, and then repeated applications of the proposition do the job.

Next we give an application to good reduction in the CM case, generalizing the classical case of CM elliptic curves. Recall that for an abelian variety X of dimension g over a field F, $\mathrm{End}_F^0(X)$ is a semisimple $\mathbf{Q}$-algebra of dimension at most $(2g)^2$, and the commutative semisimple $\mathbf{Q}$-subalgebras have $\mathbf{Q}$-dimension at most $2g$. When equality holds for some such subalgebra $L = \prod L_i$ (with fields L_i) we say that X has *sufficiently many complex multiplications*, and we call X a *CM abelian variety* over F. (In the literature, the distinction between satisfying this condition over F or over some extension is not always made clear. It certainly makes a difference; e.g., elliptic curves over $\mathbf{Q}$ cannot support an action by an imaginary quadratic field on their tangent line over $\mathbf{Q}$, but there are plenty of such elliptic curves that acquire CM over an extension field.) It is true but nontrivial over general F that in the CM case, the fields L_i can be arranged to all be CM fields. We will not need this fact.

Proposition 6.3. *An abelian variety that acquires sufficiently many complex multiplications over an extension of K becomes good at all places of any finite separable extension K'/K where it is everywhere semistable.*

By considering quadratic twists of elliptic curves, we see that having sufficiently many complex multiplications over K does not imply good reduction over K.

Proof. It is a general fact due to Chow that for an extension K' of K_s, the map $\mathrm{End}_{K_s}(A_{K_s}) \to \mathrm{End}_{K'}(A_{K'})$ is bijective; see [C2, Thm. 3.19] for a proof via descent theory. Thus, if A acquires sufficiently many complex multiplications over some extension of K then it does so over K_s and hence over a finite separable extension of K. Since the formation of the relative identity component of the Néron model commutes with finite base change on K in the semistable case (Corollary 4.5), semistable reduction that is potentially good must already be good (since we know from Sam's lecture that if the identity component of the special fiber of the Néron model is an abelian variety, then the Néron model is in fact an abelian scheme). It remains to show that if A of dimension $g > 0$ with sufficiently many complex multiplications has semistable reduction then it has good reduction.

Choose $L = \prod L_i \subseteq \mathrm{End}^0_K(A)$ with fields L_i such that $[L : \mathbf{Q}] = 2g$. The primitive idempotents in L define an isogeny decomposition $A \sim \prod A_i$ such that $L_i \subseteq \mathrm{End}^0_K(A_i)$, so $[L_i : \mathbf{Q}] \leq 2 \dim A_i$ for all i. But summing over i gives the equality $[L : \mathbf{Q}] = 2 \dim A$, so in fact each A_i has sufficiently many complex multiplications over K using the field L_i. It suffices to treat each A_i in place of A (they are all semistable, by Example 6.2), so we may and do assume that L is a field.

Let $\mathscr{A}$ be the Néron model, and let T be the maximal torus in $\mathscr{A}^0_k$. It suffices to prove $T = 0$. There is a natural ring homomorphism

$$\mathrm{End}^0_K(A) = \mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{End}_K(A) \to \mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{End}_R(\mathscr{A}) \to \mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{End}_k(T) \subseteq \mathrm{End}_{\mathbf{Q}}(\mathrm{X}(T_{k_s})^*_{\mathbf{Q}}),$$

so if $T \neq 0$ then we get a representation of the number field L of degree $2g$ on a nonzero $\mathbf{Q}$-vector space of dimension $\dim T \leq g$. This is absurd, so $T = 0$. ■

Here is a refinement of Proposition 4.1.

Proposition 6.4. *Let A and B be abelian varieties over K with semistable reduction. The natural map $\mathrm{Hom}_K(A, B) \to \mathrm{Hom}_k(\mathscr{A}^0_k, \mathscr{B}^0_k)$ is injective.*

Proof. By Proposition 4.1, it is harmless to change A and B by an isogeny. By Example 6.2, we can reduce to the case when A and B are K-simple. If A and B are not K-isogenous then $\mathrm{Hom}_K(A, B) = 0$ and there is nothing to do, so we may assume they are isogenous. Hence, we can even assume $B = A$. The semi-abelian hypothesis implies that $\mathrm{End}_k(\mathscr{A}^0_k)$ is torsion-free as an abelian group, so it is equivalent to check injectivity of the $\mathbf{Q}$-algebra homomorphism $\mathrm{End}^0_K(A) \to \mathrm{End}^0_k(\mathscr{A}^0_k)$ (i.e., injectivity after tensoring with $\mathbf{Q}$). But by K-simplicity, $\mathrm{End}^0_K(A)$ is a division algebra. Hence, it has no nonzero proper two-sided ideals, so we get the desired injectivity. ■

Proposition 6.5. *Pick $N \geq 3$ not divisible by* $\mathrm{char}(k)$. *Then A acquires semistable reduction at all places of the finite Galois splitting field $K(A[N])/K$ of $A[N]$.*

Proof. By replacing K with $K(A[N])$ and R with any of the localizations (at a maximal ideal) of its R-finite integral closure in $K(A[N])$, we may reduce to the case when $A[N]$ is K-split (i.e., has trivial Galois action). The aim is to prove that A has semistable reduction, so we can replace R with its henselization. Since N is divisible by 4 or an odd prime, we may replace N with such a divisor to arrange that $N = \ell$ is an odd prime or $N = \ell^2$ with $\ell = 2$. Either way, the ℓ-adic logarithm and exponential define inverse isomorphisms between $1 + N\overline{\mathbf{Z}}_\ell$ and $N\overline{\mathbf{Z}}_\ell$, so $1 + N\overline{\mathbf{Z}}_\ell$ contains no nontrivial roots of unity.

Consider the action on $\mathrm{T}_\ell(A)$ by an element $\sigma \in I_K$. By Grothendieck's criterion, it suffices to prove that the action by σ is unipotent. By the semistable reduction theorem and the easier direction of Grothendieck's criterion (i.e., the unipotence of the inertial action in the semistable case), for some $n > 0$ the action of σ^n is unipotent. But σ acts trivially on $A[N]$, so the action of σ is given by a matrix $1 + NM$ with $M \in \mathrm{Mat}_{2g}(\mathbf{Z}_\ell)$. Hence, the eigenvalues of σ have the form $1 + N\lambda$ for eigenvalues λ of M. Any such λ lies in $\overline{\mathbf{Z}}_\ell$, so the eigenvalues of σ lie in $1 + N\overline{\mathbf{Z}}_\ell$. But σ^n is unipotent, so these eigenvalues are roots of unity. Since $1 + N\overline{\mathbf{Z}}_\ell$ is torsion-free as a multiplicative group, it follows that all eigenvalues of σ are equal to 1, which is to say that σ acts unipotently. ■

The preceding result is quite striking, for the following reason. Consider an abelian variety A over the fraction field K of a Dedekind domain R, and choose $N \in \{12, 15, 20\}$ not divisible by $\mathrm{char}(K)$. For every maximal ideal of R, there is a factor of N distinct from its residue characteristic and at least 3. Hence, over $K(A[N])$ semistable reduction is attained at all places, yet $\mathrm{Gal}(K(A[N])/K)$ is a subgroup of $\mathrm{GL}_{2g}(\mathbf{Z}/N\mathbf{Z})$, so $[K(A[N]) : K]$ is bounded in terms of g alone (and N, which is an absolute constant). This is very interesting, since $K(A[N])/K$ is unramified away from the finite set S of bad places for A as well as the places dividing N (of which there can be only finitely many, since $N \neq 0$ in K).

In terms of g, S, and the absolute constant N we have produced a finite Galois extension K'/K of bounded degree unramified away from S and N over which A acquires semistable reduction at all places. When K is a global field there are only finitely many such extensions of K, so this proves that if K is a global field and S a non-empty finite set of places containing the archimedean places then for any $g > 0$ there is a finite Galois extension K'/K depending only on S and g such that *every* g-dimensional abelian variety A over K with good reduction outside A acquires semistable reduction at all non-archimedean places of K'. This uniform global result is crucial in Faltings' paper, since it leads to:

Corollary 6.6. *For any number field K, non-empty finite set S of places of K containing the archimedean places, and positive integer g, let* $\mathrm{Shaf}(K, S, g)$ *denote Shafarevich's conjecture that the set of isomorphism classes of g-dimensional abelian varieties over K with good reduction outside S is finite. Let* $\mathrm{Shaf}_{\mathrm{sst}}(K, S, g)$ *denote the analogous result with the additional requirement that the reduction type at all non-archimedean $v \in S$ is semistable.*

The conjecture $\mathrm{Shaf}(K, S, g)$ *is a consequence of the conjecture* $\mathrm{Shaf}_{\mathrm{sst}}(K', S', g)$ *for some finite Galois extension K'/K and the full preimage S' of S on K'. It is also sufficient to restrict attention to everywhere semistable abelian varieties over K' that admit a principal polarization at the cost of replacing g with* $8g$.

Proof. In view of the preceding discussion, for the sufficiency of $\mathrm{Shaf}_{\mathrm{sst}}(K', S', g)$ (setting aside the principal polarization aspect for dimension $8g$) it suffices to show that for a finite Galois extension k'/k of fields and an abelian variety A over k, there are only finitely many isomorphism classes of abelian varieties B over k such that $B_{k'} \simeq A_{k'}$. This will rest on two general results, the proofs of which will be discussed in the spring. The first result we need is that an abelian variety X over a field F admits only finitely many polarizations of a given square degree (over F), up to the action of $\mathrm{Aut}_F(X)$. The second result we need is that an abelian variety X over a field F admits only finitely many direct factors (over F), up to abstract F-isomorphism. These results are respectively [Mi1, 18.1] and [Mi1, 18.7], and their proofs rest on a deep general finiteness theorem in the theory of arithmetic subgroups of algebraic groups over $\mathbf{Q}$ (vastly generalizing the classical fact that there are finitely many $\mathrm{SL}_n(\mathbf{Z})$-equivalence classes of quadratic forms in n variables over $\mathbf{Z}$ with a fixed nonzero discriminant $d \in \mathbf{Z}$). In next week's lecture we'll address these aspects of [Mi1, §18].

Let's now apply these two results. For an abelian variety B over k that descends A', it seems hopeless to construct a polarization of B over k with degree bounded independently of B. Instead, we apply Zarhin's trick [Mi1, 16.12] to the abelian variety $(B \times B^{\vee})^4$ (that descends $(A' \times A'^{\vee})^4$): this abelian variety over k admits a principal polarization ϕ. As we just noted above, the k-isomorphism class of $(B \times B^{\vee})^4$ determines B up to finitely many possibilities (up to k-isomorphism) and as we vary through all k'-isomorphisms $(B \times B^{\vee})^4_{k'} \simeq (A' \times A'^{\vee})^4$ (of which there is at least one!) we can arrange that $\phi_{k'}$ is carried to one of *finitely many* principal polarizations ϕ' on $(A' \times A'^{\vee})^4$. In other words, upon renaming $(A' \times A'^{\vee})^4$ as A' we can equip A' with a *principal polarization* ϕ' and only need to check that the pair (A', ϕ') admits just finitely many k-descents (up to k-isomorphism).

If there are no k-descents then there is nothing to do. Suppose there is such a descent (A, ϕ). By standard cocycle formalism for descent (as explained in Serre's "Galois cohomology" book), the set of k-descents of (A', ϕ') up to k-isomorphism is identified with the pointed set $\mathrm{H}^1(\mathrm{Gal}(k'/k), \mathrm{Aut}(A_{k'}, \phi_{k'}))$ (where $\mathrm{Aut}(A_{k'}, \phi_{k'})$ is a $\mathrm{Gal}(k'/k)$-set in the evident manner, using the k-structure (A, ϕ) on the pair $(A_{k'}, \phi_{k'}) = (A', \phi')$ over k'). But this pointed set is finite since $\mathrm{Gal}(k'/k)$ is finite and the automorphism group of a *polarized* abelian variety over a field is always finite. ■

Corollary 6.6 justifies Faltings' restriction to everywhere semistable abelian varieties in his proof of the Shafarevich conjecture for abelian varieties over number fields, and it is likewise harmless to consider only those abelian varieties that admit a principal polarization over the ground field. In particular, to prove the conjecture over a specific number field, this reduction step requires passing to a larger number field (and much larger dimension!)

Here is a useful uniformity result in the local case:

Proposition 6.7. *Assume R is strictly henselian. Let K' and K'' be finite Galois extensions of K inside of K_s over which A becomes semistable. Then the same holds for A over $K' \cap K''$. In particular, there is a finite Galois extension K_A/K over which A becomes semistable and that lies in all other such finite Galois extensions of K.*

We call K_A/K as the *field of semistable reduction* for A over K. Note that this concept requires that R is strictly henselian (so the Galois group coincides with the inertia group). It seems unlikely that such a minimal field exists without assuming R is strictly henselian.

Proof. Let $G_K = \mathrm{Gal}(K_s/K)$, and similarly for K' and K''. Pick a prime $\ell \neq \mathrm{char}(k)$. Since R is strictly henselian, so likewise for its local integral closures in K' and K'', the semistability over K' implies that the $G_{K'}$-action on $V = \mathrm{V}_\ell(A)$ is unipotent. Form the maximal subspace on which $G_{K'}$ acts trivially, then look at the $G_{K'}$-action on the quotient of V by that, and continue. This is a "minimal flag" $F^\bullet$ on whose successive quotients the $G_{K'}$-action is trivial. The normality of $G_{K'}$ in G_K (due to K'/K being Galois) implies that G_K preserves $F^\bullet$. Since semistability holds over K'', on each of the successive quotients of

$F^\bullet$ the $G_{K''}$-action is unipotent. In this way we can build a G_K-stable flag of V on whose successive quotients both $G_{K'}$ and $G_{K''}$ act trivially. Hence, the group $G_{K'\cap K''}$ generated by $G_{K'}$ and $G_{K''}$ has unipotent action on V, so A becomes semistable over $K' \cap K''$. ■

Finally, we take up the question of universally bounding (in terms of g) the amount of ramification required to attain semistable reduction. For the purpose of ramification bounds we lose nothing by passing to a strictly henselian base (i.e., making the inertia group equal to Galois group), and in that setting we can exploit the existence of the field of semistable reduction provided by Proposition 6.7 to prove:

Theorem 6.8. *Assume R is strictly henselian, and let A be an abelian variety over K with dimension $g > 0$.*

(1) *The field of semistable reduction K_A/K has degree dividing an absolute constant $N(g)$ whose only primes factors are $p \leq 2g+1$. Explicitly, we may take*

$$N(g) = 2^{3g+\mathrm{ord}_2(g!)} \cdot \prod_{3\leq p\leq 2g+1} p^{[2g/(p-1)]+\sum_{d\leq 2g/(p-1)} \mathrm{ord}_p(d)}.$$

In particular, semistable reduction is attained over a tamely ramified extension except possibly when the residue characteristic lies in the interval $[2, 2g+1]$.

(2) *Let $M(g)$ denote the least common multiple of the integers $\mathrm{lcm}(e_i)$ as $\{e_i\}$ varies over all partitions of $2g$. (For example, $M(1) = 2$ and $M(2) = 12$.) If the residue characteristic lies outside the interval $[2, 2g+1]$ then semistable reduction is attained over an extension of degree dividing*

$$N_{\mathrm{tame}}(g) = 2^{2+\mathrm{ord}_2(M(g))} \cdot \prod_{3\leq p\leq 2g+1, (p-1)|M(g)} p^{1+\mathrm{ord}_p(M(g))},$$

and when g is a power of 2 we can even use $N_{\mathrm{tame}}(g)/2$.

The computation of $N(g)$ involves computing group orders without regard to subgroup structure, whereas $N_{\mathrm{tame}}(g)$ takes into account the orders of elements in $\mathrm{Sp}_{2g}(\mathbf{Z}/\ell\mathbf{Z})$ (or really just in $\mathrm{SL}_{2g}(\mathbf{Z}/\ell\mathbf{Z})$). For example, $N(1) = 24$ but $N_{\mathrm{tame}}(1)/2 = 12$, and $N(2) = 2^7 3^3 5$ but $N_{\mathrm{tame}}(2)/2 = 2^3 3^2$. For $g = 1$ this is consistent with the fact that away from residue characteristics 2 and 3 (the primes in the interval $[2, 2g+1]$ for $g = 1$), the proof of the semistable reduction theorem for elliptic curves in [Si] yields good reduction over a degree-12 extension.

Proof. Note that if the residual field extension attached to K_A/K is not separable then $\mathrm{char}(k)$ must divide $[K_A : K]$. Hence, the tame ramification assertion is a formal consequence of the knowledge of the prime factors of $N(g)$.

To establish the existence and prime factors of $N(g)$, fix a polarization of A over K and pick an odd prime $\ell \neq \mathrm{char}(k)$ not dividing the degree of the polarization. Thus, the Weil pairing between $\mathrm{T}_\ell(A)$ and $\mathrm{T}_\ell(A')$ (with A' the dual abelian variety) becomes a Galois-equivariant $\mathbf{Z}_\ell(1)$-valued symplectic form on $\mathrm{T}_\ell(A)$. The ℓ-adic cyclotomic character is unramified, so the Galois action on $A[\ell]$ lands in the subgroup $\mathrm{Sp}_{2g}(\mathbf{Z}/\ell\mathbf{Z}) \subset \mathrm{SL}_{2g}(\mathbf{Z}/\ell\mathbf{Z})$.

By Proposition 6.5, $K(A[\ell])/K$ is a finite Galois extension over which A acquires semistable reduction, so K_A is contained in this field. Hence, $[K_A : K]$ divides $[K(A[\ell]) : K]$ for all but finitely many ℓ, and hence divides $\#\mathrm{Sp}_{2g}(\mathbf{Z}/\ell\mathbf{Z})$ for all but finitely many ℓ. It is classical that

$$\#\mathrm{Sp}_{2g}(\mathbf{Z}/\ell\mathbf{Z}) = \ell^{g^2} \cdot \prod_{a=1}^{g}(\ell^{2a} - 1).$$

Thus, the order of $\mathrm{Sp}_{2g}(\mathbf{Z}/\ell\mathbf{Z})$ is divisible by a positive integer d for all but finitely many ℓ precisely when the polynomial

$$f(x) = x^{g^2}\prod_{a=1}^{g}(x^{2a} - 1) \in \mathbf{Z}[x]$$

vanishes as a function on $\mathbf{Z}/d\mathbf{Z}$, in which case $d|f(\ell)$ for all large ℓ by Chebotarev. Hence, we can take $N(g)$ to be the gcd of the numbers $\prod_{a=1}^{g}(\ell^{2a} - 1)$ as ℓ varieties through all large primes, provided we show that this gcd is insensitive to dropping finitely many ℓ from consideration (i.e., it does not grow). The only primes p which divide all such products for all sufficiently large ℓ (beyond any bound) are those p such that for all large ℓ we have $\ell^{2a} \equiv 1 \bmod p$ for some $1 \le a \le g$ (depending on p). By taking ℓ to represent a generator of $(\mathbf{Z}/p\mathbf{Z})^{\times}$ it follows that the order $p - 1$ of the generator divides $2a$ for some $1 \le a \le g$. Hence, $p \le 2g + 1$ for any prime p dividing $f(\ell)$ for all large primes ℓ.

Now fix $p \le 2g + 1$, and initially take $p > 2$ (so $(\mathbf{Z}/p^r\mathbf{Z})^{\times}$ is nontrivial and cyclic for any $r \ge 1$). For each $1 \le a \le g$ such that $(p - 1)|2a$ (e.g., $a = (p - 1)/2$ always works), let $r_{a,p} \ge 0$ be maximal such that $p^{r_{a,p}}(p - 1)|2a$ (or equivalently $p^{r_{a,p}}|(a/((p - 1)/2)))$. Thus, $p^{r_{a,p}+1}|(\ell^{2a} - 1)$ for all large ℓ. Choose among the arbitrarily large primes ℓ any for which ℓ represents a generator of $(\mathbf{Z}/p^{r_{a,p}+2}\mathbf{Z})^{\times}$, so $\ell^{2a} \not\equiv 1 \bmod p^{r_{a,p}+2}$ since the p-part $p^{r_{a,p}+1}$ of $\varphi(p^{r_{a,p}+2})$ does not divide $2a$. Hence, as ℓ varies through all large primes (beyond any desired lower bound), $p^{r_{a,p}+1}$ always divides $\ell^{2a} - 1$ but infinitely often $p^{r_{a,p}+2}$ does not.

More specifically, if $r(p) = \max_{1 \le a \le g} r_{a,p}$ and we choose an arbitrarily large prime ℓ that represents a generator of $(\mathbf{Z}/p^{r(p)+2}\mathbf{Z})^{\times}$ (so ℓ represents a generator of $(\mathbf{Z}/p^{r_{a,p}+2}\mathbf{Z})^{\times}$ for all $1 \le a \le g$) then for *every* $1 \le a \le g$ we have that $\ell^{2a} - 1$ is not divisible by $p^{r_{a,p}+2}$. But we have seen that $p^{r_{a,p}+1}|(\ell^{2a} - 1)$ for *all* large ℓ. Thus, it follows that

$$p^{g+\sum_{1\le a\le g} r_{a,p}} | \prod_{a=1}^{g}(\ell^{2a} - 1) \tag{6.1}$$

for all large primes ℓ yet we have produced infinitely many ℓ for which the known divisibility $p^{r_{a,p}+1}|(\ell^{2a} - 1)$ is optimal for all $1 \le a \le g$, so (6.1) cannot be improved for those ℓ. In other words, as we vary ℓ through all primes beyond any desired lower bound, the greatest common divisor of the numbers $\prod_{a=1}(\ell^{2a} - 1)$ has odd part

$$\prod_{3\le p\le 2g+1}\ \prod_{d\le 2g/(p-1)} p^{1+r_{d(p-1)/2,p}} = \prod_{3\le p\le 2g+1} p^{[2g/(p-1)]+\sum_{d\le 2g/(p-1)}\mathrm{ord}_p(d)}.$$

Finally, we consider the 2-part. As we vary over all large ℓ, $\ell^{2a} - 1$ is always divisible by 8. Hence, we always get a factor of 2^{3g} and it remains to determine the maximal power of 2 that divides

$$F(\ell) := \prod_{a=1}^{g}\frac{\ell^{2a} - 1}{8}$$

for all large primes ℓ. For $1 \le a \le g$, let $r_a = \mathrm{ord}_2(a)$, so r_a is the maximal $r \ge 0$ such that the ath-power map kills the cyclic group $(1+8\mathbf{Z}_2)/(1+2^{3+r}\mathbf{Z}_2)$ of order 2^r, with $1+8\mathbf{Z}_2 = (\mathbf{Z}_2^\times)^2$. We can choose a primitive generator of this cyclic group and pick a square root of it in $1+4\mathbf{Z}_2$, which in turn is represented modulo $2^{3+r}\mathbf{Z}_2$ by infinitely many primes. For such primes ℓ we have that $(\ell^{2a}-1)/8$ is not divisible by 2^{r_a+1}, whereas for all large ℓ we have that $(\ell^{2a}-1)/8$ is divisible by 2^{r_a}. Hence, for all large primes ℓ the value $F(\ell)$ is divisible by $2^{\sum_{1\le a\le g} r_a} = 2^{\mathrm{ord}_2(g!)}$ but for $r := \max_{1\le a\le g} r_a$ and ℓ such that ℓ^2 is is a generator of $(1+8\mathbf{Z}_2)/(1+2^{4+r}\mathbf{Z}_2)$ no higher power of 2 divides $F(\ell)$. This completes the proof of the proposed formula for $N(g)$.

Now consider cases when the residue characteristic lies outside $[2, 2g+1]$, so in the preceding Galois representation analysis we know that the Galois group of K_A/K not only embeds into $\mathrm{Sp}_{2g}(\mathbf{Z}/\ell\mathbf{Z})$ for all large ℓ, but it even has *cyclic* image (by cyclicity of tame inertia, and the assumption that R is strictly henselian). Thus, rather than computing the greatest common divisor of the order of the entire group as ℓ varies, we should instead consider the orders of individual elements in these groups. Since $\mathbf{Z}/\ell\mathbf{Z}$ is a perfect field, for any element $\gamma \in \mathrm{Sp}_{2g}(\mathbf{Z}/\ell\mathbf{Z})$ the Jordan decomposition $\gamma = \gamma_{\mathrm{ss}}\gamma_{\mathrm{u}}$ as a product of commuting semisimple and unipotent elements is rational over $\mathbf{Z}/\ell\mathbf{Z}$. The order of γ_{u} is a power of ℓ whereas the order of γ_{ss} is relatively prime to ℓ. Thus, the order of γ is the product of the orders of γ_{ss} and γ_{u}. The Galois group $\mathrm{Gal}(K_A/K)$ is a quotient of $\mathrm{Gal}(K(A[\ell])/K)$, which is a cyclic subgroup of $\mathrm{Sp}_{2g}(\mathbf{Z}/\ell\mathbf{Z})$, and so by also ensuring $\ell > 2g+1$ (so it does not divide the order of $\mathrm{Gal}(K_A/K)$) we guarantee that $[K_A : K]$ divides the order of the semisimple part of a generator of $\mathrm{Gal}(K(A[\ell])/K)$.

To summarize, in these "tame" cases $[K_A : K]$ divides the order of a semisimple element γ of $\mathrm{Sp}_{2g}(\mathbf{Z}/\ell\mathbf{Z})$ for all large ℓ. Such an element generates a commutative semisimple subalgebra of $\mathrm{Mat}_{2g}(\mathbf{Z}/\ell\mathbf{Z})$, which must be a direct product of finite fields. Hence, inside of $\mathrm{GL}_{2g}(\mathbf{Z}/\ell\mathbf{Z})$ the element γ lies in (the norm-1 subgroup of) a subgroup of the form $\prod \mathbf{F}_{\ell^{e_i}}^\times$ where $\sum e_i \le 2g$ for some positive integers e_i. Thus, for all large ℓ the order of γ divides the least common multiple $L_{\ell,\{e_i\}}$ of the integers $\ell^{e_i}-1$ for some e_i such that $\sum e_i \le 2g$. This least common multiple is unaffected by inserting extra terms $e_i = 1$, so we may restrict attention to tuples (e_i) of positive integers for which $\sum e_i = 2g$. For fixed large ℓ, the lcm of the integers $L_{\ell,\{e_i\}}$ as $\{e_i\}$ varies through all partitions of $2g$ provides a multiplicative upper bound on the order of γ. A multiplicative upper bound on this least common multiple is $\ell^{\mathrm{lcm}(e_i)} - 1$. Hence, if $M(g)$ denotes the least common multiple of the integers $\mathrm{lcm}(e_i)$ as $\{e_i\}$ varies through the partitions of $2g$ then $[K_A : K]$ divides the least common multiple the integers $\ell^{M(g)} - 1$ for all large primes ℓ.

Suppose $[K_A : K] = \prod_{p \le 2g+1} p^{b_p}$, so the $M(g)$th-power map kills $(\mathbf{Z}/p^{b_p}\mathbf{Z})^\times$ when p is odd and kills $(\mathbf{Z}/2^{b_2-1}\mathbf{Z})^\times$ if $p=2$. Hence, if $b_p \ge 1$ then $p^{b_p-1}(p-1)|M(g)$ when $p>2$ (by cyclicity) and $2^{b_2-1}|M(g)$ when $p=2$ and $b_2 \ge 1$. In other words, if $b_p \ge 1$ and $(p-1)|M(g)$ then $b_p \le 1 + \mathrm{ord}_p(M(g))$ if p is odd and $b_p \ge 2 + \mathrm{ord}_2(M(g))$ if $p=2$. This gives the divisibility

$$[K_A : K] | 2^{2+\mathrm{ord}_2(M(g))} \cdot \prod_{3 \le p \le 2g+1, (p-1)|M(g)} p^{1+\mathrm{ord}_p(M(g))}.$$

Finally, we save a factor of 2 on the 2-part of this upper bound when $g = 2^{r-1}$ with $r \geq 1$ by using that γ lies in SL_{2g} rather than just GL_{2g}. Consider a partition $\{e_i\}$ of $2g = 2^r$ as above. If this is not the trivial partition $e_1 = 2^r$, then all e_i have 2-part at most 2^{r-1}, so if such a partition is what intervenes in the study of γ in the mod-ℓ representation then we can replace $M(g)$ with $M(g)/2$ in the mod-ℓ analysis. If instead the trivial partition $e_1 = 2^r = 2g$ intervenes then γ generates a single finite field $\mathbf{F}_{\ell^{2g}}$ and the norm-1 condition saves a factor of the 2-part of $\ell - 1$ on our bound for the order. This always saves at least a factor of 2, so again we can replace $M(g)$ with $M(g)/2$ in the mod-ℓ analysis. ■

7. Picard functors and proof of semistable reduction theorem

Now we return to the topic left unfinished after the discussion following Proposition 4.3: to complete the proof of the semistable reduction theorem for abelian varieties by explaining how to treat the special case when A is the Jacobian of a smooth proper and geometrically connected curve X over K (which we can assume has genus > 0; i.e., $A \neq 0$).

The main order of business is to discuss the theory of Picard functors, as that provides the framework for Raynaud's results which compute the identity component of the special fiber of the Néron model of the Jacobian of X when X has semistable reduction. (Raynaud's work actually goes far beyond the semistable case.) The theory of Picard functors is not specific to curves, so we begin by working somewhat more generally and later will specialize to curves. An elegant overview of both the general theory and the case of curves is given in [BLR, Ch. 8–9].

The *initial setup* we wish to consider is a proper flat surjective map $f : X \to S$ of finite presentation such that the geometric fibers are connected and reduced. (The reader is welcome to assume that S is locally noetherian, as that case contains all of the essential ideas and is entirely sufficient for our needs. One can replace "finite presentation" with the equivalent "finite type" in the locally noetherian case.)

Let's first explain how to produce natural examples. The following lemma shows how to control the geometric connectedness condition in many cases:

Lemma 7.1. *Let $f : X \to S$ be a proper flat surjective map to a noetherian scheme S, and assume that f has geometrically connected and smooth generic fibers. Then all fibers are geometrically connected.*

Proof. We may and do assume that S is reduced and irreducible (by base change to irreducible components of S, equipped with the reduced structure). For a non-generic point $s \in S$ there is a discrete valuation on the function field of S that dominates $\mathscr{O}_{S,s}$ [EGA, II, 7.1.7], so by base change to such a ring we can assume that $S = \operatorname{Spec} R$ for a discrete valuation ring R. Let $K = \operatorname{Frac}(R)$.

By R-flatness of X and smoothness and geometric connectedness of the generic fiber, the R-finite $\mathrm{H}^0(X, \mathscr{O}_X)$ injects into $\mathrm{H}^0(X_K, \mathscr{O}_{X_K}) = K$. Thus, $R = \mathrm{H}^0(X, \mathscr{O}_X)$ by the normality of R. That is, $X \to \operatorname{Spec} R$ is its own Stein factorization. But Stein factorizations always have geometrically connected fibers [EGA, III_1, 4.3.4]. ■

It follows from this lemma that if R is a discrete valuation ring with fraction field K and if X is a proper flat model of a smooth and geometrically connected curve over K then X

has geometrically connected special fiber. If X has semistable special fiber (which we can always attain after a finite separable extension on K, by the semistable reduction theorem for curves) then it fits into the initial setup described above.

Now we return to the consideration of a map $f : X \to S$ that is proper, flat, surjective and finitely presented with connected and reduced geometric fibers. Note that these assumptions are preserved by base change.

Lemma 7.2. *Under the above hypotheses on f, the natural map $\mathscr{O}_S \to f_*\mathscr{O}_X$ is an isomorphism.*

Proof. By direct limit arguments, one can reduce to the locally noetherian case (if the reader wasn't already assume S to be locally noetherian). We claim that $f_*\mathscr{O}_X$ is an invertible $\mathscr{O}_S$-module. Once this is proved, we will be done since rather generally if a map of rings $A \to B$ makes B an invertible A-module then $A \to B$ is an isomorphism (as the element $1 \in B$ is nonzero in all fiber algebras of $\operatorname{Spec} B \to \operatorname{Spec} A$, and hence is an A-module generator by Nakayama's Lemma).

To prove the invertibility of $f_*\mathscr{O}_X$, we recall the following special case of Grothendieck's theory of base change for cohomology: if $\mathscr{F}$ is an S-flat coherent $\mathscr{O}_X$-module and the natural map $(f_*\mathscr{F})_s \otimes_{\mathscr{O}_s} k(s) \to \mathrm{H}^0(X_s, \mathscr{F}_s)$ is surjective for all $s \in S$ (where $\mathscr{F}_s$ denotes the pullback of $\mathscr{F}$ along $X_s \to X$) then this map is an isomorphism for all s and $f_*\mathscr{F}$ is locally free as an $\mathscr{O}_S$-module (with rank at s equal to the $k(s)$-dimension of $\mathrm{H}^0(X_s, \mathscr{F}_s)$). This theorem applies to $\mathscr{F} = \mathscr{O}_X$ (which is S-flat since f is flat), so it suffices to show that $k(s) \to \mathrm{H}^0(X_s, \mathscr{O}_{X_s})$ is an isomorphism for all $s \in S$.

In other words, we want the finite $k(s)$-algebra $\mathrm{H}^0(X_s, \mathscr{O}_{X_s})$ to be 1-dimensional as a vector space. But the formation of this H^0 commutes with extension of the ground field, and for an algebraically closed extension $k'/k(s)$ the resulting geometric fiber over k' is a non-empty proper reduced connected k'-scheme. Such a scheme has algebra of global functions that is nonzero, k'-finite, and reduced, hence a finite product of copies of the algebraically closed k'. But then this algebra must be k', as otherwise it would have nontrivial idempotents, contradicting the geometric connectedness of X_s. ■

One of the difficulties with representing a functor classifying isomorphism classes of line bundles is that line bundles have too many automorphisms (which creates difficulties for globalization of constructions). This can be removed by imposing "rigidification along a section" (if there is a section!). The following proposition makes this precise.

Proposition 7.3. *Let $\mathscr{L}$ be an invertible sheaf on X. The natural map $\mathscr{O}(S)^\times \to \mathrm{Aut}_{\mathscr{O}_X}(\mathscr{L})$ is an isomorphism. If there exists a section $e \in X(S)$ and a trivialization $i : \mathscr{O}_S \simeq e^*\mathscr{O}_X$ then the only automorphism of the pair $(\mathscr{L}, i)$ (i.e., $\mathscr{O}_X$-linear automorphism φ of $\mathscr{L}$ such that $e^*(\varphi) \circ i = i$) is the identity.*

Proof. An automorphism of $\mathscr{L}$ is multiplication by a global unit on X. But $f_*\mathscr{O}_X = \mathscr{O}_S$, so applying $\mathrm{H}^0(S, \cdot)$ implies that the natural map $\mathscr{O}_S(S) \to \mathscr{O}_X(X)$ is an isomorphism. Since $X \to S$ is surjective, it follows that $\mathscr{O}_S^\times(S) \to \mathscr{O}_X^\times(X)$ is an isomorphism. This establishes the asserted description of automorphisms of $\mathscr{L}$.

Now consider an automorphism of a pair $(\mathscr{L}, i)$. The underlying automorphism of $\mathscr{L}$ must be scaling by the pullback $f^*(u)$ for some unit u on S, and the effect on i is scaling by $e^*(f^*(u)) = (f \circ e)^*(u) = u$ (since $f \circ e = 1_S$, due to e being a section to f). Hence, preservation of i forces $u = 1$, so the automorphism of $\mathscr{L}$ is the identity. ■

Define the functor $P_{X/S}$ on the category of S-schemes via $P_{X/S}(S') = \mathrm{Pic}(X_{S'})$, where $X_{S'} := X \times_S S'$. (The reader who is making noetherian hypotheses should restrict S' to be locally noetherian. We won't comment about this sort of issue again.) The functor $P_{X/S}$ is contravariant in S' via pullback. Unfortunately, it generally has no chance to be representable: there are problems with gluing, or in other words this functor violates the sheaf-like properties that representable functors satisfy.

Example 7.4. Suppose that S admits a nontrivial line bundle $\mathscr{L}_0$ and we define $\mathscr{L} = f^*(\mathscr{L}_0)$ on X. Then $\mathscr{L}$ becomes trivial *locally over* S, so if $U \mapsto P_{X/S}(U)$ were to be a sheaf on S (as it would have to be if $P_{X/S}$ were representable, due to gluing of morphisms) then this would force $\mathscr{L}$ to be trivial, which is to say $\mathscr{L} \simeq \mathscr{O}_X$. But the natural map $\mathscr{L}_0 \to f_*f^*\mathscr{L}_0$ is an isomorphism (by Lemma 7.2 applied locally over S where $\mathscr{L}_0$ trivializes) and hence $\mathscr{L}_0 \simeq f_*\mathscr{O}_X \simeq \mathscr{O}_S$, a contradiction.

Grothendieck's fix to the failure of sheaf-like properties for $P_{X/S}$ is to sheafify:

Definition 7.5. The *relative Picard functor* $\mathbf{Pic}_{X/S}$ is the sheafification of $P_{X/S}$ relative to the fppf topology on S-schemes.

Example 7.6. By definition, for any S-scheme S' (e.g., a single geometric point over S), the restriction of $\mathbf{Pic}_{X/S}$ to the category of S'-schemes is $\mathbf{Pic}_{X_{S'}/S'}$. Likewise, if Y is an S-scheme, then the restriction of the functor $\mathrm{Hom}_S(\cdot, Y)$ to the category of S'-schemes is represented by the S'-scheme $Y_{S'} = Y \times_S S'$ (due to the definition of fiber products). Hence, if there is a representing scheme $\mathrm{Pic}_{X/S}$ then $(\mathrm{Pic}_{X/S})_{S'}$ represents $\mathbf{Pic}_{X_{S'}/S'}$. In other words, the formation of $\mathrm{Pic}_{X/S}$ naturally commutes with base change on S' when it exists.

The definition of the relative Picard functor is much too abstract; that is, elements of $\mathbf{Pic}_{X/S}(S')$ don't have a concrete meaning. There is a special case when we can interpret the meaning of such elements:

Example 7.7. Suppose $S' = \mathrm{Spec}(k)$ for an algebraically closed field k. Let $s : \mathrm{Spec}\, k = S' \to S$ be the structural morphism, and X_s the resulting pullback $X_{S'}$. Then $\mathbf{Pic}_{X/S}(S') = \mathbf{Pic}_{X_k/k}(k)$ and I claim that the natural map of groups $\mathrm{Pic}(X_k) \to \mathbf{Pic}_{X_k/k}(k)$ is an isomorphism. (Beware that this is generally *false* when k is not algebraically closed; it is related to the fact that the Jacobian of a curve without rational points may not arise from a line bundle on the curve over the ground field.)

To see this, consider the fppf-sheafification process that constructs the relative Picard functor. The key point is that an fppf-cover of $\mathrm{Spec}\, k$ (or equivalently, any non-empty finite type k-scheme) always admits a section, since $k = \overline{k}$. Hence, such covers can always be "refined" to a cover of $\mathrm{Spec}\, k$ by the identity map. The sheafification process only requires the consideration of a cofinal system of covers, so for the computation of the k-points of this abstract fppf-sheaf we don't change anything in the original group $\mathrm{Pic}(X_k)$!

Subject to an extra condition, we can describe the "points" of the relative Picard functor in concrete terms, as follows. Assume that there exists $e \in X(S)$. Define the e-rigidified Picard functor $\mathbf{Pic}_{X/S,e}$ on the category of S-schemes by declaring $\mathbf{Pic}_{X/S,e}(S')$ to be the group of isomorphism classes of pairs $(\mathscr{L}, i)$ where $\mathscr{L}$ is a line bundle on $X_{S'}$ and $i : \mathscr{O}_{S'} \simeq e_{S'}^*(\mathscr{L})$ is an isomorphism. This is a group via tensor product and dual, with trivial object $(\mathscr{O}_X, i)$ for the trivialization i via 1, and it is a contravariant functor via base change. Also, by Proposition 7.3, the objects classified by the e-rigidified Picard functor have no nontrivial automorphisms. This underlies the proof of:

Proposition 7.8. *The forgetful map $\mathbf{Pic}_{X/S,e} \to \mathbf{Pic}_{X/S}$ that forgets the rigidification i is an isomorphism. In particular, when $X(S)$ is non-empty, $\mathbf{Pic}_{X/S}$ is representable if and only if there exists a universal e-rigidified line bundle (for a choice of $e \in X(S)$).*

The elegance of Grothendieck's more abstract notion of relative Picard functor $\mathrm{Pic}_{X/S}$ is that it makes no reference to a non-canonical choice of e, nor does it even require the existence of such an e. Descent methods sometimes allow one to reduce the representability problem for this functor to cases when some e exists, in which case the concrete interpretation provides a better handle on the problem. The ability to avoid assuming the existence of e is important in applications, since we want to consider Jacobians of curves that may not have any rational points!

Proof. By definition of $\mathbf{Pic}_{X/S}$ as an fppf sheafification, it is a sheaf for the fppf topology. But by fppf descent theory for quasi-coherent sheaves (see [BLR, Ch. 6]), $\mathbf{Pic}_{X/S,e}$ is *also* such a sheaf since it classifies isomorphism classes of rigid objects. Hence, to prove the isomorphism property for the functors we can work fppf-locally.

To prove injectivity, consider a pair $(\mathscr{L}, i)$ on $X_{S'}$ such that $\mathscr{L}$ on $X_{S'}$ becomes isomorphic to the trivial line bundle fppf-locally on S' (i.e., $(\mathscr{L}, i)$ is carried to the trivial element in the fppf-sheafified target). We want to prove that the pair $(\mathscr{L}, i)$ is isomorphic to the trivial pair (consisting of $\mathscr{O}_{X_{S'}}$ with trivializing section 1 along $e_{S'}$). By descent theory and the rigidity, it suffices to construct such an isomorphism fppf-locally over S'. Hence, we may assume that $\mathscr{L} = \mathscr{O}_{X_{S'}}$. Then the isomorphism $i : e_{S'}^*(\mathscr{L}) \simeq \mathscr{O}_{S'}$ is identified with a unit u on S', so the pair $(\mathscr{L}, i)$ is isomorphic to the trivial pair via multiplication by $1/u$ on $\mathscr{L}$.

Now we turn to surjectivity. For any S-scheme S' and $\xi \in \mathbf{Pic}_{X/S}(S')$, by working fppf-locally over S' we can arrange that ξ arises from a line bundle $\mathscr{L}$ on $X_{S'}$. The pullback $e_{S'}^*(\mathscr{L})$ is an invertible sheaf on S', so Zariski-locally it admits a rigidification i. Thus, fppf-locally on S, ξ arises from a pair $(\mathscr{L}, i)$. This proves that our injective map of fppf-sheaf functors is "locally surjective" too, and hence is an isomorphism. ■

Under some strong fibral hypothesis (which are *not* satisfied for most semistable curves), Grothendieck proved the following general result as an application of his theory of Hilbert schemes:

Theorem 7.9 (Grothendieck). *If $X \to S$ is projective and flat with geometrically integral fibers then $\mathbf{Pic}_{X/S}$ is represented by a scheme $\coprod_\Phi \mathrm{Pic}_{X/S}^\Phi$, where Φ ranges through the polynomials in $\mathbf{Q}[t]$ taking $\mathbf{Z}$-values on all $t \in \mathbf{Z}$ and $\mathrm{Pic}_{X/S}^\Phi$ is a quasi-projective S-scheme that represents the subfunctor of classes whose restriction $\mathscr{L}_{\overline{s}}$ on geometric fibers satisfies $\chi_{X_s}(\mathscr{L}_{\overline{s}}(n)) = \Phi(n)$ for $n \in \mathbf{Z}$. Here, $\chi_{X_s}(\mathscr{N}) := \sum_j (-1)^j \dim \mathrm{H}^j(X_s, \mathscr{N})$.*

Note that representability by a disjoint union of quasi-projective schemes implies that the functor is separated (i.e., satisfies the valuative criterion for separatedness). There is a famous example of Mumford [BLR, p. 210] which illustrates how essential the fibral *geometric* integrality condition is in Grothendieck's theorem: without it the functor can fail to be separated! Mumford's example is very concrete: the degenerating projective conic $U^2+V^2 = tW^2$ over $\mathbf{R}[\![t]\!]$ (or over $\mathbf{Q}[\![t]\!]$). This has integral fibers, but the special fiber is not *geometrically* irreducible (so Grothendieck's theorem does not apply, as it had better not).

Example 7.10. Suppose that $S = \operatorname{Spec} k$ for a field k, and that X is a smooth curve, say with genus g. Grothendieck's geometric integrality hypothesis holds, so we obtain a Picard scheme that is a disjoint union of quasi-projective k-schemes. In particular, the identity component $\mathrm{Pic}^0_{X/k}$ is a finite type k-group scheme. Functorial criteria (using that degree-2 coherent cohomology vanishes on the 1-dimensional X) ensure that $\mathrm{Pic}_{X/k}$ is smooth. Computing with points valued in the dual numbers k-linearly identifies the tangent space at the identity with $\mathrm{H}^1(X, \mathscr{O}_X)$, so this smooth group is g-dimensional. Any group scheme over a field is separated (as the diagonal is a base change of the identity section, and rational points are always closed), and we claim that $\mathrm{Pic}^0_{X/k}$ is *proper* (hence an abelian variety).

To verify the properness, we can use the valuative criterion, but we prefer to use another method which will give us more information. In view of the separatedness, it suffices to exhibit a proper k-scheme mapping onto $\mathrm{Pic}^0_{X/k}$. We may increase the ground field to be algebraically closed, so we can choose $e \in X(k)$. Hence, Example 7.7 and Proposition 7.8 thereby provide a concrete meaning for $\mathrm{Pic}_{X/k}$ in terms of line bundles. In particular, there is a universal line bundle over $X \times \mathrm{Pic}_{X/k}$ (rigidified along $e \times 1$).

Consider the natural map $X^g \to \mathrm{Pic}_{X/k}$ defined functorially by $(x_1, \dots, x_g) \mapsto \otimes_j \mathscr{O}(x_j - e)$. This visibly carries $(e, \dots, e)$ to the origin, so by connectedness it lands in $\mathrm{Pic}^0_{X/k}$. By *local constancy* of the fibral degree of a line bundle in proper flat families of curves, the universal line bundle over $X \times \mathrm{Pic}^0_{X/k,e}$ has degree 0 on all fibers over the *connected* $\mathrm{Pic}^0_{X/k,e}$. But the Riemann–Roch theorem implies that every degree-0 line bundle on X is represented by a degree-0 divisor of the form $\sum_{j=1}^{g}(x_j - e)$. This implies two things: the map $X^g \to \mathrm{Pic}^0_{X/k}$ is surjective (so $\mathrm{Pic}^0_{X/k}$ is an abelian variety) and the geometric points of $\mathrm{Pic}^0_{X/k}$ correspond *exactly* to degree-0 line bundles. This is how the theory of Jacobians emerges from Grothendieck's work on Picard functors.

Example 7.11. In the special case that $S = \operatorname{Spec} k$ for a field k (so X is a proper k-scheme that is geometrically connected and geometrically reduced), it was proved shortly after Grothendieck by Murre and Oort that $\mathbf{Pic}_{X/k}$ is always represented by a k-group scheme (generally not smooth when $\dim X > 1$ if $\mathrm{char}(k) > 0$). This goes beyond Grothendieck's result, since geometric integrality is not assumed (and we do need to avoid such hypotheses, for applications to semistable curves).

The reader is referred to [BLR, 9.2/9–13] for an elegant general discussion of the structure of $\mathrm{Pic}^0_{X/k}$ for proper curves over fields.

An essential new viewpoint was introduced by Artin: he created the theory of algebraic spaces, which are certain functors that are "close enough" to being representable by schemes that one can set up the usual notions of algebraic geometry for them (Zariski topology, irreducibility, fiber products, properness, flatness, smoothness, étaleness, etc.) even though they may not be representable. *Algebraic spaces are not locally ringed spaces.* The viewpoint is rather to develop a "geometry" of certain functors which are almost represented by schemes. Most of EGA carries over to algebraic spaces, with some extra care in several places, and the theory of quotients for algebraic spaces is far more robust than more schemes (making them especially well-suited for the study of moduli problems).

Failure of separatedness is quite commonplace for algebraic spaces, in contrast with schemes. Artin developed remarkable local techniques for proving that various interesting functors on schemes are algebraic spaces, and there are conditions which suffice to prove that an algebraic space is represented by a scheme. The situation is similar to that of the theory of distributions in analysis (as a replacement for functions, alongside theorems such as elliptic regularity that guarantee that certain distributions are ordinary functions).

A "defect" of the theory of algebraic spaces is that it is very hard to control quasi-compactness hypotheses. So when working over a field, one often has to work with algebraic spaces only known to be locally of finite type rather than finite type. It is a general fact that algebraic spaces groups locally of finite type over a field are necessarily (represented by) *schemes*, and the identity component is necessarily finite type by Lemma 2.5. The following result of Artin therefore recovers as a very special case the result of Murre and Oort in Example 7.11.

Theorem 7.12 (Artin)**.** *Under the above running hypothesis,* $\mathbf{Pic}_{X/S}$ *is an algebraic space; it is denoted* $\mathrm{Pic}_{X/S}$.

Using facts about Picard schemes of curves over fields (mainly that their component groups are torsion-free), it follows from [SGA6, XIII, Thm. 4.7(i), (iii)] that in Artin's theorem the subfunctor

$$\mathbf{Pic}^0_{X/S} : S' \rightsquigarrow \{\xi \in \mathbf{Pic}_{X/S}(S') \,|\, \xi_{s'} \in \mathrm{Pic}^0_{X_{s'}/k(s')} \text{ for all } s' \in S'\}$$

is represented by an *open* algebraic subspace $\mathrm{Pic}^0_{X/S} \subseteq \mathrm{Pic}_{X/S}$ that is of *finite type* over S. Its formation is easily checked to commute with any base change on S, due to Example 7.6.

Now assume $\dim(X_s) = 1$ for all $s \in S$. Cohomological vanishing results for coherent cohomology on curves (beyond degree-1 cohomology) imply the functorial smoothness criterion for $\mathrm{Pic}_{X/S}$, so this algebraic space is smooth. We conclude that for such "S-curves" X, $\mathrm{Pic}^0_{X/S}$ exists as a finite type smooth algebraic space group over S. An important result of Raynaud asserts that this algebraic space is sometimes a separated scheme (see [BLR, 9.4/2,3] for a discussion and references). We record a special case of Raynaud's theorem, sufficient for our needs (in view of the semistable reduction theorem for curves).

Theorem 7.13 (Raynaud)**.** *Let X be a proper flat semistable curve over $S = \mathrm{Spec}\, R$ for a discrete valuation ring R, with X regular and the generic fiber smooth. Then* $\mathrm{Pic}^0_{X/S}$ *is a separated scheme.*

This is fantastic: under the hypotheses of Raynaud's theorem (which can always be satisfied after a mild base change, by the semistable reduction theorem for curves of positive genus and its relation with the theory of minimal regular proper models), $\mathrm{Pic}^0_{X/S}$ is a smooth *separated* S-group scheme of finite type. Hence, if its special fiber is a semi-abelian variety then it is a semi-abelian scheme and so by Theorem 4.4 it would be the relative identity component of the Néron model of its generic fiber! In particular, we will have completed the proof the the semistable reduction theorem for Jacobians (which, as we have seen, implies the general case). Since the formation of $\mathrm{Pic}^0_{X/S}$ commutes with any base change (Example 7.6), such as passage to (geometric) fibers over S it remains to apply the following result (see [BLR, 9.2/8] for a proof):

Proposition 7.14. *Let X be a proper and geometrically connected semistable curve over a field k. The k-group $\mathrm{Pic}^0_{X/k}$ is a semi-abelian variety.*

In fact, the geometric fiber $(\mathrm{Pic}^0_{X/k})_{\overline{k}} = \mathrm{Pic}^0_{X_{\overline{k}}/\overline{k}}$ fits into a short exact sequence of k-groups

$$0 \to T \to \mathrm{Pic}^0_{X_{\overline{k}}/\overline{k}} \to \prod_i \mathrm{Pic}^0_{\widetilde{X}_i/\overline{k}} \to 0$$

where $\{X_i\}$ is the set of irreducible components of $X_{\overline{k}}$, $\widetilde{X}_i$ is the smooth normalization of X_i, and T is a torus whose character group is $\mathrm{H}_1(\Gamma, \mathbf{Z})$ for the dual graph Γ of $X_{\overline{k}}$.

Let us recall the meaning of the dual graph Γ. Its vertices correspond to the X_i, and its edges correspond to the singularities; two vertices of an edge corresponding to the X_i containing the two formal branches through the associated singularity (such formal branches may lie on the same component, in which case the edge is a loop in the graph). For example, if X is the nodal cubic then Γ is a graph with one vertex and one loop, whereas if X consists of two lines crossing transversally at a point then Γ is an edge with two vertices.

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