CHOW'S K/k-IMAGE AND K/k-TRACE, AND THE LANG–NÉRON THEOREM

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1. Introduction

Let K/k be an extension of fields, and assume that it is *primary*: the algebraic closure of k in K is purely inseparable over k. The most interesting case in practice is when K/k is a *regular* extension: K/k is separable and k is algebraically closed in K. Regularity is automatic if k is perfect. (For K/k finitely generated, regularity is equivalent to K arising as the function field of a smooth and geometrically connected k-scheme.)

In the theory of abelian varieties over finitely generated regular extensions K/k with respect to some field of "constants" k, there is a generalization of the Mordell–Weil theorem, due to Néron [26] (in his thesis) and Lang–Néron [19], and in this theorem a crucial role is played by the K/k-trace and the K/k-image of an abelian variety A over K. These constructions are also ubiquitous in many problems concerning families of abelian varieties. (The family is parameterized by a nice base V over k, and $K = k(V)$.) For an arbitrary primary extension of fields K/k, the K/k-*trace* of A is a final object in the category of pairs (B, f) consisting of an abelian variety B over k equipped with a K-map of abelian varieties $f : B_K \to A$, where B_K denotes the scalar extension $B \otimes_k K$; we write $(\mathrm{Tr}_{K/k}(A), \tau_{A,K/k})$ to denote such a final object (if it exists). Likewise, the K/k-*image* of A is an initial object in the category of pairs (B, f) consisting of an abelian variety B over k equipped with a K-map of abelian varieties $f : A \to B_K$; we write $(\mathrm{Im}_{K/k}(A), \lambda_{A,K/k})$ to denote such an object (if it exists). Roughly speaking, the K/k-image is the largest quotient of A that can be defined over k, and the K/k-trace is the largest abelian subvariety of A that can be defined over k. A precise description along these lines requires some care in positive characteristic. These concepts are due to Chow ([3], [4]).

Despite the importance of Chow's K/k-trace and K/k-image and the Lang–Néron theorem in arithmetic geometry, unfortunately no detailed general reference on these topics has been available entirely in the language of schemes. The papers of Chow ([3], [4]) and the book on abelian varieties by Lang [18] discuss the K/k-image and K/k-trace and develop their properties, but entirely in Weil's framework [34]. Similarly, in Lang's modern book [20] the Lang–Néron theorem is proved in Weil's language. In connection with my work in [5], where the Lang-Néron theorem plays a crucial role, I was motivated to write this expository account of a scheme-theoretic approach to Chow's results and the Lang–Néron theorem. In some instances the old and new methods are expressing similar ideas, but in other cases where we make extensive use of infinitesimal or flat descent methods it is less clear how much overlap there is. For example, our use of infinitesimal group schemes in the proof of the fundamental Chow regularity theorem (Theorem 5.5) replaces the ineffective "sufficiently large" aspect of the original version of the theorem (as in [3, Cor. to Thm. 8] and [18, VIII, Thm. 3]) with a simple explicit lower bound.

We begin in §2 with some intuition and examples related to Chow's work and the Lang–Néron theorem (including a precise statement of the latter). In §3 we summarize some background facts and terminology from algebraic geometry (centered largely on Grothendieck's descent theory and group schemes) and prove some other additional results for convenient reference later; some of the topics discussed in §3 are used in §2. In our development of the K/k-image in §4, we prove that the canonical map $\lambda_{A,K/k} : A \to \mathrm{Im}_{K/k}(A)_K$ is surjective with connected kernel that may be non-smooth in positive characteristic (Example 4.4). The behavior of the K/k-image with respect to extension of the ground field k is treated in §5. The key result

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here is that the formation of the K/k-image commutes with linearly-disjoint extension on k when K/k is regular. This is the most important fact in Chow's theory, and it is also the hardest to prove.

In §6 we develop the dual theory of the K/k-trace $\tau = \tau_{A,K/k} : \mathrm{Tr}_{K/k}(A)_K \to A$ whose kernel is K-finite with connected Cartier dual. We show by example (Example 6.3) that $\ker \tau$ may not be connected in positive characteristic, and we also prove the one fact that is not a trivial consequence of duality and the theory of the K/k-image: if K/k is regular then $\ker \tau$ is connected. In terms of the dual map $\lambda = \lambda_{A^\vee,K/k} : A^\vee \to \mathrm{Im}_{K/k}(A^\vee)_K$ this means that $\ker \lambda$ has vanishing multiplicative part when K/k is regular. In §7 we prove the Lang–Néron theorem, following some of the same reduction steps as in [20] and retaining the key idea of exploiting the fact that certain Hom-schemes are quasi-compact (a result known in the pre-Grothendieck era in the form of Chow coordinates). The reader is encouraged to begin with §2 and §7. We conclude in §8–§10 with a scheme-theoretic development of the theory of Néron–Tate heights for abelian varieties over rather general ground fields as in the context of the Lang–Néron theorem.

A nice application of the theory of the K/k-trace and the Lang–Néron theorem is Grothendieck's spectacular proof that an abelian variety of CM-type over an algebraically closed field must be isogenous to an abelian variety defined over a finite extension of the prime field. (In characteristic zero we can replace "isogenous" with "isomorphic", but in positive characteristic this cannot be done and hence the result really is non-trivial.) The key to constructing the right abelian variety over a finite extension of the prime field is to form a suitable K/k-trace. We refer the reader to [27] for an exposition of Grothendieck's proof. In §3 of Raynaud's Bourbaki report [28] on Grothendieck's generalization of the Ogg–Shafarevich formula, the reader can find some additional elegant applications of the Lang–Néron theorem. Some more recent papers that apply the Lang–Néron theorem and discuss constructions of the K/k-image and K/k-trace for finitely generated regular extensions K/k are [15] (which gives a construction of the K/k-image using Albanese varieties) and [13] and [29] (which give Raynaud's construction of the K/k-trace using Picard varieties).

Terminology and Notation. For any field k, a *k-variety* is a separated and geometrically integral k-scheme of finite type. If V is a finite-dimensional vector space over a field k then $\mathbf{P}(V) = \mathrm{Proj}(\mathrm{Sym} V)$ denotes the projective space classifying hyperplanes in V. The *dual* of an abelian variety A is denoted $A^\vee$. For any scheme S and S-scheme X, if $S' \to S$ is a map of schemes then $X_{S'}$ and $X_{/S'}$ denote $X \times_S S'$ considered as an S'-scheme in the usual manner; we use similar notation for base change applied to S-maps between S-schemes. If $S' = \mathrm{Spec}\, A'$ then we may write $X_{A'}$ and $X_{/A'}$ (and $X \otimes_A A'$ if also $S = \mathrm{Spec}\, A$) rather than $X_{S'}$ and $X_{/S'}$.

An extension of fields K/k is *primary* if k is separably closed in K, is *separable* if K is a direct limit of finitely generated extensions that each admit a separating transcendence basis over k (one of several equivalent definitions; see [22, Thm. 26.2]), and is *regular* if it is separable and primary (so in particular, k is algebraically closed in any regular extension of k).

We indulge in one notational convention that should not cause too much confusion: if K/k is a primary extension and E/k is an arbitrary extension, then EK denotes the fraction field of the domain $(E \otimes_k K)_{\mathrm{red}}$ obtained by passing to the quotient of $E \otimes_k K$ by its unique minimal prime ideal. Beware that if E and K are given as subextensions of an ambient extension L/k, then the domain $(E \otimes_k K)_{\mathrm{red}}$ maps to the compositum of E and K inside of L but this map is an injection if and only if E and K are linearly disjoint over the intersection of $E \cap K$ with the algebraic closure of k in L (exercise!), in which case EK maps isomorphically onto the compositum. We could alternatively speak throughout in the language of linear disjointness, but this is too cumbersome. The property that makes the notation EK useful is that EK/E is again a primary extension [7, IV_2, 4.3.2] and if E'/E is an extension then $E'(EK) = E'K$. This allows us to use transitivity arguments without having to think twice. Note also that if K/k is regular then EK/E is regular because separability of K/k is inherited by EK/E.

2. Motivation and examples

The duality theory of abelian varieties shows that the concepts of K/k-image and K/k-trace are dual to each other in an evident manner. It is not a requirement in the universal property that the universal morphism $\tau : \mathrm{Tr}_{K/k}(A)_K \to A$ be a closed immersion. Also, it is not a requirement in the universal property that the universal morphism $\lambda : A \to \mathrm{Im}_{K/k}(A)_K$ have connected (or smooth) kernel or be surjective. The behavior of the K/k-image and K/k-trace with respect to extension of the ground field and the reason for their existence will depend in an essential way on the hypothesis that K/k is a primary extension.

If K/k is finitely generated and regular then there is a way to visualize the K/k-trace, as follows. Consider an abelian variety A over K as an "algebraic family" of abelian varieties over k in the sense that $K = k(V)$ for a smooth k-variety V and (by shrinking V) A is the generic fiber of an abelian scheme $\mathscr{A}$ over V. Each fiber $\mathscr{A}_v$ has a semisimple decomposition over $k(v)$ in the sense of the Poincaré reducibility theorem, and the K/k-trace is (roughly speaking) the part of these fibral decompositions that is "the same" across all fibers (or, equivalently, is independent of the parameters in the base V). For this reason, for any primary extension K/k the abelian variety $\mathrm{Tr}_{K/k}(A)$ over k is called the *fixed part* of A relative to the extension K/k. The scheme-theoretic image of $\mathrm{Tr}_{K/k}(A)_K$ in A (for any primary K/k) is an abelian subvariety of A, called the *K/k-maximal abelian subvariety* of A, but beware that in positive characteristic it is often *not* "defined over k" (in contrast with $\mathrm{Tr}_{K/k}(A)_K$); see §6 for further discussion of this issue.

Suppose that A is an abelian variety over a field K that is finitely generated and regular over a field k, so $K = k(V)$ for a smooth k-variety V. Consider the problem of whether or not $A(K)$ is finitely generated. Shrinking V if necessary, let $\mathscr{A}$ be an abelian scheme over V whose generic fiber is A. Since $\mathscr{A}$ is V-separated and V-flat, $\mathscr{A}(V)$ is naturally a subgroup of $A(K)$. (In fact, since $\mathscr{A}$ is a smooth and proper group over the normal base V, the valuative criterion for properness and an extension lemma of Weil [1, 4.4/1] ensure that $A(K) = \mathscr{A}(V)$, so all elements of $A(K)$ may be identified with cross-sections to the structural map $\mathscr{A} \to V$.) This makes it geometrically clear that if the family of abelian varieties $\mathscr{A}_v$ has a "common isogeny factor" A_0 over k, which is to say that if $\mathscr{A}$ admits $(A_0)_V$ as an isogeny factor over V, then $A(K)$ contains "constant sections" coming from $A_0(k) \subseteq (A_0)_V(V)$. Such a subgroup $A_0(k)$ may be very large (*e.g.*, if k is algebraically closed). Algebraically, if A admits an isogeny factor $(A_0)_K$ with A_0 defined over k, then $A_0(k)$ is a subgroup of $A_0(K) = (A_0)_K(K)$ and modulo a finite subgroup it injects into $A(K)$. In this way, we see that the existence of isogeny factors defined over k is a geometric obstruction to $A(K)$ being finitely generated when k algebraically closed. This motivates consideration of the quotient

$$A(K)/\tau(\mathrm{Tr}_{K/k}(A)(k)) \tag{2.1}$$

as a more reasonable group which one may hope to prove is finitely generated, where $\tau : \mathrm{Tr}_{K/k}(A)_K \to A$ is the canonical map. Since $\ker \tau$ is an infinitesimal K-group when K/k is regular (Theorem 6.12), for such K/k we can consider $\mathrm{Tr}_{K/k}(A)(k)$ as a subgroup of $A(K)$ and so we omit τ from the notation in (2.1). The reasonableness of considering (2.1) is confirmed by:

Theorem 2.1 (Lang–Néron). *If K/k is a finitely generated regular extension and A is an abelian variety over K, then $A(K)/\mathrm{Tr}_{K/k}(A)(k)$ is a finitely generated group.*

We will prove Theorem 2.1 in §7.

Example 2.2. Let K/k be a finitely generated regular extension and let E be an elliptic curve over K. We say E is *constant* (with respect to K/k) if $E \simeq (E_0)_K$ for an elliptic curve E_0 over k, and *non-constant* (with respect to K/k) otherwise. A necessary condition for constancy is that $j(E) \in K$ lies in k, but this is not sufficient. In our development of the Chow trace we shall prove that the canonical map $\mathrm{Tr}_{K/k}(A)_K \to A$ is an isomorphism for any abelian variety A over k, so the constant case of the Lang–Néron theorem for elliptic curves is the assertion that $E_0(K)/E_0(k)$ is finitely generated for any elliptic curve E_0 over k.

Now suppose that E is non-constant. In this case we claim $\mathrm{Tr}_{K/k}(E) = 0$, and so the Lang–Néron theorem for E and K/k says that $E(K)$ is finitely generated. Letting $E_0 = \mathrm{Tr}_{K/k}(E)$, in the general theory of the Chow trace we will see that the canonical map $\tau : (E_0)_K \to E$ has finite kernel, and so if $E_0 \neq 0$ then E_0 must be 1-dimensional and τ must be an isogeny. Thus, to prove $\mathrm{Tr}_{K/k}(E) = 0$ for a non-constant elliptic curve E over K, it suffices to show that a non-constant elliptic curve E over K cannot be K-isogenous to an elliptic curve of the form E'_K with E' an elliptic curve over k. Suppose otherwise, so there is an isogeny $f : E'_K \to E$. The kernel $G \subseteq E'_K$ is a finite K-subgroup of E'_K, whence $E'_K/G \simeq E$ and so to get a contradiction it suffices to prove:

Theorem 2.3. *Let K/k be a regular extension of fields, and let E' be an elliptic curve over k. Every finite K-subgroup G in E'_K is induced from a (necessarily unique) finite k-subgroup of E'.*

The main issue in the proof of this theorem is that the connected-étale sequence of G may be non-split when K is not perfect. The connected-étale sequence and other background concerning group schemes are discussed in §3. Note also that if we consider replacing elliptic curves in Theorem 2.3 with higher-dimensional abelian varieties (such as a product of two supersingular elliptic curves) then there are counterexamples to the k-descent conclusion when $\mathrm{char}(k) > 0$ and G is not K-étale.

Proof. If the identity component G^0 is the base change of a finite k-subgroup of E' then passing to the quotient by this subgroup would reduce us to the étale case. Hence, it is enough to separately treat the cases of connected G and étale G. The connected case is trivial in characteristic 0. The étale case in any characteristic is settled by Lemma 3.11 (taking H in this lemma to be $E'[N]$ for a nonzero integer N killing G).

It remains to treat the connected case in characteristic $p > 0$. In this case G must have p-power order (Example 3.10), say p^{n_0} with $n_0 \geq 0$. The key point now is that an elliptic curve over a field with characteristic $p > 0$ (unlike higher-dimensional abelian varieties) contains a *unique* infinitesimal subgroup of length p^n for each $n \geq 0$. Indeed, for any regular curve over a field there is a unique infinitesimal closed subscheme with any desired length supported at a rational point, and in the case of elliptic curves and subgroups supported at the origin we use the kernel of the relative p^n-Frobenius map (Definition 3.15) to settle the existence aspect for order p^n for each $n \geq 1$. The unique infinitesimal subgroup of E' with order p^{n_0} therefore gives the required descent from K to k. ■

Example 2.4. Let K_0 be a global field and let $K = K_0(t_1, \ldots, t_n)$ with $n \geq 1$. If A is an abelian variety over K then $A(K)$ is finitely generated by Theorem 2.1 because $\mathrm{Tr}_{K/K_0}(A)(K_0)$ is finitely generated (by the usual Mordell–Weil theorem over K_0), and there is a nonempty open $U \subseteq \mathbf{P}^n_{K_0}$ such that A extends to an abelian scheme $\mathscr{A}$ over U. Thus, for all $u_0 \in U(K_0)$ we get an abelian variety $\mathscr{A}_{u_0}$ over K_0 and there is a natural map between finitely generated groups

$$\rho_{u_0} : A(K) = \mathscr{A}(U) \to \mathscr{A}_{u_0}(K_0).$$

If A has large rank over K and one can control the kernel of the specialization map at u_0 then one can hope to find fibers $\mathscr{A}_{u_0}$ with large rank over K_0. For example, it is a theorem of Silverman [30, Thm. C] that if $n = 1$ and $\mathrm{Tr}_{K/K_0}(A) = 0$ then $\ker \rho_{u_0} = 0$ for all but finitely many $u_0 \in U(K_0)$; Silverman's proof requires characteristic 0, due to a use of resolution of singularities, but the argument can be modified to avoid resolution and to thereby work in any characteristic (for $n = 1$). Néron [26] proved a weaker specialization result for all $n > 0$: there are infinitely many $u_0 \in U(K_0)$ for which ρ_{u_0} is injective.

3. Some preliminary results

To make our arguments as self-contained as possible, we need to review some background facts and terminology related to Grothendieck's *fpqc* descent theory (which vastly generalizes classical Galois descent) and group schemes over a base scheme (which vastly generalize classical group varieties over a field). We also give proofs for some other results that will be needed in what follows.

An excellent introduction to Grothendieck's descent theory is [1, Ch. 6] (along with [7, IV_2, §2.2–2.7]). A basic question in the theory is the following: given a faithfully flat and quasi-compact (*fpqc*) map of schemes $S' \to S$, such as $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ for a faithfully flat map of rings $A \to A'$ (the main example for us being an extension of fields $k \to K$), can we identify the category of S-schemes as a full subcategory of the category of S'-schemes? We also want to relate properties of an S-morphism $f : X \to Y$ (such as properness, surjectivity, finiteness, smoothness, etc.) with the corresponding properties of the induced S'-morphism $f_{S'} : X_{S'} \to Y_{S'}$, and to relate "structures" on an S-scheme X (such as quasi-coherent sheaves, closed subschemes, group scheme structure, etc.) with corresponding "structures" on $X_{S'}$ equipped with suitable descent data with respect to $S' \to S$. See [1, Ch. 2] and the references therein for the fundamental definitions and results related to smooth and étale morphisms of schemes.

In general the natural map $\mathrm{Hom}_S(X, Y) \to \mathrm{Hom}_{S'}(X_{S'}, Y_{S'})$ is injective, and one of the first important results in *fpqc* descent theory is to characterize the image of this injection. To formulate the answer, we introduce some notation: if Z' is an S'-scheme then we write $p_1^*(Z')$ and $p_2^*(Z')$ to denote the schemes over

$S'' = S' \times_S S'$ induced by base change along the projections $p_1, p_2 : S'' \rightrightarrows S'$. For example, consider a finite Galois extension of fields k'/k with Galois group G, and take $S' \to S$ to be $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$. The natural map of k-algebras

$$k' \otimes_k k' \to \prod_{g \in G} k' \tag{3.1}$$

defined by $a \otimes b \mapsto (ag(b))_{g \in G}$ is an isomorphism, and this identifies $p_1^*(Z')$ with the disjoint union $\coprod_{g \in G} Z'$ of copies of Z' indexed by G and it identifies $p_2^*(Z')$ with the disjoint union $\coprod_{g \in G} g^*(Z')$ of the various "Galois twists" of the k'-scheme Z' with respect to the G-action on k'. The problem of descending objects over k' to objects over k was described by Weil and his contemporaries in terms of invariance with respect to suitable Galois actions, and the preceding description of the $p_j^*(Z')$'s as disjoint unions via (3.1) provides a mechanism to translate such statements about Galois-invariance into statements concerning schemes over the fiber product $\mathrm{Spec}\, k' \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k'$. This makes Weil's theory of Galois descent fit into the framework of the following descent theorems with respect to general (not necessarily algebraic or separable) field extensions and even general *fpqc* base change $S' \to S$.

If Z is an S-scheme, then for the S'-scheme $Z' = Z_{S'}$ there is a canonical S''-isomorphism $\varphi_Z : p_1^*(Z') \simeq p_2^*(Z')$ via the common identification of each side with $S'' \times_S Z$. Using this φ_Z, the main result on descent of morphisms is the following (see [1, 6.1/6(a)] for a proof):

Theorem 3.1 (Grothendieck)**.** *If $S' \to S$ is faithfully flat and quasi-compact map of schemes and X and Y are S-schemes then an S'-morphism $f' : X_{S'} \to Y_{S'}$ has the form $f_{S'}$ for a (necessarily unique) S-morphism $f : X \to Y$ if and only if $p_1^*(f') = p_2^*(f')$ in the sense that these maps correspond under the canonical S''-isomorphisms $\varphi_X : p_1^*(X_{S'}) \simeq p_2^*(X_{S'})$ and $\varphi_Y : p_1^*(Y_{S'}) \simeq p_2^*(Y_{S'})$.*

Example 3.2. If $S' \to S$ corresponds to a finite Galois extension of fields k'/k, the isomorphism (3.1) converts the criterion in Theorem 3.1 into the classical Galois-equivariance criterion for descending a k'-morphism to a k-morphism. This is worked out in [1, 6.2/B]. In another direction, a diagram chase shows that if $X_{S'}$ is endowed with an S'-group scheme structure then this descends (necessarily uniquely) to an S-group scheme structure on X if and only if the induced S''-group scheme structures on $p_1^*(X_{S'})$ and $p_2^*(X_{S'})$ coincide via the canonical S''-isomorphism $\varphi_X : p_1^*(X_{S'}) \simeq p_2^*(X_{S'})$.

Remark 3.3. Even if one is only interested in Theorem 3.1 or other descent theorems for the special case $S' = \mathrm{Spec}\, K$ and $S = \mathrm{Spec}\, k$ corresponding to a field extension K/k, it is crucial in some proofs to apply the descent machinery to the *fpqc* morphism $T' = X_{S'} \to X = T$ that is generally not a map between spectra of fields. Thus, even for practical purposes it is useful to allow the generality of $S' \to S$ as above.

As we have noted already, in practice one does not just want to (uniquely) descend morphisms but also quasi-coherent sheaves (from $X_{S'}$ to X), closed subschemes, properties of morphisms, etc. For many standard properties $\mathbf{P}$ of morphisms of schemes (such as properness, surjectivity, finiteness, smoothness, etc.; see [7, IV$_2$, 2.7.1] and [7, IV$_4$, 17.7.3(*ii*)] for typical properties) one has that an S-map $f : X \to Y$ satisfies $\mathbf{P}$ if and only if $f_{S'} : X_{S'} \to Y_{S'}$ does. The problem of descent of an S'-scheme to an S-scheme in general is a subtle one, even for finite Galois extensions of fields, but in a special case we have a simple criterion that notably applies to abelian varieties (and is a special case of a general criterion of Grothendieck [1, 6.1/6(b)]):

Corollary 3.4. *Let k'/k be a finite Galois extension of fields and let X' be a quasi-projective k'-scheme. Let $G = \mathrm{Gal}(k'/k)$. To specify a k-scheme X equipped with a k'-isomorphism $X_{k'} \simeq X$ is equivalent to giving the data of k'-isomorphisms $\alpha_g : g^*(X') \simeq X'$ satisfying the cocycle condition $\alpha_{g_1 g_2} = \alpha_{g_1} \circ g_1^*(\alpha_{g_2})$ for all $g_1, g_2 \in G$. Such an X is necessarily quasi-projective over k.*

A k'-group scheme structure on X' descends to a k-group scheme structure on such an X if and only if each α_g is a k'-group scheme map.

To functorially descend a quasi-coherent sheaf on $X_{S'}$ to one on X there is a necessary and sufficient criterion that is natural generalization of a classical Galois-action criterion (see [1, 6.1/4], applied to the *fpqc* morphism $X_{S'} \to X$). In the case of quasi-coherent ideal sheaves this leads to the following key fact that we will often use without comment:

Theorem 3.5. *Let $S' \to S$ be faithfully flat and quasi-compact, and let X be an S-scheme. The map $Z \mapsto Z_{S'}$ from the set of closed subschemes of X to the set of closed subschemes of $X_{S'}$ is injective, and a closed subscheme $Z' \hookrightarrow X_{S'}$ descends (necessarily uniquely) to a closed subscheme $Z \hookrightarrow X$ if and only if $p_1^*(Z') = p_2^*(Z')$ as closed subschemes of $p_1^*(X_{S'}) \simeq p_2^*(X_{S'})$.*

In particular, if X is an S-group scheme and Z is a closed subscheme of X then Z is an S-subgroup scheme of X if and only if $Z_{S'}$ is an S'-subgroup scheme of $X_{S'}$.

Example 3.6. If k'/k is a finite Galois extension of fields and X is a k-scheme then the theorem says that a closed subscheme Z' in $X_{k'}$ descends to one in X if and only if the natural isomorphism $g^*(X_{k'}) \simeq X_{k'}$ for each $g \in \mathrm{Gal}(k'/k)$ carries $g^*(Z')$ to Z'; this is the classical Galois-stability criterion. If K/k is an arbitrary extension of fields, A is an abelian variety over k, and $B' \subseteq A_K$ is an abelian subvariety over K that descends to a closed subscheme $B \subseteq A$ then B is necessarily an abelian subvariety of A.

In the theory of group schemes, the main results that we require take place in the category of group schemes of finite type over a field k. We will sometimes have to work with possibly disconnected k-group schemes, but in the connected case over k there is never disconnectedness arising from extension of the base field because a connected k-scheme X with $X(k) \neq \emptyset$ is geometrically connected over k (i.e., $X \otimes_k K$ is connected for any extension field K/k); this geometric connectivity is a special case of [7, IV_2, 4.5.13].

Theorem 3.7. *Let k be a field, and let G be a k-group scheme of finite type. For any closed k-subgroup scheme H in G there is a unique H-invariant faithfully flat k-map $\pi : G \to G/H$ to a separated finite type k-scheme such that the action map $G \times H \to G \times_{G/H} G$ is an isomorphism, and π is initial for H-invariant morphisms from G to other schemes. In particular, if K/k is an extension field then the natural map $G_K/H_K \to (G/H)_K$ is an isomorphism.*

If G is a smooth k-group then G/H is k-smooth, and if in addition H is normal in G in the sense that the action map $G \times H \to G$ via $(g, h) \mapsto ghg^{-1}$ factors through $H \hookrightarrow G$ then G/H has a unique k-group structure compatible with that on G.

Proof. This follows from [12, $\mathrm{IV_A}$, 3.2] and Theorem 3.1. In the special case that H is a k-finite commutative group scheme, these results are special cases of [25, Thm. 1, p.111]. ∎

Example 3.8. If $f : G \to G'$ is a k-group morphism between finite type k-group schemes then $G/(\ker f)$ is naturally a k-group scheme of finite type and $G/(\ker f) \to G'$ is monic, hence a closed immersion [12, $\mathrm{VI_B}$, Cor. 1.4.2]. That is, $G/(\ker f)$ is naturally a closed k-subgroup of G'. In particular, if f is surjective and G' is smooth then $G/(\ker f) \simeq G'$. As a special case, if $f : A \to B$ is a map between abelian varieties over a field k then $A/(\ker f)$ is an abelian variety and so it is naturally an abelian subvariety of B.

Example 3.9. If G is a finite commutative group scheme over a field k and $H \subseteq G$ is a closed k-subgroup then $\#G = \#H \cdot \#(G/H)$ where the *order* $\#X$ of a finite k-scheme X is the k-dimension of its coordinate ring. Indeed, since $G \times H \simeq G \times_{G/H} G$ we just have to check that the finite flat map $G \to G/H$ has constant fibral degree equal to $\#H$, and this equality is clear because its geometric fibers are isomorphic to H via translation. As a simple consequence, we see that if G has prime order then $H = 0$ or $H = G$.

Example 3.10. Let G be a finite commutative group scheme over a field k, and let G^0 be its identity component; this is geometrically connected over k and (for topological reasons) is a subgroup scheme of G. Since the formation of the finite commutative k-group G/G^0 is compatible with extension on k, by extending scalars to an algebraic closure $\overline{k}$ of k and using that each connected component of $G_{\overline{k}}$ contains a unique $\overline{k}$-rational point we see that $G_{\overline{k}}$ is uniquely and functorially the product of $G^0_{\overline{k}}$ and a constant group (that in turn is canonically identified with $G_{\overline{k}}/G^0_{\overline{k}} \simeq (G/G^0)_{\overline{k}}$). Hence, G/G^0 is k-étale. By [8, Ch. I, 9.1, 9.5/2], the case $G^0 \neq 0$ can only occur in characteristic $p > 0$, in which case $G^0 \simeq \mathrm{Spec}(k[x_1, \dots, x_N]/(x_1^{p^{e_1}}, \dots, x_N^{p^{e_N}}))$ as pointed k-schemes for some $N \geq 0$ and $e_1, \dots, e_N > 0$, so the order of G^0 is a power of p.

We call the diagram

$$0 \to G^0 \to G \to G/G^0 \to 0$$

the *connected-étale sequence* of G and we call G/G^0 the *étale part* of G and denote it $G^{\text{ét}}$; the formation of this diagram is functorial in G and commutes with any field extension on k. We have just seen that the connected-étale sequence uniquely and functorially splits over an algebraic closure $\overline{k}$, so by Galois descent it uniquely and functorially splits when k is perfect (i.e., when $\overline{k}/k$ is Galois). This sequence can fail to split when k is imperfect, and this possibility will arise in a crucial step in our proof of an important result of Chow (Theorem 5.5). For this purpose, the following descent lemma (along with Lemma 3.14) will be useful.

Lemma 3.11. *Let K/k be a regular extension of fields and let H be a finite commutative k-group. If $G \subseteq H_K$ is an étale K-subgroup then it arises by base change from a unique étale k-subgroup of H.*

Note that the regularity of K/k is a crucial hypothesis in this lemma. Indeed, one gets many counterexamples in characteristic $p > 0$ for purely inseparable K/k by taking $K = k(a^{1/p})$ for $a \in k^\times$ not a pth power in k and H equal to the non-split p-torsion extension of $\mathbf{Z}/p\mathbf{Z}$ by μ_p classified by the non-trivial element $a \bmod (k^\times)^p \in k^\times/(k^\times)^p$ as in [16, 8.7.1]. In any characteristic, another source of counterexamples in the absence of a regularity hypothesis is k-étale H and K/k a finite Galois splitting field for H.

Proof. The uniqueness is clear by Theorem 3.5. Pick a separable closure k' of k, and let $K' = k' \otimes_k K$. Since k is separably closed in K we see that K' is a field and K'/K is Galois with the same Galois group as k'/k. Hence, if we can solve the descent problem for K'/k' then the k'-descent Γ' of $G_{K'} \subseteq H_{K'}$ in $H_{k'}$ is a $\mathrm{Gal}(k'/k)$-stable k'-subgroup because of the uniqueness of descent and the fact that $\Gamma' \otimes_{k'} K' = G_{K'} \subseteq H_{K'}$ is visibly $\mathrm{Gal}(K'/K)$-stable. Using Galois descent with respect to k'/k, the k'-descent Γ' in $H_{k'}$ then must descend to a k-subgroup Γ of H that solves the original problem: Γ has K-fiber in H_K that coincides with G because its K'-fiber in $H_{K'}$ is $G_{K'}$ by construction. This shows that it suffices to treat the case when k is separably closed, so we now assume k to be separably closed. In particular, $H^{\text{ét}} = H/H^0$ is a constant k-group. By expressing K as a direct limit of finitely generated regular extensions of k we can assume that K/k is finitely generated. Hence, $K = k(V)$ for a smooth k-variety V.

The composite map $G \to H_K \to H_K^{\text{ét}}$ has kernel $G \cap H_K^0$ that vanishes since G is K-étale, so G is identified with a closed K-subgroup of $H_K^{\text{ét}}$. But $H^{\text{ét}}$ is constant, so each closed K-subgroup of $H_K^{\text{ét}}$ arises by base change from a unique closed k-subgroup of $H^{\text{ét}}$. By replacing H with the preimage of this latter k-subgroup under the quotient map $H \to H^{\text{ét}}$ we can assume that G maps isomorphically to $H_K^{\text{ét}}$. In other words, the data of G amounts to a splitting of the connected-étale sequence of H_K, and we wish to prove that this forces the connected-étale sequence of H to be split. More generally, if

$$0 \to H' \to H \to H'' \to 0$$

is a short exact sequence of finite commutative k-groups (i.e., H' is closed in H and $H/H' \simeq H''$) and if there is a splitting after extending scalars to $K = k(V)$ then we claim that there is a splitting over k. By "smearing out" from the generic point $\mathrm{Spec}\, K$ of V, a K-splitting extends to a V_0-splitting of the diagram

$$0 \to H'_{V_0} \to H_{V_0} \to H''_{V_0} \to 0$$

for a suitable dense open $V_0 \subseteq V$. The set $V_0(k)$ is non-empty since V_0 is smooth over the separably closed field k, so specializing a V_0-splitting at any $v_0 \in V_0(k)$ gives a splitting of the original exact sequence over k. ∎

The methods in [25, §14] show that if k is a field and G is a finite commutative k-group then the functor $S \mapsto \mathrm{Hom}_{\mathrm{Gp}/S}(G_S, \mathbf{G}_{m,S})$ on k-schemes (where Gp/S denotes the category of group schemes over S) is represented by a finite commutative k-group $\mathbf{D}(G)$, the *Cartier dual* of G, and the canonical map $G \to \mathbf{D}(\mathbf{D}(G))$ is an isomorphism ("double duality isomorphism"). For example, $\mathbf{D}(\mathbf{Z}/n\mathbf{Z}) = \mathbf{G}_m[n] = \mu_n$. The same methods work over any base ring, so for any base scheme S_0 and any finite locally free commutative group scheme G over S_0 there is a finite locally free commutative group scheme $\mathbf{D}(G)$ representing the functor $S \mapsto \mathrm{Hom}_{\mathrm{Gp}/S}(G_S, \mathbf{G}_{m,S})$ on the category of S_0-schemes, and $G \simeq \mathbf{D}(\mathbf{D}(G))$. If $S_0' \to S_0$ is any S_0-scheme, then we have naturally $\mathbf{D}(G)_{S_0'} \simeq \mathbf{D}(G_{S_0'})$ as S_0'-groups. In the special case that the base S_0 is $\mathrm{Spec}\, k$ for a field k, since an inclusion between Hopf algebras over a field is faithfully flat [33, 14.1] it follows that that a map $f : G' \to G$ between finite commutative k-groups is a closed immersion (resp. faithfully flat) if and only if $\mathbf{D}(f)$ is faithfully flat (resp. a closed immersion). Using Nakayama's Lemma on fibers and fibral flatness criteria [7, IV_3, 11.3.10], the same assertion carries over to maps between finite locally free commutative group schemes over any base scheme S_0.

Example 3.12. A finite commutative group scheme G over a field k is *multiplicative* if $\mathbf{D}(G)$ is étale over k. If k is separably closed then this says that $\mathbf{D}(G)$ is constant, or equivalently (by double duality) that G is a finite product of groups of the form $\mathbf{D}(\mathbf{Z}/n\mathbf{Z}) \simeq \mu_n$ (hence the terminology). In particular, if k has positive characteristic p then a multiplicative group is connected if and only if it has p-power order. In the case of perfect k with characteristic $p > 0$, we may apply Cartier duality to the uniquely and functorially split connected-étale sequence of G to uniquely decompose G into a product of four kinds of finite commutative k-groups: étale with étale dual (this is the prime-to-p part of $G^{\text{ét}}$), étale with connected dual (this is $G^{\text{ét}}[p^\infty]$), connected with étale dual (this is $\mathbf{D}(\mathbf{D}(G^0)^{\text{ét}})$, the *multiplicative part* of G), and connected with connected dual (this is $\mathbf{D}(\mathbf{D}(G^0)^0)$, the *local-local* part of G). These four factors are respectively denoted G_{rr}, G_{rl}, G_{lr}, and G_{ll} since a finite scheme over a perfect field is étale if and only if it is reduced. In the case of algebraically closed k, this is all worked out in [25, p. 136].

Example 3.13. If $f : A \to B$ is an isogeny between abelian varieties over a field k and $f^\vee : B^\vee \to A^\vee$ is the dual isogeny then the finite commutative k-groups $\ker f$ and $\ker f^\vee$ are each canonically isomorphic to the Cartier dual of the other (in a manner respecting extension of the base field). This is stated over an algebraically closed field in [25, §15, Thm. 1], but the proof there works without restriction on the base field. There are more refined questions that one can ask concerning double duality for finite k-groups and abelian varieties over k, but we do not need to address such matters for our purposes.

Lemma 3.14. *Let k be a perfect field with characteristic $p > 0$ and let H be a finite commutative k-group with associated four-fold decomposition*

$$H = H_{\text{rr}} \times H_{\text{rl}} \times H_{\text{lr}} \times H_{\text{ll}}$$

as at the end of Example 3.12. *For any extension field K/k and any closed K-subgroup $G \subseteq H_K$, the natural map*

$$(G \cap (H_{\text{rr}})_K) \times (G \cap (H_{\text{rl}})_K) \times (G \cap (H_{\text{lr}})_K) \times (G \cap (H_{\text{ll}})_K) \to G \tag{3.2}$$

is an isomorphism.

Proof. If K is perfect then we have $(H_{\text{rr}})_K = (H_K)_{\text{rr}}$ and similarly for the other three factors of H, so the functoriality of the four-fold decomposition over K (applied also to G) gives the result in this case. For general K, since the formation of $G \cap (H_{\text{rr}})_K, \dots, G \cap (H_{\text{ll}})_K$ commute with arbitrary extension on K we see that the map (3.2) between finite commutative K-groups becomes an isomorphism after extension of scalars to the perfect closure of K. Hence, it is an isomorphism. ■

The final general concepts that we shall review from the theory of group schemes are the relative Frobenius and Verschiebung morphisms. Fix a prime p and consider $\mathbf{F}_p$-schemes. For any $\mathbf{F}_p$-scheme S, let $F_S : S \to S$ be the absolute Frobenius morphism (identity on underlying topological spaces, the pth-power map on $\mathscr{O}_S$); this is functorial with respect to arbitrary maps of $\mathbf{F}_p$-schemes. For any S-scheme X and $n \geq 0$, we let $X^{(p^n)}$ denote the S-scheme $S \times_{F_S^n, S} X$ obtained from X by base change through F_S^n. Roughly speaking, $X^{(p^n)}$ is obtained from X by replacing coefficients in the "defining equations" of X over S by their p^nth powers. This is well-behaved with respect to base change in the sense that if $S' \to S$ is a map of $\mathbf{F}_p$-schemes then there is a natural S'-isomorphism $(X_{S'})^{(p^n)} \simeq (X^{(p^n)})_{S'}$ due to the functoriality of F_S and $F_{S'}$ with respect to the map $S' \to S$. If $f : X \to Y$ is an S-morphism then $f^{(p^n)} : X^{(p^n)} \to Y^{(p^n)}$ denotes the induced map after base change.

Definition 3.15. For $n \geq 0$, the *relative p^n-Frobenius morphism* $F_{X/S,n} : X \to X^{(p^n)}$ is the unique S-map whose composite with the projection $X^{(p^n)} \to X$ (over $F_S^n : S \to S$) is F_X^n. For $n = 1$ we also use the notation $F_{X/S}$, and this is called the *relative Frobenius morphism* for X over S.

This definition makes sense since the absolute Frobenius morphisms F_X and F_S are compatible via the structure map $X \to S$. Note that $F_{X/S,n}$ is an S-map whereas F_X^n is generally not (unless F_S^n is the identity, such as for $S = \mathrm{Spec}\,\kappa$ with κ a finite field satisfying $[\kappa : \mathbf{F}_p] | n$). Roughly speaking, $F_{X/S,n}$ is the map induced by raising "coordinates" (over S) to the p^nth power. Explicitly, for $n \geq 1$,

$$F_{X/S,n} = F_{X^{(p^{n-1})}/S} \circ \cdots \circ F_{X/S}. \tag{3.3}$$

The map $F_{X/S,n}$ is functorial in the S-scheme X, is compatible with the formation of products in X over S, and is compatible with any base change $S' \to S$ in the sense that $(F_{X/S,n})_{S'} = F_{X_{S'}/S',n}$ via the natural isomorphism $(X^{(p^n)})_{S'} \simeq (X_{S'})^{(p^n)}$. In particular, for an S-group scheme G the map $F_{G/S,n}$ is a morphism of S-groups and $F_{G/S,n}^{(p^m)} = F_{G^{(p^m)}/S,n}$ for any $m \geq 1$.

For an S-group G that is commutative and S-flat, there is a canonical S-group map $V_{G/S} : G^{(p)} \to G$ [12, $\mathrm{VII_A}$, 4.2–4.3] called the *relative Verschiebung morphism* that satisfies $V_{G/S} \circ F_{G/S} = [p]_G$. The formation of $V_{G/S}$ commutes with any base change on S and it is functorial in the S-group G. If G is a finite locally free commutative group scheme over S then $V_{G/S} = \mathbf{D}(F_{\mathbf{D}(G)/S})$ [12, $\mathrm{VII_A}$, 4.3.3]. For $n \geq 1$, we define the S-group map

$$V_{G/S,n} \stackrel{\mathrm{def}}{=} V_{G/S} \circ \cdots \circ V_{G^{(p^{n-1})}/S} : G^{(p^n)} \to G,$$

so $V_{G/S,n} \circ F_{G/S,n} = [p^n]_G$. In particular, $[p^n]_G = 0$ if $F_{G/S,n} = 0$, and so by Examples 3.9 and 3.10 we see that any finite commutative group scheme over a field is killed by its order.

Example 3.16. The map $V_{\mathbf{G}_{a,S}/S}$ vanishes because $F_{\mathbf{G}_{a,S}/S}$ is faithfully flat and $[p]_{\mathbf{G}_{a,S}} = 0$. The subgroup $\alpha_{p,S} = \ker F_{\mathbf{G}_{a,S}/S} = \mathrm{Spec}_S(\mathscr{O}_S[T]/(T^p)) \subseteq \mathbf{G}_{a,S}$ is the S-group scheme of pth roots of 0 (with additive group structure), and it is a tautology that $F_{\alpha_{p,S}/S} = 0$ whereas $V_{\alpha_{p,S}/S} = 0$ due to the vanishing of $V_{\mathbf{G}_{a,S}/S}$.

Example 3.17. By working over an algebraic closure $\overline{k}$ of k and using the explicit description of the relative Frobenius in terms of pth-power maps, we see that (i) $F_{G/k}$ is an isomorphism if and only if G is étale over k, and (ii) $F_{G/k,n} = 0$ for large n if and only if G is connected. Hence, by (3.3) we can filter the connected part G^0 by kernels of successive iterates of relative Frobenius so that the successive quotients in the filtration have vanishing relative Frobenius. On the maximal local-local quotient of G^0 (the Cartier dual to $\mathbf{D}(G^0)^0$) we can apply the same procedure and then refine it further by using kernels of iterates of the relative Verschiebung morphism (i.e., we form kernels of Frobenius iterates on the Cartier dual, and then dualize back). In this way we can filter the local-local part of G with successive quotients whose relative Frobenius and relative Verschiebung morphisms both vanish.

This motivates the question of describing all finite commutative k-groups G for which $F_{G/k}$ and $V_{G/k}$ vanish. In case k is perfect (e.g., algebraically closed), such G's are precisely products of finitely many copies of the k-group scheme α_p. Indeed, by Dieudonné theory over k [8, Ch. III, 1.4, 3.2, 3.3] the category of G's with $F_{G/k} = 0$ and $V_{G/k} = 0$ is antiequivalent to the category of finite-dimensional k-vector spaces, with G of order p^r going over to a vector space of k-dimension r, and the k-group α_p of order p corresponds to a 1-dimensional k-vector space under this anti-equivalence. (As a special case, $\mathbf{D}(\alpha_p) \simeq \alpha_p$ over $\mathrm{Spec}(\mathbf{F}_p)$ and hence over *any* $\mathbf{F}_p$-scheme by base change.) A useful consequence of this classification is the following result that will be used in our proof of Chow's regularity theorem.

Theorem 3.18. *Let k be a perfect field of characteristic $p > 0$ and let G be a finite commutative k-group such that $F_{G/k}$ and $V_{G/k}$ vanish. For any extension field E/k, the operation $H \mapsto T_0(H)$ is a bijection from the set of closed E-subgroups of G_E to the set of E-subspaces of $T_0(G_E)$. Moreover, $H_1 \subseteq H_2$ if and only if $T_0(H_1) \subseteq T_0(H_2)$.*

Proof. Since k is perfect, we may and do fix an isomorphism $G \simeq \alpha_p^n \subseteq \mathbf{G}_a^n$ over k and then we claim (with slight abuse of notation) that the operations $W \mapsto W \cap \alpha_{p,E}^n$ and $H \mapsto T_0(H) \subseteq T_0(\mathbf{G}_{a,E}^n) \simeq \mathbf{G}_{a,E}^n$ are inverse bijections between the set of vector subgroups of $\mathbf{G}_{a,E}^n$ and the set of closed E-subgroups of $\alpha_{p,E}^n$; this claim certainly implies the theorem. It suffices to check this general claim over an arbitrary algebraically closed extension field E/k. Every closed E-subgroup of $\alpha_{p,E}^n$ for such E is a product of copies of $\alpha_{p,E}$, and

$\mathrm{Hom}_E(\alpha_{p,E}, \alpha_{p,E}) = E$ via the scaling action, so we easily get that the two operations are inverse to each other. ■

This concludes our background review of descent theory and group schemes, and now we provide proofs for a few other necessary results. Let us begin with a crucial result due to Chow (see [3] or [18, Ch. II, Thm. 5]), for which we give a Grothendieck-style proof via descent theory.

Theorem 3.19 (Chow)**.** *Let A and B be abelian varieties over a field k and let K/k be a primary extension. Any map of abelian varieties $f : A_K \to B_K$ is defined over k in the sense that the injective map $\mathrm{Hom}_k(A, B) \to \mathrm{Hom}_K(A_K, B_K)$ is bijective.*

This theorem is especially useful for separably closed k, in which case every extension K/k is primary. In the proof of Theorem 3.19 and throughout later sections we will find it useful to invoke some elementary concepts related to *abelian schemes* (i.e., smooth proper group schemes with geometrically connected fibers). In [24, Ch. 6] there is given a systematic treatment of the basics (and much more) concerning abelian schemes.

Proof. Let $K' = K \otimes_k K$. Since K is a primary extension of k, $\mathrm{Spec}\, K'$ is irreducible and in particular is connected. By Theorem 3.1, it suffices to show that the two pullbacks $p_j^*(f) : A_{K'} \to B_{K'}$ of f along the projections $p_1, p_2 : \mathrm{Spec}\, K' \rightrightarrows \mathrm{Spec}\, K$ are equal. To prove that $p_1^*(f) = p_2^*(f)$, we first check such equality on a single fiber over $\mathrm{Spec}\, K'$. Consider the canonical point $\mathrm{Spec}\, K \to \mathrm{Spec}\, K'$ defined by the diagonal. The pullback of each $p_j^*(f)$ via this point is f, so the desired equality is achieved on the fiber over the diagonal point. With equality achieved on one fiber, now consider the K'-maps induced by the $p_j^*(f)$'s on ℓ^n-torsion for $n \geq 1$, with ℓ a fixed prime distinct from the characteristic of k (so ℓ is a unit on $\mathrm{Spec}\, K'$). These torsion subschemes are finite étale over the connected base $\mathrm{Spec}\, K'$, and a map $h : Z' \to Z$ between finite étale schemes over a connected scheme S is uniquely determined by its restriction $Z'_s \to Z_s$ to fibers over a single geometric point s of the base scheme S [7, IV_4, 17.4.8]. Hence, $p_1^*(f)$ and $p_2^*(f)$ coincide on each $A_{K'}[\ell^n]$ for all $n \geq 1$.

To infer equality of $p_1^*(f)$ and $p_2^*(f)$ on $A_{K'}$, we want a map between abelian schemes over K' to be uniquely determined by its restricton to all ℓ-power torsion subgroup schemes. We shall appeal to a more general sufficient claim: if $\mathscr{A} \to \mathscr{S}$ is any abelian scheme over a scheme $\mathscr{S}$ and if ℓ is any prime then the collection of closed subschemes $\mathscr{A}[\ell^n]$ for all $n \geq 1$ is universally schematically dominant in $\mathscr{A}$ with respect to $\mathscr{S}$ in the sense of [7, IV_3, 11.10.8] (we only need the case when ℓ is a unit on $\mathscr{S}$). To prove this, by working locally on $\mathscr{S}$ one can reduce to the case of noetherian $\mathscr{S}$, in which case [7, IV_3, 11.10.4, 11.10.9] reduces the problem to the classical schematic density of such torsion-levels on geometric fibers. ■

Theorem 5.5 ensures that the concept of "defined over k" for abelian varieties over K is both well-defined and *functorial* when K/k is primary. We shall use this repeatedly without comment. An important corollary is the validity of the Poincaré reducibility theorem over an arbitrary base field:

Corollary 3.20. *Let k be a field. If Y is an abelian subvariety of an abelian variety X over k then there exists an abelian subvariety $Z \subseteq X$ such that the natural map $Y \times Z \to X$ is an isogeny. In particular, the isogeny category of abelian varieties over a field is artinian and semisimple.*

Proof. A proof is given in [23, §12] when the base field k is perfect. (The proof is inapplicable for non-perfect k because the underlying reduced scheme of a finite type k-group scheme can fail to be a subgroup scheme when k is not perfect.) In the general case, if K/k is the perfect closure and $Y \hookrightarrow X$ is an abelian subvariety then we may pick an abelian subvariety $Z' \subseteq X_K$ such that the natural map $f : Y_K \times Z' \to X_K$ is an isogeny. Let $X_K \to Y_K \times Z'$ be an isogeny whose composite with f is multiplication by a nonzero integer. The composite K-map

$$X_K \to Y_K \times Z' \overset{\mathrm{pr}_2}{\to} Z' \hookrightarrow X_K$$

descends to a k-map $X \to X$ by Theorem 3.19, and its schematic image $Z \subseteq X$ is an abelian subvariety that is an isogeny-complement to Y in X (as we may check after the faithfully flat extension of scalars $k \to K$). ■

Corollary 3.21. *Let K/k be a primary extension of fields and let A be an abelian variety over k. Any abelian subvariety of A_K has the form A'_K for a unique abelian subvariety A' of A over k. In particular, if k is separably closed then an abelian variety over k acquires no new abelian subvarieties under any extension on the ground field.*

Proof. By Theorem 5.5, passage from k to K does not change Hom-groups, and in particular does not introduce new idempotents in the isogeny category, so if $\{A_i\}$ is a collection of mutually non-isogeneous k-simple abelian varieties such that A is k-isogenous to $\prod A_i^{e_i}$ (with $e_i > 0$), then the $A_{i/K}$'s are K-simple and A_K is K-isogeneous to $\prod A_{i/K}^{e_i}$. Thus, by Poincaré reducibility over K, any abelian subvariety B in A_K is the schematic image of some K-map of abelian varieties $\prod A_{i/K}^{e'_i} \to A_K$ for suitable $e'_i \leq e_i$. By Theorem 3.19, this map descends to a k-map of abelian varieties $\prod A_i^{e'_i} \to A$. The schematic image of this map is an abelian subvariety A' in A. Since the formation of schematic image commutes with the flat extension of scalars from k to K, we conclude that $B = A'_K$ as abelian subvarieties of A_K. ■

4. The K/k-image

Throughout this section, K/k denotes a primary extension of fields. We begin with a definition:

Definition 4.1. Let A be an abelian variety over K. A *K/k-image* of A is an initial object $(\mathrm{Im}_{K/k}(A), \lambda)$ in the category of pairs (B, f) consisting of an abelian variety B over k and a K-map of abelian varieties $f : A \to B_K$.

It is obvious that a K/k-image is unique up to unique isomorphism if it exists. An important example is:

Theorem 4.2. *Let A be an abelian variety over k. A K/k-image of A_K is given by the pair $(A, 1_{A_K})$.*

Proof. The assertion is that if B is any abelian variety over k and $f : A_K \to B_K$ is a map of abelian varieties over K, then it arises as the base change of a unique k-map of abelian varieties $A \to B$. This follows from Theorem 3.19, since K/k is primary. ■

Theorem 4.3. *For any abelian variety A over K, the K/k-image exists.*

Proof. If $f : A \to B_K$ and $f' : A \to B'_K$ are maps of abelian varieties with B and B' abelian varieties over k, then $(f, f') : A \to B_K \times B'_K = (B \times B')_K$ is a map of the same sort. The image of this map is an abelian subvariety of $(B \times B')_K$, and so by Corollary 3.21 it has the form X_K for a unique abelian subvariety X in $B \times B'$. It is clear that f and f' respectively uniquely factor through the K-fibers of the natural k-maps of abelian varieties $X \to B$ and $X \to B'$, so we have shown that the collection of pairs (B, f) admits finite suprema.

Each object (B, f) is uniquely dominated by an object (C, h) where C is an abelian subvariety of B and $h : A \to C_K$ is a surjection of abelian varieties (namely, take C to be the unique abelian subvariety of B such that C_K is the image of f; here we once again use that K/k is primary). Thus, it is enough to make an initial object in the category of pairs (B, f) such that f is surjective. Any such object is determined by the K-subgroup $\ker f \subseteq A$, and the construction of finite suprema shows that this collection of kernels is stable under finite intersection in A. The descending chain condition in A thereby produces an initial object. ■

Example 4.4. We give an example such that the map to the K/k-image has non-smooth kernel. Let K/k be a non-trivial extension in positive characteristic p with k algebraically closed. Let E/k be a supersingular elliptic curve, so the self-duality of E (see the proof of [16, 2.1.2]) implies that $\ker F_{E/k} \simeq \alpha_p$ due to Examples 3.13 and 3.17. Let

$$G \subseteq (\alpha_p \times \alpha_p)_K \subseteq E_K[p] \times_{\mathrm{Spec}\, K} E_K[p]$$

be an $\alpha_{p,K}$ that is not defined over k as a K-subgroup of $(\alpha_p \times \alpha_p)_K$. (By Theorem 3.18, to pick such a G amounts to picking a K-line L in the plane $T_0(\alpha_{p,K}^2) = K \otimes_k T_0(\alpha_p^2)$ such that L does not arise from a line in the k-vector space $T_0(\alpha_p^2)$.) For $A = (E \times E)_K/G$, we have $\mathrm{Im}_{K/k}(A) = E^{(p)} \times E^{(p)}$ and the natural map

$$\lambda : A \to (E \times E)_K/(\alpha_p \times \alpha_p)_K \simeq E_K^{(p)} \times E_K^{(p)}$$

is the K/k-image. The kernel of λ is isomorphic to $\alpha_{p,K}$, so $\ker\lambda$ is not smooth.

Let us now treat some formal properties.

Theorem 4.5. *Let A be an abelian variety over K.*

1. *If k/k_0 is primary and $(\mathrm{Im}_{k/k_0}(\mathrm{Im}_{K/k}(A)), \lambda_0)$ denotes the k/k_0-image of $\mathrm{Im}_{K/k}(A)$ then*
$$(\mathrm{Im}_{k/k_0}(\mathrm{Im}_{K/k}(A)), \lambda_{0/K} \circ \lambda)$$
is a K/k_0-image of A.
2. *If K'/K is a primary extension then $(\mathrm{Im}_{K/k}(A), \lambda_{K'})$ is a K'/k-image of $A_{K'}$.*
3. *The canonical map $\lambda : A \to \mathrm{Im}_{K/k}(A)_K$ is surjective with (geometrically) connected kernel.*

Proof. The first part is a tautology. The second part follows from the first part and Theorem 4.2. For the final part, let $H = \ker\lambda$. This is a (possibly non-smooth) closed subgroup of A_K. The quotient A/H is an abelian subvariety of $\mathrm{Im}_{K/k}(A)_K$ by Example 3.8. By Corollary 3.21, A/H must have the form X_K for a unique abelian subvariety X in $\mathrm{Im}_{K/k}(A)$. For any K-map of abelian varieties $h : A \to B_K$ with B an abelian variety over k, there is a unique k-morphism of abelian varieties $f : \mathrm{Im}_{K/k}(A) \to B$ such that $h = f_K \circ \lambda$, so h uniquely factors through the K-extension of $f|_X : X \to B$ via the natural map $A \twoheadrightarrow X_K$ induced by λ. By universality, we conclude that the inclusion of X into $\mathrm{Im}_{K/k}(A)$ must be an isomorphism. Hence, λ is surjective.

It remains to show that $H = \ker\lambda$ is connected. Let H^0 be the identity component and consider the quotient A/H^0. This is an abelian variety over K and the natural map
$$\rho : A/H^0 \to A/H = \mathrm{Im}_{K/k}(A)_K$$
is a finite surjection with kernel H/H^0 that is étale over K, so ρ is a finite étale covering. Let n be the degree of this covering, so the map $[n]_K : \mathrm{Im}_{K/k}(A)_K \to \mathrm{Im}_{K/k}(A)_K$ factors through ρ. The connected part of $\ker[n]_K$ is killed by ρ, so ρ is dominated by the base-change to K of the finite étale cover
$$\mathrm{Im}_{K/k}(A)/(\ker[n])^0 \to \mathrm{Im}_{K/k}(A)$$
induced by $[n]$. We claim that the subgroup
$$\ker(\mathrm{Im}_{K/k}(A)_K/(\ker[n]_K)^0 \to A/H^0) \subseteq (\ker[n]_K)/(\ker[n]_K)^0 = (\ker[n])^{\text{ét}}_K$$
descends to a subgroup of the finite étale $(\ker[n])^{\text{ét}}$. This holds because for compatible separable closures k_s/k and K_s/K, the natural map $\mathrm{Gal}(K_s/K) \to \mathrm{Gal}(k_s/k)$ is surjective (as K/k is primary, so $k_s \otimes_k K$ is naturally a subextension of K_s/K). Thus, there exists a unique abelian variety A_0 over k equipped with a finite étale map $\pi : A_0 \to \mathrm{Im}_{K/k}(A)$ that descends the canonical map $\rho : A/H^0 \to A/H$.

For any abelian variety B over k, any K-map of abelian varieties $h : A \to B_K$ admits a unique factorization as $f_K \circ \lambda$ where $f : \mathrm{Im}_{K/k}(A) \to B$ is a k-map of abelian varieties. Writing λ as $\rho \circ \lambda^0 = \pi_K \circ \lambda^0$ for the projection $\lambda^0 : A \to A/H^0$, clearly there is also a factorization of h as $g_K \circ \lambda^0$ for a unique map of abelian varieties $g = f \circ \pi : A_0 \to B$ over k (uniqueness of g follows from surjectivity of λ^0). Thus, the pair (A_0, λ^0) has the universal property of a K/k-image, and so the map π_K carrying λ^0 to λ must be an isomorphism. This shows that $H = H^0$ is connected. ■

Remark 4.6. Theorem 4.5(2) is false if the primality condition on K'/K is dropped. To give a counterexamples with regular K/k in arbitrary characteristic, let E_0 be an elliptic curve over k such that E_0 has geometric automorphism group $\{\pm 1\}$, let K'/K be a quadratic Galois extension with k algebraically closed in K' (so K'/k is regular), and let A be the nontrivial quadratic twist of $E_{0/K}$ associated to K'/K. In this case A cannot arise from an elliptic curve E_1 over k because otherwise the resulting K'-isomorphism $E_{1/K'} \simeq A_{K'} = E_{0/K'}$ would descend to a k-isomorphism $E_1 \simeq E_0$ (since K'/k is primary) and so would give a K-isomorphism $A \simeq E_{0/K}$, a contradiction. This non-constancy of A with respect to K/k forces $\mathrm{Im}_{K/k}(A) = 0$ by Example 2.2 and duality, yet $\mathrm{Im}_{K'/k}(A_{K'}) = E_0$.

Note that the functor $\mathrm{Im}_{K/k}(\cdot)$ carries finite products to finite products (since $\mathrm{Hom}(A \times A', X) = \mathrm{Hom}(A, X) \times \mathrm{Hom}(A', X)$ for abelian varieties A, A', and X over a field). Also, it carries isogenies to isogenies since isogenies are characterized as having a two-sided "inverse" (up to multiplication by a nonzero integer). Thus, for many questions about the K/k-image that take place in the isogeny category, there is often no loss of generality by restricting attention to the case of K-simple abelian varieties. The following useful result reduces many questions about the K/k-image to the case when the canonical map $\lambda = \lambda_{A,K/k} : A \to \mathrm{Im}_{K/k}(A)_K$ is an isogeny.

Corollary 4.7. *For any abelian variety A over K there exists a unique abelian subvariety $A' \subseteq A$ such that $\mathrm{Im}_{K/k}(A') = 0$ (so $\mathrm{Im}_{K/k}(A) \to \mathrm{Im}_{K/k}(A/A')$ is an isomorphism) and $A/A' \to \mathrm{Im}_{K/k}(A/A')_K$ is an isogeny.*

Proof. Since the additive functor $\mathrm{Im}_{K/k}$ commutes with products and carries isogenies to isogenies, by Corollary 3.21 and Theorem 4.2 we see that A' is the unique maximal abelian subvariety of A whose K-simple isogeny factors are K-isogenous to an abelian variety defined over k. ■

5. The K/k-image and base change

We now consider extension of the ground field. As before, K/k is a primary extension of fields and A is an abelian variety over K. For any extension E/k, there is a unique E-map of abelian varieties

$$I_{E/k} : \mathrm{Im}_{EK/E}(A_{EK}) \to \mathrm{Im}_{K/k}(A)_E \tag{5.1}$$

characterized by the property that composing

$$\lambda' = \lambda_{A_{EK},EK/E} : A_{EK} \to \mathrm{Im}_{EK/E}(A_{EK})_{EK}$$

with $(I_{E/k})_{EK}$ yields the base change

$$\lambda_{EK} : A_{EK} \to (\mathrm{Im}_{K/k}(A)_K)_{EK} = (\mathrm{Im}_{K/k}(A)_E)_{EK}$$

of $\lambda = \lambda_{A,K/k}$. We remind the reader that EK denotes the fraction field of the domain $(E \otimes_k K)_{\mathrm{red}}$, and it is not the compositum in an arbitrary common extension of E and K over k (unless we restrict attention to composites that satisfy a linear-disjointness condition over a suitable purely inseparable extension of k).

Theorem 5.1. *The canonical map $I_{E/k}$ is a purely inseparable isogeny.*

Proof. Since λ' and λ_{EK} are surjective with connected kernels by Theorem 4.5(3), $I_{E/k}$ is surjective and the EK-group scheme $(\ker I_{E/k})_{EK} = (\ker \lambda)_{EK}/(\ker \lambda')$ is connected, so $\ker I_{E/k}$ is connected. Hence, it remains to compare dimensions. Quite generally, for a primary extension K/k we wish to give a "geometric" description of $\dim \mathrm{Im}_{K/k}(A)$ in a manner that is unaffected by replacing K/k with EK/E (and replacing A with A_{EK}).

If X_0 and X_{00} are abelian varieties over k such that $X_{0/K}$ and $X_{00/K}$ are K-isogenous then X_0 is k-isogenous to X_{00} (since K/k is primary). Thus, for any abelian variety X over K there is a well-defined k-isogeny class $\mathscr{C}_{X,K/k}$ of abelian varieties of maximal dimension that are K-isogenous to a factor of X, and any abelian variety over k admitting a K-isogeny to a factor of X is k-isogenous to a factor of any member of the distinguished k-isogeny class $\mathscr{C}_{X,K/k}$. Roughly speaking, $\mathscr{C}_{X,K/k}$ corresponds to a maximal isogeny-factor of X that can be defined over k. It is obvious that $\mathrm{Im}_{K/k}(A)$ is a distinguished member of this isogeny class for $X = A$, and so the dimension of $\mathrm{Im}_{K/k}(A)$ is equal to the common dimension of the members of $\mathscr{C}_{A,K/k}$.

The problem of finiteness of $I_{E/k}$ is thereby reduced to showing that the scalar extension $k \to E$ carries members of $\mathscr{C}_{A,K/k}$ to members of $\mathscr{C}_{A_{EK},EK/E}$. If A' denotes an isogeny-factor of A over K that is complementary to $\mathrm{Im}_{K/k}(A)_K$ then the proof of Corollary 4.7 shows that $\mathrm{Im}_{K/k}(A') = 0$. Thus, it is enough to show that if $\mathrm{Im}_{K/k}(A) = 0$ then $\mathrm{Im}_{EK/E}(A_{EK}) = 0$. That is, if A admits no nonzero maps to abelian varieties B_K with B defined over k, then we must show that A_{EK} admits no nonzero maps to abelian varieties B_{EK} with B defined over E. This property is transitive in E, so it is enough to treat separately the cases where E/k

purely inseparable, separable algebraic, and separable in general. In each case, what we will really prove is that if $\mathrm{Im}_{EK/E}(A_{EK}) \neq 0$ then $\mathrm{Im}_{K/k}(A) \neq 0$. More precisely, since

$$\lambda_{A_{EK},EK/E} : A_{EK} \to \mathrm{Im}_{EK/E}(A_{EK})_{EK}$$

is a surjection, it suffices to prove that if A_{EK} admits an EK-isogeny factor B_{EK} for a nonzero abelian variety B over E, then there is a nonzero K-map of abelian varieties $A \to X_K$ for some abelian variety X over k (and hence $\mathrm{Im}_{K/k}(A) \neq 0$).

First consider the case when E/k is purely inseparable, so EK/K is primary. By expressing E as a direct limit of subextensions of finite degree over k, we may assume E to be of finite degree over k. We can also assume that k has positive characteristic p (as otherwise $E = k$ and we are done), so some relative q-Frobenius twist $B^{(q)}$ (with $q = p^n$ for some $n \geq 0$) is defined over k. Hence, $A_{EK}^{(q)}$ admits a nonzero isogeny-factor that is defined over k. A projection to such a factor descends from EK down to K since EK/K is primary, so $A^{(q)}$ has a nonzero K-isogeny factor that is defined over k. However, the relative q-Frobenius $A \to A^{(q)}$ is a K-isogeny, so we conclude that A has a nonzero K-isogeny factor that is defined over k. This takes care of the case when E is purely inseparable.

Now assume that E is separable algebraic, so we can assume E/k is a finite Galois extension. In particular, $EK = E \otimes_k K$. The Weil restriction $\mathrm{Res}_{EK/K}(A_{EK})$ (see [1, 7.6]) is a product of copies of A, and it has a K-isogeny factor given by the nonzero abelian variety $\mathrm{Res}_{EK/K}(B_{EK}) = \mathrm{Res}_{E/k}(B)_K$; this equality is due to compatibility of Weil restriction and base change. We thereby get a nonzero K-map of abelian varieties from A to an abelian variety over K that is defined over k.

Finally, we may assume that E/k is separable, and since the separable algebraic case is settled we can use a direct limit argument with E/k to see that it is enough to treat the case when $E = k(t)$ is purely transcendental of degree 1 over k. At the expense of separable algebraic increase on k (permissible by the steps we have just settled), it may be assumed that k is separably closed and in particular infinite. Let us assume that there is an abelian variety B over $k(t)$ and a nonzero map $f : A_{K(t)} \to B_{K(t)}$ over $K(t)$. Since B extends to an abelian scheme $\mathscr{B}$ over a dense open U in $\mathbf{P}^1_k$, the infinitude of k allows us to find $t_0 \in U(k)$ such that f extends around $t = t_0$ and so may be specialized to define a *nonzero* K-map of abelian varieties from A to $(B_{t_0})_K$ with B_{t_0} an abelian variety over k. (Non-vanishing of the specialization follows from considering the finite étale ℓ^n-power torsion subschemes over U in the abelian scheme $\mathscr{B}$ for a prime $\ell \neq \mathrm{char}(k)$ and all $n \geq 1$.) ■

The following corollary gives a criterion for an abelian variety A over K to be defined over k (i.e., for $\lambda_{A,K/k}$ to be an isomorphism) via a descent hypothesis on A_{EK} relative to E for a separable extension E/k.

Corollary 5.2. *Let K/k be a primary extension of fields and let A be an abelian variety over K. If there exists an abelian variety B defined over an extension E/k such that A_{EK} is EK-isogenous to a factor of B_{EK}, then the natural map*

$$\lambda = \lambda_{A,K/k} : A \to \mathrm{Im}_{K/k}(A)_K$$

is a purely inseparable isogeny. This map is an isomorphism if A_{EK} is EK-isomorphic to B_{EK} and E/k is separable.

Proof. We first claim that A_{EK} has the same dimension as its EK/E-image; that is, we claim that the canonical surjective map

$$\lambda_{A_{EK},EK/E} : A_{EK} \to \mathrm{Im}_{EK/E}(A_{EK})_{EK}$$

is an isogeny. This property is isogeny-invariant and is inherited by direct factors, so since A_{EK} is an isogeny factor of B_{EK} the desired result follows from the fact that B_{EK} has EK/E-image equal to B (by Theorem 4.2). By Theorem 5.1 we conclude that

$$\dim \mathrm{Im}_{K/k}(A) = \dim \mathrm{Im}_{EK/E}(A_{EK}) = \dim A,$$

so the map $\lambda : A \to \mathrm{Im}_{K/k}(A)_K$ that is *a priori* surjective with connected kernel must be an isogeny and hence is purely inseparable.

Now assume that E/k is separable and that there is an EK-isomorphism of abelian varieties $\varphi : B_{EK} \simeq A_{EK}$. We want to show that λ is an isomorphism. Equivalently, in view of Theorem 4.2, we need to show that A can be defined over k. By direct limit considerations with the separable E/k we can assume $E = k'(V')$ for a smooth variety V' over a finite separable extension k'/k, and by smearing-out of φ over a dense open $U' \subseteq V'$ and specializing at a closed point $u' \in U'$ for which $k'(u')/k'$ is separable we may assume E/k is finite and separable. By increasing E/k to be normal, transitivity (as in Theorem 4.5(1)) reduces us to treating the case when E/k is finite Galois, so $EK = E \otimes_k K$ and hence we may transfer the Galois descent data on A_{EK} (via the K-structure A) into Galois descent data on B_{EK} relative to the extension EK/K. However, $\mathrm{Gal}(EK/K) = \mathrm{Gal}(E/k)$ and any EK-isomorphism $B_{EK} \simeq (B_{EK})^{\sigma} = (B^{\sigma})_{EK}$ of abelian varieties (for $\sigma \in \mathrm{Gal}(EK/E) = \mathrm{Gal}(E/k)$) uniquely descends to an E-isomorphism $B \simeq B^{\sigma}$ because EK/E is primary. Thus, we have Galois descent data on the abelian variety B relative to E/k, and so by Corollary 3.4 we conclude that $B = X_E$ for an abelian variety X over k, with this equality respecting the actions of $\mathrm{Gal}(E/k)$. Thus, A_{EK} is EK-isomorphic to $X_{EK} = (X_K)_{EK}$ in a manner that respects the actions of $\mathrm{Gal}(EK/K) = \mathrm{Gal}(E/k)$ on both sides. By Theorem 3.1, this EK-isomorphism descends to a K-isomorphism $A \simeq X_K$, so A is defined over k as desired. ■

The proofs of Theorem 5.1 and Corollary 5.2 use direct limit arguments with E/k, but they avoid the issue of how the K/k-image behaves with respect to direct limit processes. Now we address this issue; the next result reduces most questions about the K/k-image (and base change) to the case of finitely generated extensions:

Lemma 5.3. *If $E = \varinjlim E_i$ is a rising union of extensions of k, then the natural map*

$$I_{E/E_i} : \mathrm{Im}_{EK/E}(A_{EK}) \to \mathrm{Im}_{E_iK/E_i}(A_{E_iK})_E$$

is an isomorphism for large i. Also, if $K = \varinjlim K_i$ is a rising union of primary extensions of k and A_{i_0} is an abelian variety over some K_{i_0} with $A_i \stackrel{\mathrm{def}}{=} A_{i_0/K_i}$ for $i \geq i_0$ and $A \stackrel{\mathrm{def}}{=} A_{i_0/K}$, then the natural map

$$\mathrm{Im}_{K/k}(A) \to \mathrm{Im}_{K_i/k}(A_i)$$

is an isomorphism for all large i.

Proof. To show that I_{E/E_i} is an isomorphism for large i, first recall that $\mathrm{Im}_{EK/E}(A_{EK})_{EK}$ is constructed as the largest quotient of A_{EK} that is defined over E. The kernel of the quotient map $\lambda_{A_{EK},EK/E}$ is a closed subgroup scheme of A_{EK} and so is the base change of some closed E_iK-subgroup Γ of A_{E_iK} for some large i. The quotient A_{E_iK}/Γ over E_iK might not be defined over E_i, but since its EK-fiber is defined over E it is clear that by replacing i with some $i' \geq i$ and Γ with $\Gamma \otimes_{E_iK} E_{i'}K$ we may arrange that the quotient A_{E_iK}/Γ is defined over E_i. We have now shown that for sufficiently large i there is a quotient X_i of A_{E_iK} over E_iK that is defined over E_i and has EK-fiber $(X_i)_{EK}$ equal to the quotient $\mathrm{Im}_{EK/E}(A_{EK})_{EK}$ of A_{EK} that is defined over E. Consequently, the maximality of this latter quotient over EK forces the maximality of X_i as a quotient over E_iK that is defined over E_i. This implies that the E_i-descent of the abelian variety X_i, equipped with its quotient structure over E_iK, is an E_iK/E_i-image of A_{E_iK}. Hence, I_{E/E_i} is an isomorphism for such large i.

Next, we turn to the behavior with respect to limits in K. The morphism

$$\lambda : A \to \mathrm{Im}_{K/k}(A)_K$$

descends to a map

$$\lambda' : A_{i'} \to \mathrm{Im}_{K/k}(A)_{K_{i'}}$$

over some subextension $K_{i'}/K_{i_0}$. It is clear via Theorem 3.19 that this gives a $K_{i'}/k$-image of $A_{i'}$. ■

We conclude our discussion of base change by studying an important case when the formation of the K/k-image commutes with *any* (linearly disjoint) extension on k relative to K; without a doubt, this is the most important theorem in the theory and all of the difficulties in its proof are related to purely inseparable extensions in positive characteristic:

Theorem 5.4. *Let K/k be a primary extension of fields and let E/k be an arbitrary extension of fields. Assume either that E/k is separable or that K/k is regular. For any abelian variety A over K, the natural map $I_{E/k}$ in* (5.1) *is an isomorphism. In particular,* $(\mathrm{Im}_{K/k}(A)_E, \lambda_{EK})$ *is an EK/E-image of A_{EK}.*

Note that the separability and regularity assumptions both hold if k is perfect.

Proof. By transitivity, it suffices to treat two cases: when E/k is separable, and when E/k is purely inseparable with K/k regular. We first treat the separable case. By Lemma 5.3 it suffices to handle separately the cases when E/k is finite separable and when $E = k(t)$. In the finite separable case, so $EK = E \otimes_k K$, it is easy to reduce to treating the case when E/k is finite Galois. In this case we have $\mathrm{Gal}(EK/K) = \mathrm{Gal}(E/k)$, and the universality of

$$A_{EK} \to \mathrm{Im}_{EK/E}(A_{EK})_{EK}$$

gives a natural action of $\mathrm{Gal}(EK/K)$ on the target that is compatible with the action on the source. This descends to a $\mathrm{Gal}(E/k)$-action on $\mathrm{Im}_{EK/E}(A_{EK})$ because EK/E is primary; let us write X to denote the descended abelian variety over k. The natural map

$$A_{EK} \to \mathrm{Im}_{EK/E}(A_{EK})_{EK} = (X_E)_{EK} = (X_K)_{EK}$$

is equivariant with respect to the actions of $\mathrm{Gal}(EK/K)$, so it descends to a map $A \to X_K$ as abelian varieties over K. This latter map factors through the K-fiber of a unique map of abelian varieties

$$\mathrm{Im}_{K/k}(A) \to X$$

over k. Extending scalars to E thereby gives a map of abelian varieties

$$\mathrm{Im}_{K/k}(A)_E \to X_E = \mathrm{Im}_{EK/E}(A_{EK})$$

respecting projections from A_{EK}, so this is an inverse to $I_{E/k}$. Thus, $I_{E/k}$ is an isomorphism, as desired. This settles the case when E/k is finite and separable, and so when E/k is separable algebraic.

Since we have verified compatibility with separable algebraic base change, by a transitivity argument we may now assume that k is separably closed, and hence infinite. To handle $E = k(t)$, it is enough to show that for any abelian variety B over $k(t)$, any map $f : A_{K(t)} \to B_{K(t)}$ over $EK = K(t)$ uniquely factors through

$$\lambda_{K(t)} : A_{K(t)} \to \mathrm{Im}_{K/k}(A)_{K(t)}.$$

Certainly B extends to an abelian scheme $\widetilde{B}$ over a dense open U in $\mathbf{P}^1_k$, and so f extends to a map of abelian schemes $\widetilde{f} : A_W \to \widetilde{B}_K|_W$ over a nonempty open $W \subseteq U_K$. It is obvious that $U(k)$ is contained in $W(K)$ with at most finitely many exceptions (as we are working in $\mathbf{P}^1$).

For each $t_0 \in U(k) \cap W(K)$, the specialization $\widetilde{f}_{t_0}$ uniquely factors through λ. Thus, $\ker \lambda$ is contained in $\ker \widetilde{f}_{t_0}$ for all $t_0 \in U(k)$. In other words, the induced map

$$\widetilde{f} : (\ker \lambda)_W \to \widetilde{B}_K|_W$$

over W specializes to zero over $U(k) \cap W(K)$. This map factors through $\widetilde{B}_K[n]|_W$ with $n = \# \ker \lambda_{K(t)}$, and the resulting map $(\ker \lambda)_W \to \widetilde{B}_K[n]|_W$ between finite flat W-groups specializes to zero over the infinite set $U(k) \cap W(K)$. Since W is a nonempty open in $\mathbf{P}^1_K$, this implies that $\widetilde{f}$ vanishes on $(\ker \lambda)_W$, and hence $f = \widetilde{f}_{K(t)}$ kills $\ker \lambda_{K(t)}$. Thus, f uniquely factors through $\lambda_{K(t)}$ as desired.

Finally, we suppose that K/k is regular and E/k is purely inseparable (hence algebraic). Since k is separably closed, E must be separably closed. By Lemma 5.3, we can assume $[E : k]$ is finite. Clearly $EK = E \otimes_k K$ since K/k is regular. If k has characteristic 0 then $E = k$ and there is nothing to prove. Thus, we may assume that the separably closed field k has positive characteristic p.

We shall reduce to the case when the natural maps

$$\lambda = \lambda_{A,K/k} : A \to \mathrm{Im}_{K/k}(A)_K, \ \ \lambda' = \lambda_{A_{EK},EK/E} : A_{EK} \to \mathrm{Im}_{EK/E}(A_{EK})_{EK}$$

are isogenies. Let us first check that $\mathrm{Im}_{K/k}(A) = 0$ if and only if $\mathrm{Im}_{EK/E}(A_{EK}) = 0$. Since the map $A_{EK} \to \mathrm{Im}_{K/k}(A)_{EK}$ is surjective and factors through $\mathrm{Im}_{EK/E}(A_{EK})_{EK}$ (via $(I_{E/k})_{EK}$), if $\mathrm{Im}_{EK/E}(A_{EK}) = 0$ then $\mathrm{Im}_{K/k}(A) = 0$. Conversely, assuming $\mathrm{Im}_{EK/E}(A_{EK}) \neq 0$ let us show that $\mathrm{Im}_{K/k}(A) \neq 0$. By assumption, there exists a nonzero morphism $A_{EK} \to B_{EK}$ for an abelian variety B over E, so composing with a relative q-Frobenius $B \to B^{(q)}$ such that $B^{(q)}$ is defined over k (e.g., $q = [E : k]$) allows us to assume that B is defined over k. In this case we may descend to get a nonzero morphism of abelian varieties $A \to B_K$ because EK/K is primary. Thus, $\mathrm{Im}_{K/k}(A) \neq 0$.

We now reduce to the case when λ and λ' are isogenies. Since EK/K is primary, base change from K to EK carries K-simple abelian varieties to EK-simple abelian varieties. We have proved the equivalence of the vanishing of K/k- and EK/E-images for any abelian variety over K, and these "image" functors carry isogenies to isogenies and commute with the formation of products. Thus, by Corollary 4.7, we can replace A with the quotient by its unique abelian subvariety that splits (in the isogeny sense) the quotient map λ so as to reduce to the case where λ is an isogeny without changing either the K/k-image or the EK/E-image of interest. Since $\ker \lambda' \subseteq (\ker \lambda)_{EK}$, we conclude that the surjective λ' also has a finite kernel and so λ' is an isogeny. This completes the reduction to the case when λ and λ' are both isogenies. Since the map $I_{E/k}$ satisfies

$$(I_{E/k})_{EK} \circ \lambda' = \lambda_{EK},$$

$I_{E/k}$ must be a purely inseparable isogeny. In concrete terms, $G = \ker \lambda$ and $G' = \ker \lambda' \subseteq G_{EK}$ are the unique minimal connected finite subgroups of A and A_{EK} such that A/G and A_{EK}/G' are respectively defined over k and E. We wish to prove that $G' = G_{EK}$, but such a concrete formulation is not the way we will make progress since it is hard to directly exploit the minimality properties that define G and G'. Instead, we are going to indirectly show that the purely inseparable isogeny $I_{E/k}$ is smooth, and so it is an isomorphism.

By Lemma 5.3, we may assume that the regular extension K/k is finitely generated, so $K = k(V)$ for a smooth k-variety V. By shrinking V we may assume that A extends to an abelian scheme $\widetilde{A}$ over V and that the isogeny $\lambda : A \to \mathrm{Im}_{K/k}(A)_K$ extends to a map of abelian V-schemes

$$\widetilde{\lambda} : \widetilde{A} \to \mathrm{Im}_{K/k}(A)_V.$$

Thus, for all $v \in V(k)$ we have a well-defined specialization $\widetilde{\lambda}_v : \widetilde{A}_v \to \mathrm{Im}_{K/k}(A)$. Since EK is the function field of the smooth E-variety V_E, by possibly shrinking some more on V (in fact, no shrinking is needed) we also have a map

$$\widetilde{\lambda}' : \widetilde{A} \times_V V_E \to \mathrm{Im}_{EK/E}(A_{EK})_{V_E}$$

of abelian schemes over V_E that smears out the map $\lambda' : A_{EK} \to \mathrm{Im}_{EK/E}(A_{EK})_{EK}$ on generic fibers over V_E.

We now must formulate (and prove) *Chow's regularity theorem.* This theorem pleasantly disentangles the roles of E and k: it says that for any sufficiently large integer m there exists a dense open $V_{(m)}$ in the m-fold product $V^m = V \times_{\mathrm{Spec}\, k} \cdots \times_{\mathrm{Spec}\, k} V$ over $\mathrm{Spec}\, k$ such that for all extensions F/k and all $(v_j) \in V_{(m)}(F) \subseteq V(F)^m$, the specialized surjective F-map of abelian varieties

$$\sum \widetilde{\lambda}_{v_j} : \widetilde{A}_{v_1} \times \cdots \times \widetilde{A}_{v_m} \to \mathrm{Im}_{K/k}(A)_F$$

is *smooth* (or equivalently, this map induces a "regular extension" of function fields in Weil's terminology). In other words, the universal flat surjective addition morphism

$$\sum_{j=1}^{m} p_j^*(\widetilde{\lambda}) : \prod p_j^*(\widetilde{A}) \to \mathrm{Im}_{K/k}(A)_{V^m}$$

of abelian V^m-schemes is smooth on fibers over the generic point of V^m for large m.

Granting such a general result and also applying it to the situation with the EK/E-image over the separably closed E, for large m we similarly get a dense open $V'_{(m)}$ in the m-fold product of $V' = V_E$ over E with an analogous specialization property. Since E/k is a purely inseparable extension we can arrange for $V'_{(m)}$ to map into $V_{(m)}$ under the canonical morphism from ${V'}^m$ onto V^m for all large m. This has the fantastic consequence that for a common large m and an algebraic closure $\overline{E}$ of E and k, for $(v_j) \in V'_{(m)}(\overline{E}) \subseteq V_{(m)}(\overline{E})$ the $\overline{E}$-maps $\sum \widetilde{\lambda}_{v_j}$ and $\sum \widetilde{\lambda}'_{v_j}$ are both smooth. However, $(I_{E/k})_{\overline{E}}$ carries the first of these smooth surjections to the second, and hence $(I_{E/k})_{\overline{E}}$ is smooth, so $I_{E/k}$ is smooth! This forces the purely inseparable isogeny $I_{E/k}$ to be an isomorphism, as desired. The regularity theorem of Chow is presented below. ■

Theorem 5.5 (Chow's regularity theorem)**.** *Let V be a smooth variety over a field k. Let A be an abelian variety over $K = k(V)$ such that A extends to an abelian scheme $\widetilde{A}$ over V. Let*

$$\widetilde{\lambda} : \widetilde{A} \to \mathrm{Im}_{K/k}(A)_V$$

be the unique map of abelian V-schemes that extends the canonical map $\lambda : A \to \mathrm{Im}_{K/k}(A)_K$. For any $m > \dim A$ there exists a dense open $V_{(m)}$ in V^m over which the flat surjective summation morphism

$$\sum p_j^*(\widetilde{\lambda}) : p_1^*(\widetilde{A}) \times_{V^m} \cdots \times_{V^m} p_m^*(\widetilde{A}) \to \mathrm{Im}_{K/k}(A)_{V^m} \tag{5.2}$$

is smooth.

The fact that λ extends to $\widetilde{\lambda}$ over all of V is a special case of a general extension lemma of Weil [1, 4.4/1] (extending the Néron mapping property of abelian schemes to the case of a normal noetherian base), but for our purposes in the proof of Theorem 5.4 it is enough to use elementary denominator-chasing to initially shrink V to a smaller dense open over which λ extends to a map of abelian schemes, thereby bypassing the need to use Weil's lemma.

Proof. The case of characteristic 0 is trivial for any $m \geq 1$, so we may (and do) now assume that k has positive characteristic p. By using Corollary 4.7 and shrinking V, we can assume that the canonical map

$$\lambda : A \to \mathrm{Im}_{K/k}(A)_K$$

is an isogeny. By Theorem 4.5(3), $\ker\lambda$ is (geometrically) connected, so λ is a purely inseparable isogeny. Hence, $\widetilde{\lambda}$ is an isogeny.

The compatibility of $\mathrm{Im}_{K/k}$ with respect to separable extension on k has already been established in the part of above proof of Theorem 5.4 that is not conditional on Chow's regularity theorem, so we may (and do) assume that k is separably closed. Let $A^\vee$ and $\widetilde{A}^\vee$ be the duals of A and $\widetilde{A}$ (see [2, Ch. I, Thm. 1.9] for the general existence of the dual abelian scheme, or shrink V to make $A \to V$ projective so that Grothendieck's construction of the dual may be applied), and let

$$\tau : \mathrm{Tr}_{K/k}(A^\vee)_K \to A^\vee$$

denote the dual of the purely inseparable isogeny λ (since this dual map τ will later be called the K/k-trace of $A^\vee$). A key technical problem is that we do not yet know that the finite $\ker\tau$ is *connected*. The proof of such connectivity will be given later (Theorem 6.12), using the general validity of Theorem 5.4 whose proof has not yet been finished. (See Example 6.3 for examples of non-regular primary extensions K/k with $\lambda = \tau^\vee$ a purely inseparable isogeny and $\ker\tau$ disconnected.)

Duality translates the universal property of λ into a universal property of τ: it is a final object in the category of pairs (B, f) consisting of abelian varieties B over k and maps of abelian varieties $f : B_K \to A^\vee$ over K. This finality implies that the finite K-subgroup $H_\eta \stackrel{\mathrm{def}}{=} \ker\tau$ inside of $\mathrm{Tr}_{K/k}(A^\vee)_K$ cannot contain any nonzero K-subgroup defined over k in $\mathrm{Tr}_{K/k}(A^\vee)_K$, as otherwise we could replace $\mathrm{Tr}_{K/k}(A^\vee)$ with a non-trivial quotient to contradict the minimality property of τ. In particular, the connected K-subgroup $H_\eta^0 = (\ker\tau)^0$ in $\mathrm{Tr}_{K/k}(A^\vee)[N]_K^0$ (with N the order of H_η) cannot contain any nonzero K-subgroup in $\mathrm{Tr}_{K/k}(A^\vee)[N]_K^0$ that is defined over k. Let $\widetilde{\tau} : \mathrm{Tr}_{K/k}(A^\vee)_V \to \widetilde{A}^\vee$ denote the isogeny that is dual to the isogeny $\widetilde{\lambda}$. The kernel $H = \ker\widetilde{\tau}$ is a finite flat V-group, so by working on the K-fiber we see that $H \subseteq \mathrm{Tr}_{K/k}(A^\vee)[N]_V$. For any $m \geq 1$, any extension F/k, and any $(v_j) \in V(F)^m$, the F-map of abelian varieties

$$(\widetilde{\tau}_{v_1}, \ldots, \widetilde{\tau}_{v_m}) : \mathrm{Tr}_{K/k}(A^\vee)_F \to \widetilde{A}^\vee_{v_1} \times \cdots \times \widetilde{A}^\vee_{v_m} \tag{5.3}$$

is dual to $\sum \widetilde{\lambda}_{v_j}$ and its kernel is the schematic intersection $\cap H_{v_j}$ inside of $\mathrm{Tr}_{K/k}(A^\vee)_F$. If this intersection vanishes then (5.3) is a closed immersion of abelian varieties, and hence its dual $\sum \widetilde{\lambda}_{v_j}$ is smooth. This motivates us to consider the following rather concrete assertion concerning finite connected k-groups and generic specialization of certain finite K-groups.

Let B be a finite commutative connected k-group (such as $\mathrm{Tr}_{K/k}(A^\vee)[N]^0$ above) and let $G \subseteq B_V$ be a finite flat V-subgroup (such as the V-group $H \cap \mathrm{Tr}_{K/k}(A^\vee)[N]^0_V$ that is open and closed in H). Assume also that the generic fiber G_η contains no nonzero K-subgroups that are defined over k as subgroups of B_K. For $m > \dim_K T_0(G_\eta)$ we claim that there exists some dense open $V'_{(m)}$ in V^m such that for all F/k and all $(v_j) \in V'_{(m)}(F)$ the intersection $\cap G_{v_j}$ in B_F *vanishes*. Roughly speaking, the claim is that for a family of subgroups $\{G_v\}$ of B that is parameterized by a smooth k-variety V and is truly varying in the sense that the generic fiber G_η contains no nonzero subgroup arising from a k-subgroup of B (there is no nonzero "fixed part" in the family), an intersection $\cap G_{v_j}$ of sufficiently many generic specializations of the family is equal to 0 (where "sufficiently many" can be taken to mean "more than $\dim_K T_0(G_\eta)$").

Once this general claim is proved, we can apply it to the preceding situation with the k-group $B = \mathrm{Tr}_{K/k}(A^\vee)[N]^0$ and $G = H \cap B_V$ (so $G_\eta = H^0_\eta$). This gives that for $m > \dim A = \dim_K T_0(A)$ the kernel of (5.3) for any extension field F/k and $(v_j) \in V'_{(m)}(F)$ has vanishing connected part, and so is F-étale. Fix such an m and consider the special case that $F = k(V^m)$ and (v_j) is the generic point of V^m. Since F/k is regular, by Lemma 3.11 the étale kernel of (5.3) in this case arises from an étale k-subgroup Γ_m of B. Smearing out from $\mathrm{Spec}\, F = \mathrm{Spec}\, k(V^m)$ provides a dense open $U \subseteq V'_{(m)}$ such that the restriction over U of the canonical map

$$(p_1^*(\widetilde{\tau}), \dots, p_m^*(\widetilde{\tau})) : \mathrm{Tr}_{K/k}(A^\vee)_{V^m} \to p_1^*(\widetilde{A}^\vee) \times_{V^m} \cdots \times_{V^m} p_m^*(\widetilde{A}^\vee)$$

has kernel $(\Gamma_m)_U$. Letting $q : V^m \to V^{m-1}$ denote the flat projection away from the first V-factor, pick $\xi \in V^{m-1}(k)$ in the non-empty Zariski-open $q(U) \subseteq V^{m-1}$ (such ξ exists since k is separably closed and V^{m-1} is k-smooth). Specializing at the generic point $\mathrm{Spec}\, K$ of the fiber $q^{-1}(\xi) = V$ thereby realizes $(\Gamma_m)_K$ as a K-subgroup of $H_\eta = \ker \tau$ that is defined over k as a subgroup of $\mathrm{Tr}_{K/k}(A^\vee)_K$. This forces $\Gamma_m = 0$, whence (5.3) at the generic point of V^m is a closed immersion. Hence, the dual map is smooth on fibers over the generic point of V^m, and this is (5.2) over the generic point of V^m. We conclude that (5.2) is smooth over a Zariski-open neighborhood $V_{(m)}$ of the generic point in V^m as desired.

It remains to prove the above general claim concerning a connected finite k-group B and a finite flat V-subgroup $G \subseteq B_V$. We can assume $G_\eta \neq 0$. Since a nonzero finite connected commutative K-group has nonzero kernel for its relative Frobenius morphism, we have $\ker F_{G_\eta/K} \neq 0$. Thus, by shrinking V so that $\ker F_{G/V}$ is V-flat, we can replace G with $\ker F_{G/V}$ and B with $\ker F_{B/k}$ to reduce to the case when $F_{B/k} = 0$.

For any $m \geq 1$, generic flatness over the reduced V^m provides a dense open in V^m over which the universal m-fold intersection of fibers of the subgroup $G \hookrightarrow B_V$ is flat over the base. Within this dense open locus in V^m, the vanishing condition on the m-fold intersection $\cap G_{v_j}$ is a Zariski-closed condition. We seek to prove that if $m > \dim_K T_0(G_\eta)$ then this locally-closed locus in V^m contains a non-empty open and hence (by irreducibility) is Zariski-dense in V^m. Since k is separably closed, for an algebraic closure $\overline{k}/k$ we see that $V^m_{\overline{k}} \to V^m$ is a homeomorphism. Hence, it is enough to solve our finite-group problem with k replaced by $\overline{k}$ and V replaced by $V_{\overline{k}}$; that is, we can assume k is algebraically closed. Here we use crucially that extending scalars to $\overline{k}$ does not destroy the irreducibility and reducedness properties used above. With k algebraically closed, the connected finite k-group B is naturally a product of a local-local group B_1 and a multiplicative group B_2. The intersection $G_\eta \cap B_{2,K}$ must vanish because it is a K-subgroup of the multiplicative $B_{2,K}$ and all such K-subgroups arise from k-subgroups of B_2 (as we can see via Cartier duality and the constancy of $\mathbf{D}(B_2)$). Lemma 3.14 implies that $G_\eta = (G_\eta \cap B_{1,K}) \times (G_\eta \cap B_{2,K})$ inside of $B_K = B_{1,K} \times B_{2,K}$, so G_η is contained in $B_{1,K}$. Hence, $G \subseteq (B_1)_V$ and so we are reduced to the case when B is local-local.

Just as we reduced to the case $F_{B/k} = 0$, now that B is local-local we can reduce to the case when the relative Verscheibung morphism $V_{B/k}$ vanishes too. Thus, by Theorem 3.18, for any extension E/k the Lie functor on the set of E-subgroups of B_E sets up an inclusion-preserving bijective correspondence between the set of such E-subgroups and the set of E-linear subspaces of the tangent space $T_0(B)_E = T_0(B_E)$. The main consequence of interest to us is that $T_0(G_\eta)$ must be a K-subspace of $T_0(B)_K$ that contains no nonzero k-rational subspaces.

Working with the relative tangent spaces for G and B_V along their identity sections over V, our problem now translates into relative linear algebra: $T_0(G)$ is a subbundle of $T_0(B)\otimes_k \mathscr{O}_V$ whose generic fiber contains no nonzero k-rational subspaces, and we seek to prove that if $m > \dim_K T_0(G_\eta)$ then on some dense open locus of m-tuples (v_j) in V^m the intersection of the $T_0(G)_{v_j}$'s in $T_0(B)$ is equal to zero. It is obviously enough to work with v_j's that are k-points of V, as k is now algebraically closed. For any positive m at all, consider the universal map

$$\phi_m : T_0(B)_{V^m} \to (T_0(B)_{V^m}/p_1^*(T_0(G))) \oplus \cdots \oplus (T_0(B)_{V^m}/p_m^*(T_0(G)))$$

over V^m. The induced map on fibers over a point $(v_j) \in V(k)^m$ is the natural map

$$T_0(B) \to (T_0(B)/T_0(G)_{v_1}) \oplus \cdots \oplus (T_0(B)/T_0(G)_{v_m}),$$

and hence this fibral map is injective if and only if $\cap T_0(G)_{v_j} = 0$.

Since ϕ_m is a map of vector bundles on V^m, if it is injective on the fibers at some k-point ξ then it is a direct summand over a Zariski-open neighborhood of ξ in V^m. Thus, the locus of points $\xi = (v_j) \in V(k)^m$ such that $\cap T_0(G)_{v_j} = 0$ is a Zariski-open set in $V(k)^m$. Since V^m is irreducible, it therefore suffices (for any particular m) to find *some* $(v_j) \in V(k)^m$ such that $\cap T_0(G)_{v_j} = 0$. We may assume that the rank r of $T_0(G)$ is positive. To prove the existence of such a (v_j) if $m > r$, it suffices to prove (by induction on i) that for $1 \le i \le r$ and any $v_1, \dots, v_i \in V(k)$ such that $\cap_{j\le i} T_0(G)_{v_j}$ in $T_0(B)$ has dimension at most $r-(i-1)$, there exists $v_{i+1} \in V(k)$ such that $T_0(G)_{v_{i+1}}$ does not contain $\cap_{j\le i} T_0(G)_{v_j}$ in $T_0(B)$. More generally, for any nonzero subspace T in $T_0(B)$ we claim that there exists $v \in V(k)$ such that $T_0(G)_v$ does not contain T. If no such v exists then the composite map

$$T \otimes_k \mathscr{O}_V \to (T_0(B)\otimes_k \mathscr{O}_V)/T_0(G)$$

vanishes on all k-fibers and hence vanishes, so $T_0(G)$ contains $T\otimes_k \mathscr{O}_V$ and therefore the K-subspace $T_0(G_\eta)$ in $T_0(B)_K$ contains the nonzero k-rational subspace T_K, a contradiction. ∎

6. The K/k-trace

As usual, we let K/k be a primary extension of fields.

Definition 6.1. Let A be an abelian variety over K. A *K/k-trace* is a final object $(\mathrm{Tr}_{K/k}(A), \tau)$ in the category of pairs (B, f) where B is an abelian variety over k and $f : B_K \to A$ is a map of abelian varieties.

In view of the double-duality theorem for abelian varieties, the existence of the K/k-trace is obvious by dualizing the K/k-image of $A^\vee$ and using the dual of its universal morphism. Combining this with Theorem 4.5(3) we get:

Theorem 6.2. *Let K/k be a primary extension of fields, and A an abelian variety over K. The K/k-trace*

$$\tau = \tau_{A,K/k} : \mathrm{Tr}_{K/k}(A)_K \to A$$

exists, and the associated dual morphism is the K/k-image $\lambda_{A^\vee,K/k}$ of the dual abelian variety $A^\vee$.

The image of the map τ as above is an abelian subvariety of A and it is called the *K/k-maximal abelian subvariety* in [18]. By Theorem 3.19 and Theorem 4.5(3), this subvariety is defined over k if and only if $\ker\tau$ descends to a k-subgroup of $\mathrm{Tr}_{K/k}(A)$, and (by the universality of τ) this happens if and only if $\ker\tau = 0$, or equivalently τ is a closed immersion. In characteristic 0, τ is a closed immersion because it is dual to the surjective map

$$\lambda_{A^\vee,K/k} : A^\vee \to \mathrm{Im}_{K/k}(A^\vee)_K$$

whose connected kernel must be smooth (by Cartier's theorem [25, p. 101]) and hence is an abelian subvariety of $A^\vee$.

In characteristic $p > 0$, the K-subgroup $\ker\tau$ may be nonzero, or equivalently the connected kernel of the dual map $\tau^\vee$ may not be smooth. Example 4.4 gives many examples for which this possibility happens with $\ker\tau^\vee = \alpha_{p,K}$ (so τ is an isogeny and $\ker\tau \simeq \mathbf{D}(\alpha_{p,K}) \simeq \alpha_{p,K}$). For general primary extensions K/k the kernel of τ might not be connected (but see Theorem 6.12 below for the absence of this phenomenon when K/k is regular); the following class of disconnected étale examples was suggested by the referee.

Example 6.3. Let E be an ordinary elliptic curve over a field k of characteristic $p > 0$ such that the connected-étale sequence of $E[p]$ is not split. (Many examples of such E are provided by Serre–Tate theory, applied to the generic fiber over $k_0[\![q]\!]$ of a sufficiently generic deformation of an ordinary elliptic curve over a field k_0 of characteristic p. The standard Tate curve over $k_0(\!(q)\!)$ is another example of such an elliptic curve.) Since the sequence splits over a perfect closure of k, it splits over a sufficiently large finite purely inseparable extension K/k. Such a splitting over K is unique (since there are no nonzero maps from an étale commutative group scheme to a finite connected commutative group scheme over a field), and we let $G \subseteq E_K[p]$ be the unique étale K-subgroup of order p.

Define $E' = E_K/G$, and consider the degree-p étale isogeny $E_K \to E'$ over K. This isogeny factors uniquely as $\tau' \circ h_K$ where $\tau' : \mathrm{Tr}_{K/k}(E')_K \to E'$ is the K/k-image and $h : E \to \mathrm{Tr}_{K/k}(E')$ is a map of abelian varieties over k. This forces $\mathrm{Tr}_{K/k}(E')$ to be nonzero and h and τ' to be étale isogenies of elliptic curves with $\deg h \cdot \deg \tau' = p$. The map h must be an isomorphism because if it is not then it is étale with degree p and so the étale subgroup $\ker h \subseteq E[p]$ with order p defines a k-splitting of the connected-étale sequence of $E[p]$ (which we assumed is not split over k). Hence, the universal morphism $\tau' : \mathrm{Tr}_{K/k}(E')_K \to E'$ is a degree-p étale isogeny, so its kernel is disconnected.

Some basic properties of the K/k-trace with respect to extensions of fields are formal consequences of the theory of the K/k-image by means of duality. For example, dualizing Theorem 4.2, Theorem 4.5, and Corollary 4.7 gives:

Theorem 6.4. *Let K/k be a primary extension of fields, and let A be an abelian variety over K with K/k-trace $\tau = \tau_{A,K/k} : \mathrm{Tr}_{K/k}(A)_K \to A$.*

(1) *If $A = X_K$ for an abelian variety X over k then τ is an isomorphism.*
(2) *If k/k_0 is a primary extension and $(\mathrm{Tr}_{k/k_0}(\mathrm{Tr}_{K/k}(A)), \tau_0)$ denotes the k/k_0-trace of $\mathrm{Tr}_{K/k}(A)$ then*
$$(\mathrm{Tr}_{k/k_0}(\mathrm{Tr}_{K/k}(A)), \tau \circ \tau_{0/K})$$
is a K/k_0-trace of A.
(3) *If K'/K is a primary extension then $(\mathrm{Tr}_{K/k}(A), \tau_{K'})$ is a K'/k-trace of $A_{K'}$.*
(4) *The canonical map $\tau : \mathrm{Tr}_{K/k}(A)_K \to A$ has finite kernel.*

Moreover, there exists a unique abelian subvariety $A' \subseteq A$ such that $\mathrm{Tr}_{K/k}(A/A') = 0$ (so $\mathrm{Tr}_{K/k}(A') \to \mathrm{Tr}_{K/k}(A)$ is an isomorphism) and $\tau_{A',K/k} : \mathrm{Tr}_{K/k}(A')_K \to A'$ is an isogeny.

The abelian subvariety $A' \subseteq A$ at the end of Theorem 6.4 is the K/k-maximal abelian subvariety of A. Combining Theorem 6.4 with Theorem 4.5(3) gives an interesting property of the finite K-group $\ker \tau_{A,K/k}$:

Corollary 6.5. *Let K/k, A, and τ be as in Theorem 6.4. The finite K-group $\ker \tau$ has connected Cartier dual.*

Proof. By the final assertion in Theorem 6.4, we easily reduce to the case when τ is an isogeny. Hence, the Cartier dual of $\ker \tau$ is the kernel of the dual isogeny $\lambda_{A^\vee,K/k}$, and the connectedness of this latter kernel follows from Theorem 4.5(3). ■

Dualizing Theorem 5.1 gives:

Theorem 6.6. *Let K/k be a primary extension of fields and A an abelian variety over K. For any extension E/k, consider the unique E-map of abelian varieties*
$$I'_{E/k} : \mathrm{Tr}_{K/k}(A)_E \to \mathrm{Tr}_{EK/E}(A_{EK})$$
such that $\tau_{A_{EK},EK/E} \circ (I'_{E/k})_{EK} = (\tau_{A,K/k})_{EK}$. The map $I'_{E/k}$ is an isogeny and its kernel has connected Cartier dual.

Remark 6.7. Corollary 5.2 and Lemma 5.3 are of an essentially technical nature, and their analogues for K/k-traces are immediate via either dualizing from K/k-images or (better) copying the earlier proofs in our new setting (which is possible, due to the preceding results), so we do not state them formally here.

The dual of Theorem 5.4 is very useful, so we record it here for later reference:

Theorem 6.8. *Let K/k be a primary extension of fields and E/k an arbitrary extension, and assume either that E/k is separable or that K/k is regular. For any abelian variety A over K with associated K/k-trace $\tau : \mathrm{Tr}_{K/k}(A)_K \to A$, the pair $(\mathrm{Tr}_{K/k}(A)_E, \tau_E)$ is an EK/E-trace of A_{EK}.*

By working with K-isogeny factors of A that are defined over k (as in the proof of Theorem 5.1), we deduce an unsurprising relationship between the K/k-image and K/k-trace:

Theorem 6.9. *Let K/k be a primary extension of fields and A an abelian variety over K. The unique map*

$$\mathrm{Tr}_{K/k}(A) \to \mathrm{Im}_{K/k}(A)$$

of abelian varieties over k that descends the K-map $\lambda_{A,K/k} \circ \tau_{A,K/k}$ is an isogeny.

Another simple but useful consequence of duality is a dual version of Chow's regularity theorem (Theorem 5.5):

Theorem 6.10. *Let V be a smooth variety over a field k. Let A be an abelian variety over $K = k(V)$ such that A extends to an abelian scheme $\widetilde{A}$ over V. Let*

$$\widetilde{\tau} : \mathrm{Tr}_{K/k}(A)_V \to \widetilde{A}$$

be the unique map of abelian V-schemes that extends the canonical map $\tau : \mathrm{Tr}_{K/k}(A)_K \to A$. For all $m > \dim A$, there exists a dense open $V_{(m)}$ in V^m over which the morphism

$$(p_1^*(\widetilde{\tau}), \dots, p_m^*(\widetilde{\tau})) : \mathrm{Tr}_{K/k}(A)_{V^m} \to p_1^*(\widetilde{A}) \times_{V^m} \cdots \times_{V^m} p_m^*(\widetilde{A}) \tag{6.1}$$

is a closed immersion.

Remark 6.11. Theorem 6.10 is not a formal consequence of the statement of Chow's regularity theorem. Indeed, from the statement of Chow's theorem one gets smoothness of the kernel of the surjective dual of (6.1) over some dense open in V^m for all $m > \dim A$, but in general the dual of a smooth surjection between abelian varieties need not be a closed immersion. Fortunately, it is the stronger closed immersion condition for (6.1) over some dense open in V^m for all $m > \dim A$ that was established in the proof of Chow's regularity theorem.

It is natural to seek a criterion for $\ker \tau$ to be connected (and hence infinitesimal, by Theorem 6.4(4)). The proof of the following criterion requires the full strength of Theorem 6.8 (allowing E/k to be inseparable):

Theorem 6.12. *Let K/k be a regaular extension of fields. For any abelian variety A over K, the finite K-group $\ker \tau_{A,K/k}$ is connected with connected dual.*

Proof. The connectedness of the dual holds for any primary extension K/k (Corollary 6.5), and to prove connectedness when K/k is regular we first use that the formation of τ commutes with passage to EK/E for any extension E/k (by Theorem 6.8). By taking E to be an algebraic closure of k, we may assume that k is algebraically closed. In particular, for any extension K'/K the extension K'/k is regular. By Theorem 3.19, for any primary extension K'/K the map

$$\tau_{K'} : \mathrm{Tr}_{K/k}(A)_{K'} \to A_{K'}$$

is a K'/k-trace of $A_{K'}$. Thus, by taking K' to be a perfect closure of K we can assume K is perfect. This perfectness ensures that the connected-étale sequence of the finite K-group $\ker \tau$ is split, and its étale factor G descends to a finite k-subgroup of $\mathrm{Tr}_{K/k}(A)$ by Lemma 3.11 (applied to $H = \mathrm{Tr}_{K/k}(A)[n]$ with $n = \#G$). We conclude that τ factors through the K-fiber of the projection map $\mathrm{Tr}_{K/k}(A) \to \mathrm{Tr}_{K/k}(A)/G$, and so by finality of the K/k-trace it follows that G must be trivial. Hence, $\ker \tau$ is connected. ■

7. The Lang–Néron theorem

Theorem 6.12 implies that if K/k is regular and A is an abelian variety over K then the map $\tau = \tau_{A,K/k} : \mathrm{Tr}_{K/k}(A)_K \to A$ is injective on K-points, so $\mathrm{Tr}_{K/k}(A)(k)$ is naturally a subgroup of $A(K)$.

Theorem 7.1 (Lang–Néron)**.** *Let K/k be a finitely generated regular extension of fields. Let A be an abelian variety over K. The quotient group $A(K)/\mathrm{Tr}_{K/k}(A)(k)$ is finitely generated.*

The reader who is only interested in the case $K = k(C)$ with algebraically closed k and a smooth proper connected k-curve C can skip ahead to the paragraph containing (7.2). For non-constant elliptic curves E over such a K (*i.e.*, non-constant elliptic fibrations $\mathscr{E} \to C$), the K/k-trace vanishes by Theorem 2.3. The argument following (7.2) therefore gives a proof that $E(K) = \mathscr{E}(C)$ is finitely generated for such E over K without using any of the material in §4–§6.

Since an abelian variety over a finite field obviously has a finitely generated (even finite) group of rational points, and an abelian variety over a number field has a finitely generated group of rational points (the classical Mordell–Weil theorem), a special case of the Lang–Néron theorem is the main result of Néron's thesis [26]:

Corollary 7.2. *Let K be a field that is finitely generated over its prime field, and let A be an abelian variety over K. The group $A(K)$ is finitely generated.*

To prove the Lang–Néron theorem, the first step is to reduce to the special case when k is algebraically closed and K/k is finitely generated of transcendence degree 1; that is, $K = k(C)$ for a smooth proper connected curve C over k. The reader may find it interesting to compare our arguments below with those in [20, Ch. 6].

Let us now turn to the reduction steps.

Lemma 7.3. *If k'/k is an extension, it suffices to prove the Lang–Néron theorem for the regular extension $k'K/k'$ instead of K/k.*

Proof. Let $K' = k'K$ and $A' = A_{K'}$. We know that $\mathrm{Tr}_{K'/k'}(A') = \mathrm{Tr}_{K/k}(A)_{k'}$, by Theorem 6.8, so $\mathrm{Tr}_{K'/k'}(A')(k') = \mathrm{Tr}_{K/k}(A)(k')$ inside of $A(K')$ (recall that τ and τ' are injective on field-valued points, by Theorem 6.12). Thus, by hypothesis $A(K')/\mathrm{Tr}_{K/k}(A)(k')$ is finitely generated, and so it is enough to prove that the natural map

$$A(K)/\mathrm{Tr}_{K/k}(A)(k) \to A(K')/\mathrm{Tr}_{K/k}(A)(k')$$

is injective. That is, we want the natural inclusion

$$\mathrm{Tr}_{K/k}(A)(k) \subseteq A(K) \cap \mathrm{Tr}_{K/k}(A)(k')$$

inside of $A(K')$ to be an equality.

Let F_m be the fraction field of the domain $K^{\otimes m}$ (tensor product over k), and let $p_i : \mathrm{Spec}\, F_m \to \mathrm{Spec}\, K$ over $\mathrm{Spec}\, k$ be the map induced by the ith standard projection. By Theorem 6.10, for sufficiently large m the map of abelian varieties over F_m

$$\mathrm{Tr}_{K/k}(A)_{F_m} \to p_1^*(A) \times \cdots \times p_m^*(A) \tag{7.1}$$

is a closed immersion. Let F'_m denote the fraction field of $K'^{\otimes m}$ (tensor product over k'), so $F'_m = k'F_m$. Since k is algebraically closed in F_m we have $F_m \cap k' = k$ inside of F'_m, so to show that a k'-point of $\mathrm{Tr}_{K/k}(A)$ inducing a K-point of A (inside of $A(K')$) is a k-point of $\mathrm{Tr}_{K/k}(A)$ it is enough to prove that an F'_m-point of $\mathrm{Tr}_{K/k}(A)$ inducing a K-point of A is an F_m-point of $\mathrm{Tr}_{K/k}(A)$. Concretely, if we let $F_{m,i}$ and $F'_{m,i}$ denote F_m and F'_m viewed as K-algebras via the ith tensor-factor, then the assertion to be proved is that if $x \in A(K)$ is a point such that the points $p_i^*(x) \in A(F_{m,i})$ are all induced by a common point

$$y \in \mathrm{Tr}_{K/k}(A)(F'_m) = p_i^*(\mathrm{Tr}_{K/k}(A))(F'_{m,i})$$

then $y \in \mathrm{Tr}_{K/k}(A)(F_m)$.

By descent theory (Theorem 3.1), it suffices to show that y has the same image under the two maps $\mathrm{Tr}_{K/k}(A)(F'_m) \rightrightarrows \mathrm{Tr}_{K/k}(A)(F'_m \otimes_{F_m} F'_m)$. Since (7.1) is a monomorphism of functors, it is enough to check that the two natural maps

$$A(F'_{m,1}) \times \cdots \times A(F'_{m,m}) \rightrightarrows A(F'_{m,1} \otimes_{F_{m,1}} F'_{m,1}) \times \cdots \times A(F'_{m,m} \otimes_{F_{m,m}} F'_{m,m})$$

have the same composite with the diagonal embedding

$$A(K) \to A(F'_{m,1}) \times \cdots \times A(F'_{m,m}).$$

Thus, it suffices to show that for each i, the two composite maps

$$K \to F'_{m,i} \rightrightarrows F'_{m,i} \otimes_{F_{m,i}} F'_{m,i}$$

coincide. This equality of maps is obvious, since the map $K \to K'^{\otimes m}$ to the ith tensor-factor factors through the map $K \to K^{\otimes m}$ to the ith tensor-factor. ■

By the preceding lemma, if we wish to prove the Lang–Néron theorem for any specific abelian variety relative to a given finitely generated regular extension K/k then it suffices to treat the analogous situation relative to $\overline{k}K/\overline{k}$ for an algebraic closure $\overline{k}/k$.

Lemma 7.4. *For any intermediate extension $K/E/k$ such that K/E is regular, it suffices to separately treat the cases K/E and E/k.*

Note that, under the hypotheses in the lemma, K/E and E/k are automatically finitely generated and E/k is automatically regular.

Proof. Since $\mathrm{Tr}_{E/k}(\mathrm{Tr}_{K/E}(A))$ is a K/k-trace of A (Theorem 6.4(2)), via the commutative diagram

$$\begin{array}{ccc} \mathrm{Tr}_{K/E}(A)_K & \xrightarrow{\tau_{K/E}} & A \\ \uparrow{\scriptstyle (\tau_{E/k})_K} & & \uparrow{\scriptstyle \tau_{K/k}} \\ \mathrm{Tr}_{E/k}(\mathrm{Tr}_{K/E}(A)_E)_K & \underset{\simeq}{\longrightarrow} & \mathrm{Tr}_{K/k}(A)_K \end{array}$$

we are done. ■

We now may and do assume k to be algebraically closed, and we can choose a smooth k-variety V such that $K = k(V)$. The case $\dim V = 0$ is trivial (as then $K = k$). If $\dim V > 1$, then by Bertini methods we can shrink V so that there is a smooth map $f : V \to V'$ with V' a k-variety of dimension $\dim V - 1$ and *all* fibers of f geometrically connected of dimension 1. In particular, K is regular over $E = k(V')$ with $\mathrm{trdeg}_E(K) = 1$. Using Lemma 7.4, we are thereby reduced to the case when k is algebraically closed and the finitely generated extension K/k has transcendence degree equal to 1.

Let C be the proper smooth connected curve over k with function field K. Let U be a dense open in C such that A extends to an abelian scheme $\mathscr{A}$ over U. Note that $A(K) = \mathscr{A}(U)$. Letting $m > 1$ be an integer not divisible by the characteristic of k, the Kummer sequence

$$0 \to \mathscr{A}[m] \to \mathscr{A} \xrightarrow{m} \mathscr{A} \to 0 \qquad (7.2)$$

on $U_{\text{ét}}$ induces an injection $A(K)/mA(K) \hookrightarrow \mathrm{H}^1_{\text{ét}}(U, \mathscr{A}[m])$. Since k is separably closed and $\mathscr{A}[m]$ is a locally constant constructible sheaf of $\mathbf{Z}/m\mathbf{Z}$-modules on the smooth k-curve U (with m a unit in k), the group $\mathrm{H}^1_{\text{ét}}(U, \mathscr{A}[m])$ is finite by a general finiteness theorem [9, I, 8.10] for compactly supported cohomology, together with Poincaré duality [32, Thm. 4.8] on U. (See [6, Thm. 1.1] for a much deeper finiteness theorem.) Hence, $A(K)/mA(K)$ is finite. This is an analogue of the so-called weak Mordell–Weil theorem in the classical case (with K a global field).

Using the standard normalized valuations on K arising from the points of $C(k)$, we have a product formula and thereby get a logarithmic height-function on $A(K)$ via a choice of projective embedding of $A \hookrightarrow \mathbf{P}^n_K$ over K. We will show that the set of elements of $A(K)$ with height below any given bound M has finite image in $A(K)/\mathrm{Tr}_{K/k}(A)(k)$; once this is proved, the classical proof of the Mordell–Weil theorem (combining the weak Mordell–Weil theorem and the elementary parts of the theory of heights) may be easily adapted to show that $A(K)/\mathrm{Tr}_{K/k}(A)(k)$ is finitely generated.

Now choose a projective embedding $A \hookrightarrow \mathbf{P}^n_K$ and let h be the resulting logarithmic height on $A(K)$. Let $\widetilde{A} \hookrightarrow \mathbf{P}^n_k \times C$ be the closure of A; this is a projective k-variety. By the valuative criterion for properness,

$$A(K) = \{f \in \mathrm{Hom}_k(C, \widetilde{A}) \mid \mathrm{pr}_2 \circ f = 1_C\} = \widetilde{A}(C)$$

where $\mathrm{pr}_2 : \widetilde{A} \hookrightarrow \mathbf{P}^n_k \times C \to C$ is the second projection and is used to view $\widetilde{A}$ as a C-scheme.

Lemma 7.5. *Choose a projective embedding $C \hookrightarrow \mathbf{P}^m_k$ as a degree-d curve. For $P \in A(K)$, the associated Segre-map*

$$f_P : C \to \widetilde{A} \hookrightarrow \mathbf{P}^n_k \times C \hookrightarrow \mathbf{P}^{(n+1)(m+1)-1}_k$$

is a closed immersion and the projective curve $f_P(C)$ has degree $\le h(P) + d$.

In particular, as P varies with bounded height, the map f_P varies with bounded degree for its image.

Proof. The map $C \hookrightarrow \mathbf{P}^m_k$ is given by a tuple $[h_0, \cdots, h_m]$ with $h_j \in k(C) = K$ and not all h_j equal to zero. The point

$$P \in A(K) \subseteq \mathbf{P}^n(K) = \mathrm{Hom}_k(C, \mathbf{P}^n_k)$$

is given by a tuple $[g_0, \ldots, g_n]$ with $g_i \in k(C)$ not all zero, so f_P is given by the tuple of $g_i h_j$'s (by the definition of the Segre embedding). Thus, viewing f_P as a K-point of $\mathbf{P}^{(n+1)(m+1)-1}$, it has naive logarithmic height equal to

$$\begin{aligned}\sum_{x \in C(k)} \max_{i,j}(-\mathrm{ord}_x(g_i h_j)) &\le \sum_{x \in C(k)} \max_i(-\mathrm{ord}_x(g_i)) + \sum_{x \in C(k)} \max_j(-\mathrm{ord}_x(h_j)) \\ &= h(P) + \sum_{x \in C(k)} \max_j(-\mathrm{ord}_x(h_j)).\end{aligned}$$

We claim as a general identity that

$$\sum_{x \in C(k)} \max_j(-\mathrm{ord}_x(h_j)) = d; \tag{7.3}$$

this would complete the proof, since applying it to f_P would also show that the naive height just shown to be bounded by $h(P) + d$ would in fact coincide with the degree of $f_P(C)$, as desired.

Note that, by the product formula, the left side of (7.3) is unaffected by a common $k(C)^\times$-scaling on the h_j's. Hence, this left side is intrinsic to the embedding of C into $\mathbf{P}^m_k$ and is independent of the choice of representative homogeneous rational coordinate functions $h_0, \ldots, h_m$. Let $\ell = \sum a_j X_j$ be a generically chosen nonzero linear form over k, with zero-scheme H in $\mathbf{P}^m_k$. By genericity, C is not contained in H and all a_j are nonzero. Clearly $H \cap C$ is the zero-scheme of the nonzero rational function $\sum a_j h_j$ on C. Thus, d is the degree of the zero-scheme of this rational function (by the definition of d as the degree of C as a curve in $\mathbf{P}^m_k$), and so d is also the degree of the polar-scheme of the rational function $\sum a_j h_j$. For generic choices of the a_j's, $\sum a_j h_j$ will have its poles exactly where the h_j's have poles, with the pole-order of $\sum a_j h_j$ at each such point equal to the maximal pole-order among the h_j's at the point. Hence,

$$d = \sum_{x \in C(k)} \max_j(-\mathrm{ord}_x(h_j))$$

as long as the h_j's have no common zero (this lack of a common zero ensures that the contribution to the sum at each x is non-negative, and is positive at precisely the points where some h_j has a pole). By making a common $k(C)^\times$-scaling on the h_j's we may suppose some h_j is equal to 1, so this eliminates common zeros. ■

By Lemma 7.5, as P varies over $A(K)$ with $h(P) \le M$ (for fixed M), the curves

$$f_P : C \hookrightarrow \mathbf{P}^{(n+1)(m+1)-1}_k$$

have degree $\le M + d$. It is therefore enough to show that the set of points $P \in A(K)$ for which the closed immersion

$$f_P : C \hookrightarrow \mathbf{P}^{(n+1)(m+1)-1}_k$$

has a fixed degree (or *equivalently*, a fixed Hilbert polynomial) has finite image in $A(K)/\mathrm{Tr}_{K/k}(A)(k)$.

By the quasi-compactness aspects of Grothendieck's representability results on Hilbert and Hom-schemes [10], the functor of morphisms $P : C \to \widetilde{A}$ such that $\mathrm{pr}_2 \circ P = 1_C$ and f_P has degree δ in $\mathbf{P}^{(n+1)(m+1)-1}$ is represented by the "degree-δ" Hom-scheme $\mathbf{H}_\delta$ that is of *finite type* over k. Thus, it suffices to restrict attention to those P's corresponding to k-points on a common irreducible component of $\mathbf{H}_\delta$. The case of a

0-dimensional component is trivial, so we may focus attention on positive-dimensional components. Any two k-points on an irreducible finite-type k-scheme V of positive dimension lie in a common irreducible curve X in V (see the Lemma on p. 56 in [25]), so it remains to check that if $P, P' : C \rightrightarrows \widetilde{A}$ are two C-maps lying in an algebraic family of maps parameterized by an irreducible k-curve X then P and P' coincide in $A(K)/\mathrm{Tr}_{K/k}(A)(k)$. To be precise, by an algebraic family of maps we mean an $X \times C$-map

$$\mathscr{P} : X \times C \to X \times \widetilde{A},$$

and for all $x \in X(k)$ we will show that the points $\mathscr{P}_x \in \widetilde{A}(C) = A(K)$ represent a common class modulo $\mathrm{Tr}_{K/k}(A)(k)$.

Using pullback by the finite surjective normalization $\widetilde{X} \to X$, we may assume that X is k-smooth. Let $\overline{X}$ denote the k-smooth compactification of X. Passing to fibers over the generic point $\mathrm{Spec}\, K$ of C, we get a section

$$\mathscr{P}_K : X_K \to X_K \times A,$$

or equivalently a K-map $X_K \to A$, and by the valuative criterion for properness this uniquely extends to a K-map

$$F : \overline{X}_K \to A.$$

Since $X(k) \neq \emptyset$, upon choosing $x_0 \in X(k)$ we can use Albanese functoriality to find a unique factorization

$$\begin{array}{ccc} \overline{X}_K & \xrightarrow{\iota_K} & (\mathrm{Alb}_{\overline{X}/k})_K \\ & \searrow^{F} & \downarrow{\scriptstyle \eta} \\ & & A \end{array}$$

where $\eta(0) = F(x_0) \in A(K)$. Here, $\iota : (\overline{X}, x_0) \to (\mathrm{Alb}_{\overline{X}/k}, 0)$ is the universal pointed map to an abelian variety over k, and its formation commutes with extension on k. Since $\eta - F(x_0)$ respects origins, it is *a map of abelian varieties* over K. (For example, if A is a non-constant elliptic curve over K then $\eta - F(x_0)$ vanishes because $\mathrm{Tr}_{K/k}(A) = 0$ by Theorem 2.3.)

We apply the universal property of

$$\tau : \mathrm{Tr}_{K/k}(A)_K \to A$$

to get a factorization $\eta - F(x_0) = \tau \circ f_K$ for a unique map of abelian varieties $f : \mathrm{Alb}_{\overline{X}/k} \to \mathrm{Tr}_{K/k}(A)$ over k! Thus, composing with ι_K gives $F = F(x_0) + \tau \circ (f \circ \iota)_K$. Composing this identity with the map $x : \mathrm{Spec}\, K \to \overline{X}_K$ defined by $x \in X(k)$ gives that $\mathscr{P}_x : \mathrm{Spec}\, K \to A$ in $A(K)$ is equal to $F(x_0) + \tau \circ (f \circ \iota)_K(x)$, so the $\mathscr{P}_x$'s agree as elements in $A(K)/\mathrm{Tr}_{K/k}(A)(k)$: they all represent the residue class of the point $F(x_0) \in A(K)$ that has nothing to do with x. This concludes the proof of the Lang–Néron theorem.

8. Generalized global fields

In the final three sections, we give a scheme-theoretic development of the theory of heights in the "geometric" context of the Lang–Néron theorem. The theory of canonical heights on abelian varieties over a global field K provides a natural positive-definite quadratic form on $A(\overline{K})_{\mathbf{R}} \stackrel{\mathrm{def}}{=} \mathbf{R} \otimes_{\mathbf{Z}} A(\overline{K})$ for any polarized abelian variety (A, ϕ) over K such that the polarization ϕ satisfies an auxiliary symmetry condition: the ample line bundle $\mathscr{N}_\phi = (1, \phi)^*(\mathscr{P})$ on A is *symmetric* (i.e., $[-1]^*(\mathscr{N}_\phi) \simeq \mathscr{N}_\phi$), where $\mathscr{P}$ is the Poincaré bundle on $A \times A^\vee$. There are many such ϕ for any A, such as $\phi = \phi_{\mathscr{L}} : x \mapsto t_x^*(\mathscr{L}) \otimes \mathscr{L}^{-1}$ for any ample symmetric line bundle $\mathscr{L}$ on A, in which case $\mathscr{N}_\phi = [2]^*(\mathscr{L}) \otimes \mathscr{L}^{\otimes(-2)}$.

For any regular and finitely generated extension of fields K/k and any polarized abelian variety (A, ϕ) over K such that $(1, \phi)^*(\mathscr{P})$ is symmetric, we wish to put a similar structure on $(A(\overline{K})/\mathrm{Tr}_{K/k}(A)(\overline{k}))_{\mathbf{R}}$ once K is endowed with a collection of absolute values resembling the "product formula" situation in the classical special case $\mathrm{trdeg}_k(K) = 1$ (using $\|\cdot\|_v = e^{-[k(v):k]\mathrm{ord}_v}$ as v runs over the closed points of the unique regular proper k-curve with function field K). In this section we shall develop the theory of fields endowed with a "product formula" structure, and in §9 we use it to develop a theory of heights. Applications to positive-definiteness are given in §10 (and also see Corollary 9.12).

Let K be a field. Two absolute values $|\cdot|$ and $|\cdot|'$ on K are *equivalent* if they define the same topology on K. By [20, Ch. 1, 1.1], it is the same to say $|\cdot|' = |\cdot|^r$ for some $r > 0$.

Definition 8.1. A *generalized global field* is a field K equipped with an infinite set of equivalence classes v of non-trivial absolute values on K and a choice of representative absolute value $\|\cdot\|_v$ for each v such that

1. all but finitely many v are non-archimedean, each non-archimedean v is discretely-valued, and each $x \in K^\times$ is a v-unit for all but finitely of the non-archimedean v;
2. for all $x \in K^\times$ the *product formula* $\prod_v \|x\|_v^{e_v} = 1$ holds, where $e_v = 2$ if v is complex (that is, if v is archimedean and $K_v \simeq \mathbf{C}$) and $e_v = 1$ otherwise;
3. for all non-archimedean v, the discrete valuation ring $\mathscr{O}_v$ for v on K is excellent (this is equivalent to K_v/K being a separable extension, so it is always satisfied when K has characteristic 0).

Remark 8.2. Beware that for non-archimedean v the notation $\mathscr{O}_v$ denotes the discrete valuation ring for v in the field K, and it is not to be confused with the complete discrete valuation ring of the v-adic completion K_v of K; this latter valuation ring will never arise below. To keep the distinction clear, note that complete discrete valuation rings are always excellent whereas general discrete valuation rings (with positive generic characteristic) may fail to be excellent. We refer the reader to [21, Ch. 13] for a development of the basic properties of excellent rings. See [7, IV$_3$, 7.8ff] for further results concerning excellence.

Let us give two important classes of examples.

Example 8.3. The *arithmetic case* is when K is a number field. In this case, we use the traditional set of normalized absolute values $\|\cdot\|_v$: for non-archimedean v we require the value group of $\|\cdot\|_v$ in $\mathbf{R}_{>0}$ to be $q_v^{\mathbf{Z}}$ with q_v equal to the size of the finite residue field at v, and for archimedean v we use the standard absolute value on the topological field K_v (satisfying $\|q\|_v = |q|$ for $q \in \mathbf{Q}$). An element $x \in K^\times$ satisfies $\|x\|_v = 1$ for all v if and only if x is a root of unity.

Example 8.4. The *geometric case* with constant field k is when K is a finitely generated over a field k with k algebraically closed in K and $\mathrm{trdeg}_k(K) > 0$; we do *not* assume K/k is separable. In this case, let V be a proper integral k-scheme with $k(V) = K$ and assume V is regular in codimension 1 (for example, normal projective V). The codimension-1 points $v \in V$ give rise to inequivalent non-trivial discrete valuations on K with local ring $\mathscr{O}_{V,v}$ and associated normalized order function denoted $\mathrm{ord}_v : K^\times \twoheadrightarrow \mathbf{Z}$. If $\dim V > 1$ then this collection of local rings depends on the choice of V (though V is unique if $\mathrm{trdeg}_k(K) = 1$), and for each $x \in K^\times$ we have $\mathrm{ord}_v(x) = 0$ for all but finitely many v. Since schemes of finite type over a field are excellent [7, IV$_2$, 7.8.3], each $\mathscr{O}_{V,v}$ is excellent. To give K a structure of generalized global field, we want to find constants $0 < c_v < 1$ such that defining $\|\cdot\|_v = c_v^{\mathrm{ord}_v}$ makes the product formula $\prod_v \|x\|_v = 1$ hold for all $x \in K^\times$. (A special property of the generalized global field structures $\{\|\cdot\|_v\}_v$ on K arising in this way is that an element $x \in K^\times$ satisfies $\|x\|_v = 1$ for all v if and only if $x \in k^\times$, since k is algebraically closed in K and the normalization map $\widetilde{V} \to V$ is a finite birational map that is an isomorphism away from a closed subset of codimension ≥ 2 in V.)

To find such c_v's, first assume there exists a closed immersion $i : V \hookrightarrow \mathbf{P}^n_k$ over k. We can use $c_{v,i} = e^{-\deg_{k,i}(v)}$, with $\deg_{k,i}(v)$ the k-degree of the closure of $i(v)$ as an integral closed subscheme of $\mathbf{P}^n_k$: the product formula is the classical fact that on an integral closed subscheme of $\mathbf{P}^n_k$ that is regular in codimension 1, any principal Weil divisor has k-degree 0. More generally, if there exists an ample line bundle $\mathscr{N}$ on V then we can use $c_{v,\mathscr{N}} = e^{-\deg_{k,\mathscr{N}}(v)}$ where

$$\deg_{k,\mathscr{N}}(v) \stackrel{\mathrm{def}}{=} \deg_{k,V}([\overline{\{v\}}] \cap c_1(\mathscr{N})^{\dim V - 1}) = \deg_{k,\overline{\{v\}}}(c_1(\mathscr{N}|_{\overline{\{v\}}})^{\dim \overline{\{v\}}}).$$

Since $c_{v,\mathscr{N}^{\otimes n}} = c_{v,\mathscr{N}}^{n^{\dim V - 1}}$ for all positive integers n, reduction to the very ample case shows that the absolute values $\|\cdot\|_{v,\mathscr{N}} = c_{v,\mathscr{N}}^{\mathrm{ord}_v}$ satisfy the product formula.

Whenever we speak of the "geometric case" for K/k, it is always understood that we use a generalized global field structure arising from such a pair $(V, \mathscr{N})$. Note that replacing V with its normalization $\widetilde{V}$ and $\mathscr{N}$ with its ample pullback to $\widetilde{V}$ does not affect this construction, so there is no serious loss of generality in restricting attention to normal projective k-models for K. In the special case $\dim V = 1$, V is unique and both $\deg_{k,\mathscr{N}}(v) = [k(v) : k]$ and $\log \|\cdot\|_{v,\mathscr{N}} = -[k(v) : k]\mathrm{ord}_v$ are independent of $\mathscr{N}$.

Of course, when k is finite (the "overlap" of the arithmetic and geometric cases), it is traditional to use $1/\#k$ rather than $1/e$ in the above construction. In Remark 8.7 we will recall the justification for this convention, but we note here that since this change merely scales all $\log\|\cdot\|_v$'s by the universal positive constant $\log\#k$, it has essentially no impact on the theory of heights and so does not affect the meaning of any of the theorems of this paper (when applied to the geometric case with k finite).

Let us now explain the canonical procedure for extending generalized global field structures through finite extensions (and in Example 8.5 we will make it explicit in the arithmetic and geometric cases). Let K be a generalized global field and let K'/K be a finite extension. Each v on K lifts to finitely many equivalence classes v' on K', and each such v' admits a unique representative $\|\cdot\|_{v'}$ defined by the requirement that its restriction to K is $\|\cdot\|_v^{[K'_{v'}:K_v]e_v/e_{v'}}$ (where $e_v = 2$ for complex v and $e_v = 1$ otherwise, and similarly for $e_{v'}$). Note that for archimedean v we are requiring $\|\cdot\|_{v'}|_K = \|\cdot\|_v$, and obviously at most finitely many v' are archimedean. For $x' \in K'^{\times}$, if x' (resp. $1/x'$) is non-integral at a non-archimedean place v' of K' over a place v of K then one of the coefficients of the minimal polynomial of x' (resp. $1/x'$) over K is non-integral at v. Hence, x' is a v'-unit for all but finitely many non-archimedean v'. Also, for non-archimedean v the excellence requirement on the $\mathscr{O}_v$'s is inherited by the $\mathscr{O}_{v'}$'s because excellence is preserved under normalization in finite extensions [7, IV$_2$, 7.8.2]. The rings $K' \otimes_K K_v$ are reduced because K_v/K is separable for all v (thanks to the excellence hypothesis in the non-archimedean case), and hence the natural map

$$K' \otimes_K K_v \to \prod_{v'|v} K'_{v'}$$

is an isomorphism for all v. Thus, for all v and all $x' \in K'^{\times}$ we have

$$\prod_{v'|v} \|x'\|_{v'}^{e_{v'}} = \prod_{v'|v} \left(\|\mathrm{N}_{K'_{v'}/K_v}(x')\|_v^{[K'_{v'}:K_v]e_v/e_{v'}}\right)^{e_{v'}/[K'_{v'}:K_v]} = \|\mathrm{N}_{K'/K}(x')\|_v^{e_v},$$

and so the product formula holds for the $\|\cdot\|_{v'}$'s. This gives K' the sought-after natural structure of generalized global field, and the procedure is transitive in towers of finite extensions. This construction is the *algebraic method* for putting a generalized global field structure on K' (via the one given on K).

Example 8.5. In the arithmetic case, the algebraic method for endowing a finite extension K'/K of a number field K with a structure of generalized global field does give the number field K' its traditional collection of normalized absolute values as in Example 8.3.

Consider the geometric case K/k with a generalized global field structure $\{\|\cdot\|_{v,\mathscr{N}}\}_v$ as in Example 8.4, using a choice of pair $(V, \mathscr{N})$, so $c_v = e^{-\deg_{k,\mathscr{N}}(v)}$ for all codimension-1 points $v \in V$. The algebraic method as above gives any finite extension K' a structure of generalized global field via absolute values having the form $\|\cdot\|_{v'} = c_{v'}^{\mathrm{ord}_{v'}}$ on $K'^{\times}$ for suitable $0 < c_{v'} < 1$, with v' ranging over the codimension-1 points on the V-finite normalization V' of V in K'. Since V' is k-proper, integral, and normal with function field K', clearly the k-finite $\Gamma(V', \mathscr{O}_{V'}) \subseteq K'$ coincides with the algebraic closure k' of k in K'. In particular, V' is naturally a k'-scheme. The only elements $x' \in K'^{\times}$ satisfying $\|x'\|_{v'} = 1$ for all v' are the nonzero elements in k'. (Note that K'/k' need not be separable even if K/k is.) We would like to describe the $c_{v'}$'s explicitly, in a manner similar to the c_v's.

Let $\mathscr{N}'$ be the ample pullback of $\mathscr{N}$ to V'. In proofs it is sometimes necessary to replace K/k with K'/k', and so it is crucial to know that the generalized global field structure put on K' via the algebraic method (with respect to the given "geometric" generalized global field structure $\{\|\cdot\|_{v,\mathscr{N}}\}_v$ on K) is closely related to the generalized global field structure $\{\|\cdot\|_{v',\mathscr{N}'}\}_{v'}$ put on K' via k', V', and $\mathscr{N}'$, at least up to a constant factor in the exponent. First, observe that for both constructions the resulting set of equivalence classes of valuations on K' is the same, namely the equivalence classes of the discrete valuations on K' lifting the ones arising from the generalized global field structure on K. Hence, the absolute values on K' arising from the algebraic method may be denoted $\{\|\cdot\|_{v'}\}$ with index set given by the codimension-1 points $v' \in V'$. The relationship between $\|\cdot\|_{v'}$ and $\|\cdot\|_{v',\mathscr{N}'}$ is explained in the following lemma.

Lemma 8.6. *For all codimension-1 points $v' \in V'$ we have $c_{v'} = c_{v',\mathscr{N}'}^{[k':k]} = e^{-\deg_{k',\mathscr{N}'}(v')[k':k]}$. Thus, $\|\cdot\|_{v'} = \|\cdot\|_{v',\mathscr{N}'}^{[k':k]}$ for all such v'.*

Proof. Using the defining property of $\|\cdot\|_{v,\mathscr{N}}$ and the general formulas

$$\mathrm{ord}_{v'}|_{K^\times} = e(v'|v)\cdot \mathrm{ord}_v,\ \ [K'_{v'} : K_v] = [k'(v') : k(v)]e(v'|v)$$

with $e(v'|v)$ denoting the ramification degree for v' over v, the problem comes down to verifying the identity

$$[k' : k]\deg_{k',\mathscr{N}'}(v') \stackrel{?}{=} [k'(v') : k(v)]\deg_{k,\mathscr{N}}(v).$$

Letting X and X' denote the closures of v and v' in V and V' respectively, we are reduced to proving that if k'/k is a finite extension of fields, $f : X' \to X$ is a finite dominant map from an integral proper k'-scheme to an integral proper k-scheme, and $\mathscr{N}$ is a line bundle on X with pullback $\mathscr{N}'$ on X', then

$$[k' : k]\deg_{k',\mathscr{N}'}(X') \stackrel{?}{=} [k'(X') : k(X)]\deg_{k,\mathscr{N}}(X)$$

with $\deg_{k,\mathscr{N}}(X) \stackrel{\mathrm{def}}{=} \deg_k(c_1(\mathscr{N})^{\dim X})$ and likewise for $(X', \mathscr{N}', k')$.

Equivalently, since $\dim X = \dim X'$, we want the polynomials

$$[k' : k]\cdot\chi_{k'}(X', \mathscr{N}'^{\otimes n}),\ \ [k'(X') : k(X)]\cdot\chi_k(X, \mathscr{N}^{\otimes n})$$

in n to have the same leading coefficients. Since

$$[k' : k]\cdot\chi_{k'}(X', \mathscr{N}'^{\otimes n}) = \chi_k(X', \mathscr{N}'^{\otimes n}) = \chi_k(X, f_*(\mathscr{N}'^{\otimes n})) = \chi_k(X, (f_*\mathscr{N}')^{\otimes n})$$

and $f_*\mathscr{N}' = f_*f^*\mathscr{N} = (f_*\mathscr{O}_{X'})\otimes_{\mathscr{O}_X}\mathscr{N}$, with $f_*\mathscr{O}_{X'}$ generically a vector bundle of rank $[k'(X') : k(X)]$, it suffices to show that if $\mathscr{F}$ is a coherent sheaf on an integral proper k-scheme X and $\mathscr{F}$ has positive rank r at the generic point, then $\chi_k(X, \mathscr{F}\otimes\mathscr{N}^{\otimes n})$ has leading coefficient that is r times the leading coefficient of $\chi_k(X, \mathscr{N}^{\otimes n})$. This is proved in [25, §6, App.]. ■

Remark 8.7. For function fields of varieties over finite fields, the equality $(1/\#k)^{[k':k]} = 1/\#k'$ enables us to eliminate the intervention of $[k' : k]$ in Lemma 8.6 by using $1/\#k$ rather than $1/e$ in Example 8.4.

9. Review of heights

Let K be a generalized global field with associated set of absolute values $\{\|\cdot\|_v\}_v$ as in Definition 8.1, and choose an algebraic closure $\overline{K}$. For $n \geq 0$, the *standard K-height* $h_{K,n} : \mathbf{P}^n_K(\overline{K}) = (\overline{K}^{n+1} - \{0\})/\overline{K}^\times \to \mathbf{R}$ is

$$h_{K,n}([t_0, \ldots, t_n]) = \frac{1}{[K' : K]}\sum_{v'}\max_i(\log\|t_i\|_{v'}^{e_{v'}}) \geq 0$$

where $K' \subseteq \overline{K}$ is a finite subextension over K that contains the t_j's and we canonically endow K' with a structure of generalized global field via the algebraic method as in §8. This formula is independent of the choice of K' (because $[K'' : K'] = \sum_{v''|v'}[K''_{v''} : K'_{v'}]$ for all v' on K'), it is well-defined (by the product formula), and it is invariant under the action of $\mathrm{Aut}(\overline{K}/K)$ on $\mathbf{P}^n_K(\overline{K})$ (so it is essentially independent of the choice of $\overline{K}$). It would be more canonical to not choose $\overline{K}$ and to instead work with $h_{K,n}$ as a function on the set of closed points of $\mathbf{P}^n_K$. However, we are interested in applications to abelian varieties and so we prefer to work with the set of $\overline{K}$-points because for a locally finite type K-group G the set of $\overline{K}$-points $G(\overline{K})$ is naturally a group whereas the set of closed points of G is not naturally a group.

For any $T \in \mathrm{Aut}_K(\mathbf{P}^n_K)$, $h_{K,n} - h_{K,n}\circ T$ is bounded (in absolute value) on $\mathbf{P}^n_K(\overline{K})$. For proofs of this and all subsequent unattributed assertions in this section concerning K-heights, see [14, §B] and [25, §4, Appendix II], where proofs are given for number fields but carry over essentially *verbatim* to any generalized global field. Many basic proofs in [14] are written with restrictive smoothness hypotheses, though as noted in [14, B.3.6] such hypotheses can be avoided with better definitions in terms of Cartier divisors rather than Weil divisors. (The proofs of the basics in [25] make no smoothness restriction.)

For any K-vector space V of dimension $n+1 \geq 1$, transporting $h_{K,n}$ by means of any linear isomorphism $V \simeq K^{n+1}$ gives rise to a common (and hence intrinsic) residue class $h_{K,V}$ in the $\mathbf{R}$-vector space of $\mathbf{R}$-valued functions on $\mathbf{P}(V)(\overline{K})$ modulo $O(1)$ (by which we mean: modulo the $\mathbf{R}$-subspace of bounded functions). This residue class is denoted $h_{K,V}$.

Remark 9.1. In the arithmetic case it is traditional to work with $h_n = h_{K,n}/[K:\mathbf{Q}]$ and $h_V = h_{K,V}/[K:\mathbf{Q}]$ because these are invariant under finite extension on K. There is no "smallest subfield of finite index" analogous to $\mathbf{Q}$ in the geometric case, and so we must keep track of the ground field K in general.

Let X be a projective K-variety. For any very ample line bundle $\mathscr{L}$ on X, the closed immersion

$$\iota_{\mathscr{L}} : X \hookrightarrow \mathbf{P}(\mathrm{H}^0(X, \mathscr{L}))$$

defines a K*-height function* (modulo $O(1)$)

$$h_{K,\mathscr{L}} = h_{K,\mathrm{H}^0(X,\mathscr{L})} \circ \iota_{\mathscr{L}} \tag{9.1}$$

on $X(\overline{K})$. In what follows, all equations and inequalities involving $h_{K,\mathscr{L}}$ are understood to be taken modulo $O(1)$, though we may sometimes repeat this explicitly for emphasis.

Since $h_{K,\mathscr{L}\otimes\mathscr{L}'} = h_{K,\mathscr{L}} + h_{K,\mathscr{L}'}$ for any two very ample line bundles $\mathscr{L}$ and $\mathscr{L}'$ on X, and $h_{K,\mathscr{L}} = h_{K,\mathscr{L}'}$ if $\mathscr{L} \simeq \mathscr{L}'$ on X, if $\mathscr{L}$ is an arbitrary line bundle on X then we may define

$$h_{K,\mathscr{L}} = h_{K,\mathscr{L}_1} - h_{K,\mathscr{L}_2}$$

where $\mathscr{L} \simeq \mathscr{L}_1 \otimes \mathscr{L}_2^{-1}$ with very ample line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$. This is independent of the choice of $\mathscr{L}_1$ and $\mathscr{L}_2$, and $\mathscr{L} \mapsto h_{K,\mathscr{L}}$ is a homomorphism from $\mathrm{Pic}(X)$ to the $\mathbf{R}$-vector space of $\mathbf{R}$-valued functions on $X(\overline{K})$ modulo $O(1)$.

Remark 9.2. Let K'/K be finite and give K' a generalized global field structure via the algebraic method as in §8. Upon picking a K-embedding of K' into $\overline{K}$, we have

$$[K':K] h_{K,\mathscr{L}} = h_{K',\mathscr{L}_{K'}} \tag{9.2}$$

on $X(\overline{K}) = X_{K'}(\overline{K})$ (modulo $O(1)$, as always). Thus, for applications where one considers sets of bounded height it is harmless if we replace K with a finite extension K' and X with the projective K'-variety $X_{K'}$.

The identity (9.2) has a useful application for K/k as in the geometric case when $K' = K \otimes_k k'$ for an algebraic extension k'/k such that either k'/k or K/k is separable (so K' is a field and k' is algebraically closed in K'). Fix a choice of generalized global field structure on K using a pair $(V, \mathscr{N})$ as in Example 8.4. The hypotheses ensure that $V_{k'}$ is integral. Let V' be the normalization of $V_{k'}$ and let $\mathscr{N}'$ be the ample pullback of $\mathscr{N}_{k'}$ to V'. Upon choosing an algebraic closure $\overline{K}/K$, we pick a k-embedding of k' into $\overline{K}$ and thereby realize $\overline{K}$ as an algebraic closure of K'. Define $h^{\mathrm{geom}}_{K',\mathscr{L}_{K'}}$ to be the mod-$O(1)$ class of functions on $X_{K'}(\overline{K})$ defined via the line bundle $\mathscr{L}_{K'}$ and the generalized global field structure on K' corresponding to the pair $(V', \mathscr{N}')$. Beware that when $[k':k]$ is finite and larger than 1, this "geometric" generalized global field structure on K' is *not* the one assigned to K' as a finite extension of K via the algebraic method as in §8: there is a discrepancy by a factor of $[k':k]$ due to Lemma 8.6.

The advantage of this "geometric" procedure for making K' and K into generalized global fields via such pairs $(V, \mathscr{N})$ over k and $(V', \mathscr{N}')$ over k' is that it gives a variant on (9.2) in which there is no intervention of field degrees and so is well-suited to the case of algebraic extensions k'/k with possibly infinite degree (such as k' taken to be a separable closure of k):

Theorem 9.3. *With notation and hypotheses as above,*

$$h_{K,\mathscr{L}} = h^{\mathrm{geom}}_{K',\mathscr{L}_{K'}}$$

on $X(\overline{K}) = X_{K'}(\overline{K})$.

Proof. Since heights are calculated as finite sums, by descending through direct limits we may reduce to the case when $[k' : k]$ is finite. In this case, (9.2) translates the problem into that of proving the identity

$$\frac{h_{K',\mathscr{L}_{K'}}}{[k' : k]} \stackrel{?}{=} h^{\mathrm{geom}}_{K',\mathscr{L}_{K'}}$$

on $X_{K'}(\overline{K})$, where the K'-height on the left is defined using the generalized global field structure on the finite extension K'/K via the algebraic method in §8, and the K'-height on the right is defined in terms of the pair $(V', \mathscr{N}')$ as we have explained above. The desired identity is a special case of Lemma 8.6. ■

Here are some basic properties of K-heights:

- (functoriality) If $f : X \to X'$ is a map of projective K-varieties and $\mathscr{L}'$ is a line bundle on X' then $h_{K,f^*\mathscr{L}'} = h_{K,\mathscr{L}'} \circ f$. This follows from the Nullstellensatz over K.
- (positivity of ample K-heights) If $\mathscr{L}$ is an ample line bundle on X and $\mathscr{L}_0$ is an arbitrary line bundle on X then for some $c > 0$ we have $|h_{K,\mathscr{L}_0}| \le c \cdot h_{K,\mathscr{L}}$ modulo $O(1)$ on $X(\overline{K})$. This follows from the fact that the two line bundles $\mathscr{L}^{\otimes N} \otimes \mathscr{L}_0^{\otimes(\pm 1)}$ are very ample for N sufficiently large, together with the fact that the standard K-height $h_{K,n}$ on $\mathbf{P}^n(\overline{K})$ is non-negative at all points.
- (quasi-equivalence) If $\mathscr{L}$ and $\mathscr{L}'$ are algebraically equivalent (that is, they give rise to geometric points in the same connected component of the Picard scheme $\mathrm{Pic}_{X_{\overline{K}}/\overline{K}}$) and one of them is ample (so the other is also ample [17, 4.6]), then
$$\lim_{h_{K,\mathscr{L}}(x)\to\infty} \frac{h_{K,\mathscr{L}'}(x)}{h_{K,\mathscr{L}}(x)} = 1$$
as x ranges over $X(\overline{K})$ (in this limit we must choose representative functions on $X(\overline{K})$ for the mod-$O(1)$ residue classes $h_{K,\mathscr{L}}$ and $h_{K,\mathscr{L}'}$, but *a priori* these choices do not affect the limit).
- (positivity away from the base locus) If $\mathscr{L}$ is a line bundle on X then $h_{K,\mathscr{L}}$ is bounded below on $(X - B)(\overline{K}) = X(\overline{K}) - B(\overline{K})$, where
$$B = \mathrm{supp}(\mathrm{coker}(\mathrm{H}^0(X, \mathscr{L}) \otimes_K \mathscr{L} \to \mathscr{L}))$$
is the base locus of $\mathscr{L}$ (so $X - B$ is a non-empty open set in X if and only if $\mathrm{H}^0(X, \mathscr{L}) \neq 0$).

Example 9.4. Let K/k be as in the geometric case, endowed with a generalized global field structure as in Example 8.4. Let Y be a projective k-variety and X a projective K-variety, and let $f : Y_K \to X$ be a map over K. Using the algebraic closure $\overline{k} \subseteq \overline{K}$, we claim that $h_{K,\mathscr{L}} \circ f$ on $Y(\overline{K})$ is bounded on $Y(\overline{k})$ for any line bundle $\mathscr{L}$ on X. By functoriality, we may assume $X = Y_K$ and f is the identity, so the claim is that if $X = X_0 \otimes_k K$ for a projective k-variety X_0, then $h_{K,\mathscr{L}}$ is bounded on the subset $X_0(\overline{k}) \subseteq X(\overline{K})$. It suffices to check this for a single very ample $\mathscr{L}$, so we choose $\mathscr{L}$ to arise from a k-embedding $X_0 \hookrightarrow \mathbf{P}^n_k$. Since all points in the subset $\mathbf{P}^n(\overline{k}) \subseteq \mathbf{P}^n(\overline{K})$ have standard K-height 0, the claim is proved.

Example 9.5. For a proper K-variety X endowed with a projective K-embedding $\iota : X \hookrightarrow \mathbf{P}^n_K$ it is traditional to consider $h_{K,n} \circ \iota$ as the "induced height function" on $X(\overline{K})$. This *ad hoc* construction represents the mod-$O(1)$ residue class $h_{K,\iota^*\mathscr{O}_{\mathbf{P}^n_K}(1)}$ defined via the associated complete linear system (a fact we shall use below without comment). Indeed, the K-height $h_{K,\mathscr{O}_{\mathbf{P}^n_K}(1)}$ on $\mathbf{P}^n_K$ is represented by the function $h_{K,n}$, so $h_{K,n} \circ \iota$ represents the residue class $h_{K,\mathscr{O}_{\mathbf{P}^n_K}(1)} \circ \iota$, and this residue class is $h_{K,\iota^*\mathscr{O}_{\mathbf{P}^n_K}(1)}$ by functoriality of K-heights.

In the special case of abelian varieties A over K, one has a much finer theory of canonical K-heights in the sense that the mod-$O(1)$ residue class $h_{K,\mathscr{L}}$ admits a canonical representative function, the *canonical K-height function* $\widehat{h}_{K,\mathscr{L}} : A(\overline{K}) \to \mathbf{R}$ attached to $\mathscr{L}$ by Néron and Tate. Let us recall how this is constructed. For $\varepsilon = \pm 1$, a line bundle $\mathscr{L}$ on A is *ε-symmetric* if $\mathscr{L} \simeq [-1]^*(\mathscr{L})^\varepsilon$ (we also say *symmetric* if $\varepsilon = 1$ and *anti-symmetric* if $\varepsilon = -1$). If $\mathscr{L}$ is ε-symmetric, then the limit

$$\widehat{h}^{+}_{K,\mathscr{L}}(a) = \lim_{n\to\infty} \frac{h_{K,\mathscr{L}}(na)}{n^2} \in \mathbf{R} \qquad (9.3)$$

for $\varepsilon = 1$ and

$$\widehat{h}^{-}_{K,\mathscr{L}}(a) = \lim_{n\to\infty} \frac{h_{K,\mathscr{L}}(na)}{n} \in \mathbf{R} \tag{9.4}$$

for $\varepsilon = -1$ exists for all $a \in A(\overline{K})$; the formation of these limits uses a fixed choice of representative function for $h_{K,\mathscr{L}}$, the choice of which does not affect the limit. If $\mathscr{L}$ is symmetric then $\widehat{h}^{+}_{K,\mathscr{L}}$ is a quadratic form, and if $\mathscr{L}$ is anti-symmetric then $\widehat{h}^{-}_{K,\mathscr{L}}$ is additive. The dependence of $\widehat{h}^{+}_{K,\mathscr{L}}$ on symmetric $\mathscr{L}$ and of $\widehat{h}^{-}_{K,\mathscr{L}}$ on anti-symmetric $\mathscr{L}$ is additive.

For any line bundle $\mathscr{L}$ on A, define the symmetric and anti-symmetric line bundles

$$\mathscr{L}^{+} = \mathscr{L} \otimes [-1]^{*}(\mathscr{L}),\ \ \mathscr{L}^{-} = \mathscr{L} \otimes [-1]^{*}(\mathscr{L})^{-1},$$

and define the quadratic function

$$\widehat{h}_{K,\mathscr{L}} = \frac{\widehat{h}^{+}_{K,\mathscr{L}^{+}} + \widehat{h}^{-}_{K,\mathscr{L}^{-}}}{2} : A(\overline{K}) \to \mathbf{R}$$

as a sum of a quadratic form and an additive function. Strictly speaking, this "quadratic" function may have vanishing quadratic part, so it is really of degree ≤ 2 with value 0 at the origin; we shall nonetheless often refer to it as being a quadratic function. If $\mathscr{L}$ is symmetric (resp. anti-symmetric) then this quadratic function coincides with $\widehat{h}^{+}_{K,\mathscr{L}}$ (resp. $\widehat{h}^{-}_{K,\mathscr{L}}$), and $\widehat{h}_{K,\mathscr{L}_1\otimes\mathscr{L}_2} = \widehat{h}_{K,\mathscr{L}_1} + \widehat{h}_{K,\mathscr{L}_2}$ on $A(\overline{K})$ in general.

Remark 9.6. By Remark 9.2, if K'/K is a finite extension and we choose a K-embedding $K' \hookrightarrow \overline{K}$, then for $\mathscr{L}$ on A we have $\widehat{h}_{K',\mathscr{L}_{K'}} = [K' : K]\widehat{h}_{K,\mathscr{L}}$ on $A(\overline{K}) = A_{K'}(\overline{K})$ when K' is made into a generalized global field by the algebraic method in §8.

Clearly the function $\widehat{h}_{K,\mathscr{L}}$ on $A(\overline{K})$ is a representative for the residue class $h_{K,\mathscr{L}}$, and it only depends on the isomorphism class of $\mathscr{L}$. Functoriality holds for canonical K-heights in the sense that if $f : A \to B$ is a K-map of abelian varieties (so $f(0) = 0$) then for any line bundle $\mathscr{L}$ on B,

$$\widehat{h}_{K,f^{*}\mathscr{L}} = \widehat{h}_{K,\mathscr{L}} \circ f. \tag{9.5}$$

Indeed, both sides are $\mathbf{R}$-valued quadratic functions on $A(\overline{K})$ that vanish at the origin, so the boundedness of their difference (due to functoriality of the mod-$O(1)$ object $h_{K,\mathscr{L}}$) forces the difference to be zero. We can improve (9.5) by allowing f to merely be a map of K-varieties (with $f(0) \neq 0$ permitted): the general identity is

$$\widehat{h}_{K,f^{*}\mathscr{L}} = \widehat{h}_{K,\mathscr{L}} \circ f - \widehat{h}_{K,\mathscr{L}}(f(0)), \tag{9.6}$$

and to prove this we use the factorization $f = t_{f(0)} \circ (f - f(0))$ with $f - f(0)$ a homomorphism and t_b the translation by $b \in B(K)$ to reduce ourselves to treating the special case of translation morphisms by points in $B(K)$. Slightly more generally:

Lemma 9.7. *Let K be a generalized global field with algebraic closure $\overline{K}$ and let A be an abelian variety over K. For any finite subextension K'/K inside $\overline{K}$, any $a \in A(K')$, and any line bundle $\mathscr{L}$ on $A_{K'}$, we have*

$$\widehat{h}_{K',t_a^{*}\mathscr{L}} = \widehat{h}_{K',\mathscr{L}} \circ t_a - \widehat{h}_{K',\mathscr{L}}(a) \tag{9.7}$$

as functions on $A(\overline{K}) = A_{K'}(\overline{K})$. Here, K' is endowed with its canonical structure of generalized global field as a finite extension of K.

Proof. The height function $\widehat{h}_{K',\mathscr{L}}$ is defined as a sum of a quadratic form and a linear form by construction of canonical heights, and since K'-height functions as in (9.1) are functorial modulo $O(1)$ with respect to arbitrary morphisms of K'-varieties we see that the mod-$O(1)$ residue class $h_{K',\mathscr{L}} \circ t_a$ of the function $\widehat{h}_{K',\mathscr{L}} \circ t_a$ is the class $h_{K',t_a^{*}\mathscr{L}}$ that admits a representative function $\widehat{h}_{K',t_a^{*}\mathscr{L}}$. Thus, the two sides of (9.7) are functions of degree ≤ 2 that lie in the same residue class modulo $O(1)$, and so they differ by a constant. Comparing values at the origin shows that this constant is zero. ■

The property $h_{K,\mathscr{L}} \geq 0$ (modulo $O(1)$) for ample $\mathscr{L}$ implies $\widehat{h}_{K,\mathscr{L}} \geq 0$ on $A(\overline{K})$ for symmetric ample $\mathscr{L}$ because $\widehat{h}_{K,\mathscr{L}}$ is a bounded-below quadratic form for such $\mathscr{L}$.

The "quasi-equivalence" for K-height functions acquires a stronger form for canonical K-heights in the symmetric case (even without ampleness):

Theorem 9.8. *For symmetric invertible $\mathscr{L}$ on A, the quadratic form $\widehat{h}_{K,\mathscr{L}}$ on $A(\overline{K})$ only depends on $\mathscr{L}$ up to algebraic equivalence.*

Proof. Choose a symmetric ample $\mathscr{L}'$, and pick a large n so that the symmetric $\mathscr{L} \otimes \mathscr{L}'^{\otimes n}$ is ample too. Since $\widehat{h}_{K,\mathscr{L}} = \widehat{h}_{K,\mathscr{L}\otimes\mathscr{L}'^{\otimes n}} - n\widehat{h}_{K,\mathscr{L}'}$, it suffices to prove the result for symmetric ample line bundles.

Now let $\mathscr{L}$ be a symmetric ample line bundle, and $\mathscr{L}'$ another symmetric line bundle algebraically equivalent to $\mathscr{L}$, so $\mathscr{L}'$ is ample. We want to prove $\widehat{h}_{K,\mathscr{L}} = \widehat{h}_{K,\mathscr{L}'}$ on $A(\overline{K})$. By ordinary quasi-equivalence, applied to the canonical K-heights as representatives of the residue classes $h_{K,\mathscr{L}}$ and $h_{K,\mathscr{L}'}$, we have

$$\frac{\widehat{h}_{K,\mathscr{L}}(a)}{\widehat{h}_{K,\mathscr{L}'}(a)} \to 1$$

as $\widehat{h}_{K,\mathscr{L}'}(a) \to \infty$. For arbitrary $a \in A(\overline{K})$ with $\widehat{h}_{K,\mathscr{L}'}(a) \neq 0$ we have $\widehat{h}_{K,\mathscr{L}'}(na) = n^2\widehat{h}_{K,\mathscr{L}'}(a) \to \infty$ as $n \to \infty$ (since ampleness of $\mathscr{L}'$ ensures $\widehat{h}_{K,\mathscr{L}'}(a) \geq 0$), so as $n \to \infty$ we obtain

$$\frac{\widehat{h}_{K,\mathscr{L}}(a)}{\widehat{h}_{K,\mathscr{L}'}(a)} = \frac{\widehat{h}_{K,\mathscr{L}}(na)}{\widehat{h}_{K,\mathscr{L}'}(na)} \to 1$$

and hence $\widehat{h}_{K,\mathscr{L}}(a) = \widehat{h}_{K,\mathscr{L}'}(a)$. We likewise get such an equality when $\widehat{h}_{K,\mathscr{L}}(a) \neq 0$, and of course when both canonical K-heights vanish they are still equal. ■

The canonical K-height construction is important because it gives rise to a *canonical K-height pairing*

$$\langle\cdot,\cdot\rangle_{A,K} : A(\overline{K}) \times A^\vee(\overline{K}) \to \mathbf{R}$$

defined by

$$(a, [\mathscr{L}_{\overline{K}}]) \mapsto \frac{\widehat{h}_{K',\mathscr{L}}(a)}{[K' : K]}$$

for $a \in A(K')$ and $\mathscr{L}$ a representative line bundle on $A_{K'}$ for a finite extension K'/K inside of $\overline{K}$ (with K' given its canonical structure of generalized global field via the algebraic method as in §8); by Remark 9.6, the choice of $K' \subseteq \overline{K}$ adapted to the $\overline{K}$-points a and $[\mathscr{L}_{\overline{K}}]$ does not matter. This is $\mathbf{Z}$-bilinear because line bundles associated to geometric points of $A^\vee = \mathrm{Pic}^0_{A/K}$ are anti-symmetric (by the theorem of the square). Thus, we can extend scalars to $\mathbf{R}$ to get an induced $\mathbf{R}$-bilinear pairing

$$\langle\cdot,\cdot\rangle_{A,K,\mathbf{R}} : A(\overline{K})_{\mathbf{R}} \times A^\vee(\overline{K})_{\mathbf{R}} \to \mathbf{R}.$$

Also, if K'/K is a finite extension (given its generalized global field structure via the algebraic method in §8) and we choose a K-embedding $K' \hookrightarrow \overline{K}$, then under the general identification $X(\overline{K}) = X_{K'}(\overline{K})$ for K-schemes X (such as A and $A^\vee$) we have

$$\langle\cdot,\cdot\rangle_{A,K'} = [K' : K] \cdot \langle\cdot,\cdot\rangle_{A,K}. \tag{9.8}$$

The functoriality of canonical K-heights immediately implies adjointness with respect to dual maps: for $f : A \to B$ a map of abelian varieties over K,

$$\langle a, f^\vee(b')\rangle_{A,K} = \langle f(a), b'\rangle_{B,K}. \tag{9.9}$$

Remark 9.9. If $\mathscr{P}$ denotes the Poincaré line bundle on $A \times A^\vee$ then $\langle\cdot,\cdot\rangle_{A,K} : (A \times A^\vee)(\overline{K}) \to \mathbf{R}$ is also equal to $\widehat{h}_{K,\mathscr{P}}$. Indeed, consider a finite extension K'/K inside of $\overline{K}$, a point $a \in A(K')$, and a line bundle $\mathscr{L}$ on $A_{K'}$ that is algebraically equivalent to 0 (i.e., $\mathscr{L}$ is classified by a K'-point of the identity component $A^\vee$ of $\mathrm{Pic}_{A/K}$). We want to prove $\widehat{h}_{K',\mathscr{L}}(a)/[K' : K] = \widehat{h}_{K,\mathscr{P}}(a, \mathscr{L})$, and by Remark 9.6 we can rename K' as K. By the universal property of the Poincaré bundle, the slice inclusion $i : A \to A \times A^\vee$ defined by $x \mapsto (x, \mathscr{L})$

satisfies $i^*(\mathscr{P}) \simeq \mathscr{L}$. Thus, since $i(a) = (a, \mathscr{L})$, the generalized functoriality (9.6) for canonical heights gives $\widehat{h}_{K,\mathscr{P}}(a, \mathscr{L}) = \widehat{h}_{K,\mathscr{L}}(a) + \widehat{h}_{K,\mathscr{P}}(i(0))$. We therefore just need to prove that $\widehat{h}_{K,\mathscr{P}}(i(0)) = \widehat{h}_{K,\mathscr{P}}(0, \mathscr{L})$ is equal to 0. This reduces us to the special case $a = 0$. But now we can view A as dual to $A^\vee$ (retaining the fact that $\mathscr{P}$ is the universal line bundle) and so running the same calculation with roles of the factors swapped gives $\widehat{h}_{K,\mathscr{P}}(0, \mathscr{L}) = \widehat{h}_{K,\mathscr{P}}(0, \mathscr{O}_A) = 0$.

The quadratic form $\widehat{h}_{K,\mathscr{L}}$ for symmetric $\mathscr{L}$ is naturally recovered from the canonical K-height pairing $\langle \cdot, \cdot \rangle_{A,K}$, up to a factor of 2, by means of the map $\phi_{\mathscr{L}} : A \to A^\vee$ ($x \mapsto t_x^*(\mathscr{L}) \otimes \mathscr{L}^{-1}$). This reflects the correspondence between quadratic forms and symmetric bilinear forms:

Theorem 9.10. *For any invertible $\mathscr{L}$ on A we have*

$$\langle a_1, \phi_{\mathscr{L}}(a_2) \rangle_{A,K} = \widehat{h}_{K,\mathscr{L}}(a_1 + a_2) - \widehat{h}_{K,\mathscr{L}}(a_1) - \widehat{h}_{K,\mathscr{L}}(a_2) \tag{9.10}$$

for all $a_1, a_2 \in A(\overline{K})$, where $\phi_{\mathscr{L}}(x) = t_x^*(\mathscr{L}) \otimes \mathscr{L}^{-1}$. In particular, this pairing is symmetric and if $\mathscr{L}$ is symmetric then $\langle a, \phi_{\mathscr{L}}(a) \rangle_{A,K} = 2\widehat{h}_{K,\mathscr{L}}(a)$ for all $a \in A(\overline{K})$.

Proof. By functoriality of canonical K-heights,

$$\widehat{h}_{K,\mathscr{P}} \circ (1_A \times \phi_{\mathscr{L}}) = \widehat{h}_{K,(1\times\phi_{\mathscr{L}})^*(\mathscr{P})} = \widehat{h}_{K,m^*\mathscr{L}\otimes p_1^*\mathscr{L}^{-1}\otimes p_2^*\mathscr{L}^{-1}} = \widehat{h}_{K,\mathscr{L}} \circ m - \widehat{h}_{K,\mathscr{L}} \circ p_1 - \widehat{h}_{K,\mathscr{L}} \circ p_2.$$

Thus, by Remark 9.9 we get (9.10). The rest follows immediately (e.g., the final assertion for symmetric $\mathscr{L}$ holds because $\widehat{h}_{K,\mathscr{L}}$ is a quadratic form for such $\mathscr{L}$). ■

Corollary 9.11. *Let $\iota_A : A \to A^{\vee\vee}$ be the double-duality isomorphism. For any $(a, a') \in A(\overline{K}) \times A^\vee(\overline{K})$ we have $\langle a, a' \rangle_{A,K} = \langle a', \iota(a) \rangle_{A^\vee,K}$.*

Proof. Again using Remark 9.9 and the functoriality of canonical K-heights, we just have to recall that if $s : A \times A^\vee \simeq A^\vee \times A$ is the flipping isomorphism and $\mathscr{P}_{A^\vee}$ is a Poincaré bundle on $A^\vee \times A^{\vee\vee}$ then $s^*((1_{A^\vee} \times \iota_A)^*(\mathscr{P}_{A^\vee}))$ is a Poincaré bundle on $A \times A^\vee$. ■

Corollary 9.12. *For any polarization $\phi : A \to A^\vee$, the induced* **R***-bilinear pairing*

$$A(\overline{K})_{\mathbf{R}} \times A(\overline{K})_{\mathbf{R}} \to \mathbf{R}$$

defined by $(a_1, a_2)_\phi = \langle a_1, \phi(a_2) \rangle_{A,K,\mathbf{R}}$ is symmetric. If the ample line bundle $(1, \phi)^*(\mathscr{P})$ on A is symmetric then $(\cdot, \cdot)_\phi$ is positive semidefinite (i.e., $(a, a)_\phi \geq 0$ for all $a \in A(\overline{K})_{\mathbf{R}}$).

Proof. By replacing K with a finite extension and using (9.8), we can assume $\phi = \phi_{\mathscr{L}}$ for some ample $\mathscr{L}$ on A. This gives the symmetry, by Theorem 9.10. If we define $\mathscr{N} = (1, \phi)^*(\mathscr{P}) = [2]^*(\mathscr{L}) \otimes \mathscr{L}^{\otimes(-2)}$ then $\mathscr{N}$ is ample and $2\phi_{\mathscr{L}} = \phi_{\mathscr{N}}$ by the theorem of the square. Hence, in case $\mathscr{N}$ is symmetric it is harmless to replace ϕ with 2ϕ to reduce the positive semidefiniteness claim to the case $\phi = \phi_{\mathscr{L}}$ for a symmetric ample $\mathscr{L}$ on A. It therefore remains to recall our earlier observation that the quadratic form $\widehat{h}_{K,\mathscr{L}}$ on $A(\overline{K})_{\mathbf{R}}$ is non-negative for any symmetric ample line bundle $\mathscr{L}$ on A. ■

The preceding discussion of heights is valid for any generalized global field K. We now turn our attention to the geometric case. Let K/k be a finitely generated regular extension, and give K a generalized global field structure using a pair $(V, \mathscr{N})$ as in Example 8.4. This generalized global field structure on K gives rise to a theory of heights for abelian varieties over K.

Lemma 9.13. *For a generalized global field K/k as in Example* 8.4*, let A be an abelian variety over K and let $\mathscr{L}$ be a line bundle on A. For all $a \in A(\overline{K})$ and $a_0 \in \mathrm{Tr}_{K/k}(A)(\overline{k})$ we have $\widehat{h}_{K,\mathscr{L}}(a + a_0) = \widehat{h}_{K,\mathscr{L}}(a)$. In particular, the quadratic* (*or additive*) *function $\widehat{h}_{K,\mathscr{L}}$ uniquely factors as a quadratic* (*or additive*) *function*

$$\widehat{h}_{K,\mathscr{L}} : A(\overline{K})/\mathrm{Tr}_{K/k}(A)(\overline{k}) \to \mathbf{R} \tag{9.11}$$

We let $\widehat{h}_{K,\mathscr{L},\mathbf{R}} : (A(\overline{K})/\mathrm{Tr}_{K/k}(A)(\overline{k}))_{\mathbf{R}} \to \mathbf{R}$ denote the induced function after extension of scalars to $\mathbf{R}$ on the source. This is a quadratic form if $\mathscr{L}$ is symmetric.

Proof. By Theorem 6.8, Theorem 9.3, and the definition of $\widehat{h}_{K,\mathscr{L}}$, we can replace K/k with $\overline{k}K/\overline{k}$ to reduce to the case that k is algebraically closed. Since $a \in A(K')$ for some finite extension K'/K inside of $\overline{K}$, and K'/k is regular (since k is algebraically closed), by Remark 9.6 we can assume $a \in A(K)$. By Lemma 9.7, we just have to prove that $\widehat{h}_{K,t_a^*\mathscr{L}}$ vanishes on $\mathrm{Tr}_{K/k}(A)(\overline{k})$. For any line bundle $\mathscr{N}$ on A, applying Example 9.4 to $\tau : \mathrm{Tr}_{K/k}(A)_K \to A$ gives that any representative function for $h_{K,\mathscr{N}}$ on $A(\overline{K})$ is bounded on the subgroup $\mathrm{Tr}_{K/k}(A)(\overline{k}) \subseteq A(\overline{K})$. Hence, the quadratic (or additive) function $\widehat{h}_{K,\mathscr{N}}$ on $A(\overline{K})$ is bounded on $\mathrm{Tr}_{K/k}(A)(\overline{k})$ with value 0 at the origin, and therefore it vanishes on this subgroup. ∎

Remark 9.14. Assume K/k in Lemma 9.13 is regular, so $K \otimes_k k'$ is a field for any algebraic extension k'/k. By the proof of Lemma 7.3, for any algebraic extension k'/k and any extension K' of $K \otimes_k k'$, the natural map $A(K)/\mathrm{Tr}_{K/k}(A)(k) \to A(K')/\mathrm{Tr}_{K/k}(A)(k')$ is injective (clearly the key case is $K' = K \otimes_k k'$). Thus, by expressing $\overline{k}/k$ as a direct limit of finite subextensions, the source in (9.11) is a direct limit with *injective* transition maps when K/k is regular.

Recall that when K is a global field of the classical type (a number field or function field of a curve over a finite field), then for a symmetric ample line bundle $\mathscr{L}$ on an abelian variety A over K, the positive semidefinite canonical K-height $\widehat{h}_{K,\mathscr{L}}$ on $A(\overline{K})$ has positive-definite scalar extension to $A(\overline{K})_{\mathbf{R}}$. Thus, this scalar extension is also positive-definite (or equivalently, non-degenerate) on each finite-dimensional subspace $A(K')_{\mathbf{R}}$ for finite K'/K inside of $\overline{K}$, and we can use Theorem 9.10 to rephrase this non-degeneracy in more canonical terms: when K is a global field, the canonical K-height pairing

$$\langle \cdot,\cdot \rangle_{A,K,\mathbf{R}} : A(\overline{K})_{\mathbf{R}} \times A^{\vee}(\overline{K})_{\mathbf{R}} \to \mathbf{R}$$

restricts to a perfect duality between $A(K')_{\mathbf{R}}$ and $A^{\vee}(K')_{\mathbf{R}}$ for all finite K'/K inside of $\overline{K}$. In the classical global function field case with finite constant field $k \subseteq K$, the subgroup $\mathrm{Tr}_{K/k}(A)(\overline{k})$ is a torsion group and so it is killed by the operation of tensoring against $\mathbf{R}$. Thus, in this case we can equivalently say that $\widehat{h}_{K,\mathscr{L},\mathbf{R}}$ is positive-definite on $(A(\overline{K})/\mathrm{Tr}_{K/k}(A)(\overline{k}))_{\mathbf{R}}$. In general, we have:

Theorem 9.15. *Let K be a finitely generated regular extension of a field k with* $\mathrm{trdeg}_k(K) > 0$, *and endow K with a structure of generalized global field by means of a pair $(V,\mathscr{N})$ over k as in Example* 8.4. *For any abelian variety A over K and any symmetric ample line bundle $\mathscr{L}$ on A, the quadratic form*

$$\widehat{h}_{K,\mathscr{L},\mathbf{R}} : (A(\overline{K})/\mathrm{Tr}_{K/k}(A)(\overline{k}))_{\mathbf{R}} \to \mathbf{R}$$

is positive-definite.

This is proved in [20, Ch. 6, §5] using pre-Grothendieck methods, and in §10 we shall give a proof in the language of schemes. Let us now give the reduction steps that eliminate the appearance of algebraic closures, as this also leads to a reformulation of Theorem 9.15 in terms of the canonical K-height pairing.

Observe that by expressing $\overline{K}$ as a direct limit of finite extensions of $\overline{k}K$, we see that among the finite extensions of K inside of $\overline{K}$, a cofinal set is given by those K' that are regular over the algebraic closure k' of k in K' (this regularity is automatic when K'/K is separable or k is perfect). Thus, by Theorem 6.8, Example 8.5, Lemma 8.6, Remark 9.6, and Remark 9.14, by suitable renaming of the constant field it suffices to prove positive-definiteness on $(A(K')/\mathrm{Tr}_{K/k}(A)(k))_{\mathbf{R}}$ in general for finite extensions K'/K such that K'/k is regular. (In case K'/K is separable, the extension K'/k is regular if and only if it is primary.)

Lemma 9.16. *For A and K as above, let K'/K be a finite extension with K'/k regular. The natural map*

$$A(K)/\mathrm{Tr}_{K/k}(A)(k) \to A(K')/\mathrm{Tr}_{K'/k}(A_{K'})(k) \tag{9.12}$$

has finite kernel.

Before proving the lemma, let us show by example in arbitrary characteristic that the kernel of (9.12) can be nonzero. Let $K'/K/k$ and the elliptic curves E_0 over k and A over K be as in Remark 4.6, so $\mathrm{Tr}_{K/k}(A) = 0$. By construction, $A(K) \subseteq A(K') = E_0(K')$ is the -1-eigenspace $E_0(K')^-$ for the natural action by $\mathrm{Gal}(K'/K)$. Hence, (9.12) is the map $E_0(K')^- \to E_0(K')/E_0(k)$ that has kernel $E_0(k)[2]$. We can choose E_0 so that this latter group is nonzero.

Proof. Let K_0/K be the separable closure of K in K'. Since K'/K_0 is purely inseparable, Theorem 6.4(3) settles the case of K'/K_0 and so it remains to treat the case when K'/K is separable. Let K'' be a Galois closure of K'/K, and let k''/k be the algebraic closure of k in K''. The extension K''/k'' is regular, but we need to first circumvent the possibility that $k'' \neq k$. To this end, let $F = K \otimes_k k''$ and $F' = K' \otimes_k k''$ considered as subfields of K''. Theorem 6.8 and Remark 9.14 imply that the natural map

$$A(K')/\mathrm{Tr}_{K'/k}(A)(k) \to A(F')/\mathrm{Tr}_{F'/k''}(A_{F'})(k'')$$

is injective. The composite of this injection with (9.12) is equal to the composite of natural maps

$$A(K)/\mathrm{Tr}_{K/k}(A)(k) \to A(F)/\mathrm{Tr}_{F/k''}(A)(k'') \to A(F')/\mathrm{Tr}_{F'/k''}(A_{F'})(k'')$$

with injective first step (by Theorem 6.8 and Remark 9.14). Hence, we can replace K/k with F/k'' to reduce to the case when $k'' = k$ (i.e., K''/k is regular). It clearly suffices to treat K''/K instead of K'/K, so we can assume K'/K is Galois. Hence, we need to prove that when K'/K is Galois and K'/k is regular (or equivalently, finite), the quotient group $(A(K) \cap \mathrm{Tr}_{K'/k}(A_{K'})(k))/\mathrm{Tr}_{K/k}(A)(k)$ is finite.

For $\gamma \in \mathrm{Gal}(K'/K)$, there are canonical isomorphisms

$$i_\gamma : \gamma^*(\mathrm{Tr}_{K'/k}(A_{K'})_{K'}) \simeq \mathrm{Tr}_{K'/k}(A_{K'})_{K'}, \;\; j_\gamma : \gamma^*(A_{K'}) \simeq A_{K'}$$

as abelian varieties over K' (encoding the evident Galois descents to K). By the universal property of the K'/k-trace $\tau_{A_{K'},K'/k}$, there is a unique k-map of abelian varieties $[\gamma] : \mathrm{Tr}_{K'/k}(A_{K'}) \to \mathrm{Tr}_{K'/k}(A_{K'})$ such that the diagram

$$\begin{array}{ccccc} \mathrm{Tr}_{K'/k}(A_{K'})_{K'} & \xrightarrow[\simeq]{i_\gamma^{-1}} & \gamma^*(\mathrm{Tr}_{K'/k}(A_{K'})_{K'}) & & \\ \downarrow{\scriptstyle [\gamma]_{K'}} & & & \searrow{\scriptstyle \gamma^*(\tau_{A_{K'},K'/k})} & \\ \mathrm{Tr}_{K'/k}(A_{K'})_{K'} & \xrightarrow[\tau_{A_{K'},K'/k}]{} & A_{K'} & \xleftarrow[j_\gamma]{\simeq} & \gamma^*(A_{K'}) \end{array}$$

commutes. Uniqueness gives $[1] = \mathrm{id}$ and $[\gamma_1\gamma_2] = [\gamma_1] \circ [\gamma_2]$, so each $[\gamma]$ is an automorphism and we get a natural action of the finite group $\mathrm{Gal}(K'/K)$ on the abelian variety $\mathrm{Tr}_{K'/k}(A_{K'})$ over k. For $x \in \mathrm{Tr}_{K'/k}(A_{K'})(K)$ and $y \in A(K)$ we have $i_\gamma(\gamma^*(x)) = x$ and $j_\gamma(\gamma^*(y)) = y$, so this action by $\mathrm{Gal}(K'/K)$ is the identity on all points in $A(K) \cap \mathrm{Tr}_{K'/k}(A_{K'})(k)$.

The Zariski-closure Z of $A(K) \cap \mathrm{Tr}_{K'/k}(A_{K'})(k)$ in $\mathrm{Tr}_{K'/k}(A_{K'})$ is a smooth closed k-subgroup of $\mathrm{Tr}_{K'/k}(A_{K'})$, so the identity component Z^0 is an abelian variety (perhaps $Z^0 = 0$). The triviality of the $\mathrm{Gal}(K'/K)$-action on $A(K) \cap \mathrm{Tr}_{K'/k}(A_{K'})(k)$ implies (by Zariski-denseness considerations) that the map $t' : Z_{K'} \to A_{K'}$ induced by $\tau_{A_{K'},K'/k}$ is $\mathrm{Gal}(K'/K)$-equivariant with respect to the K'/K-descent data on both sides, so it descends to a K-map of K-groups $t : Z_K \to A$. The restriction $t^0 : Z^0_K \to A$ of t factors uniquely as $\tau_{A,K/k} \circ \varphi_K$ for a unique k-map of abelian varieties $\varphi : Z^0 \to \mathrm{Tr}_{K/k}(A)$. Hence, the image of $Z^0(k)$ in $A(K)$ lies in $\mathrm{Tr}_{K/k}(A)(k) \subseteq A(K)$, so by working inside of $\mathrm{Tr}_{K'/k}(A_{K'})(k)$ we have that the subgroup $A(K) \cap \mathrm{Tr}_{K'/k}(A_{K'})(k) \subseteq Z(k)$ meets $Z^0(k)$ in a subgroup of $\mathrm{Tr}_{K/k}(A)(k)$. The group

$$(A(K) \cap \mathrm{Tr}_{K'/k}(A_{K'})(k))/\mathrm{Tr}_{K/k}(A)(k)$$

is therefore a quotient of the subgroup

$$(A(K) \cap \mathrm{Tr}_{K'/k}(A_{K'})(k))/(Z^0(k) \cap A(K) \cap \mathrm{Tr}_{K'/k}(A_{K'})(k)) \hookrightarrow Z(k)/Z^0(k),$$

so finiteness of $Z(k)/Z^0(k)$ finishes the proof. ■

By Lemma 9.16, the natural map $(A(K)/\mathrm{Tr}_{K/k}(A)(k))_{\mathbf{R}} \to (A(K')/\mathrm{Tr}_{K'/k}(A_{K'})(k))_{\mathbf{R}}$ is *injective* for finite K'/K such that K'/k is regular, and so (again using Example 8.5, Lemma 8.6, and Remark 9.6) by renaming K' as K we see to prove Theorem 9.15 it is equivalent prove positive-definiteness of the positive semidefinite quadratic form $\widehat{h}_{K,\mathscr{L},\mathbf{R}}$ on the $\mathbf{R}$-vector space $(A(K)/\mathrm{Tr}_{K/k}(A)(k))_{\mathbf{R}}$ in general. This result will be proved in §10.

In view of the preceding reduction steps and Theorem 9.10, the Lang–Néron theorem enables us to reformulate Theorem 9.15 as follows:

Corollary 9.17. *With hypotheses and notation as in Theorem 9.15, the canonical K-height pairing restricts to a perfect duality $(A(K')/\mathrm{Tr}_{K/k}(A)(k'))_{\mathbf{R}} \times (A^\vee(K')/\mathrm{Tr}_{K/k}(A^\vee)(k'))_{\mathbf{R}} \to \mathbf{R}$ between finite-dimensional vector spaces for any finite extension K'/K that is regular over the algebraic closure k'/k of k in K'.*

The regularity condition on K'/k' in the corollary is satisfied for all separable finite extensions K'/K, and also for all finite extensions K'/K when k is perfect.

10. Proof of Theorem 9.15

We begin by recalling a general lemma of Minkowski that reduces the positive-definiteness problem over $\mathbf{R}$ to a finiteness assertion on a lattice.

Lemma 10.1 (Minkowski)**.** *Let Λ be a finitely generated $\mathbf{Z}$-module and let $q : \Lambda \to \mathbf{R}$ be a quadratic form such that $q(\lambda) \geq 0$ for all $\lambda \in \Lambda$. Let $q_{\mathbf{R}} : V = \mathbf{R} \otimes_{\mathbf{Z}} \Lambda \to \mathbf{R}$ be the induced quadratic form. If, for all $C > 0$, there are only finitely many $\lambda \in \Lambda$ such that $q(\lambda) < C$, then $q_{\mathbf{R}}$ is positive-definite.*

Proof. See [31, Ch. VIII, Lemma 9.5]. ■

This lemma and the reduction steps in §9 reduce us to showing that for all $C > 0$, the elements $P \in A(K)$ satisfying $\widehat{h}_{K,\mathscr{L}}(P) < C$ represent only finitely many residue classes modulo $\mathrm{Tr}_{K/k}(A)(k)$. We can replace $\mathscr{L}$ with a very ample power $\mathscr{L}^{\otimes n}$, and we can work with a K-height function arising from a choice of ordered K-basis of $\Gamma(A, \mathscr{L})$ and the associated projective K-embedding of A (as this function differs from the corresponding canonical height by a bounded amount). Thus, by the reduction steps in §9, Theorem 9.15 is reduced to:

Theorem 10.2. *Let K/k be a finitely generated regular extension of fields with $\mathrm{trdeg}_k(K) > 0$, and fix a pair $(V, \mathscr{N})$ over k giving K a structure of generalized global field as in Example 8.4. Fix a projective K-embedding $A \hookrightarrow \mathbf{P}^n_K$ and let $h_K : A(K) \to \mathbf{R}$ be the resulting K-height function. For all $M > 0$, the elements $P \in A(K)$ satisfying $h_K(P) \leq M$ represent only finitely many residue classes modulo $\mathrm{Tr}_{K/k}(A)(k)$.*

The special case $\mathrm{trdeg}_k(K) = 1$ with k algebraically closed was proved as the key ingredient in the proof of the Lang–Néron theorem in §7. The case of higher transcendence degree requires more care because we have to work systematically with rational maps f_P on V whose domain of definition in V may vary with P. The diligent reader will observe that the reduction of our task to proving Theorem 10.2 did not use the Lang–Néron theorem, nor does the following proof of Theorem 10.2 use the Lang–Néron theorem, and so (at the expense of using the foundational discussion in §8–§9) Theorem 10.2 gives a proof of the Lang–Néron theorem that avoids the need to initially reduce to the case of transcendence degree 1 with an algebraically closed constant field.

Proof. Let k'/k be a separable algebraic extension and define $K' = K \otimes_k k'$, so we get a standard K'-height on $\mathbf{P}^n(K')$ by using the generalized global field structure on K' arising from $(V_{k'}, \mathscr{N}_{k'})$ as in Example 8.4; note that $V_{k'}$ is integral and regular in codimension 1 since k'/k is separable and V is geometrically irreducible over k (and regular in codimension 1). By Lemma 8.6, if $[k' : k]$ is finite then this is generally *not* the generalized global field structure put on the finite extension K'/K via the algebraic method in §8; there is a discrepancy factor of $[k' : k] = [K' : K]$. Even worse, there is no uniform discrepancy factor when $[k' : k]$ is infinite. Fortunately, by Theorem 9.3, the standard K'-height on $\mathbf{P}^n(K')$ defined via the generalized global field structure on K' arising from $(V_{k'}, \mathscr{N}_{k'})$ has restriction to $\mathbf{P}^n(K)$ that coincides with the standard K-height defined via the generalized global field structure on K arising from $(V, \mathscr{N})$. Thus, by Remark 9.14 we can extend scalars to a separable closure of k to reduce to the case when k is infinite.

Let η be the generic point of V. Replacing the ample $\mathscr{N}$ with a very ample power $\mathscr{N}^{\otimes n}$ causes K-heights to be multiplied by a universal constant $n^{\dim V - 1} = n^{\mathrm{trdeg}_k(K)-1}$, so we can assume that there is a projective k-embedding $\iota : V \hookrightarrow \mathbf{P}^m_k$ that induces the structure of generalized global field on $K = k(\eta)$ (with field of constants k). We let $[h_0, \ldots, h_m]$ be a representative ordered $(m+1)$-tuple of rational functions on V not all of which are zero and which define ι as a rational map. We let d be the k-degree of V in $\mathbf{P}_k^m$. Here is a formula for d:

$$\deg_{\mathbf{P}_k^m}(V) \stackrel{?}{=} h_{K,m}(\iota(\eta)) = \sum_v \max_j(-\mathrm{ord}_v(h_j)\deg_{\mathbf{P}_k^m}(v)), \tag{10.1}$$

where the sum runs over all codimension-1 points $v \in V$ and $\deg_{\mathbf{P}_k^m}(v)$ is the degree of the closure of $\iota(v)$ as an integral closed subscheme of $\mathbf{P}_k^m$. For the case $\mathrm{trdeg}_k(K) = 1$ and k algebraically closed, (10.1) is the identity (7.3). In general, the right side of (10.1) is invariant under a common $k(V)^\times$-scaling on the h_j's, by the product formula, and so the argument used in the 1-dimensional case carries over essentially *verbatim* to the general case as long as we are able to find k-rational points in Zariski-dense open loci of hyperplanes (parameterized by a dual projective space). This is no problem, since k is infinite. (The reduction steps to get to the case of infinite k also show that (10.1) is valid for finite k.)

The given closed embedding $A \hookrightarrow \mathbf{P}_K^n$ identifies $P \in A(K)$ with a K-point $[g_0, \dots, g_n]$ of projective n-space with $g_i \in K$ not all zero. By definition of h_K and the generalized global field structure on K,

$$h_K(P) = \sum_v \max_i(-\mathrm{ord}_v(g_i)\deg_{\mathbf{P}_k^m}(v)).$$

Let $\widetilde{A}$ be the k-variety closure of A under the map

$$\phi : A \hookrightarrow \mathbf{P}_\eta^n = \mathbf{P}_k^n \times \eta \subseteq \mathbf{P}_k^n \times_k V \hookrightarrow \mathbf{P}_k^n \times \mathbf{P}_k^m \hookrightarrow \mathbf{P}_k^N,$$

where $N = (n+1)(m+1)-1$. The closure W_P of P in $\widetilde{A}$ is the scheme-theoretic image of the rational k-map $f_P = \phi \circ P$ on V defined by the tuple $[g_ih_j]$; the domain of f_P on V may vary with P. By construction, the projection from $W_P \subseteq \mathbf{P}_k^n \times V$ to V is a birational morphism. Thus, W_P is a projective k-variety model for K, but (unlike V) it is generally not regular in codimension 1.

We shall now bound the degree of W_P inside of $\mathbf{P}_k^N$. The generic point $f_P(\eta)$ of W_P is a K-point of $\mathbf{P}^N$ whose standard K-height has an upper bound:

$$\begin{aligned} h_{K,N}(f_P(\eta)) &= \sum_v \max_{i,j}(-\mathrm{ord}_v(g_ih_j)\deg_{\mathbf{P}_k^m}(v)) \\ &\le \sum_v \max_i(-\mathrm{ord}_v(g_i)\deg_{\mathbf{P}_k^m}(v)) + \sum_v \max_j(-\mathrm{ord}_v(h_j)\deg_{\mathbf{P}_k^m}(v)) \\ &= h_K(P) + d \end{aligned}$$

by (10.1). We claim that $h_{K,N}(f_P(\eta))^{\dim W_P}$ is an upper bound on the k-degree of W_P as a k-subvariety of $\mathbf{P}_k^N$. Rather more generally:

Lemma 10.3 (Néron). *If $f : \eta = \mathrm{Spec}\, K \to \mathbf{P}_k^N$ is a k-morphism and W denotes the k-variety closure of $f(\eta)$, then*

$$\deg_{\mathbf{P}_k^N}(W) \le h_{K,N}(f(\eta))^{\dim W}.$$

Although we are presently working under the extra property that k is infinite, the lemma makes sense for any k and it true in such generality: the preceding arguments concerning separable algebraic extension of the constant field show that both sides of the inequality are unaffected by any separable algebraic extension on k.

Proof. Let $[f_0, \dots, f_N]$ be a representative tuple of elements of K not all zero that induces the rational k-map f from V to $\mathbf{P}_k^N$. We can and do assume one of the f_j's is equal to 1. The case $\dim W = 0$ is trivial, so we suppose $r = \dim W$ is positive.

Choose dense opens $W' \subseteq W$ and $V' \subseteq V$ such that V' lies in the domain of definition of every f_j and f induces a surjective k-morphism from V' onto W'. Since k is infinite and W is generically smooth with $r = \dim W > 0$, by Bertini techniques we can find k-rational hyperplanes $H_1, \dots, H_r$ in $\mathbf{P}_k^N$ whose common intersection with W is finite étale over k and is supported in W'. In fact, we can choose the H_i's so that for $1 \le i < r$ each $H_1 \cap \cdots \cap H_i \cap W$ is geometrically integral of codimension i and H_{i+1} is "generic" in the dual projective space of hyperplanes. The k-finite étale intersection $W \cap (\cap_\alpha H_\alpha)$ has k-length $\deg_{\mathbf{P}^N_k}(W)$, and we want to bound this k-length from above by $h_{K,N}(f(\eta))^r$.

The preimage of $H_i \cap W'$ in V' is an effective Cartier divisor in V', and let D_i be its scheme-theoretic closure in V, so D_i is a k-subscheme of V with codimension 1 having its generic points in V'. The genericity of the choices of the H_i's therefore ensures that we can arrange that if $1 \le i < r$ then D_{i+1} does not contain the generic points of $D_1 \cap \cdots \cap D_i$, and so $\cap_\alpha D_\alpha$ is k-finite. This intersection contains a closed subscheme surjecting onto the k-finite étale scheme $W \cap (\cap_\alpha H_\alpha) \subseteq W'$ whose k-length is $\deg_{\mathbf{P}^n_k}(W)$, so

$$\deg_{\mathbf{P}^n_k}(W) \le \ell_k(\cap_\alpha D_\alpha).$$

Thus, it suffices to prove $\ell_k(\cap_\alpha D_\alpha) \le h_{K,N}(f(\eta))^r$. By Bézout's theorem on V in $\mathbf{P}^n_k$,

$$\ell_k(\cap_\alpha D_\alpha) = \prod_{\alpha=1}^{r} \deg_{\mathbf{P}^n_k}(D_\alpha),$$

and so it suffices to prove $\deg_{\mathbf{P}^n_k}(D_i) \overset{?}{\le} h_{K,N}(f(\eta)) = \sum_v \max_j(-\mathrm{ord}_v(f_j)\deg_{\mathbf{P}^n_k}(v))$ for each $1 \le i \le r$.

By definition, D_i is the closure in V of the zero locus on V' of some $L_i = \sum a_j^{(i)} f_j$ with $a_j^{(i)} \in k$ not all zero, and so $\deg_{\mathbf{P}^n_k}(D_i)$ is the degree in $\mathbf{P}^n_k$ for the part of the zero-scheme Weil divisor $\mathrm{div}_0(L_i) \subseteq V$ that meets the dense open $V' \subseteq V$. Hence,

$$\deg_{\mathbf{P}^n_k}(D_i) \le \deg_{\mathbf{P}^n_k}(\mathrm{div}_0(L_i)) = \deg_{\mathbf{P}^n_k}(-\mathrm{div}_\infty(L_i)) = \sum_v \max(-\mathrm{ord}_v(L_i), 0)\deg_{\mathbf{P}^n_k}(v).$$

It therefore suffices to prove that for a generic $[a_0^{(i)}, \dots, a_N^{(i)}] \in \mathbf{P}^N(k)$,

$$\max(-\mathrm{ord}_v(\sum a_j^{(i)} f_j), 0) \le \max_j(-\mathrm{ord}_v(f_j))$$

for all v. Since one of the f_j's is equal to 1, the right side is always nonnegative. We therefore just need to consider those codimension-1 points v at which $\sum_j a_j^{(i)} f_j$ (for fixed i) has a pole. The only such v are those at which some f_j has a pole, and the pole order of the sum $\sum_j a_j^{(i)} f_j$ is certainly no worse than the maximum pole order of any of the f_j's at such v. So in fact we do not even need a genericity condition on the $a_j^{(i)}$'s. ■

To summarize, for every $P \in A(K)$ with $h_K(P) \le M$, the corresponding rational k-map f_P from V to $\widetilde{A} \subseteq \mathbf{P}^N_k$ is a generic immersion whose image has k-variety closure $W_P = \overline{(\phi \circ P)(\eta)}$ with dimension $\delta = \mathrm{trdeg}_k(K)$ that is independent of P and has k-degree in $\mathbf{P}^N_k$ that is uniformly bounded above by $(M+d)^\delta$. Thus, we may now abandon K-heights and instead aim to prove that for any $M' \ge 0$, the points $P \in A(K)$ satisfying $\deg_{\mathbf{P}^N_k}(W_P) \le M'$ lie in finitely many classes in $A(K)/\mathrm{Tr}_{K/k}(A)(k)$. This statement does not involve heights, so it does not matter for this assertion that the projective k-model V is regular in codimension 1. Thus, even though the integral $\overline{k}$-scheme $V_{\overline{k}}$ may fail to be regular in codimension 1, we can nevertheless replace k and K with $\overline{k}$ and $K \otimes_k \overline{k}$ to reduce to the case when k is algebraically closed.

The W_P's are geometrically integral closed subschemes of $\mathbf{P}^N_k$ with $\dim W_P$ independent of P and $\deg_{\mathbf{P}^N_k}(W_P)$ bounded independently of P. Thus, as Grothendieck explains in the discussion of "limited families" in his work on Hilbert schemes (see [10, §2], especially Lemma 2.4 there), an application of Chow coordinates and Grothendieck's basic results on constructibility loci for fibers of morphisms ensures that there exists a k-scheme S of *finite type* and an S-flat closed subscheme $\mathscr{Z} \hookrightarrow S \times \mathbf{P}^N_k$ such that all fibers $\mathscr{Z}_s$ are geometrically integral and each W_P arises as such a fiber over some $s \in S(k)$ (here we use crucially that k is algebraically closed). By replacing S with a suitable closed subscheme without losing any of the above properties, we can impose the extra requirement that $\mathscr{Z}$ lies in $S \times_{\mathrm{Spec}\, k} \widetilde{A}$ since the fibers $W_P \subseteq \mathbf{P}^n_k$ lie in $\widetilde{A}$. We can also assume that S is a disjoint union of k-varieties.

We now claim that if W_P and $W_{P'}$ occur as fibers over the same irreducible component of S, then P and P' have the same image in $A(K)/\mathrm{Tr}_{K/k}(A)(k)$; this will certainly solve our problem. The case $P = P'$ is trivial, so we can assume we are working over an irreducible base component with positive dimension. By

[25, p. 56], on an irreducible variety of positive dimension over an algebraically closed field, any two rational points lie in a common irreducible curve in the variety. Thus, it suffices to suppose the base of our family is an irreducible curve X , which we may moreover suppose to be k-smooth by base change to its normalization (recall that k is algebraically closed). Thus, we have an X-flat closed subscheme

$$Z \hookrightarrow X \times_{\operatorname{Spec} k} \widetilde{A}$$

such that the closed subscheme $Z_x \subseteq \widetilde{A} \subseteq \mathbf{P}_k^N$ is geometrically integral for all $x \in X$, and for suitable $x_0, x_0' \in X(k)$ the fibers Z_{x_0} and $Z_{x_0'}$ in $\widetilde{A} \subseteq \mathbf{P}_k^N$ coincide with W_P and $W_{P'}$ respectively. In particular, Z is integral with dimension $\dim W_P + \dim X = \dim V + 1$.

Consider the composite map

$$Z \hookrightarrow X \times \widetilde{A} \to \widetilde{A} \to V, \tag{10.2}$$

where the final step uses that $\widetilde{A}$ is constructed inside of $\mathbf{P}_k^n \times V$. The map (10.2) is dominant, since even $W_P = Z_{x_0} \subseteq Z$ maps birationally onto V, so Z hits the generic point $\eta \in V$ with fiber Z_η that must be integral and have dimension $\dim Z - \dim V = 1$. Thus, the proper map

$$Z \hookrightarrow X \times \widetilde{A} \to X \times V$$

has restriction over X_K that is a proper map $\xi : Z_\eta \to X_K$ between integral curves over K. Since X_K is a K-smooth curve, ξ is either constant or finite and flat. The fibers of ξ over the K-points $\{x_0\} \times_{\operatorname{Spec} k} K$ and $\{x_0'\} \times_{\operatorname{Spec} k} K$ of X_K are $(Z_{x_0})_\eta = (W_P)_\eta$ and $(Z_{x_0'})_\eta = (W_{P'})_\eta$, and these are non-empty because $W_P \to V$ and $W_{P'} \to V$ are dominant (even birational) morphisms. Thus, ξ must be finite and flat. Since $W_P \to V$ is birational, so $(W_P)_\eta \to \eta$ is an isomorphism, ξ has degree 1 and thus is an isomorphism. It follows that for some dense open $V^0 \subseteq V$, the restriction of the composite $Z \hookrightarrow X \times \widetilde{A} \to X \times V$ over $X \times V^0$ is an isomorphism.

Hence, we can consider $Z|_{V^0}$ as a section $\mathscr{P}_{V^0} : X_{V^0} \to X_{V^0} \times_{V^0} \widetilde{A}_{V^0}$. Restricting this over the generic point η of V^0 and recalling that (by construction of $\widetilde{A}$) the map $\widetilde{A} \to V$ has generic fiber equal to the abelian variety A over η, we arrive at a section $\mathscr{P}_K : X_K \to X_K \times A$ over X_K such that $\mathscr{P}_K(\{x_0\}_K) \in A(K)$ is the K-point P that was used to define W_P via closure, and likewise $\mathscr{P}_K(\{x_0'\}_K) \in A(K)$ is P'. It is therefore enough to prove that for *all* $x \in X(k)$, the points $\mathscr{P}_K(x) \in A(K)$ coincide modulo $\mathrm{Tr}_{K/k}(A)(k)$. The argument with Albanese varieties that we used to conclude the proof of the Lang–Néron theorem may now be carried over *verbatim* to prove this final claim. ■

References

[1] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron models*, Springer-Verlag, Berlin, 1990.

[2] C-L. Chai, G. Faltings, *Degeneration of abelian varieties*, Springer-Verlag, New York, 1990.

[3] W. L. Chow, "Abelian varieties over function fields", *Trans. AMS* **78** (1955), pp. 253–275.

[4] W. L. Chow, "On abelian varieties over function fields", *Proc. Natl. Acad. Sci. USA* **41** (1955), pp. 582-586.

[5] B. Conrad, K. Conrad, H. Helfgott, "Root numbers and ranks in positive characteristic," *Advances in Mathematics* **198** (2005), pp. 684–731.

[6] P. Deligne, "Théorèmes de finitude en cohomologie ℓ-adique" in *Étale cohomologie. Séminaire de Géométrie Algébrique du Bois-Marie* SGA 4 1/2. LNM **569**, Springer-Verlag, New York, 1977.

[7] J. Dieudonné, A. Grothendieck, Éléments de géométrie algébrique, *Publ. Math. IHES*, **4, 8, 11, 17, 20, 24, 28, 32** (1960-7).

[8] J-M. Fontaine, *Groupes p-divisible sur les corps locaux*, Asterisque **47–48**, Soc. Math. France, 1977.

[9] E. Frietag, R. Kiehl, *Étale cohomology and the Weil conjectures*, Ergebnisse der Math. **13**, Springer-Verlag, New York, 1988.

[10] A. Grothendieck, Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert, *Séminaire Bourbaki* **6**, Exp. 221, pp. 249–276.

[11] A. Grothendieck, *Séminaire de géométrie algébrique* 1, Paris, 1961.

[12] A. Grothendieck, *Séminaire de géométrie algébrique* 3, Paris, 1963/4.

[13] M. Hindry, A. Pacheco, R. Wazir, "Fibrations et conjecture de Tate", *Journal of Number Theory* **112** no. 2 (2005), pp. 345–368.

[14] M. Hindry, J. Silverman, *Diophantine geometry*, Springer-Verlag, New York, 2000.

[15] B. Kahn, "Sur le groupe des classes d'un schéma arithmétique" to appear in *Bull. Soc. Mat. de France.*

[16] N. Katz, B. Mazur, *Arithmetic moduli of elliptic curves*, Princeton Univ. Press, Princeton, 1985.
[17] S. Kleiman, "Les théorèmes de finitude pour le foncteur de Picard", Exp. XIII in *Theorie des intersections et theoreme de Riemann–Roch* (SGA6), Lecture Notes in Mathematics **225**, Springer–Verlag, New York, 1971.
[18] S. Lang, *Abelian varieties*, Interscience tracts **7**, Interscience, New York, 1958.
[19] S. Lang, A. Néron, "Rational points of abelian varieties over function fields", *Amer. J. Math.* **81** (1959), pp. 95–118.
[20] S. Lang, *Fundamentals of Diophantine geometry*, Springer-Verlag, New York, 1983.
[21] H. Matsumura, *Commutative algebra* (2nd ed.), Lecture Note Series **56**, Benjamin/Cummings Publ. Co., Reading, Mass., 1980.
[22] H. Matsumura, *Commutative ring theory*, Cambridge studies in advanced mathematics **8**, Cambridge Univ. Press, Cambridge, 1986.
[23] J. Milne, "Abelian varieties" in *Arithmetic geometry* (Cornell/Silverman ed.), Springer-Verlag, New York, 1986.
[24] D. Mumford, *Geometric invariant theory*, Springer-Verlag, New York, 1965.
[25] D. Mumford, *Abelian varieties*, Oxford University Press, 1970.
[26] A. Néron, "Problèmes arithmétiques et géométriques rattachés à la notion de rang d'une courbe algébrique dans un corps", *Bull. Soc. Math. France* **80** (1952), pp. 101–166.
[27] F. Oort, "The isogeny class of a CM-type abelian variety is defined over a finite extension of the prime field", *Journal of pure and applied algebra*, **3** (1973), pp. 399–408.
[28] M. Raynaud, "Caractéristique d'Euler–Poincaré d'un faisceau et cohomologie des variétés abéliennes" in *Séminarie Bourbaki* **9**, Exp. 286, pp. 129–147.
[29] T. Shioda, "Mordell–Weil lattices for higher genus fibration over a curve" in *New trends in algebraic geometry*, selected papers presented at the Euro conference, Warwick, UK, July 1996. Cambridge Univ. Press, Lecture Note Series **264** (1999), pp. 359–373.
[30] J. Silverman, "Heights and the specialization map for families of abelian varieties", *J. Riene Angew. Math.* **342** (1983), pp. 197–211.
[31] J. Silverman, *The arithmetic of elliptic curves*, Springer–Verlag GTM **106**, New York, 1986.
[32] J-L. Verdier, "A duality theorem in the étale cohomology of schemes" in *Proceedings of a conference on local fields*, Springer-Verlag, New York, 1967, pp. 184–198.
[33] W. Waterhouse, *Introduction to affine group schemes*, Springer-Verlag GTM **66**, New York, 1979.
[34] A. Weil, *Foundations of algebraic geometry*, American Math. Soc. **29**, New York, 1946.

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