A Note on Height Pairings, Tamagawa Numbers, and the Birch and Swinnerton-Dyer Conjecture

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Introduction

Let $G$ be an algebraic group defined over a number field $k$. By choosing a lifting of $G$ to a group scheme over $\mathcal{O}_S \subset k$, the ring of $S$-integers for some finite set of places $S$ of $k$, we may define $G(\mathcal{O}_v)$, where $\mathcal{O}_v \subset k_v$ is the ring of integers in the $v$-adic completion of $k$ for all non-archimedean places $v \notin S$. In this way, we can define the adelic points $G(\mathcal{A}_k)$. Since different choices of lifting will change $G(\mathcal{O}_v)$ for only a finite number of $v$, $G(\mathcal{A}_k)$ is intrinsically defined independent of the choice of $\mathcal{O}_S$-scheme structure.

It may happen that $G(k) \subset G(\mathcal{A}_k)$ is discrete. This will be the case, for example, if $G$ is affine. If so, we may try to compute the volume of $G(\mathcal{A}_k)/G(k)$. Writing $\mathcal{O}_v = $ residue field at $v$, $q_v = \# \mathcal{O}_v$, $N_v = \# G(\mathcal{O}_v)$, the natural volume form gives $\text{Vol}(G(\mathcal{O}_v)) = N_v q_v^{-1}$ for all $v \notin S$. It can happen that $\prod N_v q_v^{-1}$ does not converge (example: $G = G_m$), but in many cases there is an $L$-function $L(G,s)$ available such that $L(G,s) = \prod L_v(G,s)$ where the product converges absolutely for $\text{Re } S \gg 0$ and extends meromorphically to the whole plane with $L_v(G,1) = q_v/N_v$. Suppose $\lim_{S \to 1} L(G(s-1)-r \to 0$. The Tamagawa number $\tau(G)$ is defined by modifying the measure on $G(\mathcal{A}_k)$ so $\text{Vol}(G(\mathcal{O}_v)) = 1$, all $v \notin S$, computing the measure of $G(\mathcal{A}_k)/G(k)$, and then multiplying by $\lim_{S \to 1} L(G(s-1)-r$. For more details, the reader should see [10].

The Tamagawa number has been computed for all except a few particularly stubborn affine algebraic groups, and takes the value (see [10, 4–6])

$$\tau(G) = \frac{\# \text{Pic}(G)}{\# \text{III}(G)},$$

where $\text{Pic}(G) = $ Picard group, and $\text{III}(G) = \text{Ker}(H^1(\bar{k}/k, G(\bar{k}))) \to \prod_v H^1(\bar{k}_v/k_v, G(\bar{k}))$. Moreover, $r \leq 0$, and $r = 0$ if $G(\mathcal{A}_k)/G(k)$ is compact.

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Suppose now that $G$ is not necessarily affine, but that $G(k)$ is discrete in $G(A_k)$. One conjectures that $\text{III}(G)$ is finite. (This is not known for a single abelian variety $G$!) $\text{Pic}(X)$ may be infinite but $\text{Pic}(X)_{\text{tors}}$ is finite and one may

\begin{equation}
\text{(0.1) Conjecture. } \tau(G) = \frac{\# \text{Pic}(G)_{\text{tors}}}{\# \text{III}(G)}.
\end{equation}

Moreover, $r \leq 0$ and $r = 0$ if and only if $G(A_k)/G(k)$ has finite volume.

We refer to this in the sequel as the Tamagawa number conjecture.

Consider now the case of an abelian variety $A$. Conjecture (0.1) makes sense only if $A(k)$ is finite. The Hasse-Weil $L$-function $L(A, s) = \prod_{v \notin S} L_v(A, s)$, where $S$ = set of bad reduction places, and

\begin{equation}
L_v(A, s) = \frac{1}{\det(1 - q_v^{-s} F_v H^1_{et}(A_{\overline{\mathbb{Q}}_v}, \mathbb{Q}_v))} \quad (F_v = \text{geometric frobenius}).
\end{equation}

Birch and Swinnerton-Dyer conjecture that $L(A, s)$ has a zero of order $r = r_k A(k)$ at $s = 1$ (so $r \geq 0$) and that

\begin{equation}
\lim_{s \to 1} L(A, s)(s - 1)^{-r} = \frac{\# \text{III}(A) \cdot \det \left< \cdot \right> \cdot V_x \cdot V_{\text{bad}}}{\# A(k)_{\text{tors}} \cdot \# \text{Pic}(A)_{\text{tors}}},
\end{equation}

where $V_x = \text{Volume } A(k \otimes \mathbb{Q} \mathbb{R})$ and $V_{\text{bad}} = \text{Volume } \prod_{v \notin S} A(k_v)$. Finally, $\left< \cdot \right>$ denotes the height pairing $[1, 3]$

\begin{equation}
\left< \cdot \right>: A(k) \times A'(k) \to \mathbb{R}
\end{equation}

with $A'(k) = \text{Pic}^0(A)$.

The purpose of this note is to deduce (0.2) from (0.1), and thus to give a purely volume-theoretic interpretation of Birch and Swinnerton-Dyer. An element $\alpha \in \text{Pic}(A)$ corresponds to a $\mathbb{G}_m$-torseur $X_\alpha \to A$. If $\alpha \in \text{Pic}^0(A) = A'(k)$, $X_\alpha$ is a group extension of $A$ by $\mathbb{G}_m$. We construct in this way an extension

\begin{equation}
0 \to T \to X \to A \to 0
\end{equation}

where $T$ is the split torus with character group $\simeq A'(k)_{\text{tors}}$. An important point is that the “logarithmic modulus” map factors

\begin{equation}
0 \to T(A_k) \to X(A_k) \xrightarrow{\log \text{mod.}} \text{Hom}(A'(k), \mathbb{R})
\end{equation}

The product formula shows $\log \text{mod.}(T(k)) = (0)$, so by restriction to global points, we obtain

\begin{equation}
A(k) \cong X(k)/T(k) \to \text{Hom}(A'(k), \mathbb{R}),
\end{equation}

or again

\begin{equation}
A(k) \times A'(k) \to \mathbb{R}.
\end{equation}
Using the axiomatic characterization of Neron's local pairings \([1, 3]\), we show that (0.4) is the height pairing. From this it follows without difficulty that \(X(k)\) is discrete and cocompact in \(X(A_k)\), and that (0.1) for \(X\) implies (0.2) for \(A\).

It seems likely that this technique will lead to height pairings in many new situations, e.g., for algebraic cycles other than zero cycles and divisors. I hope to return to this question in the future. I am indebted to W. Messing for several helpful discussions regarding the Neron model.

1. The Global Construction

Let \(A\) be an abelian variety over a number field \(k\). Let \(N\) be the Neron model of \(A\) over the ring of integers \(\mathcal{O}_k\), \(N^0 \subset N\) the largest open subgroup scheme whose fibres are connected. Let \(A'\) be the dual abelian variety, \(N' =\) Neron model of \(A'\). It is known (cf. [11], p. 53) that

\[
N' \cong \text{Ext}_{\text{group scheme}}^1(\mathcal{N}^0, \mathbb{G}_m).
\]

In particular, if we fix once for all a splitting

\[
A'(k) = B \oplus A'(k)_\text{tors}
\]

and use \(A'(k) = N'(\mathcal{O}_k)\), we can build an extension over \(\mathcal{O}_k\)

\[
0 \to T \to X \to N^0 \to 0,
\]

where \(T\) is the \(k\)-split torus with character group \(B\). Let \(A_k\) denote the adeles of \(k\). Since \(H^1(\text{Sp} \mathcal{R}, \mathbb{G}_m) = (0)\) for \(\mathcal{R}\) local, we get exact sequences

\[
0 \to T(k) \to X(k) \to A(k) \to 0
\]

\[
0 \to T(A_k) \to X(A_k) \to N^0(A_k) \to 0.
\]

Define \(T^1 \subset T(A_k)\) by the exact sequence

\[
0 \to T^1 \to T(A_k) \to \text{Hom}(B, \mathbb{R}) \to 0
\]

where \(l\) is induced from the usual logarithmic modulus map from the ideles to \(\mathbb{R}\). Define further, for \(v\) a place of \(k\)

\[
X^1_v = \begin{cases} 
X(\mathcal{O}_v) & v \text{ non-archimedean} \\
X(k_v)_{\text{max, compact}} & v \text{ archimedean}
\end{cases}
\]

\[
\hat{X}^1 = T^1 \cdot \prod_v X^1_v \subset X(A_k).
\]

Finally, let \(X^1\) be the rational saturation of \(\hat{X}^1\), i.e.,

\[
X^1 = \{a \in X(A_k) | \exists n \geq 1, n \in \mathbb{Z}, na \in \hat{X}^1\}.
\]
(1.8) **Lemma.** There exists a diagram with exact rows and columns

\[
\begin{array}{cccccc}
0 & \to & 0 & \to & 0 & \to \\
0 & \to & T^1 & \to & X^1 & \to & N^0(A_k) & \to 0 \\
0 & \to & T(A_k) & \to & X(A_k) & \to & N^0(A_k) & \to 0 \\
& & & \downarrow l & & & \downarrow & \\
& & Hom(B, \mathbb{R}) = Hom(B, \mathbb{R}) & \to & 0 & \to & 0 & \to \\
\end{array}
\]

**Proof.** It suffices to show \(X^1 \cap T(A_k) = T^1\), and \(X^1 \to N^0(A_k)\). The first point is straightforward, using that \(T(A_k)/T^1\) is torsion-free, and \(l\) is trivial on \(T(0,v)\) and \(T(k_v)_{\text{max,compact}}\). For the second, note that the image of \(X^1\) in \(N^0(A_k)\) contains \(A(k_v) = N^0(0_v)\) for almost all \(v\) and the cokernel

\[W = N^0(A_k)/\text{Im}(\tilde{X}^1 \to N^0(A_k))\]

is finite. It follows easily that \(X(A_k)/\tilde{X}^1 \cong Hom(B, \mathbb{R}) \oplus W\). Replacing \(\tilde{X}^1\) by \(X^1\) eliminates torsion in the quotient, and the lemma follows by diagram chasing. Q.E.D.

Combining (1.4) and (1.8), and using the fact that \(T(k) \subset T^1\) (product formula) we get

\[A(k) \cong X(k)/T(k) \to X(A_k)/T(k) \to Hom(B, \mathbb{R})\]

and hence a pairing

\[\langle \rangle : A(k) \times A(k) \to \mathbb{R}.
\]

(1.9) **Theorem.** The above pairing coincides with the height pairing.

We postpone the proof until the next section.

(1.10) **Theorem.** \(X(k) \subset X(A_k)\) is discrete and cocompact.

**Proof.** Let \(U = X(k) \cap X^1 \subset X(A_k)\). Since the height pairing is perfect, we get \(0 \to T(k) \to U \to A(k)_{\text{tors}} \to 0\), and hence exact sequences

\[0 \to T^1/T(k) \to X^1/U \to N^0(A_k)/A(k)_{\text{tors}} \to 0\]

(1.11)

\[0 \to X^1/U \to X(A_k)/X(k) \to \frac{Hom(B, \mathbb{R})}{\text{Image } A(k)} \to 0\]
The image of $A(k)$ in $\text{Hom}(B, \mathbb{R})$ is known to be discrete and cocompact (perfectness of height pairings), and compactness is known for $T^1/T(k)$ (classical theorem about ideles) and $N^0(A_k)$. The assertions of the theorem follow. Q.E.D.

What about the Tamagawa number of $X$? With notation as in the introduction, let $r = rk A'(k)$. We choose convergence factors in the sense of [10] for the measure on $X(A_k)$:

$$ (1 - q_v^{-1})^r L_v(A, 1)^{-1} \quad v \text{ non-archimedean, } A \text{ has good reduction at } v, $$

$$ (1 - q_v^{-1})^r \quad v \text{ non-archimedean, } A \text{ does not have good reduction at } v, $$

$$ 1 \quad v \text{ archimedean}. $$

These correspond to convergence factors $(1 - q_v^{-1})^r$ on $T(A_k)$ and $L_v(A, 1)^{-1}$ on $N^0(A_k)$ (in good reduction place). Writing $\zeta_k(s)$ for the zeta function of $k$ we get from (1.11)

$$ \text{Volume}(T^1/T(k)) = \lim_{s \to 1} (\zeta_k(s)(s-1))^r $$

(1.12) $$ \text{Vol. } N^0(A_k) = \text{Vol. } (A \otimes q \mathbb{R}) \cdot \prod_{v \text{ bad}} \text{Vol. } A(k_v) $$

$$ \text{Vol. } (X^1/U) = \frac{1}{\# A(k)_{\text{tors}}} \lim_{s \to 1} (\zeta_k(s)(s-1))^r V_{\infty} V_{\text{bad}} $$

(1.13) $$ \text{Vol. } (X(A_k)/X(k)) = \frac{1}{\# A(k)_{\text{tors}}} \lim_{s \to 1} (\zeta_k(s)(s-1))^r V_{\infty} V_{\text{bad}} R $$

where $R$ is the absolute value of the discriminant of the height pairing.

We assume now that $\lim_{s \to 1} \zeta_k(s)^r L(A, s) = 0$, i.e., that the $L$-function of $X$ has no zero or pole at $s = 1$ as predicted by the Tamagawa number conjecture, or equivalently that the $L$-function of $A$ has a zero of order $r = rk A(k)$ as predicted by Birch and Swimmerton-Dyer. To define the Tamagawa number $\tau(X)$ we eliminate the (non-canonical) choice of convergence factors by dividing the volume computed above by $\lim_{s \to 1} L(A, s)^{-1}$, getting

$$ \tau(X) = \frac{1}{\# A(k)_{\text{tors}}} \lim_{s \to 1} L(A, s)^{-1}(s-1)^r V_{\infty} V_{\text{bad}} R. $$

(1.14) Conjecture (0.2) is thus equivalent to

$$ \tau(X) = \frac{\# A'(k)_{\text{tors}}}{\# III(A)}. $$

(1.15) Lemma. $\text{III}(A) \cong \text{III}(X)$ and $A'(k) \cong \text{Pic}(X)_{\text{tors}}$.  

(1.16)
Proof. The first isomorphism follows from chasing the diagram

\[
\begin{array}{c}
\text{0} \\
\downarrow \\
\text{III}(X) \\
\downarrow \\
\text{0} \\
\downarrow \\
\text{H}^1(\bar{k}/k, X) \\
\downarrow \\
\text{H}^1(\bar{k}/k, A) \\
\downarrow \\
\text{H}^2(\bar{k}/k, T) \\
\end{array}
\begin{array}{c}
\text{0} \\
\downarrow \\
\text{0} \\
\downarrow \\
\text{0} \\
\downarrow \\
\text{0} \\
\end{array}
\begin{array}{c}
\text{0} \\
\downarrow \\
\text{III}(A) \\
\downarrow \\
\text{0} \\
\downarrow \\
\text{0} \\
\end{array}
\]

For the second isomorphism, note that if \( T \) is a split torus over a ring \( R \) with character group \( \hat{T} \), then taking units in the ring of regular functions on \( T \) yields an exact sequence

\[
0 \to R^* \to R[T]^* \to \hat{T} \to 0.
\]

Let \( \pi : X \to A \) be the projection. The above sequence globalizes

\[
0 \to \mathbb{G}_{m,A} \to \pi_* \mathbb{G}_{m,X} \to B_A \to 0
\]

where \( B_A \) is the constant Zariski sheaf on \( A \) with stalk \( B \). The boundary map

\[
B = \Gamma(A, B_A) \to H^1(A, \mathbb{G}_m) = \text{Pic } A
\]

is the natural inclusion, so we obtain

\[
H^1(A, \pi_* \mathbb{G}_{m,X}) \cong (\text{Pic } A)/B.
\]

Locally over \( A \), \( Z \cong \mathbb{G}_m^r \times A \), so \( R^1 \pi_* \mathbb{G}_{m,X} = (0) \) and we find

\[
\text{Pic } X \cong (\text{Pic } A)/B
\]

and a similar result holds for torsion. Q.E.D.

Combining (1.15) and (1.16) yields

(1.17) Theorem. The Birch and Swinnerton-Dyer conjecture holds for \( A \) if and only if

\[
\tau(X) = \frac{\# \text{Pic}(X)_{\text{tors}}}{\# \text{III}(X)}.
\]

2. The Local Neron Pairing

The purpose of this section is to prove (1.9). Let \( k \) be a local field, \( A \) an abelian variety over \( k \), \( N = \text{Neron model of } A \), \( N^0 \subset N \) the subgroup scheme with connected fibres. The Néron model of the dual variety \( A' = \text{Ext}^1(A, \mathbb{G}_m) \) is then \( N' = \text{Ext}^1(N^0, \mathbb{G}_m) \) ([11], p. 53). Thus given a divisor \( D \) on \( A \) defined over \( k \) and algebraically equivalent to 0, we get a corresponding extension

(2.1)

\[
0 \to \mathbb{G}_m \to X_D \to N^0 \to 0.
\]
If $\mathcal{L}_D$ is the line bundle associated to $\Delta$,

$$X_D = V(\mathcal{L}_D) - (0\text{-section})$$

as a $\mathbb{G}_m$-torsor. The extension (2.1) depends only on the linear equivalence class of $\Delta$.

Restricting to $\text{Sp} k$, the extension (2.1) is split as a torsor over $A - |\Delta|$ ($|\Delta| = \text{Supp} \Delta$)

\[(2.1) 0 \to G_{m,k} \to X_{\Delta,k} \to A \to 0 \]

where $\sigma_\Delta$ is canonical up to translation by $G_{m,k}(k) = k^*$ (choosing $\sigma_\Delta$ is tantamount to choosing a rational section of $\mathcal{L}_D$ corresponding to the divisor $\Delta$).

Let $Z_{\Delta,k} = \text{group of zero cycles } \mathcal{U} = \sum n_i(p_i) \text{ on } A \text{ defined over } k$ such that $\sum n_i \deg p_i = 0$ and $\text{Supp } \mathcal{U} \subset A - |\Delta|$. We get a homomorphism

\[(2.2) \sigma_\Delta: Z_{\Delta,k} \to X_{\Delta,k}(k).\]

Define

$$X^1_D = \begin{cases} X_D(\text{Sp } G) & k \text{ non-archimedean} \\ X_D(k)_{\text{max, compact}} & k \text{ archimedean}, \end{cases}$$

$$G^1_m = \begin{cases} G_k^* & k \text{ non-archimedean} \\ (k^*)_{\text{max, compact}} & k \text{ archimedean}, \end{cases}$$

$$F = \begin{cases} \mathbb{Z} & k \text{ non-archimedean, } N = N^0 \\ \mathbb{Q} & k \text{ non-archimedean, } N \neq N^0 \\ \mathbb{R} & k \text{ archimedean}. \end{cases}$$

(2.4) **Lemma.** Assume either $v$ archimedean or $N = N^0$. Then there is a diagram with exact rows and columns

\[
\begin{array}{ccc}
0 & \to & G^1_m \\
\downarrow & & \downarrow \\
0 & \to & X^1_D \\
\downarrow & & \downarrow \\
0 & \to & A(k) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

\[
\begin{array}{ccc}
0 & \to & k^* \\
\downarrow & & \downarrow \\
0 & \to & X_D(k) \\
\downarrow & & \downarrow \\
0 & \to & A(k) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]
Proof. The map labeled 1 is either the logarithm or the valuation map. As in the proof of (1.8), the only thing we need to show is $X_A^1 \rightarrow A(k)$. In the non-archimedean case we have $A(k) \cong N(C) = N^0(C)$. Surjectivity $X_A^1 = X(C) \rightarrow N^0(C)$ follows from $H^1(Sp C, G_m) = (0)$. In the archimedean case, the existence of an exponential implies the connected component of 0 in $A(k)$ is contained in the image of $X_A^1$. Factoring out by $X_A^1$, we obtain an extension of a finite group by $\mathbb{R}$. Such an extension is necessarily split, so we get

$$0 \rightarrow X_A^1 \rightarrow X_A(k) \rightarrow \mathbb{R} \oplus \text{(finite)} \rightarrow 0.$$ 

Since $X_A^1$ is maximal compact, $(\text{finite}) = (0)$. Q.E.D.

Suppose now $N = N^0$, and let $A(k)_0 = \text{Image}(X_A^1 \rightarrow A(k))$. Note $A(k)/A(k)_0$ is finite, and we have a diagram (defining $Y$)

$$
\begin{array}{cccc}
0 & \rightarrow & \mathbb{G}_m^1 & \rightarrow & X_A \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & k^* & \rightarrow & A(k)_0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathbb{Z} & \rightarrow & Y \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & A(k)/A(k)_0 \\
\end{array}
$$

In particular $Y \otimes \mathbb{Q} \cong \mathbb{Q}$ (canonically) so we get $X_A(k) \rightarrow \mathbb{Q}$.

In any of the above cases, let $\psi_A: X_A(k) \rightarrow F$ be the map just defined, and for $\mathfrak{a} \in Z_{A,k}$ define

$$(2.6) \quad \langle A, \mathfrak{a} \rangle_{\text{local}} = \psi_A \sigma_A(\mathfrak{a}).$$

When the given ground field is the completion of a global field at some place $v$, we write $\langle \cdot \rangle_v$ instead of $\langle \cdot \rangle_{\text{local}}$.

$$(2.7) \quad \textbf{Theorem. Let } k \text{ be a number field. Let } a \in A(k), a' \in A'(k). \text{ Let } \Delta \ (\text{resp. } \mathfrak{a}) \text{ be a divisor algebraically equivalent to zero defined over } k \ (\text{resp. a zero cycle of degree } 0 \text{ defined over } k) \text{ on } A \text{ such that } [\Delta] = a' \ (\text{resp. } \mathfrak{a} \text{ maps to } a \in A(k)). \text{ Assume further that } \text{Supp } \Delta \text{ and } \text{Supp } \mathfrak{a} \text{ are disjoint. Then}$

$$\langle a, a' \rangle = \sum_{\text{replace of } k} \langle A, \mathfrak{a} \rangle_v,$$

where $\langle a, a' \rangle$ is defined as in (1.9).
Proof. Consider the global extension

$$0 \rightarrow T \rightarrow X \rightarrow N^0 \rightarrow 0$$

as in section 1 and push out along $a' \in \hat{T}$ to get

$$0 \rightarrow \mathbb{G}_m \rightarrow X_a \rightarrow N^0 \rightarrow 0.$$ 

We can think of $\sigma_{a'} : Z_{A,k} \rightarrow X_A(k)$ just as in the local case. The problem therefore reduces to showing the map

$$(2.8) \quad X_A(A_k) \rightarrow \mathbb{R}$$

defined via the techniques of Sect. 1 coincides with the sum of the local maps

$$\psi_{A,v} : X_A(k_v) \rightarrow F = \begin{cases} \mathbb{Z} \\ \mathbb{Q} \end{cases}$$

defined in the beginning of this paragraph.

Finally, this point is clear from the diagram

$$\begin{array}{c}
\mathbb{G}_m \rightarrow X_A(A_k) / \prod_v X_{A,v} \\
\downarrow \Sigma v \\
0 \rightarrow \text{Ker(sum)} \\
\downarrow \Sigma \psi_{A,v} \\
\prod_{v \text{arch.}} \mathbb{R} \times \prod_{v \text{non-arch}} \mathbb{Q} \rightarrow \mathbb{R} \\
\downarrow \text{sum} \\
0
\end{array}$$

(2.9)

Néron has shown [3] that the height pairing can be written as a sum of local terms. With notation as in (2.7)

$$\langle a, a' \rangle = \sum_v \langle A, A \rangle_{v, \text{Neron}}.$$

(2.10) Proposition. $\langle A, A \rangle_v = \langle A, A \rangle_{v, \text{Neron}}$.

Proof. We write $v : k_v^* \rightarrow \mathbb{R}$ for the logarithmic valuation, normalized in accordance with the global product formula. Let $D_a(A)_k$ denote the group of divisors on $A$ algebraically equivalent to zero and defined over $k$. The local Néron pairing is characterized by the following properties:

1. $\langle A, A \rangle_{v, \text{Neron}} : \{(A, A) \in D_a(A)_k \times Z_k(A) | |A| \cap |A| = \emptyset \} \rightarrow \mathbb{R}$

2. $\langle A, A \rangle_{v, \text{Neron}}$ is bilinear, assuming all terms in the desired equality are defined.

3. If $A = (f)$, then $\langle A, A \rangle_{v, \text{Neron}} = v(f(A))$, where for $A = \sum n_i (p_i)$, $f(A) = \prod f(p_i)^{n_i}$.

4. $\langle A, A \rangle_{v, \text{Neron}} = \langle A, A \rangle_{v, \text{Neron}}$, where $a \in A(k_v)$ and the subscript indicates translation by $a$.  

(2.11)
(5) For $\Delta \in \mathcal{D}_a(A)_{k_v}$ and $x_0 \in A(k_v) - |\Delta|$, the map

$$x \mapsto \langle \Delta, (x) - (x_0) \rangle_{v, \text{Neron}}$$

is bounded on every $v$-bounded subset of $A(k_v) - |\Delta|$. (Here $v$-bounded subset means subset of a coordinate neighborhood on which $v$ (coordinate functions) are bounded.)

We show that the pairing $(\Delta, \mathcal{U}) \mapsto \langle \Delta, \mathcal{U} \rangle_v$ satisfies condition (1)--(5), except that (4) will be proven only for $a \in N^0(\mathcal{E}) \subset A(k)$.

(2.12) **Lemma.** $\langle \cdot \rangle_v$ satisfies (3).

**Proof.** Let $\Delta = (f)$. Then $X_\Delta \cong \mathbb{G}_m \times N^0$ and

$$\sigma_\Delta: A - |\Delta| \to \mathbb{G}_m \times A$$

$$\sigma_\Delta(a) = (f(a), a).$$

Since $\psi_\Delta = v$ on $\mathbb{G}_m(k_v) = k_v^*$, the lemma follows. Q.E.D.

(2.13) **Lemma.** $\langle \cdot \rangle_v$ satisfies (2), i.e., it is bilinear.

**Proof.** Bilinearity in $\mathcal{U} \in \mathcal{Z}_a(A)$ holds by definition. We must show

$$\langle \mathcal{U}, \Delta_1 \rangle + \langle \mathcal{U}, \Delta_2 \rangle = \langle \mathcal{U}, \Delta_1 + \Delta_2 \rangle$$

whenever $\Delta_j \in \mathcal{D}_a(A)$ and $|\mathcal{U}| \cap (|\Delta_j| \cup |\Delta_2|) = \emptyset$. Note that $\sigma_{\Delta_1 + \Delta_2}$ can be taken to be the "sum" in the sense of torseurs of $\sigma_{\Delta_1}$ and $\sigma_{\Delta_2}$, i.e., the rational section of $\mathcal{L}_{\Delta_1 + \Delta_2}$ can be taken to be the tensor or rational sections of $\mathcal{L}_{\Delta_1}$ and $\mathcal{L}_{\Delta_2}$. The diagram

$$
\begin{array}{ccccccccc}
0 & \to & \mathbb{G}_m & \to & X_{\Delta_1 + \Delta_2} & \to & N^0 & \to & 0 \\
 & & \uparrow{\text{multiply}} & \downarrow{\sigma_{\Delta_1 + \Delta_2}} & & \downarrow{A - |\Delta_1| - |\Delta_2|} & & \\
0 & \to & \mathbb{G}_m \times \mathbb{G}_m & \to & X_{X_12} & \to & N^0 & \to & 0 \\
 & & \uparrow{(\sigma_{\Delta_1}, \sigma_{\Delta_2})} & & & & & \\
0 & \to & \mathbb{G}_m \times \mathbb{G}_m & \to & X_{\Delta_1} \times X_{\Delta_2} & \to & N^0 \times N^0 & \to & 0
\end{array}
$$

commutes, where $X_{X_12}$ is the pullback as indicated. Defining $X_{X_12}$ in the same way as $X_{\Delta}$ above, one finds

$$(X_{X_12}(k)/X_{X_12}) \otimes \mathbb{Q} \cong [(k^*/\mathbb{G}_m^1) \times (k^*/\mathbb{G}_m^1)] \otimes \mathbb{Q}$$

and the map $X_{X_12}(k)/X_{X_12} \to X_{\Delta_1 + \Delta_2}(k)/X_{\Delta_1 + \Delta_2}$ corresponds to addition on $k^*/\mathbb{G}_m^1$. The assertion of the lemma now follows. Q.E.D.

(2.14) **Lemma.** Let $a \in N^0(\mathcal{E}) \subset A(k)$. Then $\langle \Delta, \mathcal{U} \rangle = \langle A_a, \mathcal{U}_a \rangle$. 
Proof. Let \( \delta_a : N^0 \to N^0 \) be translation by \( a \). There is a map of \( G_m \)-torsors \( \tau_a : X_A \to X_{\Delta_a} \) such that the diagram

\[
\begin{array}{ccc}
X_A & \xrightarrow{\tau_a} & X_{\Delta_a} \\
\sigma_a \downarrow & & \sigma_{\Delta_a} \downarrow \\
A - |A| & \xrightarrow{\delta_a} & A - |A_a| \\
\delta_a \downarrow & & \downarrow \\
A
\end{array}
\]

commutes for suitable choice of \( \sigma_a, \sigma_{\Delta_a} \).

The key point is that we may choose \( \tau_a \) such that \( \tau_a(X_1) \subset X_{1_{\Delta_a}} \). This is clear in the non-archimedean case because \( X_1 = X_\Delta(0) \) and it suffices to take \( \tau_a \) defined over \( \mathcal{O} \). In the archimedean case, choose \( \bar{a} \in X_1_{\Delta_a} \) lying over \( a \) and consider the composition

\[
\begin{array}{ccc}
X_{\Delta_a}(k) & \xrightarrow{\tau_a} & X_{\Delta_a}(k) \\
\delta_a \downarrow & & \downarrow \\
A(k) & \xrightarrow{\delta_a} & A(k)
\end{array}
\]

Modifying \( \tau_a \) by an element of \( k^* \) we may assume \( \delta_a \circ \tau_a \) is the identity on \( k^* \), whence an isomorphism of groups \( X_\Delta(k) \xrightarrow{\sim} X_A(k) \). Thus \( \delta_a \circ \tau_a(X_1) = X_{1_{\Delta_a}} \).

Since \( \bar{a} \in X_{1_{\Delta_a}} \) we get \( \tau_a(X_{1}) = X_{1_{\Delta_a}} \).

Since subtracting \( \bar{a} \) does not change the image of a point in \( X_{\Delta_a} \) under \( \psi_{\Delta_a} \), the above discussion actually shows that for any zero cycle \( z \) on \( X_A \) defined over \( k \) we have \( \psi_A(z) = \psi_{\Delta_a}(\tau_a(z)) \). Thus

\[
\langle A_a, \mathcal{U} \rangle_v = \psi_{\Delta_a}(\sigma_a, \delta_a)(\mathcal{U}) = \psi_{\Delta_a}(\tau_a, \sigma_a)(\mathcal{U}) = \langle A, \mathcal{U} \rangle_v. \quad \text{Q.E.D.}
\]

(2.15) Lemma. The pairing \( \langle \cdot, \cdot \rangle_v \) satisfies condition (5).

Proof. The assignment \( x \mapsto \langle A,(x) - (x_0) \rangle_v \) is continuous, and \( v \)-bounded sets are compact. Q.E.D.

Proof of (2.11). Let \( \{ A, \mathcal{U} \} = \langle A, \mathcal{U} \rangle_{v, \text{Neron}} - \langle A, \mathcal{U} \rangle_v \). We have \( \{(f), \mathcal{U} \} = 0 \) so we may define

\[
\{ \cdot \} : A'(k) \times Z_k(A) \to \mathbb{R}.
\]

Let \( Z_k(A)^0 \subset Z_k(A) \) be those zero cycles \( \sum n_i(p_i) \) such that \( p_i \in N^0(\mathcal{O}) \subset A(k) \). There is a natural surjection \( Z_k(A)^0 \to N^0(\mathcal{O}) \) with kernel generated by elements \((a_1 + a_2) - (a_1) - (a_2) + (0), a_i \in N^0(\mathcal{O}) \). Translation invariance implies \( \{ \cdot \} \) factors
through \{ \} : \mathcal{A}'(k) \times N^0(\mathcal{C}) \to \mathbb{R}. The image under \{}A, \cdot\{ \text{of a subgroup of } N^0(\mathcal{C})\text{ contained in a } \epsilon\text{-bounded neighborhood of } 0\text{ is trivial by (5). It follows that } \{} = 0. \quad \text{Q.E.D.} \\

References


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