FINITENESS OF BRAUER AND Ш

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1. The Classical Brauer Group

We begin with a review of basic facts over a field, which came first historically.

Definition: Let k be a field. The *Brauer group* of k is given by the set

$$\mathrm{Br}(k) = \{\text{finite-dimensional central simple algebras over } k\}/\sim$$

where $A \sim A'$ if $A \otimes \mathrm{End}(k^n) \simeq A' \otimes \mathrm{End}(k^n)$ for some $n \geq 1$.

The relation $\sim$ is transitive because $\mathrm{End}(k^n) \otimes \mathrm{End}(k^{n'}) \simeq \mathrm{End}(k^{n+n'})$. The operation $\otimes$ on algebras gives $\mathrm{Br}(k)$ a group structure, since $A \otimes A^{\mathrm{opp}} \simeq \mathrm{End}(A)$. The latter is an equivalent definition of a (finite-dimensional) central simple algebra. For any extension k'/k, there is a map $\mathrm{Br}(k) \to \mathrm{Br}(k')$ given by base change.

The Artin-Wedderburn Theorem classifies all finite-dimensional simple k-algebras as $\mathrm{End}(D^n)$ for a unique finite-dimensional central division algebra D over k. If $k = \overline{k}$, then $\mathrm{Br}(k) = 1$ since k has no non-trivial central division algebras D (as $k \subset k[x]$ is a nontrivial finite extension for any $x \in D - k$). But we can say much more: for any field k, an element of Brauer can be trivialized after a finite separable extension k'/k. Thus, if $k = k^s$ then $\mathrm{Br}(k) = \{1\}$.

We can now interpret a central simple algebra as a twisted form of $\mathrm{End}(k^n)$ over k, with respect to the étale topology on $\mathrm{Spec}\,(k)$. This brings us to

Theorem: (Skolem-Noether) *Every automorphism of A/k is inner.*

In particular, the k-automorphism group of $\mathrm{End}(k^n)$ is $\mathrm{GL}_n(k)/k^* = \mathrm{PGL}_n(k)$. By descent, a central simple algebra A corresponds to an element of $H^1_{\mathrm{et}}(\mathrm{Spec}\,k, \mathrm{PGL}_n)$. Consider the short exact sequence of algebraic groups

$$1 \to \mathbf{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$$

which induces the long exact sequence

$$\to H^1_{\mathrm{et}}(k, \mathrm{GL}_n) \xrightarrow{p} H^1_{\mathrm{et}}(k, \mathrm{PGL}_n) \xrightarrow{\delta} H^2_{\mathrm{et}}(k, \mathbf{G}_m) \to$$

A central simple algebra is isomorphic to $\mathrm{End}(k)$ if and only if its class lies in the image of p. If we let n vary, the boundary maps δ induce an injective map $i : \mathrm{Br}(k) \hookrightarrow H^2_{\mathrm{et}}(k, \mathbf{G}_m) = H^2(k, k_s^\times)$. This turns out to be an isomorphism, by explicitly constructing a central simple algebra from a Galois 2-coycle.

A more geometric interpretation of the Brauer group uses the fact that PGL_n represents the automorphism functor of $\mathbf{P}^n$, so an element of $H^1(k, \mathrm{PGL}_n)$ is a twisted form of $\mathbf{P}^n_k$, or more precisely, a k-scheme V such that $V_{k'} \simeq \mathbf{P}^n_{k'}$ for some finite separable extension k'/k. Such varieties are called *Severi–Brauer varieties*; see Lecture 7 for a discussion of real conics as Severi–Brauer varieties.

2. The Modern Brauer Group

There is a topological analogue of the above construction, which leads naturally to the common generalization to the Brauer group of a scheme.

Definition: Let X be a paracompact topological space, and $\mathcal{O}_X$ its sheaf of continuous $\mathbf{C}$-valued functions. The *Brauer group* of X is given by the set

$$\mathrm{Br}(X) = \{\mathcal{O}_X\text{-algebras locally isomorphic to } \mathrm{End}(\mathcal{O}_X^n)\}/\sim$$

where $\mathcal{A} \sim \mathcal{A}'$ if $\mathcal{A} \otimes \mathrm{End}(\mathcal{E}) \simeq \mathcal{A}' \otimes \mathrm{End}(\mathcal{E}')$ for $\mathcal{E}, \mathcal{E}'$ locally free.

The relation is transitive because $\mathrm{End}(\mathcal{E}) \otimes \mathrm{End}(\mathcal{E}') \simeq \mathrm{End}(\mathcal{E} \otimes \mathcal{E}')$. Such $\mathcal{O}_X$-algebras $\mathcal{A}$ are called Azumaya algebras, and they satisfy $\mathcal{A} \otimes \mathcal{A}^{\mathrm{opp}} \simeq \mathrm{End}(\mathcal{A})$ via the obvious map.

There is a Skolem–Noether theorem for these so-called Azumaya algebras $\mathcal{A}$. By descent, an Azumaya algebra $\mathcal{A}$ corresponds to an element of $\check{H}^1(X, \mathrm{PGL}_n(\mathbf{C}))$, and $\mathcal{A} \simeq \mathrm{End}(\mathcal{E})$ if and only if its class comes from $\check{H}^1(X, \mathrm{GL}_n(\mathbf{C}))$. Thus, we have an injection

$$\mathrm{Br}(X) \hookrightarrow H^2(X, \mathbf{C}^\times).$$

Note that the Brauer group is torsion because of the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_n & \longrightarrow & SL_n & \longrightarrow & \mathrm{PGL}_n & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{=} & & \\ 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathrm{GL}_n & \longrightarrow & \mathrm{PGL}_n & \longrightarrow & 1 \end{array}$$

and the fact that the connecting map $\delta : H^1(X, \mathrm{PGL}_n(\mathbf{C})) \to H^2(X, \mathbf{C}^\times)$ factors through $H^1(X, \mathrm{PGL}_n(\mathbf{C})) \to H^2(X, \mu_n)$, which is torsion. This fact persists in all definitions of the Brauer group.

Theorem (Serre): *For X a finite CW-complex,* $\mathrm{Br}(X) \simeq H^2(X, \mathbf{C}^\times)_{\mathrm{tor}}$.
Grothendieck generalized both of the previous constructions to the notion of Brauer group for $(X, \mathcal{O}_X)$ a locally ringed space.

Definition: Let $(R, \mathfrak{m})$ be a local ring with residue field k. An *Azumaya algebra* A over R is a free R-module such that $A \otimes_R A^{\mathrm{opp}} \simeq \mathrm{End}_R(A)$.

Remark: By Nakayama's Lemma, A/R is Azumaya if and only if $A/\mathfrak{m}$ is central simple over k.

Definition: Let X be any scheme. An *Azumaya algebra* $\mathcal{A}$ over X is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras that is locally free of finite rank as an $\mathcal{O}_X$-module such that $A \otimes \mathcal{O}_{X,x}$ is an Azumaya algebra over $\mathcal{O}_{X,x}$ for all $x \in X$. Equivalently, $\mathcal{A} \otimes k(x)$ is central simple over $k(x)$ for all x.

Definition: The *Brauer group* of a scheme X is given by

$$\mathrm{Br}(X) = \{\text{Azumaya algebras}\}/\sim .$$

By Skolem–Noether, we again have an injection

$$\mathrm{Br}(X) \hookrightarrow \check{H}^2_{\mathrm{et}}(X, \mathbf{G}_m)$$

and the latter injects into $H^2(X, \mathbf{G}_m)$, by the Čech-to-derived spectral sequence. As above, we know that the Brauer group is torsion, since $H^2(X, \mu_n)$ is torsion. Grothendieck studied when the map

$$\mathrm{Br}(X) \hookrightarrow H^2_{\mathrm{et}}(X, \mathbf{G}_m)_{\mathrm{tor}} \subset H^2_{\mathrm{et}}(X, \mathbf{G}_m) =: \mathrm{Br}'(X)$$

is an isomorphism (the latter is called the *cohomological Brauer group*). They agree in the étale local case by comparing Henselian local rings to their residue fields, but global results are more difficult. It is worth noting that $\mathrm{Br}'(X)$ is torsion for X regular, as we will see later.

Theorem (Grothendieck): *Let X be a quasi-compact scheme, and let $\gamma \in H^2_{\mathrm{et}}(X, \mathbf{G}_m)$. There exists an Azumaya algebra on $U = X - Y$ whose class agrees with $\gamma|_U$. Here, Y is a subscheme of codimension ≥ 2, or ≥ 3 when X is regular.*

Thus, in the case of a regular surface (relevant for this talk), the Brauer group is cohomological.

3. Examples and Further Properties

Theorem (Auslander-Brumer): *Let R be a discrete valuation ring, K its field of fractions, and k its residue field. Then we have a short exact sequence*

$$0 \to Br(R) \to Br(K) \to Hom(G_k, \mathbf{Q}/\mathbf{Z}) \to 0$$

.

Corollary. *If K is a non-archimedean local field, $Br(K) \simeq \mathbf{Q}/\mathbf{Z}$.*

Proof: We have $\mathrm{Br}(\mathbf{F}_q) = \{0\}$, since finite division rings are fields, and

$$\mathrm{Hom}\left(\widehat{\mathbf{Z}}, \mathbf{Q}/\mathbf{Z}\right) \simeq \mathbf{Q}/\mathbf{Z}.$$

Let X be a regular variety with function field K. For each irreducible divisor $D \subset X$, we have a valuation v_D on K, with valuation ring R_D. The Auslander-Brumer Theorem in this case reads as an exact sequence

$$0 \to \mathrm{Br}(R_D) \to \mathrm{Br}(K) \to \mathrm{Hom}(G_{k(D)}, \mathbf{Q}/\mathbf{Z}) \to 0.$$

Varying over all irreducible divisors D, we find

$$0 \to \mathrm{Br}(X) \to \mathrm{Br}(K) \to \bigoplus_D \mathrm{Hom}(G_{k(D)}, \mathbf{Q}/\mathbf{Z}) \to 0.$$

This implies that when X is a smooth proper and geometrically connected curve over $\mathbf{F}_q$ then $\mathrm{Br}(X)$ is trivial, since the right map coincides with the one from global class field theory:

$$0 \to \mathrm{Br}(K) \to \bigoplus_v \mathrm{Br}(K_v) \to \mathbf{Q}/\mathbf{Z} \to 0.$$

Lastly, I should mention the fact that $\mathrm{Br}'(X)$ is a birational invariant for smooth proper varieties in characteristic zero (or dimension ≤ 2 in any characteristic). Grothendieck proves this by explicitly computing the effect of blowing up, which is sufficient by resolution of singularities.

4. Br($\mathcal{X}$) versus Ш(X)

Let us recall the set up from last lecture. Let K be the function field of a smooth proper and geometrically connected curve C over $\mathbf{F}_q$, and let X/K be a smooth proper and geometrically connected curve over K. Then there is a regular proper model $f : \mathcal{X} \to C$, which is far from unique. The goal of this section is to relate the group $\mathrm{Br}(\mathcal{X})$ to the Tate-Shafarevich group of the Jacobian: $Ш(J_{X/K})$. We assume that X has a K-point to simplify the argument.

To access $\mathrm{Br}(\mathcal{X}) = H^2_{\mathrm{et}}(\mathcal{X}, \mathbf{G}_m)$, we would like to use the Leray spectral sequence that composes the derived functors of f_* and Γ:

$$E_2^{pq} = H^p_{\mathrm{et}}(C, R^q f_* \mathbf{G}_m) \Rightarrow H^{p+q}_{\mathrm{et}}(\mathcal{X}, \mathbf{G}_m).$$

But first, we discuss a vanishing result that simplifies the sequence substantially.

Theorem (Artin): $R^i f_* \mathbf{G}_m = 0$ *when* $i \geq 2$.

Proof: It suffices to show that the stalks $H^i_{\mathrm{et}}(\mathcal{X}_{\mathcal{O}^{\mathrm{sh}}_{C,c}}, \mathbf{G}_m)$ vanish when $i \geq 2$. By proper base change, we have an isomorphism

$$H^i_{\mathrm{et}}(\mathcal{X}_{\mathcal{O}^{\mathrm{sh}}_{C,c}}, \mu_{\ell^n}) \simeq H^i_{\mathrm{et}}(\mathcal{X}_{k(c)^s}, \mu_{\ell^n})$$

which vanishes for $i \geq 3$, since $\mathcal{X}_{k(c)^s}$ is a curve. The Kummer sequence implies that there is no ℓ^n-torsion inside $H^i_{\mathrm{et}}(\mathcal{X}_{\mathcal{O}^{\mathrm{sh}}_{C,c}}, \mathbf{G}_m)$ for $i \geq 3$. To finish up this case, we need the general:

Lemma. *Let Y/k be an irreducible regular Noetherian scheme. Then $H^i_{\mathrm{et}}(Y, \mathbf{G}_m)$ is torsion for $i \geq 1$.*

Proof: First, for any field K, a sheaf F over $\mathrm{Spec}\,(K)$ has torsion higher cohomology. Next, consider the exact sequence

$$0 \to \mathbf{G}_m \to R^*_Y \to \underline{Div}_Y \to 0$$

where R^*_Y denotes the sheaf of nonzero rational functions, and $\underline{Div}_Y$ is the sheaf of Cartier divisors. The sheaf $R^*_Y = \eta_*(\mathbf{G}_m)$ where $\eta : \mathrm{Spec}\, k(Y) \to Y$ is the inclusion of the generic point. By the Leray spectral sequence, $\eta_*(\mathbf{G}_m)$ has torsion higher cohomology.

Since Y is regular, we can describe $\underline{Div}_Y$ in terms of Weil divisors:

$$\bigoplus_{D \subset X} \iota_*(\underline{\mathbf{Z}})$$

where $\iota : \mathrm{Spec}\, k(D) \to Y$ is the inclusion, so the same argument applies. The long exact sequence then gives us the desired statement about the $\mathbf{G}_m$ cohomology.

Returning to the proof of Artin's Theorem, we must address the case where $i = 2$. We know that $\mathrm{Br}(\mathcal{X}_{k(c)^s}) = 0$, so it suffices to show that the base change map

$$\mathrm{Br}(\mathcal{X}_{\mathcal{O}^{\mathrm{sh}}_{C,c}}) \to \mathrm{Br}(\mathcal{X}_{k(c)^s})$$

is injective. If an Azumaya algebra becomes trivial upon restriction, we lift the trivialization using deformation theory of algebras. First, we can lift it to all finite Artinian extensions, and then use formal GAGA to lift to the completed local ring. A result of Greenberg allows us to lift all the way to $\mathcal{O}^{\mathrm{sh}}_{C,c}$.

Returning to the original problem, we can now write down the Leray spectral sequence for f, placing all 0's in the second row. This yields the long exact sequence

$$\to H^2(C,\mathbf{G}_m) \to H^2(\mathcal{X},\mathbf{G}_m) \to H^1(C, R^1 f_* \mathbf{G}_m) \to H^3(C,\mathbf{G}_m) \to H^3(\mathcal{X},\mathbf{G}_m) \to$$

Here, $f_*\mathbf{G}_m = \mathbf{G}_m$ because f is proper and smooth with geometrically connected fibers. Recall that $H^2(C,\mathbf{G}_m) = \mathrm{Br}(C) = 0$ because C is a smooth curve over $\mathbf{F}_q$. The map $H^3(C,\mathbf{G}_m) \to H^3(\mathcal{X},\mathbf{G}_m)$ is given by pulling back through f. Since f admits a section, this map is injective. Combining these observations, we obtain

$$\mathrm{Br}(\mathcal{X}) = H^2(\mathcal{X},\mathbf{G}_m) \simeq H^2(C, R^1 f_*\mathbf{G}_m).$$

What is the sheaf $R^1 f_*\mathbf{G}_m$? By definition, it is the sheafification of the functor

$$U \mapsto H^1(f^{-1}(U),\mathbf{G}_m) = \mathrm{Pic}(f^{-1}(U)),$$

which is also the definition of the relative Picard functor, $\mathrm{Pic}_{\mathcal{X}/C}$. Since f is not smooth, this functor is generally not represented by a separated scheme, but the subfunctor $\mathrm{Pic}^0_{\mathcal{X},C}$ parametrizing line bundles with degree 0 on every component of every geometric fiber is representable by a separable scheme of finite type. The key link between $\mathrm{Br}(\mathcal{X})$ and $Ш(J_{X/K})$ is given by

Theorem (Raynaud): We have $\mathrm{Pic}^0_{\mathcal{X}/C} \simeq \mathcal{J}^0$, where $\mathcal{J}$ is the Neron model of $J_{X/K}$.

Admitting this, we can analyze the finiteness of the two groups, as follows.

First, we claim that finiteness of $\mathrm{Br}(\mathcal{X})$ is equivalent to finiteness of $H^1(C,\mathcal{J}^0)$.

In the case where f is smooth, we have the short exact sequence

$$0 \to \mathrm{Pic}^0_{\mathcal{X}/C} \to \mathrm{Pic}_{\mathcal{X}/C} \to \underline{\mathbf{Z}} \to 0$$

which induces the long exact sequence

$$\mathrm{Pic}_{\mathcal{X}/C}(C) \to \mathbf{Z} \to H^1_{\mathrm{et}}(C,\mathcal{J}^0) \to H^1_{\mathrm{et}}(C,\mathrm{Pic}_{\mathcal{X}/C}) \to H^1_{\mathrm{et}}(C,\underline{\mathbf{Z}}) \to$$

The leftmost map is surjective since f has a section. The rightmost cohomology group vanishes because

$$H^1(X,\underline{\mathbf{Z}}) = \mathrm{Hom}_{cts}(\pi_1^{\mathrm{et}}(C),\mathbf{Z}) = 0.$$

If f has singular fibers, then $\underline{\mathbf{Z}}$ is replaced by an extension supported over the bad places. This is where the index and period enter the picture.

Next, we relate $H^1(C,\mathcal{J}^0)$ to $H^1(C,\mathcal{J})$ via the short exact sequence

$$0 \to \mathcal{J}^0 \to \mathcal{J} \to \bigoplus_{s\in\Sigma} j_{s*}\Phi_s \to 0$$

from Lecture 3, which induces the long exact sequence

$$\bigoplus_{s\in\Sigma}\Phi_s \to H^1(C,\mathcal{J}^0) \to H^1(C,\mathcal{J}) \to H^1\left(C, \bigoplus_{s\in\Sigma} j_{s*}\Phi_s\right)$$

and the two groups on the outside are finite. Lastly, we use the description of Ш due to Mazur given in Lecture 3 to deduce the desired equivalence. Any torsor for $J_{X/K}$ admits a model over C, which is a torsor for $\mathcal{J}$. This gives us a map

$$Ш(J_{X/K}) \hookrightarrow H^1(C, \mathcal{J})$$

The original definition of Ш as the kernel of

$$H^1(K, J_{X/K}) \to \prod_v H^1(K_v, J_{X/K})$$

can be restricted to the Neron model torsor space, giving a Cartesian square

$$\begin{array}{ccc} H^1(C, \mathcal{J}) & \longrightarrow & H^1(K, J_{X/K}) \\ \downarrow & & \downarrow \\ \prod_v H^1(\widehat{\mathcal{O}}_v, \mathcal{J}) & \longrightarrow & \prod_v H^1(K_v, J_{X/K}) \end{array}$$

The lower left group is finite, since it is supported at the bad places, and there we use Lang's Theorem on the Néron fiber. Thus, Ш(J) is finite if and only if $H^1(C, \mathcal{J})$ is finite.

5. Pairing on Ш

The key ingredient for the construction of the pairing is the "arithmetic Poincaré duality" statement. In spirit, it says that a smooth variety X of dimension d over a finite field k behaves like a closed $(2d+1)$-manifold with respect to étale cohomology with torsion coefficients. It should not be confused with the usual Poincaré duality statement for $X_{\overline{k}}$, which behaves like a closed $2d$-manifold.

For p not dividing m, we have a non-degenerate cup product

$$H^i_c(X, \mu_m^{\otimes n}) \times H^{2d+1-i}_{\text{et}}(X, \mu_m^{\otimes d-n}) \to H^{2d+1}_c(X, \mu_m^{\otimes d}) \xrightarrow{\sim} \mathbf{Z}/m\mathbf{Z}$$

where the last map is the trace, and the pairing is compatible with varying m. In the case of a regular proper surface, we have

$$H^i(X, \mu_m) \times H^{5-i}(X, \mu_m) \to \mathbf{Z}/m\mathbf{Z}$$

Furthermore, the Bockstein homomorphism $\delta_m : H^i(X, \mu_m) \to H^{i+1}(X, \mu_m)$ is a derivation with respect to the cup product:

$$\langle \delta_m x, y\rangle + (-1)^m \langle x, \delta_m y\rangle = \delta_m \langle x, y\rangle.$$

Since $H^5(X, \mu_m^{\otimes 2}) \to H^5(X, \mu_{m^2}^{\otimes 2})$ is isomorphic via the trace map to the natural inclusion $\mathbf{Z}/m\mathbf{Z} \hookrightarrow \mathbf{Z}/m^2\mathbf{Z}$, we have

$$\langle x, \delta_m y\rangle + \langle \delta_m x, y\rangle = 0$$

for $x, y \in H^2(X, \mu_m)$.

From the long exact Kummer sequence on X, we have

$$0 \to \text{Pic}(X)/(m) \to H^2(X, \mu_m) \to \text{Br}(X) \xrightarrow{m} \text{Br}(X) \to H^3(X, \mu_m) \to H^3(X, \mathbf{G}_m)[m] \to 0$$

Taking direct limits, this becomes

$$0 \to \text{NS}(X) \otimes \mathbf{Q}/\mathbf{Z} \to H^2(X, \mu_\infty) \to \text{Br}(X) \to 0$$

and taking inverse limits, it becomes

$$0 \to \varprojlim \text{Br}(X)/(n) \to H^3(X, T\mu) \to TH^3(X, \mathbf{G}_m) \to 0$$

In the limit, the arithmetic Poincaré pairing induces a non-degenerate pairing between $H^2(X, \mu_\infty)$ and $H^3(X, T\mu)$. It is a general fact that for a continuous non-degenerate pairing between a discrete torsion group M and a pro-finite group N, $M_{\rm div}$ and $N_{\rm tor}$ annihilate each other. Applying this to $M = H^2(X, \mu_\infty)$ and $N = H^3(X, T\mu)$, we obtain a non-degenerate pairing

$$\mathrm{Br}(X)/\mathrm{Br}(X)_{\rm div} \times \mathrm{Br}(X)/\mathrm{Br}(X)_{\rm div} \to \mathbf{Q}/\mathbf{Z},$$

since $\mathbf{Q}/\mathbf{Z}$ is divisible, and $TH^3(X, \mathbf{G}_m)$ is torsion-free. This pairing is skew-symmetric by the Bockstein property.

This pairing resembles the Cassels-Tate pairing at first glance. Its skew-symmetry implies that if $\mathrm{Br}(X)$ is finite, then its order is a square or twice a square. In fact, Liu-Lorenzini-Raynaud proved that its order is always a square. This is done by expressing any regular surface X has a fibration, after blowing up, recalling that $\mathrm{Br}(X)$ is a birational invariant. Next, they use the explicit relation between the orders of Brauer and Tate-Shafarevich, and the results of Poonon-Stoll on the Cassels-Tate pairing. As far as I know, it is not known whether the pairing on $\mathrm{Br}(X)$ is alternating.