Artin-Tate Conjecture, fibered surfaces, and minimal regular proper model

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1 Minimal Models of Surfaces

1.1 Notation and setup

Let K be a global function field of characteristic p, with finite field of constants k (the algebraic closure of $\mathbf{F}_p$ in K) of size q, so K is the function field $k(V)$ of a unique smooth, proper, geometrically connected curve V over k.

Let X_K be a smooth proper geometrically connected curve over K, with genus $g > 0$. The Jacobian $J = \mathrm{Pic}^0_{X_K/K}$ of X_K is an abelian variety over K with dimension $g > 0$.

The goal is to interpret the BSD conjecture for J in terms of the geometry of a smooth projective surface X over k equipped with a proper flat map $X \to V$ having generic fiber X_K as given. The key point is to express the conjecture in terms of such an X *without reference to the fibration.* That such an X exists at all is already a real theorem, so we begin by discussing (without proofs) the story of "nice" proper flat models for curves.

1.2 Existence of proper regular models

We have $X \subset \mathbf{P}^n_K \subset \mathbf{P}^n_V = \mathbf{P}^n_k \times V$. Suppose we take the Zariski closure $\mathcal{X}$ of X_K in $\mathbf{P}^n_V$. Then $\mathcal{X}$ is flat and proper over V, and a surface over k. The k-scheme $\mathcal{X}$ could be pretty nasty, but we can make it nicer by considering the normalization $\widetilde{\mathcal{X}} \to \mathcal{X}$. Then $\widetilde{\mathcal{X}}$ is a normal projective surface over k, and normality implies that there are no singularities in codimension 1, so $\widetilde{\mathcal{X}}$ has only finitely many singular points. (Recall that since k is perfect, regularity is the same as smoothness over k for a finite type k-scheme, and the non-smooth locus is closed.)

Remark 1.1. A proper map V-scheme with generic fiber that is a geometrically connected and regular curve (so all fibers are of pure dimension 1 by flatness and are

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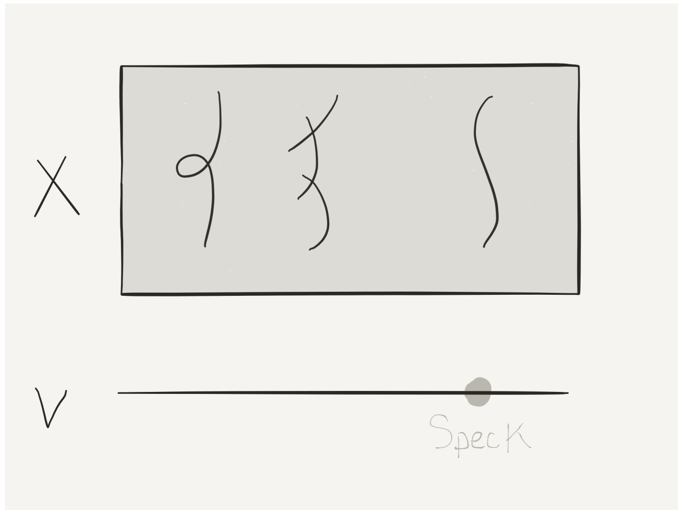

Figure 1: The fibered surface X.

geometrically connected by consideration of Stein factorization) is called a *fibered surface* over V.

Can we improve $\mathcal{X}$ to be k-smooth (equivalently, regular)? The answer is yes, thanks to the following theorem.

Theorem 1.2 (Lipman)**.** *For 2-dimensional excellent integral noetherian schemes $\mathcal{Y}$, there exists a canonical resolution of singularities*

Remark 1.3. The hypothesis of excellence guarantees certain nice properties, like the finiteness of normalization, the openness of the regular locus, preservation of normality and reducedness under completion of local rings, etc.

More precisely, one can describe an iterative process and Lipman's theorem says that it produces a resolution. The process is described as follows. Let $\mathcal{Y}_0 := \mathcal{Y}$. The first step is to normalize, so all the codimension one points are regular. The non-regular locus is closed and has no codimension one points, so by excellence it is just a finite collection of closed points. Then you set $\mathcal{Y}_1$ to be the blowup at these non-regular points. We then define $\mathcal{Y}_2$ in terms of $\mathcal{Y}_1$ exactly as we defined $\mathcal{Y}_1$ in terms of $\mathcal{Y}_0$, and so on.

[Reference: Artin's article in the book *Arithmetic Geometry* (on Faltings' proof of the Mordell Conjecture).]

In this way, we eventually obtain a surface $X \to V$ such that X is smooth (over k) with generic fiber X_K. This is very useful, as geometry is a lot nicer on *smooth* projective surfaces; e.g., intersection theory is available. (The surfaces we consider are projective by design, but actually any smooth proper surface over a field is automatically projective: it suffices to consider algebraically closed ground fields, and that case is handled early in Badescu's beautiful book "Algebraic Surfaces".)

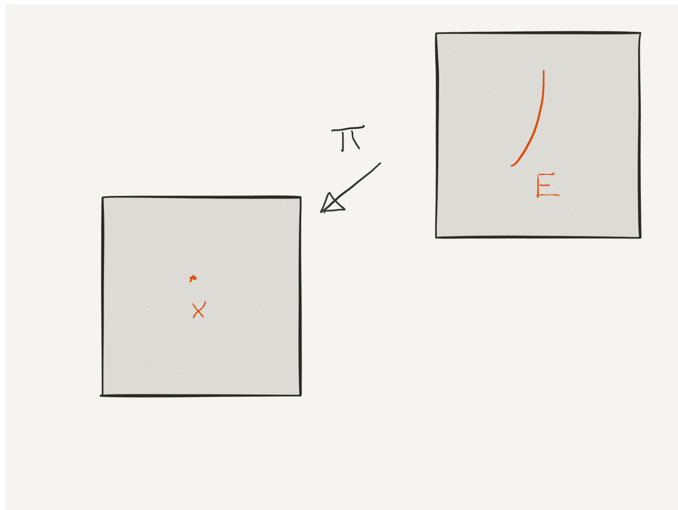

Figure 2: Blowup at a singular point.

1.3 The minimal regular proper model

The fibered surface X we have built is not canonical in terms of its generic fiber X_K. Given one such X, we can make many more by blowing up at closed points $x_0 \in X$. If you want to understand $\mathrm{Bl}_{x_0}(X)$ near the x_0 fiber (such as for regularity

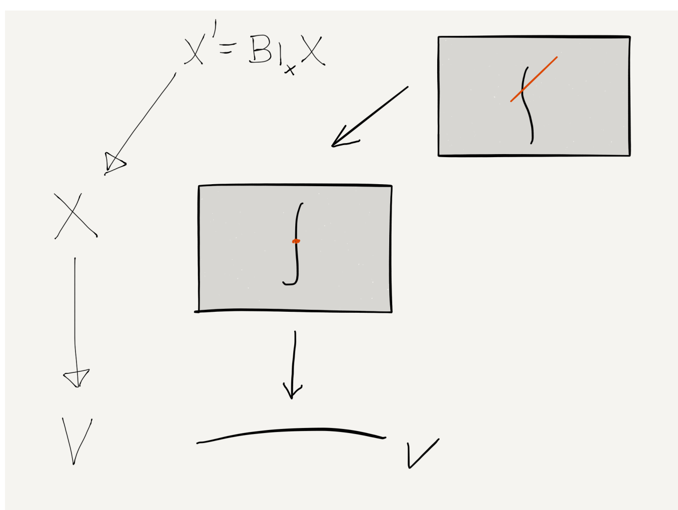

Figure 3: Another regular proper model, obtained by blowing up at a point.

properties), it suffices to compute after base change along the localization morphism $\operatorname{Spec} \mathcal{O}_{X,x_0} \to X$, or even after further base change to the completion of that local ring (since regularity can be checked on completed local rings upstairs). Since blowing up commutes with flat base change, the fibered product is the blow-up of $\mathrm{Spec}\,(R)$ at its closed point, with R equal to the local ring at x_0 (or its completion).

Exercise 1.4. If R is an n-dimensional regular local ring, then $\mathrm{Bl}_{\mathfrak{m}}(\mathrm{Spec}\ R) \to \mathrm{Spec}\ R$ has $\mathfrak{m}$-fiber $\mathbf{P}_k^{n-1}$ and is regular (where $k = R/\mathfrak{m}$). You can check this at the level of open affine charts of blowup.

Definition 1.5. We say that X' *dominates* X (as V-models of X_K) if there exists a map $f\colon X' \to X$ over V and inducing the identity on X_K

$$\begin{tikzcd} X \arrow[rr, dashed, "f"] \arrow[dr] & & X \arrow[dl] \\ & V & \end{tikzcd}$$

Note that given any two models X_1, X_2 for X_K over V, there is third model dominating both. You can just take $\mathcal{X}$ to be the closure of $(X_1)_K \cong X_K \cong (X_2)_K$ in $X_1 \times_V X_2$ and then the resolution of $\mathcal{X}$ dominates X_1, X_2:

$$\begin{tikzcd} & \mathcal{X} \arrow[dl] \arrow[dr] & \\ X_1 \arrow[dr] & & X_2 \arrow[dl] \\ & V & \end{tikzcd}$$

Is there a *minimal* such model? All regular fibered surfaces $X \to V$ admit a reasonable intersection theory in their fibers (see Chinburg's article on minimal models in *Arithmetic Geometry*), and if E is an irreducible (and reduced) component in a fiber X_v such that $\mathrm{H}^1(E, \mathscr{O}_E) = 0$ and $E.E = -1$ (intersection number computed relative to the $k(v)$-finite field $\mathrm{H}^0(E, \mathscr{O}_E)$, rather than relative to $k(v)$) then by some non-trivial work E can be blown down preserving regularity (i.e., $X \simeq \mathrm{Bl}_{y_0}(Y)$ for a *regular* fibered surface $Y \to V$ and closed point $y_0 \in Y$ with E as the exceptional fiber; such (Y, y_0) over V is uniquely determined by (X, E) if it exists). We call such an E a "-1-curve". The *Castelnuovo criterion* is that the minimal fibrations over V are precisely those without a -1-curve in its fibers. (This all works with V replaced by any connected Dedekind scheme.) Consequently, in finitely many steps we always reach a minimal regular fibered surface over V starting with *some* regular fibration having the specified generic fiber. But with this approach it is not at all clear if the minimal regular fibration thereby obtained is uniquely determined by its generic fiber, since the construction depends on an initial choice of regular fibration $X \to V$ having the given generic fiber.

Before we address the uniqueness aspect, let's mention a related notion of minimality in the resolution process itself. For a fibered surface $\mathcal{Y}$ over a Dedekind base, consider a regular resolution $\mathcal{Y}' \to \mathcal{Y}$. It is an immediate consequence of counting the number of irreducible components in the (finitely many) positive-dimensional fibers and systematic application of Castelnuovo's criterion mentioned above (contract a -1-curve that appears in a fiber over $\mathcal{Y}$, and continue until none remain) that there always exists a minimal such resolution relative to domination through proper birational maps. But it is not at all obvious that there can't be several minimal resolutions, pairwise non-isomorphic over $\mathcal{Y}$. The problem of *uniqueness* for a minimal resolution of $\mathcal{Y}$ is determining if a given minimal resolution may actually be dominated by *all* resolutions of $\mathcal{Y}$. Such uniqueness always holds, and the proof rests on the Factorization Theorem that describes *all* proper birational maps between *regular* surfaces as a composition of contractions of -1-curves (equivalently, of blow-ups at closed points); references for uniqueness include Theorem 2.2.2 of my paper "$J_1(p)$ has connected fibers" with Edixhoven and Stein and Theorem 9.3.32 of Qing Liu's book. The *minimal regular resolution* of $\mathcal{Y}$ is denoted $\mathcal{Y}^{\mathrm{reg}}$.

Let's now see why it is tempting to guess that $\mathcal{Y}^{\mathrm{reg}}$ is the output of the process in Lipman's theorem applied to $\mathcal{Y}$, and why the reasoning can fail precisely due to non-reducedness in fibers. Consider regular surfaces $\mathcal{Y}'$ equipped with a proper birational map $\pi : \mathcal{Y}' \to \mathcal{Y}$, so π uniquely factors through the normalization $\widetilde{\mathcal{Y}}$ via some $\widetilde{\pi} : \mathcal{Y}' \to \widetilde{\mathcal{Y}}$. Our problem is to determine if $\widetilde{\pi}$ factors through the blowup of $\widetilde{\mathcal{Y}}$ at the non-regular points. The map $\widetilde{\pi}$ cannot be quasifinite over a non-regular point $\xi \in \widetilde{\mathcal{Y}}$, because if it were then by semi-continuity of fiber dimension and properness (and Zariski's Main Theorem) $\widetilde{\pi}$ would be *finite* over a neighborhood of ξ, hence an isomorphism near there by normality of $\widetilde{\mathcal{Y}}$, a contradiction since $\mathcal{Y}'$ is regular. Thus, consideration of Stein factorization implies that the topological fiber $|\widetilde{\pi}^{-1}(\xi)|$ is a connected union of 1-dimensional irreducible components, so by regularity of $\mathcal{Y}'$ that topological fiber with its reduced structure is *Cartier*. This suggests that the universal property of $\mathrm{Bl}_\xi(\widetilde{\mathcal{Y}}) \to \widetilde{\mathcal{Y}}$ may provide the desired factorization for $\widetilde{\pi}$.

However, the *scheme-theoretic* fiber $\widetilde{\pi}^{-1}(\xi)$ might be non-reduced and hence perhaps not Cartier in $\mathcal{Y}'$, so there is no evident reason why $\widetilde{\pi}$ must factor through the blow-up of $\widetilde{\mathcal{Y}}$ at ξ. The the mapping property of blow-ups is very weak, so the failure of the scheme $\widetilde{\pi}^{-1}(\xi)$ to be Cartier in $\mathcal{Y}'$ doesn't imply there cannot be such a factorization. In fact, Exercise 3.27(a) in 8.3.4 of Qing Liu's book asks the reader to prove that such a factorization always exists (so Lipman's resolution is always the minimal one). Alas, an explicit counterexample to exactly that possibility is given in Exercise 9.3.7 of the same book. In that latter Exercise, $\mathcal{Y}$ over a discrete valuation ring is normal with one non-regular point and its blow-up $\mathcal{Y}_1$ at that point is normal with one non-regular point. The special fiber of $\mathcal{Y}_1$ has an additional component (with multiplicity 2) beyond the strict transform of the special fiber of $\mathcal{Y}$. Upon blowing up $\mathcal{Y}_1$ at its non-regular point and normalizing we get $\mathcal{Y}_2$ whose special fiber has an additional component (a genus-1 curve with multiplicity 1) beyond the strict transforms of the irreducible components in the special fiber of $\mathcal{Y}_1$, and $\mathcal{Y}_2$ is regular (i.e., it is the "Lipman resolution"). But the strict transform E in $\mathcal{Y}_2$ of the multiplicity-2 component from $\mathcal{Y}_1$ turns out to be a -1-curve! Hence, we can contract $\mathcal{Y}_2$ at E to get a new regular resolution of $\mathcal{Y}$, and that one has no -1-curves in fibers over $\mathcal{Y}$, so it is $\mathcal{Y}^{\mathrm{reg}}$.

Remark 1.6. According to Remark 3.34 in §9.3 of Qing Liu's book, Lipman's resolution for a normal fibered surface $\mathcal{Y}$ is minimal when the non-regular points of $\mathcal{Y}$ are of a very mild type called *rational singularities* (but no reference is given for a proof; it might be related to Theorem 9.4.15 of that book via Exercise 9.4.7, but I do not know if rational singularities always fit into the setting of that Theorem).

In the context of fibered surfaces over V, just equipped with a given generic fiber X_K but no "base surface" $\mathcal{Y}$ with generic fiber X_K over which everything is done, the preceding notion of minimality for regular resolutions is not relevant. But remarkably, if we just fix the generic fiber X_K as above then in the sense of "domination" there is a *unique* smallest such regular proper flat model:

Theorem 1.7 (Lichtenbaum-Shafarevich)**.** *Among all regular fibrations $X \to V$ extending X_K, there exists a unique minimal one* (*we need $g(X_K) > 0$ here*).

Proof. Reference: Qing Liu's book, Theorem 9.3.8 (contraction), 9.3.21 (minimal). Also see Theorem 3.1 in Chinburg's article in *Arithmetic Geometry* (Faltings). □

The moral is that there is a "best" proper regular model X for X_K, and it is characterized by the absence of -1-curves in its fibers over V. Note that there is also a separate notion of "minimal surface" over k, which is a stronger notion than relative minimality (with respect to a V-fibration). If X is a regular fibered surface over V and it is minimal as an abstract surface over k then it is certainly minimal over V, but generally the converse is false.

We shall write $X^{\min} \to V$ denote the minimal regular proper model of our fixed initial curve X_K over K with positive genus.

1.4 Properties and examples of the minimal regular proper model

Functoriality

It is a tautology (from the definition of "domination" among models for X_K over V) that $X^{\min}$ is functorial with respect to *isomorphisms* in X_K. (This is why you can't have such a model in genus 0, essentially because $\mathrm{Aut}_R(\mathbf{P}^1_R) = \mathrm{PGL}_2(R) \subsetneq \mathrm{PGL}_2(K) = \mathrm{Aut}_K(\mathbf{P}^1_K)$ for discrete valuation rings R with fraction field K.)

Example 1.8. The minimal regular proper model $X^{\min}$ is *not* functorial with respect to finite maps in the generic fiber. Consider the natural finite "forgetful" map $X_1(p) \to X_0(p)$ for modular curves over $\mathbf{Q}$ with $p \geq 11$ (so genera are positive), and let $\mathcal{X}_1(p)$ and $\mathcal{X}_0(p)$ be their minimal regular proper models over $\mathbf{Z}_{(p)}$. (We do *not* claim that these have any moduli-theoretic significance.) The scheme $\mathcal{X}_0(p)$ happens to be the minimal regular resolution of the coarse moduli scheme over $\mathbf{Z}_{(p)}$

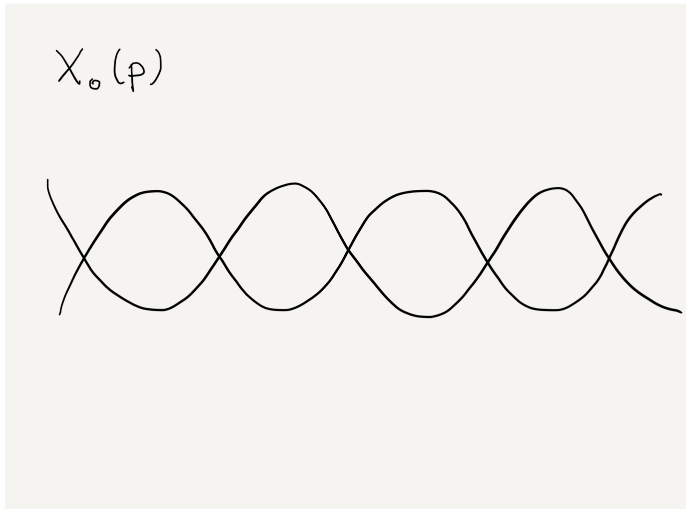

Figure 4: The special fiber of $\mathcal{X}_0(p)$ when $p \equiv 1 \bmod 12$.

(and in fact it is the output of Lipman's process applied to that coarse scheme); if $p \equiv 1 \bmod 12$ then its mod-p fiber consists of of two irreducible components crossing at supersingular geometric points, as shown above. The significance of the case $p \equiv 1 \bmod 12$ is that elliptic curves with extra geometric automorphisms are then *not* supersingular ($j = 1728$ is governed by the class of $p \in (\mathbf{Z}/4\mathbf{Z})^\times = \{\pm 1\}$ and $j = 0$ is governed by the class of $p \in (\mathbf{Z}/3\mathbf{Z})^\times = \{\pm 1\}$). In general, the geometry of the irreducible components, depending on p mod 12, is illustrated in §1 of Chapter II of Mazur's "Eisenstein ideal" paper (with additional components related to $j = 0, 1728$ when supersingular in characteristic p).

In contrast, the minimal regular resolution over $\mathbf{Z}_{(p)}$ for the coarse moduli scheme $X_1(p)$ always has a "-1-curve" in its mod-p fiber (not in its fibers over the coarse scheme), and as one successively contracts these there continue to arise yet more -1-curves, until ultimately the special fiber consists of a single irreducible component! That is, the minimal regular proper model $\mathcal{X}_1(p)$ over $\mathbf{Z}_{(p)}$ for $X_1(p)_{\mathbf{Q}}$ has *irreducible* special fiber, so there cannot be a map $\mathcal{X}_1(p) \to \mathcal{X}_0(p)$ extending the evident forgetful map between their $\mathbf{Q}$-fibers (it would have to be surjective by properness). Amusingly, by using the valuative criterion to handle codimension-1 points (e.g., generic points in the mod-p fiber) we know that there is such a map away from a non-empty finite set of closed points on $\mathcal{X}_1(p)$, but I do not know the points are at which that map is not defined. The geometry of the mod-p fiber of the minimal regular resolution over $\mathbf{Z}_{(p)}$ of $X_1(p)/H$ for *every* subgroup $H \subset (\mathbf{Z}/p\mathbf{Z})^\times/\langle -1\rangle$ is worked out in my paper "$J_1(p)$ has connected fibers" with Edixhoven and Stein, and the possibilities depend on both p mod 12 *and* $\#H$ mod 6 (sanity check: the case $X_0(p)$ is $\#H = (p-1)/2$, in which case $\#H$ mod 6 is determined by p mod 12); pictures with intersection-theoretic data for general H are given in §5.2 of that paper.

We conclude our general discussion of fibered surfaces by focusing on the case $g = 1$ with $K = \mathrm{Frac}(R)$ for a discrete valuation ring R and $X(K) \neq \emptyset$, so $X = E$ is an elliptic curve over K. In this case there is a confusing abundance of notions of "good integral model". Let's explain how they are related to each other.

Let $\mathcal{E}$ be the minimal regular proper model for E over R. By the valuative criterion for properness, any rational point on the generic fiber extends to a section to the fibration. For any regular fibered surface over a Dedekind base, a section always passes through the relative smooth locus (exercise). (In particular, the special fiber has smooth points). Thus, the maximal R-smooth open subscheme $\mathcal{E}^{\mathrm{sm}}$ has non-empty special fiber (containing the extension of the identity section). In fact, it is equal to the Néron model $N(E)$ of E over R. On the other hand, we can also

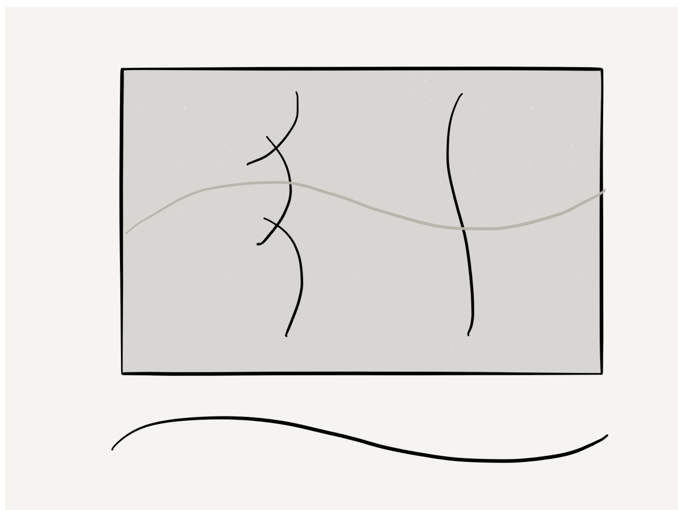

Figure 5: A section passes through the smooth locus.

consider a minimal Weierstrass model $W_{\min} \subset \mathbf{P}^2_R$ (see my notes on "Minimal models of elliptic curves" for a discussion of how to think about the conceptual meaning of Weierstrass models without equations, and proofs of everything that I am about to say). It turns out that for any two Weierstrass models W and W' of E, there is an inclusion

$$\mathrm{H}^0(W'^{\mathrm{sm}}, \Omega^1_{W'^{\mathrm{sm}}/R}) \subset \mathrm{H}^0(W^{\mathrm{sm}}, \Omega^1_{W^{\mathrm{sm}}/R})$$

as R-lines inside $\mathrm{H}^0(E, \Omega^1_{E/K})$ if and only if $\mathrm{ord}(\Delta_{W'}) \geq \mathrm{ord}(\Delta_W)$. Informally, the link between larger $\mathrm{H}^0(W^{\mathrm{sm}}, \Omega^1_{W^{\mathrm{sm}}/R})$ and smaller $\mathrm{ord}(\Delta)$ is due to the product $\Delta\omega^{\otimes 12} \in \mathrm{H}^0(E, \Omega^1_{E/K})$ being the *same* for all Weierstrass models of E over K, with Δ and ω taken from a common such model.

Hence, the minimal Weierstass model $W_{\min}$ has an intrinsic geometric meaning unrelated to playing with equations: it is *maximal* with respect to the R-line of global 1-forms over its R-smooth locus, viewed inside the K-line of global 1-forms on E. Building on this viewpoint, one gets a conceptual proof of the uniqueness of $W_{\min}$ (without equation-manipulations) and a rather non-obvious geometric characterization of minimality for a Weierstrass model: W is minimal if and only if it has rational singularities.

In the proof of Lipman's theorem one shows that for rational singularities on a normal surface, the blow-up of a non-regular point is normal and has only rational singularities. Thus, Lipman's process in such cases only involves blow-ups (i.e., no normalizations need to be computed). It follows that Lipman's resolution of $W_{\min}$ is computed *entirely via blow-ups*. But in such cases Lipman's resolution is *also* the minimal resolution $(W_{\min})^{\mathrm{reg}}$, according to Remark 1.6. This is great, due to:

Proposition 1.9. *We have* $(W_{\min})^{\mathrm{reg}} = \mathcal{E} \supset \mathcal{E}^{\mathrm{sm}} = N(E)$.

The upshot is that one can compute $\mathcal{E}$ (and hence $N(E)$) directly from $W_{\min}$ without needing to compute a normalization! That renders plausible the aim to compute $\mathcal{E}$ directly from an *arbitrary* Weierstrass model W via blow-ups, without normalization steps. Tate's algorithm achieves this (assuming the residue field is perfect when it has characteristic 2 or 3), using artful blow-ups along codimension-1 subschemes in some steps. (The algorithm is usually described in the language of coordinate changes. But to justify the correctness of the information asserted to emerge from the calculations with coordinate changes, one has to link them to the geometry of integral models via blow-ups.)

2 The Artin-Tate Conjecture

2.1 The dictionary

Now let's go back to BSD. Let $X \to V$ be a regular fibration with generic fiber X_K; it could be taken to be the minimal one, or not. One can ask if we can rephrase BSD in terms of the geometry of X (especially with the minimal model $X^{\min}$, or any such X if one is more optimistic).

The goal is to recast BSD as a conjecture about X/k (with no reference to the fibration structure over V!); then one could imagine fibering the same X over a given V (such as $V = \mathbf{P}^1_k$) in different ways, getting wildly different generic fibers (perhaps with different genera) and then hope to see that BSD for their respective Jacobians are equivalent to each other.

Here are some informal correspondences, not meant to be literal "equalities", and to be elaborated upon later in the lecture. (See Ulmer's article *Curves and Jacobians over function fields* for an extensive discussion of how the different parts of the BSD and Artin–Tate conjectures "correspond" in a precise manner, some aspects of which will be addressed in upcoming lectures possibly under simplifying hypotheses on the fibers X_v that are avoided in Ulmer's article but permit a more direct link between structures on both sides.)

$$\boxed{\mathrm{BSD}(J) \leftrightarrow \mathrm{AT}(X).}$$

- In this recasting,

  $$\boxed{L(s, J/K) \leftrightarrow \zeta_{X/k}(q^{-s})}$$

  (the link is clear at Euler factors away from bad fibers X_v).

- Next, we can interpret the order of vanishing:

  $$\boxed{\mathrm{ord}_{s=1} L(s, J/K) \leftrightarrow \text{multiplicity of } q^{-1} \text{ as a pole of } \zeta_{X/k}}.$$

- What about the height pairing?

  The group $J(K) = \mathrm{Pic}^0_{X/K}(K) \subset \mathrm{Pic}(X)$ with the height pairing "corresponds" to the Néron-Severi group $\mathrm{NS}(X) = \mathrm{Pic}_{X/k}(k)/\,\mathrm{Pic}^0_{X/k}(k)$. Let's see how this notion $\mathrm{NS}(X)$ is related to the geometric Néron–Severi group $\mathrm{NS}(X_{\overline{k}})$ given by the group of geometric points of the étale component group $\mathrm{Pic}_{X/k}/\,\mathrm{Pic}^0_{X/k}$.

  Usually rational points of Picard schemes don't mean anything, but the Brauer group of a finite field is trivial, so $\mathrm{Pic}_{X/k}(k) = \mathrm{Pic}(X)$. The group $\mathrm{Pic}^0_{X/k}(k)$ is finite, and a quotient among rational points is generally not the set of rational points of the quotient due to an obstruction in H^1 of what we quotient by. But Lang's Theorem ensures $\mathrm{H}^1(k, \mathrm{Pic}^0_{X/k}) = 1$, so $\mathrm{NS}(X) = (\mathrm{Pic}_{X/k}/\,\mathrm{Pic}^0_{X/k})(k)$ is the group of Galois-fixed points in $\mathrm{NS}(X_{\overline{k}})$. On $\mathrm{NS}(X)$ we have the intersection pairing of divisors on the smooth projective surface X.

- Finally, the Tate-Shafarevich group corresponds to the Brauer group (to be defined!):

  $$\boxed{Ш(J) \leftrightarrow \mathrm{Br}(X).}$$

Once we formulate $\mathrm{AT}(X/k)$, one can ask if it is independent of choice of X for its given function field over k. Any two regular projective surfaces with the same function field are related through a finite sequence of blow-ups and blow-downs at closed points, due to the Factorization Theorem that describes all dominating morphisms between regular projective surfaces (as discussed in Chinburg's article on minimal models in *Arithmetic Geometry* in the case of fibered surfaces, the main case of interest to us). Hence, the main task is to study the effect on $\mathrm{AT}(X/k)$ when blowing up at closed points.

2.2 The Brauer Group

Definition 2.1. Let S be a scheme. Then we define the *cohomological Brauer group of* S to be $\mathrm{Br}(S) := H^2_{\text{ét}}(S, \mathbf{G}_m)$.

Example 2.2. Let $S = \text{Spec } F$ for F a field. If A is a central simple algebra of rank n^2, then A is classified by $[A] \in H^1(F, \text{PGL}_n)$. Indeed, it is a theorem that $A_{F_s} \cong \text{Mat}_n(F_s)$ whose automorphism group is $\text{PGL}_n(F_s)$ by the Skolem-Noether Theorem.

From the exact sequence

$$1 \to \mathbf{G}_m \to \text{GL}_n \to \text{PGL}_n \to 1$$

we get a boundary map $H^1(F, \text{PGL}_n) \xrightarrow{\delta} H^2(F, \mathbf{G}_m)$. This is injective, and exhausts $H^2(F, \mathbf{G}_m)$ as n grows.

In the scheme setting, one has the notion of "Azumaya algebra" of rank n^2, which is (roughly speaking) a sheaf of twisted matrix algebras. There is also a notion of "Brauer equivalence" generalizing the equivalence relation used when defining the Brauer group of a field, and there is a map from the set of equivalence classes of Azumaya algebras modulo Brauer equivalence into $\text{Br}(S)$ via a relative version of the connecting maps mentioned above

The Brauer group is very hard to get your hands on. For instance, the Čech cohomology only injects in general: $\check{H}^2(S, \mathbf{G}_m) \hookrightarrow H^2_{\text{ét}}(S, \mathbf{G}_m)$.

One might hope to show that the natural map from the Brauer group of a reasonably nice scheme X (at least a regular curve or surface, not necessarily proper) injects into that of its function field and is characterized within that latter Brauer group by geometric conditions (e.g., "unramifiedness" along divisors, etc.).

Remark 2.3. Grothendieck studies Brauer groups of curves and surfaces in "Group de Brauer III" (building on his two previous papers I and II that studied Azumaya algebras, their deformation theory, and motivations from topology). One of his main results, elaborating on work of Artin that inspired the Artin–Tate conjecture, is that for a regular proper fibration $f : X \to V$ with generic fiber X_K, $\text{Ш}(J)$ and $\text{Br}(X)$ are closed related provided that $X_v^{\text{sm}} \neq \emptyset$ for all v; this hypothesis ensures (by deep work of Raynaud that is surveyed in Chapter 9 of "Néron Models") that the Néron model $\mathcal{J}$ of J coincides with the V-group scheme $\text{Pic}^{[0]}_{X/V}$ of line bundles fiberwise of degree 0 and that $\mathcal{J}^0 = \text{Pic}^0_{X/V}$ is the V-group scheme of line bundles with degree 0 on each irreducible component of each geometric fiber. These links between Picard schemes and Néron models are relevant because $\text{Ш}(J) = \text{Im}\,(H^1(V, \mathcal{J}^0) \to H^1(V, \mathcal{J}))$ and the low-degree parts of the Leray spectral sequence

$$\text{H}^i(V_{\text{ét}}, \text{R}^j f_*(\mathbf{G}_m)) \Rightarrow \text{H}^{i+j}_{\text{ét}}(V, \mathbf{G}_m)$$

involve $\text{Br}(X)$ and $\text{R}^1 f_*(\mathbf{G}_m) = \text{Pic}_{X/V}$; Francois will discuss these relationships in his lecture.

Conjecture 2.4 (Artin)**.** *Let Y be proper over $\mathbf{Z}$. Then $\text{Br}(Y)$ is finite.*

Brauer groups, although very hard to analyze, are in some ways more hands-on than Tate-Shafarevich groups in view of the theorem of Gabber (and independently, de Jong) that Brauer classes on many schemes (e.g., all schemes quasi-projective over an affine scheme) always arise from Azumaya algebras. For example, the recent spectacular progress on the Tate conjecture for K3 surfaces (which also proved the Artin–Tate conjecture below for all such surfaces over finite fields away from characteristic 2) relies crucially on studying the geometry of moduli stacks related to Azumaya algebras and generalizations thereof.

Some computations of Brauer groups

Let S be connected and Dedekind with generic point η and function field $F = k(\eta)$. How do you get your hands on the Brauer group? You can use the "divisor exact sequence" (for the étale topology)

$$1 \to \mathbf{G}_m \to \eta_*(\mathbf{G}_{m,\eta}) \to \bigoplus_{s \in S^0} i_{s*}(\mathbf{Z}) \to 0.$$

(We don't really need the Dedekind assumption; you could try this taking s to vary through the points of codimension 1 in any connected regular scheme S. Regularity ensures surjectivity on the right when checking locally at points of high codimension because all height 1 primes in a regular local ring are principal.)

Now let's consider the associated long exact sequence.

$$\ldots \to \bigoplus_{s \in S^0} H^1(s, \mathbf{Z}) \to \mathrm{Br}(S) \to H^2(S, \eta_* \mathbf{G}_m) \to \bigoplus_{s \in S^0} H^2(s, \mathbf{Z}) \to \ldots$$

In the Dedekind case $H^1(s, \mathbf{Z})$ is Galois cohomology of $\mathbf{Z}$, which vanishes because continuous homomorphisms from the Galois group to $\mathbf{Z}$ are trivial. Beware that this step already runs into subtleties if we try to consider a regular surface S and the divisor associated to the codimension-1 point s is not regular: a "split" nodal plane cubic has a nontrivial $\mathbf{Z}$-torsor.

On the other hand, $H^2(s, \mathbf{Z}) = H^1(s, \mathbf{Q}/\mathbf{Z}) = \mathrm{Hom}(\mathrm{Gal}(k(s)_{\mathrm{sep}}/k(s)), \mathbf{Q}/\mathbf{Z})$ by the long exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$$

and calculations as in the theory of local fields identify this latter group with the Brauer group $\mathrm{Br}(\widehat{F}_s)$ of the completion at s *provided* that $k(s)$ is perfect (e.g., finite). This identification is interesting because also $H^2(S, \eta_* \mathbf{G}_m) = H^2(\eta, \mathbf{G}_m) = \mathrm{Br}(F)$ due to a Leray spectral sequence degeneration in low degree: $R^j \eta_* \mathbf{G}_m = 1$ for all $j = 1, 2$ when the residue fields of S at its closed points are perfect.

Unraveling these identifications, if all residue fields at closed points of S are perfect then we get an exact sequence

$$0 \to \mathrm{Br}(S) \to \mathrm{Br}(F) \to \bigoplus_{s \in S^0} \mathrm{Br}(\widehat{F}_s)$$

where all maps are the natural ones. Applying this to $S = V$ we get:

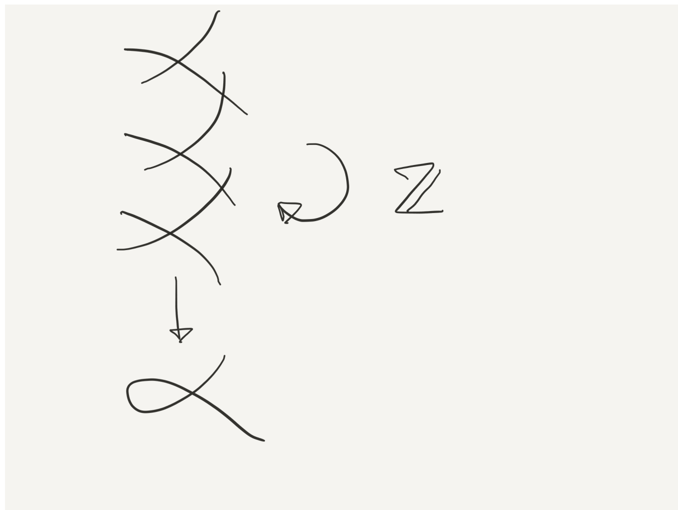

Figure 6: A non-trivial $\mathbf{Z}$-torsor of a nodal curve.

Corollary 2.5. *The Brauer group* $\mathrm{Br}(V)$ *of a smooth proper curve* V *over a finite field is trivial.*

Proof. The fundamental short exact sequence of class field theory. □

Remark 2.6. In Brauer III, Grothendieck studies the Leray spectral sequence

$$H^i(V, R^j f_* \mathbf{G}_m) \implies H^{i+j}(X, \mathbf{G}_m).$$

and deduced that if all $X_v^{\mathrm{sm}} \neq \emptyset$, then the spectral sequence provides an injection $\mathrm{Br}(X) \hookrightarrow Ш(J)$ with finite cylic cokernel whose size is govered by the gcd among K-degrees of closed points on X_K (i.e., the image of $\deg_K : \mathrm{Pic}(X) \to \mathbf{Z}$).

2.3 Statement of the Artin-Tate Conjecture

Let X be a smooth proper geometrically connected surface over k ($\#k = q$). By the Lefschetz trace formula,

$$\zeta_{X/k} = \frac{P_1(q^{-s})P_3(q^{-s})}{(1-q^{-s})P_2(q^{-s})(1-q^{2-s})}$$

where $P_i(T) = \det(1 - \phi T \mid H^i_{\text{ét}}(X_{\overline{k}}, \mathbf{Q}_\ell))$ and ϕ is the geometric Frobenius in Gal_k. By the Riemann Hypothesis (proved by Deligne in the early 1970's), P_i has roots that are algebraic integers which are q-Weil numbers of weight i, i.e. absolute value $q^{i/2}$ under all complex embeddings. Thus the roots of P_2 in $\mathbf{C}$ lie on the circle centered at 0 with radius $q^{-2/2} = q^{-1}$.

For the numerator, Poincaré duality implies that $P_3(q^{-s}) = P_1(q^{1-s})$. Also, $P_1(q^{-s})$ is related to $H^1(X_{\overline{k}}, \mathbf{Q}_\ell(1)) = T_\ell(\mathrm{Pic}_{X/k})$. This is relatively tangible, so P_2 is the "mystery piece".

Let $\rho(X) = \operatorname{rank} \mathrm{NS}(X)$; by a general theorem of Lang and Néron the Néron–Severi group of projective varieties over algebraically closed fields are always finitely generated, so the subgroup $\mathrm{NS}(X) \subset \mathrm{NS}(X_{\overline{k}})$ is finitely generated.

Let $\alpha(x) = h^0(\Omega^2_{X/k}) - (h^1(\mathcal{O}_X) - \dim \mathrm{Pic}^0_{X/k})$. The difference $h^1(\mathcal{O}_X) - \dim \mathrm{Pic}^0_{X/k}$ is always non-negative, and vanishes if and only if the Picard scheme of X over k is smooth. (Mumford's book "Lectures on curves on an algebraic surface" thoroughly analyzes the phenomenon of non-smoothness of the Picard scheme for smooth projective surfaces over algebraically closed fields of positive characteristic.)

Conjecture 2.7 (Artin-Tate)**.** $\mathrm{Br}(X)$ *is finite and* q^{-1} *is a root of* P_2 *with multiplicity* $\rho(X)$*, and*

$$P_2(q^{-s}) \sim_{s \to 1} \left(\frac{\#\,\mathrm{Br}(X) \cdot |\det(D_i \cdot D_j)|}{q^{\alpha(X)} (\#\mathrm{NS}(X)_{\mathrm{tor}})^2} \right) (1 - q^{1-s})^{\rho(X)}$$

where $\{D_i\}$ *is a basis for* $\mathrm{NS}(X)/\mathrm{NS}(X)_{\mathrm{tor}}$.

The geometry of X gives a lot of interesting structure to this conjecture, as we will see in upcoming lectures.