ÉTALE COHOMOLOGY OF ALGEBRAIC NUMBER FIELDS

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Abstract

We use étale cohomology to obtain a conceptual proof of global Tate duality by means of Artin–Verdier duality, elaborating on Zink's method of appending "real points" to $\mathrm{Spec}(\mathcal{O}_K)$ to extend Mazur's proof in the totally imaginary case.

It is organized in six sections.

After some introductory review in §1, including statements of Tate's main theorems, in §2 we discuss the foundations of the étale topology on algebraic number fields and its modification to allow consideration of real places. In §3 we deal with the cohomology theory arising from what we call extended small étale sites, introduced in §2.

In §4 we prove one of the main theorems (including a finiteness aspect) via a series of explicit computations and reductions.

In §5 we prove the basics of Artin–Verdier duality. Finally, in §6 we use Artin-Verdier duality to recover Tate's theorems, including the 9-term long exact sequence for finite ramification as a local cohomology sequence.

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1 Introduction

1.1 Some history

Given a field k and a separable closure $\overline{k}$ of k, we denote by G_k the Galois group $\mathrm{Gal}(\overline{k}/k)$. We consider the category $\mathbf{Mod}_{G_k}$ of discrete G_k-modules (called "Galois modules over k", or simply "Galois modules" when k is understood). A very interesting G_k-module is $\overline{k}^{\times}$, which plays an essential role in $\mathbf{Mod}_{G_k}$ when k is a non-archimedean local field. Our main interest will be the case when k is a number field, but first we discuss some more basic cases that feed into the study of the number field case.

Suppose first k is a non-archimedean local field. Let M be a finite Galois module over k, and define

$$M^{\vee} := \mathrm{Hom}_{\mathrm{Ab}}(M, \overline{k}^{\times}) = \mathrm{Hom}_{\mathrm{Ab}}(M, \mu_{\infty})$$

where μ_{∞} is the Galois module of roots of unity in $\overline{k}^{\times}$; $M^{\vee}$ is a finite Galois module in the evident manner. Tate local duality says that cup-product induces a perfect pairing of *finite* abelian groups

$$\mathrm{H}^i(G_k, M) \times \mathrm{H}^{2-i}(G_k, M^{\vee}) \to \mathrm{H}^2(G_k, \overline{k}^{\times}) \xrightarrow{\simeq} \mathbf{Q}/\mathbf{Z}$$

for $0 \leq i \leq 2$ and all finite M of order prime to the characteristic of k. Here, the isomorphism $\mathrm{H}^2(G_k, \overline{k}^{\times}) \simeq \mathbf{Q}/\mathbf{Z}$ comes from local class field theory. Tate also proved

$$\chi(M) := \frac{h^0(M)h^2(M)}{h^1(M)} = \frac{1}{|\#M|}$$

for the normalized $|\cdot|$ on k (with $h^i(M) := \#\mathrm{H}^i(G_K, M)$).

These results are proved in §5.2 of Chapter II of Serre's book *Galois cohomology* using local class field theory and general results on the cohomology of profinite groups; the same methods are also used there to show that for such M the cohomology $\mathrm{H}^i(G_k, M)$ vanishes for $i > 2$ (so G_k has cohomological dimension 2 relative to torsion coefficients with torsion-orders prime to $\mathrm{char}(k)$). There is also a duality result when $p = \mathrm{char}(k) > 0$ and $p|\#M$ but then it involves Cartier duality and fppf cohomology, so we do not discuss that here. The analogue for archimedean local fields is elementary but will be important in some later considerations:

Remark 1.1.1 The field $k = \mathbf{R}$ admits a duality using Tate cohomology: for a finite G_k-module M, the cup product

$$\mathrm{H}_T^i(G_k, M) \times \mathrm{H}_T^{2-i}(G_k, M^{\vee}) \to \mathrm{H}_T^2(G_k, \overline{k}^{\times}) = \mathrm{H}^2(G_k, \overline{k}^{\times}) = \mathrm{Br}(k) = \mathbf{Z}/2\mathbf{Z}$$

is a perfect pairing of finite groups (with $\mathrm{H}_T^{\bullet}(G, \cdot)$ denoting Tate cohomology for a finite group G). To prove this, note that since the Tate cohomologies are 2-torsion, $\mathrm{Hom}(\cdot, \mathbf{Z}/2\mathbf{Z})$ is exact on long exact cohomology sequences in such Tate cohomologies. We may thereby reduce to the case when M is a simple G_k-module, so it is killed by some prime ℓ. If ℓ is odd then the Tate cohomologies vanish and there is nothing to do. Hence, we can assume M is 2-torsion. Since G_k has order 2, it follows that M has a composition series of G_k-submodules whose successive quotients are $\mathbf{Z}/2\mathbf{Z}$. In this way we may assume $M = \mathbf{Z}/2\mathbf{Z}$.

Let $\omega \in \mathrm{H}^2(G_k, \mathbf{Z})$ be the unique generator (an element of order 2). By the theory of Tate cohomology for cyclic groups, cup product with ω induces and isomorphism $\mathrm{H}_T^p(G_k, N) \to \mathrm{H}_T^{p+2}(G_k, N)$ for every G_k-module N. Using the δ-functoriality of Tate cohomology cup products, the pairing

$$\mathrm{H}_T^n(G_k, M) \times \mathrm{H}_T^m(G_k, M^{\vee}) \to \mathrm{H}^{n+m}(G_k, \overline{K}^{\times})$$

depend on n and m only through their parities. Hence, it suffices to prove that

$$\mathrm{H}_T^{-n}(G_k, \mathbf{Z}/2\mathbf{Z}) \times \mathrm{H}_T^n(G_k, \mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}_T^0(G_k, \mathbf{Z}/2\mathbf{Z}) = \mathrm{H}_T^0(G_k, \overline{K}^\times)$$

(final equality via inclusion of μ_2 into $\overline{K}^\times$!) is perfect for $n = 0, 1$. This is a trivial verification.

Remark 1.1.2 If k is a finite field, so $G_k = \widehat{\mathbf{Z}}$ has cohomological dimension 1 (and strict cohomological dimension 2), the "dualizing module" for G_k is $\mathbf{Q}/\mathbf{Z}$ (playing a role analogous to that of μ_∞ for local fields above). Hence, we consider the Pontryagin dual Galois module $M^D := \mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z})$ and we get that for all finite M (no condition on $\gcd(\#M, \mathrm{char}(k))$!) there is a perfect pairing of finite abelian groups $\mathrm{H}^i(G_k, M) \times \mathrm{H}^{1-i}(G_k, M) \to \mathrm{H}^1(G_k, \mathbf{Q}/\mathbf{Z}) = \mathbf{Q}/\mathbf{Z}$ induced by cup-product. One also has $\chi(M) := h^0(M)/h^1(M) = 1$ from the classical theory of Herbrand quotients for the cohomology of (pro-)cyclic groups.

In the case k is a global field, there is a duality principle, called global Tate duality and first announced by Tate at the 1962 ICM [Tate]. He never published his proofs, which were later documented in [Mi] and [Hab]. Based entirely on techniques in Galois cohomology, the proofs (and even the some aspects of the statements) were sufficiently complicated that they did not provide a great conceptual insight into "why" they held (in contrast with the duality above for local fields, for which the statements are more concrete and calculations with μ_n have explicit meaning that leads to the general case).

As first worked out by Mazur in the totally imaginary case in [Ma], the formalism of Artin–Verdier duality provides an elegant approach to Tate's main global results; e.g., the famous "9-term exact sequence" (see Theorem 13) which seems so mysterious in the traditional formulation (as it was originally built in a somewhat ad hoc manner as a device to record a few other dualities built by prior means) is obtained directly as a long exact "local cohomology" sequence.

In these notes we use or extend arguments from [Ma] to recover Tate's results on global duality for algebraic number fields via étale cohomology on spectra of rings of S-integers augmented with some "real places at infinity", expanding on ideas of Zink introduced in the Appendix of [Hab]. This provides topological insight into Tate's results that make $\mathrm{Spec}(\mathcal{O}_{K,S})$ for a number field K (and finite set S of places of K containing the archimedean places) behave as if it is a 3-manifold.

The analogy with 3-manifolds can be seen directly in the function field case: if X is a smooth proper and geometrically connected curve over a finite field $\mathbf{F}$ of characteristic p then for a dense open $j : U \hookrightarrow X$ the structure map $f : U \to \mathrm{Spec}(\mathbf{F})$ is viewed as a "family" of open Riemann surfaces fibered over a loop, "hence" of dimension 3 for cohomological purposes with torsion coefficients away from p. More specifically, we have the Leray spectral sequence $\mathrm{H}^n(\mathbf{F}, \mathrm{R}^m f_!(\cdot)) \Rightarrow \mathrm{H}_c^{n+m}(U, \cdot)$ with $\mathbf{F}$ of cohomological dimension 1 and $f_!$ of cohomological dimension 2 on torsion objects.

1.2 Summary of results

First, we set some notation to be fixed for the entire discussion. We let K be a number field, and $\mathcal{O}_K$ its ring of integers. We set $X := \mathrm{Spec}(\mathcal{O}_K)$. Let S be a finite set of places of K that contains the archimedean places, and denote by $\{v_1, \dots, v_r\}$ the set of real places of K (which may be empty). As usual, denote by K_S the maximal algebraic extension of K which is unramified outside S, and by G_S its Galois group over K.

Let $U := \mathrm{Spec}(\mathcal{O}_{K,S}) \subset X$. Since finite extensions of K inside K_S "correspond" (via normalization) to connected finite étale covers of U, we have $G_S = \pi_1(U, \overline{\eta})$ where $\overline{\eta} : \mathrm{Spec}(\overline{K}) \to U$ is a geometric generic point (over K_S).

One of the results we will prove takes care of global cohomology beyond degree 2:

Theorem 1.2.1 *Let M be a finite G_S-module whose order is an S-unit. For $i \geq 3$ there is a canonical isomorphism:*

$$\mathrm{H}^i(G_S, M) \to \bigoplus_{j=1}^{r} \mathrm{H}^i(D_{v_j}, M)$$

where D_v is a decomposition group at a real place v.

This will emerge as a piece of a local cohomology sequence (the flanking terms vanishing due to the "3-manifold" property of $\mathrm{Spec}(\mathcal{O}_{K,S})$); see Theorem 4.3.1. Before going further, we state and prove a basic finiteness result:

Proposition 1.2.2 *If M is finite and $\#M$ an S-unit then $\mathrm{H}^i(G_S, M)$ is finite for all i.*

Finiteness also holds for cases with $\#M$ not an S-unit, and such M are certainly of much importance in Galois deformation theory (so their avoidance in Proposition 1.2.2 does not mean that such cases of are no interest). Proposition 1.2.2 also follows from a general finiteness result of Artin–Verdier (part of Theorem 1.2.6) via a local cohomology argument and a comparison isomorphism in Appendix A between continuous cohomology of the fundamental group G_S of $\mathrm{Spec}(\mathcal{O}_{K,S})$ and étale cohomology on $\mathrm{Spec}(\mathcal{O}_{K,S})$ with locally constant coefficients; the essential arithmetic input from class field theory remains the same under both approaches.

Proof. Theorem 1.2.1 settles the case $i \geq 3$, so the remaining issue is $i = 1, 2$. If K'/K is a finite Galois splitting field for M (so $K' \subset K_S$) and S' is the set of places of K' over S then the Hochschild–Serre spectral sequence

$$\mathrm{H}^i(K'/K, \mathrm{H}^j(G_{S'}, M)) \Rightarrow \mathrm{H}^{i+j}(G_S, M)$$

reduces the finiteness assertion to the case when M has trivial G_S-action, so we can assume $M = \mathbf{Z}/n\mathbf{Z}$ for some $n > 0$ that is an S-unit. In a similar manner we can replace K with $K(\zeta_n)$ so that $M = \mu_n$. Note that $\mathcal{O}_{K_S,S}^{\times}$ is n-divisible since n is an S-unit.

The extension K_S/K is the directed union of finite-degree Galois subextensions F/K, for all of which $\mathrm{Spec}(\mathcal{O}_{F,S})$ is a connected finite étale Galois cover of $\mathrm{Spec}(\mathcal{O}_{K,S})$. Hence, étale descent for line bundles identifies $\mathrm{H}^1(F/K, \mathcal{O}_{F,S}^{\times})$ with $\ker(\mathrm{Pic}(\mathcal{O}_{K,S}) \to \mathrm{Pic}(\mathcal{O}_{F,S}))$. Passing to the limit over such F/K, we get

$$\mathrm{H}^1(G_S, \mathcal{O}_{K_S,S}^{\times}) = \ker(\mathrm{Pic}(\mathcal{O}_{K,S}) \to \mathrm{Pic}(\mathcal{O}_{K_S,S})).$$

But every ideal class for $\mathcal{O}_{K,S}$ becomes trivial over the ring of S-integers of the Hilbert class field of K by the Principal Ideal Theorem, so the above kernel coincides with $\mathrm{Pic}(\mathcal{O}_{K,S})$. Hence, the exact sequence

$$1 \to M \to \mathcal{O}_{K_S,S}^{\times} \xrightarrow{t^n} \mathcal{O}_{K_S,S}^{\times} \to 1$$

of discrete G_S-modules provides an exact sequence

$$1 \to \mathcal{O}_{K,S}^{\times}/(\mathcal{O}_{K,S}^{\times})^n \to \mathrm{H}^1(G_S, M) \to \mathrm{Pic}(\mathcal{O}_{K,S})[n] \to 0.$$

The S-unit theorem and finiteness for class groups thereby settle the case $i = 1$ for this M.

Likewise, by Hilbert 90 the natural inflation map $\mathrm{H}^2(G_S, M) \to \mathrm{H}^2(K, \mu_n) = \mathrm{Br}(K)[n]$ is injective, so to prove finiteness it suffices to check that for $v \notin S$ the image of this injection has vanishing local restriction into $\mathrm{Br}(K_v)$. This is reduced to the vanishing of $\mathrm{H}^2(K_v^{\mathrm{un}}/K_v, \mu_n)$, that in turn is immediate since $\mathrm{Gal}(K_v^{\mathrm{un}}/K_v) = \widehat{\mathbf{Z}}$ has cohomological dimension 1. (Note that in $\mathrm{Br}(K_v) = \mathrm{H}^2(K_v^{\mathrm{un}}/K_v, (K_v^{\mathrm{un}})^{\times})$ the n-torsion has *nothing to do* with $\mathrm{H}^2(K_v^{\mathrm{un}}/K_v, \mu_n) = 1$ for $n > 1$ because the n-power endomorphism of $(K_v^{\mathrm{un}})^{\times}$ is not surjective! That is, there is no n-power Kummer sequence at the level of K_v^{un}/K_v when $n > 1$.) □

For each $v \in S$ we denote by $\mathrm{H}^p(K_v, M)$ the group $\mathrm{H}^p(D_v, M)$ for finite places v in S and the Tate cohomology group $\mathrm{H}^p_T(D_v, M)$ for real places v. Tate established the following 9-term exact sequence that we will recover as a local cohomology sequence in étale cohomology:

Theorem 1.2.3 *Let M be a finite G_S-module such that $\#M$ is an S-unit. Then there exists a natural exact sequence:*

$$0 \longrightarrow \mathrm{H}^0(G_S, M) \longrightarrow \bigoplus_{v\in S} \mathrm{H}^0(K_v, M) \longrightarrow \mathrm{H}^2(G_S, M^\vee)^D \xrightarrow{f_M} \mathrm{H}^1(G_S, M)$$

$$\longrightarrow \bigoplus_{v\in S} \mathrm{H}^1(K_v, M) \longrightarrow \mathrm{H}^1(G_S, M^\vee)^D \xrightarrow{f'_M} \mathrm{H}^2(G_S, M) \longrightarrow \bigoplus_{v\in S} \mathrm{H}^2(K_v, M)$$

$$\longrightarrow \mathrm{H}^0(G_S, M^\vee)^D \longrightarrow 0$$

where $M^\vee := \mathrm{Hom}(M, \overline{K}^\times)$ is the dual Galois module, $(\cdot)^D$ denotes the $\mathbf{Q}/\mathbf{Z}$-dual of a finite abelian group, and all maps aside from f_M and f'_M are the evident ones.

This exact sequence encodes some deep duality theorems. To see this, define

$$Ш^i_S(K, M) := \ker(\mathrm{H}^i(G_S, M) \to \prod_{v\in S} \mathrm{H}^i(K_v, M))$$

(with the "modified" definition for real v with Tate cohomology as noted above). A "justification" for this notation is that if A is an abelian variety over K with good reduction outside S then the analogous definition $Ш^1_S(K, A) \subset \mathrm{H}^1(G_S, A(K_S))$ actually coincides with $Ш(A)$ inside $\mathrm{H}^1(K, A)$ (because $\mathrm{H}^1(K_v^{\mathrm{un}}/K_v, A(K_v^{\mathrm{un}})) = 0$ for good v, proved using Néronian properties of abelian schemes and Lang's theorem on torsors for *connected* group varieties over finite fields; this local triviality at unramified places has no counterpart for finite coefficient modules M!)

The 9-term exact sequence provides an exact sequence

$$0 \to Ш^2_1(K, M)^D \to \mathrm{H}^2(G_S, M) \to \bigoplus_{v\in S} \mathrm{H}^2(K_v, M) \tag{1}$$

and this expresses exactly a perfect *duality* pairing

$$Ш^2_S(K, M) \times Ш^1_S(K, M^\vee) \to \mathbf{Q}/\mathbf{Z}$$

that will arise for us as an instance of "Poincaré duality" on the "3-manifold" $\mathrm{Spec}(\mathcal{O}_{K,S})$.

As a consequence of the above theorems, we can define a useful notion of Euler-Poincaré characteristic for any M as above, via an alternating product of the sizes $\#\mathrm{H}^p(G_S, M)$. There is a complication: if K has a real place and M has even size then the cohomology groups will be nonzero for an infinite number of values of p in general (see Theorem 1.2.1). We ignore this and simply define

$$\chi(G_S, M) := \frac{\#\mathrm{H}^0(G_S, M) \cdot \#\mathrm{H}^2(G_S, M)}{\#\mathrm{H}^1(G_S, M)}$$

in general (with $\#M$ an S-unit; one can drop this condition on $\#M$ but then more ideas are needed to establish useful results building on Tate's).

By adapting the technique with Grothendieck groups used to prove his local Euler characteristic formula (which is proved in §5.7 of Chapter II of Serre's book on Galois cohomology), Tate proved the following formula (again, with $\#M$ an S-unit):

$$\chi(G_S, M) = \prod_{v \text{ arch}} \frac{\#\mathrm{H}^0(G_v, M)}{|\#M|_v},$$

where $|\cdot|_v$ is the normalized absolute value (which for complex v is understood to be the square of the usual complex absolute value, as needed in the product formula); see [Mi, §5, Ch. I] for the details on this proof (which is not proved in a new way via the étale cohomology method).

Example 1.2.4 Consider $M = \mathbf{Z}/n\mathbf{Z}$ with $n > 0$ and S the set of infinite primes together with those dividing n, so $\mathcal{O}_{K,S} = \mathcal{O}_K[1/n]$. Clearly $\mathrm{H}^0(G_v, M) = M^{G_v} = \mathbf{Z}/n\mathbf{Z}$, and $|\#M|_v$ is equal to n or n^2 depending on whether v is real or complex respectively. The formula yields

$$\chi(G_S, \mathbf{Z}/n\mathbf{Z}) = n^{-s}$$

with s the number of complex places.

Example 1.2.5 To illustrate the power of Tate's global Euler characteristic formula when combined with the 9-term exact sequence, we compute a "formula" for $\#\mathrm{H}^1(G_S, M)$ in general (remembering that $\#M$ is required to be an S-unit!). We have $\#Ш^2_S(K, M) = \#Ш^1_S(K, M^\vee)$ since these two finite groups are Pontryagin dual to each other, and $\#\mathrm{H}^2(K_v, M) = \#\mathrm{H}^0(K_v, M^\vee)$ for all v (by Tate local duality for finite v, and some care for real v). Combining this with the Euler characteristic formula and (1), we get

$$\#\mathrm{H}^1(G_S, M) = \#Ш^1_S(K, M^\vee) \cdot \frac{\#\mathrm{H}^0(G_S, M)}{\#\mathrm{H}^0(G_S, M^\vee)} \cdot \prod_{v \in S} \#\mathrm{H}^0(G_S, M^\vee) \cdot \prod_{v \text{ arch}} \frac{|\#M|_v}{\#\mathrm{H}^0(G_S, M)}.$$

This formula has a mystery term, namely $Ш^1_S(K, M^\vee)$. The merit of this formula is that the groups $Ш^i_S(K, \cdot)$ are more robust than $\mathrm{H}^i(G_S, \cdot)$ for some purposes since they admit a good duality theory (which the $\mathrm{H}^i(G_S, \cdot)$'s do not); it is in this sense that the above formula for $\#\mathrm{H}^1(G_S, M)$ can be useful. (Wiles established a important generalization comparing sizes of such H^1's for M and $M^\vee$ when it is *not* assumed that $\#M$ is an S-unit.)

We have seen that the 9-term exact sequence encodes some duality results among subgroups $Ш^i_S(K, \cdot)$ of G_S-cohomologies. In Tate's original work, he first directly established the duality between $Ш^1_S(K, M^\vee)$ and $Ш^2_S(K, M)$ and then *defined* the 9-term exact sequence in terms of that global duality and his local duality theorems.

In the étale cohomology approach, we will begin with the Artin–Verdier duality theorem stated below that is of entirely different nature, relating cohomology and Ext's (much as in Grothendieck–Serre coherent duality and Verdier's topological duality), applicable to rather general constructible coefficient sheaves. From that foundation, we will obtain the 9-term exact sequence as an exact "local cohomology" sequence and *deduce* the duality between $Ш^1_S(K, M^\vee)$ and $Ш^2_S(K, M)$ from that 9-term sequence (so the logic is *opposite* to Tate's original approach).

For totally imaginary K, Mazur wrote up an account [Ma] with all of the essential arithmetic ideas. But there are many interesting number fields that are not totally imaginary, the most basic being $\mathbf{Q}$ itself. In an appendix to [Hab], Zink introduced the idea of including extra "points" for real places to extend Mazur's aproach so that one recovers the entirety of Tate's results for all number fields. We have provided some additional explanations and define an appropriate δ-functorial notion of $\mathrm{H}^\bullet_c(U, \cdot)$ for dense open $U \subset \mathrm{Spec}(\mathcal{O}_K)$ (which is not addressed in Zink's exposition) to clarify some aspects. This appears in the statement of:

Theorem 1.2.6 (Artin-Verdier) *Let $U \subset X = \mathrm{Spec}(\mathcal{O}_K)$ be a dense open subscheme. There are natural isomorphisms $\mathrm{H}^3_c(U, \mathbf{G}_{m,U}) \simeq \mathrm{H}^3_c(X, \mathbf{G}_{m,X}) \simeq \mathbf{Q}/\mathbf{Z}$ and for constructible F on U*

there is a δ-functorial perfect pairing

$$\mathrm{H}^i_c(U,F)\times \mathrm{Ext}^{3-i}_U(F,\mathbf{G}_{m,U})\to \mathrm{H}^3_c(U,\mathbf{G}_{m,U})=\mathbf{Q}/\mathbf{Z},$$

of finite abelian groups for all $i\in\mathbf{Z}$.

Beware that $\mathrm{H}^i_c(U,F)$ doesn't generally vanish for $i<0$ when there are real places (but by definition the Ext's vanish in negative degrees); see Remark 5.4.5. Also, as is noted in [Ma], there is *no* hypothesis concerning stalks of F having order that is a unit on U!

Theorem 1.2.6 is the basis upon which everything else will depend in the approach of these notes (and it will be easily reduced to the case $U=X$ once the definitions are in place). Incorporating the real places in a systematic way will require quite a bit of preparatory work. But as with any cohomological machine, once the foundations are laid we can reap many rewards from some key calculations.

Notation. For a scheme X, denote by $X_{\text{ét}}$ the small étale site of X; i.e., the category $\mathbf{\acute{E}t}_X$ consists of étale morphisms $U\to X$ together with X-morphisms as arrows, endowed with the étale topology for covers of such U. We denote by $\mathbf{Ab}_X$ the category of abelian sheaves on X_{et}. If X is of finite type over $\mathbf{Z}$, we denote by X^0 the set of closed points of X.

To avoid overload of notation, with any scheme X:
from now on, by "abelian sheaf on X" we shall means an abelian sheaf on X_{et}.

2 Étale topology on algebraic number fields

This section is organized as follows. We briefly review some definitions and basic facts from the theory of abelian sheaves on the small étale site on a scheme, for which the main case of interest is the scheme $\mathrm{Spec}(\mathcal{O}_K)$ and schemes étale over it. We also build an extension of the category of abelian étale sheaves on $\mathrm{Spec}(\mathcal{O}_K)$ so that its objects have "stalks" at real places of K. Finally, we introduce and study an appropriate notion of "finite morphism" in this extended setting, especially the exactness of an associated pushforward functor; that wil be a valuable tool to reduce some problems to the case of totally imaginary K (which don't involve the intervention of real places, so ordinary étale cohomology of schemes does the job).

2.1 Abelian sheaves represented by constant X-schemes

Given an abelian group G, we set $\underline{G}_X$ to be the functor

$$\mathbf{Sch}\to\mathbf{Grp}$$

represented by the X-group $G_X := \coprod_{g\in G} X$; this assigns to any X-scheme Y the group of locally constant functions $|Y|\to G$. This is a sheaf for the étale topology on X because of the openness of étale maps (check!), or in fancier terms because *every* representable functor on the category of X-schemes is a sheaf for the Zariski and fpqc topologies (one of the key theorems of descent theory).

Definition 2.1.1 Every abelian sheaf on X which is represented by an abelian X-group of the form G_X, for some abelian group G, is called *constant.*

We also recall the following:

Definition 2.1.2 An abelian sheaf F on X is called *locally constant* if there exists an étale covering

$$\{f_i : U_i \to X\}_{i \in I}$$

such that each $F|_{U_i} := f_i^*(F)$ is constant as an abelian sheaf on U_i.

Example 2.1.3 Let k be a field, and G its Galois group. For $X = \mathrm{Spec}(k)$, $\mathbf{Ab}_X$ is equivalent to the category of discrete G-modules. By Galois theory, a sheaf is locally constant if an open subgroup of G acts trivially on the corresponding discrete G-module.

For example, if $n \geq 1$ is not divisible by $\mathrm{char}(k)$ then μ_n is locally constant and correspondingly $\mu_n(\overline{k})$ has trivial action by the open subgroup corresponding to the finite extension $k(\zeta_n)$; in contrast, $\mathbf{G}_m$ is not locally constant in general (aside from quirky cases such as separably closed k, or $k = \mathbf{R}$).

2.2 Artin's decomposition lemma and applications

An important source of motivation for how to systematically add "real points" to $\mathrm{Spec}(\mathcal{O}_K)$ is obtained from a result of Artin that describes how to reassemble étale sheaves on a scheme from étale sheaves on constituents of a stratification of the underlying topological space.

To explain this, let X be a scheme (we only need the case of Dedekind X), $j : U \hookrightarrow X$ an open subscheme, and $Z = X - U$ its closed complement (with any closed subscheme structure we wish, such as the reduced structure; it doesn't matter since killing nilpotents has no effect on the étale site and hence no effect on étale sheaf theory). Let $i : Z \hookrightarrow X$ be the canonical closed immersion.

Let F be an abelian sheaf on X, and define $F_U = j^*(F)$ and $F_Z = i^*(F)$. Using the natural map $F \to j_*(F_U)$, applying i^* gives a natural map

$$\varphi_F : F_Z \to i^* j_*(F_U).$$

We now denote by $\mathfrak{C}_{X,U}$ the category whose objects are triples (F', F'', φ) consisting of an abelian sheaf F' on Z, an abelian sheaf F'' on U, and a morphism $\varphi : F' \to i^* j_*(F'')$. We define a *morphism* between two such triples in the evident manner (involving a commutative square). This yields a "decomposition" functor

$$\mathrm{dec} : F \mapsto (F_Z, F_U, \varphi_F)$$

from $\mathbf{Ab}_X$ to the category $\mathfrak{C}_{X,U}$. Artin proved the crucial *decomposition lemma*, which is so important that we record it as a Theorem:

Theorem 2.2.1 (Artin, [Art, §3.2.5]) *The functor* dec *is an equivalence of categories.*

The key to the proof is to answer the puzzle: how to recover F from F_Z and F_U? It will be built as an appropriate fiber product.

Proof. Using the deep theorem that formation of strict henselization of a local ring commutes with passage to quotients by ideals, for an abelian sheaf F' on Z we naturally have

$$(i_*(F'))_{\overline{x}} = \begin{cases} 0 & \text{if } x \notin i(Z) \\ F'_{\overline{x_0}} & \text{if } x = i(x_0) \end{cases}$$

For an abelian sheaf F'' on U we have the more trivial identification $(j_*(F''))_{\overline{x}} = F''_{\overline{x_0}}$ if $x = j(x_0)$, but the stalk at points of Z is rather less easy to write down (it is the group of global sections of the pullback of F'' over the pullback of U along $\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,\overline{x}}) \to X$). We now use these two basic calculations.

Given some F on X_{et}, we claim that the commutative square

$$\begin{array}{ccc} F & \longrightarrow & j_*(F_U) \\ \downarrow & & \downarrow \\ i_*(F_Z) & \longrightarrow & i_*i^*j_*(F_U) \end{array}$$

is Cartesian. In this diagram the vertical maps express the natural transformation $\mathrm{id} \to i_*i^*$ (applied to F and to $j_*(F_U)$), and commutativity just expresses naturality with respect to $F \to j_*(F_U)$. It suffices to check the Cartesian property on stalks at geometric points $\overline{x}$ over $x \in X$.

The situation at stalks over U is trivial because the bottom of the square has vanishing stalks at such points and the top collapses to an isomorphism on such stalks. If instead $x \in Z$ then applying $\overline{x}$-stalks to the natural transformation $\mathrm{id} \to i_*i^*$ always yields an isomorphism between *functors* on abelian sheaves on X_{et}. The Cartesian property is then a tautology as well.

Inspired by this construction, for a triple (F', F'', φ) in $\mathfrak{C}_{X,U}$ we define

$$F := i_*(F') \times_{i_*(\varphi), i_*i^*j_*(F'')} j_*(F'').$$

We need to show this gives an inverse to the "decomposition" functor.

The formation of such a fiber product sheaf commutes with any pullback (left-exactness of pullback), so the restriction F_U over U is naturally identified with F'' (more precisely: the natural projection $F \to j_*(F'')$ restricts over U to an isomorphism $F_U \simeq F''$). Likewise,

$$F_Z = F' \times_{\varphi, i^*i_*i^*(j_*(F''))} i^*j_*(F''),$$

where the second component of the fiber product is given by i^* applied to the natural transformation $h_i : \mathrm{id} \to i_*i^*$ on abelian sheaves on X. But by consideration of stalks we see that $i^*(h_i)$ is always an *isomorphism*, so the fiber-product description of F_Z collapses to make the projection $F_Z \to F'$ an isomorphism.

The upshot is that under the fiber product definition of F, the first projection restricts to an isomorphism over Z and the second restricts to one over U. In particular, the natural map

$$F \to i_*(F_Z) \times j_*(F_U)$$

is injective, and more specifically recovers exactly the *definition* of F as a fiber product inside a direct product! By its definition, F is thereby identified with the fiber product along the two maps

$$i_*(F_Z) \xrightarrow{i^*(\varphi)} i_*i^*j_*(F_U) \xleftarrow{\mathrm{can}} j_*(F_U)$$

where $\varphi : F_Z = F' \to i^*j_*(F'') = i^*j_*(F_U)$ is the initially given map, so we have to show that $\varphi = \varphi_F$. It is equivalent to prove equality of such maps over Z after applying i_*, and it is harmless to check equality after composing with the *surjection* $F \twoheadrightarrow i_*(F_Z)$. This becomes a problem of comparing two composite maps $F \rightrightarrows i_*i^*j_*(F_U)$.

Ah, but by *definition* of F as a fiber product, the composite map resting on φ is equal to the composition of the *canonical* second projection $F \to j_*(F_U)$ with the effect of $\mathrm{id} \to i_*i^*$ applied to $j_*(F_U)$. However, this latter composition is *exactly* what arises in the initial considerations we made when realizing every abelian sheaf H on X as a fiber square using φ_H. Thus, indeed $\varphi = \varphi_F$, so we are done! □

Example 2.2.2 If F'' is an abelian sheaf on U then $j_!(F'')$ corresponds to $(0, F'', \varphi = 0)$. By contrast, $j_*(F'')$ corresponds to the triple $(j_*(F'')|_Z, F'', \varphi)$ where φ is the identity map on $j_*(F'')|_Z$.

Before specializing the decomposition lemma to the global setting of interest, we first discuss an instructive local example.

Example 2.2.3 Let R be a complete (or henselian) discrete valuation ring with fraction field L, and consider $X := \mathrm{Spec}(R)$. Let $G := \mathrm{Gal}(L^{\mathrm{sep}}/L)$, with I the inertia subgroup. Let k be the residue field of R and $G_k := \mathrm{Gal}(k^{\mathrm{sep}}/k)$. Then we claim that the decomposition lemma provides an equivalence of $\mathbf{Ab}_{X_{\text{ét}}}$ onto the category of triples (M, N, φ) with M a discrete G_k-module, N a discrete G-module, and $\varphi : M \to N^I$ a G_k-module homomorphism.

Setting $Z := \mathrm{Spec}(k) \to X$ and $U := \mathrm{Spec}(L) \to X$, clearly $\mathbf{Ab}_Z$ is the category of discrete G_k-modules and $\mathbf{Ab}_U$ is the category of discrete G-modules. The only issue with content is to show that the functor i^*j_* from discrete G-modules to discrete G_k-modules is precisely the formations of I-invariants.

Using the link between stalks at a geometric point and global sections of the pullback to a strictly henselian local ring, the discrete G_k-module associated to $i^*(j_*(N))$ is precisely the G_k-module of global sections of the pullback of $j_*(N)$ along $\pi : X^{\mathrm{sh}} := \mathrm{Spec}(R^{\mathrm{sh}}) \to \mathrm{Spec}(R) = X$ (where G_k acts through its identification with $\mathrm{Aut}(R^{\mathrm{sh}}/R) = G/I!$). But expressing R^{sh} as a direct limit of local finite étale extensions R' inside L^{sep} and using the compatibility of the formation of global sections with passage to limits of qcqs schemes (with affine transition maps) shows that

$$\Gamma(X^{\mathrm{sh}}, \pi^*(j_*(N))) = \varinjlim \Gamma(\mathrm{Spec}(R'), j_*(N)) = \varinjlim \Gamma(L', N) = \varinjlim N^{\mathrm{Gal}(L^{\mathrm{sep}}/L')} = N^I,$$

the final equality because N is a discrete G-module.

Now we finally come to the situation of interest: *for the remainder of these notes* we write $X = \mathrm{Spec}(\mathcal{O}_K)$ for a number field K. Let $\eta : \mathrm{Spec}(K) \to X$ be the generic point, and likewise

$$\overline{\eta} : \mathrm{Spec}(\overline{K}) \to X$$

for a fixed separable closure $\overline{K}$ of K. We set $G := \mathrm{Gal}(\overline{K}/K)$, and for any closed point

$$x : \mathrm{Spec}(\kappa) \to X,$$

we set $G_x := \mathrm{Gal}(\overline{\kappa}/\kappa)$ when an algebraic closure $\overline{\kappa}$ of κ is specified.

For each closed point x, we fix a decomposition subgroup $D_x \hookrightarrow G$, and recall D_x is unique up to conjugation inside G. The choice of D_x amounts to a choice of place on $\overline{K}$ extending the x-adic place on K, and relative to this choice we get an inclusion of $\overline{K}$ into an algebraic closure of the completion K_x at x, thereby identifying D_x with $\mathrm{Gal}(\overline{K_x}/K_x)$. We denote by I_x the inertia subgroup of D_x. This data identifies the residue field of $\overline{K_x}$ (or equivalently of $\overline{K}$ at the chosen place over x) with an algebraic closure of the residue field κ at x, so it defines $\overline{\kappa}$ and identifies G_x with D_x/I_x.

By Example 2.1.3, for any abelian sheaf F on X we identify $F_\eta := \eta^*(F)$ with a discrete G-module, each $F_x := x^*(F)$ with a discrete G_x-module, and the pullback of F over $\mathrm{Spec}(\mathscr{O}_{X,x})$ corresponds to the triple (F_x, F_η, φ_x) where

$$\varphi_x : F_x \to F_\eta^{I_x}$$

is called the *specialization map* at x. The decomposition lemma (Theorem 2.2.1) now becomes:

Lemma 2.2.4 *Let $S = \{x_1, \ldots, x_n\}$ be a finite set of closed points of X, and $U := X - S$. Then for any abelian sheaf F on X, the functor* dec : $\mathbf{Ab}_X \to \mathfrak{C}_{X,U}$ *takes the form:*

$$\mathrm{dec}(F) = \left(\bigoplus_{k=1}^{n} (x_k)_* F_{x_k}, F_U, \bigoplus_{k=1}^{n} \varphi_{x_k} \right)$$

for the specialization maps $\varphi_{x_k} : F_{x_k} \to F_\eta^{I_{x_k}}$ where $F_\eta := \eta^*(F_U)$.

This lemma says that F is determined by: the skyscraper sheaves F_{x_k} ($1 \leq k \leq n$), the sheaf F_U on U, and the specialization maps from each F_{x_k} into the I_{x_k}-invariants of $\eta^*(F_U) = F_\eta$. The case when every φ_{x_k} vanishes corresponds to

$$F = j_!(F_U) \oplus \bigoplus_{k=1}^{n} (x_k)_*(F_{x_k}).$$

Recall that in general a *constructible* abelian étale sheaf on a noetherian scheme is one that becomes locally constant with finite geometric fibers on the constituents of a finite stratification by locally closed sets (and these are *characterized* as the noetherian objects in the category of abelian étale sheaves). Thus, it is a simple exercise to check that an abelian sheaf F on X is *constructible* if it is locally constant with finite fibers on some dense open $U \subset X$ and has stalks F_x that are finite G_x-modules for all $x \in X = U$. If F is locally constant and constructible (equivalently, locally constant with finite stalks) we say F is *lcc*.

Example 2.2.5 A constructible abelian sheaf F is locally constant on an étale neighborhood of a closed point $x \in X$ if and only if two conditions hold: I_x acts trivially on F_η and $\varphi_x : F_x \to F_\eta^{I_x} = F_\eta$ is an isomorphism.

Hence, a constructible abelian etale sheaf F is characterized in terms of the associated data F_η, F_x for closed $x \in X$, and specialization maps $\varphi_x : F_x \to F_\eta^{I_x}$ by the conditions: F_η is finite (hence by G-discreteness it is unramified at all but finitely many x!), every F_x is finite, and φ_x is an isomorphism for all but finitely many x.

Turning this around, if we are given such data

$$(F_\eta, \{F_x\}_{x \in X^0}, \{\varphi_x : F_x \to F_\eta^{I_x}\}_{x \in X^0})$$

satisfying these finiteness and isomorphism conditions then we claim that it arises from a canonically determined constructible F on X. Indeed, the finite F_η comes from a locally constant sheaf F_U on some U_{et} governed by the finitely many ramified places for F_η as a finite discrete G-module, so then by removing from U any closed $u \in U$ for which φ_u isn't an isomorphism, we we equivalently package this data in terms of $(F_U, \{F_x\}_{x \in X-U}, \{\varphi_x\}_{x \in X-U})$ to which the decomposition lemma really is applicable.

2.3 Adjoining the real places to $X = \mathrm{Spec}(\mathcal{O}_K)$

We define $\overline{X}$ to be the union of X and the set $X_\infty := \{v_1, \ldots, v_r\}$ of all real places of K, and we endow $\overline{X}$ with the topology whose closed sets are finite subsets of $\overline{X} - \{\eta\}$ (so X is an open subspace of $\overline{X}$).

Notation. Let $\overline{v}$ be a fixed extension of a real place v to a place on a fixed algebraic closure $\overline{K}$ of K. Denote by D_v the decomposition group of $\overline{v}$, so uniquely $D_v \simeq \mathbf{Z}/2\mathbf{Z}$ since v is real. We define $I_v := D_v$ due to the convention in algebraic number theory that a real place is "unramified" when it is totally split (such as in the definition of Hilbert class fields), so we define $G_v := 1$. In particular, *real points should be treated as if they are geometric points*!

For any dense open subscheme $U \subset X$, we denote by $\overline{U}$ the union of U and X_∞; this is an open subspace of $\overline{X}$ (i.e., it is the result of applying to $U = \mathrm{Spec}(\mathcal{O}_{K,S})$ the same formalism we applied to X when defining $\overline{X}$). We also use the notation $\widetilde{U}$ for a typical open subset of $\overline{X}$ (which may contains several or no real points). Given a choice of $\widetilde{U}$, we denote by $\widetilde{U}_\infty = \{\widetilde{v}_1, \ldots, \widetilde{v}_s\}$ its overlap with X_∞, and define $U := \widetilde{U} \cap X$ (so if $\widetilde{U}$ is *non-empty* then $\overline{U}$ makes sense and contains $\widetilde{U}$).

Next, inspired by Artin's decomposition lemma, we shall now define notions of: "abelian étale sheaf" on non-empty open subsets $\widetilde{U}$ of $\overline{X}$, "stalk" at real places (simply an abelian group, since we view real points as geometric points), and constructibility for such "sheaves":

Definition 2.3.1 An *abelian sheaf* $\widetilde{F}$ on a non-empty open subset $\widetilde{U} \subset \overline{X}$ is a tuple

$$\widetilde{F} := (\{\widetilde{F}_{\widetilde{v}}\}, F, \{\varphi_{\widetilde{v}}\})$$

where $\widetilde{v}$ varies through $\widetilde{U}_\infty$, the $\widetilde{F}_{\widetilde{v}}$ are abelian groups, F is an abelian sheaf on $U := \widetilde{U} \cap X$, and $\varphi_{\widetilde{v}} : \widetilde{F}_{\widetilde{v}} \to F_\eta^{I_{\widetilde{v}}}$ is a group homomorphism. We call $\widetilde{F}_{\widetilde{v}}$ the *stalk* of $\widetilde{F}$ at $\widetilde{v}$, and we also denote by $\widetilde{F}_x$ the stalk F_x at any $x \in U$. The "sheaf" $\widetilde{F}$ is called *constructible* if F is constructible and each $\widetilde{F}_{\widetilde{v}}$ is finite.

Remark 2.3.2 The fixed field $\overline{K}^{I_v}$ is the algebraic closure K_v^{alg} of K inside its completion $K_v \simeq \mathbf{R}$ (isomorphism unique!). In particular, K_v^{alg} is *intrinsic* to K (given v), in contrast with $\overline{K}$. Since $F_\eta^{I_v}$ is identified with the group $F(K_v^{\mathrm{alg}})$ of global sections of the pullback of F over $\mathrm{Spec}(K_v^{\mathrm{alg}})$, we see that it is intrinsic to (K, v) in a way that $F_\eta^{I_x} = F(\overline{K}^{I_x}) = F(\mathrm{Frac}(\mathscr{O}^{\mathrm{sh}}_{X,x}))$ is not for $x \in U^0$ (a geometric point over x is needed to define the strict henselization $\mathscr{O}^{\mathrm{sh}}_{X,x}$).

We can likewise identify $F_\eta^{I_v}$ with the group $F(K_v)$ of global sections of the pullback of F over $\mathrm{Spec}(K_v)$, saving a bit of notation. Consequently, we will sometimes prefer to speak in terms of $F(K_v)$ or $F(K_v^{\mathrm{alg}})$ rather than $F_\eta^{I_v}$ to emphasize that it doesn't depend on base points. However, the notation $F_\eta^{I_v}$ is closer to that at closed points x as in the decomposition lemma (with $D_x \neq I_x$ for $x \in U^0$!).

The abelian category of such "sheaves" is denoted $\mathfrak{Ab}_{\widetilde{U}}$. The reason that we do not (yet) denote it as $\mathbf{Ab}_{\widetilde{U}}$ is that later we will define an *étale site* for $\widetilde{U}$, so the category of abelian sheaves on that site will deserve the name $\mathbf{Ab}_{\widetilde{U}}$; in Theorem 2.4.3 we will show that this naturally coincides with $\mathfrak{Ab}_{\widetilde{U}}$ (after which we shall pass between the two notions interchangeably). At this moment, objects in $\mathfrak{Ab}_{\widetilde{U}}$ are not yet genuinely realized as sheaves on an actual site, so the usual Grothendieck cohomological formalism is *not yet applicable* to them.

In the special case that $\widetilde{U}$ contains no real points, $\mathfrak{Ab}_{\widetilde{U}}$ trivially recovers $\mathbf{Ab}_U$. Our aim is, for now, to gain information on $\widetilde{F}$ from knowledge of information of $F \in \mathbf{Ab}_U$ and the real stalks. In effect, we shall establish a special case of the decomposition lemma in our new framework (but later we will have to do work to establish a general version of the decomposition lemma in our new setting, appropriately interpreted as sheaves on a site).

In the setting of étale sheaf theory on schemes, the functors of pushforward and pullback are related by an adjunction, as are extenson by zero $j_!$ and pullback j^* relative to an open immersion j, and pushforward i_* and extraordinary pullback $i^!$ relative to a closed immersion i (with $\mathrm{Hom}(i_*(F'), \widetilde{F}) = \mathrm{Hom}(F', i^!(\widetilde{F}))$; i.e., $i^!$ is a right adjoint to i_*).

Remark 2.3.3 For usual étale sheaf theory and the inclusion i of a *geometric* closed point, a right-adjoint $i^!$ to i_* is given by the functor "global sections vanishing away from the point".

If instead i is the inclusion of a general closed point and its open complement has associated inclusion j then $\ker(i^* \to i^* j_* j^*) = i^*(\ker(\mathrm{id} \to j_* j^*))$ gives such a right-adjoint $i^!$ to i_*.

Motivated by how these functors are described in terms of the decomposition lemma (e.g., see Example 2.2.2 for extension by zero and pushforward from an open set), for the open embedding of topological spaces $j : U \to \widetilde{U}$ and the closed embedding of topological spaces $i_{\widetilde{v}} : \{\widetilde{v}\} \to \widetilde{U}$ for a real point $\widetilde{v}$ of $\widetilde{U}$ we *define* functors

$$\mathbf{Ab}_U \;\begin{array}{c} \xrightarrow{\;j_!\;} \\ \xleftarrow{\;j^*\;} \\ \xrightarrow{\;j_*\;} \end{array}\; \mathfrak{Ab}_{\widetilde{U}} \;\begin{array}{c} \xrightarrow{\;(i_{\widetilde{v}})^!\;} \\ \xleftarrow{\;(i_{\widetilde{v}})_*\;} \\ \xrightarrow{\;(i_{\widetilde{v}})^*\;} \end{array}\; \mathbf{Ab}$$

as follows (with our notation conventions as used above):

$$\begin{aligned}
j_!(F) &:= (0, \dots, 0, F, 0 \to F_\eta^{I_{\widetilde{v}_1}}, \dots, 0 \to F_\eta^{I_{\widetilde{v}_s}}) \\
j^*(\widetilde{F}) &:= F \\
j_*(F) &:= (F_\eta^{I_{\widetilde{v}_1}}, \dots, F_\eta^{I_{\widetilde{v}_s}}, F, \mathrm{id}_{F_\eta^{I_{\widetilde{v}_1}}}, \dots, \mathrm{id}_{F_\eta^{I_{\widetilde{v}_s}}}) \\
i_{\widetilde{v_k}}^*(\widetilde{F}) &:= \widetilde{F}_{\widetilde{v_k}} \\
i_{\widetilde{v_k}*}(A_k) &:= (0, \dots, 0, A_k, \dots, 0, 0 \to 0, \dots, A_k \to 0, \dots, 0 \to 0) \\
i_{\widetilde{v_k}}^!(\widetilde{F}) &:= \ker(\varphi_{\widetilde{v_k}}).
\end{aligned}$$

It is easy to check that the first three satisfy the expected adjunctions, as do the $i_{\widetilde{v}*}$ and $i_{\widetilde{v}}^*$, and that $i_{\widetilde{v}}^!$ is right adjoint $i_{\widetilde{v}*}$. Hence, these notations are all "justified".

Remark 2.3.4 It is *irrelevant* to try to make sense of such j and $i_{\widetilde{v}}$ as "morphisms" of geometric objects; all that actually matters are the 6 associated functors as defined above. Speaking in terms of j and $i_{\widetilde{v}}$ is mainly for ease of exposition when we want to argue similarly to what is done with actual open immersions and closed immersions between schemes.

Example 2.3.5 A moment's reflection yields a canonical map $\widetilde{F} \to j_* j^*(\widetilde{F})$ that realizes the adjunction between j_* and j^*.

Remark 2.3.6 We can reformulate the map $\varphi_{\widetilde{v}}$ appearing in the definition of $i_{\widetilde{v}}^!$ as follows: for $F := j^*(\widetilde{F})$ clearly

$$i_{\widetilde{v}}^* j_* j^*(\widetilde{F}) = i_{\widetilde{v}}^* j_*(F) = i_{\widetilde{v}}^*(F_\eta^{I_{\widetilde{v}_1}}, \dots, F_\eta^{I_{\widetilde{v}_s}}, F, \mathrm{id}_{F_\eta^{I_{\widetilde{v}_1}}}, \dots, \mathrm{id}_{F_\eta^{I_{\widetilde{v}_s}}}) = F_\eta^{I_{\widetilde{v}}},$$

so $i_{\widetilde{v}}^!(\widetilde{F})$ is identified with the kernel of a map $\varphi_{\widetilde{v}} : i_{\widetilde{v}}^*(\widetilde{F}) \to i_{\widetilde{v}}^* j_*(F)$, and we easily identify this map as $i_{\widetilde{v}}^*$ applied to the *canonical* map $\widetilde{F} \to j_* j^*(\widetilde{F})$ as in Example 2.3.5.

This calculation will ensure that once we have realized $\mathfrak{Ab}_{\widetilde{U}}$ as a genuine sheaf category for some site, the map $\varphi_{\widetilde{v}}$ are built exactly as in the decomposition lemma on X.

Remark 2.3.7 The functors $j_!, j^*, i^*$ and i_* are exact, and therefore their respective right adjoints $j^*, j_*, i_*, i^!$ preserve injectives. It is also clear from the termwise construction of kernels that a map in $\mathfrak{Ab}_{\widetilde{U}}$ has vanishing kernel if the same holds after applying j^* and $i_{\widetilde{v}}^*$ for all $\widetilde{v} \in \widetilde{U}_\infty$. If we let J be an injective object in $\mathbf{Ab}_U$ and $J_{\widetilde{v_k}}$ be an injective abelian group for $k = 1, \dots, s$, then

$$j_*(J) \oplus \bigoplus i_{\widetilde{v_k}*}(J_{\widetilde{v_k}})$$

is injective in $\mathfrak{Ab}_{\widetilde{U}}$, so it is clear that $\mathfrak{Ab}_{\widetilde{U}}$ has enough injectives (and more specifically of this special form). Of course, the existence of enough injectives will also follow from general sheaf theory on sites once we define an appropriate site whose category of abelian sheaves is $\mathfrak{Ab}_{\widetilde{U}}$;

however, for some later theoretical calculations it will be useful to know that "enough" injectives can be found in this explicit form.

Remark 2.3.8 The collection of functors $\{i_x^*\}_{x\in\widetilde{U}}$ has the familiar properties one wants for a collection of "enough stalk functors" on $\mathfrak{Ab}_{\widetilde{U}}$ (with i_x^* valued in $\mathbf{Ab}$ for real x, and valued in the category of abelian sheaves on $\mathrm{Spec}(\kappa(x))$ for $x \in U$): each such functor is exact, we can detect equality for a pair of maps between abelian sheaves by checking equality after applying every i_x^*, and a complex in $\mathfrak{Ab}_{\widetilde{U}}$ is exact if and only if it is exact after applying every i_x^*. These assertions are all obvious by inspection how i_x^* is defined for real x and how the formation of kernels and cokernels and images is expressed in the language of triples that *define* objects of $\mathfrak{Ab}_{\widetilde{U}}$.

2.4 Completed small étale sites

We now wish to show that the category $\mathfrak{Ab}_{\widetilde{U}}$ of abelian sheaves on $\widetilde{U}$ is actually equivalent to the category of abelian sheaves for a suitable Grothendieck topology. This will *provide us with a cohomology theory* on $\widetilde{U}$. One could try to avoid this and impose the later explicit formula for the global sections functor given in Example 2.4.5 out of thin air as a definition, and then define cohomology as its derived functors (since we have seen directly that $\mathfrak{Ab}_{\widetilde{U}}$ has enough injectives). That would be a mistake: avoiding the completed site introduced below would entail a lack the insight and techniques that come for free once we can view $\mathfrak{Ab}_{\widetilde{U}}$ as sheaves on an actual site.

We first note that the set $\{\widetilde{v_1}, \ldots, \widetilde{v_s}\} = \widetilde{U}_\infty$ can be regarded as a subset of

$$U(\mathbf{R}) = \mathrm{Hom}(\mathrm{Spec}(\mathbf{R}), U).$$

In Definition 2.3.1 we rigged the notion of abelian sheaf on $\widetilde{U}$ as if $\widetilde{U}$ were "stratified" by the pair $(U, \widetilde{U}_\infty)$. Therefore, let us introduce the category $\widetilde{\mathbf{Sch}}_X$, whose objects are pairs (Y, Y_∞) for an étale X-scheme Y and a subset $Y_\infty \subset Y(\mathbf{R})$, and morphisms $(Y, Y_\infty) \to (Y', Y'_\infty)$ are morphisms of X-schemes $Y \to Y'$ sending Y_∞ into Y'_∞ (at the level of $\mathbf{R}$-points).

Remark 2.4.1 We are forcing into the definition of $\widetilde{\mathbf{Sch}}_X$ that there is no "residue field extension" at real points (as the latter are meant to behave like geometric points; we want to treat $\mathbf{C}$ as "ramified" over $\mathbf{R}$, and more algebraically $\overline{K}$ as "ramified" over K_v^{alg} for real v).

Remark 2.4.2 The category $\widetilde{\mathbf{Sch}}_X$ has fibered products: for $(Y, Y_\infty) \to (T, T_\infty) \leftarrow (Y', Y'_\infty)$ two morphisms in $\widetilde{\mathbf{Sch}}_X$, by writing $\widetilde{Y} := (Y, Y_\infty)$ and likewise for $\widetilde{T}$ and $\widetilde{Y'}$ we have:

$$\widetilde{Y} \times_{\widetilde{T}} \widetilde{Y'} = (Y \times_T Y', Y_\infty \cup Y'_\infty).$$

By definition, the category $\acute{\mathbf{E}}\mathbf{t}_{\widetilde{U}}$ has as its objects the morphisms $\widetilde{Y} \to \widetilde{U}$ in $\widetilde{\mathbf{Sch}}_X$ as its objects, and the morphisms in this category are defined over $\widetilde{U}$ in the evident manner. A family of morphisms $\{\widetilde{f_i} : \widetilde{Y_i} \to \widetilde{Y}\}$ is a *covering* in $\acute{\mathbf{E}}\mathbf{t}_{\widetilde{U}}$ precisely when $\{f_i : Y_i \to Y\}$ is a covering in $\acute{\mathbf{E}}\mathbf{t}_U$ and moreover

$$\bigcup_{i\in I} f_i(\widetilde{Y}_{i\infty}) = \widetilde{Y}_\infty.$$

These coverings define the *étale topology* on $\widetilde{U}$, so in this way we have defined the small étale site $\widetilde{U}_{\text{ét}}$. We denote by $\mathbf{Ab}_{\widetilde{U}}$ the category of abelian sheaves on $\widetilde{U}_{\text{ét}}$.

The main theorem of this section is:

Theorem 2.4.3 *The categories $\mathfrak{Ab}_{\widetilde{U}}$ and $\mathbf{Ab}_{\widetilde{U}}$ are equivalent.*

We first need to set some notation. First, for a real point $\widetilde{v}$ of $\widetilde{U}$ let $T_{\widetilde{v}}$ be a copy of the category of sets, equipped with the topology generated by jointly surjective families of morphisms. Let

$$j_{\widetilde{U}} : \mathbf{\acute{E}t}_{\widetilde{U}} \to \mathbf{\acute{E}t}_U$$

be the forgetful functor, and

$$i_{\widetilde{v}} : \mathbf{\acute{E}t}_{\widetilde{U}} \to T_{\widetilde{v}}$$

the functor associating to $\widetilde{Y}$ the inverse image of $\widetilde{v}$ under the map

$$\widetilde{Y}_\infty \to \widetilde{U}_\infty.$$

Manifestly, the category $\mathbf{Ab}_{\widetilde{v}}$ of abelian sheaves on $T_{\widetilde{v}}$ is the category of abelian groups. From the functors $j := j_{\widetilde{U}}$ and $i_{\widetilde{v}}$ we obtain functors:

$$\mathbf{Ab}_U \underset{j^*}{\overset{j_*}{\rightleftarrows}} \mathbf{Ab}_{\widetilde{U}} \underset{(i_{\widetilde{v}})_*}{\overset{(i_{\widetilde{v}})^*}{\rightleftarrows}} \mathbf{Ab}_{\widetilde{v}}$$

where j_* is defined by composing a sheaf with the functor j, likewise for $(i_{\widetilde{v}})_*$, and j^* and $i^*_{\widetilde{v}}$ are their respective left adjoints.

Explicitly, j^* is restriction of a sheaf to the full subcategory $\mathbf{\acute{E}t}_U \subset \mathbf{\acute{E}t}_{\widetilde{U}}$, and $(i_{\widetilde{v}})^*(\widetilde{F})$ is the sheaf on $T_{\widetilde{v}}$ whose value on the singleton is the direct limit of the diagram of abelian groups $\widetilde{F}(\widetilde{Y})$ for all $\widetilde{Y} \to \widetilde{U}$ with non-empty $\widetilde{v}$-fiber (and with maps in the diagram arising from morphisms among such $\widetilde{Y}$'s); the same limit is attained by limiting attention to those $\widetilde{Y}$'s with a singleton over $\widetilde{v}$.

Proof. (of Theorem 2.4.3) We want to define a functor $\tau : \mathbf{Ab}_{\widetilde{U}} \to \mathfrak{Ab}_{\widetilde{U}}$ yielding an equivalence of categories. Given M in $\mathbf{Ab}_{\widetilde{U}}$, we shall cook up a triple

$$(\{\widetilde{F}_{\widetilde{v}}\}, F, \{\varphi_{\widetilde{v}}\})$$

with group homomorphisms $\varphi_{\widetilde{v}} : \widetilde{F}_{\widetilde{v}} \to F_\eta^{I_{\widetilde{v}}} = F(K_{\widetilde{v}})$ for all $v \in \widetilde{U}_\infty$.

We denote by $j : U \to \widetilde{U}$ and $i_x : x \to \widetilde{U}$ the natural maps (where x is a closed point, possibly "real"), so we get the triple

$$(\{i^*_{\widetilde{v}}(M)\}, j^*(M), \{\varphi_{\widetilde{v}}\})$$

with $\varphi_{\widetilde{v}} : i^*_{\widetilde{v}}(M) \to i^*_{\widetilde{v}} j_* j^*(M)$ the canonical "specialization" maps. Clearly $i^*_{\widetilde{v}}(M)$ is an abelian group for all $\widetilde{v}$, $j^*(M)$ an abelian sheaf on U, and $\varphi_{\widetilde{v}}$ is a group homomorphism.

We now check that $\varphi_{\widetilde{v}}$ has target of the required form. Clearly

$$i^*_{\widetilde{v}} j_* j^*(M)(\{\widetilde{v}\}) = \varinjlim_{(W,w)} j^*(M)(W)$$

with objects W in $\mathbf{\acute{E}t}_{\widetilde{U}}$ and $w \in W(\mathbf{R})$ mapping to $\widetilde{v}$; the limit can be interpreted as evaluating $j^*(M)$ at a maximal extension of K *unramified* at $\widetilde{v}$. Since $\overline{K}^{I_{\widetilde{v}}}$ is such extension, we get

$$i^*_{\widetilde{v}} j_* j^*(M)(\{\widetilde{v}\}) = (j^*M)_\eta^{I_{\widetilde{v}}}$$

as desired. We can now define

$$\tau(M) := (\{i^*_{\widetilde{v}}(M)\}, j^*(M), \{\varphi_{\widetilde{v}} : i^*_{\widetilde{v}}(M) \to (j^*M)_\eta^{I_{\widetilde{v}}}\})$$

and it is clear that τ is functorial in M with values in the category $\mathfrak{Ab}_{\widetilde{U}}$.

We seek a quasi-inverse. For ease of notation, suppose $\widetilde{U}$ contains one real point $\widetilde{v}$. (The general case will go similarly, just with more notation.) Let $\widetilde{F}$ be the object of $\mathfrak{Ab}_{\widetilde{U}}$ given by a triple

$$(\{\widetilde{F}_{\widetilde{v}}\}, F, \{\varphi_{\widetilde{v}}\}).$$

In the spirit of the proof of the decomposition lemma, define

$$M := i_*(\widetilde{F}_{\widetilde{v}}) \times_{i_*(\varphi_{\widetilde{v}}), i_* i^* j_*(F)} j_*(F)$$

where we ease the notation by writing $i := i_{\widetilde{v}}$ and $j := j_U$. Heuristically, $\widetilde{F}_{\widetilde{v}}$ is viewed as an abelian sheaf on the punctual site $\{v\}$ and the inclusion i "yields" the functor $i_* : \mathbf{Ab} \to \mathbf{Ab}_{\widetilde{U}}$.

We would like to argue exactly the same way as in the proof of Theorem 2.2.1, and to do so we need stalk functors for every x point in $\widetilde{U}$, with the usual desired properties. For $x \in U$ and an abelian sheaf $\widetilde{F}$ on $\widetilde{U}$ we define

$$\widetilde{F}_x := (j^* \widetilde{F})_x;$$

this inherits the usual exactness properties from the classical theory of stalks for étale sheaves.

Consider x a real point of $\widetilde{U}$, so a cofinal system of étale neighbourhoods of x is of the form: (W, w), with $w \in W(\mathbf{R})$ mapping to x and $W \to U$ étale. Maps between such pairs are defined in the evident manner, and by definition

$$\widetilde{F}_x := \varinjlim_{(W,w)\to\widetilde{U}} \widetilde{F}(W, w),$$

for which exactness is immediate. By design of the pullback functors i_x^*, it is readily checked that $\widetilde{F}_x = i_x^*(\widetilde{F})$ in accordance with étale sheaf theory on schemes. Now the argument in the proof of Theorem 2.2.1 applies. □

Next, we record the Decomposition Lemma adapted to the study of sheaves on the étale site of $\widetilde{U}$. (This goes beyond the version for schemes, but is easily deduced from that due to what has been shown above, including Remark 2.3.6).

Lemma 2.4.4 *Let $j : \widetilde{U}' \to \widetilde{U}$ be a non-empty open subset, and $S = \{v_1, \ldots, v_n\}$ the complement of $\widetilde{U}'$ in $\widetilde{U}$. For any abelian sheaf $\widetilde{F}$ on $\widetilde{U}$, the assignment:*

$$\mathrm{dec}(\widetilde{F}) := (\widetilde{F}_{v_1}, \ldots, \widetilde{F}_{v_n}, j^*(\widetilde{F}), \varphi_{v_1} : \widetilde{F}_{v_1} \to \widetilde{F}_\eta^{I_{v_1}}, \ldots, \varphi_{v_n} : \widetilde{F}_{v_n} \to \widetilde{F}_\eta^{I_{v_n}})$$

with $\widetilde{F}_{v_k}$ being the discrete G_{v_k}-module $i_{v_k}^*(\widetilde{F})$ for each k, and

$$\varphi_{v_k} : \widetilde{F}_{v_k} = i_{v_k}^*(\widetilde{F}) \to i_{v_k}^* \eta_* \eta^*(\widetilde{F}) = \widetilde{F}_\eta^{I_{v_k}}$$

the natural G_{v_k}-module homomorphism for all k, is a functor which yields an equivalence between the category $\mathbf{Ab}_{\widetilde{U}}$ and the category of tuples:

$$(M_1, \ldots, M_n, F', \varphi_1, \ldots, \varphi_n)$$

with M_k a discrete G_{v_k}-module for each k, F' an abelian sheaf on $\widetilde{U}'$, and $\varphi_k : M_k \to F'^{I_{v_k}}_\eta$ a G_{v_k}-module homomorphism for all k.

To appreciate the power of Theorem 2.4.3, we discuss an example.

Example 2.4.5 The constant sheaf $\mathbf{Z}_{\widetilde{U}}$ on the site $\widetilde{U}$ corresponds to $(\{\mathbf{Z}\}_{\widetilde{v}}, \mathbf{Z}_U, \{\mathrm{id}_{\mathbf{Z}}\}_{\widetilde{v}}) \in \mathfrak{Ab}_{\widetilde{U}}$. Since $\widetilde{F}(\widetilde{U}) = \mathrm{Hom}(\mathbf{Z}_{\widetilde{U}}, \widetilde{F})$, we immediately obtain the formula

$$\widetilde{F}(\widetilde{U}) = F(U) \times_{F_\eta} \widetilde{F}_{\widetilde{v_1}} \times_{F_\eta} \cdots \times_{F_\eta} \widetilde{F}_{\widetilde{v_s}},$$

where the fibre product is taken with respect to the natural map $F(U) \to F_\eta$ and the specialization maps $\varphi_{\widetilde{v_k}} : \widetilde{F}_{\widetilde{v_k}} \to F_\eta$. The right side thereby *defines* the "global sections" functor $\Gamma(\widetilde{U}, \cdot)$ on the category of triples $\mathfrak{Ab}_{\widetilde{U}}$.

2.5 Push/pull functors, finite morphisms, and traceable sheaves

We finish this preliminary discussion by introducing pushforward and pullback functors beyond the setting open (and closed) immersions; they will be especially useful in the case of *finite* morphisms (appropriately defined!). Fix a pair of number fields K' and K and the associated schemes $X' = \mathrm{Spec}(\mathcal{O}_{K'})$ and $X = \mathrm{Spec}(\mathcal{O}_K)$. We have the corresponding "spaces" $\overline{X}$ and $\overline{X}'$ and associated étale sheaf theories.

[It would be more natural to permit K and K' to be finite étale $\mathbf{Q}$-algebras, allowing disconnectedness of the spaces, as disconnectedness arises when forming fiber products. However, this will not be necessary for our needs and it overloads the notation too much to keep track of the connected components and associated abundance of generic points. Hence, we impose connectedness throughout.]

For non-empty open subsets $\widetilde{U}' \subset \overline{X}'$ and $\widetilde{U} \subset \overline{X}$ with respective "finite parts" $U' \subset X'$ and $U \subset X$ (complement of the respective real points in each), a *morphism* $\widetilde{f} : \widetilde{U}' \to \widetilde{U}$ is defined to be a morphism $f : U' \to U$ such that on $\mathbf{R}$-points f carries $\widetilde{U}'$ into $\widetilde{U}$. Composition of such morphisms is defined in the evident manner.

Definition 2.5.1 Let $\widetilde{f} : \widetilde{U}' \to \widetilde{U}$ be a morphism. The *pullback* $\widetilde{f}^* : \mathfrak{Ab}_{\widetilde{U}} \to \mathfrak{Ab}_{\widetilde{U}'}$ is

$$\widetilde{f}^*(\{\widetilde{F}_{\widetilde{v}}\}, F, \{\varphi_{\widetilde{v}}\}) = (\{\widetilde{F'}_{\widetilde{v}'}\}, F', \{\varphi'_{\widetilde{v}'}\})$$

where $F' = f^*(F)$, $\widetilde{F'}_{\widetilde{v}'} = \widetilde{F}_{\widetilde{f}(\widetilde{v}')}$, and $\varphi'_{\widetilde{v}'} = \varphi_{\widetilde{f}(\widetilde{v}')}$. The *pushforward* $\widetilde{f}_* : \mathfrak{Ab}_{\widetilde{U}'} \to \mathfrak{Ab}_{\widetilde{U}}$ is

$$\widetilde{f}_*(\{\widetilde{F'}_{\widetilde{v}'}\}, F', \{\varphi'_{\widetilde{v}'}\}) = (\{\widetilde{F}_{\widetilde{v}}\}, F, \{\varphi_{\widetilde{v}}\})$$

where $F = f_*(F')$,

$$\widetilde{F}_{\widetilde{v}} = (\bigoplus_{\widetilde{v}' \in \widetilde{f}^{-1}(\widetilde{v})} \widetilde{F'}_{\widetilde{v}'}) \times_{\bigoplus_{\widetilde{v}' \in \widetilde{f}^{-1}(\widetilde{v})} F'(K'_{v'})} (f_*F')(K_v) = (\bigoplus_{\widetilde{v}' \in \widetilde{f}^{-1}(\widetilde{v})} \widetilde{F'}_{\widetilde{v}'}) \times_{F'(\prod_{v' \mapsto v} K'_{v'})} F(K_v)$$

(using the maps $\varphi'_{\widetilde{v}'}$ for the first component of the fiber product and the natural projection from $(f_*F')(K_v) = F'(K' \otimes_K K_v) = F(K_v)$ onto its "real factors" for the second component), and for $\widetilde{v} \in \widetilde{U}$ the map $\varphi_{\widetilde{v}} : \widetilde{F}_{\widetilde{v}} \to F_\eta^{I_v} = F(K_v)$ is the second projection from the fiber product.

Remark 2.5.2 If w' varies through the *non-real* places of K' over a real place v of K then

$$(\widetilde{f}_*\widetilde{F'})_{\widetilde{v}} = \bigoplus_{\widetilde{v}' \mapsto \widetilde{v}} \widetilde{F'}_{\widetilde{v}'} \times \bigoplus_{w'} F'(K'_{w'}).$$

The appearance of such places w' in general will underlie the adjointness between $\widetilde{f}_*$ and $\widetilde{f}^*$! (Think about the case when K' has *no* real places to see that complex places of K' over v must be taken into account to possibly have adjoint functors over v.)

Example 2.5.3 For open $\widetilde{V} \subset \widetilde{U}$ there is an evident notion of *preimage* $\widetilde{f}^{-1}(\widetilde{V}) \subset \widetilde{U}'$, and building on Example 2.4.5 it is very instructive (and a good exercise!) to build a natural isomorphism

$$(\widetilde{f}_*(\widetilde{F'}))(\widetilde{V}) = \widetilde{F'}(\widetilde{f}^{-1}(\widetilde{V})).$$

This is rather interesting: the appearance of complex stalks in the definition of $\widetilde{f}_*$ is wiped out at the level of global sections as on the right side; this apparent paradox is resolved by the role of fiber products over the generic stalk in the description of the global-sections functors. This formula for $(\widetilde{f}_*(\widetilde{F'}))(\widetilde{V})$ with varying open $\widetilde{V} \subset \widetilde{U}$ cannot be used to *define* $\widetilde{f}_*$ (why not?).

For a second morphism $\widetilde{F'} : \widetilde{U}'' \to \widetilde{U}'$ involving a third number field K'', we define the isomorphism $\widetilde{f}_* \circ \widetilde{F'}_* \simeq (\widetilde{f} \circ \widetilde{F'})_*$ in the evident manner, and this satisfies an obvious associativity condition.

Lemma 2.5.4 *The functor* $\widetilde{f}^*$ *is exact and is a left adjoint to* $\widetilde{f}_*$.

Proof. It is elementary to check that exactness of $\widetilde{f}^*$ by computing on stalks (see Remark 2.3.8!), and the left-adjoint property is a straightforward diagram-chase (underlying the reason that we incorporate the stalks of F' at *all* points of $\mathrm{Spec}(K' \otimes_K K_v)$ in the definition of $\widetilde{f}_*(\widetilde{F'})_{\widetilde{v}}$!) $\square$

Example 2.5.5 The $\widetilde{v}$-stalk of the kernel of $\widetilde{F} \to \widetilde{f}_* \widetilde{f}^*(\widetilde{F})$ vanishes if v lifts to a real place of K' and otherwise it is $\ker \varphi_{\widetilde{v}} = i_{\widetilde{v}*} i^!_{\widetilde{v}}(\widetilde{F})$ (since the stalk of $f^*(F)$ at any complex place over v computes the geometric points of the generic fiber F_η). That the kernel can have a nonzero stalk at a real place that does not lift to a real place is geometrically reasonable.

Definition 2.5.6 Let $\widetilde{f} : \widetilde{U}' \to \widetilde{U}$ be a morphism, with $K \to K'$ the associated injection arising from the dominant map $f : U' \to U$. We say that $\widetilde{f}$ is *finite* if f is finite and $\widetilde{U}'_\infty$ coincides with the preimage of $\widetilde{U}_\infty$ under the map $X'(\mathbf{R}) \to X(\mathbf{R})$ induced by K'/K.

Put in other words, if K'/K is a given finite extension and $\widetilde{U}$ is a dense open subset of $\overline{X}$ then there is an evident notion of "preimage" of $\widetilde{U}$ in $\overline{X}'$ and finiteness says exactly that $\widetilde{U}'$ is that preimage (informally, it says that $\widetilde{U}'$ is the "normalization" of $\widetilde{U}$ in $\overline{X}'$). One subtlety is that if K has real places but K' is totally imaginary then $\widetilde{f}^*(\widetilde{F})$ loses all information about the stalks of $\widetilde{F}$ at real points of $\widetilde{U}$, so the natural map $\widetilde{F} \to \widetilde{f}_* \widetilde{f}^*(\widetilde{F})$ is the zero map on stalks at real points of $\widetilde{U}$ that do not lift to real points of $\widetilde{U}'$.

The purpose of the preceding abstract nonsense is so that we can obtain:

Proposition 2.5.7 *If* $\widetilde{f} : \widetilde{U}' \to \widetilde{U}$ *is finite then* $\widetilde{f}_*$ *is exact and*

$$\ker(\widetilde{F} \to \widetilde{f}_* \widetilde{f}^*(\widetilde{F})) = \bigoplus_{v \in \Sigma} i_{\widetilde{v}*} i^!_{\widetilde{v}}(\widetilde{F})$$

where Σ *is the set of real places of* $\widetilde{U}$ *that do not lift to a real place of* $\widetilde{U}'$.

In particular, if $U \subset X$ *and* $U' \subset X'$ *are dense opens with* $f : U' \to U$ *a finite map and* $\overline{f} : \overline{U}' \to \overline{U}$ *the associated finite morphism then*

$$\ker(\overline{F} \to \overline{f}_* \overline{f}^* \overline{F}) = \bigoplus_{v \in U(\mathbf{R}) - f(U'(\mathbf{R}))} i_{\widetilde{v}*} i^!_{\widetilde{v}}(\overline{F}).$$

Proof. This is an elementary computation on stalks (see Remark 2.3.8), requiring just a bit of attention to the role of complex stalks in the definition of $\widetilde{f}_*$. The determination of the kernel is a matter of unwinding definitions. $\square$

Example 2.5.8 If $\widetilde{F}_{\widetilde{v}} = 0$ for all real points $\widetilde{v} \in \widetilde{U}$ then $\widetilde{F} \to \widetilde{f}_* \widetilde{f}^* \widetilde{F}$ has trivial kernel. But beware that $\widetilde{f}_* \widetilde{f}^* \widetilde{F}$ generally has nonzero stalks at such $\widetilde{v}$ (if K' has complex places over v).

Finally, we consider $f : U' \to U$ that is a connected finite étale cover and ask when the usual trace map $\mathrm{tr} : f_* f^* \to \mathrm{id}$ extends to a "trace map" $\overline{f}_* \overline{f}^* \to \mathrm{id}$. Over U we know what to do, but how should we make a definition for stalks at a real point $\widetilde{v}$? If Σ is the set of real places on K' over v and Σ' is the set of non-real places over such a v then on $\widetilde{v}$-stalks we seek a suitable map

$$\prod_{v' \in \Sigma} \widetilde{F}_{\widetilde{v}'} \times \prod_{w' \in \Sigma'} F_\eta^{I_{w'}} \to \widetilde{F}_{\widetilde{v}}.$$

There is no apparent such map when Σ' is non-empty except if $\varphi_{\widetilde{v}} : \widetilde{F}_{\widetilde{v}} \to F_\eta^{I_v} = F(K_v)$ is an isomorphism, in which case $(\overline{f}_* \overline{f}^* \overline{F})_{\widetilde{v}} = F(K' \otimes_K K_v)$ and hence we may *define* the $\widetilde{v}$-stalk map

$$\mathrm{tr}_{\widetilde{v}} : F(K' \otimes_K K_v) \to F(K_v)$$

to be the map on K_v-points by the usual trace map $f_* f^*(F) \to F$. This motivates:

Definition 2.5.9 A sheaf $\overline{F}$ on $\overline{U}$ is *traceable* if $\varphi_{\widetilde{v}} : \overline{F}_{\widetilde{v}} \to F_\eta^{I_v} = F(K_v)$ is an isomorphism for all real points v of U. For such $\overline{F}$ that are locally constant and constructible on U, and connected finite étale cover $f : U' \to U$, the *trace map*

$$\mathrm{tr}_{f,\overline{F}} : \overline{f}_* \overline{f}^* (\overline{F}) \to \overline{F}$$

is defined to be the usual trace $f_* f^* F \to F$ over U and its effect on K_v-points at each real place v (so the necessary diagrams commute, hence $\mathrm{tr}_{f,\overline{F}}$ is a morphism of sheaves on $\overline{U}$).

Remark 2.5.10 It is clear that $\overline{F} \rightsquigarrow \overline{F}|_U$ is an equivalence between the categories of traceable sheaves on $\overline{U}$ and abelian sheaves on U. The inverse functor is j_* where $j : U \to \overline{U}$ is the natural "map"; explicitly, $j_*(F) = (\{F(K_v)\}, F, \{\mathrm{id}\})$. Note that $\overline{F}$ is locally constant and constructible if and only if F is.

For such $\overline{F}$ that are locally constant and constructible and any connected finite étale cover $f : U' \to U$, the composition $\overline{F} \to \overline{F}$ of $\mathrm{tr}_{f,\overline{F}}$ with the natural map $\overline{F} \to \overline{f}_* \overline{f}^* \overline{F}$ coincides with multiplication by the degree of f, as we may verify by computing over U since $\overline{F}$ is lcc (or can do directly on stalks separately at real points and usual points of U). This will be *very useful* to reduce some later cohomological problems with K to the case of a totally imaginary extension K'/K that is easier to work with (e.g., no real points).

3 Cohomology theory

We now develop what is necessary for étale cohomology over $\widetilde{U}$ with real points $\widetilde{v}_1, \ldots, \widetilde{v}_s$.

3.1 Two basic δ-functors

Since $\Gamma(\widetilde{U}, \cdot)$ is left-exact, we can make the habitual definition:

Definition 3.1.1 For any abelian sheaf $\widetilde{F}$ in $\mathbf{Ab}_{\widetilde{U}}$, we set:

$$\mathrm{H}^p(\widetilde{U}, \widetilde{F}) := \mathrm{R}^p\Gamma(\widetilde{U}, \widetilde{F}), \ p \geq 0.$$

For $x \in \widetilde{U}_\infty$, recall from Remark 2.3.6 that the functor

$$\widetilde{F} \rightsquigarrow \ker(\varphi_x : \widetilde{F}_x \to i_x^* j_* j^*(\widetilde{F}))$$

has the right-adjunction property to deserve the notation $i_x^!$. More importantly, in view of the explicit formula describing $\Gamma(\widetilde{U}, \cdot)$ in Example 2.4.5, we see that if $\widetilde{V} = \widetilde{U} - \{x\}$ then the functor

$$\Gamma_x(\widetilde{U}, \cdot) := \ker(\Gamma(\widetilde{U}, \cdot) \to \Gamma(\widetilde{V}, \cdot))$$

of "global sections with supports in x" coincides with $i_x^!$. Thus, the functor

$$\mathrm{H}_x^\bullet(\widetilde{U}, \cdot) = \mathrm{R}^\bullet\Gamma_x(\widetilde{U}, \cdot)$$

of *local cohomology with supports in* $x \in \widetilde{U}_\infty$ coincides with $\mathrm{R}^\bullet i_x^!$.

For a usual closed point $x \in U$, the functor $\Gamma_x(\widetilde{U}, \cdot)$ of "global sections with supports in x" is the composition of $\Gamma(x, \cdot)$ with $i_x^!$. The modified étale site treats the real points as if they are geometric points, but the usual closed points of U are cohomologically *nontrivial* (so the output of $i_x^!$ must be viewed as a sheaf or Galois module, not just an abelian group, when $x \notin \widetilde{U}_\infty$).

3.2 The local cohomology sequence

Let $j : \widetilde{V} \hookrightarrow \widetilde{U}$ be a non-empty open subset, and call S its complement.

For any abelian sheaf $\widetilde{F}$ on $\widetilde{U}$, we set:

$$\Gamma_S(\widetilde{U}, \widetilde{F}) := \ker(\widetilde{F}(\widetilde{U}) \to \widetilde{F}(\widetilde{V})).$$

It is easy to see that the functor

$$\Gamma_S(\widetilde{U}, \cdot) : \mathbf{Ab}_{\widetilde{U}} \to \mathbf{Ab}$$

from the category of abelian sheaves on $\widetilde{U}$ to the category of abelian groups is a left-exact functor. We denote by $\mathrm{H}_S^p(\widetilde{U}, \cdot)$ the right-derived functors $\mathrm{R}^p\Gamma_S(\widetilde{U}, \cdot)$.

For $\widetilde{I}$ an injective sheaf on $\widetilde{U}$, we have that the sequence:

$$0 \to \Gamma_S(\widetilde{U}, \widetilde{I}) \to \Gamma(\widetilde{U}, \widetilde{I}) \to \Gamma(\widetilde{V}, \widetilde{I})$$

is also right-exact. To see this, we claim it is enough to consider sheaves of the form described in Remark 2.3.7. Indeed, if $\widetilde{I}$ is contained into another abelian sheaf $\widetilde{J}$, then it is a direct summand of this latter, by injectivity, and it suffices to treat $\widetilde{J}$. But by Remark 2.3.7, every abelian sheaf on $\widetilde{U}$ can be embedded into one of the injectives described there as an abelian subsheaf, and we conclude. The right-exactness of the above sequence for this explicit class of injectives is clear, because it boils down to the analogous claim for injectives on $U_{\text{ét}}$.

We now let $\widetilde{F} \to \widetilde{I}^\bullet$ be an injective resolution, and take cohomology of the short-exact sequence of complexes:

$$0 \to \Gamma_S(\widetilde{U}, \widetilde{I}^\bullet) \to \Gamma(\widetilde{U}, \widetilde{I}^\bullet) \to \Gamma(\widetilde{V}, \widetilde{I}^\bullet) \to 0$$

It yields the following *local cohomology sequence*

$$\cdots \to \bigoplus_{x \in S} \mathrm{H}_x^p(\widetilde{U}, \widetilde{F}) \to \mathrm{H}^p(\widetilde{U}, \widetilde{F}) \to \mathrm{H}^p(\widetilde{V}, \widetilde{F}|_{\widetilde{V}}) \to \cdots$$

Remark 3.2.1 A consequence of the local cohomology sequence is that étale cohomology of $\widetilde{U}$ commutes with filtered colimits. Indeed, it reduces this to the case of usual étale cohomology of U, and cohomology of I_v, which are both known. This will be used without comment!

Remark 3.2.2 It is natural to wonder if the connecting map $\mathrm{H}^p(\widetilde{V}, \widetilde{F}|_{\widetilde{V}}) \to \mathrm{H}^{p+1}_S(\widetilde{U}, \widetilde{F})$ is δ-functorial in the sense that for a short exact sequence

$$0 \to \widetilde{F}' \to \widetilde{F} \to \widetilde{F}'' \to 0$$

of abelian sheaves on $\widetilde{V}$, the diagram of connecting maps

$$\begin{array}{ccc} \mathrm{H}^p(\widetilde{V}, \widetilde{F}''|_{\widetilde{V}}) & \longrightarrow & \mathrm{H}^{p+1}_S(\widetilde{U}, \widetilde{F}'') \\ \downarrow & & \downarrow \\ \mathrm{H}^{p+1}(\widetilde{V}, \widetilde{F}'|_{\widetilde{V}}) & \longrightarrow & \mathrm{H}^{p+2}_S(\widetilde{U}, \widetilde{F}') \end{array}$$

commutes. This is not an entirely idle question: it will arise later when we need to convert an abstract isomorphism into *concrete terms* to recover one of Tate's theorems.

The diagram actually does not commute, but it comes close: it *anti-commutes*. This sign issue is a well-known phenomenon for the analogous situation in usual topological and étale cohomologies, and by going back to the definitions of the maps it expresses a basic fact in homological algebra: given a commuting diagram of short exact sequences of complexes

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

in an abelian category, the resulting diagram of connecting maps from the snake lemma

$$\begin{array}{ccc} \mathrm{H}^n(C'') & \longrightarrow & \mathrm{H}^{n+1}(A'') \\ \downarrow & & \downarrow \\ \mathrm{H}^{n+1}(C') & \longrightarrow & \mathrm{H}^{n+2}(A) \end{array}$$

anti-commutes; see Exercise 10.2.6 in Weibel's book on homological algebra for a broader context on the significance of this fact.

Setting $\widetilde{V} := \widetilde{U} - \widetilde{U}_\infty$, we have

$$\mathrm{H}^p(\widetilde{V}, \widetilde{F}|_{\widetilde{V}}) = \mathrm{H}^p(U, F).$$

It follows that problems for $\mathrm{H}^p(\widetilde{U}, \widetilde{F})$ (such as finiteness properties) can sometimes be reduce to separate consideration of usual étale cohomology groups and of local cohomology with support at real points. We deal with this latter case in the following:

Lemma 3.2.3 *For $\widetilde{v} \in \widetilde{U}_\infty$, we have canonical isomorphisms:*

$$\mathrm{H}^0_{\widetilde{v}}(\widetilde{U}, \widetilde{F}) = \ker(\varphi_{\widetilde{v}}), \quad \mathrm{H}^1_{\widetilde{v}}(\widetilde{U}, \widetilde{F}) = \mathrm{coker}(\varphi_{\widetilde{v}})$$

$$\mathrm{H}^p_{\widetilde{v}}(\widetilde{U}, \widetilde{F}) = \mathrm{H}^{p-1}(I_{\widetilde{v}}, \widetilde{F}_\eta), \quad p \geq 2.$$

In this latter isomorphism, the right side is group cohomology.

Proof. We only need to check that the right sides form a δ-functor such that

$$\mathrm{H}^0_{\widetilde{v}}(\widetilde{U}, \widetilde{F}) = \ker(\varphi_{\widetilde{v}})$$

and the "erasability" property

$$\mathrm{coker}(\varphi_{\widetilde{v}}) = \mathrm{H}^{p-1}(I_{\widetilde{v}}, \widetilde{F}_\eta) = 0, \quad p \geq 2$$

holds for all injective sheaves $\widetilde{F}$.

In §3.1 we reviewed the reason that $\mathrm{H}^0_{\widetilde{v}}(\widetilde{U}, \widetilde{F}) = \ker(\varphi_{\widetilde{v}})$. Likewise, if $\widetilde{F}$ is injective then to show that $\mathrm{coker}(\varphi_{\widetilde{v}}) = 0$ it suffices to show that $\widetilde{F} \to j_* j^*(\widetilde{F})$ is an epimorphism. If $\widetilde{F}$ is embedded in another abelian sheaf then it splits off as a direct summand, and so this epimorphism problem is reduced to the "cofinal" system of injectives as built in Remark 2.3.7. For those specific injectives the epimorphism property is obvious.

Now consider a short-exact sequence of abelian sheaves on $\widetilde{U}$:

$$0 \to \widetilde{F'} \to \widetilde{F} \to \widetilde{F''} \to 0$$

Using the long-exact sequence for group cohomology, we find a commutative diagram with exact lines:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{F'}_{\widetilde{v}} & \longrightarrow & \widetilde{F}_{\widetilde{v}} & \longrightarrow & \widetilde{F''}_{\widetilde{v}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi'_{\widetilde{v}}} & & \downarrow{\scriptstyle \varphi_{\widetilde{v}}} & & \downarrow{\scriptstyle \varphi''_{\widetilde{v}}} & & \\
0 & \longrightarrow & \mathrm{H}^0(I_{\widetilde{v}}, \widetilde{F'}_\eta) & \longrightarrow & \mathrm{H}^0(I_{\widetilde{v}}, \widetilde{F}_\eta) & \longrightarrow & \mathrm{H}^0(I_{\widetilde{v}}, \widetilde{F''}_\eta) & \longrightarrow & \mathrm{H}^1(I_{\widetilde{v}}, \widetilde{F'}_\eta) \longrightarrow \cdots
\end{array}$$

Taking the ker-coker sequence on each column, we deduce that the right sides in the assertions of the Lemma indeed form a δ-functor.

Using Remark 2.3.7 once again, and noting that tautologically $i^*_{\widetilde{v}}(M)_\eta = 0$ for any abelian group M, we are reduced to showing that any injective sheaf F on U satisfies

$$\mathrm{H}^p(I_{\widetilde{v}}, F_\eta) = 0$$

for all $p \geq 1$. Here we are computing cohomology on the field $\overline{K}^{I_{\widetilde{v}}}$ of real algebraic numbers over K. This field is the directed limit of Zariski-localized rings of integers of number fields inside $\overline{K}^{I_{\widetilde{v}}}$. This corresponds via Spec with the inverse system of connected étale affine U-schemes Y whose function field is equipped with an embedding into $\overline{K}^{I_{\widetilde{v}}}$. Since étale cohomology of qcqs schemes commutes with inverse limits having affine transition maps,

$$\mathrm{H}^p(I_{\widetilde{v}}, F_\eta) = \varinjlim_{Y} \mathrm{H}^p(Y, F|_Y).$$

Pullback along étale maps $Y \to U$ preserves injectives, so the colimit on the right side vanishes for all $p \geq 1$. □

As an immediate corollary of the local cohomology sequence and the fact that $\mathrm{H}^p(I_{\widetilde{v}}, M) = 0$ for $p \geq 1$ when M is an $I_{\widetilde{v}}$-module on which 2 acts invertibly (e.g., a torsion group with trivial 2-primary part), we get:

Corollary 3.2.4 *If* 2 *acts invertibly on* F_η *then*

$$\mathrm{H}^p(\widetilde{U}, \widetilde{F}) = \mathrm{H}^p(U, F)$$

for all $p \geq 2$.

3.3 Cohomology of $\mathbf{G}_{m,\overline{U}}$

Let $U \subset X$ be a dense open subscheme. We shall define the multiplicative group $\mathbf{G}_{m,\overline{U}}$ over $\overline{U}$ and compute its cohomology. Recall that $\overline{U}$ contains all real points. In §5.1 we will motivate a new canonical procedure to extend abelian sheaves F on U to abelian sheaves $\widehat{F}$ on $\overline{U}$ (not extension-by-zero from U, nor pushforward from U; it will be right-exact and generally not left-exact!). The definition given directly below will be an instance of that more general formalism applied to the sheaf $F = \mathbf{G}_{m,U}$ (see Example 5.1.2); here we give the direct definition without any broader context and see how to work with it.

For each $v \in X_\infty = \overline{U}_\infty$, denote by K_v^{alg} the algebraic closure of K inside its completion K_v at the real place v. We write $K_v^{\mathrm{alg},+}$ for its positive part, positivity being determined via the unique isomorphism $K_v \simeq \mathbf{R}$ (or by virture of K_v^{alg} being a real closed field, so it has a unique order structure as a field). Clearly $K_v^{\mathrm{alg},+}$ is a multiplicative subgroup of $K_v^{\mathrm{alg},\times}$, and we denote by φ_v this inclusion of groups.

Definition 3.3.1 The sheaf $\mathbf{G}_{m,\overline{U}} \in \mathfrak{Ab}_{\overline{U}} = \mathbf{Ab}_{\overline{U}}$ is defined to be the triple

$$((K_v^{\mathrm{alg},+})_{v \in X_\infty}, \mathbf{G}_{m,U}, (\varphi_v)_{v \in X_\infty}).$$

We want to calculate the cohomology of $\mathbf{G}_{m,\overline{U}}$ over $\overline{U}$, by adapting Artin's method for usual curves. To that end, we need some modifications of constructions that arise in Artin's method. Given the embedding of the generic point $\eta_U : \mathrm{Spec}(K) \to U$ and the embedding $j_U : U \to \overline{U}$, we have functors

$$\eta_{U*} : \mathbf{Ab}_\eta \to \mathbf{Ab}_U, \quad j_{U*} : \mathbf{Ab}_U \to \mathbf{Ab}_{\overline{U}}.$$

Denote by μ_* their composition. From the definitions we get at once

$$\mu_*(\mathbf{G}_{m,\eta}) = ((K_v^{\mathrm{alg}})_{v \in X_\infty}, \eta_{U*}\mathbf{G}_{m,\eta}, (K_v^{\mathrm{alg}} \xrightarrow{\mathrm{id}} K_v^{\mathrm{alg}})_{v \in X_\infty})$$

and an evident exact sequence of abelian shaves on $\overline{U}$

$$0 \to \mathbf{G}_{m,\overline{U}} \to \mu_*(\mathbf{G}_{m,\eta}) \to \bigoplus_{x \in U^0} i_{x*}(\mathbf{Z}) \oplus \bigoplus_{k=1}^{r} i_{\widetilde{v_k}*}(\mathbf{Z}/2\mathbf{Z}) \to 0$$

in which the first map corresponds to the identification of $\mathbf{G}_{m,\eta}$ with $\mu^*(\mathbf{G}_{m,\overline{U}})$. If we compute the cohomology of the latter two terms in this exact sequence then we can hope to compute the cohomology of $\mathbf{G}_{m,\overline{U}}$. This will be achieved by using a lot of class field theory.

For $p > 0$ we have $\mathrm{R}^p\eta_{U*}(\mathbf{G}_{m,\eta}) = 0$ by applying Proposition A.2.1 after base change to the henselization (not completion!) of $\mathcal{O}_K$ at closed points of U. We deduce, as in the proof of Lemma 3.2.3:

$$\mathrm{R}^p j_{U*}(\eta_{U*}\mathbf{G}_{m,\eta}) = \bigoplus_{k=1}^{r} i_{\widetilde{v_k}*}\mathrm{H}^p(I_{v_k}, \mathbf{G}_{m,\eta}), \quad p \geq 1.$$

Using Grothendieck's composition of functors spectral sequence, we thereby get:

$$\mathrm{R}^p\mu_*(\mathbf{G}_{m,\eta}) \simeq \bigoplus_{k=1}^{r} i_{\widetilde{v_k}*}\mathrm{H}^p(I_{v_k}, \mathbf{G}_{m,\eta}), \quad p \geq 1.$$

Now we use the Leray spectral sequence for the functor μ_*:

$$E_2^{p,q} = \mathrm{H}^p(\overline{U}, \mathrm{R}^q\mu_*(\mathbf{G}_{m,\eta})) \Rightarrow \mathrm{H}^{p+q}(\eta, \mathbf{G}_{m,\eta}).$$

Since $\mathrm{R}^q\mu_*(\mathbf{G}_{m,\eta})$ is concentrated in the real places for $q \geq 1$, we deduce $E_2^{p,q} = 0$ for all $p, q \geq 1$. Therefore, we have the following exact sequence:

$$0 \longrightarrow \mathrm{H}^1(\overline{U}, \mu_*(\mathbf{G}_{m,\eta})) \longrightarrow \mathrm{H}^1(\eta, \mathbf{G}_{m,\eta}) \longrightarrow \textstyle\bigoplus_{k=1}^r \mathrm{H}^1(I_{v_k}, \overline{K}^\times)$$

$$\longrightarrow \mathrm{H}^2(\overline{U}, \mu_*(\mathbf{G}_{m,\eta})) \longrightarrow \mathrm{H}^2(\eta, \mathbf{G}_{m,\eta}) \longrightarrow \textstyle\bigoplus_{k=1}^r \mathrm{H}^2(I_{v_k}, \overline{K}^\times)$$

$$\longrightarrow \mathrm{H}^3(\overline{U}, \mu_*(\mathbf{G}_{m,\eta})) \longrightarrow \mathrm{H}^3(\eta, \mathbf{G}_{m,\eta}).$$

By Hilbert's Theorem 90, we have

$$\mathrm{H}^1(\eta, \mathbf{G}_{m,\eta}) = 0 = \mathrm{H}^1(I_{v_k}, \overline{K}^\times)$$

and by class field theory we get an exact sequence:

$$0 \to \mathrm{H}^2(\eta, \mathbf{G}_{m,\eta}) \to \bigoplus_{v} \mathrm{Br}(K_v) \xrightarrow{\text{sum}} \mathbf{Q}/\mathbf{Z} \to 0 \tag{2}$$

where the direct sum runs over all places v of K. The sequence from before becomes:

$$0 \longrightarrow \mathrm{H}^1(\overline{U}, \mu_*(\mathbf{G}_{m,\eta})) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{H}^2(\overline{U}, \mu_*(\mathbf{G}_{m,\eta})) \longrightarrow \mathrm{H}^2(\eta, \mathbf{G}_{m,\eta}) \longrightarrow \textstyle\bigoplus_{k=1}^r \mathrm{Br}(K_{v_k})$$

$$\longrightarrow \mathrm{H}^3(\overline{U}, \mu_*(\mathbf{G}_{m,\eta})) \longrightarrow \mathrm{H}^3(\eta, \mathbf{G}_{m,\eta}).$$

From (2) we get an exact equence:

$$0 \to \mathrm{H}^2(\overline{U}, \mu_*(\mathbf{G}_{m,\eta})) \xrightarrow{\text{inv}} \bigoplus_{x \in X^0} \mathrm{Br}(K_x) \xrightarrow{\text{sum}} \mathbf{Q}/\mathbf{Z} \to 0$$

and that

$$\mathrm{H}^2(\eta, \mathbf{G}_{m,\eta}) \to \bigoplus_{k=1}^{r} \mathrm{Br}(K_{v_k})$$

is surjective. But $\mathrm{H}^3(\eta, \mathbf{G}_{m,\eta}) = 0$ by class field theory, so we obtain exact sequences:

$$0 \longrightarrow \mathrm{H}^1(\overline{U}, \mu_*(\mathbf{G}_{m,\eta})) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{H}^2(\overline{U}, \mu_*(\mathbf{G}_{m,\eta})) \longrightarrow \mathrm{H}^2(\eta, \mathbf{G}_{m,\eta}) \longrightarrow \bigoplus_{k=1}^r \mathrm{Br}(K_{v_k}) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{H}^3(\overline{U}, \mu_*(\mathbf{G}_{m,\eta})) \longrightarrow 0$$

We summarize what we have shown so far in the following:

Proposition 3.3.2 *We have*

$$\mathrm{H}^0(\overline{U}, \mu_*(\mathbf{G}_{m,\eta})) = K^\times, \quad \mathrm{H}^1(\overline{U}, \mu_*(\mathbf{G}_{m,\eta})) = 0$$

the exact sequence:

$$0 \to \mathrm{H}^2(\overline{U}, \mu_*(\mathbf{G}_{m,\eta})) \to \bigoplus_{x \in X^0} \mathbf{Q}/\mathbf{Z} \to \mathbf{Q}/\mathbf{Z} \to 0$$

and

$$\mathrm{H}^3(\overline{U}, \mu_*(\mathbf{G}_{m,\eta})) = 0.$$

We also have

$$\mathrm{H}^p\left(\overline{U}, \bigoplus_{x \in U^0} i_{x*}(\mathbf{Z}) \oplus \bigoplus_{k=1}^r i_{\widetilde{v_k}*}(\mathbf{Z}/2\mathbf{Z})\right) = 0$$

if $p \neq 0, 2$, and it is equal to $\bigoplus_{x \in U^0} \mathbf{Q}/\mathbf{Z}$ if $p = 2$. Indeed, by Remark 3.2.1,

$$\mathrm{H}^p\left(\overline{U}, \bigoplus_{x \in U^0} i_{x*}(\mathbf{Z}) \oplus \bigoplus_{k=1}^r i_{\widetilde{v_k}*}(\mathbf{Z}/2\mathbf{Z})\right) = \bigoplus_{x \in U^0} \mathrm{H}^p(\overline{U}, i_{x*}(\mathbf{Z})) \oplus \bigoplus_{k=1}^r \mathrm{H}^p(\overline{U}, i_{\widetilde{v_k}*}(\mathbf{Z}/2\mathbf{Z})),$$

and exactness of the pushforwards from the (finite and real) closed points identifies the cohomologies on the right side with derived functors of the composition of $\Gamma(\overline{U}, \cdot)$ with the pushforward. At real points such composite functors are the identity, so their higher derived functors vanish. At finite closed points such composite functors are global sections for étale sheaves on the spectrum of a finite field, so the higher derived functors are Galois cohomology of $\widehat{\mathbf{Z}}$. This gives that $\mathrm{H}^p(\overline{U}, i_{x*}(\mathbf{Z}))$ is $\mathbf{Q}/\mathbf{Z}$ for $p = 2$ and vanishes for all other positive p, and that $\mathrm{H}^p(\overline{U}, i_{\widetilde{v_k}*}(\mathbf{Z}/2\mathbf{Z})) = 0$ for all $p > 0$. The case $p = 0$ is trivial.

3.4 Putting everything together

Given the exact sequence:

$$0 \to \mathbf{G}_{m,\overline{U}} \to \mu_*(\mathbf{G}_{m,\eta}) \to \bigoplus_{x \in U^0} i_{x*}(\mathbf{Z}) \oplus \bigoplus_{k=1}^r i_{\widetilde{v_k}*}(\mathbf{Z}/2\mathbf{Z}) \to 0$$

we apply $\mathrm{H}^0(\overline{U},\cdot)$ to get an exact sequence

$$0 \to \mathrm{H}^0(\overline{U}, \mathbf{G}_{m,\overline{U}}) \to K^\times \xrightarrow{f} \bigoplus_{x\in U^0} \mathbf{Z} \oplus \bigoplus_{k=1}^{r} \mathbf{Z}/2\mathbf{Z} \to \mathrm{H}^1(\overline{U}, \mathbf{G}_{m,\eta}) \to 0$$

where

$$f : K^\times \to \bigoplus_{x\in U^0} \mathbf{Z} \oplus \bigoplus_{k=1}^{r} \mathbf{Z}/2\mathbf{Z}$$

has x-component given by the normalized valuation for each $x \in U^0$ and has v_k-component given by the sign relative to v_k. Thus, $\mathrm{H}^0(\overline{U}, \mathbf{G}_{m,\overline{U}})$ is the group $\mathcal{O}^{\times,+}_{K,X-U}$ of totally positive elements of K which are units over U; for $U = X$ we shall denote this as $\mathcal{O}^{\times,+}_K$.

By classical weak approximation, the composition of f with projection onto the direct sum of the real components is surjective with kernel $K^\times_+$ consisting of the totally positive elements of K. Thus,

$$\mathrm{H}^1(\overline{U}, \mathbf{G}_{m,\overline{U}}) = \bigoplus_{x\in U^0} \mathbf{Z}/\pi(K^\times_+)$$

where $\pi : K^\times_+ \to \bigoplus_{x\in U^0} \mathbf{Z}$ is the evident map. Hence, $\mathrm{H}^1(\overline{U}, \mathbf{G}_{m,\overline{U}})$ is a refinement of the Picard group of U via a positivity condition, so we denote it as $\mathrm{Pic}^+(U)$.

Combining the preceding calculations with the commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^2(\overline{U}, \mu_*\mathbf{G}_{m,\eta}) & \longrightarrow & \mathrm{H}^2(\overline{U}, i_{x*}\mathbf{Z}) & = & \mathrm{H}^2(x, \mathbf{Z}) \\ \downarrow & & & & \uparrow \simeq \\ \mathrm{H}^2(\eta, \mathbf{G}_{m,\eta}) & \longrightarrow & \mathrm{Br}(K_x) & \xleftarrow{\simeq} & \mathrm{H}^2(K_x^{\mathrm{un}}/K_x, \mathbf{G}_m) \end{array}$$

(in which the two maps emanating from the lower-right corner are shown to be isomorphisms in local class field theory), we obtain:

Proposition 3.4.1 *For $U \neq X$, we have*

$$\mathrm{H}^0(\overline{U}, \mathbf{G}_{m,\overline{U}}) = \mathcal{O}^{\times,+}_{K,X-U}, \;\; \mathrm{H}^1(\overline{U}, \mathbf{G}_{m,\overline{U}}) = \mathrm{Pic}^+(U),$$

an exact sequence

$$0 \to \mathrm{H}^2(\overline{U}, \mathbf{G}_{m,\overline{U}}) \to \bigoplus_{x\in(X-U)^0} \mathbf{Q}/\mathbf{Z} \to \mathbf{Q}/\mathbf{Z} \to 0,$$

and $\mathrm{H}^3(\overline{U}, \mathbf{G}_{m,\overline{U}}) = 0$. *For $U = X$, we have:*

$$\mathrm{H}^0(\overline{X}, \mathbf{G}_{m,\overline{X}}) = \mathcal{O}^{\times,+}_K, \;\; \mathrm{H}^1(\overline{X}, \mathbf{G}_{m,\overline{X}}) = \mathrm{Pic}^+(X), \;\; \mathrm{H}^2(\overline{X}, \mathbf{G}_{m,\overline{X}}) = 0, \;\; \mathrm{H}^3(\overline{X}, \mathbf{G}_{m,\overline{X}}) = \mathbf{Q}/\mathbf{Z}.$$

4 Finiteness and vanishing

4.1 Main goal

The purpose of this section is to show the following:

Theorem 4.1.1 *Let U be an non-empty open subscheme of X, and $\overline{F}$ a constructible sheaf on $\overline{U}$. Then the groups $\mathrm{H}^p(\overline{U}, \overline{F})$ are finite abelian groups, and $\mathrm{H}^p(\overline{U}, \overline{F}) = 0$ for all $p \geq 4$.*

First consider when K is totally imaginary or the stalk F_η has odd order. In these cases, the assertions amount to the same for usual étale cohomology on U: this is obvious if K is totally imaginary (as then there are no real places), and in the second case it is Corollary 3.2.4.

Upon decomposing the constructible F into its ℓ-primary parts for varying primes ℓ (only finitely many of which have nontrivial contribution to F), the general vanishing for each ℓ-primary part beyond degree 3 in the totally imaginary case is part of [SGA4, Exp. X, §6.1], the proof of which amounts to many applications of results in Serre's book on Galois cohomology and also applies to ordinary étale cohomology of ℓ-primary parts for odd ℓ *without* a totally imaginary hypothesis (as will be useful shortly).

Suppose instead that F_η has odd order but K may have real places. In that case the local cohomology sequence attached to the dense open $U \subset \overline{U}$ and its complement consisting of the real points has vanishing local cohomologies at the real points in degrees ≥ 3 by Lemma 3.2.3. Hence, the restriction map $\mathrm{H}^p(\overline{U}, \overline{F}) \to \mathrm{H}^p(U, F)$ is an isomorphism for $p \geq 3$ in such cases, so the vanishing in degree ≥ 4 reduces to the case of ordinary étale cohomology that we have noted is established in [SGA4] without a totally imaginary hypothesis.

Next, consider the finiteness assertion if either K is totally imaginary or F_η has odd order. In both cases the finiteness concerns ordinary étale cohomology, so for those cases the finiteness assertion is a consequence of:

Proposition 4.1.2 *For an arbitrary number field K and dense open subscheme $U \subset \mathrm{Spec}(\mathcal{O}_K)$, the étale cohomology groups $\mathrm{H}^p(U, F)$ are finite for all constructible sheaves F on U.*

Before we prove this result, we require a technical lemma that justifies the idea of local cohomology at a point only depending on a "small open set" around the point:

Lemma 4.1.3 *Let F be a sheaf on $X_{\text{ét}}$ for a scheme X. Let $x \in X$ be a closed point, and denote by X_x^{h} the henselization of X at x. For the natural map $f_x^{\mathrm{h}} : X_x^{\mathrm{h}} \to X$, there is a natural isomorphism*

$$\mathrm{H}_x^p(X, F) = \mathrm{H}_x^p(X_x^{\mathrm{h}}, (f_x^{\mathrm{h}})^* F), \quad p \geq 0.$$

Proof. It is a general fact (used all the time when working with étale cohomology) that if $\{Y_i\}$ is an inverse system of qcqs schemes with affine transition maps and $\{F_i\}$ is a compatible system of étale sheaves on them (i.e., for $i' \geq i$, the pullback of F_i along $Y_{i'} \to Y_i$ is identified with $F_{i'}$ in a "transitivie" manner) then for the inverse limit Y of the Y_i's (i.e., over any affine open U in a fixed Y_{i_0} we form Spec of the direct limit of the corresponding coordinate rings from the affine open preimages in each Y_i of U) and the common pullback F of the F_i's to Y, the natural map $\varinjlim \mathrm{H}^p(Y_i, F_i) \to \mathrm{H}^p(Y, F)$ is an isomorphism for all $p \geq 0$.

Applying this to cohomology over X and $X - Z$ for closed $Z \subset X$, the local cohomology sequence shows that $\mathrm{H}_Z^p(X, \cdot)$ satisfies the same limiting behavior. In particular, for $X_x := \mathrm{Spec}(\mathscr{O}_{X,x})$, to show that $\mathrm{H}_x^p(X, F) \to \mathrm{H}_x^p(X_x, F|_{X_x})$ is an isomorphism it suffices to show that $\mathrm{H}_x^p(X, F) \to \mathrm{H}_x^p(U, F|_U)$ is an isomorphism for every open $j : U \subset X$ containing X. But this is obvious since $\Gamma_x(X, \cdot) = \Gamma_x(U, (\cdot)|_U)$ and j^* preserves injectivity (as its left adjoint $j_!$ is exact).

Since X_x^{h} is the inverse limit of the spectra of the residually trivial local-étale extensions of $\mathscr{O}_{X,x}$, each of which is Zariski-localizing at a $k(x)$-point over x on an étale $\mathscr{O}_{X,x}$-scheme, it remains suffices to check that H_x^p is unaffected by pullback to a residually trivial pointed étale neighborhood of (X, x). That is, if $\pi : X' \to X$ is étale and $x' \in \pi^{-1}(x)$ is a $k(x)$-point of X' over x then we claim that the natural map

$$\mathrm{H}_x^p(X, \cdot) \to \mathrm{H}_{x'}^p(X', \pi^*(\cdot)).$$

The functor π^* has left adjoint $\pi_!$ that is exact, the same reasoning as in the proof of Zariski-local invariance reduces the task to checking that $\Gamma_x(X, F) \to \Gamma_{x'}(X', F)$ is an isomorphism for any étale abelian sheaf F on X.

The settled passage to the local case allows us to now assume X is local, so $X' \to X$ is surjective. We may also replace X' with an open around x' so that x' is the only point over x. In particular, $X' - \{x'\}$ maps onto $X - \{x\}$. Hence, $F(X) \to F(X')$ is injective and $F(X) \cap \Gamma_{x'}(X', F) = \Gamma_x(X, F)$. For any $\sigma' \in \Gamma_{x'}(X', F) \subset F(X')$, to show that σ' comes from $\Gamma_x(X, F)$ it is therefore equivalent to show that σ' comes from the subgroup $F(X) \subset F(X')$. In other words, the two pullbacks of σ' in $F(X' \times_X X')$ coincide. Both pullbacks vanish over the part of $X' \times_X X'$ lying over $X - \{x\}$ since $X' - \{x'\}$ is the preimage of $X - \{x\}$ in X' (check!). Consequently, our problem is concentrated in the stalks at points of $X' \times_X X'$ over x. But we have arranged that x' is the unique point of X' over x and that $k(x') = k(x)$, so the x-fiber of $X' \times_X X'$ is a single $k(x)$-point too. Since this schematic point (as a morphism from $\mathrm{Spec}(k(x))$ over X) is *invariant* under the X-automorphism of $X' \times_X X'$ that swaps the two factors (thereby interchanging the two projections), the desired equality of pullbacks follows. □

We turn to the proof of Proposition 4.1.2. Let's first see that for a given F we may shrink U as much as we wish. Let $j : V \hookrightarrow U$ be a non-empty open subscheme with finite complement S, so the local cohomology sequence

$$\cdots \to \mathrm{H}^p_S(U, F) \to \mathrm{H}^p(U, F) \to \mathrm{H}^p(V, F|_V) \to \dots$$

allows us to replace U with V provided that the local cohomologies are finite. Since $\Gamma_S = \bigoplus_{x \in S} \Gamma_x(U, \cdot)$, for the finiteness of local cohomology we may use Lemma 4.1.3 to reduce to showing the finiteness of $\mathrm{H}^p_{\mathfrak{m}}(\mathrm{Spec}(A), F)$ for a henselian discrete valuation ring A with maximal ideal $\mathfrak{m}$ and a constructible abelian étale sheaf F on $\mathrm{Spec}(A)$. This in turn reduces to finiteness of the étale cohomology of constructible sheaves on $\mathrm{Spec}(A)$ and on its open generic point $\mathrm{Spec}(K)$.

Since A is henselian, the Galois theory of K coincides with that of its completion (due to Krasner's Lemma), so finiteness for the Galois cohomology of K on finite Galois modules is reduced to the known case of local fields of characteristic 0. The henselian property also ensures that cohomology on $\mathrm{Spec}(A)$ coincides with that at its closed point (for the stalk of a sheaf) and hence the elementary finiteness properties of the Galois cohomology of finite fields completes the proof of Proposition 4.1.2.

Remark 4.1.4 Since finite fields have cohomological dimension 1 (hence strict cohomological dimension ≤ 2) and non-archimedean local fields of characteristic 0 have cohomological dimension 2 (hence strict cohomological dimension ≤ 3), the preceding considerations relating local cohomology to Galois cohomology shows that for any $x \in U^0$ the functor $\mathrm{H}^p_x(U, \cdot)$ vanishes for $p \geq 4$ on constructible sheaves (hence on all torsion sheaves, as they are always a limit of constructible subsheaves) and for $p \geq 5$ on arbitrary abelian sheaves.

Now we may shrink U so that F is locally constant on U. For such F the finiteness in degree 0 is clear. Let $f : U' \to U$ is a connected finite étale cover and define $F' = f^*(F)$. In the exact sequence

$$0 \to F \to f_*(F') \to H \to 0$$

all terms are locally constant. Since $\mathrm{H}^p(U, f_*(\cdot)) = \mathrm{H}^p(U', \cdot)$, we can induct upwards on p to reduce to replacing (U, F) with (U', F'). In this way we can arrange that F is constant, and then even $F = \mathbf{Z}/n\mathbf{Z}$ for some $n > 0$. Further use of the local cohomology sequence allows us to shrink U to make n a unit on U, and then to replace U with a connected finite étale cover so that $\mu_n = \mathbf{Z}/n\mathbf{Z}$ on U and K is totally imaginary. The Kummer sequence on $U = \overline{U}$ and

Proposition 3.4.1 now do the job in view of standard finiteness properties of unit groups and class groups for rings of S-integers.

We have established Theorem 4.1.1 when either K is totally imaginary or F_η has odd order. In fact, we can now establish the finiteness assertion in general (but will have to work harder for the vanishing beyond degree 3 in general). Indeed, by Proposition 4.1.2 (which had no "totally imaginary" hypothesis) the groups $\mathrm{H}^p(U,F)$ are finite for all p, so to prove that $\mathrm{H}^p(\overline{U},\overline{F})$ is finite for all p we may use the local cohomology sequence to reduce to proving the finiteness of $\mathrm{H}^p_{\widetilde{v}}(\overline{U},\overline{F})$ for all p. But $\overline{F}$ is constructible, so the finiteness of these local cohomologies at the real points is an immediate consequence of Lemma 3.2.3.

It remains to prove the vanishing assertion for $p \geq 4$ when we do not assume either that K is totally imaginary or F_η has odd size. For this we need some preliminary results.

4.2 Preliminaries and reductions

We need one more lemma before we can complete the proof of Theorem 4.1.1.

Lemma 4.2.1 *Let $\overline{F}$ be a constructible sheaf on $\overline{U}$. For fixed $p > 0$ there exists an injection $\overline{F} \to \overline{F}'$ of $\overline{F}$ into a constructible sheaf $\overline{F}'$ such that the induced map*

$$\mathrm{H}^p(\overline{U},\overline{F}) \to \mathrm{H}^p(\overline{U},\overline{F}')$$

vanishes.

Proof. By Remark 3.2.1, $\mathrm{H}^p(\overline{U},\cdot)$ commutes with direct limits. Also recall that $\overline{F}$ is a subsheaf of an injective J that is a direct sum of $j_{U*}(I)$ and $i_{\widetilde{v_k}*}(I_k)$ for an injective I on U_{et} and injective abelian groups I_k. More specifically, we may pick an integer $m > 0$ killing $\overline{F}$ and take such I and I_k's to be injective as $\mathbf{Z}/m\mathbf{Z}$-module sheaves. But *every* torsion abelian sheaf on a noetherian scheme is a direct limit of its constructible subsheaves, and the analogue for torsion abelian groups (as a direct limit of finite subgroups) is clear. Hence, since j_{U*} and each $i_{\widetilde{v_k}*}$ commute with direct limits (clearly from their definitions), and the constructible $\overline{F}$ satisfies the ascending chain condition for directed systems of (necessarily constructible!) subsheaves, we can find a sufficiently large constructible subsheaf $\overline{F}' \subset J$ containing $\overline{F}$ such that the vanishing of the image of the *finite* group $\mathrm{H}^p(\overline{U},\overline{F})$ in $\mathrm{H}^p(\overline{U},J)$ already occurs in $\mathrm{H}^p(\overline{U},F')$. □

We are ready to complete the proof of Theorem 4.1.1, whose statement the reader is encouraged to review.

Proof of Theorem 4.1.1. Recall that it remains to prove the vanishing assertion for $p \geq 4$. Consider a dense open subscheme $V \to U$ and the local cohomology sequence:

$$\bigoplus_{x\in(U-V)^0} \mathrm{H}^p_x(\overline{U},\overline{F}) \to \mathrm{H}^p(\overline{U},\overline{F}) \to \mathrm{H}^p(\overline{V},\overline{F}) \to \bigoplus_{x\in(U-V)^0} \mathrm{H}^{p+1}_x(\overline{U},\overline{F}).$$

Note that none of the points x here are real; they are all ordinary closed points. Hence, $\mathrm{H}^p_x(\overline{U},\overline{F}) = 0$ for all $p \geq 4$ by Remark 4.1.4. Therefore, we can replace U by any dense open subscheme V.

By Lemma 4.2.1, there exists an injection $\alpha : \overline{F} \to \overline{F}'$ into a constructible sheaf $\overline{F}'$ with the property that $\mathrm{H}^p(\overline{U},\overline{F}) \to \mathrm{H}^p(\overline{U},\overline{F}')$ is the zero map. Call $\overline{F}''$ the cokernel of α. From the long exact sequence we obtain a surjection

$$\mathrm{H}^{p-1}(\overline{U},\overline{F}'') \twoheadrightarrow \mathrm{H}^p(\overline{U},\overline{F}).$$

By induction on $p \geq 4$ we thereby reduce to the case $p = 4$.

Running the preceding argument for $\overline{F}$ with $p = 4$, we can shrink U so that: $U \neq X$, F'' is locally constant on U and the order of F''_η is a unit on U. It suffices to show that in this case $\mathrm{H}^3(\overline{U}, \overline{F}'') = 0$. This is achieved in the following result (applied to $\overline{F}''$). □

Proposition 4.2.2 *Let $\overline{F}$ be a constructible sheaf on $\overline{U}$ such that F is locally constant on U with order that is a unit on U. Then $\mathrm{H}^3(\overline{U}, \overline{F}) = 0$.*

Let us first reduce to the case that $\overline{F}$ is traceable (in the sense of Definition 2.5.9). For $F = \overline{F}|_U$, consider the traceable sheaf $j_*(F)$ as defined in Remark 2.5.10. There is an evident map $\overline{F} \to j_*(F)$ whose kernel and cokernel are supported at the real points of U; in fact, by Lemma 3.2.3 we have an exact sequence

$$0 \to \bigoplus_{v \in X_\infty} i_{\widetilde{v}*} i^!_{\widetilde{v}}(\overline{F}) \to \overline{F} \to j_*(F) \to \bigoplus_{v \in X_\infty} i_{\widetilde{v}*} \mathrm{R}^1 i^!_{\widetilde{v}}(\overline{F}) \to 0.$$

The cohomology functors $\mathrm{H}^m(\overline{U}, \cdot)$ kill all pushforwards from real points v for $m > 0$ because $\Gamma(\overline{U}, i_{\widetilde{v}*}(\cdot))$ is the identity functor. It follows that the natural map

$$\mathrm{H}^p(\overline{U}, \overline{F}) \to \mathrm{H}^p(\overline{U}, j_*(F))$$

is an isomorphism for all $p > 0$. Hence, for our purposes we may replace $\overline{F}$ with $j_*(F)$ to reduce to the case that $\overline{F}$ is traceable.

To go further, we need a lemma that is an application of Sylow subgroups:

Lemma 4.2.3 *Let U be a connected normal noetherian scheme. Let ℓ be a prime number and F a locally constant sheaf of finite-dimensional $\mathbf{F}_\ell$-vector spaces on U_{et}. There exists a connected finite étale cover $f : V \to U$ of degree prime to ℓ such that $f^*(F)$ has a finite filtration whose successive quotients are $\underline{\mathbf{Z}/\ell\mathbf{Z}}_V$.*

Proof. Since U is connected and F is locally constant and constructible, it has constant fiber-rank r. We may therefore choose a Galois connected finite étale cover $f : V \to U$ such that $f^*(F)$ is isomorphic to $\underline{\mathbf{Z}/\ell\mathbf{Z}}^{\oplus r}$. Let G be the Galois group of this cover; this is the Galois group of the field extension at the generic points. Observe that the action of G on V lifts to an action of G on $f^*(F) \simeq \underline{\mathbf{Z}/\ell\mathbf{Z}}^{\oplus r}$. Looking at the stalk in the generic point gives a representation $\rho : G \to \mathrm{GL}_r(\mathbf{F}_\ell)$.

Let $H \subset G$ be an ℓ-Sylow subgroup. We claim that $V/H \to U$ works. The action of $\pi_1(V/H)$ (based at its generic point) associated to the pullback of F to V/H corresponds to $\rho|_H$, but H is a finite ℓ-group and hence $\rho|_H$ is a successive extension of copies of $\mathbf{Z}/(\ell)$ with trivial H-action. Since $V/H \to U$ has degree $[G : H]$ that is prime to ℓ, it does the job. □

Continuing with the proof of Proposition 4.2.2, since $K(\zeta_\ell)$ is unramified over K away from ℓ and has degree at most $\ell - 1 < \ell$ we can find a connected finite étale cover $f : U' \to U$ of degree prime to ℓ over which μ_ℓ becomes constant. Using Lemma 4.2.3, we can moreover arrange that $f^*(F)$ has a filtration whose successive quotients are copies of the constant sheaf $\mathbf{Z}/(\ell) = \mu_\ell$. Since $\overline{F}$ is now traceable, so it coincides with $j_*(F)$, we have the composite map

$$\overline{F} \to \overline{f}_* \overline{f}^*(\overline{F}) \to \overline{F}$$

that is multiplication by $\deg(f)$ (see Remark 2.5.10).

This degree acts invertibly on the ℓ-torsion $\overline{F}$, so likewise $\mathrm{H}^p(\overline{U}, \overline{F})$ is thereby realized as a direct summand of $\mathrm{H}^p(\overline{U}', \overline{f}^*(\overline{F}))$. It therefore suffices to prove that the latter vanishes for

$p = 3$. This permits us to replace $(\overline{U}, \overline{F})$ with $(\overline{U}', \overline{f}^*(\overline{F}))$; note that this step preserves the "traceable" condition (as it is clear by inspection that the functors $\overline{f'}^*$ and $F' \rightsquigarrow \widehat{F'}$ naturally commute for any connected finite étale cover $f' : V \to U$). The filtration provided by Lemma 4.2.3 thereby reduces us to the case $\overline{F} = j_*(\mu_\ell)$.

If ℓ is odd then there are no real places in K (as we have arranged that K contains a primitive ℓth root of unity), so in such cases $\overline{U} = U$ and $j_*(\mu_\ell) = \mu_\ell$. But if $\ell = 2$ then K might have real places, and so then $j_*(\mu_\ell)$ does not naturally lie inside $\mathbf{G}_{m,\overline{U}}$ (there are problems at any real place since $-1 < 0$). Thus, rather than work with $j_*(\mu_\ell)$, it is better to work with $j_!(\mu_\ell)$. Indeed, for odd ℓ this is the same as $\mu_\ell = j_*(\mu_\ell)$ and so for any ℓ the natural map $j_!(\mu_\ell) \to j_*(\mu_\ell)$ has kernel and cokernel that are pushforwards from real points. Hence, these two sheaves have the same higher cohomology over $\overline{U}$, whence it is equivalent to prove the vanishing of $\mathrm{H}^3(\overline{U}, j_!(\mu_\ell))$.

The reason it is advantageous to consider $j_!(\mu_\ell)$ for all ℓ is that it fits into an exact "Kummer sequence"

$$0 \to j_!(\mathbf{Z}/\ell\mathbf{Z}) \to \mathbf{G}_{m,\overline{U}} \to \mathbf{G}_{m,\overline{U}} \to 0 \tag{3}$$

over $\overline{U}$. Now we may conclude via Proposition 3.4.1 as follows. We arranged by preliminary shrinking that $U \neq X$, so $\mathrm{H}^3(\overline{U}, \mathbf{G}_{m,\overline{U}}) = 0$. Thus,

$$\mathrm{H}^3(\overline{U}, j_!(\mu_\ell)) = \mathrm{H}^2(\overline{U}, \mathbf{G}_{m,\overline{U}})/(\ell).$$

Thus, we just need to show that $\mathrm{H}^2(\overline{U}, \mathbf{G}_{m,\overline{U}})$ is ℓ-divisible. But we have an exact sequence

$$0 \to \mathrm{H}^2(\overline{U}, \mathbf{G}_{m,\overline{U}}) \to \bigoplus_{x \in (X-U)^0} \mathbf{Q}/\mathbf{Z} \to \mathbf{Q}/\mathbf{Z} \to 0,$$

so the snake lemma applied to the ℓ-power endomorphism of this short exact sequence identifies the cokernel of ℓ on $\mathrm{H}^2(\overline{U}, \mathbf{G}_{m,\overline{U}})$ with the cokernel of

$$\bigoplus_{x \in (X-U)^0} (\mathbf{Q}/\mathbf{Z})[\ell] \to (\mathbf{Q}/\mathbf{Z})[\ell].$$

This latter map is surjective, so we are done. This completes the proof of Proposition 4.2.2.

4.3 Global Galois cohomology beyond degree 2

We are ready to show Theorem 1.2.1, which we restate here for convenience of the reader:

Theorem 4.3.1 *Let S be a finite set of places of K that contains the archimedean places, and let $\{v_1, \ldots, v_r\}$ be the real places (with corresponding decomposition groups $D_{v_k} = I_{v_k} \subset G_S$). Let M be a finite discrete G_S-module whose order is an S-unit. For $p \geq 3$ the natural restriction map*

$$\mathrm{H}^p(G_S, M) \to \bigoplus_{k=1}^{r} \mathrm{H}^p(D_{v_k}, M)$$

is an isomorphism.

Proof. Let $U = \mathrm{Spec}(\mathcal{O}_{K,S})$, so $M = F_\eta$ for a locally constant constructible sheaf F on U whose fiber-rank is a unit on U. Hence, for the traceable sheaf $j_*(F)$ we have $\mathrm{H}^p(\overline{U}, j_*(F)) = 0$ for all

$p \geq 3$ by Proposition 4.2.2 (for $p = 3$) and Theorem 4.1.1 (for $p \geq 4$). The local cohomology sequence arising from the inclusion $U \hookrightarrow \overline{U}$ thereby takes the form

$$\cdots \to \bigoplus_{k=1}^{r} \mathrm{H}^p_{\widetilde{v_k}}(\overline{U}, j_*(F)) \to \mathrm{H}^p(\overline{U}, j_*(F)) \to \mathrm{H}^p(U, F) \to \ldots.$$

The vanishing of $\mathrm{H}^p(\overline{U}, j_*(F))$ for $p \geq 3$ therefore implies that the connecting map

$$\mathrm{H}^p(U, F) \to \bigoplus_{k=1}^{r} \mathrm{H}^{p+1}_{\widetilde{v_k}}(\overline{U}, j_*(F))$$

is an isomorphism for all $p \geq 3$.

Lemma 3.2.3 identifies the local cohomology in degree $p + 1$ at a real point $\widetilde{v}$ with the I_v-cohomology in degree p for the generic fiber $(j_*(F))_\eta = F_\eta = M$. Putting it all together, we have natural isomorphisms

$$\mathrm{H}^p(U, F) \simeq \bigoplus_{k=1}^{r} \mathrm{H}^p(I_v, M)$$

for all $p \geq 3$. An application of Theorem A.1.1 (whose proof uses Theorem 4.1.1 and Proposition 4.2.2, but not the present result we are aiming to prove!) then identifies $\mathrm{H}^p(U, F)$ with $\mathrm{H}^p(G_S, M)$, so we get isomorphisms

$$f^p_M : \mathrm{H}^p(G_S, M) \to \bigoplus_{k=1}^{r} \mathrm{H}^p(D_{v_k}, M) \tag{4}$$

for $p \geq 3$. It remains to check that f^p_M is the natural restriction map for $p \geq 3$. This is verified in Proposition B.1.1 as special case of a more general compatibility argument. □

5 Artin-Verdier duality for algebraic number fields

5.1 A first look

Let us recall that our aim is to show Tate's global duality theorem, together with the 9-term exact sequence for finite ramification, by means of the Duality Theorem (Theorem 1.2.6). Such exact sequence is intended to be an instance for $\mathrm{Spec}(\mathcal{O}_{K,S}) \subset \overline{X}$ of the local cohomology sequence which for dense open $\widetilde{V}$ inside $\widetilde{U}$ with complement denoted S takes the form:

$$\cdots \to \bigoplus_{x \in S} \mathrm{H}^p_x(\widetilde{U}, \widetilde{F}) \to \mathrm{H}^p(\widetilde{U}, \widetilde{F}) \to \mathrm{H}^p(\widetilde{V}, \widetilde{F}|_{\widetilde{V}}) \to \cdots$$

As a first step, since we will be given only a sheaf F on $\mathrm{Spec}(\mathcal{O}_{K,S})$ (such as arising from a finite discrete G_S-module M) yet we want to exploit local cohomology relative to the inclusion of U into $\overline{X}$ (accounting for all the places in S, including the real places!), we need an appropriate functorial extension of abelian sheaves on U to abelian sheaves on $\overline{X}$. Extending to X will be achieved via extension-by-zero, but to extend from X to $\overline{X}$ involves an entirely different idea.

More generally, for any dense open subscheme U of X and abelian sheaf F on U we wish to naturally extend to it to an abelian sheaf $\widehat{F}$ on $\overline{U}$ in a manner that interacts well with local cohomology (and will then mainly need for $U = X$, though considering more general U is illuminating and useful too). We will also need the functors $\mathrm{H}^p(\overline{X}, \widehat{F})$ to assemble into a

δ-functor of F; this will be a bit trick since it turns out that the appropriate functor $\widehat{F}$ will *not* be exact in F (due to complications in stalks at real points) but merely right-exact. In particular, calling j the "open embedding" of U into $\overline{U}$, neither of the left-exact functors $j_!$ nor j_* will do the job we need.

For motivation, assume we have some such functor $F \rightsquigarrow \widehat{F}$, so for a real point v we have an exact sequence

$$0 \to i_v^!(\widehat{F}) \to \widehat{F}_v \xrightarrow{\varphi_v} i_v^* j_* j^*(\widehat{F}) \to \mathrm{R}^1 i_v^!(\widehat{F}) \to 0 \tag{5}$$

(we used i_v^* is exact and that $i_v^* i_{v*} \to \mathrm{id}$ is an isomorphism). Note that the third term $i_v^* j_* j^*(\widehat{F}) = i_v^* j_*(F) = (j_*(F))_v$ is equal to $F(K_v) = \mathrm{H}^0(I_v, F_\eta)$ by the definition of j_*. By Lemma 3.2.3, we also have

$$i_v^!(\widehat{F}) = \mathrm{H}_v^0(\overline{U}, \widehat{F}), \;\; \mathrm{R}^1 i_v^!(\widehat{F}) = \mathrm{H}_v^1(\overline{U}, \widehat{F}).$$

What should these be *as functors of* F?

To guess correctly, we will use *Tate cohomology* (as developed in [CF, Chapter IV]). Since Lemma 3.2.3 yields canonical isomorphisms

$$\mathrm{H}_v^p(\overline{X}, \widehat{F}) \simeq \mathrm{H}^{p-1}(I_v, F_\eta) = \mathrm{H}_T^{p-1}(I_v, F_\eta)$$

for $p \geq 2$ ($(\cdot)_T$ denoting Tate cohomology), the δ-functor $F \rightsquigarrow \mathrm{H}_v^\bullet(\overline{X}, \widehat{F})$ we hope to have in degrees ≥ 0 would extend *two degrees to the left* the δ-functor of Tate cohomology of F_η in degrees ≥ 1. Hence, whatever $\widehat{F}$ is to be (as a functor of F), it is natural to want

$$\mathrm{H}_v^p(\overline{U}, \widehat{F}) \simeq \mathrm{H}_T^{p-1}(I_v, F_\eta)$$

for $p = 0, 1$ since Tate cohomology in degrees ≥ -1 provides the only erasable δ-functorial extension to degrees ≥ -1 of Tate cohomology in degrees ≥ 1. In particular, (5) would then take the form of an exact sequence

$$0 \to \mathrm{H}_T^{-1}(I_v, F_\eta) \to \widehat{F}_v \overset{\varphi_v}{\to} \mathrm{H}^0(I_v, F_\eta) \to \mathrm{H}_T^0(I_v, F_\eta) \to 0$$

for some unknown maps relating the terms in the sequence.

But H_T^p is *defined* for $p = 0, 1$ via a tautological exact sequence

$$0 \to \mathrm{H}_T^{-1}(I_v, F_\eta) \to \mathrm{H}_0(I_v, F_\eta) \overset{\mathrm{N}_v}{\to} \mathrm{H}^0(I_v, F_\eta) \to \mathrm{H}_T^0(I_v, F_v) \to 0$$

where N_v is the norm map for the I_v-module F_η. Hence, this motivates us to *define* $\widehat{F} \in \mathfrak{Ab}_{\overline{U}}$ via real stalks $\widehat{F}_v := \mathrm{H}_0(I_v, F_\eta)$ with $\varphi_v := \mathrm{N}_v$ for all v. In other words, we now make:

Definition 5.1.1 The *modified sheaf* $\widehat{F}$ on $\overline{U}$ is given by:

$$(\mathrm{H}_0(I_{v_1}, F_\eta), \ldots, \mathrm{H}_0(I_{v_r}, F_\eta), F, \mathrm{H}_0(I_{v_1}, F_\eta) \xrightarrow{\mathrm{N}_v} \mathrm{H}^0(I_{v_1}, F_\eta), \ldots, \mathrm{H}_0(I_{v_r}, F_\eta) \xrightarrow{N} \mathrm{H}^0(I_{v_r}, F_\eta)).$$

Example 5.1.2 Since the norm map $\overline{K} \to K_v^{\mathrm{alg}}$ has image $K_v^{\mathrm{alg},+}$ (or upon completion, the norm map $\mathbf{C}^\times \to \mathbf{R}^\times$ has image $\mathbf{R}_{>0}^\times$), by computing $\mathrm{H}_0(I_v, \mathbf{G}_{m,\eta})$ and the norm maps for all real places v one readily sees that

$$\widehat{\mathbf{G}_{m,U}} = \mathbf{G}_{m,\overline{U}}.$$

The first basic feature of $\widehat{F}$ as a functor of F is that it is not left-exact. Instead, we have:

Lemma 5.1.3 *The functor $\mathfrak{Ab}_U \to \mathfrak{Ab}_{\overline{U}}$ (or $\mathbf{Ab}_U \to \mathbf{Ab}_{\overline{U}}$) given by $F \mapsto \widehat{F}$ is right-exact and preserves injectives; explicitly, for $j : U \hookrightarrow \overline{U}$ naturally $\widehat{F} \simeq j_*(F)$ for injective F.*

Proof. Right exactness can be checked over U and on the real stalks. Over U we have $\widehat{F}|_U = F$, and the real stalks are group homologies in degree 0, which are right-exact.

Now assuming that F is injective, so $j_*(F)$ is injective as an abelian sheaf on $\overline{U}$ (as its left adjoint j^* is exact), we will show that the natural map $\widehat{F} \to j_*(F)$ adjoint to the isomorphism $j^*(\widehat{F}) \simeq F$ is itself an isomorphism (so $\widehat{F}$ is indeed then injective). In view of how j_* is defined, the content is that the maps φ_v associated to $\widehat{F}$ are all isomorphisms, or in other words that the kernel and cokernel of each φ_v vanishes. This is precisely the assertion that the Tate cohomology groups $\mathrm{H}^p_T(I_{v_k}, F_\eta)$ vanish for $p = 0, 1$ for any injective F on U.

The injectivity of F and $j_*(F)$ implies that the cohomologies

$$\mathrm{H}^p(\overline{U}, j_*(F)), \;\; \mathrm{H}^p(U, F)$$

vanish for all $p > 0$, so the local cohomology sequence relating them gives that $\mathrm{H}^p_v(\overline{U}, j_*(F)) = 0$ for $p \geq 2$ and real points v. Hence, $\mathrm{H}^p(I_v, F_\eta) = 0$ for all $p \geq 1$ by Lemma 3.2.3. Since the cohomology of I_v is doubly periodic, it follows that all $\mathrm{H}^p_T(I_v, F_\eta)$ are zero. □

5.2 Modified étale cohomology

As with Tate cohomology as a refinement of cohomology of finite groups, we are going to define a modified δ-functorial cohomology theory $\widehat{\mathrm{H}}^\bullet(U, \cdot)$ that coincides with $\mathrm{H}^\bullet(\overline{U}, \widehat{(\cdot)})$ in degrees ≥ 0 but extends to negative degrees accounting for the fact that $\mathrm{H}^0(\overline{U}, \widehat{F})$ is generally *not* left-exact in F. One subtlety is that since $\widehat{F}$ is not left-exact in F (the real stalks can fail to be left-exact), some argument is needed to explain how $\mathrm{H}^\bullet(\overline{U}, \widehat{(\cdot)})$ is at all a δ-functor in degrees ≥ 0!

Consider a short-exact sequence of abelian sheaves on U,

$$0 \to F' \to F \to F'' \to 0,$$

and apply the modification functor $\widehat{(\cdot)}$. By means of the homology long-exact sequence for I_v with real points v and the equality of Tate cohomology in degrees ≤ -2 with group homology in degrees > 0 up to a re-indexing with a sign, we obtain the following long-exact sequence extending on the left:

$$\cdots \longrightarrow \bigoplus_{k=1}^r i_{v_k *}\mathrm{H}^{-2}(I_{v_k}, F_\eta) \longrightarrow \bigoplus_{k=1}^r i_{v_k *}\mathrm{H}^{-2}(I_{v_k}, F''_\eta)$$

$$\longrightarrow \widehat{F'} \longrightarrow \widehat{F} \longrightarrow \widehat{F''} \longrightarrow 0$$

We let $M = \ker(\widehat{F} \to \widehat{F''})$, so there is an induced map $\widehat{F'} \twoheadrightarrow M$. We call N the kernel of this latter map, so also $N = \ker(\widehat{F'} \to \widehat{F})$. Hence, $N = \bigoplus_v i_{v*}(N_v)$ for

$$N_v := \ker(\mathrm{H}_0(I_v, F'_\eta) \to \mathrm{H}_0(I_v, F_\eta)) = \operatorname{coker}(\mathrm{H}^{-2}_T(I_v, F_\eta) \to \mathrm{H}^{-2}_T(I_v, F''_\eta)).$$

Hence, $\mathrm{H}^p(\overline{U}, N) = 0$ for $p \geq 1$ since $\Gamma(\overline{U}, i_{v*}(\cdot))$ is the identity functor for real v. We deduce the following exact sequences:

$$0 \to \mathrm{H}^0(\overline{U}, N) \to \mathrm{H}^0(\overline{U}, \widehat{F'}) \to \mathrm{H}^0(\overline{U}, M) \to 0$$

$$0 \to \mathrm{H}^0(\overline{U}, M) \to \mathrm{H}^0(\overline{U}, \widehat{F}) \to \mathrm{H}^0(\overline{U}, \widehat{F''}) \to \mathrm{H}^1(\overline{U}, M) \to \cdots$$

together with the canonical isomorphism:

$$\mathrm{H}^p(\overline{U},\widehat{F'}) \simeq \mathrm{H}^p(\overline{U},M), \quad p \geq 1.$$

Hence, we get a long exact sequence

$$0 \to \mathrm{H}^0(\overline{U},N) \to \mathrm{H}^0(\overline{U},\widehat{F'}) \to \mathrm{H}^0(\overline{U},\widehat{F}) \to \mathrm{H}^0(\overline{U},\widehat{F''}) \to \mathrm{H}^1(\overline{U},\widehat{F'}) \to \mathrm{H}^1(\overline{U},\widehat{F}) \to \dots$$

The description of N in terms of N_v's also gives an exact sequence

$$\bigoplus_{k=1}^{r} \mathrm{H}_T^{-2}(I_{v_k},F_\eta) \to \bigoplus_{k=1}^{r} \mathrm{H}_T^{-2}(I_{v_k},F''_\eta) \to \mathrm{H}^0(\overline{U},N) \to 0$$

This motivates the following:

Definition 5.2.1 For all $p \in \mathbf{Z}$, the *modifield étale cohomology groups* are:

$$\widehat{\mathrm{H}}^p(U,F) := \begin{cases} \mathrm{H}^p(\overline{U},\widehat{F}), & p \geq 0 \\ \bigoplus_{k=1}^r \mathrm{H}_T^{p-1}(I_{v_k},F_\eta), & p < 0. \end{cases}$$

The preceding considerations make the modified étale cohomology groups into a δ-functor: to each the short-exact sequence of abelian sheaves

$$0 \to F' \to F \to F'' \to 0$$

on U we have naturally associated a long exact sequence

$$\cdots \to \widehat{\mathrm{H}}^p(U,F') \to \widehat{\mathrm{H}}^p(U,F) \to \widehat{\mathrm{H}}^p(U,F'') \to \widehat{\mathrm{H}}^{p+1}(U,F') \to \cdots$$

with $p \in \mathbf{Z}$. Voila, $\widehat{\mathrm{H}}^\bullet(U,\cdot)$ has been made into a δ-functor on $\mathbf{Ab}_U$ despite the failure of $\widehat{F}$ to be exact in F.

5.3 Modified cohomology with support

Using Lemma 3.2.3 and the definition of $\widehat{(\cdot)}$, for all real v and $p \geq 0$

$$\mathrm{H}_v^p(\overline{U},\widehat{F}) = \mathrm{H}_T^{p-1}(I_v,F_\eta). \tag{6}$$

This fails for $p = 0,1$ if instead of $\widehat{F}$ we use a general abelian sheaf $\overline{F} = (\{\overline{F}_v\},F,\{\varphi_v\})$ on $\overline{U}$ (rather than one of the form $\widehat{F}$). Indeed, for such $\overline{F}$ we have

$$\mathrm{H}_v^0(\overline{U},\overline{F}) = \ker(\varphi_v : \overline{F}_v \to F_\eta^{I_v}), \ \ \mathrm{H}_v^1(\overline{U},\overline{F}) = \mathrm{coker}(\varphi_v).$$

Hence, if $\overline{F} = j_!(F)$ for the inclusion j of U into $\overline{U}$ then $\overline{F}_v = 0$ for all real v, so $\mathrm{H}_v^0(\overline{U},\overline{F}) = 0$, but generally $\mathrm{H}_T^0(I_v,F_\eta) \neq 0$. Hence, motivated by (6) as a functor of F, we are led to make:

Definition 5.3.1 For all $p \in \mathbf{Z}$ and real v, the *modified local cohomology groups* on U are

$$\mathrm{H}_v^p(U,F) := \begin{cases} \mathrm{H}_v^p(\overline{U},\widehat{F}), & p \geq 0 \\ \mathrm{H}_T^{p-1}(I_v,F_\eta), & p < 0 \end{cases}$$

We make the convention that the functors $\mathrm{H}^p_x(U,\cdot)$ for $x \in U^0$ and $\mathrm{H}^p(U,\cdot)$ vanish for $p < 0$ since they are left-exact for $p = 0$ (whereas for real v the functor $\mathrm{H}^0_v(U,\cdot) := \mathrm{H}^0_v(\overline{U},\widehat{(\cdot)})$ is *not* left-exact, so its non-vanishing extension to negative degrees is appropriate). Using that convention, for any dense open subset $V \subset U$ we get a long exact "local cohomology" sequence

$$\to \cdots \bigoplus_{s\in\overline{U}-V} \mathrm{H}^p_s(U,F) \to \widehat{\mathrm{H}}^p(U,F) \to \mathrm{H}^p(V,F|_V) \to \bigoplus_{s\in\overline{U}-V} \mathrm{H}^{p+1}_s(U,F) \to \cdots \tag{7}$$

with $p \in \mathbf{Z}$. This is just a reformulation of the local cohomology sequence for $\widehat{F}$ on $\overline{U}$ relative to V for $p \geq 0$ and tautological identifications for $p < 0$ via the considerations in §5.2.

5.4 Compactly supported cohomology and duality preparations

We let j be the open immersion of an open subscheme U into X, and F be an abelian sheaf on U. We make use of the notation developed so far with no further mention.

Definition 5.4.1 For $p \in \mathbf{Z}$, the *compactly supported cohomology groups* are

$$\mathrm{H}^p_c(U,F) := \widehat{\mathrm{H}}^p(X, j_!(F))$$

for abelian sheaves F on U.

First of all, we observe that $\mathrm{H}^\bullet_c(U,\cdot)$ is a covariant δ-functor because $\widehat{\mathrm{H}}^\bullet(X,\cdot)$ is such a δ-functor and $j_!$ is exact. For $U = X$ we have $j_! = \mathrm{id}$, by definition for every abelian sheaf F on X and every integer p we have

$$\mathrm{H}^p_c(X,F) = \widehat{\mathrm{H}}^p(X,F).$$

Likewise, if $h : V \hookrightarrow U$ is an inclusion of dense open subsets of X then for any sheaf F on U there are natural maps

$$\mathrm{H}^p_c(V, F|_V) \to \mathrm{H}^p_c(U,F) \tag{8}$$

defined by applying $\mathrm{H}^p_c(U,\cdot)$ to the map $h_!(F|_V) \to F$. Here is an important example for arithmetic duality:

Example 5.4.2 Let $j : U \hookrightarrow X$ be a dense open subscheme. We claim that naturally

$$\mathrm{H}^3_c(U,\mathbf{G}_{m,U}) \simeq \mathrm{H}^3_c(X,\mathbf{G}_{m,X}) \simeq \mathbf{Q}/\mathbf{Z}. \tag{9}$$

For the second isomorphism, we note that by definition $\mathrm{H}^3_c(X,\mathbf{G}_{m,X}) = \mathrm{H}^3(\overline{X},\mathbf{G}_{m,\overline{X}})$ by Example 5.1.2, so it is identified with $\mathbf{Q}/\mathbf{Z}$ by Proposition 3.4.1.

The first map in (9) is the map (8) for the inclusion $U \hookrightarrow X$ and the sheaf $\mathbf{G}_{m,X}$ on X; explicitly, it is obtained by applying $\mathrm{H}^3(\overline{X},\cdot)$ to the inclusion

$$(j_!(\mathbf{G}_{m,U}))^\wedge \hookrightarrow \mathbf{G}_{m,\overline{X}} \tag{10}$$

over $\overline{X}$ whose cokernel is supported on $X - U$ (no real points!) with stalk at $x \in X - U$ given by the Galois module $(\mathscr{O}^{\mathrm{sh}}_{X,x})^\times$ for $G_x = \mathrm{Gal}(\overline{\kappa(x)}/\kappa(x)) \simeq \widehat{\mathbf{Z}}$. Since G_x has cohomological dimension 1 and hence strict cohomological dimension ≤ 2, the map obtained from applying $\mathrm{H}^3(\overline{X},\cdot)$ to (10) is surjective with kernel that is a quotient of $\bigoplus_{x\in X-U} \mathrm{H}^2(\kappa(x),(\mathscr{O}^{\mathrm{sh}}_{X,x})^\times)$.

Now it suffices to show that if A is a henselian (not necessarily complete!) excellent discrete valuation ring with finite residue field k then $\mathrm{H}^2(k,(A^{\mathrm{sh}})^\times) = 1$. This is the limit of the finite-layer cohomology groups $\mathrm{H}^2(k'/k, A'^\times)$ for finite extensions k'/k and the corresponding local finite étale extension A' of A. But the Tate cohomology of cyclic groups is doubly periodic, so this degree-2 cohomology coincides with the degree-0 Tate cohomology that is the cokernel of the norm map on local unit groups $A'^{\times} \to A^{\times}$. Hence, it suffices to show that such norm maps are surjective. If A were complete then such surjectivity is clear by a successive approximation argument. In general surjectivity is a consequence of the complete case via the Artin approximation theorem, which applies to the existence of solutions to system of polynomial equations over any henselian excellent discrete valuation ring. (A similar use of Artin approximation occurs in the proof of Proposition A.2.1.)

Remark 5.4.3 Since the trace $\mathrm{Tr}_U : \mathrm{H}^3_c(U, \mathbf{G}_m) \to \mathbf{Q}/\mathbf{Z}$ is *defined* to be a composition

$$\mathrm{H}^3_c(U, \mathbf{G}_m) \to \mathrm{H}^3_c(X, \mathbf{G}_m) \xrightarrow{\mathrm{Tr}} \mathbf{Q}/\mathbf{Z}$$

(with Tr an isomorphism), for dense open $h : V \hookrightarrow U$ we wish to express a compatibility between Tr_U and Tr_V.

Calling j_U the inclusion of U into X, and j_V the inclusion of V into X, by definition

$$\mathrm{H}^3_c(U, \mathbf{G}_m) = \mathrm{H}^3(\overline{X}, (j_{U!}\mathbf{G}_m)^{\wedge}), \;\; \mathrm{H}^3_c(V, \mathbf{G}_m) = \mathrm{H}^3(\overline{X}, (j_{V!}\mathbf{G}_m)^{\wedge}).$$

The identification $j_{V!} = j_{U!} \circ h_!$ implies that the inclusion $(j_{V!}\mathbf{G}_{m,V})^{\wedge} \hookrightarrow \mathbf{G}_{m,\overline{X}}$ factors through $(j_{U!}\mathbf{G}_{m,U})^{\wedge} \hookrightarrow \mathbf{G}_{m,\overline{X}}$, so we obtain a natural map

$$f : \mathrm{H}^3_c(V, \mathbf{G}_{m,V}) \to \mathrm{H}^3_c(U, \mathbf{G}_{m,U})$$

by applying $\widehat{\mathrm{H}}^3(\overline{X}, \cdot)$ to the inclusion $(j_{V!}\mathbf{G}_{m,V})^{\wedge} \hookrightarrow (j_{U!}\mathbf{G}_{m,U})^{\wedge}$. By functoriality of $\mathrm{H}^3(\overline{X}, \cdot)$, the diagram

$$\begin{array}{ccc} \mathrm{H}^3_c(V, \mathbf{G}_m) & \longrightarrow & \mathrm{H}^3_c(X, \mathbf{G}_m) \\ {\scriptstyle f}\downarrow & & \| \\ \mathrm{H}^3_c(U, \mathbf{G}_m) & \longrightarrow & \mathrm{H}^3_c(X, \mathbf{G}_m) \end{array}$$

commutes. It follows that f is an isomorphism and $\mathrm{Tr}_V = \mathrm{Tr}_U \circ f$.

We have established most of the necessary terminology and preliminary results to begin to prove Theorem 1.2.6, but we require one more definition in order for the entire statement to make sense. Let us first recall the statement of the result to be proved:

Theorem 5.4.4 (Artin-Verdier) *Let F be a constructible abelian étale sheaf on a dense open $U \subset X$. The Yoneda pairing*

$$\mathrm{H}^p_c(U, F) \times \mathrm{Ext}^{3-p}_U(F, \mathbf{G}_{m,U}) \to \mathrm{H}^3_c(U, \mathbf{G}_{m,U}) = \mathbf{Q}/\mathbf{Z}$$

is a perfect pairing of finite abelian groups for all $p \in \mathbf{Z}$.

Remark 5.4.5 In this statement, the Ext-group is defined to be 0 when $p > 3$ (so part of the assertion is that $\mathrm{H}^p_c(U, F) = 0$ if $p > 3$). Also, $\mathrm{H}^p_c(U, F)$ doesn't generally vanish for $p < 0$ when there are real places since $\widehat{\mathrm{H}}^p(\overline{X}, \cdot)$ doesn't generally vanish for $p < 0$ when there are real places: see Definition 5.2.1 (and Definition 5.4.1). It may seem totally ridiculous to comtemplate $p < 0$, but if there are real places then H^0_c is *not* left-exact and so to push through the method of proof (which is an induction from the left) it is actually essential to incorporate negative degrees in the presence of real places (in which case H^p_c for $p < 0$ is Tate cohomology for decomposition groups at real places; these can be handled quite concretely, at the cost of some actual work to ensure that the connecting map from degree -1 to degree 0 interacts well with the constructions we'll make!).

The gap which remains before we can begin the proof is to explain what the "Yoneda pairing" is; the technical problem is that the definition of $\mathrm{H}_c^\bullet(U,\cdot)$ involves the intervention of the non-exact functor $\widehat{(\cdot)}$. The role of such a non-exact functor made it a bit surprising that $\mathrm{H}_c^\bullet(U,\cdot)$ admits a δ-functor structure, but $\widehat{(\cdot)}$ applied to complexes of sheaves does not preserve quasi-isomorphisms and hence doesn't make sense at the level of derived categories. Consequently, the standard method of defining Yoneda pairings via the identification of $\mathrm{Ext}^n(F,G)$ with the set of derived-category homomorphisms $F\to G[n]$ (where $G[n]$ is the one-term complex G supported in degree $-n$) cannot be used. We shall proceed in another way.

Let $j:U\hookrightarrow X$ be the open immersion. By definition $\mathrm{H}_c^p(U,F)\simeq \mathrm{H}_c^p(X,j_!(F))$ for any abelian sheaf F (even δ-functorial!), $\mathrm{H}_c^3(U,\mathbf{G}_{m,U})\simeq \mathrm{H}_c^3(X,\mathbf{G}_{m,X})$ (see Example 5.4.2), and

$$\mathrm{Ext}_X^i(j_!(F),F')\simeq \mathrm{Ext}_U^i(F,j^*(F'))$$

induced by restriction along j for any F on U and any F' on X (δ-functorial in F). Setting $F'=\mathbf{G}_{m,X}$, we see that to define the pairing over U for F in a manner that is δ-functorial it suffices to treat the situation over X and then apply that to $j_!(F)$ (as $j_!$ is exact). Hence, for the rest of the proof we may focus on X.

If we let $\overline{j}:X\hookrightarrow\overline{X}$ be the inclusion then for abelian sheaves F and F' on X we have

$$\mathrm{Ext}_X^m(F,F')=\mathrm{Ext}_X^m(\overline{j}^*(\widehat{F}),F')=\mathrm{Ext}_{\overline{X}}^m(\widehat{F},\overline{j}_*(F'))$$

due to the adjunction between restriction $\overline{j}^*$ and pushforward $\overline{j}_*$. Hence, the usual derived-category Yoneda method gives a natural pairing

$$\mathrm{H}^n(\overline{X},\widehat{F})\times \mathrm{Ext}_{\overline{X}}^m(\widehat{F},\overline{j}_*(F'))\to \mathrm{H}^{n+m}(\overline{X},\overline{j}_*(F'))$$

for $n\geq 0$. The natural isomorphism $\overline{j}^*(\widehat{F'})\simeq F'$ is adjoint to a natural homomorphism $\widehat{F'}\to\overline{j}_*(F')$ whose kernel and cokernel are supported at real points. Thus, applying $\mathrm{H}^p(\overline{X},\cdot)$ to this gives an isomorphism for $p>0$, so with $p=3$ we have a natural isomorphism

$$\mathrm{H}_c^3(X,F')\simeq \mathrm{H}^3(\overline{X},\overline{j}_*(F')).$$

Using the inverse of this gives a pairing

$$\mathrm{H}^p(\overline{X},\widehat{F})\times \mathrm{Ext}_{\overline{X}}^{3-p}(\widehat{F},\overline{j}_*(F'))\to \mathrm{H}^3(\overline{X},\overline{j}_*(F'))=\mathrm{H}_c^3(X,F')$$

for $p\geq 0$.

Setting $F'=\mathbf{G}_{m,X}$, we get a natural pairing

$$\mathrm{H}_c^p(X,F)\times \mathrm{Ext}_X^{3-p}(F,\mathbf{G}_{m,X})\to \mathrm{H}^3(X,\mathbf{G}_{m,X})=\mathbf{Q}/\mathbf{Z}$$

for $p\geq 0$. That is the Yoneda pairing *by definition* for $p\geq 0$, and applying it to the extension-by-zero to X of an abelian sheaf on U defines such a pairing over any dense open U in X (for any abelian sheaf F on U).

For $p<0$, by definition we seek a pairing

$$(\bigoplus_v \mathrm{H}_T^{p-1}(I_v,F_\eta))\times \mathrm{Ext}_X^{3-p}(F,\mathbf{G}_{m,X})\to\mathbf{Q}/\mathbf{Z}.$$

To define this, we need to analyze the Ext's in degree ≥ 4. This require a lemma:

Lemma 5.4.6 *For every constructible sheaf F on a dense open $U \subset X$, the edge map*

$$\mathrm{H}^i(U, \underline{\mathrm{Hom}}_U(F, \mathbf{G}_m)) \to \mathrm{Ext}^i_U(F, \mathbf{G}_m)$$

in the local-to-global spectral sequence $\mathrm{H}^m(U, \underline{\mathrm{Ext}}^n_U(F, \mathbf{G}_m)) \Rightarrow \mathrm{Ext}^{m+n}_U(F, \mathbf{G}_m)$ is an isomorphism if either $i \geq 4$ or if F is locally constant with order that is a unit on U.

Proof. First consider the case of general F and $i \geq 4$. Let $j : V \hookrightarrow U$ be a dense open such that $F|_V$ is locally constant with order that is a unit on V. By Remark 4.1.4, the local cohomology functor $\mathrm{H}^i_x(U, \cdot)$ vanishes on torsion abelian sheaves for $i \geq 4$. Hence, the restriction map $\mathrm{H}^i(U, F') \to \mathrm{H}^i(V, F'|_V)$ is an isomorphism for all $i \geq 4$ and all torsion abelian sheaves F'. We shall apply that isomorphism with F' equal to any of the $\underline{\mathrm{Ext}}$-sheaves in the local-to-global Ext spectral sequence.

The formation of that spectral sequence is pullback-functorial with respect to working locally on U, so it follows that the formation of the edge map is not only compatible with restriction from U to V but moreover, those restriction maps between edge terms as well as between the global abutments are *isomorphisms* in all degrees ≥ 4. Hence, we may assume F is locally constant with order that is a unit on U.

It suffices to show that for those special class of F's, the higher Ext-sheaves $\underline{\mathrm{Ext}}^i_U(F, \mathbf{G}_m)$ vanish for all $i > 0$. This assertion is of étale-local nature on U, so by passing to a suitable connected finite étale cover we may assume $F = \mathbf{Z}/n\mathbf{Z}$ for some $n > 0$ that is a unit on U. Then the exact sequence of constant sheaves

$$0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} \to 0$$

reduces the task to showing two facts:

(i) $\underline{\mathrm{Ext}}^i_U(\mathbf{Z}, \mathbf{G}_m) = 0$ for $i > 0$,

(ii) multiplication by n on $\underline{\mathrm{Ext}}^0(\mathbf{Z}, \mathbf{G}_m) = \mathbf{G}_m$ is a surjection of étale sheaves.

Assertion (ii) is clear since n is a unit on U, and in general $\underline{\mathrm{Ext}}^i_U(\mathbf{Z}, \mathbf{G}_m)$ is the sheafification of

$$V \rightsquigarrow \mathrm{Ext}^i_V(\mathbf{Z}, \mathbf{G}_m) = \mathrm{H}^i(V, \mathbf{G}_m).$$

This sheafification clearly vanishes when $i > 0$ (as for cohomology of any sheaf), settling (i). □

For $i \geq 4$ and any *constructible* F on U, Lemma 5.4.6 allows us to define a composite map

$$\mathrm{Ext}^i_U(F, \mathbf{G}_m) = \mathrm{H}^i(U, \underline{\mathrm{Hom}}(F, \mathbf{G}_m)) \to \mathrm{H}^i(\eta, \underline{\mathrm{Hom}}(F_\eta, \mathbf{G}_{m,\eta})) = \mathrm{H}^i(K, F^\vee_\eta)$$

via localization to the generic point, where $F^\vee_\eta = \mathrm{Hom}(F_\eta, \mathbf{G}_{m,\eta})$ is the dual Galois module. (The formation of $\underline{\mathrm{Hom}}(F, \cdot)$ commutes with pullback to η because F is *constructible*; in general the formation of $\underline{\mathrm{Hom}}$-sheaves does *not* commute with non-étale localization.) Pulling back further to cohomology at the real places $v_1, \dots, v_r$ of K thereby provides a canonical map

$$\mathrm{Ext}^i_U(F, \mathbf{G}_m) \to \bigoplus_{k=1}^{r} \mathrm{H}^i(I_{v_k}, F^\vee_\eta).$$

Corollary 5.4.7 *The preceding canonical map is an isomorphism for all $i \geq 4$.*

Proof. In the proof of Lemma 5.4.6, we saw that since $i \geq 4$ we can shrink U without affecting the Ext's in degree ≥ 4. Hence, we may assume F is locally constant with order that is a unit on U. Our task is equivalent to showing that the collective restriction map

$$\mathrm{H}^i(U, F^\vee) \to \bigoplus_v \mathrm{H}^i(I_v, F^\vee_\eta)$$

is an isomorphism for $i \geq 4$, with $F^\vee = \underline{\mathrm{Hom}}(F, \mathbf{G}_m)$ the Cartier dual sheaf (also lcc, and naturally $F \simeq F^{\vee\vee}$). Writing $U = \mathrm{Spec}(\mathcal{O}_{K,S})$ (where S contains the archimedean places of K) and focusing on the finite discrete G_S-module corresponding to $F_\eta^\vee$, this was established in the proof of Theorem 4.3.1. □

Remark 5.4.8 For later purposes we need to address a compatibility with finite pushforward. Consider a finite extension K'/K and the finite normalization $f : U' \to U$ of U in K'/K. Let $\eta' \in U'$ be the generic point. For a constructible abelian sheaf F' on U'_{et}, so F' is lcc over the preimage of a dense open in U, naturally $(f_*F')_\eta^\vee \simeq f_*((F'_{\eta'})^\vee)$. We claim that the isomorphism in Corollary 5.4.7 is compatible with f_* in the sense that for $i \geq 4$ the diagram

$$\begin{array}{ccc} \mathrm{Ext}^i_{U'}(F', \mathbf{G}_{m,U'}) & \xrightarrow{\simeq} & \bigoplus_v \bigoplus_{w|v} \mathrm{H}^i(I_w, (F'_{\eta'})^\vee) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^i_U(f_*(F'), \mathbf{G}_{m,U}) & \xrightarrow[\simeq]{} & \bigoplus_v \mathrm{H}^i(I_v, (f_*F')_\eta^\vee) \end{array}$$

commutes, where the left side rests on the exactness of f_* and the norm map $f_*\mathbf{G}_{m,U'} \to \mathbf{G}_{m,U}$.

To prove this commutativity, we can use the compatibility of the *isomorphisms* in Lemma 5.4.6 and Corollary 5.4.7 with respect to Zariski localization to reduce to the case when F' is lcc on U' and f is étale. Hence, the natural composite map

$$f_*\underline{\mathrm{Hom}}_{U'}(F', \mathbf{G}_{m,U'}) \to \underline{\mathrm{Hom}}_U(f_*F', f_*\mathbf{G}_{m,U'}) \to \underline{\mathrm{Hom}}_U(f_*F, \mathbf{G}_{m,U})$$

is an *isomorphism* (here we use both that F' is lcc and that f is finite étale). Since f_* is exact and preserves injectives (so it carries Cartan–Eilenberg resolutions to Cartan–Eilenberg resolutions), it follows that for $i \geq 0$ the edge maps in Lemma 5.4.6 for F' on U' and for f_*F' on U are compatible in the sense that the diagram

$$\begin{array}{ccc} \mathrm{H}^i(U', \underline{\mathrm{Hom}}_{U'}(F', \mathbf{G}_{m,U'})) & \longrightarrow & \mathrm{Ext}^i_{U'}(F', \mathbf{G}_{m,U'}) \\ \downarrow & & \downarrow \\ \mathrm{H}^i(U, \underline{\mathrm{Hom}}_U(f_*F', \mathbf{G}_{m,U})) & \longrightarrow & \mathrm{Ext}^i_U(f_*F', \mathbf{G}_{m,U}) \end{array}$$

commutes (as we check by chopping the diagram into two smaller squares via an intermediate row involving $f_*\mathbf{G}_{m,U'}$, in terms of which the bottom square commutes by functoriality of the edge map over U relative the norm map $f_*\mathbf{G}_{m,U'} \to \mathbf{G}_{m,U}$).

Returning to the task of defining the Yoneda pairing for negative p with *constructible* F, by Corollary 5.4.7 it suffices to define pairings

$$\mathrm{H}_T^{p-1}(I_v, F_\eta) \times \mathrm{H}^{3-p}(I_v, F_\eta^\vee) \to (1/2)\mathbf{Z}/\mathbf{Z} \subset \mathbf{Q}/\mathbf{Z} \tag{11}$$

for $p < 0$. But Tate cohomology coincides with group cohomology in positive degrees, so the usual cup product for Tate cohomology gives a pairing valued in $\mathrm{H}^2(I_v, \mathbf{G}_m) = \mathrm{H}^0_T(I_v, \mathbf{G}_m)$, and this latter group is the cokernel of the norm map $\overline{K}^\times \to (K_v^{\mathrm{alg}})^\times$ (with K_v^{alg} the real closed field given by the algebraic closure of K in $K_v = \mathbf{R}$). The image of the norm is the subgroup of positive elements in the real closed field K_v^{alg}, so its cokernel has order 2. That defines the Yoneda pairing for negative degrees.

In order to do nontrivial things with the Yoneda pairing as just defined, we need:

Proposition 5.4.9 *The pairing* $\mathrm{H}^p_c(U, F) \times \mathrm{Ext}^{3-p}_U(F, \mathbf{G}_{m,U}) \to \mathbf{Q}/\mathbf{Z}$ *for constructible abelian* F *on* U_{et} *is* δ*-functorial for* $p \in \mathbf{Z}$.

Proof. Since extension by zero from U to X is exact, by design the problem reduces to the case $U = X$. Consider a short exact sequence of abelian sheaves on X

$$0 \to F' \xrightarrow{f} F \xrightarrow{f'} F'' \to 0$$

and apply the right-exact $\widehat{(\cdot)}$ to get a pair of exact sequences

$$0 \to N \to \widehat{F'} \to M \to 0, \;\; 0 \to M \to \widehat{F} \to \widehat{F''} \to 0$$

with $M := \ker \widehat{f}'$ and $N := \ker \widehat{f}$, with N supported at the real points.

The δ-functor structure of $\mathrm{H}^\bullet_c(X, \cdot)$ was defined in degrees ≥ 0 to be the maps

$$\delta^p : \mathrm{H}^p(\overline{X}, \widehat{F''}) \to \mathrm{H}^{p+1}(\overline{X}, M) \simeq \mathrm{H}^{p+1}(\overline{X}, \widehat{F'})$$

for $p \geq 0$. Since $M \to \widehat{F}$ restricts to an isomorphism over X, applying $\mathrm{Ext}^\bullet_{\overline{X}}(\cdot, \overline{j}_*(G))$ to that map gives an isomorphism for any G on X (such as $G = \mathbf{G}_{m,X}$). Hence, the δ-functoriality of the usual Yoneda pairing does the job for connecting maps in non-negative degrees. Up to here, the constructibility of F has not been used.

Now let's consider the connecting maps in degrees ≤ 0; i.e., from degree p to degree $p+1$ for $p < 0$. Recall that the pairing in negative degrees has only been defined for constructible F. For $p < -1$ this expresses the basic fact that cup product in Tate cohomology is δ-functorial (in all degrees). Hence, it remains to analyze the connecting map from degree -1 to degree 0. This is proved in Proposition B.2.1. □

The proof of Artin–Verdier duality will involve pulling down results from a finite extension of K that is totally imaginary (over which the theorem has been proved in [Ma, Thm. 2.4]). In addition to the δ-functoriality already established, we also require compatibility with pushforward along finite maps. More specifically, let K'/K be a finite extension, $U \subset \mathrm{Spec}(\mathcal{O}_K)$ a dense open subset, and $U' \subset \mathrm{Spec}(\mathcal{O}_{K'})$ the preimage of U. We get a finite morphism $f : U' \to U$ and will use the following preliminary result:

Lemma 5.4.10 *For any abelian sheaf F' on U' and $p \geq 0$ we have canonical isomorphisms*

$$\widehat{\mathrm{H}}^p(U', F') \simeq \widehat{\mathrm{H}}^p(U, f_*(F')).$$

This isomorphism in nonnegative degrees is also δ-functorial by construction, but we won't need that. Such isomorphisms also exist for $p < 0$ (and δ-functoriality holds in all integral degrees). We discuss these further aspects in §B.2, where we establish a result that is crucial in subsequent arguments: the isomorphism is compatible with the Yoneda pairing in Proposition 5.4.9 (under a constructibility hypothesis when $p < 0$).

Proof. We need to compare $\mathrm{H}^p(\overline{U}', \widehat{F'})$ and $\mathrm{H}^p(\overline{U}, \widehat{f_*(F')})$. Using the map $\overline{f} : \overline{U}' \to \overline{U}$ that is "finite" in the sense of Definition 2.5.6, the functor $\overline{f}_*$ as in Definition 2.5.1 is exact by Proposition 2.5.7 and it preserves injectivity (since its left adjoint $\overline{f}^*$ is exact), so

$$\mathrm{H}^p(\overline{U}', \cdot) \simeq \mathrm{H}^p(\overline{U}, \overline{f}_*(\cdot)).$$

Hence, it suffices to construct a natural isomorphism

$$\overline{f}_*(\widehat{F'}) \simeq (f_*(F'))^\wedge.$$

Over U this is tautological, so we just need to compare stalks at real points v of K along with the associated maps φ_v. With the help of Remark 2.5.2, this amounts to elementary bookkeeping with real and complex places (such as factor fields of $K' \otimes_K \overline{K_v} = (K' \otimes_K K_v) \otimes_{K_v} \overline{K_v}$ for real places v of K) that we leave to the reader. □

5.5 Proof of Artin–Verdier duality

We are ready to prove Theorem 1.2.6. By definition, for the case $p < 0$ this amounts to the statement that if v is a real place and M is a finite I_v-module with dual $M^\vee = \mathrm{Hom}(M, \overline{K}^\times)$ then

$$\mathrm{H}_T^{2-p}(I_v, M) \times \mathrm{H}_T^p(I_v, M^\vee) \to \mathrm{H}^2(I_v, \overline{K}^\times) = \mathrm{H}^2(I_v, \mu_2) = (1/2)\mathbf{Z}/\mathbf{Z}$$

is a perfect pairing. This is exactly Remark 1.1.1.

Now consider the case $p \geq 0$. This was settled by Mazur for totally imaginary K in [Ma] using a massive amount of class field theory. (In such cases $\mathrm{H}_c^0(X, \cdot) = \mathrm{H}^0(X, \cdot)$ is left exact!) We will establish the general case by induction on $p \geq 0$, using the settled cases of $p < 0$ and of totally imaginary K (for all $p \in \mathbf{Z}$) to get the induction started. That induction is the following lemma, which concludes the proof.

Lemma 5.5.1 *Assume that for some $p_0 \geq 0$ the pairing is perfect for any K and any constructible sheaf F with any $p < p_0$. Then it is also perfect in general for $p = p_0$.*

Proof. We choose a finite extension L/K which is totally imaginary, and let $Y := \mathrm{Spec}(\mathcal{O}_L)$. Let $\nu : Y \to X$ the projection map. Consider the natural map over X:

$$F \to \nu_*\nu^*(F).$$

This has trivial kernel, so it fits into a short exact sequence

$$0 \to F \to \nu_*\nu^*(F) \to G \to 0.$$

For any abelian group M, denote by M^D its Pontryagin dual $\mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$. The pairing gives a natural map:

$$\mu^p(F) : \widehat{\mathrm{H}}^p(X, F) \to \mathrm{Ext}_X^{3-p}(F, \mathbf{G}_{m,X})^D$$

that is *δ-functorial* by Proposition 5.4.9. We know the source is a finite group, and the Ext-group is torsion (since F is constructible), so the Ext is finite if and only if its Pontryagin dual is finite, in which case their sizes agree. Thus, our task is equivalent to proving that $\mu^{p_0}(F)$ is always an isomorphism.

Since ν_* is exact (as ν is finite) and carries injectives to injectives, we have for any abelian sheaf F' on Y a natural map

$$\mathrm{Ext}_Y^p(F', \mathbf{G}_{m,Y}) \to \mathrm{Ext}_X^p(\nu_*(F'), \nu_*(\mathbf{G}_{m,Y}))$$

and then we compose this with the natural norm map $\nu_*(\mathbf{G}_{m,Y}) \to \mathbf{G}_{m,X}$ on X_{et} to get

$$\mathrm{Ext}_Y^p(F', \mathbf{G}_{m,Y}) \to \mathrm{Ext}_X^p(\nu_*(F'), \mathbf{G}_{m,X});$$

this is an isomorphism for any F' by [Ma, Thm. 2.7].

Remark 5.5.2 The proof of [Ma, Thm. 2.7] uses δ-functoriality excision arguments to reduce to separate arguments for finite étale f and F' with supported at finitely many closed points. The latter case ultimately reduces to Shapiro's Lemma for Galois cohomology via Tate local duality. There is a fair amount of work involved in this proof!

By Lemma 5.4.10 we have naturally

$$\widehat{\mathrm{H}}^p(X, \nu_*F') = \widehat{\mathrm{H}}^p(Y, F').$$

These identifications are *compatible with the Yoneda pairings* for X and Y by Proposition B.2.4, and the Duality Theorem is true for Y (as L is totally imaginary), so $\mu^p(\nu_* F')$ is an isomorphism for all $p \in \mathbf{Z}$ (this has no content for $p < 0$).

From the above short exact sequences of sheaves on X we get a commutative diagram with exact rows:

$$\begin{array}{ccccc}
\cdots \to \widehat{\mathrm{H}}^{p_0-1}(X, \nu_*\nu^*(F)) & \longrightarrow & \widehat{\mathrm{H}}^{p_0-1}(X, G) & \longrightarrow & \\
\downarrow & & \downarrow & & \\
\cdots \to \mathrm{Ext}_X^{3-p_0}(\nu_*\nu^*(F), \mathbf{G}_m)^D & \longrightarrow & \mathrm{Ext}_X^{3-p_0}(G, \mathbf{G}_m)^D & \longrightarrow &
\end{array}$$

$$\begin{array}{ccccc}
\cdots \to \widehat{\mathrm{H}}^{p_0}(X, F) & \longrightarrow & \widehat{\mathrm{H}}^{p_0}(X, \nu_*\nu^*(F)) & \longrightarrow & \widehat{\mathrm{H}}^{p_0}(X, G) \longrightarrow \\
\downarrow{\scriptstyle \mu^{p_0}(F)} & & \downarrow{\scriptstyle \mu^{p_0}(\nu_*\nu^*(F))} & & \downarrow{\scriptstyle \mu^{p_0}(G)} \\
\cdots \to \mathrm{Ext}_X^{3-p_0}(F, \mathbf{G}_m)^D & \longrightarrow & \mathrm{Ext}_X^{3-p_0}(\nu_*\nu^*(F), \mathbf{G}_m)^D & \longrightarrow & \mathrm{Ext}_X^{3-p_0}(G, \mathbf{G}_m)^D \to
\end{array}$$

The first and fourth vertical arrows are isomorphisms (setting $F' = \nu^*(F)$ above). The second arrow $\mu^{p_0-1}(G)$ is an isomorphism by induction. It follows by a diagram chase that $\mu^{p_0}(F)$ is injective. That proves the injectivity *in general* for μ^{p_0}, so therefore $\mu^{p_0}(G)$ is injective! We run the same diagram chase with this extra information to deduce that the injective $\mu^{p_0}(F)$ is bijective. □

6 Global duality and the finite ramification exact sequence

In this section we prove that Theorem 1.2.6 is basically equivalent to Tate's global duality, and we deduce Tate's long-exact sequence for finite ramification. Equivalence with Tate's duality is an immediate consequence of the following comparison result between étale cohomology and continuous cohomology.

6.1 Tate's theorems

We fix a non-empty open set U in X and let S be the finite set of places of K consisting of the set X_∞ of archimedean places and those corresponding to points in $X - U$. We denote by G_S the étale fundamental group $\pi_1^{\text{ét}}(U)$ of U; this coincides with the Galois group of the maximal extension of K unramified outside S.

Theorem 6.1.1 *Let F be a locally constant and consrtructible sheaf on U whose fiber-rank is a unit on U. There is a canonical isomorphism of δ-functors*

$$\mathrm{H}^p(G_S, F_\eta) = \mathrm{H}^p(U, F)$$

for $p \geq 0$, and it is compatible with cup products.

In Appendix A we give a proof of Theorem 6.1.1, which relies on class field theory, among the other ingredients.

We now construct Tate's 9-term long-exact sequence for finite ramification from Artin–Verdier duality. Let M be a finite discrete G_S-module. For any closed point $s \in \overline{X}$, denote by $\mathrm{H}^p(K_s, M)$ the group $\mathrm{H}^p(D_s, M)$ if s is a finite point and the Tate cohomology group $\mathrm{H}_T^p(I_s, M)$

if s is a real point (this is a quotient of $\mathrm{H}^0(I_s, M)$ for $p = 0$ and agrees with usual I_s-cohomology if $p > 0$).

Theorem 6.1.2 *Let M be a finite discrete G_S-module whose order is a unit on U. Then there exists an exact sequence:*

$$0 \longrightarrow \mathrm{H}^0(G_S, M) \longrightarrow \bigoplus_{s\in S} \mathrm{H}^0(K_s, M) \longrightarrow \mathrm{H}^2(G_S, M^\vee)^D \longrightarrow \mathrm{H}^1(G_S, M)$$

$$\longrightarrow \bigoplus_{s\in S} \mathrm{H}^1(K_s, M) \longrightarrow \mathrm{H}^1(G_S, M^\vee)^D \longrightarrow \mathrm{H}^2(G_S, M) \longrightarrow \bigoplus_{s\in S} \mathrm{H}^2(K_s, M)$$

$$\longrightarrow \mathrm{H}^0(G_S, M^\vee)^D \longrightarrow 0$$

naturally in M, where $M^\vee := \mathrm{Hom}(M, \mathbf{G}_{m,\eta})$ is the Cartier dual Galois module and we denote by $(\cdot)^D$ the Pontryagin dual.

Proof. Since G_S is the fundamental group of U, M defines a locally constant sheaf F_M on U. We denote by $j : U \to X$ the inclusion. By Lemma 5.4.6, we have canonical isomorphisms:

$$\mathrm{Ext}^p_X(j_!(F_M), \mathbf{G}_{m,X}) \simeq \mathrm{Ext}^p_U(F_M, \mathbf{G}_{m,U}) \simeq \mathrm{H}^p(U, \underline{\mathrm{Hom}}(F_M, \mathbf{G}_{m,U})).$$

We denote the dual sheaf $\underline{\mathrm{Hom}}(F_M, \mathbf{G}_{m,U})$ by $F_M^\vee$. Since the order of F_M is a unit on U, $F_M^\vee$ is locally constant on U; its generic fiber is $M^\vee$. By Theorem 6.1.1 applied to F_M, we get an isomorphism:

$$\mathrm{Ext}^p_X(j_!(F_M), \mathbf{G}_{m,X}) \simeq \mathrm{H}^p(G_S, M^\vee). \tag{12}$$

The local cohomology sequence of §3.2 becomes

$$\cdots \to \bigoplus_{s\in S} \mathrm{H}^p_s(X, j_!(F_M)) \to \widehat{\mathrm{H}}^p(X, j_!(F_M)) \to \mathrm{H}^p(U, F_M) \to \bigoplus_{s\in S} \mathrm{H}^{p+1}_s(X, j_!(F_M)). \tag{13}$$

This sequence will be the desired 9-term long-exact sequence for finite ramification upon unraveling its terms and some maps (e.g., to ensure that $\text{Ш}^1_S(K, \cdot)$ and $\text{Ш}^2_S(K, \cdot)$ and the perfect duality between them can be extracted from the long exact sequence).

By Theorem 6.1.1 we have

$$\mathrm{H}^p(U, F_M) = \mathrm{H}^p(G_S, M).$$

Moreover, by Artin–Verdier duality (!) and dualizing (12) we get:

$$\widehat{\mathrm{H}}^p(X, j_!(F_M)) \simeq \mathrm{Ext}^{3-p}_X(j_!(F_M), \mathbf{G}_{m,X})^D \simeq \mathrm{H}^{3-p}(G_S, M^\vee)^D.$$

We are left to interpret $\mathrm{H}^p_s(X, j_!(F_M))$ and some of the maps in the long exact sequence (13).

Suppose s is a real point. From Definition 5.3.1, we have:

$$\mathrm{H}^p_s(X, j_!(F_M)) = \mathrm{H}^{p-1}_T(I_s, M) = \mathrm{H}^{p-1}(K_s, M)$$

(where the final equality is our *definition* of $\mathrm{H}^\bullet(K_s, \cdot)$ for real s, coinciding with usual Galois cohomology in positive degrees). Let us now assume $s \in S$ is a finite point, and denote by $f_s : X_s^{\mathrm{h}} \to X$ the henselization of X at s. By Lemma 4.1.3 we have:

$$\mathrm{H}^p_s(X, j_!(F_M)) = \mathrm{H}^p_s(X_s^{\mathrm{h}}, f_s^* j_!(F_M)).$$

The latter group can be computed from the local cohomology sequence for a henselian discrete valuation ring A (such as that of X at s), giving an isomorphism

$$\mathrm{H}^p_s(X, j_!(F_M)) \simeq \mathrm{H}^{p-1}(D_s, M) = \mathrm{H}^{p-1}(K_s, M)$$

because of the *vanishing* of the the flanking terms $\mathrm{H}^p(\mathrm{Spec}(A), j'_!(G))$ for the open immersion $j' : \mathrm{Spec}(L) \hookrightarrow \mathrm{Spec}(A)$ of the generic point and an étale sheaf G on the generic point (since the degree-p cohomology of a sheaf on $\mathrm{Spec}(A)$ coincides with the degree-p cohomology of the fiber sheaf at the closed point due to A being henselian, so it vanishes when that fiber sheaf is 0).

Next, we describe some of the maps that emerge via these identifications:

Lemma 6.1.3 *For M as in Theorem 6.1.2 and $p \geq 0$, the composite isomorphism*

$$\mathrm{H}^p(G_S, M) \simeq \mathrm{H}^p(U, F_M) \to \bigoplus_{s \in S} \mathrm{H}^{p+1}_s(X, j_!(F_M)) \simeq \bigoplus_{s \in S} \mathrm{H}^p(K_s, M)$$

is the restriction map and moreover the map $\bigoplus_{s \in S} \mathrm{H}^1(K_s, M) \to \mathrm{H}^1(U, M^\vee)^D$ defined via identification with the outer terms in

$$\bigoplus_{s \in S} \mathrm{H}^2_s(X, j_!(F_M)) \to \widehat{\mathrm{H}}^2(X, j_!(F_M)) \simeq \mathrm{Ext}^1_X(j_!(F_M), \mathbf{G}_{m,X})^D \simeq \mathrm{H}^1(G_S, M^\vee)^D$$

is Pontryagin dual to the map $\mathrm{H}^1(G_S, M^\vee) \to \bigoplus_{s \in S} \mathrm{H}^1(K_s, M)^D$ defined by Tate local duality $\mathrm{H}^1(K_s, M) \times \mathrm{H}^1(K_s, M^\vee) \to \mathbf{Q}/\mathbf{Z}$ for all $s \in S$ (including real s!).

These descriptions ensure that the 9-term exact sequence we will build yields perfect pairings between $Ш^1_S(K, M^\vee)$ and $Ш^2_S(K, M)$ (entirely opposite to Tate's approach that directly constructed such perfect pairings and used that to *define* parts of the 9-term exact sequence). The degree-1 identification is not entirely a tautology, since (for non-real s) the degree-1 local duality pairing changes by a sign if we swap the factors in the pairing!

Proof. Consider the s-component of the first assertion. If s is real then is reduced to the description for $p \geq 0$ of connecting maps from global to local cohomology at a real point for arbitrary abelian sheaves on X as in the proof of Theorem 4.3.1. The case of finite s is handled by a simpler version of the same limit trick as in the case of real points, namely we use compatibility with residually trivial pointed étale neighborhoods of s to reduce to the analogous identification when X is replaced with $\mathrm{Spec}(\mathscr{O}^{\mathrm{h}}_{X,s})$, in which case our task is a tautology in view of how $\mathrm{H}^{p+1}_s(X, j_!(F_M))$ is identified with $\mathrm{H}^p(K_s, M)$ via a local cohomology sequence on $\mathrm{Spec}(\mathscr{O}^{\mathrm{h}}_{X,s})$. The link to Tate local duality in degree 1 lies deeper, and we address it in Proposition B.3.1. □

It remains to check exactness at the endpoints of the 9-term sequence; i.e., that (13) is injective from $\mathrm{H}^0(U, F_M)$ and is surjective onto $\widehat{\mathrm{H}}^3(X, j_!(F_M))$. By Lemma 6.1.3, the injectivity expresses the obvious fact that the natural map $\mathrm{H}^0(G_S, M) \to \bigoplus_{s \in S} \mathrm{H}^0(K_s, M)$ is injective: if S contains a finite point s_0 then even the projection to the s_0-factor is injective, and if S consists only of the real places then $U = X$ yet the order of M is a unit on U, forcing $M = 0$. By Lemma 6.1.3 the surjectivity is the assertion that the natural restriction map

$$\mathrm{H}^3(G_S, M) \to \bigoplus_{s \in S} \mathrm{H}^3(K_s, M)$$

is injective, and even its projection to the direct sum over the real points is injective (in fact, an isomorphism!) due to Theorem 4.3.1. This completes the proof. □

A Relating Galois cohomology and étale cohomology

A.1 A global comparison

We begin by proving Theorem 6.1.1, which we restate here for convenience.

Theorem A.1.1 *Let F be a locally constant constructible sheaf on U whose fiber-rank is a unit on U. There is a natural isomorphism of δ-functors*

$$\mathrm{H}^p(G_S, F_\eta) = \mathrm{H}^p(U_{\mathrm{et}}, F)$$

for $p \geq 0$, and it is compatible with cup products.

Proof. We consider U_{et}, which is equivalent to the category of finite étale K-algebras unramified outside S. Therefore, $\mathrm{H}^\bullet(U_{\text{fét}}, \cdot)$ is G_S-cohomology on discrete G_S-modules. We consider the functor

$$U_{\text{fét}} \to U_{\mathrm{et}}$$

assigning to each finite étale U-scheme the same scheme viewed as an étale U-scheme. This yields a morphism of categories of sheaves of sets

$$\mu = (\mu_*, \mu^*) : \mathrm{Shv}(U_{\mathrm{et}}) \to \mathrm{Shv}(U_{\text{fét}})$$

for which μ_* sends an étale sheaf of sets on $U_{\text{ét}}$ its restriction on $U_{\text{fét}}$ and the corresponding pullback μ^* assigns to each sheaf of sets F on $U_{\text{fét}}$ the sheafification of the presheaf $V \mapsto F(V')$ where V' is the unique maximal finite étale U-scheme dominated by V. (If $V = \emptyset$ then $V' = \emptyset$.)

Since μ^* is exact, implying that μ_* preserves injectives, the equality

$$\Gamma(U_{\text{ét}}, \cdot) = \Gamma(U_{\text{fét}}, \cdot) \circ \mu_*$$

yields a Grothendieck spectral sequence

$$\mathrm{H}^p(U_{\text{fét}}, \mathrm{R}^q\mu_*(A)) \Rightarrow \mathrm{H}^{p+q}(U_{\text{ét}}, A)$$

for any étale abelian sheaf A on U. We shall prove that the edge map

$$\mathrm{H}^p(G_S, \mu_*(F)_\eta) = \mathrm{H}^p(U_{\text{fét}}, \mu_*(F)) \to \mathrm{H}^p(U_{\mathrm{et}}, F)$$

is an isomorphism when F is lcc with order a U-unit (and then at the end we will address why this is a map of δ-functors compatible with cup products). It suffices to prove that for every such F, $\mu_*(F)_\eta = F_\eta$ and $\mathrm{R}^q\mu_*(F)$ vanishes for all $q > 0$. The equality $\mu_*(F)_\eta = F_\eta$ is immediate from the definition of μ_* since the lcc hypothesis on F implies that F_η as a $\mathrm{Gal}(K_s/K)$-module is a G_S-module. (In general, if F is an arbitrary abelian sheaf on U_{et} then $\mu_*(F)_\eta = F_\eta^{\mathrm{Gal}(\overline{K}/K_S)}$.)

Higher direct images are always computed as "sheafified cohomology", so upon renaming a connected finite étale cover of U as U it suffices to show that

$$\varinjlim \mathrm{H}^p(U'_{\text{fét}}, F) = 0$$

for all $p > 0$, the colimit running over all connected finite étale covers $U' \to U$ dominated by the chosen K_S.

Up to replacing again U with a suitable connected finite étale cover, we reduce to the case $F = \mathbf{Z}/n\mathbf{Z}$ with n invertible on U such that the function field K of U contains the n-th roots of unity and is totally imaginary. For any connected finite étale $U' \to U$ we will use the Kummer sequence

$$0 \to \mathbf{Z}/n\mathbf{Z} \to \mathbf{G}_{m,U'} \xrightarrow{\times n} \mathbf{G}_{m,U'} \to 0$$

that yields the exact sequence:

$$0 \longrightarrow \mathrm{H}^0(U, \mathbf{Z}/n\mathbf{Z}) \longrightarrow \mathcal{O}_{K,S}^{\times} \xrightarrow{t^n} \mathcal{O}_{K,S}^{\times} \longrightarrow$$

$$\mathrm{H}^1(U, \mathbf{Z}/n\mathbf{Z}) \longrightarrow \mathrm{Pic}(U) \xrightarrow{\times n} \mathrm{Pic}(U) \longrightarrow \mathrm{H}^2(U, \mathbf{Z}/n\mathbf{Z}) \longrightarrow$$

$$\mathrm{H}^2(U, \mathbf{G}_{m,U}) \xrightarrow{\times n} \mathrm{H}^2(U, \mathbf{G}_{m,U}) \longrightarrow \mathrm{H}^3(U, \mathbf{Z}/n\mathbf{Z})$$

Since K is totally imaginary, so $\overline{U} = U$, Theorem 4.1.1 and Proposition 4.2.2 imply $\mathrm{H}^p(U, \mathbf{Z}/n\mathbf{Z}) = 0$ for all $p \geq 3$.

To show $\varinjlim \mathrm{H}^p(U', \mathbf{Z}/n\mathbf{Z}) = 0$ for $p = 1, 2$ as U' varies through the connected finite étale covers of U dominated by K_S, first note that $\varinjlim \mathcal{O}_{U'}^{\times} = \mathcal{O}_{K_S,S}^{\times}$ is n-divisible (as n is an S-unit), and $\varinjlim \mathrm{Pic}(U') = 1$ due to the Principal Ideal Theorem (as we saw in the proof of Proposition 1.2.2); this gives the vanishing for $p = 1$.

To settle $p = 2$ it suffices to prove

$$\varinjlim_{U'} \mathrm{H}^2(U'_{\text{ét}}, \mathbf{G}_{m,U'})[n] = 0,$$

By Grothendieck's work on Brauer groups in the case of Dedekind schemes (applying the end of [Mi, Ch. 13] to a connected Dedekind scheme), $\mathrm{H}^2(U'_{\text{ét}}, \mathbf{G}_{m,U'})$ is the group of Brauer classes of the function field K' which are locally trivial at all places corresponding to closed points. Hence, it suffices to show that every class in $\mathrm{Br}(K')[n]$ locally trivial outside S is split by some finite extension of K' unramified outside S.

Let L/K' be an extension which is unramified along U'. Let w be a place of K' over a place $u \in S$. By local class field theory we get a commutative diagram:

$$\begin{array}{ccc} \mathrm{Br}(K'_u) & \xrightarrow{\text{inv}} & \frac{1}{n}\mathbf{Z}/\mathbf{Z} \\ \downarrow & & \downarrow{\scriptstyle [L_w:K'_u]} \\ \mathrm{Br}(L_w) & \xrightarrow[\text{inv}]{} & \frac{1}{n}\mathbf{Z}/\mathbf{Z} \end{array}$$

It is sufficient to construct an extension L/K' such that it is unramified along U' and all local degrees $[L_w : K'_u]$ are divisible by n for all $u \in S$ and $w|u$. If we let H' be the Hilbert class field of K' then all maximal ideals $q \in S$ become principal in H' by the Principal Ideal Theorem , say generated by some f_q. The extension

$$L := H'(\{f_q^{1/n}, q \in S\})$$

does the job. Indeed, passing to L makes all classes in $\mathrm{Br}(K')[n]$ locally trivial at all places (he archimedean ones being complex), hence globally trivial. This completes the proof of the desired natural isomorphism.

It remains to show that the edge-map isomorphisms $f_F^p : \mathrm{H}^p(G_S, F_\eta) \simeq \mathrm{H}^p(U_{\mathrm{et}}, F)$ for lcc F constitute a map of δ-functors compatible with cup products. The compatibility with δ-functoriality is a special case of a general property of edge maps along the bottom in the Grothendieck spectral sequence: if $G : C \to C'$ and $G' : C' \to C''$ are left-exact functors between abelian categories with enough injectives and G carries injectives to G'-acyclics then we claim that the edge map

$$\mathrm{R}^p(G') \circ G \to \mathrm{R}^p(G' \circ G)$$

respects the δ-functor structure on both sides when applied to a short exact sequence $0 \to F_1 \to F_2 \to F_3 \to 0$ in C *for which* applying G preserves short-exactness (we apply this with $G = \mu_*$, $G' = \Gamma(U_{\mathrm{f\acute{e}t}}, \cdot)$, and the F_j an abelian sheaf on U_{et} arising from a discrete G_S-module M_j). This is an elementary diagram-chasing exercise in reviewing how the Grothendieck spectral sequence is constructed by means of Cartan–Eilenberg resolutions, so it is left to the interested reader.

For *any* discrete G_S-module M and the associated abelian sheaf F_M on U_{et}, we have $\mu_*(F_M)_\eta = M$ as discrete modules over $\mathrm{Gal}(\overline{K}/K)$ (acting on M through its quotient G_S). Hence, we have edge maps

$$f_M^p : \mathrm{H}^p(G_S, M) = \mathrm{H}^p(U_{\mathrm{f\acute{e}t}}, \mu_*(F_M)) \to \mathrm{H}^p(U_{\mathrm{et}}, F_M)$$

for any such M. (This edge map is *not* claimed to be an isomorphism for all such M, especially without torsion hypotheses.) The δ-functoriality argument just given applies to all such M; i.e., there no need to assume M is finite with order a unit on U. But in that generality the class of M's under consideration is far richer than just finite discrete G_S-modules with order a unit on U. In particular, $\mathrm{H}^\bullet(G_S, \cdot)$ is a *derived functor* on this huge class of such M's, so we can apply Grothendieck's theorem on the universality of erasable δ-functors to conclude that the collection of edge maps $\{f_M^p\}_p$ is the *unique* such map of δ-functors. Hence, to establish compatibility with cup products it suffices to produce *some* map of δ-functors $\mathrm{H}^\bullet(G_S, M) \to \mathrm{H}^\bullet(U_{\mathrm{et}}, F_M)$ in discrete G_S-modules M such that it is compatible with cup products. (The point is that for M of finite order that is a unit on U such an alternative map must recover the isomorphisms that are the focus of interest in the statement of Theorem A.1.1, so these isomorphisms would be compatible with cup products as desired.)

The key observation is that if we view M as a sheaf on $U_{\mathrm{f\acute{e}t}}$ then via the natural isomorphism $M \simeq \mu_*(F_M)$ we get an adjoint map $h_M : \mu^*(M) \to F_M$ and the latter is *also* an isomorphism. To check this latter isomorphism property we may use compatibility with direct limits in M to reduce to the case when M is finitely generated as a discrete G_S-module, so M is split by a connected finite étale cover of U. The formation of the map h_M is compatible with pullback to a connected finite étale cover of U, so we may reduce to the case when the G_S-action on M is trivial, so M on $U_{\mathrm{f\acute{e}t}}$ is a constant sheaf and hence $\mu^*(M)$ is the "same" constant sheaf on U_{et} because if $V \to U$ is étale with V connected (and non-empty) then the maximal finite étale U-scheme V' through which $V \to U$ factors is also connected. Since F_M is also the "same" constant sheaf, in this way we see that h_M is identified with the identity map on that constant sheaf and so it is an isomorphism.

Now our aim is to give an "alternative" construction of a map of δ-functors $\mathrm{H}^\bullet(U_{\mathrm{f\acute{e}t}}, \cdot) \to \mathrm{H}^p(U_{\mathrm{et}}, \mu^*(\cdot))$ which is compatible with cup products. But for any map of topoi (such as (μ_*, μ^*)!) there is an associated δ-functorial pullback map in abelian cohomology, and it is *always* compatible with cup products (as cup products may always be computed in terms of composition of maps in derived categories, as an instance of Ext-pairings). □

A.2 Local computations

Now let A be a discrete valuation ring with fraction field K and residue field k. For the open immersion $j : \mathrm{Spec}(K) \to \mathrm{Spec}(A)$ and its closed complement $i : \mathrm{Spec}(k) \to \mathrm{Spec}(A)$ we have the usual exact sequence

$$(*) \quad 0 \to \mathbf{G}_{m,A} \to j_*(\mathbf{G}_{m,K}) \to i_*(\mathbf{Z}) \to 0$$

as in Artin's calculation of cohomology on smooth curves over a field.

Proposition A.2.1 *Assume A is henselian, its residue field k is perfect, and $\widehat{K}$ is separable over K. Then*

$$\mathrm{R}^p j_*(\mathbf{G}_{m,K}) = 0, \quad p > 0.$$

The separability of $\widehat{K}$ over K is automatic in characteristic 0 and in general says that A is excellent.

Proof. The stalk at the generic point vanishes, ultimately because j_* carries injectives to injectives and $j^* j_* \simeq \mathrm{id}$ (good exercise). Thus, we need to compute the stalk at a geometric closed point. As for higher direct images in general, this is the direct limit of the cohomologies of the K-fibers of local-etale neighborhoods of the closed point, which is to say $\mathrm{H}^p(K^{\mathrm{sh}}, \mathbf{G}_m)$ where K^{sh} is the fraction field of A^{sh}, or equivalently the maximal unramified extension of K. We may rename A^{sh} as A to reduce to the case that k is algebraically closed. Hilbert 90 gives vanishing for $p = 1$, and for $p = 2$ the assertion is that $\mathrm{Br}(K) = 1$.

Granting the vanishing of the Brauer group of such K in general, the same then holds for all of its finite separable extensions too. Consequently, by the criterion in §3.1 of Chapter II of Serre's *Galois Cohomology*, the field K has cohomological dimension ≤ 1 (i.e., its Galois cohomology vanishing beyond degree 1 on torsion discrete Galois modules). But then it is strict cohomological dimension ≤ 2 (i.e., vanishing Galois cohomology beyond degree 2 on *all* discrete Galois modules), by §3.2 in Chapter I of Serre's book. Hence, we would get the desired vanishing for all $p > 2$.

It remains to prove that $\mathrm{Br}(K) = 1$ when k is algebraically closed. If A is complete then this is Example (b) in §7 of Chapter X of Serre's "Local Fields". To settle the general case, first note that since A is henselian, by Krasner's Lemma completion defines an equivalence between the finite separable extensions of K and $\widehat{K}$. Hence, the Galois groups of K and $\widetilde{K}$ coincide. In particular, for any $n > 0$ not divisible by $\mathrm{char}(K)$ the natural map

$$\mathrm{H}^2(K, \mu_n) \to \mathrm{H}^2(\widehat{K}, \mu_n)$$

is an isomorphism. But this is the map between n-torsion in the Brauer groups, so it follows that the map

$$\mathrm{Br}(K) \to \mathrm{Br}(\widehat{K})$$

is an equality if $\mathrm{char}(K) = 0$ and away from p-primary parts if $\mathrm{char}(K) = p > 0$.

To settle the case $p = \mathrm{char}(K) > 0$, we will use Artin approximation (which applies to systems of polynomial equations over the henselian local A since it is excellent); one can avoid Artin approximation by instead appealing to the deep result in Lang's thesis that such fields K are C_1-fields. More specifically, one of the several sufficient conditions for the Brauer groups of all finite separable extensions of K to vanish is that for every finite separable extension K'/K and finite Galois extension L/K' the norm map $L^\times \to {K'}^\times$ is surjective. It is harmless to rename such K' as K and we want to show that the norm map $L^\times \to K^\times$ is surjective. The completion $\widehat{L}$ coincides with $\widehat{K} \otimes_K L$, and the settled result in the complete case ensures that the norm map $\widehat{L}^\times \to \widehat{K}^\times$ is surjective. In particular, for every $c \in K^\times$ the norm equation $\mathrm{N}_{L/K}(x) = c$ viewed as a polynomial equation over K (via a K-basis of L) has a solution in $\widehat{K} = K \otimes_A \widehat{A}$. This norm equation is easily rewritten as a polynomial equation over A which has a solution in $\widehat{A} - \{0\}$, so by Artin approximation there exists a nearby solution in A. □

Finally, we recall that there is a right adjoint $i^!$ to i_* (i.e., $\mathrm{Hom}(i_*(M), F) = \mathrm{Hom}(M, i^!(F))$ for a discrete G_k-module F and étale sheaf F on $\mathrm{Spec}(A)$). In the language of Artin's decomposition lemma applied to $\mathrm{Spec}(A)$, it is easily checked that

$$i^!(M, N, \varphi : M \to N^I) = \ker \varphi$$

has the correct adjunction property. Since $j_*(N) = (N^I, N, \mathrm{id}_{N^I})$ it follows that $i^! \circ j_* = 0$. Likewise, since $i_*(M) = (M, 0, 0)$, clearly $i^! i_* = \mathrm{id}$. Using this, we shall prove:

Proposition A.2.2 *We have:*

$$\mathrm{R}^p i^!(\mathbf{G}_{m,A}) = \begin{cases} 0 & \text{if } p \neq 1 \\ \mathbf{Z} & \text{if } p = 1 \end{cases}$$

Proof. To prove this we use the exact sequence $(*)$, and compute the outer terms. We claim:

$$(\mathrm{R}^p i^!) j_*(\mathbf{G}_{m,K}) = 0, \quad p \geq 0.$$

Indeed, j_* preserves injectives, and hence by Grothendieck's composition of functors sequence:

$$(\mathrm{R}^p i^!)(\mathrm{R}^q j_*)(\mathbf{G}_{m,K}) \Rightarrow \mathrm{R}^{p+q}(i^! j_*)(\mathbf{G}_{m,K}).$$

Next we use the fact that $i^! j_* = 0$, and deduce the claim from Proposition A.2.1.

Now it suffices to show

$$(\mathrm{R}^p i^!) i_*(\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } p = 0 \\ 0 & \text{if } p \neq 0. \end{cases}$$

Indeed, i_* is exact and preserves injectives, and then we have:

$$(\mathrm{R}^p i^!) i_* = \mathrm{R}^p(i^! i_*) = \mathrm{R}^p(\mathrm{id}).$$

This gives the result. □

B Homological compatibilities

To avoid disrupting the exposition with long compatibility checks that may not be so interesting, we place some of those verifications in this appendix.

B.1 Global to local restriction maps

Proposition B.1.1 *The isomorphism f^p_M in (4) for $p \geq 3$ is the natural restriction map.*

Proof. To determine f^p_M, we consider it as a special case of a much more general problem, going far beyond isomorphisms and locally constant sheaves. In Lemma 3.2.3 we computed the δ-functor $\mathrm{H}^\bullet_{\widetilde{v}}(\overline{U}, G)$ in a general abelian sheaf G on $\overline{U}$. The local cohomology sequence for $U \hookrightarrow \overline{U}$ provides connecting maps

$$\mathrm{H}^p(U, G|_U) \to \bigoplus_{\widetilde{v}} \mathrm{H}^{p+1}_{\widetilde{v}}(\overline{U}, G)$$

for $p \geq 0$ that are δ-functorial by Remark 3.2.2 (where the sign problem doesn't arise at real points since Tate cohomology at real points is 2-torsion and hence insensitive to signs!). In view of how f^p_M was built for $p \geq 3$, it therefore suffices to show that for all such G and all $p \geq 1$ the composite map

$$\mathrm{H}^p(U, G|_U) \to \bigoplus_{\widetilde{v}} \mathrm{H}^{p+1}_{\widetilde{v}}(\overline{U}, G) \simeq \bigoplus_{\widetilde{v}} \mathrm{H}^p(I_v, G_\eta)$$

is the natural restriction map.

By standard dimension-shifting arguments (using crucially that we allow *arbitrary* G so that we may avail ourselves of the existence of enough injectives in the category of all abelian sheaves on $\widetilde{U}$), the δ-functoriality in Remark 3.2.2 (again, no sign problem with Tate cohomology at real places) and the δ-functoriality in Lemma 3.2.3 reduce this task to the case $p = 1$. We need to push one step further into degree 0, as follows.

In Lemma 3.2.3 we identified the δ-functor $\mathrm{H}^\bullet_{\widetilde{v}}(\overline{U}, G)$ in degree 1 with the quotient

$$\mathrm{coker}(\varphi_{\widetilde{v}} : G_{\widetilde{v}} \to \mathrm{H}^0(I_v, G_\eta))$$

of $\mathrm{H}^0(I_v, G_\eta)$. Moreover, by *construction*, the δ-functor structure on the concrete side of the statement of Lemma 3.2.3 is induced in degree 0 by the usual connecting map in group cohomology. Hence, running the same argument once again reduces our task to checking for *arbitrary* abelian sheaves G on $\overline{U}$ that the composite map

$$f_G : \mathrm{H}^0(U, G|_U) \to \bigoplus_{\widetilde{v}} \mathrm{H}^1_{\widetilde{v}}(\overline{U}, G) \simeq \bigoplus_{\widetilde{v}} \mathrm{coker}(\varphi_{\widetilde{v}} : G_{\widetilde{v}} \to \mathrm{H}^0(I_v, G_\eta))$$

is equal to the composition of the natural restriction map $\mathrm{H}^0(U, G|_U) \to \bigoplus_{\widetilde{v}} \mathrm{H}^0(I_v, G_\eta)$ followed by the natural quotient map.

We may and do focus on describing the composition $f_{G,v}$ of f_G with projection to the v-component for a *fixed* choice of real place v of K. It is easy to check from the construction that $f_{G,v}$ is compatible with pullback to a connected étale U-scheme U' arising from a number field K'/K inside K_v^{alg} (equipped with the induced replace place over v), and the concrete map we want to prove coincides with $f_{G,v}$ is also compatible with such pullback. Hence, we can pass to the limit over such U' to reduce to the analogous compatibility question when U is replaced with $\mathrm{Spec}(K_v^{\mathrm{alg}})$, we use the real place v' on the real closed field K_v^{alg} arising from its tautological inclusion into the completion $K_v = \mathbf{R}$, and $\mathrm{H}^\bullet_{\widetilde{v}}(\overline{U}, \cdot)$ is replaced with the derived functor of the analogue of $\mathrm{H}^0_{\widetilde{v}}(\overline{U}, \cdot)$ for $\mathrm{Spec}(K_v^{\mathrm{alg}})$ equipped with v'. Note that I_v is naturally identified with the Galois group of K_v^{alg}, so the role of $G_\eta^{I_v}$ collapses to the global sections over K_v^{alg}!

Our new setup is as follows. We have a (real closed) field E, the category $\mathbf{Ab}_Y$ of abelian étale sheaves on $Y = \mathrm{Spec}(E)$, and an "extended" category $\mathbf{Ab}_{\overline{Y}}$ consisting of triples $G = (G_0, G_\eta, \varphi)$ where $G_\eta \in \mathbf{Ab}_Y$, G_0 is an abelian group, and $\varphi : G_0 \to G_\eta(E)$ is a homomorphism into the global sections. By the analogue of Example 2.4.5, the functor $\Gamma(\overline{Y}, \cdot)$ assigns to G the fiber product $G_\eta(E) \times_{G_\eta(\overline{E}), \varphi} G_0 = G_0$ and the restriction functor $\Gamma(\overline{Y}, G) \to \Gamma(Y, G)$ is $\varphi : G_0 \to G_\eta(E)$. Hence, the role of $\mathrm{H}^0_{\widetilde{v}}(U, \cdot)$ is played by the functor $\mathrm{H}^0_* : G \rightsquigarrow \ker \varphi$. Our task is to prove that via the isomorphism $\mathrm{H}^1_*(G) \simeq \mathrm{coker}(\varphi)$ provided by (the proof of) the analogue of Lemma 3.2.3 for the present setting, the natural connecting map $G_\eta(E) = \Gamma(Y, G) \to \mathrm{H}^1_*(G) \simeq \mathrm{coker}(\varphi)$ is *exactly* the canonical quotient map.

Consider the initial short exact sequence $0 \to G \to I \to N \to 0$ arising from an injective resolution of G. This yields a commutative diagram with exact rows and columns as indicated:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
0 & \longrightarrow & \ker\varphi_N & \longrightarrow & N_0 & \xrightarrow{\varphi_N} & N_\eta(F) & & \\
 & & \uparrow{\scriptstyle f} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \ker\varphi_I & \longrightarrow & I_0 & \xrightarrow{\varphi_I} & I_\eta(F) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \ker\varphi_G & \longrightarrow & G_0 & \xrightarrow{\varphi_G} & G_\eta(F) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Applying the snake lemma to the diagram given by the right two columns provides a connecting isomorphism $\mathrm{coker} f \simeq \mathrm{coker}\varphi_G$ that is exactly the isomorphism $\mathrm{H}^1_*(G) \simeq \mathrm{coker}\varphi_G$ provided by the proof of Lemma 3.2.3. Applying the snake lemma to the diagram given by the upper two rows provides a connecting isomorphism $\mathrm{coker}\varphi_G \simeq \mathrm{coker} f$ whose composition with the natural quotient map $G_\eta(F) \to \mathrm{coker}\varphi_G$ is the definition of the connecting map $G_\eta(F) \to \mathrm{H}^1_*(G)$ in the "local cohomology" exact sequence.

Our task is now reduced to showing that the connecting isomorphisms that we have just constructed in opposite directions between $\mathrm{coker}\varphi_G$ and $\mathrm{coker} f$ (one using the right two columns, the other using the upper two rows) are inverse to each other. As an assertion in homological algebra for any such commutative diagram with the indicated exactness in the rows and columns, this is a trivial diagram chase (traversing in opposite directions along the same zig-zag staircase path between $G_\eta(F)$ and $\ker\varphi_N$ passing through $I_\eta(F)$, I_0, and N_0). □

B.2 Properties of some pairings

For a dense open $U \subset X = \mathrm{Spec}(\mathcal{O}_K)$ and an abelian sheaf F on U_{et}, in §5.4 we defined general pairings

$$\mathrm{H}^p_c(U, F) \times \mathrm{Ext}^{3-p}_U(F, \mathbf{G}_{m,U}) \to \mathbf{Q}/\mathbf{Z} \tag{14}$$

naturally in F for any $p \in \mathbf{Z}$, assuming F is constructible when $p < 0$. This relied on Ext-pairings on $\overline{X}$ for $p \geq 0$ (using the extension-by-zero of F to X) and an entirely different procedure with Tate cohomology when $p < 0$ (using Corollary 5.4.7 and (11)). In this section we establish some properties of (14) that underlie how it is analyzed in proofs (such as for Artin–Verdier duality).

The proof of Proposition 5.4.9 addresses the δ-functoriality of (14) except for the case that links up the two aspects of how the pairing is defined (i.e., relating $p \geq 0$ and $p < 0$), which we now address:

Proposition B.2.1 *The pairing (14) is δ-functorial from degree -1 to degree 0 for constructible abelian F.*

Proof. [1] □

[1] To be filled in.

Now we turn to refinements of the isomorphism in Lemma 5.4.10: defining it in all degrees $p \in \mathbf{Z}$ (not just $p \geq 0$), establishing its δ-functoriality, and especially proving that it is compatible with pairings as in (14) on U and U' via the composite map

$$\mathrm{Ext}^i_{U'}(F', \mathbf{G}_{m,U'}) \to \mathrm{Ext}^i_U(f_*(F'), f_*(\mathbf{G}_{m,U'})) \to \mathrm{Ext}^i_U(f_*(F'), \mathbf{G}_{m,U})$$

(first step using exactness of f_*, and second step using the evident "norm" map); this composite map is an isomorphism for constructible F' by [Ma, Thm. 2.7] (see Remark 5.5.2) but that is not relevant to the compatibility assertion. This compatibility with finite pushforward is essential for bootstrapping the proof of Artin–Verdier duality from the totally imaginary case.

The definition of (14) rests on the global trace isomorphism $\mathrm{Tr}_U : \mathrm{H}^3_c(U, \mathbf{G}_{m,U}) \simeq \mathbf{Q}/\mathbf{Z}$ whose compatibility with shrinking U was addressed in Remark 5.4.3. A further compatibility we shall require involves comparing local and global trace maps, in the sense of a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(K_v) & \xrightarrow{\mathrm{inv}_v} & \mathbf{Q}/\mathbf{Z} \\ \downarrow & & \| \\ \mathrm{H}^3_c(U, \mathbf{G}_m) & \xrightarrow[\simeq]{\mathrm{Tr}_U} & \mathbf{Q}/\mathbf{Z} \end{array} \tag{15}$$

for all points $v \in \overline{X} - U$. This will be largely a matter of review of definitions (such as of Tr_U in terms of Tr_X), and is especially useful for finite v (i.e., $v \in X - U$) since in such cases inv_v is an isomorphism (thereby making the left side of (15) an isomorphism).

We first define the map along the left side of this diagram before we prove that it commutes. It is convenient to initially work with a general abelian sheaf F on U rather than specifically with $\mathbf{G}_{m,U}$. Letting $j : U \hookrightarrow X$ be the open immersion, the local cohomology sequence (7) for $j_!F$ relative to the dense open inclusion of U into X has the form

$$\cdots \to \bigoplus_{v \in \overline{X} - U} \mathrm{H}^p_v(X, j_!(F)) \to \mathrm{H}^p_c(U, F) \to \mathrm{H}^p(U, F) \to \cdots \tag{16}$$

where the local term at real v coincides with $\mathrm{H}^p_v(U, F)$ as in Definition 5.3.1 (using (6)).

For any $v \in \overline{X} - U$, the v-term in this local cohomology sequence is $\mathrm{H}^{p-1}(K_v, F_\eta)$ by arguments treating real points and points of $X - U$ separately; this is shown the proof of Theorem 6.1.2 in a special case, and for the convenience of the reader we now reproduce the argument in general. If v is real then by Definition 5.3.1 we have

$$\mathrm{H}^p_v(X, j_!(F)) = \mathrm{H}^{p-1}_T(I_s, F_\eta) = \mathrm{H}^{p-1}(K_s, F_\eta)$$

(where the final equality is our *definition* of the notation $\mathrm{H}^\bullet(K_s, \cdot)$ for real s, coinciding with usual Galois cohomology in positive degrees). For $x \in X - U$ and $f_x : X^{\mathrm{h}}_x \to X$ the henselization of X at x, by Lemma 4.1.3 we have

$$\mathrm{H}^p_x(X, j_!(F)) = \mathrm{H}^p_x(X^{\mathrm{h}}_x, f_x^* j_!(F)).$$

The latter group can be computed from the local cohomology sequence for a henselian discrete valuation ring A (such as that of X at x), giving an isomorphism

$$\mathrm{H}^p_x(X, j_!(F)) \simeq \mathrm{H}^{p-1}(D_x, F_\eta) = \mathrm{H}^{p-1}(K_x, F_\eta)$$

because of the *vanishing* of the the flanking terms $\mathrm{H}^p(\mathrm{Spec}(A), j'_!(G))$ for the open immersion $j' : \mathrm{Spec}(L) \hookrightarrow \mathrm{Spec}(A)$ of the generic point and an étale sheaf G on the generic point (since the degree-p cohomology of a sheaf on $\mathrm{Spec}(A)$ coincides with the degree-p cohomology of the fiber sheaf at the closed point due to A being henselian, so it vanishes when that fiber sheaf is 0).

The left vertical arrow in (15) is *defined* to be the "v-summand" of the resulting map

$$\bigoplus_{v\in\overline{X}-U}\mathrm{Br}(K_v)=\bigoplus_{v\in\overline{X}-U}\mathrm{H}^2_v(X,j_!(\mathbf{G}_{m,U}))\to\mathrm{H}^3_c(U,\mathbf{G}_m).$$

Via the definition of Tr_U in terms of Tr_X and the computation of $\mathrm{H}^3(\overline{X},\mathbf{G}_{m,\overline{X}})$ performed in §3.3, the invariant map $\mathrm{inv}_v:\mathrm{Br}(K_v)\to\mathbf{Q}/\mathbf{Z}$ makes (15) commute. This local-global compatibility will have a very useful consequence for the global trace relative to finite pushforward, as we shall see in Example B.2.3.

Now let K'/K be a finite extension, and U' the K'-normalization of U, with $\nu:U'\to U$ the resulting finite surjection. For any abelian sheaf A on $U'_{\text{ét}}$ and $p\in\mathbf{Z}$ we aim to construct an isomorphism

$$\omega_{\nu,A,p}:\mathrm{H}^p_c(U,\nu_*A)\to\mathrm{H}^p_c(U',A)$$

that is δ-functorial in A.

For $X':=\mathrm{Spec}(\mathcal{O}_{K'})$, the open immersion $j':U'\to X'$, and the normalization map $\nu_X:X'\to X$, the following diagram is cartesian:

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\nu_X} \\ U & \xrightarrow[j]{} & X \end{array}$$

Since the natural map $j_!\nu_*\to\nu_{X*}j'_!$ is an isomorphism (as can be checked on stalks) and ν_X is finite, we obtain a composition of natural maps

$$\mathrm{H}^p_c(U,\nu_*A):=\widehat{\mathrm{H}}^p(X,j_!\nu_*A)=\widehat{\mathrm{H}}^p(X,\nu_{X*}j'_!A)\simeq\widehat{\mathrm{H}}^p(X',j'_!A)=:\mathrm{H}^p_c(U',A)$$

(using Lemma 5.4.10 for the third identification). For $p\geq 0$, we define $\omega_{\nu,A,p}$ to be this composite isomorphism.

Now consider $p<0$, so by definition $\mathrm{H}^p_c(U,\nu_*A)=\bigoplus_{v\in X_\infty}\mathrm{H}^{p-1}_T(I_v,(\nu_*A)_\eta)$. Denoting by $\eta'\in U'$ the generic point, we have $\mathrm{H}^p_c(U',A)=\bigoplus_{w\in X'_\infty}\mathrm{H}^{p-1}_T(I_w,A_{\eta'})$. Tate cohomology in degree <-1 is positive-degree group homology, and for any real place v the étale sheaf $(\nu_*A)_\eta$ viewed as an I_v-module is the pushforward of A through

$$\coprod_{v'|v}\mathrm{Spec}(K'_{v'})=\mathrm{Spec}(K'\otimes_K K_v)\to\mathrm{Spec}(K_v).$$

The contributions from complex places over v are induced I_v-modules, so their higher group homology vanishes. The contributions to this pushforward from real places w of K' over v contribute A viewed as an I_w-module. Thus, for $p<0$ we have natural isomorphisms

$$\omega_{\nu,A,p}:\mathrm{H}^p_c(U,A):=\mathrm{H}^{p-1}_T(I_v,(\nu_*A)_\eta)=\bigoplus_{w\in X'_\infty}\mathrm{H}^{p-1}_T(I_w,A_{\eta'})=:\mathrm{H}^p_c(U',A)$$

(dropping the vanishing contribution from complex places of K' over real places of K!).

Lemma B.2.2 *The isomorphisms $\omega_{\nu,A,p}$ for $p\in\mathbf{Z}$ are δ-functorial in A.*

Proof. We need to check compatibility with connecting maps $\mathrm{H}_c^p \to \mathrm{H}_c^{p+1}$ arising from every short exact sequence of abelian sheaves

$$0 \to A' \to A \to A'' \to 0$$

on U' and its short exact ν-pushforward on U. The case $p < -1$ is a trivial assertion in higher group homology. For $p \geq 0$, since $j_!\nu_* \to \nu_{X*}j'_!$ is an isomorphism of exact functors and the isomorphism $\mathrm{H}^\bullet(\overline{X}, \overline{\nu_X}_*(\cdot)) \simeq \mathrm{H}^\bullet(\overline{X}', \cdot)$ (as in the proof of Lemma 5.4.10) is visibly δ-functorial, a review of how $\widehat{\mathrm{H}}^\bullet(X, \cdot)$ and $\widehat{\mathrm{H}}^\bullet(X', \cdot)$ are made into δ-functors in §5.2 gives the result.

It remains to consider $p = -1$. By definition of $\mathrm{H}_c^p(U, \cdot)$ and $\mathrm{H}_c^p(U', \cdot)$ we reduce to the case $U = X$ (so $U' = X'$). Let

$$0 \to A' \to A \to A'' \to 0$$

be a short exact sequence of abelian sheaves on X', so we get a diagram of long exact sequences

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \bigoplus_v \mathrm{H}_T^{-2}(I_v, (\nu_* A'')_\eta) & \longrightarrow & \widehat{\mathrm{H}}^0(X, \nu_* A') & \xrightarrow{\alpha} & \widehat{\mathrm{H}}^0(X, \nu_* A) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \omega_{\nu, A'', -1}} & & \downarrow{\scriptstyle \omega_{\nu, A', 0}} & & \downarrow{\scriptstyle \omega_{\nu, A, 1}} & \\
\cdots \longrightarrow & \bigoplus_v \bigoplus_{w|v} \mathrm{H}_T^{-2}(I_w, A''_{\eta'}) & \longrightarrow & \widehat{\mathrm{H}}^0(X', A') & \xrightarrow[\beta]{} & \widehat{\mathrm{H}}^0(X', A) & \longrightarrow \cdots
\end{array}$$

where v and w vary through real places. Defining $M := \ker(\alpha)$ and $M' := \ker(\beta)$, it suffices to check commutativity of the resulting diagram

$$\begin{array}{ccc}
\bigoplus_v \mathrm{H}_T^{-2}(I_v, (\nu_* A'')_\eta) & \longrightarrow & M \\
{\scriptstyle \omega_{\nu, A'', -1}}\downarrow & & \downarrow{\scriptstyle \omega_{\nu, A', 0}} \\
\bigoplus_v \bigoplus_{w|v} \mathrm{H}_T^{-2}(I_w, A''_{\eta'}) & \longrightarrow & M'
\end{array}$$

As we saw in §5.2 when making $\widehat{\mathrm{H}}^\bullet(X, \cdot)$ into a δ-functor, by design of $\widehat{(\cdot)}$ we have

$$M \subset \bigoplus_v \mathrm{H}_T^{-1}(I_v, (\nu_* A')_\eta), \quad M' \subset \bigoplus_v \bigoplus_{w|v} \mathrm{H}_T^{-1}(I_w, A'_{\eta'})$$

intertwining $\omega_{\nu, A', 0}$ with the naturally induced map between H_T^{-1}'s (via the description of $(\nu_* A')_\eta$ as a Galois module for K_v using pushforward through $\mathrm{Spec}(K' \otimes_K K_v) \to \mathrm{Spec}(K_v)$ and the vanishing of $\mathrm{H}_T^\bullet(I_v, \cdot)$ on induced I_v-modules arising from complex places of K' over v). Thus, it suffices to check the commutativity of the diagram

$$\begin{array}{ccc}
\bigoplus_v \mathrm{H}_T^{-2}(I_v, (\nu_* A'')_\eta) & \longrightarrow & \bigoplus_v \mathrm{H}_T^{-1}(I_v, (\nu_* A')_\eta) \\
\downarrow & & \downarrow \\
\bigoplus_v \bigoplus_{w|v} \mathrm{H}_T^{-2}(I_w, A''_{\eta'}) & \longrightarrow & \bigoplus_v \bigoplus_{w|v} \mathrm{H}_T^{-1}(I_w, A'_{\eta'})
\end{array}$$

where the maps are the obvious ones. Hence, for each real place w of K' and its induced real place v of K it remains to note that the diagram of group homologies

$$\begin{array}{ccc}
\mathrm{H}_1(I_v, (\nu_* A'')_\eta) & \longrightarrow & \mathrm{H}_0(I_v, (\nu_* A')_\eta) \\
\downarrow & & \downarrow \\
\mathrm{H}_1(I_w, A''_{\eta'}) & \longrightarrow & \mathrm{H}_0(I_w, A'_{\eta'})
\end{array}$$

(using evident vertical maps, via the isomorphism $I_w \simeq I_v$) obviously commutes. $\square$

Example B.2.3 The natural "norm" map $\nu_*\mathbf{G}_{m,U'} \to \mathbf{G}_{m,U}$ yields a map

$$\mathrm{N}_{U'/U} : \mathrm{H}^3_c(U', \mathbf{G}_{m,U'}) \to \mathrm{H}^3_c(U, \mathbf{G}_{m,U})$$

given by composing the inverse of $\omega_{\nu,\mathbf{G}_{m,U'},3} : \mathrm{H}^3_c(U, \nu_*\mathbf{G}_{m,U'}) \xrightarrow{\simeq} \mathrm{H}^3_c(U', \mathbf{G}_{m,U'})$ with the map $\mathrm{H}^3_c(U, \nu_*\mathbf{G}_{m,U'}) \to \mathrm{H}^3_c(U, \mathbf{G}_{m,U})$ induced by $\mathrm{H}^3_c(U, \cdot)$-functoriality. It will be important that this is compatible with global trace maps:

$$\mathrm{Tr}_U \circ \mathrm{N}_{U'/U} = \mathrm{Tr}_{U'}.$$

To prove that identity, we shall use the local-global compatibility in (15) to reduce the problem to a fact in local class field theory. For dense open $V \subset U$ and its preimage $V' \subset U'$, it is easy to see from the definitions of the isomorphisms

$$\mathrm{H}^3_c(V, \mathbf{G}_{m,V}) \simeq \mathrm{H}^3_c(U, \mathbf{G}_{m,U}),\ \ \mathrm{H}^3_c(V', \mathbf{G}_{m,V'}) \simeq \mathrm{H}^3_c(U', \mathbf{G}_{m,U'}) \tag{17}$$

that they intertwine $\mathrm{N}_{U'/U}$ and $\mathrm{N}_{V'/V}$. The isomorphisms (17) also respectively intertwine global traces by Remark 5.4.3, so to prove the desired compatibility of $\mathrm{N}_{U'/U}$ with global traces it is equivalent to do the same for $\mathrm{N}_{V'/V}$. Hence, we may assume that $U \neq X$, so there exists a point $x \in X - U$.

Rather generally, for any $v \in X - U$ (which exists!) and $w \in X' - U'$ over v, we claim that the diagram

$$\begin{array}{ccc} \mathrm{Br}(K'_w) & \longrightarrow & \mathrm{H}^3_c(U', \mathbf{G}_{m,U'}) \\ {\scriptstyle \mathrm{Norm}}\downarrow & & \downarrow{\scriptstyle \mathrm{N}_{U'/U}} \\ \mathrm{Br}(K_v) & \longrightarrow & \mathrm{H}^3_c(U, \mathbf{G}_{m,U}) \end{array} \tag{18}$$

commutes, where the left vertical map is induced by the usual norm map between local Brauer groups (defined similarly to $\mathrm{N}_{U'/U}$, using Shapiro's Lemma) and the horizontal maps are the ones defined in (15). Once this is proved, since we have *isomorphisms* along the horizontal sides of (18) (due to the commutativity of (15)) it will follow that the compatibility of $\mathrm{N}_{U'/U}$ with global traces is reduced to the compatibility of norm maps between Brauer groups with local invariant maps in local class field theory. This local compatibility is a well-known fact: see Prop. 1(ii) in §2 of Chapter XI in Serre's book "Local Fields" (applicable since the norm map between Brauer groups is an instance of "corestriction" in Galois cohomology).

To check commutativity of (18), if we go back to how the horizontal maps in (18) are *defined* via local cohomology sequences and we use that

$$X' \times_X X^{\mathrm{h}}_v \simeq \coprod_{w|v} X'^{\mathrm{h}}_w$$

then we are reduced to proving the commutativity of

$$\begin{array}{ccc} \mathrm{H}^2(K'^{\mathrm{h}}_w, \mathbf{G}_m) & \xrightarrow{\simeq} & \mathrm{H}^3_w(\mathrm{Spec}(\mathcal{O}^{\mathrm{h}}_{X',w}), j_{w!}(\mathbf{G}_m)) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{N}} \\ \mathrm{H}^2(K^{\mathrm{h}}_v, \mathbf{G}_m) & \xrightarrow[\simeq]{} & \mathrm{H}^3_v(\mathrm{Spec}(\mathcal{O}^{\mathrm{h}}_{X,v}), j_{v!}(\mathbf{G}_m)) \end{array}$$

where $j_v : \mathrm{Spec}(K^{\mathrm{h}}_v) \hookrightarrow \mathrm{Spec}(\mathcal{O}^{\mathrm{h}}_{X,v})$ is the open generic point and likewise for j_w. This final commutativity is an immediate consequence of the compatibility of the local cohomology sequence with respect to *finite* pushforward and the functoriality of the local cohomology sequence in the sheaf (such as with respect to the norm map $\nu_{v*}(j_{w!}\mathbf{G}_m) \to j_{v!}\mathbf{G}_m$ defined by applying $j_{v!}$ to the "w-component" of the K^{h}_v-pullback of the norm $\nu_*\mathbf{G}_{m,U'} \to \mathbf{G}_{m,U}$).

The following refinement of Lemma B.2.2 is crucial in the proof of Artin–Verdier duality beyond the totally complex case:

Proposition B.2.4 *For any abelian sheaf F' on U', the natural δ-functorial isomorphisms*

$$\omega_{\nu,F',p} : \mathrm{H}^p_c(U, \nu_*(F')) \xrightarrow{\simeq} \mathrm{H}^p_c(U', F')$$

are compatible with the Yoneda pairing (14) and its analogue on U' for all $p \in \mathbf{Z}$, assuming F' is constructible when $p < 0$.

The Yoneda pairing for $p < 0$ is only defined for constructible F'.

Proof. In Example B.2.3 we defined a natural "norm" map

$$\mathrm{N}_{U'/U} : \mathrm{H}^3_c(U', \mathbf{G}_{m,U'}) \to \mathrm{H}^3_c(U, \mathbf{G}_{m,U})$$

and proved that it is compatible with global traces. To define what it means to say that $\omega_{\nu,F,p}$ is compatible with the Yoneda pairing, we shall use $\mathrm{N}_{U'/U}$ and the natural composite map

$$\mathrm{Ext}^q_{U'}(F', \mathbf{G}_{m,U'}) \to \mathrm{Ext}^q_U(\nu_*F', \nu_*\mathbf{G}_{m,U'}) \to \mathrm{Ext}^q_U(\nu_*F', \mathbf{G}_{m,U})$$

(with $q = 3 - p$) that we also call the "norm" and denote as N. We claim commutativity (in the evident sense) of the diagram

$$\begin{array}{ccccccc}
\mathrm{H}^p_c(U', F') & \times & \mathrm{Ext}^{3-p}_{U'}(F', \mathbf{G}_{m,U'}) & \longrightarrow & \mathrm{H}^3_c(U', \mathbf{G}_{m,U'}) & \xrightarrow[\simeq]{\mathrm{Tr}} & \mathbf{Q}/\mathbf{Z} \\
\uparrow{\scriptstyle \omega_{\nu,F',p}}\ \simeq & & \downarrow{\scriptstyle \mathrm{N}} & & \downarrow{\scriptstyle \mathrm{N}} & & \| \\
\mathrm{H}^p_c(U, \nu_*F') & \times & \mathrm{Ext}^{3-p}_U(\nu_*F', \mathbf{G}_{m,U}) & \longrightarrow & \mathrm{H}^3_c(U, \mathbf{G}_{m,U}) & \xrightarrow[\simeq]{\mathrm{Tr}} & \mathbf{Q}/\mathbf{Z}
\end{array}$$

where the abelian sheaf F' is assumed to be constructible when $p < 0$. Commutativity of the right square allows us to reduce our considerations to checking commutativity of

$$\begin{array}{ccccc}
\mathrm{H}^p_c(U', F') & \times & \mathrm{Ext}^{3-p}_{U'}(F', \mathbf{G}_{m,U'}) & \longrightarrow & \mathrm{H}^3_c(U', \mathbf{G}_{m,U'}) \\
\uparrow{\scriptstyle \omega_{\nu,F',p}}\ \simeq & & \downarrow{\scriptstyle \mathrm{N}} & & \downarrow{\scriptstyle \mathrm{N}} \\
\mathrm{H}^p_c(U, \nu_*F') & \times & \mathrm{Ext}^{3-p}_U(\nu_*F', \mathbf{G}_{m,U}) & \longrightarrow & \mathrm{H}^3_c(U, \mathbf{G}_{m,U})
\end{array}$$

where F' is any abelian sheaf on U', with F' constructible when $p < 0$.

For $p \geq 4$ the Ext's vanish and there is nothing to do. Next, consider $0 \leq p \leq 3$. In view of how the Yoneda pairing and $\mathrm{H}^p_c(U, \cdot)$ (for $p \geq 0$) are *defined* by using an extension by zero to X', the problem for F' is equivalent to the same for its extension by zero to X'. Hence, we may assume $U = X$.

The Yoneda pairing over X is initially valued in $\mathrm{H}^3(\overline{X}, \overline{j}_*(\mathbf{G}_{m,X}))$ (with $\overline{j} : X \to \overline{X}$ the canonical "morphism"), and is then made to take values in $\mathrm{H}^3_c(X, \mathbf{G}_{m,X}) = \mathrm{H}^3(\overline{X}, \mathbf{G}_{m,\overline{X}})$ via inverting the *isomorphism* obtained by applying $\mathrm{H}^3(\overline{X}, \cdot)$ to the map $\mathbf{G}_{m,\overline{X}} \to \overline{j}_*(\mathbf{G}_{m,X})$ adjoint to the natural isomorphism $\overline{j}^*\mathbf{G}_{m,\overline{X}} \simeq \mathbf{G}_{m,X}$.

Using $\overline{j}' : X' \hookrightarrow \overline{X}'$, we get a natural "norm" homomorphism

$$\overline{\mathrm{N}} : \overline{\nu}_*(\overline{j}'_*\mathbf{G}_{m,X'}) = \overline{j}_*(\nu_*\mathbf{G}_{m,X'}) \to \overline{j}_*\mathbf{G}_{m,X}$$

over $\overline{X}$, and applying $\mathrm{H}^3(\overline{X}, \cdot)$ defines the left side of the diagram

$$\begin{array}{ccc} \mathrm{H}^3(\overline{X'}, \overline{j'}_*\mathbf{G}_{m,X'}) & \xleftarrow{\simeq} & \mathrm{H}^3_c(X', \mathbf{G}_{m,X'}) \\ \downarrow{\scriptstyle \mathrm{N}} & & \downarrow{\scriptstyle \mathrm{N}} \\ \mathrm{H}^3(\overline{X}, \overline{j}_*\mathbf{G}_{m,X}) & \xleftarrow[\simeq]{} & \mathrm{H}^3_c(X, \mathbf{G}_{m,X}) \end{array}$$

that is easily verified to commute. Thus, we are reduced to proving the commutativity of the diagram

$$\begin{array}{ccccc} \mathrm{Ext}^p_{\overline{X'}}(\underline{\mathbf{Z}}, \widehat{F'}) & \times & \mathrm{Ext}^{3-p}_{\overline{X'}}(\widehat{F'}, \overline{j'}_*\mathbf{G}_{m,X'}) & \longrightarrow & \mathrm{Ext}^3_{\overline{X'}}(\underline{\mathbf{Z}}, \overline{j'}_*\mathbf{G}_{m,X'}) \\ {\scriptstyle \omega_{\nu,F',p}}\uparrow \simeq & & \downarrow{\scriptstyle \mathrm{N}} & & \downarrow{\scriptstyle \mathrm{N}} \\ \mathrm{Ext}^p_{\overline{X}}(\underline{\mathbf{Z}}, \overline{\nu}_*\widehat{F'}) & \times & \mathrm{Ext}^{3-p}_{\overline{X}}(\overline{\nu}_*\widehat{F'}, \overline{j}_*\mathbf{G}_{m,X}) & \longrightarrow & \mathrm{Ext}^3_{\overline{X}}(\underline{\mathbf{Z}}, \overline{j}_*\mathbf{G}_{m,X}) \end{array}$$

whose vertical maps are defined via composition in the derived categories of abelian sheaves on $\overline{X'}$ and $\overline{X}$. The inverse to $\omega_{\nu,F',p}$ is the map that applies the exact (injective-preserving) functor $\overline{\nu}_*$ to a morphism $\underline{\mathbf{Z}} \to F'[p]$ in the derived category over $\overline{X}'$, and the two other maps are computed by applying $\overline{\nu}_*$ and then composing with (a shift of) $\overline{\mathrm{N}}$, so the desired commutativity is just a special case of the functoriality of $\overline{\nu}_*$ between the derived categories.

Finally, we address the case $p < 0$, so assume F' is constructible. Letting $\eta' \in U'$ be the generic point, by the *definition* of the Yoneda pairing in negative degrees via cup products in Tate cohomology (using Corollary 5.4.7!) we are reduced to checking that for $v \in X_\infty$ and $w \in X'_\infty$ over v, the diagram

$$\begin{array}{ccccc} \mathrm{H}^{p-1}_T(I_w, F'_{\eta'}) & \times & \mathrm{H}^{3-p}_T(I_w, (F'_{\eta'})^\vee) & \longrightarrow & \mathrm{H}^2_T(I_w, \mathbf{G}_{m,\eta'}) \\ \uparrow & & \downarrow{\scriptstyle \mathrm{N}} & & \downarrow{\scriptstyle \mathrm{N}} \\ \mathrm{H}^{p-1}_T(I_v, (\nu_*F')_\eta) & \times & \mathrm{H}^{3-p}_T(I_v, (\nu_*F')^\vee_\eta) & \longrightarrow & \mathrm{H}^2_T(I_v, \mathbf{G}_{m,\eta}) \end{array}$$

commutes, where the left map is the projection onto the w-summand as in Shapiro's Lemma and we have used Remark 5.4.8 to identify the map between H^{3-p}_T's.

The H^i_T's on the bottom are identified as a direct sum of analogous terms for $F'_{\eta'}$ relative to the real places of K over v, and the two norm maps are isomorphisms onto the w-factors of such direct sums via the *isomorphism* $I_w \simeq I_v$. The pushforward of the duality between $F'_{\eta'}$ and $(F'_{\eta'})^\vee$ has K_v-pullback that is the direct sum of the duality pairing for $F'_{\eta'}$ over $K'_{v'}$ indexed by the real places v' over v, so the desired commutativity is now obvious. □

B.3 A duality comparison

The construction of Tate's 9-term exact sequence via local cohomology requires identifying many of the abstract maps with something concrete. This is recorded in Lemma 6.1.3, whose proof has a loose end that we address here:

Proposition B.3.1 *In the setting of Lemma 6.1.3, the map* $\mathrm{H}^1(G_S, M^\vee) \to \bigoplus_{s\in S} \mathrm{H}^1(K_s, M)^D$ *is the one defined by the local duality pairings* $\mathrm{H}^1(K_s, M) \times \mathrm{H}^1(K_s, M^\vee) \to \mathbf{Q}/\mathbf{Z}$.

Proof. [2] □

[2] To be filled in, treating real s separately.

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