The Bloch-Kato Tamagawa Number Conjecture

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1 Introduction

Bloch and Kato originally thought of their conjectures as a version of the Tamagawa number conjecture for algebraic groups, replacing the algebraic group by a pure motive.

algebraic groups \(\rightarrow\) abelian varieties \(\rightarrow\) motives

Abelian varieties are the prototypical motive. We recall how the BSD conjecture for abelian varieties equals a Tamagawa number conjecture.

First, we recall the theorem for algebraic groups:

Theorem 1.1 (?) For a connected algebraic group \(G\) over a number field \(K\), we have

\[
\tau(G) = \frac{|Pic^0(G)|}{|\Xi(G)|}.
\]

Here, \(\tau(G)\) is roughly the volume of \(G(\mathbb{A}_K)/G(K)\) with respect to Haar measures on \(G(\mathbb{A}_K)\) for all places \(v\), where we have to use \(L\)-functions to make the product measure converge, and also have to restrict to measuring some “compact part” of \(G(\mathbb{A}_K)\), by taking the kernel of all \(|\chi|_v, \chi: G \to \mathbb{G}_m\).

Now, let’s formulate a version of this for abelian varieties. For simplicity, assume that \(E\) is an elliptic curve over \(\mathbb{Q}\), with \(E(\mathbb{Q})\) finite. There is a Neron model \(E\) for \(E\), with Neron form \(\omega\). This induces a measure on \(E(\mathbb{Q}_p)\): one way to formulate this is that the map \(\log = \int \omega: E(\mathbb{Z}_p) \to Lie(E)_{\mathbb{Z}_p}\) (multiply till you land in “kernel of reduction” \(E(\mathbb{Z}_p)\)), then evaluate power series) induces a measure on \(E(\mathbb{Z}_p)\) by declaring that it preserves measure and that \(Lie(E)_{\mathbb{Z}_p}\) has volume 1.

A calculation shows that \(\text{vol}(E(\mathbb{Q}_p)) = \frac{|E(\mathbb{F}_p)|}{\Phi_p(\mathbb{F}_p)}\), where \(\Phi_p\) is the component group-scheme. Define \(c_p = |\Phi_p(\mathbb{F}_p)|\), the Tamagawa factor at \(p\). Also, \(\text{vol}(E(\mathbb{R})) = \int_{E(\mathbb{R})} \omega\) is the real period.

The product \(\prod_p \text{vol}(E(\mathbb{Q}_p))\) does not converge. However, note that \(L(E,1) = \prod_p \det(1 - p^{-1}f|((V_1E)^*)^L)\) \(-1 = \prod_p \det(1 - f|(V_1E)^L)\) \(-1 = \prod_p \frac{\text{det}^p}{|E(\mathbb{F}_p)|}\), using the Cartier duality \(T_p(E)^*(1) \cong T_p(E)\).

Thus we can define the renormalized adelic volume to be

\[
\text{vol}(E(\mathbb{A})) = L(E,1)^{-1} \text{vol}(E(\mathbb{R})) \prod_p c_p
\]

The Tamagawa number conjecture then becomes

\[
\text{vol}\left(\frac{E(\mathbb{A})}{E(\mathbb{Q})}\right) = \frac{L(E,1)^{-1} \text{vol}(E(\mathbb{R})) \prod_p c_p}{|E(\mathbb{Q})|} \approx \frac{|E(\mathbb{Q})|}{\Xi(E)}.
\]
This is evidently equivalent to BSD. More generally, if \(E(\mathbb{Q})\) is not finite, \(L(E, 1)\) should vanish, and \(|E(\mathbb{Q})|, |\text{Pic}^0(E)|\) are not finite. However, if we replace \(L(E, 1)\) by the leading term \(L^*(E, 1)\) and introduce height pairings to measure, not the covolume of \(E(\mathbb{Q})/\text{tors}\) in \(E(\mathbb{A})\), but its “density”, we again recover BSD.

How do we generalize this to other motives?

- Global points modulo torsion via K-theory
- Torsion in global points, \(\text{Pic}^0\), \(\text{III}\), via global etale cohomology
- Local nonarchimedean points via local etale cohomology
- Local nonarchimedean volumes via Bloch-Kato exponential
- Local real volumes via period map, real regulators, as in Beilinson conjecture
- Height pairings to make sense of quotienting adelic points by global points, when not just torsion

Once we make precise what all this means, we will have, for a motive \(M = h^i(X)(j)\), the exact same conjecture:

\[
\frac{\text{vol}(M(\mathbb{A}))\text{Reg}(M)}{|M(\mathbb{Q})_{\text{tors}}|} = \frac{L^*(M, 0)^{-1}\text{Reg}(M)\text{vol}(M(\mathbb{R})) \prod_p c_p}{|M(\mathbb{Q})_{\text{tors}}|^2} = \frac{|(M^*(1))(\mathbb{Q})_{\text{tors}}|}{|\text{III}(M)_{\text{tors}}|}.
\]

One convenient way to formalize this was found by Fontaine and Perrin-Riou.

First, recall that, in formulating Beilinson’s conjecture, we had, for a motive \(M\) of weight \(w < -1\), an injective real period map

\[
\alpha: (M^+_B)_{\mathbb{R}} \to (M_{\text{dR}}/F^0 M_{\text{dR}})_{\mathbb{R}} =: \text{Lie}(M)_{\mathbb{R}}
\]

and a conjectural isomorphism

\[
H^{i+1}_{M, Z}(X, \mathbb{Q}(j))_{\mathbb{R}} \cong \text{coker}(\alpha) =: H^{i+1}_D(X_{\mathbb{R}}, \mathbb{R}(j)).
\]

Using these, we obtain, denoting \([\cdot] := \text{det}(\cdot)\) for the top exterior power of a vector space, division meaning tensor with dual,

\[
\theta_\infty: \mathbb{R} \cong \left( \begin{array}{c} [\text{Lie}(M)] \\ [H^{i+1}_{M, Z}(X, \mathbb{Q}(j))] [M^+_B] \end{array} \right) =: \Xi(M) \otimes \mathbb{R}
\]

Beilinson’s conjecture (not the rank part) is equivalent to the claim that

\[
\theta_\infty(1/L^*(M)) \in \Xi(M).
\]

In other words \(L^*(M)\) should measure how far \(\theta_\infty\) is from respecting the rational structures \(\mathbb{Q} \subset \mathbb{R}\), \(\Xi(M) \subset \Xi(M) \otimes \mathbb{R}\).

Fontaine and Perrin-Riou generalize this for all \(w\), defining \(Q\)-vs \(H^*_f(M)\), \(* = 0, 1, 2, 3\), and define

\[
\Xi(M) := \frac{[H^*_f(M)][\text{Lie}(M)]}{[M^+_B]},
\]
as well as an isomorphism
\[ \theta_\infty : \mathbb{R} \to \Xi(M)_{\mathbb{R}}, \]
and still conjecture
\[ \theta_\infty(1/L^*(M)) \in \Xi(M). \]

We think of this canonical element in \( \Xi(M) \) as defining a \( \mathbb{Z} \)-integral structure. Using etale cohomology, we can define \( \mathbb{Z}_p \)-integral structures
\[ \theta_p : \Lambda \hookrightarrow \Xi(M)_{\mathbb{Q}_p}, \]
(\( \Lambda \cong \mathbb{Z}_p \), but not canonically so)
The Bloch-Kato conjecture is then:

**Conjecture 1.2** ([2]) The \( \mathbb{Z}_p \)-integral structures
\[ \mathbb{Z}_p \cdot \theta_\infty(1/L^*(M)) \subset \Xi(M)_{\mathbb{Q}_p} \supset \theta_p(\Lambda) \]
agree for all \( p \).

In terms of volumes, this says, roughly, that

**Conjecture 1.3** For all primes \( p \),
\[ \text{ord}_p \left( \frac{\text{Reg}(M) \text{vol}(M(\mathbb{R}))}{L^*(M)} \right) = \text{ord}_p \left( \frac{|M(\mathbb{Q})_{\text{tors}}||M^*(1)(\mathbb{Q})_{\text{tors}}|}{|\text{III}(M)||\prod_v c_v} \right) \]

**Remark 1.4** We could make sense of the \( p \)-adic valuation of all the invariants on the RHS in terms of \( p \)-adic etale cohomology.

## 2 Determinants

**Motivation** This is mostly just book-keeping.

Given a (finite dimensional) vector space \( V \), define \([V] = \bigwedge^{\text{top}} V\). Note that if \( V = 0 \), then \([V] \cong \mathbb{Q}_p \) canonically. Given an exact sequence \( 0 \to A \to B \to C \to 0 \), we have \([B] \cong [A][C]\). We will need to keep track of isomorphisms, or else this is useless. We consider integral structures \( T \) on \( \mathbb{Q}_p \)-vector spaces \( V \), by which we mean finitely-generated \( \mathbb{Z}_p \) modules \( T \) with a canonical isomorphism \( T \otimes \mathbb{Q}_p \cong V \). An integral structure \( T \) on \( V \) determines a \( \mathbb{Z}_p \)-submodule \([T] \subset [V]\) as follows:

If \( T \subset V \) is torsion-free, then \([T] \subset [V]\) is what you expect. If \( V = 0 \), then \([T] \cong \mathbb{Z}_p \subset \mathbb{Q}_p \cong [V]\). If \( V = 0 \), \( T = \mathbb{Z}/p\mathbb{Z} \), then \([\mathbb{Z}/p\mathbb{Z}] = \frac{1}{p}\mathbb{Z}_p \subset \mathbb{Q}_p \cong [V]\). i.e. torsion groups have larger volumes than trivial groups.

For a general integral structure \( T \), \([T] = [T/\text{tors}]|T_{\text{tors}}| = |T_{\text{tors}}|p[T/\text{tors}] \subset [V]\), where \(| \cdot |_p \) is the valuation with \(|p|_p = \frac{1}{p}\).

Given a a finite complex \( C : A_0 \to A_1 \to \ldots \to A_n \) of \( \mathbb{Q}_p \)-vs, we define \([C] \cong \frac{[A_0][A_2] \cdots}{[A_1][A_3] \cdots} \).

We can deduce: \([C] = [H^*(C)]\).

Consider an integral structure \( X \subset C \). The cohomology complex \( H^*(X) \) is an integral structure of \( H^*(X) \). Thus we obtain \([H^*(X)] \subset [H^*(C)] \cong [C]\).
Consider $f \in \text{Aut}(V)$, such that $f(T) \subset T$ for $T \subset V$ a lattice. Consider the complexes $(T \xrightarrow{f} T) \subset (V \xrightarrow{f} V)$. Now, $[V \rightarrow V] = [H^*(V \rightarrow V)] = [0 \rightarrow 0] \cong \mathbb{Q}_p \supset \mathbb{Z}_p$ has a canonical integral structure. We compare this to the integral structure $[H^*(T \rightarrow T)]:$

$$[H^*(T \rightarrow T)] = [\text{coker}(f[T])]^{-1} = |\text{det}(f)|^{-1}_p \cdot \mathbb{Z}_p$$

Similarly, the complex $(T \xrightarrow{f} f(T)) \subset (V \xrightarrow{f} V)$ has $[H^*(T \rightarrow f(T))] \subset [H^*(V \rightarrow V)] \cong [0] = \mathbb{Q}_p \subset \mathbb{Z}_p$.

3 Motivic $f$-cohomology

Motivation We need rational structures to compare the $p$-adic, $\infty$-adic computations.

For a motive $M = h^i(X, \mathbb{Q}(j))$, with weight $w = i - 2j$, we define

- $H^0_f(M) = CH^j(X)_\mathbb{Q}/\text{hom.equiv.}$ if $i = 2j$, 0 otherwise

- $H^1_f(M) = \begin{cases} H^{i+1}_M(X, \mathbb{Q}(j)) = \text{Im}(K_{2j-i-1}(X)_{\mathbb{Q}}^{(2j)} \rightarrow K_{2j-i-1}(X)_\mathbb{Q}), & i \neq 2j-1, \\ CH^j(X)_{\text{hom.equiv.}}, & i = 2j-1 \end{cases}$

for $X$ a regular proper model of $X$ over $\mathbb{Z}$ (what if this doesn’t exist?)

- $H^2_f(M) = (H^1_f(M^*(1)))^*$

- $H^3_f(M) = (H^0_f(M^*(1)))^*$

Note that $H^0_f = 0$ if $w \neq 0$, $H^2_f = 0$ if $w \neq -2$ (immediately right and left of the point of symmetry $w = -1$).

Conjecture 3.1 ([2])

$$\text{ord}_{s=0}(L(M, s)) = \dim_{\mathbb{Q}} H^1_f(M^*(1)) - \dim H^0_f(M^*(1))$$

Remark 3.2 This is, conjecturally on the isomorphism of the $p$-adic regulator (see below), the same conjecture as in Tony’s talk in terms of Bloch-Kato selmer groups.

Remark 3.3 This conjectures possible poles for $w = -2$, possible zeros for $w \geq -1$, and ord = 0 for $w < -2$. Note, for example, that $\zeta(r)$ relates to the motives $\mathbb{Q}(r)$ of weight $-2r$.

We use these groups to define the fundamental $\mathbb{Q}$-line

$$\Xi(M) = \frac{[H^*_f(M)] [\text{Lie}(M)_\mathbb{R}]}{[M_B^+]}.$$ 

Remark 3.4 The definition of $H^2_f$ is convenient, but it is bad: $\text{Ext}^2_f(\text{Spec(}\mathbb{Q})_{\text{mot}}, M) = 0$ according to Beilinson’s conjectures (Scholl says this). Further, these groups definitely do not have the correct torsion even if you decide not to $\otimes \mathbb{Q}$: Should have class groups for number fields.
4 Real Volumes

Motivation: Incorporate Beilinson’s conjecture, including height pairings.

We have a real period map

\[ \alpha: (M_B^+)_R \to (M_{dR}/F^0 M_{dR})_R = (\text{Lie}(M))_R. \]

A motive is called “critical” when \( \alpha \) is an isomorphism. For example, motives of weight -1, such as \( H^1(E, \mathbb{Z}(1)) = H_1(E, \mathbb{Z}) \), are always critical.

In this case, we obtain an isomorphism

\[ \mathbb{R}\llbracket \alpha \rrbracket \cong \frac{[\text{Lie}(M)_R]}{[(M_B^+)_R]}. \]

Also, when the weight is -1, we have the possibility of height pairings:

**Conjecture 4.1** When \( w = -1 \), the height pairing

\[ h: H^1_f(M)_R \times H^1_f(M\ast(1))_R \to \mathbb{R} \]

is nondegenerate.

Assuming the conjecture, we obtain

\[ \mathbb{R}\llbracket h \rrbracket \cong \frac{[H^1_f(M\ast(1))]^*}{[H^1_f(M)]}. \]

In combination, these give an isomorphism

\[ \theta_\infty: \mathbb{R} \cong \Xi(M)_R. \]

Now we deal with the noncritical case, and assume the weight is \( < -1 \). Here \( H^2_f(M) = H^3_f(M) = 0 \).

**Conjecture 4.2** When \( w < -1 \), the real regulator

\[ H^1_f(M)_R \to \text{coker}(\alpha) \]

is an isomorphism.

Since \([\text{coker}(\alpha)] = \frac{[\text{Lie}(M)_R]}{[(M_B^+)_R]}\), we again obtain

\[ \theta_\infty: \mathbb{R} \cong \Xi(M)_R. \]

**Remark 4.3** When the weight is \( > -1 \), we need to use factors from the functional equation to define the map \( \theta_\infty \) in terms of that for its dual motive \( M\ast(1) \). See Fontaine and Perrin-Riou.

**Remark 4.4** All cases can be combined into the conjectural exactness of the sequence

\[ 0 \to H^0_f(M) \to \ker(\alpha) \to H^1_f(M\ast(1))^* \xrightarrow{h} H^1_f(M) \to \text{coker}(\alpha) \to H^0_f(M\ast(1))^* \to 0, \]

which perhaps suggests that \( H^*_f, \ast = 0,1 \), is dual to a cohomology theory which is “compactly supported at infinity”. See Deninger-Nart.

The map \( \theta_\infty \) can also be defined for \( w > -1 \). For all weights \( w \) we have the following conjecture:

**Conjecture 4.5 (Beilinson)**

\[ \theta_\infty(1/L^*(M)) \in \mathbb{Q}. \]
5 Local f-cohomology and the Bloch-Kato Exponential

Motivation Local conditions, being unramified, analogous to $H^*_M,\mathbb{Z}$.

Fix a prime $p$. We define complexes

$$R\Gamma_f(\mathbb{Q}_v, M_p) = \begin{cases} v = \infty : & R\Gamma(\mathbb{R}, M_p) \\ v \neq p : & M_p^{\ell \frac{1-f}{1-f}} \\ v = p : & D_{cris}(M_p) \to D_{cris}(M_p) \oplus D_{dR}(M_p)/F^0 D_{dR}(M_p) \end{cases},$$

with $f$ the geometric Frobenius.

Their cohomology groups $H^i_f$ are the same as those in Tony’s talk, as we will shortly see.

Local L-complexes (This is just notation for later.)

We have the complexes for $v \neq \infty$:

$$L^v(T_p) = \begin{cases} v \neq p : & T_p^{\ell v \frac{1-f}{1-f}} \\ v = p : & D_{cris}(T_p) \to D_{cris}(T_p) \end{cases}$$

We define $L^v(M_p) = L^v(T_p) \otimes \mathbb{Q}_p$.

We also define $[L^S(M_p)] = \otimes_{v \in S_{-\{\infty\}}} [L^v(M_p)]$, with integral structure $[L^S(T_p)] = \otimes_{v \in S_{-\{\infty\}}} [L^v(T_p)]$.

Note that if $L^v(M_p)$ is acyclic, then $[L^v(T_p)] = [\det(1 - f|M_p^{\ell v})]^{-1}$, like a local L-factor. This explains the notation.

Recall that f-cohomology is a “self-dual Selmer condition”:

**Proposition 5.1** $H^1_f(\mathbb{Q}_p, M_p)$ is the exact annihilator of $H^1_f(\mathbb{Q}_p, M_p^*(1))$ under the Tate local duality pairing.

We want to define the Bloch-Kato exponential

$$exp_{BK} : D_{dR}(M_p)/F^0 D_{dR}(M_p) \to H^1_f(\mathbb{Q}_p, M_p)$$

It arises from the “fundamental exact sequence of p-adic Hodge theory”:

$$0 \to \mathbb{Q}_p \to B_{cris} \to B_{cris} \oplus B_{dR}/B_{dR}^+ \to 0.$$  

A sequence similar to this was in Tony’s talk.

Tensoring this with our representation $M_p$ (which is assumed to be de Rham), and taking the LES of Galois cohomology

$$0 \to H^0(M_p) \to D_{cris}(M_p) \to D_{cris}(M_p) \oplus D_{dR}(M_p)/F^0 D(M_p) \to \ker(H^1(M_p) \to H^1(M_p \otimes B_{cris})) \to 0.$$  

Note that this verifies that the definition of $H^1_f$ in Tony’s talk agrees with the 1st cohomology of the above complex.

We can also express the BK exponential in terms the $Ext^1$-consequence of the crystalline comparison theorem.

**Proposition 5.2** (Π) For $M_p$ crystalline, we have the following isomorphism:

$$D(M_p)/(1 - f)F^0 D(M_p) \cong Ext^1_{f,Fil}(\mathbb{Q}_p, D(M_p)) \cong Ext^1_{f}(\mathbb{Q}_p, M_p).$$

In other words, crystalline extensions of galois representations are identified with extensions of $(f,Fil)$-modules.
An aside on Fontaine-Lafaille Theory

If the lattice \(D(T_p) \subset D(M_p)\) is “Fontaine-Lafaille” (strongly divisible and with weights in \([0, p - 1]\)), we have an integral comparison theorem

\[
D(T_p)/(1 - f)F^0D(T_p) \cong Ext^1_{f,Fil}(\mathbb{Z}_p, D(T_p)) \cong Ext^1_{K_v}(\mathbb{Z}_p, T_p)_{f}.
\]

In this case, we have the following:

\[
D(T_p)/(1 - f)F^0D(T_p) \xrightarrow{exp_{BK}} H^1(K_v, T_p)
\]

\[
\cong
\]

\[
D(T_p)/(1 - f)F^0D(T_p)
\]

This implies that when a lattice is Fontaine-Lafaille, that the local volume agrees with the local L-factor. Morally, this means that we have good reduction, in some strange new sense, since the Tamagawa factor at \(p\) is then 1.

For example, Bloch-Kato shows that the lattice \(D(\mathbb{Z}_p(r))\) is not Fontaine-Lafaille for \(p < r\), contributing an extra factor of \(1/(r - 1)!\) to the adelic volume as we vary over all such primes.

**Bloch-Kato Exponential and Kummer Theory**

For abelian varieties and tori, the Bloch-Kato exponential agrees with the Kummer map. We first show it for \(Gm\), using the following diagram:

\[
\begin{array}{cccccc}
0 & \to & \mathbb{Z}_p(1) & \xrightarrow{\lim_p} & \mathcal{O}_{\mathbb{C}_p}^* & \xrightarrow{\log} & 0 \\
\downarrow & & \downarrow{\log[\cdot]} & & \downarrow{\log} & & \\
0 & \to & \mathbb{Q}_p(1) & \xrightarrow{B_{cris}^{f=p} \cap B_{dR}^+} & \mathbb{C}_p & \xrightarrow{0} & 0 \\
\downarrow & & \downarrow{\theta} & & \downarrow & & \\
0 & \to & \mathbb{Q}_p(1) & \xrightarrow{(B_{cris}^{f=1})(1)} & (B_{dR}^+/B_{dR}^+)(1) & \xrightarrow{0} & 0
\end{array}
\]

To get the result for abelian varieties, use that \(\text{Hom}_{\text{FormalGroup}}(\hat{A}, \hat{G_m})(\mathcal{O}_{\mathbb{C}_p}) \cong T_p(A)^*(1)\) by Cartier duality. For any choice of \(\chi \in T_p(A)^*(1)\), we get a map (not galois equivariant) from the sequence

\[
0 \to T_p(A) \to \lim_p \text{A}(\mathcal{O}_{\mathbb{C}_p}) \to \text{A}(\mathcal{O}_{\mathbb{C}_p}) \to 0
\]

to the last row of the above diagram, i.e. we get a (galois equivariant) map from this sequence to the last row tensor \(V_p(A)(-1)\).

Bloch-Kato claim this proof works, in some sense, for abelian varieties with bad reduction.

### 6 Global f-cohomology

There is a homological algebra construction, which, given a map of complexes, formally create a complex fitting into a long-exact sequence:

\[
\ldots \to H^i(A) \to H^i(B) \to H^i(\text{Cone}(A \to B)) \to \ldots,
\]

Note that this implies the determinant formula

\[
[\text{Cone}(A \to B)] = \frac{[B]}{[A]}.
\]
Let $S = \{ \infty, p, v \text{ s.t. } V^L \neq V \}$. Let $R\Gamma(Z[1/S], N)$ be the complex computing global galois cohomology, for $N$ any reasonable Galois module. Similarly we use $R\Gamma(Q_v, V_p)$ for local galois cohomology.

We first define the “quotient” of local cohomology by local f-cohomology, $R\Gamma_f(Q_v, M_p)$, as

$$R\Gamma_f(Q_v, M_p) = Cone(R\Gamma_f(Q_v, M) \to R\Gamma(Q_v, M_p))$$

We define compactly supported cohomology, global f-cohomology, as

$$R\Gamma_c(Z[1/S], N) = Cone(R\Gamma(Z[1/S], N) \to \bigoplus_{v \in S} R\Gamma(Q_v, N))[-1]$$

$$R\Gamma_f(Z[1/S], M_p) = Cone(R\Gamma(Z[1/S], M_p) \to \bigoplus_{v \in S} R\Gamma_f(Q_v, M_p))[-1]$$

Note that we defined compactly-supported cohomology for any reasonable coefficients but f-cohomology only for the galois representation $V_p$ associated to our motive.

We obtain, beyond the defining triangles, a triangle relating $H^*_f$ and $H^*_c$ (Flach)

$$R\Gamma_c(Z[1/S]) \to R\Gamma_f(Q_v) \to \bigoplus_{v \in S} R\Gamma_f(Q_v)$$

We also have compactly supported cohomology with integral coefficients $R\Gamma_c(Z[1/S], T_p)$, using that on local etale cohomology $R\Gamma(Q_v, T_p)$.

**Proposition 6.1**

1. For $N$ finite, the Euler characteristic of $H^*_c(Z[1/S], N)$ is 1.

2. The integral structure

$$[H^*_c(Z[1/S], T_p)] \subset [H^*_c(Z[1/S], M_p)]$$

is independent of choice of lattice $T_p \subset V_p$.

3. The integral structure

$$[L^S(T_p)] \subset [L^S(M_p)]$$

is independent of choice of lattice $T_p \subset M_p$.

**Proof.**

i) We use Tate’s Euler Characteristic formula. $\chi(N) = \frac{|H^0(R, N)|}{|N|}$, for $\chi$ the Euler characteristic $\frac{H^0(Z[1/S], N)H^0(Z[1/S], N)}{H^0(Z[1/S], N)}$.

The local Euler characteristic formula, for $v \neq \infty$, says $\chi_v(N) = \frac{|H^0(Q_v, N)|}{|H^0(Q_v, N)|} = |N|_v = 1/|N_v|\cdot|N_v^\infty|$.

For $v = \infty$, $\chi_\infty(N) = |H^0(R, N)| \cdot \frac{|H^2(R, N)|}{|H^2(R, N)|} = |H^0(R, N)|$, where the last equality is because the Herbrand quotient is 1 for finite modules.

ii) We can assume that $T_p \subset T'_p$. Then

$$\frac{[H^*_c(T'_p)]}{[H^*_c(T_p)]} = [H^*_c(T'_p/T_p)] \cong \mathbb{Z}_p,$$

where the final isomorphism is not because $H^*_c(T'_p/T_p)$ is torsion, but because its Euler characteristic is 1. A little thought shows that this means the integral structures agree, not up to finite difference, but exactly, with changes in an individual $H^1_c(T_p)$, say, being cancelled by changes in $H^0_c(T_p)$, $H^2_c, H^3_c$ as well.
iii) When the $L$-complex $L^v(M_p)$ is acyclic, note that $[L^v(T_p)] = [\det(1 - f|M^v_p)]^{-1}$ does not depend on the lattice at all.

More generally, we can use the exact sequence

$$0 \rightarrow T_p^{f=1} \rightarrow T_p \rightarrow T_p/T_p^{f=1} \rightarrow 0$$

to obtain $[L^v(T_p)] = [L^v(T_p/T_p^{f=1})] \cdot [L^v(T_p^{f=1})]$. By the acyclic case, we have that $[L^v(T_p/T_p^{f=1})] \subset [L^v(M_p/M_p^{f=1})]$ is independent of choice of $T_p$.

Further, the determinants $[L^v(T_p^{1-f})]$ and $[L^v(M_p^{1-f})]$ have canonical elements due to the morphism $1 - f$ in the complexes being zero. These canonical elements are the same, hence the integral structure $[L^v(T_p^{1-f})] \subset [L^v(M_p^{1-f})]$ is independent of $T_p$.

□

**Conjecture 6.2** The map

$$H^1_{f}(M) \rightarrow H^1(M_p)$$

lands in the subspace $H^1_f(M_p)$.

A preprint by Nekovar ([4]) claims to prove the above conjecture for $p$ a prime of potentially good reduction.

**Conjecture 6.3** The $p$-adic regulators

$$H^i_f(M)_{\mathbb{Q}_p} \rightarrow H^i_f(M_p), i = 0, 1, 2, 3$$

are isomorphisms.

Recall that

$$\Xi(M) = \frac{[H^*_f(M)][\text{Lie}(M)]}{[M^+_B]}$$

Assuming these conjectures, we have the following isomorphism

$$\theta_p: [H^*_c(Z[1/S], M_p)][L^S(M_p)] \cong \Xi(M)_{\mathbb{Q}_p}.$$ 

This uses the isomorphism

$$[H^*_c(Z[1/S], M_p)] = \frac{[H^*_f(\mathbb{Q}, M_p)][L^S(M_p)]^{-1}[H^1_f(\mathbb{Q}_p, M_p)]}{[(M^+_B)_{\mathbb{Q}_p}]}$$

followed by the Bloch-Kato exponential

$$\exp_{BK}: D(M_p)/F^0D(M_p) \cong H^1_f(\mathbb{Q}_p, M_p)$$

and the de Rham comparison theorem

$$D(M_p) \cong (M_{dR})_{\mathbb{Q}_p}, F^0D(M_p) \cong (F^0M_{dR})_{\mathbb{Q}_p}.$$
7 Statement of Conjecture

Recall that Beilinson’s conjecture predicts that $\theta_\infty : \mathbb{R} \to \Xi(M)_{\mathbb{R}}$ has $\theta_\infty(L(M)^{-1}) \in \Xi(M)_{\mathbb{Q}}$.

Conjecture 7.1 (Bloch-Kato) For all $p$, the following holds:

Let $S = \{p, \text{ primes of bad reduction } \}$. Then the following $\mathbb{Z}_p$-integral structures agree:

$$\theta_\infty([H^*_c(\mathbb{Z}[1/S], T_p)][L^S(T_p)]) \subset \Xi(M)_{\mathbb{Q}_p} \supset \theta_\infty(L(M)^{-1}) \cdot \mathbb{Z}_p$$

Note that both integral structures are isogeny-invariant: the LHS by Euler characteristic and the RHS by definition.

8 Comparison with BSD

Let $E/\mathbb{Q}$ be an elliptic curve.

Assumption 8.1 $\text{III}(E)$ is finite.

Remark 8.2 There is no reason to restrict to $E$ an elliptic curve, except to avoiding discussing Neron models. (This is silly, and we should change it, especially since we use Neron forms below)

We consider the motive $T = H^1(E, \mathbb{Z}(1)) = H_1(E, \mathbb{Z})$. We will show that the Bloch-Kato conjectures for the motive $M = T \otimes \mathbb{Q}$ is equivalent to BSD.

The associated L-function is $L(E, s)$ at the point $s = 1$. The $l$-adic representation is the Tate module $T_p = T_p(E)$, and the Hodge realization is the first homology $H_1(E, \mathbb{Z})$, which has type $(-1,0)+(0,-1)$. This implies that $M_{dR}/F^0 = \text{Lie}(E) = (H^0(E, \Omega^1))^*$.

Note that $H^1_f(M) = E(\mathbb{Q}) \otimes \mathbb{Q}$. This shows that

$$\Xi(M) = \frac{[(E(\mathbb{Q})/\text{tors})^*_{\mathbb{Q}}]}{[(H^0(E(\mathbb{C}), \mathbb{Z}))^+]}. \frac{[\text{Lie}(E)]}{[(E(\mathbb{Q})/\text{tors})^*_{\mathbb{Q}}]}.$$  

Note that $\Xi(M)$ actually has a $\mathbb{Z}$-integral structure we do not have for the general motive, by using a canonical integral structure on de Rham cohomology. It is generated by $\beta = (\wedge v_i^*) \otimes (\wedge v_i)^{-1} \otimes \omega^* \otimes \gamma^{-1} \in \Xi(M)$, where $\{v_i\}$ is a basis for $E(\mathbb{Q})/\text{tors}$, $\{v_i^*\}$ the dual basis, $\omega^*$ is dual to a Neron form, and $\mathbb{Z} \cdot \gamma = H_1(E(\mathbb{C}), \mathbb{Z})^+$.

With respect to this integral structure, we will (roughly) measure both the real volumes and $v$-adic volumes, and, assuming the BK conjecture, show that their product is $\pm 1$, by comparing $p$-adic valuations.

8.1 Real Stuff

We have two maps:

$$\alpha : H_1(E, \mathbb{Z})^+ \to \text{Lie}(E)$$

with $\alpha(\gamma) = (\int_\gamma \omega)\omega^*$, and

$$h : E(\mathbb{Q})/\text{tors} \times E(\mathbb{Q})/\text{tors} \to \mathbb{R}$$

the canonical height pairing.

Together, these give a canonical element $\text{Reg}(E)\Omega_\mathbb{R} \cdot \beta \in \Xi(M)_{\mathbb{R}}.$
8.2 Integral Structures

For the sake of computation, we must find some ad-hoc integral structures on the $H^*_f(V_p)$ groups. Abusing notation, we will denote them as $H^*_f(T_p)$.

We define, for all places $v$, all primes $p$,

$$H^1_f(Q_v, T_p) = E(Q_v)_p,$$

and

$$H^1_f(Q, T_p) = E(Q)_p.$$

We can also define a “co-integral structure” $H^1_f(Q, V_p/T_p)$ to be the direct limit of the Selmer groups

$$Sel_{p^n}(E) = \{ x \in H^1(Q, E[p^n]) \mid x \in Im(E/p^n E(Q_v) \rightarrow H^1(Q_v, E[p^n])) \text{ for all places } v \}.$$

Then, using the global duality $H^2_f(Q, V_p) \times H^1_f(Q, V_p) \rightarrow Q_p$, we verify that $H^2_f(Q, V_p) := (H^1_f(Q, V_p/T_p))^\wedge$ is an integral structure on $H^2_f(Q, V_p)$.

We similarly define $H^3_f(Q, T_p) := H^0(Q, V_p/T_p)$.

Remark 8.3 It would have been preferable to have define these integral structures at the level of complexes, but there are issues with doing this when $p$ is a prime of bad reduction.

Theorem 8.4 ([3]) $[H^*_c(Z_S, T_p)] = [H^*_f(Q, T_p)]^{\oplus \in S} H^*_f(Q_v, T_p)]^{-1}$.

Proof. The point is to use local Tate duality for abelian varieties to show that the ad-hoc Selmer conditions above are “integrally self-dual”. As we have stated it, we are also using the compatibility of Cartier duality with local Tate duality ([5]), but that is just for convenience. □

We also need the exact sequence

$$0 \rightarrow E(Q)/\text{tors} \rightarrow H^1_f(Q, V_p/T_p) \rightarrow \Pi[p^\infty] \rightarrow 0,$$

noting that the direct limit along $E/p^nE(F) \xrightarrow{[p]} E/p^nE(F)$ is $E(F)/\text{tors} \otimes Q_p/\mathbb{Z}_p$ for any field $F$.

8.3 Computation

$$[H^*_c(Z_S, T_p)] = \frac{[H^*_f(Q, T_p)]}{\bigoplus_{v \in S} H^*_f(Q_v, T_p)}/\Pi[p^\infty]^{-1}.$$

Global f-cohomology:

• $H^0_f(Q, T_p) = 0$

• $[H^1_f(Q, T_p)] = [E(Q)_p] = [E(Q)_{\text{tors}}][(E(Q)/\text{tors})_p]$
• $[H_f^2(\mathbb{Q}, T_p)] = [\text{III}(E)][(E(\mathbb{Q})/\text{tors})_{\mathbb{Q}_p}]^*$

• $[H_f^3(\mathbb{Q}, T_p)] = [E(\mathbb{Q})_{\text{tors}}]$

Local $f$-cohomology:

• $v = \infty$: $[H_f^1(\mathbb{R}, T_p)] = [\Phi_\infty]$

• $v \neq p$: $[H_f^1(\mathbb{Q}_v, T_p)] = [E(\mathbb{Q}_v)_{\mathbb{Z}_p}] = [\Phi_v][E^0(\mathbb{F}_v)]$

• $v = p$: $[H_f^1(\mathbb{Q}_v, T_p)] = [E(\mathbb{Q}_p)_{\mathbb{Z}_p}] = [\Phi_p][E^0(\mathbb{F}_p)][\hat{E}(p\mathbb{Z}_p)] = \frac{|\phi_p||E^0(\mathbb{F}_p)|}{|p|}[D(T_p)/F^0D(T_p)]$

\[
\frac{[H_c^1(\mathbb{Z}_S, T_p)]}{|[\text{III}(E)][(E(\mathbb{Q})/\text{tors})_{\mathbb{Q}_p}]^*|} = \frac{[H_f^1(\mathbb{R}, T_p)]}{[H_f^0(\mathbb{R}, T_p)]} \prod_{v \in S, v \neq p, \infty} [\Phi_v] \cdot [E^0(\mathbb{F}_v)]
\]

\[
= \frac{|D(T_p)/F^0D(T_p)|}{[T_p^+]} \prod_{v \in S, v \neq \infty} [\Phi_v] \cdot [E^0(\mathbb{F}_v)]
\]

Note $[L^S(E, 1)] = \left(\prod_{v \in S, v \neq \infty} \frac{[E^0(\mathbb{F}_v)]}{[\mathbb{Z}/p\mathbb{Z}]}\right)^{-1}$.

\[
\frac{[H_c^1(\mathbb{Z}_S, T_p)]/[L^S(E, 1)]}{[\text{III}(E)][(E(\mathbb{Q})/\text{tors})_{\mathbb{Q}_p}]^*} = \frac{[(E(\mathbb{Q})/\text{tors})_{\mathbb{Q}_p}][D(T_p)/F^0D(T_p)]}{[(E(\mathbb{Q})/\text{tors})_{\mathbb{Q}_p}]} [\tau_p^+]
\]

\[
\subset \frac{[(E(\mathbb{Q})/\text{tors})_{\mathbb{Q}_p}][\text{Lie}(E)_{\mathbb{Q}_p}]}{[(E(\mathbb{Q})/\text{tors})_{\mathbb{Q}_p}][H_1(E(\mathbb{C}), \mathbb{Z})^+\mathbb{Q}_p]} = \Xi(M)_{\mathbb{Q}_p}
\]

Thus Bloch-Kato reduces to the claim that, for each $p$, the integral structure given by $[H_c^1(\mathbb{Z}_S, T_p)]/[L^S(E, 1)]$ agrees with the integral structure given by $\frac{\text{Reg}(E)\Omega_\infty}{L(E, 1)} \cdot \alpha$. This is equivalent to

\[
\text{ord}_p \left( \frac{[\text{III}(E)][\prod_{v \in S} [\Phi_v]]}{[E(\mathbb{Q})_{\text{tors}}]^2} \right) = \text{ord}_p \left( \frac{L(E, 1)}{\text{Reg}(E)\Omega_\infty} \right) \forall p,
\]

which implies the BSD conjecture.

**Remark 8.5** Some formulations of BSD do not use the component group at infinity $\Phi_\infty$, combining it into the period integral:

\[
\Omega_\mathbb{R} \cdot |\Phi_\infty| = \int_{E(\mathbb{R})} \omega
\]
References


