The Bloch-Kato Tamagawa Number Conjecture

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1 Introduction

Bloch and Kato originally thought of their conjectures as a version of the Tamagawa number conjecture for algebraic groups, replacing the algebraic group by a pure motive.

algebraic groups < − − −− > abelian varieties < − − −− > motives

Abelian varieties are the prototypical motive. We recall how the BSD conjecture for abelian varieties equals a Tamagawa number conjecture.

First, we recall the theorem for algebraic groups:

Theorem 1.1 (?) *For a connected algebraic group G over a number field K, we have*

$$\tau(G) = \frac{|Pic^0(G)|}{|Ш(G)|}.$$

Here, $\tau(G)$ is roughly the volume of $G(\mathbb{A}_K)/G(K)$ with respect to Haar measures on $G(K_v)$ for all places v, where we have to use L-functions to make the product measure converge, and also have to restrict to measuring some "compact part" of $G(K_v)$, by taking the kernel of all $|\chi|_v$, $\chi\colon G \to \mathbb{G}_m$.

Now, let's formulate a version of this for abelian varieties. For simplicity, assume that E is an elliptic curve over $\mathbb{Q}$, with $E(\mathbb{Q})$ finite. There is a Neron model $\mathcal{E}$ for E, with Neron form ω. This induces a measure on $E(\mathbb{Q}_p)$: one way to formulate this is that the map $log = \int \omega\colon \mathcal{E}(\mathbb{Z}_p) \to Lie(\mathcal{E})_{\mathbb{Q}_p}$ (multiply till you land in "kernel of reduction" $\mathcal{E}(p\mathbb{Z}_p)$, then evaluate power series) induces a measure on $\mathcal{E}(\mathbb{Z}_p)$ by declaring that it preserves measure and that $Lie(\mathcal{E})_{\mathbb{Z}_p}$ has volume 1.

A calculation shows that $vol(E(\mathbb{Q}_p)) = \frac{|\tilde{E}^0(\mathbb{F}_p)|}{p} \cdot |\Phi_p(\mathbb{F}_p)|$, where Φ_p is the component group-scheme. Define $c_p = |\Phi_p(\mathbb{F}_p)|$, the Tamagawa factor at p. Also, $vol(E(\mathbb{R})) = \int_{E(\mathbb{R})} \omega$ is the real period.

The product $\prod_p vol(E(\mathbb{Q}_p))$ does not converge. However, note that $L(E,1) = \prod_p det(1 - p^{-1} f|((V_l E)^*)^{I_v})^{-1} = \prod_p det(1 - f|(V_l(E))^{I_v})^{-1} = \prod_p \frac{p}{|\tilde{E}^0(\mathbb{F}_p)|}$, using the Cartier duality $T_p(E)^*(1) \cong T_p(E)$.

Thus we can define the renormalized adelic volume to be

$$vol(E(\mathbb{A})) = L(E,1)^{-1} vol(E(\mathbb{R})) \prod_p c_p$$

The Tamagawa number conjecture then becomes

$$vol\left(\frac{E(\mathbb{A})}{E(\mathbb{Q})}\right) = \frac{L(E,1)^{-1} vol(E(\mathbb{R})) \prod_p c_p}{|E(\mathbb{Q})|} \stackrel{?}{=} \frac{|E(\mathbb{Q})|}{Ш(E)}.$$

This is evidently equivalent to BSD. More generally, if $E(\mathbb{Q})$ is not finite, $L(E,1)$ should vanish, and $|E(\mathbb{Q})|$, $|Pic^0(E)|$ are not finite. However, if we replace $L(E,1)$ by the leading term $L^*(E,1)$ and introduce height pairings to measure, not the covolume of $E(\mathbb{Q})/tors$ in $E(\mathbb{A})$, but its "density", we again recover BSD.

How do we generalize this to other motives?

- Global points modulo torsion via K-theory
- Torsion in global points, Pic^0, Ш, via global etale cohomology
- Local nonarchimedian points via local etale cohomology
- Local nonarchimedean volumes via Bloch-Kato exponential
- Local real volumes via period map, real regulators, as in Beilinson conjecture
- Height pairings to make sense of quotienting adelic points by global points, when not just torsion

Once we make precise what all this means, we will have, for a motive $M = h^i(X)(j)$, the exact same conjecture:

$$\frac{vol(M(\mathbb{A}))Reg(M)}{|M(\mathbb{Q})_{tors}|} = \frac{L^*(M,0)^{-1}Reg(M)vol(M(\mathbb{R}))\prod_p c_p}{|M(\mathbb{Q})_{tors}|} \stackrel{?}{=} \frac{|(M^*(1))(\mathbb{Q})_{tors}|}{|Ш(M)_{tors}|}.$$

One convenient way to formalize this was found by Fontaine and Perrin-Riou.

First, recall that, in formulating Beilinson's conjecture, we had, for a motive M of weight $w < -1$, an injective real period map

$$\alpha \colon (M_B^+)_{\mathbb{R}} \to (M_{dR}/F^0 M_{dR})_{\mathbb{R}} =: Lie(M)_{\mathbb{R}}$$

and a conjectural isomorphism

$$H^{i+1}_{M,\mathbb{Z}}(X, \mathbb{Q}(j))_{\mathbb{R}} \cong coker(\alpha) =: H^{i+1}_D(X_{\mathbb{R}}, \mathbb{R}(j)).$$

Using these, we obtain, denoting $[\cdot] := det(\cdot)$ for the top exterior power of a vector space, division meaning tensor with dual,

$$\theta_\infty \colon \mathbb{R} \cong \left(\frac{[Lie(M)]}{[H^{i+1}_{M,\mathbb{Z}}(X, \mathbb{Q}(j))][M_B^+]} \right) =: \Xi(M) \otimes \mathbb{R}$$

Beilinson's conjecture (not the rank part) is equivalent to the claim that

$$\theta_\infty(1/L^*(M)) \in \Xi(M).$$

In other words $L^*(M)$ should measure how far θ_∞ is from respecting the rational structures $\mathbb{Q} \subset \mathbb{R}$, $\Xi(M) \subset \Xi(M) \otimes \mathbb{R}$.

Fontaine and Perrin-Riou generalize this for all w, defining $\mathbb{Q}$-vs $H^*_f(M)$, $* = 0,1,2,3$, and define

$$\Xi(M) := \frac{[H^*_f(M)][Lie(M)]}{[M_B^+]},$$

as well as an isomorphism

$$\theta_\infty \colon \mathbb{R} \to \Xi(M)_\mathbb{R},$$

and still conjecture

$$\theta_\infty(1/L^*(M)) \in \Xi(M).$$

We think of this canonical element in $\Xi(M)$ as defining a $\mathbb{Z}$-integral structure. Using etale cohomology, we can define $\mathbb{Z}_p$-integral structures

$$\theta_p \colon \Lambda \hookrightarrow \Xi(M)_{\mathbb{Q}_p},$$

($\Lambda \cong \mathbb{Z}_p$, but not canonically so)

The Bloch-Kato conjecture is then:

Conjecture 1.2 ([2]) *The $\mathbb{Z}_p$-integral structures*

$$\mathbb{Z}_p \cdot \theta_\infty(1/L^*(M)) \subset \Xi(M)_{\mathbb{Q}_p} \supset \theta_p(\Lambda)$$

agree for all p.

In terms of volumes, this says, roughly, that

Conjecture 1.3 *For all primes p,*

$$ord_p\left(\frac{Reg(M)vol(M(\mathbb{R}))}{L^*(M)}\right) = ord_p\left(\frac{|M(\mathbb{Q})_{tors}||M^*(1)(\mathbb{Q})_{tors}|}{|Ш(M)|\prod_v c_v}\right)$$

Remark 1.4 *We could make sense of the p-adic valuation of all the invariants on the RHS in terms of p-adic etale cohomology.*

2 Determinants

Motivation This is mostly just book-keeping.

Given a (finite dimensional) vector space V, define $[V] = \bigwedge^{top} V$. Note that if $V = 0$, then $[V] \cong \mathbb{Q}_p$ canonically. Given an exact sequence $0 \to A \to B \to C \to 0$, we have $[B] \cong [A][C]$. We will need to keep track of isomorphisms, or else this is useless. We consider integral structures T on $\mathbb{Q}_p$-vector spaces V, by which we mean finitely-generated $\mathbb{Z}_p$ modules T with a canonical isomorphism $T \otimes \mathbb{Q}_p \cong V$. An integral structure T on V determines a $\mathbb{Z}_p$-submodule $[T] \subset [V]$ as follows:

If $T \subset V$ is torsion-free, then $[T] \subset [V]$ is what you expect. If $V = 0$, $T = 0$, then $[T] \cong \mathbb{Z}_p \subset \mathbb{Q}_p \cong [V]$. If $V = 0$, $T = \mathbb{Z}/p\mathbb{Z}$, then $[\mathbb{Z}/p\mathbb{Z}] = \frac{1}{p}\mathbb{Z}_p \subset \mathbb{Q}_p \cong [V]$. i.e. torsion groups have larger volumes than trivial groups.

For a general integral structure T, $[T] = [T/tors][T_{tors}] = |T_{tors}|_p[T/tors] \subset [V]$, where $|\cdot|_p$ is the valuation with $|p|_p = \frac{1}{p}$.

Given a a finite complex $C : A_0 \to A_1 \to \ldots \to A_n$ of $\mathbb{Q}_p$-vs, we define $[C] \cong \frac{[A_0][A_2]\cdots}{[A_1][A_3]\cdots}$.

We can deduce: $[C] = [H^*(C)]$.

Consider an integral structure $X \subset C$. The cohomology complex $H^*(X)$ is an integral structure of $H^*(X)$. Thus we obtain $[H^*(X)] \subset [H^*(C)] \cong [C]$.

Consider $f \in Aut(V)$, such that $f(T) \subset T$ for $T \subset V$ a lattice. Consider the complexes $(T \xrightarrow{f} T) \subset (V \xrightarrow{f} V)$. Now, $[V \to V] = [H^*(V \to V)] = [0 \to 0] \cong \mathbb{Q}_p \supset \mathbb{Z}_p$ has a canonical integral structure. We compare this to the integral structure $[H^*(T \to T)]$:

$$[H^*(T \to T)] = [coker(f|T)]^{-1} = |det(f)|_p^{-1} \cdot \mathbb{Z}_p$$

Similarly, the complex $(T \xrightarrow{f} f(T)) \subset (V \xrightarrow{f} V)$ has $[H^*(T \to f(T))] \subset [H^*(V \to V)] \cong [0] = \mathbb{Q}_p \subset \mathbb{Z}_p$.

3 Motivic f-cohomology

Motivation We need rational structures to compare the p-adic, ∞-adic computations.

For a motive $M = h^i(X, \mathbb{Q}(j))$, with weight $w = i - 2j$, we define

- $H_f^0(M) = CH^j(X)_{\mathbb{Q}}/hom.equiv.$ if $i = 2j$, 0 otherwise
- $H_f^1(M) = \begin{cases} H^{i+1}_{M,\mathbb{Z}}(X, \mathbb{Q}(j)) = Im(K_{2j-i-1}(\mathfrak{X})^{(2j)}_{\mathbb{Q}} \to K_{2j-i-1}(X)_{\mathbb{Q}}), & i \neq 2j-1, \\ CH^j(X)_{hom\sim 0}, & i = 2j-1 \end{cases}$,

  for $\mathfrak{X}$ a regular proper model of X over $\mathbb{Z}$ (what if this doesn't exist?)
- $H_f^2(M) = (H_f^1(M^*(1)))^*$
- $H_f^3(M) = (H_f^0(M^*(1)))^*$

Note that $H_f^0 = 0$ if $w \neq 0$, $H_f^3 = 0$ if $w \neq -2$ (immediately right and left of the point of symmetry $w = -1$).

Conjecture 3.1 ([2])

$$ord_{s=0}(L(M, s)) = \dim_{\mathbb{Q}} H_f^1(M^*(1)) - \dim H_f^0(M^*(1))$$

Remark 3.2 *This is, conjecturally on the isomorphism of the p-adic regulator (see below), the same conjecture as in Tony's talk in terms of Bloch-Kato selmer groups.*

Remark 3.3 *This conjectures possible poles for $w = -2$, possible zeros for $w \geq -1$, and $ord = 0$ for $w < -2$. Note, for example, that $\zeta(r)$ relates to the motives $\mathbb{Q}(r)$ of weight $-2r$.*

We use these groups to define the fundamental $\mathbb{Q}$-line

$$\Xi(M) = \frac{[H_f^*(M)][Lie(M)_{\mathbb{R}}]}{[M_B^+]}.$$

Remark 3.4 *The definition of H_f^2 is convenient, but it is bad: $Ext_f^2(\mathrm{Spec}(\mathbb{Q})_{mot}, M) = 0$ according to Beilinson's conjectures (Scholl says this). Further, these groups definitely do not have the correct torsion even if you decide not to $\otimes\mathbb{Q}$: Should have class groups for number fields.*

4 Real Volumes

Motivation: Incorporate Beilinson's conjecture, including height pairings.

We have a real period map

$$\alpha\colon (M_B^+)_{\mathbb{R}} \to (M_{dR}/F^0 M_{dR})_{\mathbb{R}} = (Lie(M))_{\mathbb{R}}.$$

A motive is called "critical" when α is an isomorphism. For example, motives of weight -1, such as $H^1(E, \mathbb{Z}(1)) = H_1(E, \mathbb{Z})$, are always critical.

In this case, we obtain an isomorphism

$$\mathbb{R} \overset{[\alpha]}{\cong} [Lie(M)_{\mathbb{R}}]/[(M_B^+)_{\mathbb{R}}].$$

Also, when the weight is -1, we have the possibility of height pairings:

Conjecture 4.1 *When $w = -1$, the height pairing*

$$h\colon H_f^1(M)_{\mathbb{R}} \times H_f^1(M^*(1))_{\mathbb{R}} \to \mathbb{R}$$

is nondegenerate.

Assuming the conjecture, we obtain

$$\mathbb{R} \overset{[h]}{\cong} [H_f^1(M^*(1))^*]/[H_f^1(M)].$$

In combination, these give an isomorphism

$$\theta_\infty\colon \mathbb{R} \cong \Xi(M)_{\mathbb{R}}$$

Now we deal with the noncritical case, and assume the weight is < -1. Here $H_f^2(M) = H_f^3(M) = 0$.

Conjecture 4.2 *When $w < -1$, the real regulator*

$$H_f^1(M)_{\mathbb{R}} \to coker(\alpha)$$

is an isomorphism.

Since $[\text{coker}\,(\alpha)] = \frac{[Lie(M)_{\mathbb{R}}]}{[(M_B^+)_{\mathbb{R}}]}$, we again obtain

$$\theta_\infty\colon \mathbb{R} \cong \Xi(M)_{\mathbb{R}}.$$

Remark 4.3 *When the weight is > -1, we need to use factors from the functional equation to define the map θ_∞ in terms of that for its dual motive $M^*(1)$. See Fontaine and Perrin-Riou.*

Remark 4.4 *All cases can be combined into the conjectural exactness of the sequence*

$$0 \to H_f^0(M) \to ker(\alpha) \to H_f^1(M^*(1))^* \overset{h}{\to} H_f^1(M) \to coker(\alpha) \to H_f^0(M^*(1))^* \to 0,$$

which perhaps suggests that H_f^*, $* = 0, 1$, is dual to a cohomology theory which is "compactly supported at infinity". See Deninger-Nart.

The map θ_∞ can also be defined for $w > -1$. For all weights w we have the following conjecture:

Conjecture 4.5 (Beilinson)

$$\theta_\infty(1/L^*(M)) \in \mathbb{Q}.$$

5 Local f-cohomology and the Bloch-Kato Exponential

Motivation Local conditions, being unramified, analogous to $H^*_{M,\mathbb{Z}}$.

Fix a prime p. We define complexes

$$R\Gamma_f(\mathbb{Q}_v, M_p) = \begin{cases} v = \infty : & R\Gamma(\mathbb{R}, M_p) \\ v \neq p : & M_p^{I_v} \stackrel{1-f}{\to} M_p^{I_v} \\ v = p : & D_{cris}(M_p) \stackrel{(1-f,\pi)}{\to} D_{cris}(M_p) \oplus D_{dR}(M_p)/F^0 D_{dR}(M_p) \end{cases},$$

with f the geometric Frobenius.

Their cohomology groups H^i_f are the same as those in Tony's talk, as we will shortly see.

Local L-complexes (This is just notation for later.)

We have the complexes for $v \neq \infty$:

$$L^v(T_p) = \begin{cases} v \neq p : & T_p^{I_v} \stackrel{1-f}{\to} T_p^{I_v} \\ v = p : & D_{cris}(T_p) \stackrel{1-f}{\to} D_{cris}(T_p) \end{cases},$$

We define $L^v(M_p) = L^v(T_p) \otimes \mathbb{Q}_p$.

We also define $[L^S(M_p)] = \otimes_{v \in S - \{\infty\}}[L^v(M_p)]$, with integral structure $[L^S(T_p)] = \otimes_{v \in S - \{\infty\}}[L^v(T_p)]$.

Note that if $L^v(M_p)$ is acyclic, then $[L^v(T_p)] = [det(1 - f|M_p^{I_v})]^{-1}$, like a local L-factor. This explains the notation.

Recall that f-cohomology is a "self-dual Selmer condition":

Proposition 5.1 *$H^1_f(\mathbb{Q}_p, M_p)$ is the exact annihilator of $H^1_f(\mathbb{Q}_p, M_p^*(1))$ under the Tate local duality pairing.*

We want to define the Bloch-Kato exponential

$$exp_{BK} \colon D_{dR}(M_p)/F^0 D_{dR}(M_p) \to H^1_f(\mathbb{Q}_p, M_p)$$

It arises from the "fundamental exact sequence of p-adic Hodge theory":

$$0 \to \mathbb{Q}_p \to B_{cris} \stackrel{(1-f,\pi)}{\to} B_{cris} \oplus B_{dR}/B^+_{dR} \to 0.$$

A sequence similar to this was in Tony's talk.

Tensoring this with our representation M_p (which is assumed to be de Rham), and taking the LES of Galois cohomology

$$0 \to H^0(M_p) \to D_{cris}(M_p) \to D_{cris}(M_p) \oplus D_{dR}(M_p)/F^0 D(M_p) \to ker(H^1(M_p) \to H^1(M_p \otimes B_{cris})) \to 0,$$

Note that this verifies that the definition of H^1_f in Tony's talk agrees with the 1st cohomology of the above complex.

We can also express the BK exponential in terms the Ext^1-consequence of the crystalline comparison theorem.

Proposition 5.2 ([1]) *For M_p crystalline, we have the following isomorphism:*

$$D(M_p)/(1-f)F^0 D(M_p) \cong Ext^1_{f,Fil}(\mathbb{Q}_p, D(M_p)) \cong Ext^1_{K_p}(\mathbb{Q}_p, M_p)_f.$$

In other words, crystalline extensions of galois representations are identified with extensions of (f,Fil)-modules.

An aside on Fontaine-Lafaille Theory([1])

If the lattice $D(T_p) \subset D(M_p)$ is "Fontaine-Lafaille" (strongly divisible and with weights in $[0, p-1]$), we have an integral comparison theorem

$$D(T_p)/(1-f)F^0D(T_p) \cong Ext^1_{f,Fil}(\mathbb{Z}_p, D(T_p)) \cong Ext^1_{K_v}(\mathbb{Z}_p, T_p)_f.$$

In this case, we have the following:

$$\begin{array}{ccc} D(T_p)/F^0D(T_p) & \xrightarrow{exp_{BK}} & H^1(K_v, T_p) \\ & \searrow^{1-f} & \downarrow \cong \\ & & D(T_p)/(1-f)F^0D(T_p) \end{array}$$

This implies that when a lattice is Fontaine-Lafaille, that the local volume agrees with the local L-factor. Morally, this means that we have good reduction, in some strange new sense, since the Tamagawa factor at p is then 1.

For example, Bloch-Kato shows that the lattice $D(\mathbb{Z}_p(r))$ is not Fontaine-Lafaille for $p < r$, contributing an extra factor of $1/(r-1)!$ to the adelic volume as we vary over all such primes.

Bloch-Kato Exponential and Kummer Theory([1])

For abelian varieties and tori, the Bloch-Kato exponential agrees with the Kummer map. We first show it for $\mathbb{G}_m$, using the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow & \varprojlim_p \mathcal{O}^*_{\mathbb{C}_p} & \longrightarrow & \mathcal{O}^*_{\mathbb{C}_p} & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow log[\cdot] & & \downarrow log & & \\ 0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & B^{f=p}_{cris} \cap B^+_{dR} & \xrightarrow{\theta} & \mathbb{C}_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & (B^{f=1}_{cris})(1) & \longrightarrow & (B_{dR}/B^+_{dR})(1) & \longrightarrow & 0 \end{array}$$

To get the result for abelian varieties, use that $Hom_{FormalGroup}(\widehat{A}, \widehat{\mathbb{G}_m})(\mathcal{O}_{\mathbb{C}_p}) \cong T_p(A)^*(1)$ by Cartier duality. For any choice of $\chi \in T_p(A)^*(1)$, we get a map (not galois equivariant) from the sequence

$$0 \to T_p(A) \to \varprojlim_p A(\mathcal{O}_{\mathbb{C}_p}) \to A(\mathcal{O}_{\mathbb{C}_p}) \to 0$$

to the last row of the above diagram, i.e. we get a (galois equivariant) map from this sequence to the last row tensor $V_p(A)(-1)$.

Bloch-Kato claim this proof works, in some sense, for abelian varieties with bad reduction.

6 Global f-cohomology

There is a homological algebra construction, which, given a map of complexes, formally create a complex fitting into a long-exact sequence:

$$\ldots \to H^i(A) \to H^i(B) \to H^i(Cone(A \to B)) \to \ldots,$$

Note that this implies the determinant formula

$$[Cone(A \to B)] = \frac{[B]}{[A]}.$$

Let $S = \{\infty, p, v \text{ s.t. } V^{I_v} \neq V\}$. Let $R\Gamma(\mathbb{Z}[1/S], N)$ be the complex computing global galois cohomology, for N any reasonable Galois module. Similarly we use $R\Gamma(\mathbb{Q}_v, V_p)$ for local galois cohomology.

We first define the "quotient" of local cohomology by local f-cohomology, $R\Gamma_{/f}(\mathbb{Q}_v, M_p)$, as

$$R\Gamma_{/f}(\mathbb{Q}_v, M_p) = Cone(R\Gamma_f(\mathbb{Q}_v, M_p) \to R\Gamma(\mathbb{Q}_v, M_p))$$

We define compactly supported cohomology, global f-cohomology, as

$$R\Gamma_c(\mathbb{Z}[1/S], N) = Cone(R\Gamma(\mathbb{Z}[1/S], N) \to \oplus_{v\in S} R\Gamma(\mathbb{Q}_v, N))[-1]$$

$$R\Gamma_f(\mathbb{Z}[1/S], M_p) = Cone(R\Gamma(\mathbb{Z}[1/S], M_p) \to \oplus_{v\in S} R\Gamma_{/f}(\mathbb{Q}_v, M_p))[-1]$$

Note that we defined compactly-supported cohomology for any reasonable coefficients but f-cohomology only for the galois representation V_p associated to our motive.

We obtain, beyond the defining triangles, a triangle relating H^*_f and H^*_c (Flach)

$$R\Gamma_c(\mathbb{Z}[1/S]) \to R\Gamma_f(\mathbb{Q}) \to \oplus_{v\in S} R\Gamma_f(\mathbb{Q}_v)$$

We also have compactly supported cohomology with integral coefficients $R\Gamma_c(\mathbb{Z}[1/S], T_p)$, using that on local etale cohomology $R\Gamma(\mathbb{Q}_v, T_p)$.

Proposition 6.1

1. *For N finite, the Euler characteristic of $H^*_c(\mathbb{Z}[1/S], N)$ is* 1.
2. *The integral structure*
$$[H^*_c(\mathbb{Z}[1/S], T_p)] \subset [H^*_c(\mathbb{Z}[1/S], M_p)]$$
is independent of choice of lattice $T_p \subset V_p$.
3. *The integral structure*
$$[L^S(T_p)] \subset [L^S(M_p)]$$
is independent of choice of lattice $T_p \subset M_p$.

Proof. i) We use Tate's Euler Characteristic formula. $\chi(N) = \frac{|H^0(\mathbb{R},N)|}{|N|}$, for χ the Euler characteristic $\frac{H^0(\mathbb{Z}[1/S],N)H^0(\mathbb{Z}[1/S],N)}{H^1(\mathbb{Z}[1/S],N)}$

The local Euler characteristic formula, for $v \neq \infty$, says $\chi_v(N) = \frac{|H^0(\mathbb{Q}_v,N)|}{|H^1(\mathbb{Q}_v,N)|} = |N|_v = 1/|N[v^\infty]|$.

For $v = \infty$, $\chi_\infty(N) = |H^0(\mathbb{R}, N)| \cdot \frac{|H^2(\mathbb{R},N)|}{|H^1(\mathbb{R},N)|} = |H^0(\mathbb{R}, N)|$, where the last equality is because the Herbrand quotient is 1 for finite modules.

ii) We can assume that $T_p \subset T'_p$. Then

$$\frac{[H^*_c(T'_p)]}{[H^*_c(T_p)]} = [H^*_c(T'_p/T_p)] \cong \mathbb{Z}_p,$$

where the final isomorphism is not becaue $H^*_c(T'_p/T_p)$ is torsion, but because its Euler characteristic is 1. A little thought shows that this means the integral structures agree, not up to finite difference, but exactly, with changes in an individual $H^1_c(T_p)$, say, being cancelled by changes in $H^0_c(T_p)$, H^2_c, H^3_c as well.

iii) When the L-complex $L^v(M_p)$ is acyclic, note that $[L^v(T_p)] = [det(1-f|M_p^{I_v})]^{-1}$ does not depend on the lattice at all.

More generally, we can use the exact sequence

$$0 \to T_p^{f=1} \to T_p \to T_p/T_p^{f=1} \to 0$$

to obtain $[L^v(T_p)] = [L^v(T_p/T_p^{f=1})] \cdot [L^v(T_p^{f=1})]$. By the acyclic case, we have that $[L^v(T_p/T_p^{f=1})] \subset [L^v(M_p/M_p^{f=1})]$ is independent of choice of T_p.

Further, the determinants $[L^v(T_p^{1-f})]$ and $[L^v(M_p^{1-f})]$ have canonical elements due to the morphism $1-f$ in the complexes being zero. These canonical elements are the same, hence the integral structure $[L^v(T_p^{1-f})] \subset [L^v(M_p^{1-f})]$ is independent of T_p.

□

Conjecture 6.2 *The map*

$$H_f^1(M) \to H^1(M_p)$$

lands in the subspace $H_f^1(M_p)$.

A preprint by Nekovar ([4]) claims to prove the above conjecture for p a prime of potentially good reduction.

Conjecture 6.3 *The p-adic regulators*

$$H_f^i(M)_{\mathbb{Q}_p} \to H_f^i(M_p), i = 0, 1, 2, 3$$

are isomorphisms.

Recall that

$$\Xi(M) = \frac{[H_f^*(M)][Lie(M)]}{[M_B^+]}$$

Assuming these conjectures, we have the following isomorphism

$$\theta_p \colon [H_c^*(\mathbb{Z}[1/S], M_p)][L^S(M_p)] \cong \Xi(M)_{\mathbb{Q}_p}.$$

This uses the isomorphism

$$[H_c^*(\mathbb{Z}[1/S], M_p)] = \frac{[H_f^*(\mathbb{Q}, M_p)][L^S(M_p)]^{-1}[H_f^1(\mathbb{Q}_p, M_p)]}{[(M_B^+)_{\mathbb{Q}_p}]}$$

followed by the Bloch-Kato exponential

$$exp_{BK} \colon D(M_p)/F^0D(M_p) \cong H_f^1(\mathbb{Q}_p, M_p)$$

and the de Rham comparison theorem

$$D(M_p) \cong (M_{dR})_{\mathbb{Q}_p}, F^0D(M_p) \cong (F^0M_{dR})_{\mathbb{Q}_p}.$$

7 Statement of Conjecture

Recall that Beilinson's conjecture predicts that $\theta_\infty : \mathbb{R} \to \Xi(M)_\mathbb{R}$ has $\theta_\infty(L(M)^{-1}) \in \Xi(M)_\mathbb{Q}$.

Conjecture 7.1 (Bloch-Kato) *For all p, the following holds:*
Let $S = \{p,\ \text{primes of bad reduction}\ \}$. Then the following $\mathbb{Z}_p$-integral structures agree:

$$\theta_\infty([H_c^*(\mathbb{Z}[1/S], T_p)][L^S(T_p)]) \subset \Xi(M)_{\mathbb{Q}_p} \supset \theta_\infty(L(M)^{-1}) \cdot \mathbb{Z}_p$$

Note that both integral structures are isogeny-invariant: the LHS by Euler characteristic and the RHS by definition.

8 Comparison with BSD

Let $E/\mathbb{Q}$ be an elliptic curve.

Assumption 8.1 $\text{Ш}(E)$ *is finite.*

Remark 8.2 *There is no reason to restrict to E an elliptic curve, except to avoiding discussing Neron models. (This is silly, and we should change it, especially since we use Neron forms below)*

We consider the motive $T = H^1(E, \mathbb{Z}(1)) = H_1(E, \mathbb{Z})$. We will show that the Bloch-Kato conjectures for the motive $M = T \otimes \mathbb{Q}$ is equivalent to BSD.

The associated L-function is $L(E, s)$ at the point $s = 1$. The l-adic representation is the Tate module $T_p = T_p(E)$, and the Hodge realization is the first homology $H_1(E, \mathbb{Z})$, which has type $(-1, 0)+(0, -1)$. This implies that $M_{dR}/F^0 = Lie(E) = (H^0(E, \Omega^1))^*$.

Note that $H^1_f(M) = E(\mathbb{Q}) \otimes \mathbb{Q}$. This shows that

$$\Xi(M) = \frac{[(E(\mathbb{Q})/tors)^*_\mathbb{Q}]}{[(E(\mathbb{Q})/tors)_\mathbb{Q}]} \frac{[Lie(E)]}{[(H_1(E(\mathbb{C}), \mathbb{Z}))^+)]}.$$

Note that $\Xi(M)$ actually has a $\mathbb{Z}$-integral structure we do not have for the general motive, by using a canonical integral structure on de Rham cohomology. It is generated by $\beta = (\wedge v_i^*) \otimes (\wedge v_i)^{-1} \otimes \omega^* \otimes \gamma^{-1} \in \Xi(M)$, where $\{v_i\}$ is a basis for $E(\mathbb{Q})/tors$, $\{v_i^*\}$ the dual basis, ω^* is dual to a Neron form, and $\mathbb{Z} \cdot \gamma = H_1(E(\mathbb{C}), \mathbb{Z})^+$.

With respect to this integral structure, we will (roughly) measure both the real volumes and v-adic volumes, and, assuming the BK conjecture, show that their product is ± 1, by comparing p-adic valuations.

8.1 Real Stuff

We have two maps:

$$\alpha \colon H_1(E, \mathbb{Z})^+ \to Lie(E)$$

with $\alpha(\gamma) = (\int_\gamma \omega)\omega^*$, and

$$h : E(\mathbb{Q})/tors \times E(\mathbb{Q})/tors \to \mathbb{R}$$

the canonical height pairing.

Together, these give a canonical element $Reg(E)\Omega_\mathbb{R} \cdot \beta \in \Xi(M)_\mathbb{R}$.

8.2 Integral Structures

For the sake of computation, we must find some ad-hoc integral structures on the $H_f^*(V_p)$ groups. Abusing notation, we will denote them as $H_f^*(T_p)$.

We define, for all places v, all primes p,

$$H_f^1(\mathbb{Q}_v, T_p) = E(\mathbb{Q}_v)_{\mathbb{Z}_p},$$

and

$$H_f^1(\mathbb{Q}, T_p) = E(\mathbb{Q})_{\mathbb{Z}_p}.$$

We can also define a "co-integral structure" $H_f^1(\mathbb{Q}, V_p/T_p)$ to be the direct limit of the Selmer groups

$$Sel_{p^n}(E) = \{x \in H^1(\mathbb{Q}, E[p^n]) \mid x \in Im(E/p^nE(\mathbb{Q}_v) \to H^1(\mathbb{Q}_v, E[p^n])) \text{ for all places } v\}.$$

Then, using the global duality $H_f^2(\mathbb{Q}, V_p) \times H_f^1(\mathbb{Q}, V_p) \to \mathbb{Q}_p$, we verify that $H_f^2(\mathbb{Q}, T_p) := (H_f^1(\mathbb{Q}, V_p/T_p))^\wedge$ is an integral structure on $H_f^2(\mathbb{Q}, V_p)$.

We similarly define $H_f^3(\mathbb{Q}, T_p) := H^0(\mathbb{Q}, V_p/T_p)^\wedge$.

Remark 8.3 *It would have been preferable to have define these integral structures at the level of complexes, but there are issues with doing this when p is a prime of bad reduction.*

Theorem 8.4 ([3]) $[H_c^*(\mathbb{Z}_S, T_p)] = [H_f^*(\mathbb{Q}, T_p)][\oplus_{v\in S} H_f^*(\mathbb{Q}_v, T_p)]^{-1}$.

Proof. The point is to use local Tate duality for abelian varieties to show that the ad-hoc Selmer conditions above are "integrally self-dual". As we have stated it, we are also using the compatibility of Cartier duality with local Tate duality ([5]), but that is just for convenience. ☐

We also need the exact sequence

$$0 \to E(\mathbb{Q})/tors \to H_f^1(\mathbb{Q}, V_p/T_p) \to \text{Ш}[p^\infty] \to 0,$$

noting that the direct limit along $E/p^nE(F) \xrightarrow{[p]} E/p^nE(F)$ is $E(F)/tors \otimes \mathbb{Q}_p/\mathbb{Z}_p$ for any field F.

8.3 Computation

$$\begin{aligned}[H_c^*(\mathbb{Z}_S, T_p)] &= \frac{[H_f^*(\mathbb{Q}, T_p)]}{[\oplus_{v\in S} H_f^*(\mathbb{Q}_v, T_p)]} \\ &= \frac{[H_f^2(\mathbb{Q}, T_p)][H_f^1(\mathbb{Q}, T_p)]^{-1}[H_f^3(\mathbb{Q}, T_p)]^{-1}}{[T_p^+][\oplus_{v\in\S} H_f^1(\mathbb{Q}_v, T_p)]^{-1}}\end{aligned}$$

Global f-cohomology:

- $H_f^0(\mathbb{Q}, T_p) = 0$
- $[H_f^1(\mathbb{Q}, T_p)] = [E(\mathbb{Q})_{\mathbb{Z}_p}] = [E(\mathbb{Q})_{tors}][(E(\mathbb{Q})/tors)_{\mathbb{Z}_p}]$

- $[H^2_f(\mathbb{Q},T_p)] = [\text{Ш}(E)][((E(\mathbb{Q})/tors)_{\mathbb{Z}_p})^*]$
- $[H^3_f(\mathbb{Q},T_p)] = [E(\mathbb{Q})_{tors}]$

Local f-cohomology:

- $v=\infty$: $[H^1_f(\mathbb{R},T_p)] = [\Phi_\infty]$
- $v\neq p$: $[H^1_f(\mathbb{Q}_v,T_p)] = [E(\mathbb{Q}_v)_{\mathbb{Z}_p}] = [\Phi_v][E^0(\mathbb{F}_v)]$
- $v=p$: $[H^1_f(\mathbb{Q}_v,T_p)] = [E(\mathbb{Q}_p)_{\mathbb{Z}_p}] = [\Phi_p][E^0(\mathbb{F}_p)][\widehat{E}(p\mathbb{Z}_p)] = \frac{[\phi_p][E^0(\mathbb{F}_p)]}{[p]}[D(T_p)/F^0D(T_p)]$

$$\begin{aligned}\frac{[H^*_c(\mathbb{Z}_S,T_p)]}{\left(\frac{[\text{Ш}(E)][(E(\mathbb{Q})_{\mathbb{Z}_p}/tors)^*]}{[E(\mathbb{Q})_{tors}]^2[E(\mathbb{Q})_{\mathbb{Z}_p}/tors]}\right)} &= [H^1_f(\mathbb{Q}_p,T_p)]\frac{[H^1_f(\mathbb{R},T_p)]\prod_{v\in S,v\neq p,\infty}[\Phi_v]\cdot[E^0(\mathbb{F}_v)]}{[H^0_f(\mathbb{R},T_p)]} \\ &= \frac{[D(T_p)/F^0D(T_p)]}{[T_p^+]}\frac{[\Phi_\infty]\prod_{v\in S,v\neq\infty}[\Phi_v]\cdot[E^0(\mathbb{F}_v)]}{[\mathbb{Z}/p\mathbb{Z}]}\end{aligned}$$

Note $[L^S(E,1)] = \left(\prod_{v\in S,v\neq\infty}\frac{[E^0(\mathbb{F}_v)]}{[\mathbb{Z}/p\mathbb{Z}]}\right)^{-1}$.

$$\begin{aligned}\frac{[H^*_c(\mathbb{Z}_S,T_p)][L^S(E,1)]}{\left(\frac{[\text{Ш}(E)]\prod_{v\in S}[\Phi_v]}{[E(\mathbb{Q})_{tors})]^2}\right)} &= \frac{[(E(\mathbb{Q})/tors)^*_{\mathbb{Z}_p}]}{[(E(\mathbb{Q})/tors)_{\mathbb{Z}_p}]}\frac{[D(T_p)/F^0D(T_p)]}{[T_p^+]} \\ &\subset \frac{[(E(\mathbb{Q})/tors)^*_{\mathbb{Q}_p}]}{[(E(\mathbb{Q})/tors)_{\mathbb{Q}_p}]}\frac{[Lie(E)_{\mathbb{Q}_p}]}{[(H_1(E(\mathbb{C}),\mathbb{Z}))^+)_{\mathbb{Q}_p}]} \\ &= \Xi(M)_{\mathbb{Q}_p}\end{aligned}$$

Thus Bloch-Kato reduces to the claim that, for each p, the integral structure given by $[H^*_c(\mathbb{Z}_S,T_p)][L^S(E,1)]$ agrees with the integral structure given by $\frac{Reg(E)\Omega_{\mathbb{R}}}{L(E,1)}\cdot\alpha$. This is equivalent to

$$ord_p\left(\frac{|\text{Ш}(E)|\prod_{v\in S}|\Phi_v|}{|E(\mathbb{Q})_{tors}|^2}\right) = ord_p\left(\frac{L(E,1)}{Reg(E)\Omega_{\mathbb{R}}}\right)\forall p,$$

which implies the BSD conjecture.

Remark 8.5 *Some formulations of BSD do not use the component group at infinity Φ_∞, combining it into the period integral:*

$$\Omega_{\mathbb{R}}\cdot|\Phi_\infty| = \int_{E(\mathbb{R})}\omega$$

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