# EXAMPLES OF BEILINSON'S CONJECTURE

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Abstract. We explain explicit Beilinson's conjecture for $H^3_M(X_0(p)\times X_0(p),\mathbf{Q}(2))$ and $H^2_M(X_1(N),\mathbf{Q}(2))$.

## 1. Introduction

Recall that Akshay discussed the motivic cohomology group $H^3_M(S,\mathbf{Q}(2))$, where $S$ is a smooth projective surface over $\mathbf{Q}$, and the group $H^2_M(X,\mathbf{Q}(2))$, for a smooth projective curve $X$ over $\mathbf{Q}$. The Beilinson conjecture says that regulators of elements in these motivic cohomology group should be related to special values of certain $L$-functions, at least when the motivic classes are "integral". Our goal is to verify such a statement, ignoring the integrality, in some particular examples.

For the case of the surface $S$, we will take $X\times X$, where $X$ is a complete modular curve $X_0(p)$. In this setting, we will construct an explicit element in $H^3_M(S,\mathbf{Q}(2))$, and see that its regulator to Deligne cohomology is related to special values of Rankin-Selberg $L$-functions $L(f\times g,s)$ of two weight two newforms $f,g\in S_2(\Gamma_0(p))$. The special value in question is the one that is $\frac{1}{2}$ to the left (or right, by the functional equation) of the center point. The motivic cohomology class here that we compute with is often called a "Beilinson-Flach" element.

In the second case, $H^2_M(X,\mathbf{Q}(2))$, we will explain the existence of some elements in this motivic cohomology group, and again see that their regulator to Deligne cohomology sees the special values of certain $L$-functions. In this case, the $L$-value is $L(f,s=2)$ for a newform $f$ in $S_2(\Gamma_1(N))$. This is the special value that is 1 to the right of the center point.

Here are some references that treat $H^3_M(X\times X,\mathbf{Q}(2))$: [2], [1], [6]. Here are some references that treat $H^2_M(X,\mathbf{Q}(2))$: [2], [3], [7], [5], [14]. We begin with the discussion of $H^3_M(S,\mathbf{Q}(2))$, and then consider the case of $H^2_M(X,\mathbf{Q}(2))$.

## 2. Construction of class in $H^3_M(S,\mathbf{Q}(2))$

Suppose $S$ is a smooth surface. Recall that elements in $H^3_M(S,\mathbf{Q}(2))$ are given by finite sums of the form $\sum_i(C_i,f_i)$, where $C_i$ is a curve on $S$, $f_i$ is a rational function on $C_i$, and the sum of the divisors of the $f_i$ is 0 on $S$, $\sum_i div(f_i)=0$.

We will take $S=X_0(N)\times X_0(N)$, where $N=p$ is prime, and construct an explicit element in $H^3_M(S,\mathbf{Q}(2))$. The idea is simple: $X_0(p)$ has two cusps (if $N$ is sqaurefree, the number of cusps of $X_0(N)$ is the number factors of $N$), labeled 0 and $\infty$. The divisor $\infty-0$ is degree 0, and supported on the cusps. Thus by the Manin-Drinfeld theorem, there is modular unit with divisor $r(\infty-0)$, some $r\in\mathbf{Z}$.

**Proposition 1** (Manin-Drinfeld)**.** [11] *Suppose $X$ is a modular curve, and $D$ is a degree 0 divisor supported on the cusps of $X$. Then $D$ is torsion in the Jacobian of $X$.*

We will discuss the Manin-Drinfeld theorem in more detail below. For now, we can be very explicit: Simply consider $u_p(z):=\Delta(pz)/\Delta(z)$. From the matrix equation

$$\begin{pmatrix} N & \\ & 1\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} N & \\ & 1\end{pmatrix}^{-1}=\begin{pmatrix} a & Nb\\ N^{-1}c & d\end{pmatrix}$$

it follows that $\Delta(Nz)$ is a modular form for $\Gamma_0(N)$. Thus since $\Delta(z) = q + O(q^2)$ at the cusp $\infty$, $u_p(z)$ is a unit on $Y_0(p)$ with divisor $(p-1)(\infty - 0)$.

Now consider $S$ again. Then we set $c \in H^3_M(S, \mathbf{Q}(2))$ the class represented by

$$(X_0(p)^\Delta, u_p) + (\{\infty\} \times X_0(p), u_p^{-1}) + (X_0(p) \times \{0\}, u_p^{-1}).$$

The condition $\sum_j div(f_j) = 0$ is indeed satisfied.

## 3. Definition of regulator on $H^3_M(S, \mathbf{Q}(2))$

Suppose $S$ surface over $\mathbf{Q}$. There is a regulator map

$$r_D : H^3_M(S, \mathbf{Q}(2)) \to H^3_D(S(\mathbf{C}), \mathbf{R}(2))$$

There is a pairing

$$H^{1,1}(S(C), \mathbf{C}) \otimes H^3_D(S(\mathbf{C}), \mathbf{R}(2)) \to \mathbf{C}. \tag{1}$$

To explicate the pairing, we give explicit understanding of $H^3_D(S, \mathbf{R}(2))$. One has

$$\frac{H^2_B(S(\mathbf{C}), \mathbf{R}(1))}{pr_1(Fil^2 H^2_{dR}(S/\mathbf{C}))} \simeq \frac{H^2_B(S(\mathbf{C}), \mathbf{C})}{Fil^2 H^2_{dR}(S/\mathbf{C}) + H^2_B(S(\mathbf{C}), \mathbf{R}(2))} \simeq H^3_D(S, \mathbf{R}(2)).$$

Here $pr_1$ is the map

$$pr_1 : H^2_{dR}(S/\mathbf{C}) \to H^2_B(S(\mathbf{C}), \mathbf{C} = \mathbf{R}(1) \oplus \mathbf{R}(2)) \to H^2_B(S(\mathbf{C}), \mathbf{R}(1)).$$

Let $c$ denote the complex conjugation on the coefficients of $H^2_B(S(\mathbf{C}), \mathbf{C})$. Then $c$ conjugates the Hodge filtration, and $H^2_B(S(\mathbf{C}), \mathbf{R}(1))$ is the subspace of $H^2_B(S(\mathbf{C}), \mathbf{C})$ where $c$ acts by $(-1)$. It follows that the natural map

$$H^2_B(S(\mathbf{C}), \mathbf{R}(1)) \cap H^{1,1}(S(\mathbf{C})) \to \frac{H^2_B(S(\mathbf{C}), \mathbf{R}(1))}{pr_1(Fil^2 H^2_{dR}(S/\mathbf{C}))}$$

is an isomorphism, and thus we have a natural inclusion $H^3_D(S(\mathbf{C}), \mathbf{R}(2)) \hookrightarrow H^{1,1}(S(\mathbf{C}))$, hence the pairing (1).

The following proposition says what this pairing is on the image of a class $c$ in $H^3_M(S, \mathbf{Q}(2))$.

**Proposition 2.** *Suppose $\sum_j (Y_j, f_j)$ represents a class $c$ in $H^3_M(S, \mathbf{Q}(2))$, and $\omega \in H^{1,1}(S(\mathbf{C}), \mathbf{C})$. Then*

$$\langle \omega, r_D(c) \rangle = \frac{1}{2\pi i} \sum_j \int_{Y_j(\mathbf{C})} \log |f_j| \omega$$

*where $r_D(c)$ is the regulator of $c$ in Deligne cohomology $H^3_D(S(\mathbf{C}), \mathbf{R}(2))$, and the pairing on the left is the pairing described above.*

*Proof.* See Beilinson [4, section 6, pg. 61]. □

## 4. Classical regulator computation

Suppose $f, g$ are two newforms for $S_2(\Gamma_0(p))$. We will later assume that the Petersson inner product of $f$ and $g$ is 0.

*Remark* 3. According to the *L-functions and modular forms database*, $S_2(\Gamma_0(p))$ has at least 2 Galois orbits of newforms for $p = 37$ and $p = 43$.

Set $\omega_f = f(q)\frac{dq}{q} = 2\pi i f(z)dz$ and similarly for $g$. Then $\omega_f(z_1) \wedge \overline{\omega_g(z_2)} =: \omega_{f,g}$ is a $(1,1)$ form on $S = X_0(p) \times X_0(p)$. Our goal is to compute $\langle \omega_{f,g}, r_D(c) \rangle$. As a first step, note that we only need to consider the "Main term" $(X_0(p)^\Delta, u_p)$ from $c$, and not the other terms, which are supported on the boundary of $S$. We state this in the following lemma.

**Lemma 4.** *The "boundary terms"* $(\{\infty\} \times X_0(p), u_p^{-1}) + (X_0(p) \times \{0\}, u_p^{-1})$ *do not contribute to the regulator* $\langle \omega_{f,g}, r_D(c)\rangle$. *More precisely,*

$$\langle \omega_{f,g}, r_D(c)\rangle = \frac{1}{2\pi i}\int_{X_0(p)} \log\left|\frac{\Delta(pz)}{\Delta(z)}\right| \omega_f \wedge \overline{\omega_g}. \tag{2}$$

*Proof.* The other terms from the definition of $c$ vanish, since $\omega_f$ and $\omega_g$ vanish along the boundary, since $f$ and $g$ are cusp forms. □

According to the Beilinson conjecture, the above regulator should have something to do with an $L$-value. $L$-values are, by definition, the values at $s = s_0$ of $L$-functions, which depend on a complex parameter $s$. Thus to prove that (2) has something to do with an $L$-value, it would be great if we could put this integral in a family of integrals $I(f, g, s)$ that depend on $s \in \mathbf{C}$. We do this now. The key step is the Kronecker limit formula, which identifies a special value of an Eisenstein series with a modular unit.

Recall

$$\eta(z) = e^{\pi i z/12} \prod_{m \geq 1} \left(1 - e^{2\pi i m z}\right).$$

Thus $\eta(z)^{24} = \Delta(z)$. Define

$$E(\tau, s) = {\sum_{m,n}}' \frac{y^s}{|m\tau + n|^{2s}}.$$

The sum is over all pairs of integers $(m, n) \neq (0, 0)$.

**Theorem 5** (Kronecker's First Limit formula)**.** *At* $s = 1$, *one has the Taylor expansion*

$$E(\tau, s) = \frac{\pi}{s-1} + 2\pi(\gamma - \log 2 - \log(y^{1/2}|\eta(z)|^2)) + O(s-1),$$

*where* $\gamma$ *is the Euler-Macheroni constant.*

*Proof.* One can prove this just with explicit computation of Fourier series, with the help of Poisson summation. See Lang [13, pg. 273]. □

Set $E^*(\tau, s) = \pi^{-s}\Gamma(s)E(\tau, s) = \Gamma_{\mathbf{R}}(2s)E(\tau, s)$, where $\Gamma_{\mathbf{R}}(s) = \pi^{-s}\Gamma(s/2)$. Then

$$E^*(\tau, s) = \Gamma_{\mathbf{R}}(2s)\zeta(2s) \sum_{\gamma \in B(\mathbf{Z})\backslash \mathrm{SL}_2(\mathbf{Z})} Im(\gamma z)^s.$$

This *normalized* Eisenstein series satisfies the exact functional equation $E^*(\tau, s) = E^*(\tau, 1-s)$. Via this functional equation, we rewrite the Kronecker limit formula in terms of the expansion at $s = 0$. We have

$$\begin{aligned}
E^*(\tau, s) &= \pi^{-s}\Gamma(s)\left(\frac{\pi}{s-1} + 2\pi(\gamma - \log 2 - \log(y^{1/2}|\eta(z)|^2)) + O(s-1)\right)\\
&= \left(\pi^{-1} - \pi^{-1}\log(\pi)(s-1) + O(s-1)^2\right)\left(1 - \gamma(s-1) + O(s-1)^2\right)\\
&\quad \times \left(\frac{\pi}{s-1} + 2\pi(\gamma - \log 2 - \log(y^{1/2}|\eta(z)|^2)) + O(s-1)\right)\\
&= \frac{1}{s-1} + \gamma - \log(4\pi) - log(y|\eta(z)|^4) + O(s-1)
\end{aligned}$$

Hence we obtain

**Theorem 6** (Kronecker's limit formula, again)**.** *At* $s = 0$, *one has the Taylor expansion*

$$E^*(\tau, s) = -\frac{1}{s} + \gamma - \log(4\pi) - log(y|\eta(z)|^4) + O(s).$$

Since $\Delta(z) = \eta(z)^{24}$, using the KLF we can relate $\log|\Delta(pz)/\Delta(z)|$ to

$$E^{p,*}(z,s) := p^{-s}E^*(pz,s) - p^{-2s}E^*(z,s).$$

We have

$$\begin{aligned}
E^{p,*}(z,s) &= \left(1 - \log(p)s + O(s^2)\right)\left(-\frac{1}{s} + \gamma - \log(4\pi) - \log(py|\eta(pz)|^4) + O(s)\right) \\
&\quad - \left(1 - 2\log(p)s + O(s^2)\right)\left(-\frac{1}{s} + \gamma - \log(4\pi) - \log(y|\eta|^4) + O(s)\right) \\
&= \log\left(\left|\frac{\eta(pz)}{\eta(z)}\right|^4\right) + 2\log(p) + O(s) \\
&= \frac{1}{6}\log\left(\left|\frac{\Delta(pz)}{\Delta(z)}\right|\right) + 2\log(p) + O(s) \qquad (3)
\end{aligned}$$

Set

$$I(f,g,s) = \int_{\Gamma_0(p)\backslash\mathcal{H}} E^{p,*}(z,s)(y^2 f(z)\overline{g(z)})\frac{dx\wedge dy}{y^2}.$$

We normalize the Peterson inner product of $f$ and $g$ as

$$\langle f,g\rangle = \frac{1}{2\pi i}\int_{X_0(p)} \omega_f \wedge \overline{\omega_g}.$$

We have proved the following proposition.

**Proposition 7.** *Suppose $f,g \in S_2(\Gamma_0(p))$, $\omega_f = 2\pi i f(z)dz$, $\omega_g = 2\pi i g(z)dz$, and $\omega_{f,g} = \omega_f(z_1)\wedge\overline{\omega_g(z_2)}$ a $(1,1)$ form on $X_0(p)\times X_0(p)$. Then for the class $c \in H^3_M(X_0(p)\times X_0(p), \mathbf{Q}(2))$ and the regulator defined above, one has*

$$\frac{1}{6}\langle \omega_{f,g}, r_D(c)\rangle = (-4\pi)I(f,g,s)|_{s=0} - 2\log(p)\langle f,g\rangle.$$

## 5. The Rankin-Selberg integral

The integral $I(f,g,s)$ is the classical Rankin-Selberg integral. We calculate it now.

**5.1. The Eisenstein series.** First, we rewrite the Eisenstein series. Recall the normalized Eisenstein series $E^*(z,s) = \pi^{-s}\Gamma(s)\sum_{(m,n)\neq 0}\frac{y^s}{|mz+n|^{2s}}$. Thus

$$\begin{aligned}
E^{p,*}(z,s) &= p^{-s}E^*(pz,s) - p^{-2s}E^*(z,s) \\
&= \Gamma_{\mathbf{R}}(2s)p^{-s}\left(\sum_{(m,n)\neq 0}\frac{(py)^s}{|mpz+n|^{2s}}\right) - \Gamma_{\mathbf{R}}(2s)\left(\sum_{(m,n)\neq 0}\frac{y^s}{|mpz+pn|^{2s}}\right) \\
&= \Gamma_{\mathbf{R}}(2s)\left(\sum_{(m,n),p\nmid n}\frac{y^s}{|(pm)z+n|^{2s}}\right) \\
&= \Gamma_{\mathbf{R}}(2s)\zeta^{(p)}(2s)\sum_{(m,n),gcd(m,n)=1,p|m,p\nmid n}\frac{y^s}{|mz+n|^{2s}} \\
&= \Gamma_{\mathbf{R}}(2s)\zeta^{(p)}(2s)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(p)} Im(\gamma z)^s
\end{aligned}$$

where $\Gamma_\infty = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\cap \mathrm{SL}_2(\mathbf{Z})$ and $\zeta^{(p)}(s) = (1-p^{-s})\zeta(s)$ is the zeta function with the Euler factor at $p$ removed. Also $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$.

5.2. **Unfolding.** We now "unfold" the Rankin-Selberg integral $I(f,g,s)$. Write $f(z)=\sum_{n\ge1}a_ne^{2\pi inz}$ for the Fourier expansion of $f$ and $g(z)=\sum_{n\ge1}b_ne^{2\pi inz}$ for the Fourier expansion of $g(z)$. Set $d\mu=\frac{dx\wedge dy}{y^2}$. Then

**Proposition 8.** *The integral $I(f,g,s)$ unfolds as*

$$I(f,g,s)=\frac{1}{8}\Gamma_{\mathbf{C}}(s)\Gamma_{\mathbf{C}}(s+1)\zeta^{(p)}(2s)\left(\sum_{n\ge1}\frac{a_n\overline{b_n}}{n^{s+1}}\right)$$

*where $\Gamma_{\mathbf{C}}(s)=2(2\pi)^{-s}\Gamma(s)$.*

*Proof.* We compute

$$\begin{aligned}
I(f,g,s)&=\int_{\Gamma_0(p)\backslash\mathcal{H}}(y^2f(z)\overline{g(z)})E^{p,*}(z,s)\,d\mu\\
&=\Gamma_{\mathbf{R}}(2s)\zeta^{(p)}(2s)\int_{\Gamma_\infty\backslash\mathcal{H}}(y^2f(z)\overline{g(z)})Im(z)^s\,d\mu\\
&=\Gamma_{\mathbf{R}}(2s)\zeta^{(p)}(2s)\int_y y^{s+1}\left(\sum_{n\ge1}a_n\overline{b_n}e^{-4\pi ny}\right)\frac{dy}{y}\\
&=\Gamma_{\mathbf{R}}(2s)\zeta^{(p)}(2s)\left(\sum_{n\ge1}a_n\overline{b_n}\int_0^\infty y^{s+1}e^{-4\pi ny}\frac{dy}{y}\right)\\
&=\Gamma_{\mathbf{R}}(2s)\zeta^{(p)}(2s)(4\pi)^{-(s+1)}\Gamma(s+1)\left(\sum_{n\ge1}\frac{a_n\overline{b_n}}{n^{s+1}}\right)\\
&=\frac{1}{8}\Gamma_{\mathbf{C}}(s)\Gamma_{\mathbf{C}}(s+1)\zeta^{(p)}(2s)\left(\sum_{n\ge1}\frac{a_n\overline{b_n}}{n^{s+1}}\right).
\end{aligned}$$

This completes the proof of the proposition. □

5.3. **Euler product.** From now on we assume $f,g\in S_2(\Gamma_0(p))$ are newforms. Note that since the Hecke operators are self-adjoint, the $b_n$ are real, and thus $b_n=\overline{b_n}$, so we drop the complex conjugate from now on. Set $L_\infty(f\times g,s)=\Gamma_{\mathbf{C}}(s)\Gamma_{\mathbf{C}}(s+1)$. For $\ell\ne p$, set

$$I_\ell(f\times g,s)=(1-\ell^{-2s})^{-1}\sum_{n\ge0}a_{\ell^n}b_{\ell^n}\ell^{-n(s+1)}.$$

Also set

$$I_p(f\times g,s)=\sum_{n\ge0}a_{p^n}b_{p^n}p^{-n(s+1)}.$$

Since the $a_n,b_n$ are eigenvalues of the Hecke operators, and these eigenvalues are weakly multiplicative, we have proved the following Euler product for $I(f,g,s)$.

**Proposition 9.** *For $I(f,g,s)$ one has the Euler product*

$$I(f,g,s)=\frac{1}{8}L_\infty(f\times g,s)\prod_v I_v(f,g,s).$$

5.4. **Local integrals $I_v(f,g,s)$.** We still must relate the local factors $I_v(f,g,s)$ to (local) $L$-functions. To do this, define $\alpha_1(\ell), \alpha_2(\ell)$ via the equality

$$(1-\alpha_1(\ell)X)(1-\alpha_2(\ell)X) = 1 - a_\ell \ell^{-1/2} X + X^2$$

and similarly $\beta_1(\ell), \beta_2(\ell)$ via the equality

$$(1-\beta_1(\ell)X)(1-\beta_2(\ell)X) = 1 - b_\ell \ell^{-1/2} X + X^2.$$

The standard Hecke identity is

$$\frac{1}{1-a_\ell X + \ell X^2} = \sum_{n\geq 0} a_{\ell^n} X^n,$$

and thus

$$\frac{1}{(1-\alpha_1(\ell)X)(1-\alpha_2(\ell)X)} = \frac{1}{1-a_\ell \ell^{-1/2}X + X^2} = \sum_{n\geq 0} a_{\ell^n} \ell^{-n/2} X^n.$$

Now, we have

$$I_\ell(f,g,s) = (1-\ell^{-2s})^{-1} \sum_{n\geq 0} \left(\frac{a_{\ell^n}}{\ell^{n/2}}\right)\left(\frac{b_{\ell^n}}{\ell^{n/2}}\right) \ell^{-ns}.$$

The following is the so-called *Cauchy identity* for $\mathrm{GL}_2$.

**Lemma 10.** *If*

$$\sum_{r\geq 0} A(r)x^r = (1-\alpha_1 x)^{-1}(1-\alpha_2 x)^{-1}$$

*and*

$$\sum_{r\geq 0} B(r)x^r = (1-\beta_1 x)^{-1}(1-\beta_2 x)^{-1}$$

*then*

$$(1-\alpha_1\alpha_2\beta_1\beta_2 x^2)^{-1} \sum_{r\geq 0} A(r)B(r)x^r = \prod_{i,j=1,2} (1-\alpha_i\beta_j x)^{-1}. \tag{4}$$

If we set

$$L_\ell(f\times g, s) = \prod_{i,j=1,2} (1-\alpha_i\beta_j \ell^{-s})^{-1},$$

it follows immediately from the Cauchy identity that $I_\ell(f,g,s) = L_\ell(f\times g,s)$.

*Proof.* For a simple direct proof of the Cauchy identity, see [8, Lemma 1.6.1]. Let me explain a different proof. From the definition of $A(r)$, expanding power series, one has

$$A(r) = \alpha_1^r + \alpha_1^{r-1}\alpha_2 + \cdots \alpha_1^{r-1}\alpha_2 + \alpha_2^r.$$

Set $\alpha = \left(\begin{smallmatrix}\alpha_1 & \\ & \alpha_2\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{C})$. Then the above equality shows that $A(r) = \mathrm{tr}(\alpha|Sym^r(V_2))$, where $V_2$ is the standard two-dimensional representation of $\mathrm{GL}_2(\mathbf{C})$. Similarly, $B(r) = \mathrm{tr}(\beta|Sym^r(V_2))$, where $\beta = \begin{pmatrix}\beta_1 & \\ & \beta_2\end{pmatrix} \in \mathrm{GL}_2(\mathbf{C})$.

Going further, for integers $\lambda_1 \geq \lambda_2$, set $V_{\lambda_1,\lambda_2} = Sym^{\lambda_1-\lambda_2}(V_2)\otimes \det(V_2)^{\lambda_2}$, a representation of $\mathrm{GL}_2(\mathbf{C})$. This is the representation parametrized by the partition $(\lambda_1,\lambda_2)$. Multiplying out the power series on the left-hand side of (4), one sees that this left-hand side is

$$\sum_{\lambda_1\geq\lambda_2\geq 0} \mathrm{tr}(\alpha|V_{\lambda_1,\lambda_2})\,\mathrm{tr}(\beta|V_{\lambda_1,\lambda_2}) x^{\lambda_1+\lambda_2}. \tag{5}$$

But now, Schur-Weyl duality implies that

$$\sum_{\lambda_1 \geq \lambda_2 \geq 0} V_{\lambda_1,\lambda_2} \boxtimes V_{\lambda_1,\lambda_2} x^{\lambda_1+\lambda_2} = \sum_{N \geq 0} Sym^N(V_2 \boxtimes V_2) x^N \tag{6}$$

where the symbol $\boxtimes$ means external tensor product. I.e., $V_\lambda \boxtimes V_\mu$ means the representation of the group $\mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C})$ where the first $\mathrm{GL}_2(\mathbf{C})$ factor acts on $V_\lambda$ and the second acts on $V_\mu$.

Taking the trace of $(\alpha, \beta) \in \mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C})$ on the left-hand side of (6) gives (5), while taking the trace of $(\alpha, \beta)$ on the right-hand side of (6) gives the right-hand side of (4). This completes the proof. □

Finally, we evaluate (the easier) sum $I_p(f, g, s)$. Recall that since $f$ and $g$ are new at $p$, it follows that $a_{p^n} = a_p^n$ and similarly $b_{p^n} = b_p^n$. Hence

$$\begin{aligned} I_p(f, g, s) &= \sum_{n \geq 0} a_{p^n} b_{p^n} p^{-n} p^{-ns} \\ &= \sum_{n \geq 0} \alpha_p^n \beta_p^n p^{-ns} \\ &= (1 - \alpha_p \beta_p p^{-s})^{-1} \end{aligned}$$

where $\alpha_p = a_p p^{-1/2}$ and $\beta_p = b_p p^{-1/2}$. Set

$$L'_p(f \times g, s) = (1 - \alpha_p \beta_p p^{-s})^{-1}.$$

Hence we have proved

**Theorem 11.** *The Rankin-Selberg convolution $I(f, g, s)$ represents the $L$-function $L(f \times g, s)$, i.e.,*

$$\begin{aligned} I(f, g, s) &= \frac{1}{8} L_\infty(f \times g, s) L'_p(f \times g, s) \prod_{\ell \neq p} L_\ell(f \times g, s) \\ &=: \frac{1}{8} L_\infty(f \times g, s) \tilde{L}^p(f \times g, s), \end{aligned}$$

*where $\tilde{L}^p(f \times g, s) = L'_p(f \times g, s) \prod_{\ell \neq p} L_\ell(f \times g, s)$.*

*Remark* 12. Our local $L$-factor at $p$ $L'_p(f \times g, s)$ is not the correct local factor to define the $L$-function. One needs to multiply by an extra term.

Putting everything together, we will obtain an expression relating the regulator of our motivic cohomology class $reg_D(c)$ to some $L$-value for the $L$-function $L(f \times g, s)$. It is instructive to center this relationship around the Taylor expansion of $L(f \times g, s)$ at $s = 0$. To do this, first note that the expression (3) shows that the Eisenstein series $E^{p,*}(z, s)$ has no pole at $s = 0$. Thus, the integral $I(f, g, s)$ has no pole at $s = 0$, and hence $\Gamma_{\mathbf{C}}(s)\Gamma_{\mathbf{C}}(s+1)\tilde{L}^p(f \times g, s)$ is regular at $s = 0$. But since $\Gamma_{\mathbf{C}}(s)$ has a simple pole at $s = 0$, we deduce that $\tilde{L}^p(f \times g, s)$ has a zero at $s = 0$: $\tilde{L}^p(f \times g, s) = L(f \times g)^* s + O(s^2)$ for some number $L(f \times g)^*$ in $\mathbf{C}$. (We haven't shown $L(f \times g)^*$ is nonzero, but one can do this using the functional equation.)

Taking care of the $2\pi$ factors and such, one gets $I(f, g, s) = \frac{1}{4\pi} L(f \times g)^* + O(s)$. Thus we have proved the following theorem.

**Theorem 13.** *Suppose $f, g \in S_2(\Gamma_0(p))$ are new-forms. Let $c$ be the class in motivic cohomology $H^3_M(X_0(p) \times X_0(p), \mathbf{Q}(2))$ constructed above, $r_D(c)$ its regulator to Deligne cohomology $H^3_D((X_0(p) \times X_0(p))(\mathbf{C}), \mathbf{R}(2))$, and*

$$\omega_{f,g} = (2\pi i f(z_1) dz_1) \wedge \overline{(2\pi i g(z_1) dz_1)},$$

*a* $(1,1)$ *form on* $X_0(p)\times X_0(p)$. *Define*

$$\tilde{L}^p(f\times g,s)=L_p'(f\times g,s)\prod_{\ell\neq p}L_\ell(f\times g,s).$$

*Then* $\tilde{L}(f\times g,s)$ *vanishes to at least order* $1$ $s=0$, *and one has the Taylor expansion*

$$\tilde{L}(f\times g,s)=-\left(\frac{1}{6}\langle\omega_{f,g},reg_D(c)\rangle+2\log(p)\langle f,g\rangle\right)s+O(s^2).$$

*Consequently, if the Peterrson inner product of* $f$ *and* $g$ *is* $0$, *then* $\tilde{L}(f\times g,s)=-\frac{1}{6}\langle\omega_{f,g},reg_D(c)\rangle s+O(s^2)$.

## 6. General Kronecker limit formula

In the work above, we constructed a motivic cohomology class in $H^3_M(X\times X,\mathbf{Q}(2))$, and saw that its regulator to Deligne cohomology was connected to the Taylor expansion of a certain Rankin-Selberg $L$-functions $L(f\times g,s)$ $\frac{1}{2}$ to the left of the central point. We will now describe the existence of some elements in the motivic cohomology group $H^2_M(X_1(N),\mathbf{Q}(2))$, and see that their regulators are related to the values of $L$-functions $L(f,s)$ for a newform $f$ in $S_2(\Gamma_1(N))$. The special value here is the one that is 1 to the right of the center point.

For our discussion of $H^3_M(X_0(p)\times X_0(p),\mathbf{Q}(2))$, the key facts that enabled us to construct motivic cohomology classes and compute their regulator were the existence of modular units, and the Kronecker limit formula, which related these units to special values of Eisenstein series. We begin by describing a general formulation of this result.

6.1. **Manin-Drinfeld.** We first recall the Manin-Drinfeld theorem. For a subfield $k$ of $\mathbf{C}$, set $C_N$ the cusps of $X_1(N)$, and $\mathrm{Div}^0_k[C_N]$ the degree 0 divisors on $C_N$ that are $Aut(\mathbf{C}/k)$-invariant.

**Theorem 14** (Manin-Drinfeld)**.** *The degree* $0$ *divisors on the cusps of* $X_1(N)$ *are torsion in* $Jac(X_1(N))$ *over* $\mathbf{C}$. *More precisely, suppose* $k$ *is a subfield of* $\mathbf{C}$, *algebraic over* $\mathbf{Q}$. *Then the sequence of* $\mathbf{Q}$*-modules*

$$0\to k^\times\otimes_{\mathbf{Z}}\mathbf{Q}\to\mathcal{O}(Y_1(N)_k)^\times\otimes_{\mathbf{Z}}\mathbf{Q}\to\mathrm{Div}^0_k[C_N]\otimes_{\mathbf{Z}}\mathbf{Q}\to 0,$$

a priori *exact on the left, is exact on the right as well.*

*Proof.* One proves the result over $\mathbf{C}$ by using the action of Hecke operators on the cusps and on $\Omega^1(X_1(N))$. For this, see the 2 page paper of Drinfeld, [11]. One can then descend the result over $\mathbf{C}$ to get the result over $k$ using Hilbert Theorem 90. See [7, Lemma 5.7]. $Aut(\mathbf{C}/k)$ acts on the cusps through the cyclotomic character; see [14, 3.0.2] for this action. □

Last time, we gave ourselves an explicit degree 0 divisor $\infty-0$ supported on the cusps of $X_0(p)$, and, consistent with the Manin-Drinfeld theorem, found an element $u_p\in\mathcal{O}(Y_0(p))^\times$ whose divisor was a multiple of $\infty-0$. We then related $\log|u_p|$ to the special value of an Eisenstein series. The construction of $u_p$ and its relation to the Eisenstein series were the key components of our success in Part 1. We now explain how to generalize this relationship to arbitrary modular curves. The cost of this generalization is that we will prove less precise of a relationship between modular units and Eisenstein series.

6.2. **Cusps of a general modular curve.** Before stating this general Kronecker limit formula, we must explicitly describe the cusps of a general modular curve. Suppose $K$ is an open compact subgroup of $\mathrm{GL}_2(\mathbf{A}_f)$. Then the complex points of the modular curve $Y_K$ of level $K$ is

$$Y_K(\mathbf{C}) = \mathrm{GL}_2(\mathbf{Q})\backslash\mathcal{H}^{\pm} \times \mathrm{GL}_2(\mathbf{A}_f)/K.$$

The cusps have a similarly nice adelic description. The (complex points) of the cusps $C_K$ of $Y_K$ are

$$C_K(\mathbf{C}) = \mathrm{GL}_2(\mathbf{Q})\backslash\mathbf{P}^1(\mathbf{Q}) \times \mathrm{GL}_2(\mathbf{A}_f)/K. \tag{7}$$

(Maybe I need a $\pm$ copy of $\mathbf{P}^1(\mathbf{Q})$? I'm not sure...) First note $\mathrm{GL}_2(\mathbf{Q})$ acts transitively on $\mathbf{P}^1(\mathbf{Q})$ with stabilizer of a point a Borel subgroup $B(\mathbf{Q})$, so the double coset (7) is in bijection with $B(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A}_f)/K$. (That this latter double coset parametrizes the cusps is explained in [12, Proof of Proposition 3.1].) Before moving on, let us see that the description (7) of the cusps agrees with what we expect.

**Lemma 15.** *Suppose that the inclusion* $\det(K) \hookrightarrow \mathrm{GL}_1(\widehat{\mathbf{Z}})$ *is an equality. (For example, this holds for the usual congruence subgroup* $K_1(n)$ *but not the "full" congruence subgroup* $K(n)$*.) Then the natural inclusion*

$$(\mathrm{GL}_2(\mathbf{Z}) \cap K)\backslash\mathbf{P}^1(\mathbf{Q}) \to \mathrm{GL}_2(\mathbf{Q})\backslash\mathbf{P}^1(\mathbf{Q}) \times \mathrm{GL}_2(\mathbf{A}_f)/K$$

*is a bijection.*

*Proof.* Using strong approximation for $\mathrm{SL}_2$, one gets $\mathrm{GL}_2(\mathbf{A}_f) = \mathrm{GL}_2(\mathbf{Q})\,\mathrm{GL}_2(\widehat{\mathbf{Z}})$. Thus the inclusion

$$\mathrm{GL}_2(\mathbf{Z})\backslash\mathbf{P}^1(\mathbf{Q}) \times \mathrm{GL}_2(\widehat{\mathbf{Z}})/K \to \mathrm{GL}_2(\mathbf{Q})\backslash\mathbf{P}^1(\mathbf{Q}) \times \mathrm{GL}_2(\mathbf{A}_f)/K$$

is a bijection. Using $\det(K) = \mathrm{GL}_1(\widehat{\mathbf{Z}})$ and that $\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/N)$ is surjective, one gets $\mathrm{GL}_2(\widehat{\mathbf{Z}}) = \mathrm{GL}_2(\mathbf{Z})K$. The lemma follows. □

Now suppose $\phi : C_K(\mathbf{C}) \to \mathbf{Q}$ has degree 0, and $u_\phi$ is a modular unit with $div(u_\phi) = \phi$, whose existence is guaranteed by the Manin-Drinfeld theorem. To what Eisenstein series is $u_\phi$ related? First, via the bijection of $C_K(\mathbf{C})$ with $B(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A}_f)/K$, consider $\phi$ as a function

$$\phi : B(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A}_f)/K \to \mathbf{Q}.$$

Now, for $g \in \mathrm{GL}_2(\mathbf{A})$ and $Re(s) >> 0$, set

$$E_\phi(g, s) = -2\pi \sum_{\gamma \in B(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{Q})} \phi(\gamma_f g_f)|Im(\gamma_\infty g_\infty \cdot i)|^s.$$

Here is a "general" Kronecker limit formula.

**Theorem 16.** *Suppose* $\phi$ *has degree* 0*, and* $Y_K$ *has a single connected component. (The connected components are parametrized by* $\mathrm{GL}_1(\widehat{\mathbf{Z}})/\det(K)$*.) The Eisenstein series* $E_\phi(g, s)$ *is regular at* $s = 1$ *(i.e., no pole), and one has* $E(g, 1) = C + \log|u_\phi|$ *for a constant* $C$*.*

*Proof.* This is just a slight elaboration on the proof in [14, Proposition 3.5.1]. The idea of the proof is simple: one shows that the difference $E(g, 1) - \log|u_\phi|$ is harmonic and bounded and thus constant. First, that the Eisenstein series is regular at $s = 1$ is a consequence of the fact that $\phi$ has degree 0. In general, the Eisenstein series has a simple pole at $s = 1$ with residue a constant function of $g_\infty$, and this constant is proportional to $deg(\phi)$.

That $E_\phi(g, 1)$ is harmonic can be checked as follows. Set $\Delta = -y^2(\frac{\partial}{\partial y^2} + \frac{\partial}{\partial x^2})$, the hyperbolic Laplacian. One has $\Delta y^s = s(1-s)y^s$. Since $\Delta$ is $\mathrm{SL}_2(\mathbf{R})$ invariant, i.e. $\gamma^*\Delta(f) = \Delta(\gamma^* f)$ for $\gamma \in \mathrm{SL}_2(\mathbf{R})$, one has $\Delta E_\phi(g, s) = s(1-s)E_\phi(g, s)$ for $Re(s) > 1$, where the sum defining $E_\phi(g, s)$ converges absolutely. Since $E_\phi(g, s)$ has analytic continuation to a neighborhood of $s = 1$, we have $\Delta E_\phi(g, 1) = 0$, and thus $E_\phi(g, 1)$ is harmonic. See [8, pg. 104] for facts about $\Delta$.

To see that the difference $E_\phi(g,1)-\log|u_\phi|$ is bounded amounts to the fact that the two functions have the same asymptotics at the cusps. For $E_\phi(g,1)$, one has the asympotic $E_\phi(g,1)\sim -2\pi\phi(g_f)y$, $y=Im(g_\infty\cdot i)$, as $y\to\infty$. (One computes the constant term of the Eisenstein series along the unipotent radical of the upper triangular Borel, and the difference between the Eisenstein series and its constant term is of rapid decay.) Since $\log|q^r|=-2\pi ry$, we get the same asymptotic for $u_\phi$, proving that the difference is bounded. □

## 7. The motivic cohomology group $H^2_M(X,\mathbf{Q}(2))$

Before moving on to our specific case of the curve $X_1(N)$, let us discuss $H^2_M(X,\mathbf{Q}(2))$ for general smooth curves $X$ over a field $k$.

7.1. $K_2$ **of a curve.** Recall that this motivic cohomology group $H^2_M(X,\mathbf{Q}(2))$ is identified with $K_2(X)$, and $K_2(X)$ can be defined as the kernel, inside $K_2(k(X))$, of various tame symbols. More precisely,

$$K_2(k(X))=\frac{k(X)^\times\otimes k(X)^\times}{\{f\otimes(1-f):f\neq 0,1\in k(X)\}},$$

and if $u,v\in k(X)^\times$, and $p$ a point of $X$, then the tame symbol $\{u,v\}_p$ of $u$ and $v$ at $p$ is

$$\{u,v\}_p=(-1)^{ord_p(u)ord_p(v)}\left(\frac{u^{ord_p(v)}}{v^{ord_p(u)}}\right)(p)\in k(p)^\times.$$

With this definition, $K_2(X)$ is defined to be the kernel of the tame symbol maps $\{\cdot,\cdot\}_p$ for all points $p$ on $X$.

When there is a finite subset $C$ of $X$ such that $\mathbf{Z}[C]^0$, the degree 0 divisors supported on $C$, are torsion in the Jacobian of $X$, it is easy to construct elements of $K_2(X)$. The following lemma explains this fact.

**Lemma 17** ([9], Lemma 5.2)**.** *Suppose $X/k$ is a curve, and $C$ is a finite subset of $X$, all of whose points are defined over the field $k$. Set $U=X\setminus C$, and suppose moreover that $\mathbf{Z}[C]^0$ is finite order in the Jacobian of $X$. Then $H^2_M(U,\mathbf{Q}(2))=H^2_M(X,\mathbf{Q}(2))+\{\mathcal{O}(U)^\times,k^\times\}$.*

*Remark* 18. Note that the assumption that $\mathbf{Z}[C]^0$ is torsion in the Jacobian of $X$ means there is a plentiful supply of elements of $\mathcal{O}(U)^\times=H^1_M(U,\mathbf{Q}(1))$. Taking a cup product of two elements of $\mathcal{O}(U)^\times$ gives an element of $H^2_M(U,\mathbf{Q}(2))$. Thus it is easy to construct elements of $H^2_M(U,\mathbf{Q}(2))$ under the assumptions of the lemma.

*Proof of lemma.* The key point is that the tame symbol map on $\{\mathcal{O}(U)^\times,k^\times\}$ is essentially the divisor map, and thus one can kill off any "bad" tame symbols from the elements of $H^2_M(U,\mathbf{Q}(2))$ at points $c\in C$. See the proof [9, Lemma 5.2] for the details. □

7.2. **Regulator.** There is a regulator on $K_2(k(X))$, and thus on $K_2(X)$ by restriction. We now discuss this regulator, following [10, section 3].

The regulator is a pairing $\langle\cdot,\cdot\rangle: K_2(k(X))\otimes\Omega^1(X)\to\mathbf{C}$. Suppose $\omega\in\Omega^1(X)$, and $\{u,v\}\in K_2(k(X))$. Set $\eta(u,v)=\log|u|d(argv)-\log|v|d(argu)$. Then

$$\begin{aligned}\langle\{u,v\},\omega\rangle &:= \frac{1}{2\pi i}\int_{X(\mathbf{C})}\eta(u,v)\wedge\omega\\ &=\frac{1}{2\pi i}\int_{X(\mathbf{C})}\log|u|\overline{d\log(v)}\wedge\omega.\end{aligned}$$

That this is the "right" regulator, i.e., that it factors through Deligne cohomology, can be deduced from the formula for the cup product in Deligne cohomology. See [2, Assertion 4.2].

*Remark* 19. Note that regulators of elements of the form $\{u, \lambda\}$, with $\lambda \in k^\times$, are 0. Thus if we are in the situation of Lemma 17, to compute regulators on $H^2_M(X, \mathbf{Q}(2))$ it suffices to compute regulators on $H^2_M(U, \mathbf{Q}(2))$.

## 8. Regulator computation

In this section, we will define some elements in $H^2_M(X_1(N), \mathbf{Q}(2)) \otimes L$, for an abelian number field $L$, and compute their regulators. We begin with a discussion of some Eisenstein series.

8.1. **Eisenstein series.** Suppose $\chi : (\mathbf{Z}/N)^\times \to \mathbf{C}^\times$ is a non-trivial Dirichlet character, and $k \geq 0$ is an even integer. Set

$$E_{k,\chi}(z, s) := L(\chi^{-1}, 2s)\pi^{-(s+k/2)}\Gamma(s + k/2) \left( \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{\chi^{-1}(d)}{(cz + d)^k} \left( \frac{y}{|cz + d|^2} \right)^{s-k/2} \right),$$

where, as always, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Set $E^{hol}_{k,\chi}(z) := E_{k,\chi}(z, k/2)$. If $k \geq 2$, this Eisenstein is holomorphic.

Now, set $\delta_k = \frac{1}{2\pi i} \left( \frac{d}{dz} + \frac{k}{2iy} \right)$. This is called a Maass-Shimura differential operator. The Eisenstein series $E_{k,\chi}(z, s)$, for different $k$, are moved around by the operators $\delta_k$. That is,

**Lemma 20.** *One has the identity* $\delta_k E_{k,\chi}(z, s) = E_{k+2}(z, s)$.

*Proof.* Rather than do a stupid computation to prove this identity (and plus, I might be off by some $2\pi$'s or something), it would be better if I told you how to write down your Eisenstein series in the first place so that one gets such a nice relationship as in the statement of the lemma. I will do this below, assuming I have enough time. □

8.2. **Units on** $Y_1(N)$**.** Let our Dirichlet character $\chi$ be as above, and set $L = \mathbf{Q}(\chi)$ the field generated by the values of $\chi$. According to our general Kronecker limit formula, there should be a unit $u_\chi$ in $\mathcal{O}(Y_1(N)_{\overline{\mathbf{Q}}})^\times \otimes \overline{\mathbf{Q}}$ and a constant $C$ so that $E_{0,\chi}(z, 1) = C + \log |u_\chi|$. (Because $\chi$ is nontrivial, the function "$\phi$" of Theorem 16 will be degree 0.) The following lemma of Brunault says that $u_\chi$ can be chosen to that it is defined over $\mathbf{Q}$, the "coefficient field" is small, and $C = 0$.

**Lemma 21** (Brunault [7], Lemme 5.7)**.** *There is* $u_\chi \in \mathcal{O}(Y_1(N)_\mathbf{Q})^\times \otimes L$ *so that* $E_{0,\chi}(z, 1) = \log |u_\chi|$.

*Proof.* To see that $u_\chi$ is defined over $\mathbf{Q}$, one must analyze the action of $Aut(\mathbf{C}/\mathbf{Q})$ on the cusps, and ensure that the divisor of the $u_\chi$ determined by the Eisenstein series is defined over $\mathbf{Q}$. See [7, Lemme 5.7]. □

Suppose $\epsilon : (\mathbf{Z}/N)^\times \to \mathbf{C}^\times$ is another nontrivial Dirichlet character. Then we have an associated unit $u_\epsilon$, and thus $\{u_\chi, u_\epsilon\} \in H^2_M(Y_1(N), \mathbf{Q}(2)) \otimes L$, for $L$ the field generated by the values of $\epsilon$ and $\chi$. By Lemma 17 and Remark 19, we can content ourselves with computing the regulator of $\{u_\chi, u_\epsilon\}$. The following theorem says that the regulator is related to special values of $L$-functions of newforms $f \in S_2(\Gamma_1(N))$ at a point 1 to the right of the central point. This was proved in a less explicit way by Beilinson [2], and made explicit by Brunault [7].

**Theorem 22.** *Suppose* $\psi : (\mathbf{Z}/N)^\times \to \mathbf{C}^\times$ *is a Dirichlet character and* $f \in S_2(\Gamma_0(N), \psi)$ *is a newform with nebentype* $\psi$*. Set* $\omega_f = 2\pi i f(z)\, dz \in \Omega^1(X_1(N))$*, let* $\{u_\chi, u_\epsilon\}$ *be as above, and assume* $\chi\bar{\epsilon}\psi = 1$*. Then*

$$\langle \{u_\chi, u_\epsilon\}, \omega_f \rangle = L(f, \chi, 1)L(f, 2),$$

*where* $L(f, s)$ *is the* $L$*-function of* $f$ *normalized to have center point* $s = 1$*.*

*Remark* 23. It is the $L(f,2)$ part of this equality that carries the interesting transcendental information. By Brian's talk before, $L(f,\chi,1)$ is essentially algebraic (i.e., algebraic up to factors of $2\pi$ and stuff). Also, the equality in the theorem is probably off by factors of $2\pi$ and Gauss sums and the like, which I obviously couldn't be bothered to try to get correct.

*Proof of theorem.* The regulator is

$$\langle \{u_\chi, u_\epsilon\}, \omega_f \rangle = \int_{\Gamma_1(N)\backslash\mathcal{H}} \log|u_\chi(z)| \overline{d\log u_\epsilon(z)} f(z) dz.$$

Now $E_{0,\chi}(z,1) = \log|u_\chi|$, and

$$\begin{aligned} d\log u_\epsilon &= \frac{d}{dz}\log|u_\epsilon| dz \\ &= \frac{d}{dz} E_{0,\epsilon}(z,1) dz \\ &= \delta_0\left(E_{0,\epsilon}(z,1)\right) dz \\ &= E^{hol}_{2,\epsilon}(z) dz. \end{aligned}$$

Thus, to compute the regulator, one may compute

$$I(f,\chi,\epsilon,s) := \int_{\Gamma_0(N)\backslash\mathcal{H}} E_{0,\chi}(z,s) \overline{E^{hol}_{2,\epsilon}(z)} f(z)\, dx dy$$

and specialize to $s=1$. The integral $I(f,\chi,\epsilon,s)$ is just like the Rankin-Selberg integral we computed last time, except now we have replaced one of the cusp forms with the Eisenstein series $E^{hol}_{2,\epsilon}(z)$. It turns out that this makes little difference, and we can again compute $I(f,\chi,\epsilon,s)$ by the Rankin-Selberg method. One finds $I(f,\chi,\epsilon,s) = L(f,\chi,s)L(f,s+1)$ in the normalization that puts the center point at $s=1$. Specializing to $s=1$ gives the result. ☐

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