Beilinson's conjectures II

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1 Review of Deligne cohomology

Last time we defined *motivic cohomology* groups $H^p_{\mathcal{M}}(X, \mathbf{Q}(q))$. In a special case motivic cohomology is the Chow group:

$$H^p_{\mathcal{M}}(X, \mathbf{Q}(p)) = CH^p(X).$$

This fits into the story of Beilinson's conjecture via a *regulator map* to Deligne cohomology:

$$\text{reg}\colon H^p_{\mathcal{M}}(X, \mathbf{Q}(q)) \to H^p_{\mathcal{D}}(X(\mathbf{C}), \mathbf{Q}(q))$$

Definition 1.1. Let X be a smooth complex projective variety. We define $H^p_{\mathcal{D}}(X, \mathbf{Z}(q))$ to be " a class in $H^p(X, (2\pi i)^q\mathbf{Z})$ which lies in F^q", or more formally as the hypercohomology of a certain complex:

$$H^p_{\mathcal{D}}(X, \mathbf{Z}(q)) = \mathbf{H}^*(X, (2\pi i)^q\mathbf{Z} \to \mathcal{O} \to \Omega^1 \to \ldots \to \Omega^{q-1}).$$

We think of $H^*(\mathcal{O} \to \Omega^1 \to \ldots \to \Omega^{q-1})$ as corresponding to $H^*(X)/F^qH^*(X)$, which is the sense in which the class in $H^p(X, (2\pi i)^q\mathbf{Z})$ lies in F^q. However, more precisely we think of a Deligne cohomology class as *specifying* an antiderivative.

So the Deligne cohomology fits into a long exact sequence

$$\begin{aligned} &H^{p-1}_{\mathcal{D}}(X, \mathbf{Z}(q)) \to H^{p-1}(X, (2\pi i)^q\mathbf{Z}) \to H^{p-1}(X)/F^qH^{p-1} \\ \to &H^p_{\mathcal{D}}(X, \mathbf{Z}(q)) \to H^p(X, (2\pi i)^q\mathbf{Z}) \to H^p(X)/F^qH^p \to \ldots \end{aligned}$$

One of the important properties is that there is a "cycle class map" which recovers the usual one under the map $H^p_{\mathcal{D}}(X, \mathbf{Z}(q)) \to H^p(X, (2\pi i)^q\mathbf{Z})$, but even if this is trivial there is some additional "Abel-Jacobi" information. (We'll elaborate on this point later.)

*Notes taken by Tony Feng

2 Examples

We recall some of the examples from last time.

Example 2.1. For $q = 1$, the complex in question is quasi-isomorphic to $\mathcal{O}^*$, by the diagram

$$\begin{array}{ccc} 2\pi i\mathbf{Z} & \longrightarrow & \mathcal{O} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}^* \end{array}$$

induced by the exponential short exact sequence

$$0 \to 2\pi i\mathbf{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

so $H^1_{\mathcal{D}}(X, \mathbf{Z}(1)) \cong H^0(X, \mathcal{O}^*)$ and $H^2_{\mathcal{D}}(X, \mathbf{Z}(1)) = \mathrm{Pic}(X)$.

Example 2.2. By definition, $H^2_{\mathcal{D}}(X, \mathbf{Z}(2))$ is the cohomology of

$$(2\pi i)^2\mathbf{Z} \to \mathcal{O} \to \Omega^1$$

which is quasi-isomorphic to the complex $\mathcal{O}^* \to \Omega^1$ sending $f \mapsto df/f$:

$$\begin{array}{ccccc} (2\pi i)^2\mathbf{Z} & \longrightarrow & \mathcal{O} & \longrightarrow & \Omega^1 \\ & & \Big\downarrow{\scriptstyle \exp} & & \Big\downarrow \\ & & \mathcal{O}^* & \longrightarrow & \Omega^1 \\ & & f & \longmapsto & \frac{df}{f} \end{array}$$

A Cech representative for a class in $H^2_{\mathcal{D}}(X, \mathbf{Z}(1))$ with respect to a two-term cover $X = U \cup V$ consists of an element $f_{U\cap V} \in H^0(U \cap V, \mathcal{O}^*)$ together with a "certificate of triviality" in $H^1(\Omega^1)$, i.e. $\omega_U \in H^0(U, \Omega^1)$ and $\omega_V \in H^0(V, \Omega^1)$ such that

$$\omega_U - \omega_V = \frac{df_{U\cap V}}{f_{U\cap V}}.$$

This means that the connections $d+\omega_U$ and $d+\omega_V$ patch together to give a connection on the line bundle. So $H^2_{\mathcal{D}}(X, \mathbf{Z}(2))$ classifies line bundles *plus* a holomorphic connection.

3 Chern classes

For a L a line bundle over a curve X, we have a *Chern class* $c_1(L) \in H^2(X, 2\pi i\mathbf{Z})$. What is it?

There are many constructions, but one analytic one goes as follows. Choose a metric on L. Thinking locally at first, we can choose a holomorphic section $s \in$

$H^0(L)$. Then $\partial\overline{\partial}\log\langle s,s\rangle$ is a $(1,1)$-form on X, independent of a choice of section since replacing $s \mapsto hs$ for h a holomorphic function changes the form by

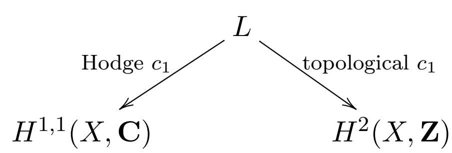
$$\begin{aligned}\partial\overline{\partial}\log\langle s,s\rangle &\mapsto \partial\overline{\partial}\log\langle hs,hs\rangle\\ &= \partial\overline{\partial}(||h||^2||s||^2)\\ &= \partial\overline{\partial}(h\cdot\overline{h}) + \partial\overline{\partial}||s||^2\\ &= \partial\overline{\partial}h + \partial\overline{\partial}\overline{h} + \partial\overline{\partial}||s||^2\\ &= 0 + 0 + \partial\overline{\partial}||s||^2.\end{aligned}$$

Thus our definition actually globalizes to a global $(1,1)$-form which represents a certain class in de Rham cohomology. This gives a definition of first Chern class which obviously lies in the first step of the Hodge filtration, although it's not obviously integral. We might call this the"Hodge" representative of $c_1(L)$.

Let's give another definition which is manifestly integral. Pick s a global holomorphic section, and take a small triangulation $\mathcal{T}$ of X fine enough so that each zero of s lies in the interior of a triangle, and each triangle contains at most one zero. Then $c_1(L)$ is the count

$$\sum_{T\in\mathcal{T}} \begin{cases} 2\pi i & s \text{ has a zero in } T \\ 0 & \text{otherwise.} \end{cases}$$

Let's call this the "topological" representative. This definition is manifestly integral, but it's not clear how it makes contact with the previous one.



The first Chern class of L in *Deligne cohomology* can be thought of as a "certificate" ofthe equivalence of these two constructions. In other words, to lift c_1 to a class in $H^2_{\mathcal{D}}(\mathbf{Z}(1))$ you should give an explicit homotopy between these constructions.

What is the homotopy? Well the $(1,1)$-form $\partial\overline{\partial}\log||s||^2$ also defines a function on the triangulation,

$$T \mapsto \int_T \partial\overline{\partial}\log||s||^2.$$

On the Hodge side, changing the metric changes the cohomology class in by an explicit coboundary. We change the metric so that $||s||^2$ is *constant* outside a small neighborhood of the zeros. Then the $(1,1)$-form vanishes away from a tiny neighborhood of the zeros. In particular, this argument shows that our two "Chern classes" agree.

Remark 3.1. This calculation is due to Chern, actually predating Chern-Weil theory, in a proof of Gauss-Bonnet:

$$\chi(M) = \int_M (\text{Euler form}).$$

Chern's argument was as follows: choosing a vector field on M, we can make an explicit antiderivative of the Euler form away from the zeros, which localizes the computation to small neighborhoods of the zeros in an analogous manner.

4 Properties of Deligne cohomology

Product structure. There is a product

$$H^p_{\mathcal{D}}(X, \mathbf{Z}(q)) \times H^{p'}_{\mathcal{D}}(X, \mathbf{Z}(q')) \to H^{p+p'}(X, \mathbf{Z}(q+q')).$$

Remark 4.1. The definition is a little tricky. It involves an asymmetric choice.

Cycle class map. For Z of codimension p, there is a *cycle class*

$$[Z] \in H^{2p}_{\mathcal{D}}(X, \mathbf{Z}(p)).$$

Let's ignore the definition. Under the map

$$H^{2p}_{\mathcal{D}}(X, \mathbf{Z}(p)) \to H^{2p}(X, (2\pi i)^p \mathbf{Z})$$

it maps to the usual cycle class. However there is more information, thanks to the part on the left of the long exact sequence

$$H^{2p-1}_{\mathcal{D}}(X, \mathbf{Z}(p)) \to H^{2p-1}(X, \mathbf{Z}(p))/F^p H^{2p-1} \to H^{2p}_{\mathcal{D}}(X, \mathbf{Z}(p)) \to H^{2p}(X, (2\pi i)^p \mathbf{Z})$$

For instance, if the usual cycle class of Z is zero, you get a class

$$[Z]_{\mathcal{D}} \in \frac{H^{2p-1}(X, \mathbf{C})}{F^p H^{2p-1} + H^{2p-1}(X, (2\pi i)^p \mathbf{Z})}$$

which is the Abel-Jacobi map to the intermediate Jacobian.

Example 4.2. If X is a curve and $p = 1$ this group is

$$H^1(X, \mathcal{O})/H^1(X, 2\pi i \mathbf{Z}) \cong \mathrm{Jac}(X).$$

For Z of degree 0, this is the classical Abel-Jacobi map.

Grothendieck pointed out that if you can define the Chern class of an arbitrary line bundle, then you can define the Chern class of an arbitrary vector bundle. So we can construct the cycle class from the Chern class of a line bundle, in terms of the Chern classes of the structure sheaf.

5 The regulator map

5.1 Examples

We discussed two examples of motivic cohomology last time.

1. For C a curve, we looked at $H^2_{\mathcal{M}}(C, \mathbf{Q}(2))$.

   A class is represented by $\sum f_i \otimes g_i$ where $f_i, g_i \in \mathbf{Q}(C)^*$ where $\prod_i \{f_i, g_i\}_P = 1$ for all tame symbols P.

2. For S a surface, we looked at $H^3_{\mathcal{M}}(S, \mathbf{Q}(2))$.

   A class is represented by $\sum (D_i, f_i)$ where D_i is a divisor and $f_i \in \mathbf{Q}(D_i)^*$ such that $\sum \mathrm{Div}(f_i) = 0$.

In the second case, the regulator is a map

$$\mathrm{reg}\colon H^3_{\mathcal{M}}(S, \mathbf{Q}(2)) \to H^3_{\mathcal{D}}(S, \mathbf{Q}(2)).$$

In this case the target $H^3_{\mathcal{D}}(S, \mathbf{Q}(2))$ maps to $H^3(S, \mathbf{Z})$ by 0, since it's too far up in the Hodge filtration. (It is supposed to consist of classes in $F^2 H^3$, but this intersects its complex conjugate trivially, hence can't map to a non-zero class in $H^3(S, \mathbf{Z})$ since that consists of elements which are in particular *real*, i.e. stable under complex conjugation.) So

$$H^3_{\mathcal{D}}(S, \mathbf{Q}(2)) = \frac{H^2(S, \mathbf{C})}{F^2 H^2 + H^2(S, (2\pi i)^2 \mathbf{Z})}.$$

Given $\omega \in F^1 H^2$, pairing with $\alpha \in \frac{H^2(S,\mathbf{C})}{F^2H^2 + H^2(S,(2\pi i)^2\mathbf{Z})}$ defines an element of $\mathbf{C}/(\text{periods of } \omega)$. (It pairs to 0 with the $F^2 H^2$, but there are periods coming from pairing with $H^(S, (2\pi i)^2 \mathbf{Z})$.) So to give a regulator map

$$H^3_{\mathcal{M}}(S, \mathbf{Q}(2)) \to H^3_{\mathcal{D}}(S, \mathbf{Q}(2)) \otimes \mathbf{R}$$

we need to give, for an element $\sum (D_i, f_i) \in H^3_{\mathcal{M}}(S, \mathbf{Q}(2))$, linear functional taking in $\omega \in F^1 H^2$ and spiting out some invariant in $\mathbf{C}/(\text{periods of } \omega)$.

We have $\omega \in F^1 H^2$, and we want to integrate over some 2-cycle. Take $\eta = \coprod f_i^{-1}[0, \infty)$. Because of the cancellation condition on $\sum (D_i, f_i)$ this is a *loop*: $\partial \eta = 0$. We claim that η is trivial in H_1. To check that, we can integrate against a one-form. Since holomorphic forms and their conjugates generate H^1_{dR}, it suffices to consider the integral against a holomorphic 1-form θ: it is enough to see that

$$\int_\eta \theta = 0 \text{ for all } \theta \in H^0(S, \Omega^1).$$

Let $D = \bigcup D_i$. Consider integrating

$$\int_{D - \eta} \theta \wedge d \log f_i$$

where $\log f_i$ is defined by choosing a branch cut along η. This vanishes because the integrand is of type $(2,0)$, which automatically vanishes on the holomorphic curve D. On the other hand,

$$\int_{D-\eta} \theta \wedge d\log f_i = \int_{D-\eta} d(\theta \log f_i) = 2\pi i \int_{\eta} \theta$$

because $\log f_i$ changes by $2\pi i$ winding around η (where we choose the branch cut of the logarithm).

So $\eta = \partial\beta$, where β is a 2-dimensional cycle. We might try to define

$$\omega \mapsto \int_{\beta} \omega.$$

Unfortunately, this doesn't work. The right thing is instead

$$\omega \mapsto \int_{\beta} \omega - \frac{1}{2\pi i} \int_{D-\eta} \omega \cdot \log f_i.$$

If we choose a different branch of the logarithm, this changes by $2\pi i \int \omega$, which is a period of ω. If you change the choice of β, then this also changes by a period of ω.

At this point we only have a functional on 2-forms, and we have to check that it descends to cohomology. That's why we have to make this modification. Suppose $\omega \leftarrow \omega + d\varphi$ where $\varphi \in \Omega^{1,0}$. (This is enough to descend to F^1H^2.) Then the integral changes by

$$\int_{\beta} d\varphi - \frac{1}{2\pi i} \int_{D-\eta} d\varphi \log f_i.$$

By Stoke's Theorem, and the fact that $\int \varphi d\log f_i$ vanishes because the integrand is of type $(2,0)$, this is equal to

$$\int_{\partial\beta=\eta} \varphi - \frac{1}{2\pi i} \int_{D-\eta} d(\varphi \log f_i).$$

This localizes to a small neighborhood $N(\eta)$ of η:

$$\int_{\eta} \varphi - \frac{1}{2\pi i} \int_{N(\eta)} d(\varphi \log f_i)$$

and arguing by branch cuts as before, we have

$$\frac{1}{2\pi i} \int_{N(\eta)} d(\varphi \log f_i) = \int_{\eta} \varphi.$$

This shows that our functional descends to $\frac{\Omega^{2,0} \oplus \Omega^{1,1}}{d\Omega^{1,0}} \cong F^1H^2$, as desired.

6 Beilinson's Conjecture: the critical case

Let X be a (proper smooth) variety over $\mathbf{Q}$. We consider $L(H^i X, q)$. For $p = i+1$, the functional equation relates

$$L(H^i X, s) \leftrightarrow L(H^i X, p-s).$$

Thanks to this we can always assume that $p \leq 2q$, i.e. we are considering points "at the center or to its right".

Example 6.1. For $X = C$ a curve, we are interested in $L(H^1 C, 2)$. For $X = S$ a surface, we are interested in $L(H^2 X, 2)$.

In the *critical* case, the map

$$H^i_B(X, (2\pi i)^q \mathbf{Q})^+ \otimes \mathbf{C} \to H^i_{\mathrm{dR}}(X, \mathbf{Q})/F^q H^i_{\mathrm{dR}}(X, \mathbf{Q}) \otimes \mathbf{C}$$

is an isomoprhism. The prediction is then that

$$L(H^i X, q) \sim_{\mathbf{Q}^*} \det.$$

Remark 6.2. Previously we were a little sloppy about real structures and integral models; we'll be more careful about them this time.

7 Motivic cohomology

7.1 Recollections

If we are not in the critical case, the map in question is still injective, and you can "fill out" $H^i_{\mathrm{dR}}/F^q H^i_{\mathrm{dR}}$ using motivic cohomology. The sequence

$$H^i_B(X, (2\pi i)^q \mathbf{Z}) \to H^i_{\mathrm{dR}}(X, \mathbf{Q})/F^q H^i_{\mathrm{dR}}(X, \mathbf{Q}) \otimes \mathbf{C} \to H_{\mathcal{D}}(X, \mathbf{Z}(q)) \to \dots$$

The sequence continues, but terminates if $p < 2q$.

The motivic cohomology is equipped with a regulator map

$$H^p_{\mathcal{M}}(X, \mathbf{Z}(q)) \xrightarrow{\text{regulator}} H^p_{\mathcal{D}}(X_{\mathbf{C}}, \mathbf{Z}(q)).$$

Let's recall some features of motivic cohomology.

$$H^0_{\mathcal{M}}(X, \mathbf{Z}(0))$$

$$H^0_{\mathcal{M}}(X, \mathbf{Z}(1)) \qquad H^1_{\mathcal{M}}(X, \mathbf{Z}(1)) \qquad H^2_{\mathcal{M}}(X, \mathbf{Z}(2))$$

$$H^0_{\mathcal{M}}(X, \mathbf{Z}(2)) \qquad H^1_{\mathcal{M}}(X, \mathbf{Z}(2)) \qquad H^2_{\mathcal{M}}(X, \mathbf{Z}(2)) \qquad H^3_{\mathcal{M}}(X, \mathbf{Z}(2)) \qquad H^4_{\mathcal{M}}(X, \mathbf{Z}(2))$$

- The rightmost slanted line are Chow groups, e.g. $H^0(X, \mathbf{Z}(0)) = \mathbf{Z}$, $H^2(X, \mathbf{Z}(1)) = \mathrm{Pic}(X) = CH^1(X)$, $H^4(X, \mathbf{Z}(2)) = CH^2(X)$, etc.

$$\begin{array}{ccccc} & & H^0_{\mathcal{M}}(X, \mathbf{Z}(0)) & & \\ & H^0_{\mathcal{M}}(X, \mathbf{Z}(1)) & H^1_{\mathcal{M}}(X, \mathbf{Z}(1)) & H^2_{\mathcal{M}}(X, \mathbf{Z}(2)) & \\ H^0_{\mathcal{M}}(X, \mathbf{Z}(2)) & H^1_{\mathcal{M}}(X, \mathbf{Z}(2)) & H^2_{\mathcal{M}}(X, \mathbf{Z}(2)) & H^3_{\mathcal{M}}(X, \mathbf{Z}(2)) & H^4_{\mathcal{M}}(X, \mathbf{Z}(2)) \end{array}$$

- The next diagonal describes relations among cycles, i.e. is represented by cycles with functions attached,

$$\begin{array}{ccccc} & & H^0_{\mathcal{M}}(X, \mathbf{Z}(0)) & & \\ & H^0_{\mathcal{M}}(X, \mathbf{Z}(1)) & H^1_{\mathcal{M}}(X, \mathbf{Z}(1)) & H^2_{\mathcal{M}}(X, \mathbf{Z}(2)) & \\ H^0_{\mathcal{M}}(X, \mathbf{Z}(2)) & H^1_{\mathcal{M}}(X, \mathbf{Z}(2)) & H^2_{\mathcal{M}}(X, \mathbf{Z}(2)) & H^3_{\mathcal{M}}(X, \mathbf{Z}(2)) & H^4_{\mathcal{M}}(X, \mathbf{Z}(2)) \end{array}$$

- The next diagonal describes relations among relations among cycles, i.e. is represented by cycles with two functions attached, etc.

$$\begin{array}{ccccc} & & H^0_{\mathcal{M}}(X, \mathbf{Z}(0)) & & \\ & H^0_{\mathcal{M}}(X, \mathbf{Z}(1)) & H^1_{\mathcal{M}}(X, \mathbf{Z}(1)) & H^2_{\mathcal{M}}(X, \mathbf{Z}(2)) & \\ H^0_{\mathcal{M}}(X, \mathbf{Z}(2)) & H^1_{\mathcal{M}}(X, \mathbf{Z}(2)) & H^2_{\mathcal{M}}(X, \mathbf{Z}(2)) & H^3_{\mathcal{M}}(X, \mathbf{Z}(2)) & H^4_{\mathcal{M}}(X, \mathbf{Z}(2)) \end{array}$$

There is a *cycle class map*

$$H^p_{\mathcal{M}}(X, \mathbf{Z}(q)) \to H^p_{\text{ét}}(X, \mathbf{Z}_\ell(q)).$$

Remark 7.1. The parameteres p, q in the construction of motivic cohomology are confusing; you should remember them as matching up with the degree and twist parameters in étale cohomology under the cycle map.

Remark 7.2. The left half of motivic cohomology is described by the other Bloch-Kato conjecture (proved by Voevodsky) using the comparison with étale cohomology; unfortunate the right half is more interesting, incorporating for instance the Tate conjecture.

Remark 7.3. Everything on the strict right half is 0 for Spec K.

Example 7.4. For $X = \text{Spec}\ \mathbf{Q}$, we have $H^1(X, \mathbf{Z}(1)) = \mathbf{Q}^*$. This is bad, because we would like something finitely generated. This is the reason why we want to work with integral models, where one has $H^1(\mathcal{X}, \mathbf{Z}(1)) = \mathbf{Z}^*$.

Remark 7.5. There is a spectral sequence relating motivic cohomology to K-theory, which degenerates rationally. The first slanted line becomes K^0, the second becomes K^1, etc.

7.2 The Gysin sequence

For $Z \hookrightarrow X$ and $U = X - Z$, we have an exact sequence

$$\ldots \to H^{p-2c}_{\mathcal{M}}(Z, \mathbf{Z}(q-c)) \to H^p_{\mathcal{M}}(X, \mathbf{Z}(q)) \to H^p_{\mathcal{M}}(U, \mathbf{Z}(q)) \to \ldots$$

Example 7.6. For $X = \text{Spec}\ \mathcal{O}_K$, $Z = \text{Spec}\ \mathbf{F}_v$ included in X as a point $\mathfrak{p}$, and $U = \text{Spec}\ \mathcal{O}_K[1/\mathfrak{p}]$, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_{\mathcal{M}}(\mathcal{X}, \mathbf{Z}(1)) & \longrightarrow & H^1_{\mathcal{M}}(U, \mathbf{Z}(1)) & \longrightarrow & H^0_{\mathcal{M}}(\mathbf{F}_v, \mathbf{Z}) & \longrightarrow & H^2_{\mathcal{M}}(\mathcal{X}, \mathbf{Z}(1)) \\ & & \| & & \| & & \| & & \| \\ 0 & \longrightarrow & \mathcal{O}_K^* & \longrightarrow & \mathcal{O}_K[1/\mathfrak{p}]^* & \longrightarrow & \mathbf{Z} & \longrightarrow & \text{Pic}(\mathcal{X}) \end{array}$$

Here the map $\mathcal{O}_K[1/\mathfrak{p}]^* \to \mathbf{Z}$ is valuation at $\mathfrak{p}$.

Example 7.7. Let $X = E \times E$. Let $\mathcal{E}/\mathcal{O}_K$ be an integral model and $\mathcal{X} = \mathcal{E} \times \mathcal{E}/\mathcal{O}_K$. From the Gysin sequence we get

$$\begin{array}{ccccccc} H^3_{\mathcal{M}}(\mathcal{X}, \mathbf{Z}(2)) & \longrightarrow & H^3_{\mathcal{M}}(U, \mathbf{Z}(2)) & \longrightarrow & H^2_{\mathcal{M}}(Z, \mathbf{Z}(1)) & \longrightarrow & H^4_{\mathcal{M}}\mathcal{X}, \mathbf{Z}(2)) \\ & & & & \| & & \| \\ & & & & \text{Pic}(Z) & \longrightarrow & CH^2(\mathcal{X}) \end{array}$$

the map sending a divisor D on Z (a surface over $\mathbf{F}_q$) to D viewed as a codimension 2 cycle on $\mathcal{X}$.

Mildenhall showed (using primes $\mathfrak{p}$ where $E/\mathbf{F}_{\mathfrak{p}}$ is supersingular) for a CM elliptic curve E/K that $H^3_{\mathcal{M}}(E \times E, \mathbf{Z}(2))$ is infinite-dimensional. Flach, following up on Mildenhall's paper, constructed for E a modular curve *explicit* elements in $H^3_{\mathcal{M}}(E \times E, \mathbf{Z}(2))$ to show that that $CH^2(\mathcal{X})$ is torsion. Flach then used the étale cohomology class in $H^3_{\text{ét}}(E \times E, \mathbf{Z}_\ell(2))$ to annihilate the Selmer group of $\text{Sym}^2 E$.

For the correct version of Beilinson's conjecture, you should use not motivic cohomology $H^p_{\mathcal{M}}(X, \mathbf{Z}(q))$ but the *image* in $H^p(X, \mathbf{Z}(q))$ of the integral version:

$$\text{Im}\,(H^p(\mathcal{X}, \mathbf{Z}(q)) \to H^p(X, \mathbf{Z}(q))).$$

By the way, where do Flach's classes come from? If you want to produce elements in $H^3(X \times X, \mathbf{Z}(2))$ you can produce a curve inside $X \times X$. There are natural choices for these, namely Hecke correspondence. And there are natural functions on these guys whose poles we know how to control, namely modular units.

8 Beilinson's conjecture in general

8.1 Statement of the conjecture

For $X/\mathbf{C}$ a smooth projective variety, the Deligne cohomology with *real* coefficients $H^p_{\mathcal{D}}(X, \mathbf{R}(q))$ fits into an exact sequence

$$0 \to H^i(X, (2\pi i)^q \mathbf{R}) \to H^i_{\mathrm{dR}}(X_{\mathbf{C}})/F^q H^i_{\mathrm{dR}} \to H^p_{\mathcal{D}}(X, \mathbf{R}(q)) \to \dots$$

where the remaining terms are 0 if $p < 2q$. This makes sense for $X/\mathbf{C}$, but now we define a version for $X/\mathbf{R}$.

If X is a (smooth projective) variety over $\mathbf{R}$, then we extend the story by taking "conjugation fixed points".

$$0 \to \left[H^i(X, (2\pi i)^q \mathbf{R})\right]^+ \to \left[H^i_{\mathrm{dR}}(X_{\mathbf{C}})/F^q H^i_{\mathrm{dR}}\right]^+ \to H^p_{\mathcal{D}}(X, \mathbf{R}(q)) \to \dots \tag{8.1}$$

Remark 8.1. This is probably best ignored on a first reading.

Once this is done, the Beilinson regulator can be made with this as target. For X a projective smooth $\mathbf{Q}$-variety, we have a map

$$H^p_{\mathcal{M}}(X, \mathbf{Q}(q)) \to H^p_{\mathcal{D}}(X/\mathbf{R}, \mathbf{R}(q)).$$

This map is conjecturally an isomorphism after tensoring with $\mathbf{R}$ *if* you use the $\mathcal{X}$-version of the left hand side. *Assuming this*, we have an exact sequence

$$0 \to H^i(X, (2\pi i)^q \mathbf{Q})^+ \otimes \mathbf{R} \to \frac{H^i_{\mathrm{dR}}(X_{\mathbf{Q}})}{F^q H^i_{\mathrm{dR}}} \otimes \mathbf{R} \to H^p_{\mathcal{M}}(X, \mathbf{Q}(q)) \otimes \mathbf{R} \to 0.$$

All three spaces have $\mathbf{Q}$-structures, and we can compare them.

Conjecture 8.2.

$$\binom{\textit{det middle}}{\mathbf{Q}-\textit{structure}} = \binom{\textit{det left}}{\mathbf{Q}-\textit{structure}} \binom{\textit{det right}}{\mathbf{Q}-\textit{structure}}$$

That is, given a short exact sequence

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

we get a canonical isomorphism

$$\wedge^{\dim V_2} V_2 \cong (\wedge^{\dim V_1} V_1) \otimes (\wedge^{\dim V_3} V_3).$$

8.2 Real structure

There are 3 involutions on $H^*(X_{\mathbf{C}}, \mathbf{C})$ where X is defined over $\mathbf{R}$:

1. c_B: the complex conjugation on the coefficient group $\mathbf{C}$. This is an antiholomorphic involution fixing $H^*_B(X_{\mathbf{C}}, \mathbf{R})$.

2. F_∞: the complex conjugation on complex points $X(\mathbf{C}) \to X(\mathbf{C})$, which induces $F^*_\infty : H^*_B(X_{\mathbf{C}}, \mathbf{C})$. This is a $\mathbf{C}$-*linear* involution.

3. $c_{\mathrm{dR}} := c_B F_\infty$, the product of the two. This is the complex structure on $H^*_{\mathrm{dR}}(X_{\mathbf{C}}, \mathbf{C})$ with respect to the real structure $H^*_{\mathrm{dR}}(X_{\mathbf{R}}, \mathbf{R})$.

The $+$ in (**??**) means *fixed points under* c_{dR}, which produces $H^i_{\mathrm{dR}}(X_{\mathbf{R}})/F^q H^i_{\mathrm{dR}}(X_{\mathbf{R}})$.

Example 8.3. $H^i(X, (2\pi i)^q \mathbf{R})^+$ is the fixed points for $c_B F_\infty$, hence the $(-1)^q$ eigenspace of F_∞ on $H^i(X, (2\pi i)^q \mathbf{R})$.

Remark 8.4. What happens when $p = 2q$? Here the L-function can vanish, so you want to compute its derivative. You also need to modify using the height pairing on $CH^q(X)$.